UGPHS - 10

Block - 1

Ordinary Differential Equations

UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGPHS-10

Mathematical Methods in Physics-II

Block

1

ORDINARY DIFFERENTIAL EQUATIONS

MATHEMATICAL METHODS IN PHYSICS-II

It is a truism that nothing is permanent except change. As you know, change in the physical world manifests itself in a variety of forms. Whether you consider seasonal variation in atmospheric temperature, objects in motion, or the flow of current in electrical circuits, you come across parameters that change in time and/or space. You can think of many more examples of change: for instance, the volume of a gas changes with applied pressure, the fuel in an automobile's tank changes with distance travelled, and so on. As a student of physics, you know by now that the change in such functions is represented in terms of the 'rates of change' or the derivatives of these functions with respect to some variables.

In your school courses you have studied Newton's laws of motion and the law of universal gravitation. By postulating the laws of mechanics, Newton was able to study the motion of a particle by an equation which involved an unknown function (displacement or velocity) and its first or higher order derivatives. For instance, consider an object falling near the earth's surface. If we consider that only gravity is acting on the object, Newton's laws lead to a model of its motion in which the object's *acceleration is constant.* The mathematical statement of this model is

$$\frac{dv}{dt} = -g$$

where v is the object's speed and g, the magnitude of acceleration due to gravity.

If the object is at a distance x from the earth's surface at an instant t, then $v = \frac{dx}{dt}$ and the equation can be written as

$$\frac{d^2x}{dt^2} = -g.$$

You know that we can solve such an equation to determine the object's position (*which is an unknown variable*) as a function of time (*which is a known variable*). So the function $x(t)$ is an unknown function which can be determined by solving this equation.

Such equations involving unknown functions and their derivatives are called **differential equations**.

Differential equations serve as useful tools in the study of change in the physical world. Most of the general laws of nature in physics, chemistry, biology, astronomy, engineering and many other areas find their most natural expression in the language of differential equations. It is precisely for these reasons that in this 2-credit course on Mathematical Methods in Physics-II, we have focused our attention entirely on **differential equations.**

This course is presented in two blocks. Block 1 deals with **ordinary differential equations,** i.e. differential equations in which the unknown function depends on only one variable. Differential equations involving unknown functions which depend on more than one variable are called **partial differential equations.** You will study such equations in Block 2.

Our emphasis in this course will be on studying various methods of solving ordinary and partial differential equations with particular reference to their applications in physics. After studying this course, you may like to go in for a more rigorous mathematical treatment of differential equations. In that case, our advice to you is to study the mathematics course MTE-08 entitled 'Differential Equations'. Finally, a word about how best to study this course.

Study Guide

In order to be able to study this course effectively, you must have an adequate background of calculus. Integration is an important tool in solving differential equations. You may, therefore, find it useful to offer the mathematics course MTE-01 on calculus before studying this course. Or else, you must brush up the calculus you studied in your +2 classes. We would also advise you to keep the course materials of the physics courses PHE-01, PHE-02 and PHE-04 handy, as we will be referring to them time and again in this course. It will also be helpful if you study the mathematics course MTE - 07 entitled ' Advanced Calculus' along with this course.

Once again, we repeat what we have said in our earlier courses. You must *acquire the skill* of effectively using the knowledge imparted here *through constant practice*. Only solving problems yourself will instil enough confidence. Always study with a paper and pencil in hand. Work through the text and the solved examples. Solve self-assessment questions (SAQs) and terminal questions given in the unit. **You will need logarithmic and trigonometric tables or a calculator for your calculations.** Resist the temptation of glancing at the answers of SAQs and terminal questions given at the end of each unit before working out the problems !

You are expected to put in a total of 60 hours work in this course. Of these, 45 to 50h, on an average, would be needed to study the print material, which includes solving the SAQs and terminal questions in the units. The two blocks are almost equally demanding in terms of content density. So you would need to spend about 22 to 25h on each one of them. The remaining time is intended for assignments, counselling sessions, audio and video programmes. In the margin beside the SAQs and the terminal questions of each unit, we have also indicated the time you should allow for solving these problems. However, all these numbers are for the sake of guidance only. Your background knowledge and capability would determine your actual study time.

We hope that you will enjoy studying this course. We wish you success.

BLOCK 1 ORDINARY DIFFERENTIAL EQUATIONS

Introduction

How is the velocity of a rocket affected by the amount of fuel burned? What is common between the suspension system of an automobile and an electric circuit? What is the shape of a power line hanging between two poles? How long will it take a polluted lake to return to its natural state, once man-made pollution is stopped? If we wish to seek the answers to such questions, we would need to study ordinary differential equations.

Indeed, we can answer such real-life questions by constructing appropriate mathematical descriptions of the phenomena with the help of ordinary differential equations. For example, consider the motion of a pendulum. From this concrete physical situation, we pass on to an '*idealised*' physical model (a bob attached to a massless string, a pivot of zero friction and zero air resistance, etc.). Next, we use physical laws (Newton's laws of motion in this case) to construct a mathematical description of the idealised model. Solutions of the mathematical problems and their comparison with physical results measured by careful experiments lead us to reasonable answers. This process is termed as '**mathematical modelling**'.

In order to effectively model physical phenomena (using ordinary differential equations), you must acquire sufficient understanding of the concepts related to ordinary differential equations. Moreover, you should be able to solve the equations so obtained. To enable you to master this art, we present the appropriate mathematics needed for this purpose in the first three units of this block.

In Unit 1 we present the basic definitions and the classification of differential equations. You will also learn some of the methods of solving what are known as the 'first order' ordinary differential equations. In Units 2 and 3, you will study the methods of solving 'second-order' ordinary differential equations. Armed with the necessary skills and the knowledge of these methods, you should be able to use them to model simple phenomena in the changing physical world. This forms the subject of Unit 4.

Ordinary differential equations originated in the works of Isaac Newton (1642–1727) and Wilhelm Leibnitz (1646–1716). This area of knowledge was further enriched by several mathematicians in the past three centuries. Among others, Fermat, the Bernoullis, Euler, Riccatti and Clairaut need special mention. Block 1 brings to you a relevant version of a body of knowledge produced by these great minds.

Before you start studying the units, we would like to refresh your memory about the abbreviations being used in the text. Sec. x.y stands for Section y of Unit x. Similarly, Fig. x.y stands for Figure y of Unit x and Eq. (x.y) for Equation y of Unit x. Thus, Sec. 1.4 is the fourth section in Unit 1, Fig. 2.1 is the first figure in Unit 2, Eq. (3.10) is the tenth equation in Unit 3, and so on.

The units are not of equal length. On an average, Unit 1 should take 8h, Units 2 and 3, 5h each and Unit 4, 6h to study. We hope you will enjoy studying this block.

Acknowledgment

We wish to thank Sh. Sunder Singh and Gopal Krishan for secretarial assistance and word processing.

UNIT 1 FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Structure

1.1 INTRODUCTION

You have read about the motion of a projectile and the motion of a rocket in your Elementary Mechanics (PHE-01) course. You know that the velocity of the projectile is affected due to air resistance and the velocity of the rocket is affected by the quantity of fuel burnt. But can we find out mathematically, *how* the velocity of a projectile is affected due to air resistance? And *how* does the quantity of fuel burnt by a rocket affect its velocity? A similar question may also be raised regarding an environmental issue. You may be aware that the oil slick formed in the Persian Gulf during the Gulf War of 1991 posed the danger of serious environmental pollution. Now, how long will it take the Persian Gulf to return to its natural state once the oil slick is completely checked? We can obtain an answer to these and many other questions pertaining to different situations by framing and subsequently solving what is known as the first order ordinary differential equation for the concerned system. In this unit you will be studying first order differential equations as they find many applications in physics. You have already dealt with a few differential equations in the "Oscillations and Waves" course (PHE-02).

Here we will first discuss what is meant by a differential equation (henceforth referred to as a DE) through some simple examples. You will then learn to classify DEs in various ways. Next you will learn what is meant by the solution of a DE.

Our ultimate aim is to learn the methods of solving DEs. In this unit we will discuss various methods for solving first order ordinary differential equations. You will learn to solve them by the method of separation of variables and the method of substitution. You will also learn to solve exact equations. Next you will learn the technique of converting an inexact equation into an exact equation. This will enable you to solve first order linear ODEs. In Unit 4, you will study about the applications of some of these equations in physics.

In the next unit we will take up the study of second order ordinary differential equations.

Objectives

After studying this unit you should be able to

- define the general solution and the particular solution of a differential equation
- solve first order ordinary differential equations reducible to separable forms
- solve an exact equation
- solve first order linear ordinary differential equations by the method of integrating factors
- solve ordinary differential equations reducible to first order.

1.2 WHAT IS A DIFFERENTIAL EQUATION?

Ernest Rutherford (1871–1937) did much of the early work on characterising radioactivity. Recall from Unit 8 of PHE-01 that Rutherford also proposed the nuclear model of the atom.

You must have read about the phenomenon of radioactivity in your school science courses. The principle of radioactive decay was discovered by the French scientist Henri Becquerel in the year 1896. He was able to establish experimentally that the rate at which the atoms of a radioactive substance disintegrate is proportional to the number of atoms (N) present in it. Now let us try to express this idea mathematically. We can express the rate of disintegration of atoms as $\left(-\frac{dN}{dt}\right)$, where t represents time. The negative sign appears because N decreases with t (so that $\frac{dN}{dt}$ is negative). Now according to Rutherford this rate is proportional to N. So we have

$$-\frac{dN}{dt} = \lambda N, \qquad \text{where } \lambda \text{ is a constant}$$

or

$$\frac{dN}{dt} + \lambda N = 0 \tag{1.1}$$

Using Eq. (1.1) we may obtain a relation between the independent variable t and the dependent variable N. Notice that Eq. (1.1) contains terms involving N and its **ordinary derivative** with respect to time. You have also come across equations involving higher order ordinary derivatives of the dependent variable with respect to the independent variable. For example, you have studied the one-dimensional equation of motion of a linear harmonic oscillator in Block 1 of PHE-02 given as

$$m\,\frac{d^2x}{dt^2} + kx = 0 \tag{1.2}$$

w.r.t. stands for 'with respect to'.

where m is the mass of the oscillator and k is the force constant. Eq. (1.2) has terms involving x and its second derivative w.r.t. time. Notice that Eqs. (1.1) and (1.2) involve only ordinary derivatives. Let us consider another example. Suppose a current i flows through an electric circuit for an infinitesimal duration of time, dt. Then the charge that flows during this time is given by

$$dq = i\,dt \tag{1.3}$$

Eq. (1.3) involves the differentials dq and dt. Equations like (1.1) to (1.3) are called **ordinary differential equations** (ODEs). More precisely,

> An equation which contains differentials or only ordinary derivatives of one or more dependent variables w.r.t. a *single independent variable* is said to be an **ordinary differential equation.**

Now, you have also studied the one-dimensional wave equation in Unit 6 of PHE-02 given by

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2\psi}{\partial t^2} \tag{1.4}$$

You can know more about partial derivatives by studying Units 2 and 3 of PHE-04 (Mathematical Methods in Physics-I) or Sec. 5.2 of Block 2 of this course.

Here ψ is a wave function and v is the wave speed. Eq. (1.4) involves second order **partial derivatives** of ψ with respect to the variables x and t. Equations like (1.4) are called **partial**

differential equations (PDEs) as they involve partial derivatives of one or more dependent variables w.r.t. *two or more independent variables.* In this block we shall deal only with ODEs. We shall take up the study of PDEs in Block 2.

We can now give a general definition of a differential equation:

> An equation containing the derivatives or differentials of one or more dependent variables with respect to *one or more independent variables* is said to be a **differential equation.**

You must now identify a few ODEs and PDEs in the following SAQ.

SAQ 1

Spend 5 min

Ten equations from various areas of physics are listed below. Identify the ordinary and partial differential equations.

i) $$\frac{d^2y}{dt^2} = -g$$

ii) $$y = u_0t - \frac{1}{2}gt^2$$

iii) $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\,\theta = 0$$

iv) $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

v) $$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{q}{C} = E(t)$$

vi) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\alpha^2}\,\frac{\partial u}{\partial t}$$

vii) $$u = A\sin(x-\omega t) + B\cos(x-\omega t)$$

viii) $$\frac{dT}{dt} = K(T-T_0)$$

ix) $$m\,\frac{dv}{dt} = mg - kv$$

x) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{\partial^2 u}{\partial z^2} = 0$$

Now there are three major aspects in the study of ordinary differential equations. These are the formation of an ODE, its solution and its application. These aspects are different for different kinds of ODEs. So before you study these aspects you must know how ODEs are classified. This is the subject of Sec. 1.3. **We expect that after studying Sec. 1.3 you ought to be able to classify an ODE just by looking at it. So study it carefully and thoroughly.**

1.3 CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equations are classified in a number of ways as shown in Table 1.1. We will be using Table 1.1 quite a lot in our discussion. So, first a word about how to use it. As you can see, the second column of the table lists several examples of ODEs. And the first row lists the various ways in which ODEs are classified. Now as you study this section it would be better if you concentrate on the particular example and the way of classification being discussed in the text. During that discussion, ignore the rest of the information presented in the table. Do you see the blank spaces in the table? You will be asked to fill them up once you have studied this section! Let us now continue our study.

The most fundamental way of classifying ODEs is on the basis of their order and degree.

Order and degree of an ordinary differential equation

> The order of an ODE is the order of the highest derivative appearing in it.
>
> The degree of an ODE is the power of the highest order derivative appearing in the equation, after it has been expressed in a form such that no derivatives have fractional or negative powers.

Let us consider equation(4) in Table 1.1. What are the order and degree of this equation? The highest derivative is the second order derivative y''. So its order is 2. Now let us remove the fractional power 1/2 in the equation by squaring it. Then, the power of y'' is 2, so that the degree of this equation is 2. Likewise you can verify the order and degree of the equations (1) and (3) in Table 1.1.

So you have learnt to classify an ODE in terms of its order and degree. We can also classify ODEs as linear or non-linear.

Linear and nonlinear ordinary differential equations

Consider equation (3) of Table 1.1. In this equation the function y and its derivatives are all of degree 1. It does not contain products like yy', yy'', $y'y''$ etc. It also does not involve any transcendental functions like $\sin y$, $\ln y$ etc. It is an example of a **linear** ODE. We call an ordinary differential equation **linear** when the following conditions are fulfilled:

Any function which cannot be expressed as a solution of a polynomial equation of the form

$P_0(x)u^n + P_1(x)u^{n-1} + \ldots$

$P_{n-1}(x)u + P_n(x) = 0$

is called a **transcendental function.** The logarithmic, trigonometric, hyperbolic functions and their corresponding inverses are examples of transcendental functions.

> i) The unknown function and its derivatives occur only to the first degree
> ii) In the equation there are no products involving either the unknown function and its derivatives or two or more derivatives.
> iii) There are no transcendental functions involving the unknown function or any of its derivatives.

An nth-order ordinary differential equation, linear in y, may be expressed as

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \ldots + a_1(x)y' + a_o(x)y = f(x) \qquad (1.5)$$

Here f and the coefficients $a_0, a_1, \ldots, a_n$ are functions of x only, on some interval of x, and $a_n(x) \neq 0$ on that interval. In writing Eq. (1.5), we have adopted the notation

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \ldots, y^{(n)} = \frac{d^ny}{dx^n}.$$

A differential equation that is *not* linear is said to be **nonlinear.** You can verify that ODE(1) in Table 1.1 is linear and ODEs(4) and (7) are nonlinear. A linear ODE can further be classified as homogeneous or nonhomogeneous.

Homogeneous and nonhomogeneous ordinary differential equations

> If in the RHS of Eq. (1.5), we have $f(x) = 0$, then it is a **homogeneous** linear ODE and if $f(x) \neq 0$, then it is called **nonhomogeneous.**

For example, ODEs (1) and (6) are nonhomogeneous because $f(x) = E$ and $f(x) = e^x$, respectively. Since $f(x) = 0$ for ODE (3), it is homogeneous.

Note: The term homogeneous has another meaning when used for a first order ODE. We will explain what a homogeneous first order ODE means in Sec. 1.5.2 of this unit.

In many books on ODEs, you will come across the term 'inhomogeneous ODEs'. It has the same meaning as the term 'non-homogeneous ODEs'.

So you have learnt to classify ODEs in four ways: by way of (i) order (ii) degree (iii) linearity/nonlinearity (iv) homogeneity/nonhomogeneity. Henceforth, in your study of ODEs, you must make it a habit to classify an ODE the moment you see it. This means that you must be able to tell what its order and degree is, and whether it is linear or nonlinear. Moreover, if it is linear you must be able to say whether it is homogeneous or nonhomogeneous. For example, ODE (1) in Table 1.1 is a linear, nonhomogeneous first order ordinary differential equation of degree 1.

Table 1.1 : Classification of ODEs

No.	ODE	Order	Degree	Linear (L)/ Non-linear (NL)	Non-homo-geneous (NH)/ Homo-geneous (H)	Remarks (if any)
(1)	$L\frac{di}{dt} + Ri = E$	1	1	L	NH	This can be made homogeneous by taking E to the LHS and making the substitution $i_1 = i - \frac{E}{R}$.
(2)	$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F\cos\omega t$					
(3)	$x^2y'' + 2xy' + y = 0$	2	1	L	H	
(4)	$(1 + (y')^2)^{1/2} = y''$	2	2	NL	-	It is nonlinear because its degree is 2. Since it is nonlinear, the question of classifying it as homogeneous or non-homogeneous does not arise.
(5)	$(y'')^3 + xy' - y = 0$					
(6)	$y''' + y = e^x$				NH	It is nonhomogeneous because of e^x on the RHS.
(7)	$y'' + 7y = \sin y$	2	1	NL	-	It is nonlinear because $\sin y$ is a transcendental function in y.
(8)	$y'' - 2y' + 3y = 0$					

You may now like to work out an SAQ on what you have learnt so far.

SAQ 2

Spend 10 min

Fill the blank boxes in Table 1.1. You need not fill in the 'Remarks' column.

So far you have learnt the basic terminology associated with ordinary differential equations. In the process you have also learnt to classify ODEs. Now your major goal in this course is to be able to solve differential equations appearing in physics. But before trying to solve DEs you must understand what is meant by the solution of a differential equation.

1.4 WHAT IS A SOLUTION OF A DIFFERENTIAL EQUATION?

Let us consider the following ODE:

$$y'' + y = 0 \tag{1.6}$$

Now if we put

$$y = \sin x \tag{1.7}$$

we have

$$y' = \cos x \quad \text{and} \quad y'' = -\sin x$$

and Eq. (1.6) becomes an identity. In that case we call the function $y = \sin x$ a **solution** of Eq. (1.6). This solution exists for every x in the interval $(-\infty, \infty)$. Now consider another example. The equation

$$y^2 + x = 4 \tag{1.8}$$

is a solution of the ODE $2yy' = -1$. We can verify this by differentiating Eq. (1.8). We get

$$2yy' + 1 = 0$$

which is identical to the given ODE. Now Eq. (1.8) may also be expressed as

$$y = \pm\sqrt{4 - x}$$

which essentially defines two functions. Each one of them is defined for every x in the interval $(-\infty, 4)$. You can see that this interval is different from the previous one. So depending on the context, the interval on which the solution of a DE exists can be any of the intervals $(-\infty, \infty)$, (a, ∞), $(-\infty, a)$, $[a, b]$, (a, b), and so on. We can now define the solution of an ODE as follows:

A function

$$y = \phi(x)$$

is called a **solution** of a differential equation in y on some interval, say, $a \le x \le b$, if $\phi(x)$ is defined and differentiable throughout that interval and is such that the equation becomes an identity when y is replaced by $\phi(x)$ in the DE.

We also say that the differential equation is satisfied by $y = \phi(x)$. The two types of solutions, (1.7) and (1.8), are typical of those we encounter in ODEs. In Eq. (1.7), we have y given as an explicit function of x. Such a solution is called an **explicit solution.** And Eq. (1.8) is an implicit relation between x and y. We say that Eq. (1.8) is an **implicit solution.** In other words, a solution of a differential equation in the form

$$G(x,y) = 0 \tag{1.9}$$

is called an **implicit solution.**

Now if you look back at Eq. (1.6), you will be able to verify easily that $y = \cos x$ is also a solution of that differential equation. In fact, a differential equation may have many solutions. The principal task of the theory of differential equations is to find all the solutions of a given differential equation. Then we investigate the physical significance of these solutions. Let us study about that in some detail now.

1.4.1 General Solution and Particular Solution

We have already illustrated through Eq. (1.6) that a differential equation may have many solutions. Let us take another example. We consider the differential equation

$$y' = \cos x \tag{1.10}$$

You may easily verify that each of the functions

$$y = \sin x,\ y = \sin x + 5,\ y = \sin x - 9,\ y = \sin x + \frac{5}{8}$$

is a solution of Eq. (1.10). You can express them generally as

$$y = \sin x + C \tag{1.11}$$

where C is an *arbitrary constant.*

Eq. (1.11) is called a **general solution** of Eq. (1.10). Eq. (1.11) can yield any number of

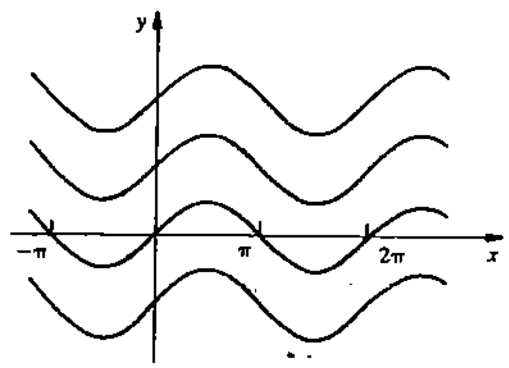


Fig.1.1 : Solutions of the differential equation: $y' = \cos x$

solutions of Eq. (1.10). We have represented a family of such solutions graphically in Fig. 1.1.

Likewise, you may also verify that

$$y = A\cos x + B\sin x \qquad (1.12)$$

where A and B are arbitrary constants, satisfies Eq. (1.6).

> A solution involving arbitrary constant(s) is known as the **general solution.**

You must have observed that Eq. (1.10) is a first order differential equation and its general solution (1.11) has one arbitrary constant. Eq. (1.6) is a second order differential equation and its general solution (1.12) has two arbitrary constants. So **the number of arbitrary constants appearing in the solution of a differential equation is equal to its order.**

Let us now impose the following condition on Eq. (1.11): $y = 0$ when $x = 0$. Then we get from Eq. (1.11) that

$$0 = 0 + C \quad \text{or} \quad C = 0 \quad \text{and} \quad y = \sin x.$$

You will encounter the term 'particular integral' in Unit 2. Do not confuse it with the term 'particular solution' being discussed here.

So by imposing a condition on Eq. (1.11), we can assign a specific value to C. The solution thus obtained is called a **particular solution.**

> If a definite value can be assigned to each arbitrary constant appearing in a general solution, then we get a **particular solution.**

For example, $y = \sin x + 2$ is a particular solution of Eq. (1.10) and $y = 2\cos x + 3\sin x$ is a particular solution of Eq. (1.6).

As you have just seen, particular solutions are determined from a general solution by imposing condition(s) on the solution function. Now, two questions arise in this connection.

(i) Does a particular solution always exist?

(ii) If it exists, is the solution unique?

Let us now discuss these questions briefly. This discussion is just to make you aware of such questions. We will not be going into the details here.

In the text books on ODEs, you will also come across another kind of solution of an ODE, the **singular solution**. It is that solution of an ODE which contains no arbitrary constant itself. Moreover, it cannot be obtained by assigning any value to the arbitrary constant in the general solution. For example, $y = \frac{x^2}{4}$ is a singular solution of the ODE $y'^2 - xy' + y = 0$. It contains no arbitrary constant and cannot be obtained by imposing a condition on the general solution $y = cx - c^2$ of this ODE.

1.4.2 Existence and Uniqueness of a Particular Solution

In this discussion we will be using certain new terms which we would like to explain first. You have learnt that a general solution of an nth-order ODE contains n arbitrary constants. So to obtain a particular solution of an nth order ODE, we have to impose n conditions on the solution function and its derivatives. We can then solve the n simultaneous linear equations so obtained for the n arbitrary constants. Now there are two common methods of specifying the conditions. We will mention them briefly.

> 1) If the conditions on the solution of a DE, or its derivatives, are specified for a **single** value of the independent variable, they are called **initial conditions.** The DE with its initial conditions is called an **initial-value problem (IVP).**
>
> 2) If the conditions on the solution of a DE, or its derivatives are specified for **two or more values** of the independent variable, they are called **boundary conditions.** The DE with its boundary conditions is called a **boundary-value problem (BVP).**

For example,

a) $y' + 2y = 3$, with the initial condition $y(0) = 1$, is a first- order initial-value problem.

b) $y'' + 3y = 0$, with the initial conditions $y(1) = 2$, and $y'(1) = -8$, is a second-order initial-value problem.

c) $y'' - 2y' + 6y = x^3$ with the boundary conditions $y(0) = 2$, $y(1) = -1$ is a second-order boundary-value problem.

Let us now deal with the questions of existence and uniqueness. Let us consider an example. You have seen that the general solution of $y'' + y = 0$ is given by

$$y = A\cos x + B\sin x$$

Now what can we say about the solution of the boundary-value problem $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 2$? Using the given boundary condition in the general solution, we get

$$0 = A\cos 0 + B\sin 0 \qquad \text{and} \qquad 2 = A\cos\pi + B\sin\pi$$

The first equation yields $A = 0$, while the second equation yields $A = -2$. Since A cannot be equal to both 0 and -2, no solution is possible for this boundary-value problem (see Fig. 1.2a). You may easily verify that the boundary value problem

$$y'' + y = 0, \qquad y(0) = y(\pi) = 0,$$

will yield $A = 0$ unambiguously. However, no value gets assigned to B. Thus, there are infinitely many solutions represented by $y = B\sin x$ (see Fig. 1.2b).

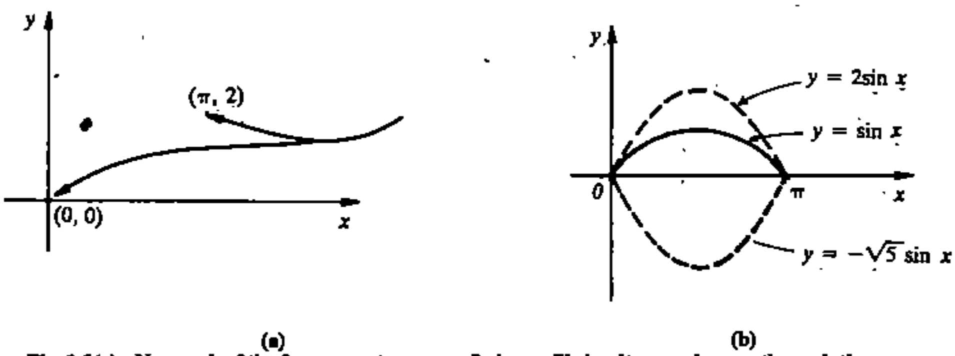


Fig. 1.2(a) : No graph of the form $y = A\cos x + B\sin x$ will simultaneously pass through the points $(\pi, 2)$ and $(0, 0)$; (b) curves of the form $y = B\sin x$ on the interval $[0, \pi]$.

From the above examples we understand that a solution may not exist for a boundary-value problem. And if it does, the solution may not be unique. In fact, there is no simple theory which can ensure a unique solution to a boundary-value problem. However, there exists a theorem that specifies necessary conditions for which a unique solution will exist for a first order initial value problem. Since our aim is just to sensitise you to these concepts, we will not go into these details here. If you are interested in such details you may like to study Sec. 1.3 (Unit 1) of the Mathematics course MTE-08 entitled Differential Equations. As a matter of fact, a more advanced course in differential equations at the post graduate level would focus on considerations of existence, uniqueness and general behaviour of solutions of DEs. So, henceforth in this block we shall consider only those DEs for which a solution exists.

So far you have learnt what is meant by the general and particular solution of an ODE. You also have some idea of what is meant by the existence and uniqueness of solutions. We will now discuss some properties associated with the solutions of linear ODEs. You will find these properties very useful when you actually solve linear ODEs.

1.4.3 General Properties of the Solutions of Linear ODEs

Let us consider the following ODEs

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0 \tag{1.13a}$$

and

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x) \tag{1.13b}$$

Eqs. (1.13a) and (1.13b) are both linear second order ODEs. The former is homogeneous and the latter is nonhomogeneous. We shall discuss the properties with reference to these second order ODEs primarily for the sake of simplicity. The other reason behind this is that you will come across linear second order ODEs quite often in physics. You will realise that *the properties being discussed are true for linear ODEs of any order.*

Properties of the solutions of linear ODEs

i) $y = 0$ is a solution of Eq. (1.13a). This is called the **trivial solution.**

ii) If y_1 and y_2 are linearly independent solutions of Eq.(1.13a), then $u = c_1y_1 + c_2y_2$ is also a solution of Eq. (1.13a), where c_1 and c_2 are constants.

iii) If y_1 is a solution of (1.13a) and y_2 is a solution of (1.13b), then $z = y_1 + y_2$ is a solution of (1.13b).

iv) The difference $(y_1 - y_2)$ of two solutions y_1 and y_2 of (1.13b) is a solution of (1.13a).

Two functions $y_1(x)$ and $y_2(x)$ are said to be **linearly dependent** on an interval I where both functions are defined, if and only if we can find nonzero constants k_1 and k_2 such that $k_1 y_1(x) + k_2 y_2(x) = 0$ for all x on I. Thus, linearly dependent functions are proportional on I. If the functions are not proportional on I, they are said to be **linearly independent.** Thus, for linearly independent functions the relation: $c_1 y_1 + c_2 y_2 = 0$ is satisfied only for $c_1 = c_2 = 0$.

The proofs of these properties are fairly straightforward. You can work them out yourself if time permits. You may now like to work out an SAQ on what you have learnt in this section.

SAQ 3

Spend 10 min

(a) Verify that $x^2 + y^2 - 1 = 0$ is a solution of the differential equation $yy' = -x$ on the interval $[-1,1]$. State whether this solution is implicit or explicit.

(b) Verify that

$$y = Ax + \cos A, \text{ for constant } A$$

is the solution of the ODE

$$y = xy' + \cos y'$$

Identify the type of the solution (i.e., whether general or particular).

Now that you have learnt the meaning and the basic properties of the solutions of linear ODEs, you can study the different methods of solving first order ODEs. We have given several SAQs in the subsequent discussion for the sake of practice. You must do them if you want to grasp these methods. We shall start with the ODEs that can be reduced to separable forms.

1.5 EQUATIONS REDUCIBLE TO SEPARABLE FORM

For several first order ODEs, you will find that the equation may be rewritten so that the concerned variables stand separated. It can then be solved by working out the integrals of the separated parts. Let us see how.

1.5.1 Method of Separation of Variables

Let us consider a general first order ordinary differential equation of the form

$$y' = f(x, y) \tag{1.14}$$

If we can write $f(x, y)$ as

$$f(x, y) = \frac{M(x)}{N(y)} \tag{1.15}$$

then Eq. (1.14) takes the form

$$M(x)dx - N(y)\,dy = 0 \tag{1.16}$$

The forms (1.14) and (1.16) are interchangeable. For example, the equations

$$y' = \frac{y}{1+x} \quad \text{and} \quad (1+x)\,dy - y\,dx = 0$$

mean the same thing. An ODE of the form $y' = \dfrac{M(x)}{N(y)}$ **is said to be a separable equation.**

In the form (1.16), the variables x and y are separated. On integrating Eq. (1.16), we get

$$\int M(x)\,dx - \int N(y)\,dy = C \tag{1.17}$$

where C is an arbitrary constant. Eq. (1.17) is the required solution of the ODE and it can be obtained if we can work out the integrals. Let us consider an example.

Example 1

Solve the equation $\frac{dy}{dx} = -\frac{5(y^2+2)}{xy}$

Solution

Comparing $f(x, y)$ of this equation with the form (1.15), we have

$$M(x) = -\frac{5}{x}, N(y) = \frac{y}{y^2+2}$$

So, we can rewrite it in the form (1.17) as

$$5\int \frac{dx}{x} + \int \frac{y\,dy}{y^2+2} = C$$

or
$$5 \ln|x| + \frac{1}{2}\ln|y^2+2| = C$$

$$\therefore \ln|x|^5 |y^2+2|^{1/2} = C$$

or
$$x^5(y^2+2)^{1/2} = C_1, \text{ where } C_1 = \exp(C)$$

is the required solution.

Let $I = \int \frac{y\,dy}{y^2+2}$

We put $u = y^2 + 2$

$\therefore du = 2y\,dy$

$I = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|y^2+2|$

Remember that

$\int \frac{dx}{x} = \ln|x|$.

Checking the solution

Let us quickly check the solution of the ODE in Example 1. Differentiating the solution we get

$5x^4(y^2+2)^{1/2} + \frac{x^5}{2}(y^2+2)^{-1/2}\,2yy' = 0$

or $y' = -\frac{5x^4(y^2+2)^{1/2}}{x^5 y (y^2+2)^{-1/2}}$

$= -\frac{5}{xy}(y^2+2)$

which is the given ODE. Hence, the solution is correct.

Note : An important step in solving an ODE is to **check the solution.** You should always substitute the solution back into the ODE and check whether you get an identity. Sometimes, you get the ODE by simply differentiating the solution as in the case of Example 1.

Thus, we see that the method of separation of variables essentially consists of the two steps summarised below.

> **The method of separation of variables**
>
> **Step 1:** Write the first order ODE in the form $y' = \frac{M(x)}{N(y)}$ or
>
> $$M(x)\,dx - N(y)\,dy = 0$$
>
> **Step 2:** Integrate to obtain the solution.

SAQ 4

Spend 5 min

(a) Find the general solution of the ODE $(y + 1)\,y' + x = 0$.

(b) Solve the IVP $y' = -2xy,\ y(0) = 3$.

Remember to check the solutions.

Some ordinary differential equations may look non-separable. But on making some substitution they become separable.

Solution by the method of substitution

First we shall take up the case where substitution can be done by mere inspection of the equation. For example, let us consider the ODE, $\frac{dy}{dx} = \cos(x + y)$. The given equation is non-separable because of the factor $(x + y)$. So we put,

$$u = x + y$$

$$\therefore \quad \frac{du}{dx} = 1 + \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

Hence,

$$\frac{du}{dx} = 1 + \cos u = 2\cos^2\frac{u}{2}$$

$$\therefore \quad \frac{du}{2\cos^2\frac{u}{2}} = dx$$

Thus, we have separated the variables u and x. Now, the above equation may be rewritten as

$$\frac{1}{2}\sec^2\frac{u}{2}\,du - dx = 0$$

or $$\frac{1}{2}\int \sec^2\frac{u}{2}\,du - \int dx = C$$

or $$\tan\frac{u}{2} - x = C$$

or $$\tan\frac{x+y}{2} - x = C$$

This is the required solution. Now how about trying an SAQ?

Checking the solution

Differentiating the solution w.r.t. x, we have

$$\sec^2\left(\frac{x+y}{2}\right)\left(\frac{1+y'}{2}\right) = 1$$

or $$y' = 2\cos^2\left(\frac{x+y}{2}\right) - 1$$

$$= \cos(x+y)$$

which is the ODE being solved.

SAQ 5

Spend 10 min

Solve the ODEs:

(a) $(x-2y-1) = (x-2y+7)y'$

(b) $(1+\cos\theta)\,dr = r\sin\theta\,d\theta$

Remember to check the solutions.

Let us now study a very typical case of substitution suitable for ODEs of the form $y' = f(y/x)$, where f is a function of y/x, e.g., $(y/x)^3$, $\sin(y/x)$, etc. Let us consider an example.

Example 2

Solve the differential equation

$$(x^2+y^2)\,dx - xy\,dy = 0$$

Solution

We can rearrange the equation as

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x}$$

Now, this form suggests a substitution, $\frac{y}{x} = v$ where v is a function of x. Thus, we get

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{1}{v} + v$$

or $$v\,dv - \frac{dx}{x} = 0$$

Thus v and x are separated. On integrating, we get

$$\frac{v^2}{2} - \ln|x| = C$$

or $$x = C_1\exp(v^2/2) = C_1\exp(y^2/2x^2)$$

Checking the solution

Differentiating the solution w.r.t. x, we get

$$1 = C_1\exp(y^2/2x^2) \times \left[\frac{2yy'}{2x^2} - \frac{2y^2}{2x^3}\right]$$

or $$1 = \frac{xy}{x^2}\left[y' - \frac{y}{x}\right]$$

or $$y' = \frac{x}{y} + \frac{y}{x}$$

which is the ODE being solved.

Having solved Example 2, you may well ask : How can we find out whether an ODE can be made separable by the substitution $y = vx$? We will answer this question in the following section.

Remember that we have defined higher order homogeneous ODEs in a different manner in Sec. 1.3.

1.5.2 Homogeneous Differential Equations of the First Order

The substitution $y = vx$ can be made to separate variables in **homogeneous first order ODEs**. What is a homogeneous first order ODE? A first order ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1.18}$$

is also called **homogeneous if M and N are homogeneous functions of the same degree.** Now, what is a homogeneous function? A **function** $f(x, y)$ is said to be **homogeneous** of degree n in x and y, if, for every k,

$$f(kx, ky) = k^n f(x, y),$$

where k is a real parameter. For example,

a) $f(x, y) = x^2 + xy + y^2$ is a homogeneous function of degree 2 since

$$f(kx, ky) = (kx)^2 + (kx)(ky) + (ky)^2$$
$$= k^2(x^2 + xy + y^2) = k^2 f(x, y)$$

b) $f(x, y) = \sqrt{x - y}$ is homogeneous of degree $\frac{1}{2}$ since

$$f(kx, ky) = \sqrt{(kx - ky)} = k^{1/2}\sqrt{x-y} = k^{1/2} f(x, y)$$

c) $f(x, y) = e^{x/y} + \tan\frac{y}{x}$ is homogeneous of degree zero since

$$f(x, y) = e^{kx/ky} + \tan\frac{ky}{kx} = e^{x/y} + \tan\frac{y}{x} = k^0 f(x, y).$$

d) $f(x, y) = x^3 + y^3 + 4$ is *not* homogeneous since

$$f(kx, ky) = k^3x^3 + k^3y^3 + 4.$$

Since M and N are homogeneous functions of the same degree, say n, $\frac{M}{N}$ is a homogeneous function of degree zero, as

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{k^n M(x, y)}{k^n N(x, y)} = \frac{M(x, y)}{N(x, y)}$$

Thus, we can say for Eq. (1.18) that $\frac{dy}{dx}$ = a homogeneous function of degree zero, and it is a homogeneous first order ODE which can be solved by making the substitution $y = vx$. For example,

(a) the differential equation $y\,dx + (x + y)\,dy = 0$ is homogeneous as $M(x, y)\{= y\}$ and $N(x, y)\{= x + y\}$ are homogeneous functions of degree 1,

(b) the differential equation $y' = e^{x/y} + \cos(y/x)$ is homogeneous as y' has been expressed as a homogeneous function of degree zero.

To sum up, this method consists of the following steps.

Method of solving a homogeneous first order ODE

Step 1 : Write the ODE in the form
$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Step 2 : Determine whether $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Step 3 : Separate variables by making the **substitution** $y = vx$.

Indeed, you can now see that the ODE of Example 2 was a first order homogeneous ODE. This method of solving a first order homogeneous differential equation can also be applied to ODEs having linear coefficients. Such ODEs can be made homogeneous by a typical substitution called **linear substitution.**

ODEs with linear coefficients

Suppose the functions M and N in Eq. (1.18) are linear functions of x and y, i.e.

$$M(x, y) = a_1 x + b_1 y + c_1 \quad \text{and} \quad N(x, y) = a_2 x + b_2 y + c_2, \qquad (1.19)$$

where a_1, a_2, b_1, b_2, c_1, c_2 are constants. We can make such an equation homogeneous by making a linear substitution: $x = x' + h$, $y = y' + k$, where h and k are constants to be determined. In this case x' and y' are variables, and not derivatives. Let us illustrate this with the help of an example.

You can see that if $c_1 = c_2 = 0$ in Eq. (1.19), Eq. (1.18) is homogeneous of degree zero.

Again, if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, you can make the substitution $u = a_1 x + b_1 y$, just by inspection, as you have done in SAQ 5(a).

Example 3

Solve the differential equation

$$(y - x - 2)\, dx + (x + y + 1)\, dy = 0$$

Solution

We put $x = x' + h$ and $y = y' + k$, so that we have

$$dx = dx' \text{ and } dy = dy'$$

The equation takes the form

$$(y' - x' + C_1)\, dx' + (x' + y' + C_2)\, dy' = 0$$

where $\quad C_1 = -h + k - 2 \qquad C_2 = h + k + 1$

We take the values of h and k to be such as to simultaneously satisfy the equations $C_1 = 0$ and $C_2 = 0$, i.e.,

$$-h + k - 2 = 0 \quad \text{and } h + k + 1 = 0$$

The solution to this system of equations is $h = -\frac{3}{2}$, $k = \frac{1}{2}$. And we have

$$(y' - x')\, dx' + (x' + y')\, dy' = 0$$

or

$$\frac{dy'}{dx'} = \frac{x' - y'}{x' + y'}$$

This equation is homogeneous in x' and y'.

You may now complete this example by putting $y' = vx'$. Do not forget to express the final solution in terms of x and y using the values of h and k.

SAQ 6

Spend 10 min

(a) Complete the solution to the differential equation given in Example 3. Do not forget to check your solution.

(b) Identify the homogeneous first order ODEs from the following :

i) $x^2 \frac{dy}{dx} = y^2 - 3xy + 5x^2$

ii) $(x^2 + y^2)\, dx + (x + y)\, dy = 0$

iii) $\{y + x \sin(y/x)\}\, dx - x\, dy = 0$

iv) $xy\, dx + (x^2 + 4)\, dy = 0$

v) $x\, dx + (y - 2x)\, dy = 0$

vi) $x\, dy - (y + \sqrt{x^2 - y^2})\, dx = 0$

So far we have discussed quite a few methods of solving first order differential equations. In all cases, we had taken the general form of the equation as

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Now if the left-hand side is such that it can be expressed as $d[z(x, y)]$, then we get

$$d[z(x, y)] = 0 \qquad (1.20)$$

and the solution is

$$z(x,y) = C = \text{a constant.}$$

A first order ODE which can be expressed in the form (1.20) is called an **exact equation.** Let us study such equations.

1.6 EXACT EQUATIONS

From the definition of an exact equation, we can see that an equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

will be exact if there exists a function $z(x,y)$ for which

$$\frac{\partial z(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial z(x,y)}{\partial y} = N(x,y) \tag{1.21}$$

Then we can express the ODE as

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\partial y = 0 \tag{1.22a}$$

You must have studied about partial derivatives and total differential of a function in Unit 2, Block 1 of Mathematical Methods in Physics-I (PHE-04). Recall that the total differential of a function $f(x,y)$ is $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

or $$d[z(x,y)] = 0 \tag{1.22b}$$

and the required solution of the ODE is

$$z(x,y) = C = \text{a constant.}$$

Now, how do we find out whether a given ODE is exact or not? From Eq. (1.21), we have

$$M = \frac{\partial z}{\partial x} \quad \text{and} \quad N = \frac{\partial z}{\partial y}$$

or $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

and $$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Now we know that if $z = z(x,y)$, then $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$. In other words

$M\,dx + N\,dy = 0$ is an exact ODE if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.23}$$

Eq. (1.23) is a necessary and sufficient condition for $M\,dx + N.dy$ to be exact, provided M and N are continuous and have continuous partial derivatives.

The question now is how to solve an exact equation.

The method of solving exact equations

We can solve an exact equation in the following way. Integrating the first of Eq. (1.21) with respect to x while holding y constant we have

$$z(x,y) = \int M(x,y)\,dx + f(y) \tag{1.24a}$$

Here the arbitrary function $f(y)$ is the 'constant' of integration. To determine $f(y)$, we differentiate Eq. (1.24a) w.r.t. y and use the second of Eq. (1.21).

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{df}{dy} = N(x,y)$$

This gives $$\frac{df}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \tag{1.24b}$$

Finally, we integrate Eq. (1.24b) w.r.t. y and substitute the result in Eq. (1.24a). The solution of the ODE is $z(x,y) = C$.

Note : In the above method we could as well have started from the second of Eq. (1.21): $\frac{\partial z}{\partial y} = N(x,y)$. The analogues of Eq. (1.24a) and (1.24b) would be, respectively,

$$z(x,y) = \int N(x,y)\,dy + g(x) \tag{1.24c}$$

and
$$\frac{dg}{dx} = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \tag{1.24d}$$

Let us now illustrate this method with an example.

Example 4

Show that the differential equation

$$3x(xy-2)\,dx + (x^3+2y)\,dy = 0 \text{ is exact.}$$

Hence, solve it.

Solution

Here, $\quad M = 3x^2y - 6x, \quad N = x^3 + 2y$

$$\therefore \frac{\partial M}{\partial y} = 3x^2,\ \frac{\partial N}{\partial x} = 3x^2,\ \text{i.e. } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So the equation is exact.

Now, we have to solve the ODE. Since the ODE is exact, there exists a function $z(x,y)$, such that $dz(x,y) = M\,dx + N\,dy = 0$. We can now use either Eqs. (1.24a) and (1.24b) or Eqs. (1.24c) and (1.24d). From Eq. (1.24a) we get

$$z = \int M(x,y)\,dx + f(y) = \int (3x^2y - 6x)\,dx + f(y) = x^3y - 3x^2 + f(y)$$

Since $\quad \frac{\partial z}{\partial y} = N(x,y)$, we have

$$x^3 + \frac{df}{dy} = x^3 + 2y$$

$$\therefore \frac{df}{dy} = 2y \text{ or } f(y) = y^2 + k,$$

where k is an arbitrary constant.

Thus, $\quad z = x^3y - 3x^2 + y^2 + k$

So the required solution is $x^3y - 3x^2 + y^2 + k = C$ = a constant

or $\quad x^3y - 3x^2 + y^2$ = a constant.

Checking the solution
Differentiating it w.r.t. x, we get
$3x^2y + x^3y' - 6x + 2yy' = 0$
or $3x(xy-2) + (x^3+2y)y' = 0$
or $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$
which is the ODE being solved.

Let us summarise the method of solving an exact equation.

The method of solving an exact equation

Step 1 : Write the differential equation in the form

$M(x,y)\,dx + N(x,y)\,dy = 0$ and check to make sure that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step 2 : Evaluate (i) $z(x,y) = \int M(x,y)\,dx + f(y)$ or (ii) $z(x,y) = \int N(x,y)\,dy + g(x)$ (treating y and x, as constants in the integration processes (i) and (ii), respectively.).

Step 3 : Evaluate the arbitrary functions $f(y)$ or $g(x)$ that occur in Step 2 by putting $\frac{\partial z}{\partial y} = N(x,y)$ or $\frac{\partial z}{\partial x} = M(x,y)$.

Step 4 : Write your solution in the form $z(x,y) = C$.

You may now like to work out an SAQ on solving exact equations.

Spend 10 min

SAQ 7

Check each of the following ODEs for exactness and solve the one that is exact

(a) $(x\cos y - y)\,dx + (x\sin y + x)\,dy = 0$

(b) $(e^x + y - 1)\,dx + (3e^y + x - 7)\,dy = 0$

Remember to check the solution.

You have now learnt to solve an exact equation. But what to do when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, i.e., the equation is inexact?

In fact, an inexact ODE can be made exact by multiplying it by a suitable function $P(x,y)$ $(\neq 0)$. Such a function is called an **integrating factor** of the ODE. We can determine integrating factors for **linear** first order ODEs in a systematic way.

1.6.1 First Order Linear Differential Equations

You may recall from Sec. 1.3.2 that a first order linear nonhomogeneous differential equation defined on an interval in x has the form

$$a_1(x)\,y' + a_0(x)\,y = f(x), \tag{1.25}$$

where $a_1(x) \neq 0$. On dividing both sides of Eq. (1.25) by $a_1(x)$, we get

$$y' + p(x)y = q(x) \tag{1.26}$$

This is the **standard form** of a first order linear nonhomogeneous differential equation. We can now show that Eq. (1.26) can be solved by obtaining an **integrating factor** $\nu(x)$ which depends only on x. Now, if such a factor exists, then on multiplying Eq. (1.26) by $\nu(x)$, we should get an exact equation. In other words

$$\nu(x)y' + \nu(x)p(x)y = \nu(x)q(x)$$

must be an exact equation. We rewrite it as

$$[\nu(x)p(x)y - \nu(x)q(x)]\,dx + \nu(x)dy = 0$$

From the condition of exactness [Eq. (1.23)], we get

$$\frac{\partial}{\partial y}[\nu(x)p(x)y - \nu(x)q(x)] = \frac{\partial}{\partial x}[\nu(x)] = \frac{d\nu(x)}{dx} \tag{1.27}$$

Hence, from Eq. (1.27), we get

$$\frac{d\nu(x)}{dx} = \nu(x)p(x)$$

or

$$\frac{d[\nu(x)]}{\nu(x)} = p(x)dx$$

Integrating both sides, we get

$$\ln|\nu(x)| = \int p(x)\,dx$$

$$\therefore\ \nu(x) = \exp[h(x)], \quad \text{where } h(x) = \int p(x)\,dx \tag{1.28}$$

We have deliberately left out the constant of integration as we wish to have only one, the integrating factor $\nu(x)$. Now multiplying Eq. (1.26) by the integrating factor we get

$$e^h[y' + py] = e^h q$$

Since from Eq. (1.28), $h' = p$, we can write this equation as

$$\frac{d}{dx}[y e^h] = e^h q$$

Integrating both sides of the equation and dividing by e^h, we obtain

$$y = e^{-h}\left[\int e^h q\,dx + C\right], \quad h = \int p(x)dx \tag{1.29}$$

This is the general solution of a first order linear nonhomogeneous ODE of the form (1.26). Let us apply this method to an example.

Example 5

Solve the equation

$$L\frac{di}{dt} + Ri = E_0 \sin \omega t$$

Solution

We may rewrite the equation as

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t$$

$$\therefore \text{ Integrating factor } = \exp\left[\int \frac{R}{L}\, dt\right] = e^{Rt/L}$$

On multiplying the ODE by this factor, we get

$$e^{Rt/L}\left[\frac{di}{dt} + \frac{Ri}{L}\right] = \frac{E_0}{L} e^{Rt/L} \sin \omega t$$

$$\therefore \frac{d}{dt}\left(i\, e^{Rt/L}\right) = \frac{E_0}{L} e^{Rt/L} \sin \omega t$$

$$\therefore i\, e^{Rt/L} = \frac{E_0}{L} \int e^{Rt/L} \sin \omega t\, dt + C$$

where C is an arbitrary constant.

$\therefore$ The required solution is

$$i = \frac{E_0 \sin(\omega t - \theta)}{\sqrt{R^2 + \omega^2 L^2}}$$

Let $I = \int e^{Rt/L} \sin \omega t\, dt$

Integrating by parts, we get

$$I = e^{Rt/L}\left(-\frac{\cos \omega t}{\omega}\right) + \frac{R}{L\omega}\int e^{Rt/L} \cos \omega t\, dt$$

$$= -e^{Rt/L}\frac{\cos \omega t}{\omega} + \frac{R}{L\omega}\left[e^{Rt/L}\frac{\sin \omega t}{\omega} - \frac{R}{L\omega}\int e^{Rt/L} \sin \omega t\, dt\right]$$

or $I = \frac{e^{Rt/L}}{\omega^2 L}\left[R \sin \omega t - \omega L \cos \omega t\right] - R^2 I/\omega^2 L^2$

or $I = \frac{L e^{Rt/L}}{(R^2 + \omega^2 L^2)}\left(R \sin \omega t - \omega L \cos \omega t\right)$

Putting $\cos\theta = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$

and $\sin\theta = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$

$I = \frac{L e^{Rt/L} \sin(\omega t - \theta)}{(R^2 + \omega^2 L^2)^{1/2}}$

where $\theta = \tan^{-1}(\omega L/R)$

The procedure for solving a first order linear non-homogeneous ODE can be summarised as follows.

> **The method of solving first order linear ODEs**
>
> **Step 1 :** Put the equation into the standard form $y' + p(x)\,y = q(x)$. (Note: The coefficient of y' must be 1).
>
> **Step 2 :** Identify $p(x)$ and compute $v(x) = \exp[\int p(x)\, dx]$
>
> **Step 3 :** Multiply the standard form of the equation by $v(x)$. The LHS of the equation will always be an ordinary derivative of the product $[y\, v(x)]$, w.r.t. the independent variable.
>
> **Step 4 :** Integrate both sides of the modified equation and solve for y.

With some experience you can also find the integrating factors by inspection for first order ODEs of other types. However, *we will not test you on this count.*

You may now like to work out an SAQ on the above method.

SAQ 8

Spend 10 min

Solve $xy' + 2y = x^3$

Note: Check the solution you obtain.

Thus, you have learnt some commonly used methods of solving first order ODEs. There can be higher order ODEs which can be reduced to first order and solved by applying any of these methods. We shall discuss some of them before ending this discussion.

1.7 EQUATIONS REDUCIBLE TO FIRST ORDER

Here, we shall consider two cases, each corresponding to a second order ODE.

i) If a second order ODE in x and y is devoid of y, then it can be expressed as

$$F(y'', y', x) = 0 \tag{1.30}$$

We make the substitution $w = y' = \frac{dy}{dx}$. Thus, Eq. (1.30) takes the form of a first order ODE

$$F(w', w, x) = 0 \tag{1.31}$$

To illustrate this technique, we consider the ODE

$$y'' + 2y' = 0 \tag{1.32}$$

We put $w = y'$, so that

$$\frac{dw}{dx} + 2w = 0 \tag{1.33}$$

You can solve this equation by the method of separation of variables. Let us now consider the second case.

ii) If a second order ODE in x and y is devoid of x, then it can be expressed as

$$F(y'', y', y) = 0 \tag{1.34}$$

We again make the substitution $w = y'$. Then we express y'' as follows:

$$y'' = \frac{dy'}{dx} = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = w\frac{dw}{dy}$$

Thus, Eq. (1.34) becomes

$$F\left(w\frac{dw}{dy}, w, y\right) \tag{1.35}$$

which is a first order ODE in w with y as the independent variable. To illustrate the method, we consider the following ODE

$$yy'' + (y')^2 = 0 \tag{1.36}$$

We put $w = y'$, so that $y'' = w\frac{dw}{dy}$

So, we get

$$y\,w\frac{dw}{dy} + w^2 = 0 \tag{1.37}$$

Now this can be solved by the method of separation of variables.

Spend 5 min

SAQ 9

Complete the solutions of the ODEs (1.33) and (1.37).

Note: Check the solutions.

In this unit you have studied various methods of solving first order ODEs. Henceforth, whenever you come across a first order ODE, begin by classifying it. Then you could consider the following questions:

- Do the variables separate?
- Is there an obvious substitution which simplifies the ODE?
- Is the ODE exact?
- Is the equation linear, nonhomogeneous?

More often than not, the answers to these questions will tell you what method to use for solving the given ODE. Of course, some ODEs may be solved by more than one method. Then you could opt for the easiest one!

Let us now sum up what we have learnt in this unit.

1.8 SUMMARY

- Equations that contain ordinary or partial derivatives or differentials of one or more dependent variables w.r.t. independent variables are called **differential equations.**

- We classify a DE by its type: **ordinary** or **partial**; by its **order** and **degree**, and by whether it is **linear** or **nonlinear**. A **linear** ODE may be **homogeneous** or **nonhomogeneous.**

- A function $y = \phi(x)$ is a **solution** of a differential equation on some interval if $\phi(x)$ is defined and differentiable throughout that interval and is such that the DE becomes an identity when y is replaced by $\phi(x)$ in the DE. A solution involving arbitrary constant(s) is called a **general solution.** If definite value(s) can be assigned to the arbitrary constant(s) in a general solution by specifying certain conditions then it becomes a **particular solution.** Depending on the way the conditions are specified we get an **initial value problem** or a **boundary value problem.**

- The method of solution for a **first order ODE** depends on an appropriate classification of the equation. We summarise four methods.

  - An equation is **separable** if it can be put into the form $N(y)dy = M(x)dx$. The solution is obtained by integrating both sides of the equation.
  - The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be **homogeneous** of first order if $M(x,y)$ and $N(x,y)$ are **homogeneous functions** of the same degree. It can be made separable by making the substitution $y = vx$. Further, if M and N are linear functions of x and y, the ODE can also be solved by making the substitutions $x = x' + h, y = y' + k, y' = vx'$, where x' and y' are variables and not derivatives.
  - The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be **exact** if $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential [$dz(x,y)$]. When M and N are continuous and have continuous partial derivatives, then $\partial M/\partial y = \partial N/\partial x$ is a necessary and sufficient condition for $M\,dx + N\,dy$ to be exact. Then there exists some function z for which $M(x,y) = \partial z/\partial x$ and $N(x,y) = \partial z/\partial y$. The method of solution of an exact ODE starts by integrating either of these latter expressions.
  - If a first order linear ODE can be put in the form

    $$y' + p(x)y = q(x),$$

    it can be reduced to the exact form by multiplying it with the **integrating factor** $\exp\left[\int p(x)dx\right]$. We can solve this equation by integrating both sides of the equation $\frac{d}{dx}\left[\left\{\exp\left(\int p(x)dx\right)\right\}y\right] = \left\{\exp\left(\int p(x)dx\right)\right\}q(x)$.

- An ODE may be reduced to one of the familiar forms by an appropriate **substitution** or **change of variables.**

- Second order ODEs of the forms $F(y'', y', x) = 0$ and $F(y'', y', y) = 0$ may be reduced to first order and hence solved by making the substitution $y' = w$.

1.9 TERMINAL QUESTIONS

Spend 30 min

1) Obtain the general solution of the first order ODE

$$2y' - 4y = 16e^x.$$

or $\int \frac{dr}{r} + \int \frac{(-\sin\theta)\, d\theta}{1+\cos\theta} = \ln|C|$

or $\ln|r| + \ln|1+\cos\theta| = \ln|C| \quad \left[\because \frac{d}{d\theta}(1+\cos\theta) = -\sin\theta\right]$

$\therefore \quad r(1+\cos\theta) = C$

6) a) $\frac{dy'}{dx'} = \frac{x'-y'}{x'+y'} = \frac{1-y'/x'}{1+y'/x'}$

We put $y' = vx' \quad \therefore \frac{dy'}{dx'} = v + x'\frac{dv}{dx'}$

$\therefore v + x'\frac{dv}{dx'} = \frac{1-v}{1+v}$

Let $I = \int \frac{(1+v)\,dv}{v^2+2v-1}$

We put $u = v^2+2v-1$

$\therefore du = (2v+2)\,dv$

$= 2(v+1)\,dv$

or $I = \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u|$

or $x'\frac{dv}{dx'} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$

or $\frac{dx'}{x'} = \frac{(1+v)\,dv}{1-2v-v^2}$

or $\frac{dx'}{x'} + \frac{(1+v)\,dv}{v^2+2v-1} = 0$

or $\int \frac{dx'}{x'} + \int \frac{(1+v)\,dv}{v^2+2v-1} = \ln|C|$

or $\ln|x'| + \frac{1}{2}\ln|u| = \ln|C|, \quad u = v^2+2v-1$

or $x' u^{1/2} = C_1$

$\therefore \quad x'(v^2+2v-1)^{1/2} = C_1$

$\therefore \quad (y'^2 + 2y'x' - x'^2)^{1/2} = C_1$

or $y'^2 + 2y'x' - x'^2 = C_1^2$

or $\left(y-\frac{1}{2}\right)^2 + 2\left(y-\frac{1}{2}\right)\left(x+\frac{3}{2}\right) - \left(x+\frac{3}{2}\right)^2 = C_1^2$

i.e., $x^2 - y^2 - 2xy + 4x - 2y + A = 0$ is the required solution, where $A = C_1^2 + \frac{7}{2}$

b) (i), (iii), (v), (vi)

7) a) $M = x\cos y - y, \qquad N = x\sin y + x$

$\therefore \quad \frac{\partial M}{\partial y} = -x\sin y - 1, \qquad \frac{\partial N}{\partial x} = \sin y + 1$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence, the equation is inexact.

b) $M = e^x + y - 1, N = 3e^y + x - 7$

$\therefore \quad \frac{\partial M}{\partial y} = 1, \ \frac{\partial N}{\partial x} = 1$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the equation is exact.

$\therefore \quad \frac{\partial z}{\partial x} = e^x + y - 1$ and $\frac{\partial z}{\partial y} = 3e^y + x - 7$

From the former, we get

$$z = e^x + xy - x + f(y)$$

Hence, we have

$$\frac{\partial z}{\partial y} = 3e^y + x - 7 = x + \frac{df}{dy}$$

$$\therefore \quad \frac{df}{dy} = 3e^y - 7$$

or $\quad f(y) = 3e^y - 7y + C_1$

The required solution is $z(x, y) = C'$

or $\quad e^x + xy + 3e^y - x - 7y + C = 0.$

8) The given ODE may be expressed as

$$y' + \frac{2}{x}y = x^2$$

The integrating factor = $\exp\left(\int \frac{2}{x}dx\right) = \exp[2\ln|x|] = \exp[\ln|x^2|] = x^2$

So, we have

$$\frac{d}{dx}(x^2y) = x^4$$

or $\quad x^2y = \int x^4 dx + C$

or $\quad x^2y - \frac{x^5}{5} = C \quad$ is the required solution.

9) From Eq. (1.33), we get

$$\frac{dw}{2w} + dx = 0$$

or $\quad \frac{1}{2}\int \frac{dw}{w} + \int dx = C, \qquad$ where C is an arbitrary constant.

or $\quad \frac{1}{2}\ln|w| + x = C$

$\therefore \quad w = Ae^{-2x}, \qquad$ where $A = e^{2c}$

or $\quad \frac{dy}{dx} = Ae^{-2x}$. The required solution is $y = -\frac{A}{2}e^{-2x} + B,$

where B is an arbitrary constant.

From Eq. (1.37), we get

$$\frac{dw}{w} + \frac{dy}{y} = 0$$

On integration, we get $\ln(wy) = C$ or $wy = e^C$

$\therefore \quad \frac{dy}{dx} = \frac{A}{y}, \qquad$ where $A = e^C$

or $\quad \int y\,dy = \int A\,dx$

So, the required solution is $\frac{y^2}{2} = Ax + B.$

Terminal Questions

1) You can see that the given ODE is a linear non-homogeneous first order ODE. So we can use the method discussed in Sec. 1.6.1. Rewriting the equation in the standard form, we get

$$y' - 2y = 8e^x$$

We note that $p(x) = -2$. So the integrating factor is

$$v(x) = \exp\left[-\int 2\,dx\right] = \exp(-2x)$$

Multiplying the given ODE (in standard form) by e^{-2x}, we get

$$e^{-2x}y' - 2y\,e^{-2x} = 8\,e^{-x}$$

or $$\frac{d}{dx}(y\,e^{-2x}) = 8\,e^{-x}$$

or $$d[y\,e^{-2x}] = 8\,e^{-x}dx$$

Integrating both sides yields

$$y\,e^{-2x} = -8\,e^{-x} + C$$

So the general solution is

$$y = -8\,e^{x} + C\,e^{2x}$$

2) The equation may be written as

$\dfrac{dy}{dx} = \dfrac{y - \sqrt{x^2+y^2}}{x}$ which is a first order homogeneous ODE.

We put $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

And the right hand side $= \dfrac{vx - x\sqrt{1+v^2}}{x} = v - \sqrt{1+v^2}$

So we get,

$$v + x\frac{dv}{dx} = v - \sqrt{1+v^2}$$

or $$\frac{dv}{\sqrt{1+v^2}} = -\frac{dx}{x}$$

$$\therefore \int \frac{dv}{\sqrt{1+v^2}} + \int \frac{dx}{x} = \ln|C|$$

or $$\ln|v + \sqrt{v^2+1}| + \ln|x| = \ln|C|$$

or $$\ln|x(v + \sqrt{v^2+1})| = \ln|C|$$

$$\therefore x[v + \sqrt{v^2+1}] = C$$

i.e., $y + \sqrt{y^2+x^2} = C$ is the required general solution.

It is given that $y = 4$ for $x = 3$

$$4 + \sqrt{4^2+3^2} = C \quad \text{or} \quad C = 9$$

Hence, the particular solution is

$$y + \sqrt{y^2+x^2} = 9$$

Checking the Solution

We can rewrite the general solution as $y + \sqrt{y^2+x^2} = C$ (i)

or $x^2 + y^2 = (C-y)^2$
$= C^2 + y^2 - 2Cy$

or $x^2 = C^2 - 2Cy$ (ii)

This gives us

$$2x\,dx = -2C\,dy$$

or $dy = -\dfrac{x}{C}dx$ (iii)

Substituting $\sqrt{x^2+y^2}$ from (i), and dy from (iii) in the ODE, we get

$$x\left(-\frac{x}{C}dx\right) - dx(y - C + y) = 0$$

or $dx\left[-\dfrac{x^2}{C} - 2y + C\right] = 0$ (iv)

Substituting (ii) in (iv) gives us an identity. Thus, (i) is a solution of the given ODE.

3) $$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dx}{dt}\left(\frac{dv}{dx}\right) = v\left(\frac{dv}{dx}\right) \text{ and } \frac{k}{m} = \omega^2$$

Eq. (1.2) may thus be written as

$$v\frac{dv}{dx}+\omega^2 x = 0$$

or $\quad v\,dv+\omega^2 x\,dx = 0$

On integrating, we get

$$\frac{v^2}{2}+\frac{\omega^2 x^2}{2} = C, \text{ where } C \text{ is an arbitrary constant, i.e.,}$$

$$v^2+\omega^2 x^2 = C', \quad \text{where } C' = 2C$$

But $\quad \frac{dx}{dt} = v = 0$, when $x = a$

$\therefore \quad C' = \omega^2 a^2$

$\therefore \quad v^2 = \omega^2(a^2-x^2)$

or $\quad v = \frac{dx}{dt} = \pm\omega\sqrt{a^2-x^2}$

or $\quad \frac{dx}{\pm\sqrt{a^2-x^2}} = \omega\,dt$

$\therefore \quad \int\frac{dx}{\pm\sqrt{a^2-x^2}} = \omega t+\delta \quad$ where δ is an arbitrary constant.

or $\quad \left.\begin{matrix}\sin^{-1}\frac{x}{a}\\ \\ \cos^{-1}\frac{x}{a}\end{matrix}\right| = \omega t+\delta$

Thus, $\quad \frac{x}{a} = \begin{matrix}\sin(\omega t+\delta)\\ \text{or}\\ \cos(\omega t+\delta)\end{matrix}$

Thus, the required solutions are

$$x = a\sin(\omega t+\delta) \quad \text{and} \quad x = a\cos(\omega t+\delta)$$

UNIT 2 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Structure

2.1 INTRODUCTION

In Unit 1 you have learnt to solve first order ordinary differential equations. These equations provide a useful means of studying physical systems. For instance, radioactive decay, free fall of a body, fluid flow, current growth in electrical circuits, etc could be studied using these equations. But many systems require solutions of ordinary differential equations of order higher than one. From Unit 1 of PHE-02 course on Oscillations and Waves, you would recall that the equation of motion of an undamped harmonic oscillator is a homogeneous second order differential equation with constant coefficients. Similarly, to determine the depression in horizontal beams, we have to solve a **second order differential equation with constant coefficients**. You will learn to solve such equations in Sec. 2.3. But can you study time-variation of charge in maintained *RLC* circuits or the phenomenon of resonance using a homogeneous second order differential equation? In such cases we have to solve **nonhomogeneous** second order differential equations with constant coefficients. But even this is not true in general. For instance, when we wish to study field distribution around a charged sphere or a cylinder, we have to solve second order differential equations **with variable coefficients**. Similar situations are also encountered in heat, optics, electromagnetic theory, energy production in a nuclear reactor, quantum mechanics, etc. In such cases we seek power series solutions or use Frobenius method. In the next unit you will learn these methods. But in this unit, we have discussed the basic techniques of solving second order differential equations with constant coefficients. Some of their applications will be discussed in Unit 4.

Objectives

After studying this unit you should be able to

- compute the Wronskian of a given ODE
- obtain linearly independent solutions of homogeneous second order ordinary differential equations with constant coefficients
- use the method of undetermined coefficients and variation of parameters to obtain linearly independent solutions of non-homogeneous second order ordinary differential equations with constant coefficients.

2.2 SOME TERMINOLOGY

While studying first order ordinary differential equations (ODEs) in Unit 1 you have learnt some basic terminology. You would come across some common terms, which you do not know as yet, in the context of second order differential equations as well. It is important for you to be familiar with them. This section is intended for this purpose.

You know that a second order linear ordinary differential equation can be written as

$$\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_0(x)y = g(x) \qquad (2.1)$$

The function $g(x)$ is termed as **the forcing function**, and $p_1(x)$ and $p_0(x)$ are **coefficient functions**. These are continuous over the interval where the solution exists.

Linearly Independent Solutions and the Wronskian

From Unit 1 (Sec. 1.4.3) we recall that if y_1 and y_2 are linearly independent solutions of the homogeneous equation

$$y''+p_1(x)y'+p_0(x)y = 0 \qquad (2.2)$$

then their linear combination

$$y = C_1 y_1 + C_2 y_2 \qquad (2.3)$$

where C_1 and C_2 are arbitrary constants, is a general solution of Eq. (2.2). For example, you know that $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$ are linearly independent solutions of the ODE for an undamped harmonic oscillator: $\frac{d^2y}{dt^2}+\omega^2 y = 0$. So the general solution of this equation is

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

You may ask: What do we mean by linearly independent solutions? How do we test linear independence? Will a linear combination of linearly independent solutions necessarily lead to a different solution? When does a set of solutions constitute the general solution of a linear differential equation? and so on. Let us now discover answers to these questions for ODEs of second order. We say that two solutions y_1 and y_2 are linearly independent on an interval if the identity

$$C_1 y_1 + C_2 y_2 = 0 \qquad (2.4)$$

is satisfied only when $C_1 = C_2 = 0$. For, if C_1 and C_2 were non-zero constants, Eq. (2.4) would yield y_2/y_1 = constant, i.e., y_1 and y_2 would be proportional on some interval. Then, by definition, y_1 and y_2 would be linearly dependent functions on that interval. In other words, linear independence of y_1 and y_2 means that the ratio y_2/y_1 is not a constant. This implies that the differential of this ratio

$$\frac{y_2' \, y_1 - y_1' \, y_2}{y_1^2} \qquad (2.5)$$

is not identically equal to zero. Therefore, we can write the condition of linear independence of two solutions y_1 and y_2 as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \qquad (2.6)$$

The determinant $W(y_1, y_2)$ is called the **Wronski determinant** or the **Wronskian** of the given differential equation. We may, therefore, conclude that

> Two solutions y_1 and y_2 are linearly independent on an interval [a , b], if and only if, their Wronskian is non-zero for $a \le x \le b$.

For a harmonic oscillator, this means that

$$W(x) = \begin{vmatrix} \sin \omega t & \cos \omega t \\ \omega \cos \omega t & -\omega \sin \omega t \end{vmatrix} = -\omega$$

showing that sin ωt and cos ωt are linearly independent. We also say that y_1 and y_2 are linearly dependent solutions on an interval I, if and only if their Wronskian is zero for some $x = x_0$ in I

Spend
5 min

SAQ 1

The solutions of the equation

$$y'' + 4y = 0$$

are given by $y_1 = \sin 2x$ and $y_2 = \cos 2x$. Are these solutions linearly independent?

Particular integral and complementary function

From Sec. 4.2 of the course Oscillations and Waves (PHE-02), you would recall that the equation of motion of a forced damped harmonic oscillator is

$$my'' + \gamma y' + ky = F_0 \cos \omega t$$

which is usually rewritten as

$$y'' + 2by' + \omega_0^2 y = f_0 \cos \omega t \tag{2.7}$$

where $2b = \gamma/m$, $\omega_0^2 = k/m$ and $f_0 = F_0/m$.

Physically, the actual motion of this system is a sum of two oscillations: one of the frequency of damped oscillations and the other of the frequency of the driving force. Mathematically, we express it as

$$y(t) = y_1 + y_2 \tag{2.8}$$

where y_1 is a solution of the homogeneous equation

$$y_1'' + 2by_1' + \omega_0^2 y_1 = 0 \tag{2.9}$$

On substituting Eq. (2.8) in Eq. (2.7) and using Eq. (2.9) in the resultant expression, you will find that y_2 satisfies the equation

$$y_2'' + 2by_2' + \omega_0^2 y_2 = f_0 \cos \omega t$$

In the language of mathematics, y_1 is called the **complementary function** and y_2 is called the **particular integral**. We can write the general solution of a second order non-homogeneous linear differential equation with constant coefficients as the sum of a complementary function and the particular integral:

$$y(x) = y_c(x) + y_p(x) \tag{2.10}$$

You know that the solution of a second order differential equation consists of only two arbitrary constants. This implies that the particular integral will not contain any arbitrary constant.

We hope that you are now equipped with all the necessary basic terminology. Let us now proceed to solve homogeneous linear ODEs of second order with constant coefficients.

2.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A second order homogeneous linear ordinary differential equation with constant coefficients finds wide applications in engineering, biological and physical systems. In particular, you know its use in mechanical and electrical vibrations. Many techniques of solving such equations have been developed. These include the reduction of order technique, which you learnt in Unit 1, and the method of exponential functions, which we will discuss now.

A homogeneous second order ordinary differential equation with constant coefficients can be expressed in the form

$$ay'' + by' + cy = 0 \tag{2.11}$$

where a, b and c are real constants.

From Unit 1, you would recall that the solution of the first order homogeneous linear ordinary differential equation ($y' + y = 0$) is an exponential function of the form

$$y = A\exp(-kx)$$

Let us, therefore, seek a solution of Eq. (2.11) of the form

$$y = A\exp(mx) \tag{2.12}$$

where dimensions of m are inverse of those of x. This ensures that the power of exponential is dimensionless.

From Eq. (2.11), you would note that y, y' and y'' are linearly dependent. This demands that y be an exponential function. For, the derivative of any order of an exponential function is linearly dependent with itself, i.e., some multiple of the original exponential.

Substituting this and its derivatives

$$y' = A\,m\exp(mx)$$

and

$$y'' = A\,m^2\exp(mx)$$

in Eq. (2.11), you will obtain

$$(am^2 + bm + c)A\exp(mx) = 0$$

Since $A\exp(mx)$ is finite, this equation will be satisfied only if

$$am^2 + bm + c = 0 \tag{2.13}$$

This quadratic equation is called the **characteristic equation** (or **auxillary equation**). Its roots are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

For example, the auxiliary equation for

$$y'' + 5y' - 7y = 0 \quad \text{is} \quad m^2 + 5m - 7 = 0$$

So you will agree that

$$y_1(x) = A\exp(m_1 x) \tag{2.14a}$$

and

$$y_2(x) = A\exp(m_2 x) \tag{2.14b}$$

are solutions of Eq. (2.11). Using the principle of superposition, you can write its most general solution as

$$y(x) = C_1\exp(m_1 x) + C_2\exp(m_2 x) \tag{2.14c}$$

for a suitable choice of constants C_1 and C_2 determined by initial or boundary conditions.

The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} A\exp(m_1 x) & A\exp(m_2 x) \\ m_1 A\exp(m_1 x) & m_2 A\exp(m_2 x) \end{vmatrix}$$

$$= (m_2 - m_1)B\exp[(m_1 + m_2)x] \tag{2.15}$$

where B is a constant. This shows that for $m_1 \neq m_2$, the solutions will be linearly independent.

You must have noticed that the process of solving a homogeneous linear second order ordinary differential equation with constant coefficients using an exponential function as a solution reduces to finding the roots of a quadratic equation. The roots of this equation can be

1. real and distinct for $b^2 - 4ac > 0$ or $b^2 > 4ac$
2. real and equal when $b^2 - 4ac = 0$, or $b^2 = 4ac$ and
3. complex conjugate for $b^2 - 4ac < 0$ or $b^2 < 4ac$

Let us now discover solutions corresponding to these roots.

Distinct Real Roots

For distinct real roots, $\exp(m_1 x)$, and $\exp(m_2 x)$ are linearly independent and the general solution is given by

$$y = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$
$$= \exp\left[-\left(\frac{bx}{2a}\right)\right][\, C_1 \exp(\alpha x) + C_2 \exp(-\alpha x)\,] \qquad (2.16)$$
$$\text{where } \alpha = \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The constants C_1 and C_2 can be determined by using given initial and boundary conditions. We now illustrate this method with the following example.

Example 1

Solve the equation

$$y'' + 3y' + 2y = 0$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 2$.

Solution

In this case, the auxiliary equation is

$$m^2 + 3m + 2 = 0$$

which has roots $m = -1$ and $m = -2$. Therefore, the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} \qquad \text{(i)}$$

To determine C_1 and C_2, we first use the condition that at $x = 0, y = 1$. This gives

$$1 = C_1 + C_2 \qquad \text{(ii)}$$

Further, since

$$y' = -C_1 e^{-x} - 2C_2 e^{-2x}$$

we find that

$$y'(0) = 2 = -C_1 - 2C_2 \qquad \text{(iii)}$$

You can readily solve (ii) and (iii) for C_1 and C_2 to obtain $C_1 = 4$ and $C_2 = -3$. Hence, the desired particular solution is

$$y = 4e^{-x} - 3e^{-2x}$$

When two roots are equal $(m_1 = m_2)$, $W(x) = 0$. This means that $e^{m_1 x}$ and $e^{m_2 x}$ are linearly dependent. What does this imply? It implies that (i) Eq. (2.14) does not hold and (ii) our starting assumption is false. You may now ask: How can we obtain two linearly independent functions when auxiliary equation of a second order differential equation has two equal roots? In such a situation, we use the method of reduction of order to construct a second linearly independent solution. This is illustrated below.

Repeated Real Roots

When a second order differential equation has two equal roots, we obtain the correct form of the second solution by assuming that

$$y_2 = u(x) \exp(mx) \qquad (2.17)$$

where m is a root of the auxiliary equation (Eq. (2.13)). Differentiating Eq. (2.17) with respect to x, we get

$$y_2' = u' \exp(mx) + mu \exp(mx)$$

and

$$y_2'' = u'' e^{mx} + 2m\, u' e^{mx} + m^2 u\, e^{mx}$$

Substituting these in Eq. (2.11), we have

$$(a m^2 + b m + c)\, u(x)\, e^{mx} + (2ma + b)\, e^{mx} u' + a\, e^{mx} u'' = 0$$

The first term in this expression vanishes in view of Eq. (2.13). The coefficient of u' is zero since $m = -b/2a$ in this case. Hence, the above expression simplifies to

$$\exp(mx)\, a\, u'' = 0$$

Multiplying by $\exp(-mx)$ and integrating, you will get

$$u' = K$$

where K is an arbitrary constant of integration.
Integrating again, you will get

$$u = Kx + C$$

Hence, the desired solution is

$$y_2 = x\, e^{mx} = x\, e^{-bx/2a} \qquad (2.18)$$

Alternatively, you can also arrive at Eq. (2.18) by constructing the following linear combination of $e^{m_1 x}$ and $e^{m_2 x}$

$$\frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

In the limit $m_1 \to m_2$ $(= m)$, this takes the form

$$\lim_{h \to 0} \frac{e^{(m+h)x} - e^{mx}}{h}; \; h = m_1 - m_2$$

$$= \frac{d}{dm} e^{mx}$$

$$= x\, e^{mx}$$

where the arbitrary constants K and C have been dropped (since we are seeking only a second linearly independent solution). Hence, the general solution of a second order differential equation, when auxiliary equation has repeated real roots, is

$$y(x) = C_1 e^{-bx/2a} + C_2\, x\, e^{-bx/2a}$$

$$= (C_1 + C_2 x) \exp\left(-\frac{bx}{2a}\right) \qquad (2.19)$$

To test that $e^{-bx/2a}$ and $x\, e^{-bx/2a}$ are linearly independent, you can compute their Wronskian

$$W(x) = \begin{vmatrix} \exp\left(-\frac{bx}{2a}\right) & x \exp\left(-\frac{bx}{2a}\right) \\ -\frac{b}{2a} \exp\left(-\frac{bx}{2a}\right) & -\frac{b}{2a} x \exp\left(-\frac{bx}{2a}\right) + \exp\left(-\frac{bx}{2a}\right) \end{vmatrix}$$

$$= e^{-(bx/a)} > 0 \quad \text{for} \quad a \le x \le b \qquad (2.20)$$

It implies that $e^{-(bx/a)}$ and $x\, e^{-bx/2a}$ are acceptable solutions. The arbitrary constants C_1 and C_2 occurring in Eq. (2.19) can be determined using specified initial or boundary conditions.

You may, therefore, conclude as follows:

> When the auxiliary equation for a second order ODE with constant coefficients has repeated real roots ($m_1 = m_2 = m$), the general solution is given by
> $$y = (C_1 + C_2 x) \exp(mx)$$
> where C_1 and C_2 are arbitrary constants.

You may now like to solve an SAQ to be sure that you have grasped this method.

SAQ 2

Spend 10 min

Solve the initial value problem

$$y'' + 6y' + 9y = 0; \; y(0) = 2 \text{ and } y'(0) = 1$$

decaying exponential. As a result, the displacement increases initially, attains a maximum and thereafter decays.

The hyperbolic functions are so named because $x = a \cosh\theta$, and $y = a \sinh\theta$ define a rectangular hyperbola $x^2 - y^2 = a^2$. Compare it with the parametric equation of the circle $x^2 + y^2 = a^2$ which is defined by $x = a\cos\theta$ and $y = a\sin\theta$.

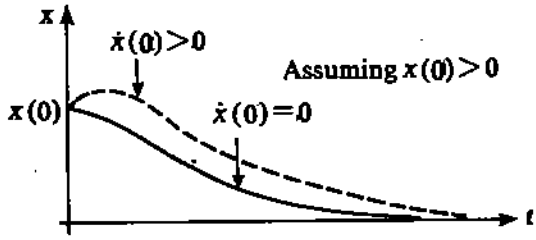


Fig.2.2: Displacement-time graph for an overdamped spring-mass system

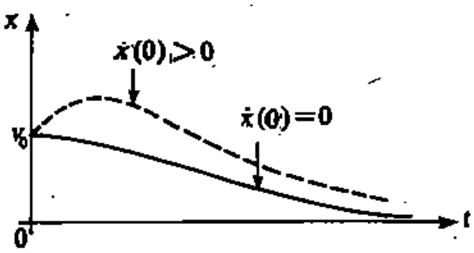


Fig 2.3: Displacement-time graph for a critically damped spring-mass system

Fig 2.2 shows two typical over-damped motions for $\frac{dx(0)}{dt} = 0$ and $\frac{dx(0)}{dt} > 0$

Case 2

When we have repeated real roots, the general solution of Eq. (2.24) is given by

$$x(t) = (C_1 + C_2 t)\exp(-bt) \qquad (2.26)$$

Note that here C_1 has dimensions of length and C_2 those of velocity. As before, these constants can be determined by specifying initial conditions. You can easily verify that for initial conditions given in SAQ 3, $C_1 = 0$ and $C_2 = v_0$ so that the complete solution is

$$x(t) = v_0\, t \exp(-bt) \qquad (2.27)$$

Such a system is said to be **critically damped.** Typical graph of a critically damped system for $\frac{dx(0)}{dt} = 0$ and $\frac{dx(0)}{dt} > 0$ is shown in Fig. 2.3.

Case 3

When the roots are imaginary, let us write

$$\sqrt{b^2 - \omega_0^2} = \sqrt{-1}\,(\omega_0^2 - b^2)^{1/2} = i\,\omega_d$$

where $i = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - b^2}$ is a real positive quantity. Hence, the displacement is given by

$$x(t) = \exp(-bt)\,[C_1 \exp(i\,\omega_d t) + C_2 \exp(-i\,\omega_d t)]$$

$$= C\exp(-bt)\cos(\omega_d t + \phi) \qquad (2.28)$$

where $C = \sqrt{C_1^2 + C_2^2}$ and $\phi = \cos^{-1}\left(\frac{C_1 + C_2}{2\sqrt{C_1 C_2}}\right)$.

You will note that Eq. (2.28) represents oscillatory motion whose amplitude decreases exponentially at a rate governed by b. Such a system is said to be **weakly damped.** The displacement of a weakly damped system is depicted in Fig. 2.4.

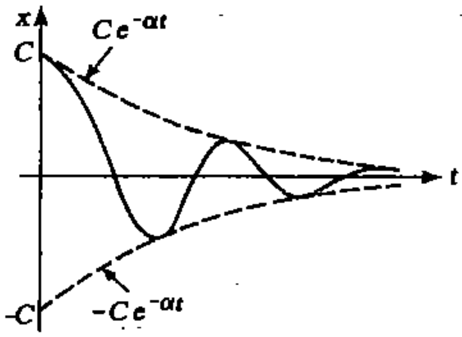


Fig. 2.4 : Oscillations of a weakly damped spring-mass system

Let us sum up what you have learnt so far:

1. We can solve a second order homogeneous linear ordinary differential equation with constant coefficients using exponential functions. The form of the solution depends on the **roots of the characteristic equation.**
2. When we have **distinct real roots**, there exist two linearly independent functions of the form $\exp(m_1 x)$ and $\exp(m_2 x)$ and the general solution is given by

   $$y(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$

3. When we have **real equal roots**, the two linearly independent functions in the general solution are of the form $\exp(mx)$ and $x\exp(mx)$, i.e.,

   $$y(x) = (C_1 + C_2 x)\exp(mx)$$

4. When we have a **complex conjugate pair of roots**, the two linearly independent solutions are of the form $\exp(\alpha x)\sin\beta x$ and $\exp(\alpha x)\cos\beta x$, and the general solution can be written as

   $$y(x) = \exp(\alpha x)(C_1 \sin\beta x + C_2 \cos\beta x)$$
   $$= C\exp(\alpha x)\cos(\beta x - \phi)$$

So far we have considered homogeneous linear equations with constant coefficients. These equations do not satisfactorily model forced mechanical and electrical systems. In fact, such systems can be fairly accurately represented by nonhomogeneous second order linear equations. We now wish to obtain solutions of such equations. Can you use the method discussed in the preceding section to solve non-homogeneous equations? No, we have to look for new methods. Let us learn some of these now.

2.4 NONHOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

From Sec. 2.2, you would recall that the general solution of a second order nonhomogeneous linear ordinary differential equation with constant coefficients is composed of the particular integral (PI) and the complementary function (CF). You can easily verify this by substituting $y = y_p + y_c$ in the equation

$$a y'' + b y' + Cy = g(x)$$

You can obtain $y_c(x)$ by using the method of the preceding section. For instance, CF for the equation

$$y'' - 2y' - 3y = \sin x$$

is given by

$$y_c = C_1 e^{-3x} + C_2 e^{-x}$$

(You can check this by direct substitution.)

This means that finding the particular integral is at the heart of the method of solving a non-homogeneous equation. But how to get y_p? One systematic approach to find y_p is based on the method of reduction of order, which you have learnt in Unit 1. The other commonly used methods are **the method of undetermined coefficients and the variation of parameters.** Let us now learn these.

2.4.1 The Method of Undetermined Multipliers

The basic idea of this method is to first construct the general form of the particular integral from the forcing function. Then we determine coefficients for y_p that allow it to satisfy the given differential equation. In the following example, we have illustrated this concept.

Spend 20 min

SAQ 5

Find the general solutions of the following differential equations:

(i) $y'' + y = x^2$

(ii) $y'' + 4y = 3\cos x$

(iii) $y'' + y' + 2y = 4e^x + 2x^2$

From Unit 4 of PHE-02 course, you would recall that non-homogeneous linear differential equations find an immediate application to damped spring-mass systems acted upon by external forces. Suppose that the applied force causes the weight in a spring-mass system to move up and down in some prescribed manner. Denoting the external applied force by $F(t)$, the differential equation describing one-dimensional motion of such a system is

$\ddot{x} = \frac{d^2x}{dt^2}$

$\dot{x} = \frac{dx}{dt}$

$$m\ddot{x} + \gamma\dot{x} + kx = F(t)$$

Suppose the forcing function is given by $F(t) = F_0 \cos \omega t$, where F_0 is the constant amplitude and ω is the angular frequency. Then, you can write

$$m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos \omega t \tag{2.29}$$

Since the solution of this equation depends on the damping force, we have to consider separately the cases $\gamma = 0$ (undamped) and $\gamma > 0$ (damped). Let us consider the first case now.

Undamped Forced Vibrations

If there is no damping force, the differential equation describing the motion of the spring-mass system becomes

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{2.30}$$

Let us assume the weight to be initially at rest and that $\omega \neq \omega_0 = \sqrt{k/m}$. By the method of undetermined coefficients, you can show that the general solution of this equation is

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

Then, using $\dot{x}(0) = x(0) = 0$, we get

$$C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad C_2 = 0$$

The desired solution is then

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

If we use the identity $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$, this expression can be rewritten in the form

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2}\right) t \sin\left(\frac{\omega_0 - \omega}{2}\right) t \tag{2.31}$$

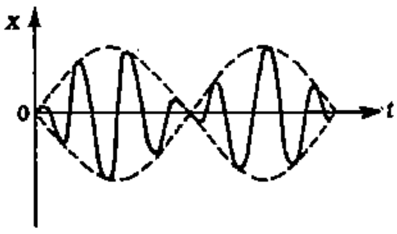


Fig. 2.5 : Displacement time plot of undamped forced vibrations

Since the two sine functions are of different frequencies, there will be occasions, especially when ω is close to ω_0, when their amplitudes will either magnify or cancel one another (see Fig. 2.5). This magnification and cancellation occurs at regular intervals and is called **beat**. In acoustics, these fluctuations can be heard when two tuning forks of slightly different frequencies are set into vibration simultaneously. The same phenomenon occurs in electronics, where it is called **amplitude modulation**.

You must have noticed that the method of undetermined coefficients is limited to those equations whose driving functions are of very special form. Let us now discuss the so-called **variation of parameters** method that is applicable to all linear differential equations with constant coefficients.

2.4.2 The Method of Variation of Parameters

Let y_1 and y_2 be any linearly independent solutions of the homogeneous equation corresponding to the given non-homogeneous equation. To find the particular integral, let us assume that

$$y_p = u\,y_1 + v\,y_2 \tag{2.32}$$

where u and v are unknown functions of x. To determine these, we differentiate Eq. (2.32) with respect to x. This gives

$$y_p' = u'\,y_1 + u y_1' + v' y_2 + v\,y_2' \tag{2.33}$$

We seek a solution such that

$$u'\,y_1 + v'\,y_2 = 0 \tag{2.34}$$

Using this condition in Eq. (2.33), you will get

$$y_p' = u\,y_1' + v\,y_2' \tag{2.35}$$

Differentiate this expression again. The result is

$$y_p'' = u\,y_1'' + v\,y_2'' + u'\,y_1' + v'\,y_2' \tag{2.36}$$

On substituting y_p, y_p' and y_p'' for y, y' and y'', respectively in the equation

$$a\,y'' + b\,y' + c\,y = g(x)$$

you will obtain

$$a\,(u\,y_1'' + v\,y_2'' + u'\,y_1' + v'\,y_2') + b\,(u\,y_1' + v\,y_2') + c\,(u\,y_1 + v\,y_2) = g(x)$$

This can be rearranged as

$$u\,(ay_1'' + b\,y_1' + cy_1) + v\,(ay_2'' + b\,y_2' + cy_2) + a\,(u'\,y_1' + v'\,y_2') = g(x) \tag{2.37}$$

Since y_1 and y_2 are solutions of the homogeneous equation

$$a\,y'' + b\,y' + c\,y = 0$$

the terms in both parentheses drop out. Hence, Eq.(2.37) reduces to

$$a\,(u'\,y_1' + v'\,y_2') = g(x) \tag{2.38}$$

This means that u' and v' satisfy the system of Eqs. (2.34) and (2.38). These can be solved by Cramer's rule. Thus

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{g(x)}{a} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2\,g(x)}{aW}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{g(x)}{a} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1\,g(x)}{aW} \tag{2.39}$$

For two simultaneous linear equations of the form $a_{11}x_1 + a_{12}x_2 = b_1$ and $a_{21}x_1 + a_{22}x_2 = b_2$

Cramer's rule tells us that the solutions for x_1 and x_2 are

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}$$

and

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is a non-zero determinat.

You will note that the denominator in this equation is the Wronskian of two linear independent functions y_1 and y_2 and is non-zero.

These equations can be integrated to obtain

2. Solve the differential equations

i) $\frac{d^2y}{dx^2} + y = \sec x$

ii) $\frac{d^2y}{dx^2} - y = x\,e^x$

2.7 SOLUTIONS AND ANSWERS

SAQs

1. The Wronskian for these functions is

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2\sin^2 2x - 2\cos^2 2x$$

$$= -2$$

Since $W(x) \neq 0$ for all x, the functions $\sin 2x$ and $\cos 2x$ are linearly independent.

2. The auxiliary equation corresponding to the given ODE is

$$m^2 + 6m + 9 = 0$$

which has a double root $m = -3$. Hence, the general solution is

$$y(x) = (C_1 + C_2 x)\, e^{-3x} \qquad \text{(i)}$$

The condition $y(0) = 2$ gives

$$2 = C_1 \qquad \text{(ii)}$$

Differentiate (i) with respect to x. This gives

$$\frac{dy}{dx} = C_2 e^{-3x} - 3\,(C_1 + C_2 x)\, e^{-3x}$$

Using the condition $\frac{dy(0)}{dx} = 1$, we get

$$1 = C_2 - 3C_1$$

or

$$C_2 = 1 + 3C_1 = 1 + 6 = 7 \qquad \text{(iii)}$$

Hence, the desired solution is

$$y(x) = (2 + 7x)\, e^{-3x}$$

3. $$x(t) = \exp(-bt)\,[C_1 \exp(\beta t) + C_2 \exp(-\beta t)] \qquad \text{(i)}$$

At $t = 0, x = 0$. This gives

$$0 = C_1 + C_2$$

or

$$C_1 = -C_2$$

Differentiate the given expression with respect to time. The result is

$$\frac{dx}{dt} = -b\exp(-bt)\,[\,C_1 \exp(\beta t) + C_2 \exp(-\beta t)\,]$$

$$+ \exp(-bt)\,[\,\beta C_1 \exp(\beta t) - \beta C_2 \exp(-\beta t)\,]$$

Using the condition $\frac{dx(0)}{dt} = v_0$, we find that

$$v_0 = -b(C_1 + C_2) + \beta(C_1 - C_2)$$

or

$$C_1 = \frac{v_0}{2\beta} = -C_2$$

Hence

$$x(t) = \frac{v_o}{2\beta}\exp(-bt)\,[\exp(\beta t) - \exp(-\beta t)]$$

$$= \frac{v_0}{\beta}\exp(-bt)\sinh\beta t$$

This shows that the resultant motion of a heavily damped oscillator is determined by the interplay of a decaying exponential and a hyperbolic function.

4. Assume that the PI is of the form

$$y_p = C_0 + C_1 x + C_2 x^2 \qquad \text{(i)}$$

Substituting it and its second derivative

$$\frac{d^2 y_p}{dx^2} = 2C_2$$

in the given ODE, you would get

$$2C_2 - (C_0 + C_1 x + C_2 x^2) = x + \frac{x^2}{2} \qquad \text{(ii)}$$

For (i) to be an acceptable solution, (ii) should be an identity. This gives

$$C_0 = -1$$

$$C_1 = -1$$

and

$$C_2 = -\frac{1}{2}$$

Therefore

$$y_p(x) = -\frac{x^2}{2} - x - 1$$

5. (i) The solution of the homogeneous equation

$$\frac{d^2 y}{dx^2} + y = 0$$

is found to be

$$y_c(x) = C_1\cos x + C_2\sin x$$

A particular integral is assumed to have the form

$$y_p(x) = Ax^2 + Bx + C$$

This is substituted into the original differential equation to give

$$2A + Ax^2 + Bx + C = x^2$$

Equating coefficients of the various powers of x, we have

Coefficient of x^0: $2A + C = 0$

Coefficient of x^1: $B = 0$

Coefficient of x^2: $A = 1$

These equations are solved simultaneously to give the particular solution

$$y_p(x) = x^2 - 2$$

Finally, the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 \cos x + C_2 \sin x + x^2 - 2$$

(ii) The solution of the corresponding homogeneous equation $y'' + 4y = 0$ which has roots $\pm 2i$, is

$$y_c = C_1 \sin 2x + C_2 \cos 2x$$

To find a particular solution of the given equation, we assume that the general form of y_p is

$$y_p = A \sin x + B \cos x$$

This is the correct expression since neither of these functions is in y_c. Substituting y_p and y_p'' into the given differential equation, we obtain

$$(-A \sin x - B \cos x) + 4\,(A \sin x + B \cos x) = 3 \cos x$$

Expanding and collecting like terms yields

$$3A = 0\,, \text{and} \quad 3B = 3$$

which has the solution $A = 0$ and $B = 1$. Hence, $y_p = \cos x$ and the general solution is

$$y = C_1 \sin 2x + C_2 \cos 2x + \cos x$$

(iii) Assume the particular solution to have the form

$$y_p(x) = Ae^x + Bx^2 + Cx + D$$

Substitute this into the given differential equation. The result is

$$Ae^x + 2B + Ae^x + 2Bx + C + 2Ae^x + 2Bx^2 + 2Cx + 2D = 4e^x + 2x^2$$

Equating the various coefficients, you would obtain

Coefficient of e^x: $\quad A + A + 2A = 4$

Coefficient of x^0: $\quad 2B + C + 2D = 0$

Coefficient of x^1: $\quad 2B + 2C = 0$

Coefficient of x^2: $\quad 2B = 2$

From the above equations, we find that $A = 1$, $B = 1$, $C = -1$ and $D = -½$. Thus,

$$y_p(x) = e^x + x^2 - x - ½$$

Terminal Questions

1. The given differential equation is

$$m\ddot{x} + \gamma\,\dot{x} + kx = F_0 \cos\omega t \qquad \text{(i)}$$

where dot over x denotes derivative with respect to time. As such, you have learnt to solve it in your PHE-02 course. But we repeat the solution for the sake of completeness. The corresponding homogeneous equation is

$$m\,\ddot{x} + \gamma\,\dot{x} + k\,x = 0$$

which is the same as the equation describing damped vibration without a forcing function. Its solution depends upon the sign of $\gamma^2 - 4mk$. Thus

If $\gamma^2 - 4mk > 0$

$$x_c(t) = C_1 e^{-(\alpha-\beta)t} + C_2\, e^{-(\alpha+\beta)t}$$

If $\gamma^2 - 4mk = 0$

$$x_c(t) = e^{-\alpha t}(C_1 t + C_2)$$

If $\gamma^2 - 4mk < 0$

$$x_c(t) = e^{-\alpha t}(C_1 \cos \omega' t + C_2 \sin \omega' t)$$

$$= Ce^{-\alpha t} \cos (\omega' t - \delta)$$

where $\alpha = \gamma/2m$, $\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$, $\omega' = \frac{1}{2m}\sqrt{4mk - \gamma^2}$, $C = \sqrt{C_1^2 + C_2^2}$

and $\tan \delta = C_2/C_1$.

Since no constant multiple of the driving function $F_0 \cos \omega t$ is a term of $x_c(t)$, the particular solution is of the form

$$x_p(t) = A \cos \omega t + B \sin \omega t$$

Differentiating twice with respect to time, we get

$$\dot{x}_p(t) = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$\ddot{x}_p(t) = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

Substituting these in (i) and collecting the coefficient of sine and cosine terms, we have

$$[(k - m\omega^2)A + \omega\gamma B] \cos \omega t + [-\omega\gamma A + (k - m\omega^2) B] \sin \omega t = F_0 \cos \omega t$$

Equating the coefficients of the sine and cosine terms on both sides of this equality, we get

$$-\omega \gamma A + (k - m\omega^2)B = 0$$

and $\quad (k - m\omega^2)A + \omega \gamma B = F_0$

Solving these for A and B, you will get

$$A = \frac{F_0 (k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2\gamma^2} \quad \text{and} \quad B = \frac{\gamma\omega F_0}{(k - m\omega^2)^2 + \omega^2\gamma^2}$$

Recalling that $\sqrt{k/m} = \omega_0$, we can write A and B as

$$A = \frac{F_0 m (\omega_o^2 - \omega^2)}{m^2 (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \quad \text{and} \quad B = \frac{\gamma\omega F_0}{m^2 (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

We choose to write $x_p(t)$ in the form

$$x_p(t) = C \cos (\omega t - \delta)$$

where $C = F_0 \Big/ \sqrt{m^2 (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$ and $\tan \delta = \omega\gamma / m (\omega_0^2 - \omega^2)$.

For large values of t, the motion is essentially described by $x_p(t)$. For this reason, $x_p(t)$ is called the **steady-state solution.**

2.(i) Since two linearly independent solutions of the corresponding homogeneous equation are $\cos x$ and $\sin x$, the general solution of the given equation is

$$y = C_1 \cos x + C_2 \sin x + y_p$$

where $y_p = u \cos x + v \sin x$ and u' and v' are, respectively given by

$$\frac{du}{dx} = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} \quad \text{and} \quad \frac{dv}{dx} = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}$$

Therefore, $\frac{du}{dx} = -\tan x$ and $\frac{dv}{dx} = 1$, from which it readily follows that $u = \ln |\cos x|$

and $v' = x$. The general solution is, therefore,

$$y = C_1 \cos x + C_2 \sin x + \ln|\cos x| \cos x + x \sin x$$

Note that the method of undetermined coefficients could not be used to obtain y_p because $\sec x$ is not a solution of the homogeneous linear differential equation.

(ii) The corresponding homogeneous equation $y'' - y = 0$ has the general solution $y = C_1 e^x + C_2 e^{-x}$. Because of the nature of the driving function, y_p could not be found by the method of undetermined coefficients. In the method of variation of parameters, we put $y_1 = e^x$ and $y_2 = e^{-x}$ in the expressions for u' and v' to obtain

$$u' = \frac{\begin{vmatrix} 0 & e^{-x} \\ xe^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} \quad \text{and} \quad v' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & xe^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}$$

Therefore, $u' = x/2$ and $v' = \dfrac{-x\exp(2x)}{2}$ from which it readily follows that $u = x^2/4$ and $v = -(xe^{2x}/4) + (e^{2x}/8)$. The general solution is then

$$y = C_1 e^x + C_2 e^{-x} + \frac{1}{4}x^2 e^x - \frac{x}{4}e^x + \frac{1}{8}e^x$$

Finally, we note that $C_1 e^x$ and $\frac{1}{8}e^x$ can be combined as $(C_1 + \frac{1}{8})e^x = Ce^x$ and the general solution may be written as

$$y = C_1 e^x + C_2 e^{-x} - \frac{1}{4}xe^x + \frac{1}{4}x^2 e^x .$$

UNIT 3 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Structure

3.1 INTRODUCTION

In Unit 2, you have learnt how to solve second order ODEs with constant coefficients. The solutions of these equations are simple exponentials, trigonometric or hyperbolic functions known from calculus. But in many physical and engineering problems, we have to solve second order ordinary differential equations with variable coefficients. For example, we have to solve such equations to study the field distribution around a charged sphere or a cylinder, and energy production in a reactor. Similarly, when we wish to know how high a vertical column of uniform cross-section can be extended upward until it buckles under its own weight, we have to solve a second order ODE with variable coefficients. In such cases, simple algebraic or transcendental solutions do not exist and methods discussed in Unit 2 do not work. We, therefore, look for other methods.

One of the most elegant and efficient methods of solving such ODEs is the **power series method**. This is so particularly because it facilitates numerical computations. Even so, it has limited utility when coefficients of the given differential equation are not well defined at some point. In such cases, we use an extension of the power series method, called the **Frobenius' method**. You will learn these two methods in this unit. The properties of power series and certain other mathematical concepts are given in an Appendix at the end of this unit. It would be better if you study the appendix before studying this unit.

Objectives

After studying this unit, you should be able to

- define ordinary and singular points
- locate and classify the type of singularity
- use power series method to solve a second order ODE about an ordinary point
- use Frobenius' method to solve a second order ODE about a regular singular point.

3.2 SOME TERMINOLOGY

While studying first and second order ODEs with constant coefficients, you have learnt some basic terminology. You would come across some more common terms, which you do not know as yet, in reference to second order ODEs with variable coefficients. This section is intended to familiarise you with these concepts.

Solution

On comparing the given equation with Eq. (3.2), you will note that $p(x) = 0$ and $q(x) = x^2$ so that $x = 0$ is an ordinary point. So we assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \quad \text{(i)}$$

and differentiate it with respect to x. This gives

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

In the summation notation, we can write

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

The lower limit on the summation index has been shifted from $n = 0$ to $n = 1$. This is because the term corresponding to $n = 0$ in (i) is constant and its derivative with respect to x is zero. Similarly, you can write

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \dots$$

or
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} \quad \text{(ii)}$$

Substituting for y and y'' in the given differential equation, we find that

$$(2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \dots) + (a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots) = 0$$

or
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Next, let us collect like powers of x. This gives

$$2a_2 + 6a_3 x + (12 a_4 + a_0) x^2 + (20 a_5 + a_1) x^3 + \dots = 0$$

Since the RHS is identically zero, we equate the coefficient of every power of x on the LHS to zero.

Here 'coeff' is being used as an abbrevation of 'coefficient'.

Coeff. of x^0 : $2a_2 = 0 \Rightarrow a_2 = 0$

Coeff. of x^1 : $6a_3 = 0 \Rightarrow a_3 = 0$

Coeff. of x^2 : $12a_4 + a_0 = 0 \Rightarrow a_4 = -\frac{a_0}{12}$

Coeff. of x^3 : $20a_5 + a_1 = 0 \Rightarrow a_5 = -\frac{a_1}{20}$

Coeff. of x^4 : $30a_6 + a_2 = 0 \Rightarrow a_6 = 0$

Coeff. of x^5 : $42a_7 + a_3 = 0 \Rightarrow a_7 = 0$

⋮

Coeff. of x^{n+2} : $(n+3)(n+4) a_{n+4} + a_n = 0$

or
$$a_{n+4} = -\frac{1}{(n+3)(n+4)} a_n \quad \text{for } n \ge 0$$

With $a_2 = 0$, it follows that alternative even coefficients beyond a_4 ($a_6, a_{10},$) will be zero. Similarly, since $a_3 = 0$, it readily follows that $a_7 = a_{11} = \dots = 0$. Hence, you can write

$$y(x) = a_0 + a_1 x + a_4 x^4 + a_5 x^5 + a_8 x^8 + a_9 x^9 + \dots$$

$$= a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + \frac{a_0}{672} x^8 + \frac{a_1}{1440} x^9 + \dots$$

$$= a_0\left(1-\frac{x^4}{12}+\frac{x^8}{672}+\dots\right)+a_1\left(x-\frac{x^5}{20}+\frac{x^9}{1440}+\dots\right)$$

$$= a_0\,y_1(x)+a_1\,y_2(x)$$

Are these solutions linearly independent? You should check it by computing their Wronskian.

You may now ask : Can we not use the power series method to solve first or second order ODEs with constant coefficients? The answer is: Yes, we can use the above steps as outlined for second order equations with variable coefficients. You can easily convince yourself about this by solving the following SAQ.

SAQ 2

Spend 15 min

Use power series method to solve the following equations.

a) $y'' + \omega_0^2 y = 0$

b) $y' + xy = x^2 - 2x$

As pointed out earlier, the equation given in Example 1 is of particular interest in physics and nuclear engineering. It is known as **Legendre's equation**. Let us obtain its two linearly independent solutions using the power series method.

Example 3

Obtain series solution of the Legendre's equation given in Example 1

$$(1-x^2)\,y'' - 2xy' + m\,(m+1)\,y = 0$$

Solution

To solve Legendre's equation using the power series method, we first rewrite it as

$$y'' - \frac{2x}{1-x^2}y' + \frac{m\,(m+1)}{1-x^2}y = 0 \qquad \text{(i)}$$

From Example 1, you would recall that the functions $p(x) = -\dfrac{2x}{1-x^2}$ and $q(x) = \dfrac{m\,(m+1)}{1-x^2}$ have regular singularities at $x = \pm 1$. However, they are analytic at $x = 0$ and we can, therefore, use power series method to solve Legendre's equation in the range $-1 < x < 1$. Let us write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \qquad \text{(ii)}$$

Substituting this and its derivatives

$$y'(x) = a_1 + 2\,a_2 x + 3\,a_3 x^2 + \dots$$

and

$$y''(x) = 2\,a_2 + 6\,a_3 x + 12\,a_4 x^2 + \dots$$

in Legendre's equation, we get

$$(1-x^2)\,[\,2a_2 + 6a_3 x + 12a_4 x^2 + \dots\,] - 2x\,[\,a_1 + 2a_2 x + 3a_3 x^2 + \dots\,]$$
$$+\,k\,[\,a_0 + a_1 x + a_2 x^2 + \dots\,] = 0$$

or

$$(1-x^2)\sum_{n=2}^{\infty} n\,(n-1)\,a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n\,a_n x^{n-1} + k\sum_{n=0}^{\infty} a_n x^n = 0$$

where we have put $m\,(m+1) = k$.

You can rewrite it as

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + k\sum_{n=0}^{\infty} a_n x^n = 0 \qquad \text{(iii)}$$

In the expanded form, you can write

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) - (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + \dots)$$
$$-2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots) + k(a_0 + a_1 x + a_2 x^2 + \dots) = 0$$

As before, let us collect the coefficients of each power of x. This gives

$$(2a_2 + ka_0) + (6a_3 - 2a_1 + ka_1)x + (12a_4 - 2a_2 - 4a_2 + ka_2)x^2 + \dots = 0$$

Again the coefficient of each power of x is zero. Thus,

Coeff. of x^0 : $\quad 2a_2 + ka_0 = 0 \Rightarrow a_2 = -\frac{k}{2}a_0 \qquad$ (iv)

Coeff. of x^1 : $\quad 6a_3 + (k-2)a_1 = 0 \Rightarrow a_3 = -\frac{k-2}{6}a_1 \qquad$ (v)

Coeff. of x^2 : $\quad 12a_4 + (k-6)a_2 = 0 \Rightarrow a_4 = -\frac{k-6}{12}a_2 \qquad$ (vi)

In general, for the n th power of x, you can write

Coeff. of x^n: $\quad (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + ka_n = 0$

or

$$a_{n+2} = \frac{n(n-1) + 2n - k}{(n+1)(n+2)}a_n$$
$$= \frac{n(n+1) - k}{(n+1)(n+2)}a_n\,; \quad n = 0, 1, 2, 3, \dots$$

You can readily verify by putting $k = m(m+1)$ that the numerator can be written as $(m-n)(n+m+1)$. Hence, this expression takes the form

$$a_{n+2} = -\frac{(m-n)(m+n+1)}{(n+1)(n+2)}a_n\,; \quad n = 0, 1, 2, \dots \qquad \text{(vii)}$$

This equality enables us to determine each expansion coefficient in terms of the second one preceding it, except for a_0 and a_1 (which are arbitrary). Such a relation between the coefficients is called a **recurrence relation** or **recursion formula.**

The recurrence relation (vii) implies that coefficients with even subscripts can be expressed in terms of a_0 and those with odd subscripts in terms of a_1. That is,

$$a_2 = -\frac{m(m+1)}{2!}a_0 \qquad a_3 = -\frac{(m-1)(m+2)}{3!}a_1$$

$$a_4 = -\frac{(m-2)(m+3)}{12}a_2 \qquad a_5 = -\frac{(m-3)(m+4)}{20}a_3$$

$$= \frac{(m-2)m(m+1)(m+3)}{4!}a_0 \qquad = \frac{(m-3)(m-1)(m+2)(m+4)}{5!}a_1$$

and so on.

By inserting these values for the coefficients in (ii), you can write

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \qquad \text{(viii)}$$

where

$$y_1 = 1 - \frac{m(m+1)}{2!}x^2 + \frac{(m-2)m(m+1)(m+3)}{4!}x^4 - \dots + \dots$$

and

$$y_2 = 1 - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!}x^5 - \dots + \dots$$

are two solutions over the interval $-1 < x < 1$. You may now again ask: Are y_1 and y_2 linearly independent? To answer this question, we note that y_1 contains only even powers of x whereas y_2 contains only odd powers of x. As a result, the ratio y_1/y_2 will not be constant, implying that y_1 and y_2 are linearly independent solutions. And (viii) is a general solution of Legendre's equation.

SAQ 3

Spend 15 min

The equation

$$y'' - 2xy' + 2my = 0$$

plays a particularly important role in statistics. Its solutions are known as **Hermite polynomials**. We would like you to obtain the coefficients of the power series solution of this equation.

Before proceeding, let us summarise the steps you should follow to solve a second order ODE for which $x = 0$ is an ordinary point.

Power Series Method

Step1 : Assume a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Step2 : Substitute the assumed solution and its derivatives into the given differential equation.

Step 3 : Equate the coefficients of each power of x to zero for a homogeneous equation. This results in a recursion relation, which helps us in determining successively the coefficients occurring in the power series in terms of two arbitrary constants.

Step 4 : Write the explicit form of the series solution which satisfies the given equation.

So far we have refrained from mathematical rigour. It may, however, be pointed out here that the mathematical justification of power series method is contained in Fuchs' Theorem. We state it without giving proof:

If $x = x_0$ is an ordinary point of the equation

$$y'' + p(x)y' + q(x)y = 0$$

then there exists a unique function $y(x)$ which is analytic and satisfies the given equation in the neighbourhood of x_0 as well as the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$ where a_0 and a_1 are two arbitrary constants.

Several second order ODEs with variable coefficients appearing in many important physical problems have coefficients which are not analytic functions. In particular, these equations may have a regular singularity at $x = x_0$. For such equations, power series solution of the form given by Eq. (3.2) is not physically acceptable and we use an extended power series which always provides at least one solution around a regular singular point:

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \qquad (3.4)$$

Such a series is also called **Frobenius series** with index r. Here r may be any (real or complex) number so that $a_0 \neq 0$. You will note that the series given in Eq. (3.4) reduces to a power series (Eq. (3.3)) if r is a non-negative integer.

3.4 THE FROBENIUS' METHOD

We illustrate the use of Frobenius' method when $x = 0$ is a regular singular point of Eq. (3.2):

$$y'' + p(x)y' + q(x)y = 0$$

Since $p(x)$ and $q(x)$ are not analytic at $x = 0$ but $x = 0$ is a regular singularity, let us rewrite this equation in terms of $b(x) = x\,p(x)$ and $c(x) = x^2 q(x)$, which will be analytic at $x = 0$. So, we multiply throughout by x^2 to obtain

$$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$$

or

$$x^2y'' + x\,b(x)y' + c(x)y = 0 \tag{3.5}$$

Since $b(x)$ and $c(x)$ are analytic at $x = 0$, we can write

$$b(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots = \sum_{n=0}^{\infty} b_n x^n \tag{3.6a}$$

and

$$c(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n \tag{3.6b}$$

You will note that if it happens that $b_0 = c_0 = c_1 = 0$, then $x = 0$ defines an ordinary point rather than a regular singular point.

Differentiating the series expansion given in Eq. (3.4) for $x_0 = 0$ term by term, you will get

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting for $y(x)$, $y'(x)$, $y''(x)$, $b(x)$ and $c(x)$ in Eq.(3.5), we get

$$x^r\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^n + x^r\Big[\sum_{n=0}^{\infty} (n+r)\,a_n x^n\Big]\Big[\sum_{m=0}^{\infty} b_m x^m\Big]$$

$$x^r\Big[\sum_{m=0}^{\infty} c_m x^m\Big]\Big[\sum_{n=0}^{\infty} a_n x^n\Big] = 0$$

As before, let us equate the sum of the coefficients of each power of x to zero. This yields a system of equations involving the unknown coefficients a_n's. The coefficient of x^r, which is the lowest power of x, is obtained from the $n = 0$ term. Equating this coefficient to zero, you will obtain

Coefficient of x^r: $\quad [\,r(r-1) + r\,b_0 + c_0\,]\,a_0 = 0$

Since $a_0 \neq 0$, this equality will be satisfied if

$$r^2 + (b_0 - 1)\,r + c_0 = 0 \tag{3.7}$$

Eq.(3.7) is called the **indicial equation** corresponding to the given differential equation. Since the indicial equation is quadratic, it will have two roots. This means that there should be two Frobenius series solutions. Will these solutions always be linearly independent? Not necessarily. In fact, the roots of the indicial equation give us some idea of the nature of solutions of the ODE of interest. In practice, the desired solutions are obtained using the steps outlined for the power series method. However, before plunging into these details, we would like you to go through the following example.

Example 4

Determine the roots of the indicial equation around the origin for the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

Solution

To be able to compare the given equation with Eq. (3.2), we divide throughout by x^2. This gives

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - 1/9}{x^2}\right)y = 0 \qquad \text{(i)}$$

You would readily recognise that this equation has a singularity at $x = 0$. Is it regular or irregular? To discover this, compare (i) with Eq. (3.2). You will note that $p(x) = \frac{1}{x}$ and $q(x) = \frac{x^2 - \frac{1}{9}}{x^2}$ so that $xp(x) = 1$ and $x^2q(x) = x^2 - (1/9)$. In the limit $x \to 0$, $xp(x) = 1$ and $x^2q(0) = -(1/9)$. That is, $b(x)$ and $c(x)$ are analytic at $x = 0$. Thus, $x = 0$ is a regular singular point.

We can, therefore, assume a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Differentiating it with respect to x, we get

$$y'(x) = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1}$$

and

$$y''(x) = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$$

On substituting these in the given equation, we get

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \frac{1}{9}\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

To arrive at the indicial equation, we equate the coefficients of the lowest power of x, i.e., x^r to zero. This gives

$$a_0\left[r(r-1) + r - \frac{1}{9}\right] = 0$$

For $a_0 \neq 0$, the indicial equation takes the form

$$r^2 - \frac{1}{9} = 0$$

which has roots $r = \pm\frac{1}{3}$. That is, the roots of the indicial equation are distinct. Moreover, they do not differ by an integer.

Let us now pause for a minute and ask: Will the roots always be distinct? Certainly not. Moreover, even when they are distinct, they may differ by an integer. In fact, there are three possibilities. The indicial equation may have (i) distinct roots not differing by an integer, (ii) double roots, and (iii) distinct roots differing by an integer. To discover these, you should solve the following SAQ.

SAQ 4

Spend 10 min

Determine the roots of the indicial equations corresponding to the following ODEs about $x = 0$.

$$y_1(x) = a_0 x^{5/6}\left[1+\sum_{p=1}^{\infty}(-1)^p\left(\frac{3}{4}\right)^p \frac{x^{2p}}{p\,!\,1\times 4\times 7\,....\,(3p+1)}\right]$$

For $r = r_2 = 1/6$ also, all odd coefficients will vanish and we leave it as an exercise for you to verify that

$$y_2(x) = d_0 x^{1/6} + \sum_{p=1}^{\infty} d_{2p}\, x^{2p+(1/6)}$$

where

$$d_{2p} = (-1)^p\left(\frac{3}{4}\right)^p \frac{d_0}{p\,!\,2\times 5\times 8\ldots(3p-1)}$$

Case 2 : Double Root of the Indicial Equation

When roots of the indicial equation are equal, we cannot obtain two linearly independent solutions using the procedure outlined above. In fact, there can be only one Frobenius series solution. To determine this solution, we first find r from Eq. (3.7) :

$$r = \frac{-(b_0-1)\pm\sqrt{(b_0-1)^2-4c_0}}{2}$$

The condition for equal roots of a quadratic equation $ax^2+bx+c = 0$ is $b^2-4ac = 0$. This means that the term under the radical sign will vanish.

Using the condition for equal roots, you will find that the common root is $-\dfrac{b_0-1}{2} = \dfrac{1-b_0}{2}$.

Hence, it readily follows from Eq. (3.4) that one of the solutions will be of the form

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{(1-b_0)/2} \sum_{n=0}^{\infty} a_n x^n \qquad (3.9)$$

where a_n's are unknown constants.

The second linearly independent solution is

$$y_2(x) = y_1(x)\ \ln\ x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)\left(\sum_{m=1}^{\infty} a_m x^m\right)$$

$$= y_1(x)\ \ln\ x + x^r \sum_{i=1}^{\infty} A_i x^i \qquad (3.10)$$

where A_i is some other constant.

Case 3 : Roots of the Indicial Equation Differing by an Integer

When the roots (r_1, r_2) of the indicial equation are distinct but differ by an integer, we can always determine the first Frobenius series solution as before. If r_1 ($= r$) and r_2 ($= r-p$, where p is a positive integer) are the two roots, then

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$

The other linearly independent Frobenius solution is

$$y_2(x) = k_p\, y_1(x)\ \ln x + x^{r_2}\sum_{m=0}^{\infty} D_m x^m$$

Let us now summarise what you have studied in this unit.

3.5 SUMMARY

- A second order ODE with variable coefficients

$$y'' + p(x)y' + q(x)y = 0$$

is said to have an ordinary point at $x = x_0$ if $p(x)$ and $q(x)$ are analytic at $x = x_0$. Otherwise, the point is said to be singular. The singularity is regular if $\lim_{x \to x_0} (x - x_0)p(x)$ and $\lim_{x \to x_0} (x - x_0)^2 q(x)$ are finite at $x = x_0$.

- We can solve a second order ODE with variable coefficients around an ordinary point at $x = 0$ by assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. The constants a_n's are determined using the recursion relation.

- The solution around a regular singularity at $x = 0$ is obtained using Frobenius method by assuming a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

- The indicial equation is obtained by equating the sum of the coefficients of the lowest power of x to zero.

- The roots of the indicial equation give us an idea of the nature of solutions of the ODE. The roots of the indicial equation may be distinct, repeated or differ by an integer.

- For distinct roots, two linearly independent solutions are

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

and

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} d_n x^n$$

3.6 TERMINAL QUESTIONS

1. Like Legendre's equation, another ODE that arises in advanced studies in physics and applied mathematics is the Bessel's equation of order m:

$$x^2 y'' + xy' + (x^2 - m^2) y = 0$$

In Example 4, we solved Bessel's equation of order 1/3. Use Frobenius' method to solve this equation.

2. Solve the ODE

$$y'' + y = e^x$$

around the point $x = 0$.

3.7 SOLUTIONS AND ANSWERS

SAQs

1(a) $x^2 y'' + 3xy' + y = 0$

To compare it with the standard form ($y'' + p(x)y' + q(x)y = 0$), you should divide it by x^2. This gives

Coeff. of x^2 : $3a_3 + a_1 = 1 \Rightarrow a_3 = \frac{1-a_1}{3} = \frac{1}{3}$

Coeff. of x^3 : $4a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4} = \frac{a_0+2}{8}$

:

Coeff. of x^{n-1} : $na_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{1}{n}a_{n-2}$ for $n \geq 4$

Iteration of this formula for $n \geq 4$ yields

$$a_4 = -\frac{1}{4}a_2 = -\frac{1}{4}\left(-\frac{a_0+2}{2}\right) = \frac{a_0+2}{4\times 2}$$

$$a_5 = -\frac{1}{5}a_3 = -\left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = -\frac{1}{5\times 3}$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{a_0+2}{6\times 4\times 2}$$

$$a_7 = -\frac{1}{7}a_5 = -\frac{a_0+2}{7\times 5\times 3}$$

$$a_8 = -\frac{1}{8}a_6 = \frac{a_0+2}{8\times 6\times 4\times 2}$$

$$a_9 = -\frac{1}{9}a_7 = -\frac{1}{9\times 7\times 5\times 3}$$

:

The solution can then be written as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 - \frac{a_0+2}{2}x^2 + \frac{1}{3}x^3 + \frac{a_0+2}{4\times 2}x^4 - \frac{1}{5\times 3}x^5 - \frac{a_0+2}{6\times 4\times 2}x^6 + \frac{1}{7\times 5\times 3}x^7 + \ldots$$

$$= a_0 - (a_0+2)\left(\frac{1}{2}x^2 - \frac{1}{4\times 2}x^4 + \frac{1}{6\times 4\times 2}x^6 + \ldots\right)$$

$$+ \left(\frac{1}{3}x^3 - \frac{1}{5\times 3}x^5 + \frac{1}{7\times 5\times 3}x^7 - \ldots\right)$$

3. You would readily recognise that $x = 0$ is an ordinary point of the given equation. So we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substituting these in the given equation, we find that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} na_n x^n + 2m\sum_{n=0}^{\infty} a_n x^n = 0$$

In the expanded form,

$$(2a_2 + 6a_3 x + 12a_4 x^2 + ... + n(n-1)a_n x^{n-2} + ...)$$

$$-2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + ... + na_n x^n + ...)$$

$$+ 2m(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...) = 0$$

Collecting the coefficients of each power of x, we get

$$(2a_2 + 2m a_0) + (6a_3 - 2a_1 + 2m a_1)x$$

$$+ (12a_4 - 4a_2 + 2m a_2)x^2 + ...$$

$$+ [(n+2)(n+1)a_{n+2} - 2n a_n + 2m a_n]x^n + ... = 0$$

Next we equate the coefficient of each power of x to zero. The result is

Coeff. of x^0: $2a_2 + 2m a_0 = 0 \Rightarrow a_2 = -\dfrac{2m a_0}{2\times 1}$

Coeff. of x^1: $6a_3 + (2m-2)a_1 = 0 \Rightarrow a_3 = \dfrac{2(1-m)}{3\times 1}a_1$

Coeff. of x^2: $12a_4 + (2m-4)a_2 = 0 \Rightarrow a_4 = \dfrac{(2-m)}{3\times 2}a_2 = -\dfrac{2m(2-m)}{4\times 3}a_0$

:

Coeff. of x^n : $(n+2)(n+1)a_{n+2} - 2(n-m)a_n = 0$

or $\quad a_{n+2} = \dfrac{2(n-m)}{(n+2)(n+1)}a_n$

4(a) $\quad x(x-1)y'' + (3x-1)y' + y = 0$

Let us first rewrite this equation in the standard form by dividing throughout by $x(x-1)$:

$$y'' + \frac{(3x-1)}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

You can readily identify that

$$p(x) = \frac{(3x-1)}{x(x-1)} \quad \text{and } q(x) = \frac{1}{x(x-1)}$$

Both functions diverge at $x = 0$ as well as $x = 1$. But $\lim_{x\to 0} xp(x) = 1$, $\lim_{x\to 0} x^2 q(x) = 0$, $\lim_{x\to 1}(x-1)p(x) = 2$ and $\lim_{x\to 1}(x-1)^2 q(x) = 0$ so that $x = 0$ and $x = 1$ are regular singularities. For $x = 0$, let us, therefore, assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

so that $\quad y'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$

and $\quad y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

On substituting for $y(x)$, $y'(x)$ and $y''(x)$ in the given equation, we find that

$$x(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$+(3x-1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}$$

$$+3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

You will note that the lowest power of x is x^{r-1}. By equating the sum of its coefficients to zero, we have

$$[-r(r-1)-r]a_0=0$$

Since $a_o \neq 0$, we must have $r^2 = 0$

Hence, this indicial equation has a double root: $r = 0$.

(b) $$(x^2-1)x^2y''-(x^2+1)xy'+(x^2+1)y=0$$

Let us divide throughout by $(x^2-1)x^2$ to put it in the standard form ((Eq. (3.2)):

$$y''-\frac{x^2+1}{x(x^2-1)}y'+\frac{x^2+1}{x^2(x^2-1)}y=0$$

You will readily recognise that $x = 0$ is a regular singular point of the given ODE. Let us, therefore, assume a solution of the form

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}$$

so that $$y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

and

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Substituting these in the given equation, we find that

$$(x^2-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-(x^2+1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+(x^2+1)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Performing the multiplication, we find that

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+2}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+2}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}a_n x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Combining the first, third and fifth series together, and second and last series together, we find that

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_n x^{n+r+2}$$

$$-\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-1]a_n x^{n+r} = 0$$

On simplification, we find that

$$\sum_{n=0}^{\infty}(n+r-1)^2 a_n x^{n+r+2} - \sum_{n=0}^{\infty}(n+r-1)(n+r+1)a_n x^{n+r} = 0$$

The lowest power of x in this equation is x^r. By equating its power to zero, you will obtain the required indicial equation:

$$(r+1)(r-1) = 0$$

whose roots are $r_1 = 1$ and $r_2 = -1$. You will readily recognise that these roots differ by an integer.

Terminal Questions

. The family of ODEs

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

is known as **Bessel's equations.** The parameter m is real and non-negative. You would readily note that $x = 0$ is a regular singular point of the equation. So we assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{(i)}$$

Substitute $y(x)$ and its derivatives in the given equation. This yields

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} m^2 a_n x^{n+r} = 0$$

Changing the first summation so that the exponent on x is $n+r$ and collecting other series, we have

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-m^2]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0 \quad \text{(ii)}$$

The smallest power of x is x^r ($n = 0$). Equating the coefficient of x^r to zero, we get

$$[r(r-1)+r-m^2]a_0 = 0$$

Since $a_o \neq 0$, we have the indicial equation

$$r^2 - m^2 = 0 \quad \text{(iii)}$$

which has roots $r_1 = m$ and $r_2 = -m$.

Depending on the value of m, the solutions can differ vastly:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \quad \text{(iv)}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-m} \quad \text{(v)}$$

To find y_1, let us write (ii) in the expanded form

$$x^r[(r^2-m^2)a_0 + \{(r+1)^2-m^2\}a_1 x + \{(r+2)^2-m^2\}a_2x^2 + \ldots + \{(n+r)^2-m^2\}a_n x^n + \ldots]$$
$$+ x^r[a_0x^2 + a_1x^3 + a_2x^4 + \ldots + a_{n-2}x^n + \ldots] = 0$$

Equating the coefficient of each power of x to zero, we get

Coeff. of x^r : $(r^2 - m^2)\, a_0 = 0$

Coeff. of x^{r+1} : $[(r+1)^2 - m^2]\, a_1 = 0$

Coeff. of x^{r+2} : $[(r+2)^2 - m^2)\, a_2 - a_0 = 0 \Rightarrow a_2 = -\dfrac{1}{(r+2)^2 - m^2} a_0$

Coeff. of x^{r+n} : $[(n+r)^2 - m^2]\, a_n + a_{n-2} = 0 \Rightarrow a_n = -\dfrac{1}{(n+r)^2 - m^2} a_{n-2}$

For $r = m$, we find that $a_1 = 0$ since the bracketed quantity does not vanish. Then recursion relation implies that $a_3 = a_5 = a_7 = 0 = ...$ That is, all odd subscripted coefficients vanish. For even subscripted coefficients, we find that

$$a_2 = -\frac{a_0}{(m+2)^2 - m^2}$$

$$= -\frac{a_0}{(m+2-m)(m+2+m)}$$

$$= -\frac{a_0}{2^2 (m+1)}$$

Similarly,

$$a_4 = -\frac{a_2}{2^2 \times 2 (m+2)} = \frac{a_0}{2^4 \times 2 (m+1)(m+2)}$$

$$a_6 = -\frac{a_4}{2^2 \times 3 (m+3)} = -\frac{a_0}{2^6 \times 3 \times 2 (m+1)(m+2)(m+3)}$$

In general,

$$a_{2n} = \frac{(-)^n a_0}{2^{2n} n!\, (m+1)(m+2) \ldots (m+n)} \qquad n = 0, 1, 2, \ldots$$

The solution corresponding to $r_2 = -m$ is found by simply replacing m by $-m$ provided m is not an integer.

2. Letting $y = \sum_{n=0}^{\infty} a_n x^n$, we find that

$$y' = \sum_{n=0}^{\infty} a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2}$$

From Table 3.1, you would recall that series expansion for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substituting in the given differential equation, we obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Rearranging the terms, you will obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \left(a_n - \frac{1}{n!} \right) x^n = 0$$

Since the right-hand side of the equality is identically zero, we equate each coefficient of every power of x on the left-hand side to zero. To evaluate a_n, the exponents in the two series must be made the same by shifting the index. Here, we choose to replace n with $n-2$ in the second series. Thus,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty}\left(a_{n-2} - \frac{1}{(n-2)!}\right)x^{n-2} = 0$$

We combine the two series to obtain

$$\sum_{n=2}^{\infty} x^{n-2}\left[n(n-1)a_n + a_{n-2} - \frac{1}{(n-2)!}\right] = 0$$

The coefficients of x^{n-2} for $n \geq 2$ must be equal to zero. Thus,

$$n(n-1)a_n + a_{n-2} - \frac{1}{(n-2)!} = 0$$

Solving for a_n, we obtain the recursion formula

$$a_n = \frac{1}{n!} - \frac{1}{n(n-1)}a_{n-2}, \quad n = 2, 3, 4, ..$$

Iteration of this formula yields

$$a_2 = \frac{1}{2!} - \left(\frac{1}{2\times 1}\right)a_0 = \frac{1}{2!} - \frac{1}{2!}a_0$$

$$a_3 = \frac{1}{2!} - \left(\frac{1}{3\times 2}\right)a_1 = \frac{1}{3!} - \frac{1}{3!}a_1$$

$$a_4 = \frac{1}{4!} - \left(\frac{1}{4\times 3}\right)a_2 = \frac{1}{4!}a_0$$

$$a_5 = \frac{1}{5!} - \left(\frac{1}{5\times 4}\right)a_3 = \frac{1}{5!}a_1$$

and so on

Since $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and a_0 and a_1 are arbitrary, we get

$$y(x) = a_0 + a_1 x + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \left(\frac{1}{2!} - \frac{1}{2!}a_0\right)x^2 + \left(\frac{1}{3!} - \frac{1}{3!}a_1\right)x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 + \ldots$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ..\right)$$

$$+ \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$= a_0 y_1(x) + a_1 y_2(x) + y_p$$

where y_1 and y_2 are, respectively, infinite series of even and odd terms, and

$$y_p = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \ldots$$

Since this differential equation is linear and of the second order, the form of the solution is $y = y_c + y_p$, where y_c consists of a linear combination of two linearly independent functions and y_p is a particular solution to the given nonhomogeneous equation.

APPENDIX A: POWER SERIES

A power series about a point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{(A.1)}$$

where the numbers $a_o, a_1, a_2, \ldots, a_n, \ldots$ are called the coefficients of the power series. A power series does not include terms with negative or fractional powers.

We say that the power series converges if

$$\lim_{n \to \infty} \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{(A.2)}$$

exists. The value of the limit is called the sum of the power series at the point $x = x_0$. If the limit does not exist, the power series is said to **diverge**. The interval of values of x for which a power series converges is called the **interval of convergence** and is denoted as $|x - x_0| < R$, where x_0 is called the centre of the power series and R is called the **radius of convergence**. If $R = 0$, the series converges only at x_0; if $R = \infty$, the series converges for all values of x.

Within a common interval of convergence, two power series may be added term by term. That is,

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \qquad \text{(A.3)}$$

Further, within the interval of convergence, the power series represents a function whose derivative and integral may be found from term-by-term differentiation and integration. That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

and

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1}$$

A power series expansion of $f(x)$ around $x = 0$ is called a **Maclaurin series**. The Maclaurin series of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}$$

where $f^{(n)}(0)$ means the value of the nth derivative of f at $x = 0$. Recall that $f(0) = f$ and $0! = 1$

UNIT 4 SOME APPLICATIONS OF ODEs IN PHYSICS

Structure

4.1 INTRODUCTION

In Units 1 to 3 you have learnt various techniques of solving first and second order ODEs. Recall that we have studied these techniques so that we are able to answer real-life questions such as the following: Does the quantity of fuel burned by a rocket affect its velocity? How long does it take a polluted gulf to return to its natural state, once man-made pollution is stopped? What is the response of an *LCR* circuit to an applied signal? You are now on the verge of being able to answer such questions. This unit on applications of ODEs will further help you in this respect.

This unit is primarily concerned not with *how to do* mathematics but with *how to use* it. Here we shall be applying the techniques you have learnt so far to solve a variety of real problems involving differential equations. These problems have both their origin and solutions outside mathematics. Doing the mathematics is only a part of the process called mathematical modelling. Today, as scientists seek to further our understanding of nature, the technique of representing our "real- world" in mathematical terms has become an invaluable tool. Indeed, *the process of mimicking reality by using the language of mathematics is known as mathematical modelling.*

In this unit, you will first learn what is involved in the process of mathematical modelling, especially with ODEs. As the unit progresses, you will realise that mathematical modelling with ODEs finds immense use in various areas of physics. We hope that having studied the unit, you will be able to answer the kind of questions posed above and many others you are likely to come across as you delve deeper in physics.

In this block you have studied various methods of solving first and second order ordinary differential equations and their applications in physics. In the next block, we shall discuss partial differential equations (PDEs). You will learn how to model physical systems with PDEs and various methods of solving them.

Objectives

After studying this unit you should be able to

- use first and second order ODEs to mathematically model known physical phenomena
- solve coupled differential equations pertaining to coupled oscillations, coupled electrical circuits and charged-particle motion in electric and magnetic fields.

4.2 MATHEMATICAL MODELLING

Are you familiar with the process of mathematical modelling? If you are, you may skip the following discussion. Just go through Fig. 4.1 and study Example 1. If not, you may like to understand the process of mathematical modelling with the help of an example. It may appear simple to you but it will help you to concentrate on the modelling process itself. Consider the following real world problem.

Suppose you and a couple of your friends have to attend a counselling session at the Study Centre. However, the bus in which you are travelling gets stuck up at a railway crossing with about 25 km still to go. The session is to start in about 45 min. The problem is: At what time would it be pointless to continue on to the counselling session if you are still stationary? You know that there is a wayside restaurant about 5 km ahead. Would it be better, instead, to get down at the restaurant for a self-help meeting over a cup of tea or coffee?

Now look at the problem closely. Wouldn't you first ask yourself *how long* should you remain in the stationary queue before it becomes pointless to travel further? You have taken the first step in the process of mathematical modelling by **specifying the real problem.**

The next thing that you would like to know is whether you can reach the Study Centre if and when the bus gets moving. You ask the driver about this. He estimates that once the bus gets going its average speed would be about 50 km h^{-1}. And a margin of about 10 min should be kept for the remaining stops en route. In this way, the journey can be simplified into two parts —one in which the bus actually moves, second in which time taken at bus-stops is accounted for. For each part numerical values are assumed for the speed and time. The driver's estimate could be called a **model** of the remaining journey. So the **second step is to set up a model.**

You can see that the model has helped in representing the real situation (the remaining journey) in a simple manner, so that you can solve your problem specified in Step 1. Thus, *a model is a simplified representation of some aspect of reality.* It is constructed for a specific purpose, such as to solve a real world problem.

This simple model leads you to a related question: How long would it take to complete the remaining journey? It is a mathematical problem. So the **third step is to formulate the mathematical problem.**

The next step obviously is to calculate the time needed for the remaining journey in the framework of this model: Time taken to travel 25 km is 60 min $\times \dfrac{25 \text{ km}}{50 \text{ km}} = 30$ min, time for stops en route = 10 min, giving a total of 40 min. The **fourth step then is to solve the mathematical problem.**

Now you would like to interpret your result. You may do so in different ways. You could wait in the bus for another 5 min if you want to reach on time. If you are prepared to arrive late (say by 10 min) you can wait for 15 min. So it is worth going on if you are not stuck at the crossing for more than 5 min (or 15 min if you don't mind being 10 min late). **The fifth step then is to interpret the result.**

Next you would like to compare the model and the solution with reality. Are the speed and distance correct? You could question the model: Could not the driver drive faster and spend less time at stops? Are the simplifications in this model suitable? You could revise the simplifications and repeat the process with different data. So the **sixth step is to compare the solution with reality.**

Finally, when the bus starts moving you would use your results to decide on whether to proceed to the Study Centre or not. If you have been stuck for less than 15 min, you can go to the Study Centre. Otherwise you could get off at the restaurant for the self-help session. You would of course have only 6 min to decide! **The final step in the process is to use the results.**

These modelling steps can be represented by a modelling process shown in Fig. 4.1. It also **shows the salient features of each step in the modelling process, which you should keep in mind.** Study it carefully before going further.

We hope you are clear about the various steps in the mathematical modelling process. Since we are considering applications of ODEs, we will be modelling with differential equations, i.e., in step 3 of the process, we simply **formulate the mathematical problems in terms of differential equations.**

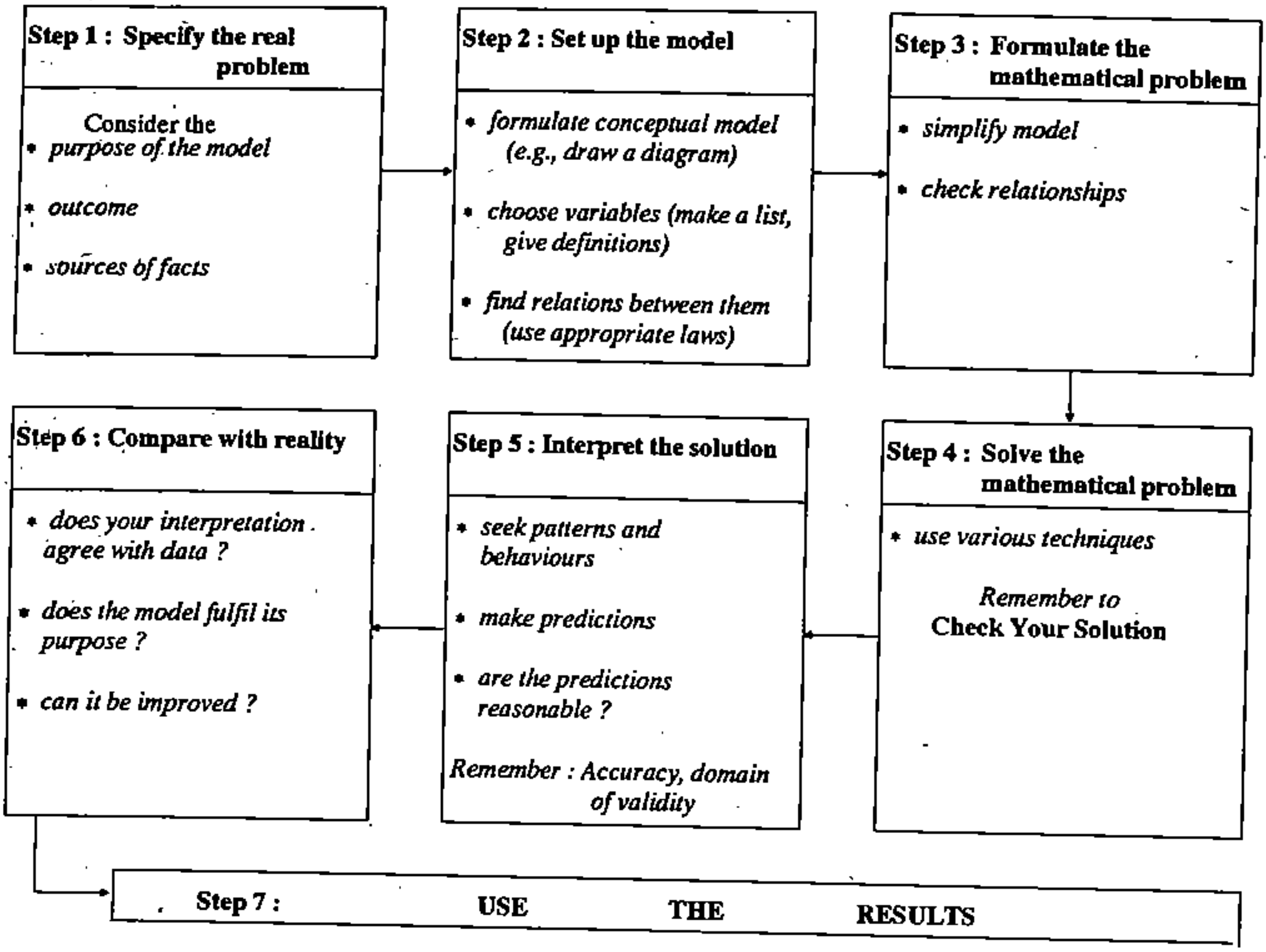


Fig. 4.1: The process of mathematical modelling showing the salient features of each step.

Let us now illustrate the modelling process with an example from physics, which is of use in everyday life. In working this example, we shall follow the steps listed in Fig. 4.1.

Example 1: Heating and cooling of buildings

A godown for storing cement has to be built. It will have no external heating or cooling arrangement. How can it be designed so that its temperature changes by a specified amount in a given period of time?

Solution

Let us follow the modelling cycle stepwise to solve this 'real world' problem.

Step 1: Specify the real problem

We have to answer the following question: How long does it take to change the building temperature by a specific amount?

Step 2: Set up a model

Our model must describe the 24-h temperature variation inside the building as a function of time and the outside temperature. In the simplest model we can view the building as a single entity, i.e., we do not take individual rooms into account. Now, recall from Fig. 4.1 that we must identify the variables important to our problem. We can immediately see that the inside temperature T and time t are the two main variables. We have to find $T(t)$, i.e., the temperature inside the building at time t. What are the factors that affect $T(t)$? Two factors come to mind immediately:

In setting up a model, you have to discard those variables that have very little or no effect on the process. For example, in studying the motion of a falling body, its colour is of no interest.

1) The heat produced by people, lights, machines inside the building will increase $T(t)$. Let $H(t)$ be the rate of increase in $T(t)$ due to this factor.

2) The effect of outside temperature $T_s(t)$ on $T(t)$.

Newton's law of cooling states that the rate of change of temperature of a cooling object is proportional to the difference in temperature of the object and the temperature of the surrounding medium, provided that this difference is not very large.

Having identified the variables, we must find the relationship between them. To model the effect of $T_s(t)$ on $T(t)$, we can apply **Newton's law of cooling**, if $T_s(t) < T(t)$ at all times. We get that the rate of change in the inside temperature $T(t)$ is proportional to the difference between $T(t)$ and the outside temperature $T_s(t)$. Thus, the time rate of change in $T(t)$ due to $T_s(t)$ is $\{-K[T(t) - T_s(t)]\}$. K is a positive constant which depends on the physical properties of the building, such as the number of doors and windows, colour scheme, type of insulation, etc. K does not depend on T_s, T or t. Note that K has the dimension of reciprocal of time. The minus sign is used to show that the temperature of the building will decrease. We can now formulate the mathematical problem and solve it.

Step 3: Formulate the mathematical problem

From our model, we get the following first order ODE:

$$\frac{dT(t)}{dt} = K[T_s(t) - T(t)] + H(t) \tag{4.1}$$

Step 4: Solve the mathematical problem

Eq. (4.1) is a first order linear nonhomogeneous ODE. It can be solved using the method of Sec. 1.6 of Unit 1. We rewrite Eq. (4.1) in the standard form

$$\frac{dT(t)}{dt} + P(t)T(t) = Q(t) \tag{4.2a}$$

where

$$P(t) = K, \quad Q(t) = KT_s(t) + H(t) \tag{4.2b}$$

The integrating factor is $\exp\left(\int K\,dt\right) = e^{Kt}$. The solution of Eq. (4.2a) is

$$T(t) = e^{-Kt}\left[\int e^{Kt}[KT_s(t) + H(t)]\,dt + C\right] \tag{4.3}$$

Let us simplify our model further to solve Eq. (4.3). Let $H(t)$ be negligibly small and $T_s(t)$ be a constant T_{s0}. So with $H(t) = 0$ and $T_s(t) = T_{s0}$, we get

$$T(t) = e^{-Kt}\left[\int e^{Kt}\,KT_{s0}\,dt + C\right]$$

$$= e^{-Kt}[T_{s0}e^{Kt} + C] = T_{s0} + Ce^{-Kt} \tag{4.4a}$$

Eq. (4.4a) is a general solution of Eq. (4.1) in our simplified model. You can check the solution by substituting Eq. (4.4a) into Eq. (4.1) with $H(t) = 0$ and $T_s(t) = T_{s0}$. Let us now specify the initial condition to determine C and get a particular solution. Let $T = T_0$ at $t = 0$. Then

$$T_0 = T_{s0} + C, \text{ or } C = T_0 - T_{s0}$$

Checking the Solution
Using Eq. (4.4a), the L.H.S. of Eq. (4.1) becomes (for the simplified model)

$$\frac{dT}{dt} = -KCe^{-Kt}$$

The R.H.S. of Eq. 4.1 is

$$KT_{s0} - KT_{s0} - KCe^{-Kt} = -KCe^{-Kt}$$

Thus, LHS = RHS

Thus, the particular solution is

$$T(t) = T_{s0} + (T_0 - T_{s0})\,e^{-Kt} \tag{4.4b}$$

Let us now interpret this solution.

Step 5: Interpret the solution

Since $T_{s0} < T_0$, $T(t)$ decreases exponentially from T_0. As t increases, the exponential term falls off and $T(t)$ tends to T_{s0} (see Fig. 4.2). Recall the problem we specified in Step 1. Let us determine the time it takes for the temperature difference $(T - T_{s0})$ to change from $(T_0 - T_{s0})$ to $\left(\frac{T_0 - T_{s0}}{e}\right)$. This time is called the **time constant of the building**. From Eq. (4.4b)

at $\quad t = 0,\ T - T_{s0} = T_0 - T_{s0}$

and at $\quad t = \frac{1}{K},\ T - T_{s0} = \frac{T_0 - T_{s0}}{e}$

So the time constant of the building is $1/K$. Thus, we can interpret the solution as follows: The building temperature decreases exponentially with a time constant $1/K$.

A typical value of the time constant of a building is 2 to 4 h. It can be made much shorter if the windows are open or if there are fans circulating the air. It can be made much longer if the building is well insulated. Let us now look at the last two steps of the process.

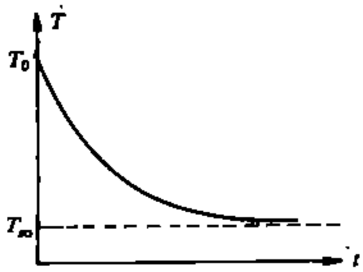


Fig. 4.2

Steps 6: Compare with reality

This can be done by collecting data on several similar buildings and comparing the results of this model for different time constants.

Step 7: Use the result

This simple model may be used to design buildings like warehouses, godowns or garages. The model can be further refined to take into account the heat inside or time variation of outside temperature. The effect of external heating (heaters) and cooling (coolers or air conditioners) can also be incorporated.

Now let us quickly summarise what you have studied in this section. We have introduced the idea of **mathematical modelling — it is a simplified mathematical representation of reality created to solve a specific problem.** The process can be broken down into seven steps shown in Fig. 4.1. However, **modelling need not always follow such a logical sequence.** *These steps only suggest the kind of things you should be doing in a logical sequence.* A good strategy is to begin with a simple model which produces a solvable mathematical problem. Then go around the modelling process again, refining your model until the comparison with reality is satisfactory and you can use the results.

Using these ideas you should now attempt an SAQ on modelling with differential equations. You will need a log table or a calculator for the calculations.

SAQ 1

Spend 10 min

A steel casting at a temperature of 20°C is put into an oven that has a temperature of 200 °C. One minute later, the temperature of the casting is 30°C. Determine the temperature profile of the casting. How long will it take the temperature of the casting to reach 190°C? [Hint: The rate at which heat is absorbed by the steel casting is proportional to the temperature difference of the casting and the surroundings.]

Now that you have grasped the process of modelling, you would like to apply it to some more physics problems. Let us take up the applications involving some first order ODEs.

4.3 FIRST ORDER ODEs IN PHYSICS

In Unit 1 you have studied two simple applications of first order ODEs in physics. For instance, you have studied about radioactive decay and the LR circuit (Example 5). In Example 1 of Sec. 4.2 you have modelled a process using Newton's law of cooling, which also involves a first order ODE. Newtonian mechanics and electric circuits are some other areas where modelling with first order ODEs is often used. Let us study applications of first order ODEs in these areas.

4.3.1 Applications in Newtonian Mechanics

Mechanics, as you know, is the study of motion of objects and the effect of forces acting on objects. Newtonian mechanics deals with the motion of objects that are large compared to atoms, and move with speeds much less than the speed of light. A model for Newtonian mechanics, can be based on Newton's laws of motion. For details of these laws and other concepts of Newtonian mechanics you can refer to the course Elementary Mechanics (PHE-01). Here we shall consider two specific examples on (i) motion under resistive forces that depend on velocity and (ii) velocity of escape.

Thus $C = v_0^2 - \frac{2GM}{R}$ and the particular solution becomes

$$v^2 = \frac{2GM}{r} + v_0^2 - \frac{2GM}{R} \tag{4.9b}$$

We do not take the negative root of Eq. (4.9b) as it does not give a physically meaningful solution.

Now, in order that the projectile escapes from the earth, v should remain positive for all values of r. On examining the right-hand side of Eq. (4.9b), we find that $v > 0$ always, if and only if,

$$v_0^2 - \frac{2GM}{R} \geq 0 \tag{4.10}$$

Let $v_0^2 - \frac{2Gm}{R} = v_1$

If $v_1 < 0$, then in Eq. (4.9b) $v = 0$ for

$v_1 = \frac{2GM}{r}$.

For, if $v_0^2 - \frac{2GM}{R} < 0$, then there will be a value of r for which $v = 0$. In such a situation the projectile would stop, its velocity **v** would change from positive to negative and it would return to the earth.

Thus, if a projectile is launched with an initial velocity v_0, such that $v_0 \geq \sqrt{2GM/r}$, it will escape from the earth. The minimum velocity of projection [with magnitude $v_e\,(= \sqrt{2GM/R})$] is called the **velocity of escape.**

Let us now study the application of first order ODEs in simple electrical circuits.

4.3.2 Simple Electrical Circuits

In your school courses you have studied about some simple electrical circuits in which we have an emf source, along with resistors, capacitors or inductors. If $i\,(t)$ denotes the current in the circuit at time t, the voltage drops across the resistor, capacitor and the inductor in an electrical circuit are shown in Fig. 4.6. Here, $q\,(t)$ is the charge on the capacitor at any instant t.

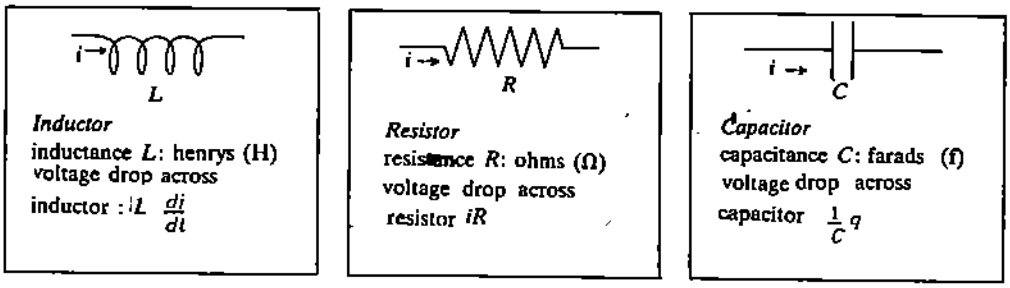


Fig.4.6: Elements in an electrical circuit.

Thus, voltage drop across the resistor, $V_R = iR$

voltage drop across the inductor, $V_L = L\,\frac{di}{dt}$

and voltage drop across the capacitor, $V_C = \frac{q(t)}{C}$

Kirchoff's voltage law:

At any instant, the algebraic sum of the voltage drops around any closed circuit is zero. Or, the voltage impressed on a closed circuit is equal to the sum of the voltage drops in the rest of the circuit.

Here, R is the resistance of the resistor, L the self-inductance of the inductor and C the capacitance of the capacitor. You know that any electrical circuit can be modelled by a differential equation which results from **Kirchoff's voltage law.** Using these basic ideas, let us consider an **ac circuit** containing a capacitor and resistor, better known as the RC circuit.

Example 3: RC **Circuits**

A sinusoidally varying source of emf is applied to a series RC circuit (Fig. 4.7). Determine the current through the circuit as a function of time.

Solution

Using Kirchoff's law, we can model the circuit by the following differential equation:

$$Ri(t) + \frac{q(t)}{C} = E(t) \tag{4.11a}$$

Since $E(t)$ varies sinusoidally, we have

$$E(t) = E_0 \sin \omega t$$

Since $i(t) = \frac{dq}{dt}$, on dividing Eq. (4.11a) by R, we can write it as

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E_0}{R} \sin \omega t \qquad (4.11b)$$

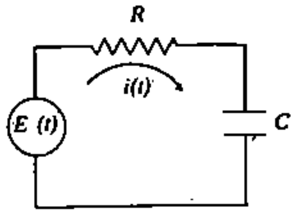


Fig. 4.7 : A series RC circuit (in which the resistor and capacitor are connected in series)

You have solved an equation analogous to Eq. (4.11b) in Unit 1 (Example 5 in Sec. 1.6.1). So you can write down the solution of Eq. (4.11b) as

$$q(t) = \frac{E_0}{R} e^{-t/RC} \int e^{t/RC} \sin \omega t \, dt + C' e^{-t/RC}$$

or

$$q(t) = \frac{E_0 C}{\sqrt{1 + \omega^2 R^2 C^2}} \sin(\omega t - \theta) + C_1 e^{-t/RC} \qquad (4.12a)$$

where

$$\theta = \tan^{-1}(\omega CR) \qquad (4.12b)$$

Hence, $$i(t) = \frac{dq}{dt} = \frac{E_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t - \theta) + C_2 e^{-t/RC} \qquad (4.13)$$

The second term on the right hand side of Eq. (4.13) decreases exponentially as t increases. It is called the transient term. The first term represents the **steady state current** which is sinusoidal.

SAQ 3

Spend 5 min

Let $E(t)$ be zero in the circuit of Fig. 4.7. The capacitor will discharge its charge through the resistor. Let its initial charge be q_0. Determine the charge $q(t)$ on it in terms of q_0, t, C and R.

In this section you have studied some applications of first order ODEs in physics. Let us now take up some situations in which we need second order ODEs to model physical phenomena.

4.4 SECOND ORDER ODEs IN PHYSICS

In Unit 2, you have modelled the natural, damped and forced motion of the spring-mass system using second order ODEs. You have also studied applications of second order ODEs in the physics courses of Elementary Mechanics (PHE-01), and Oscillations and Waves (PHE-02). In this section, we take up some more applications of second order ODEs, which you may not have yet studied.

4.4.1 Rotational Mechanical Systems

Let us consider a mechanical system in which the only motion is rotation about a fixed axis.

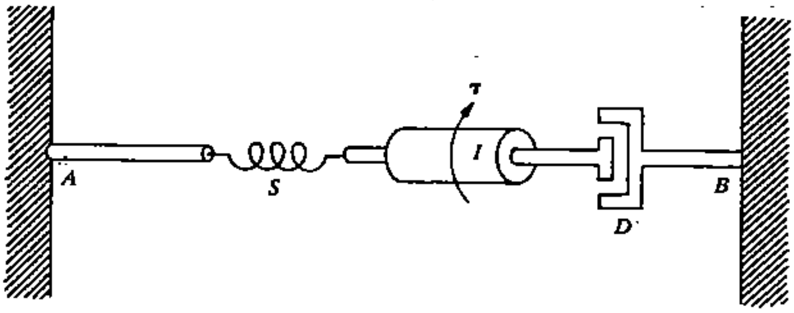


Fig. 4.8 : Rotational mechanical system consisting of a spring (S) governed by Hooke's Law, an inertial cylindrical element (I) and a damping element (D). AB is the axis of rotation.

This system consists of a linear spring S, an inertial cylindrical element having a constant moment of inertia I and a linear damper D (Fig. 4.8). The brakes in automobiles and the suspension type galvanometer can be modelled by similar systems. For this system, we have to deal with angular quantities only.

Let us set up a coordinate system about the fixed axis of rotation AB. Let θ be the angular displacement with $\theta = 0$ corresponding to the fixed axis. Now suppose an external torque τ is applied to the inertial element (the cylinder) as shown in the figure.

The rotational analogue of Newton's second law tells us that the net external torque on the cylinder equals the rate of change of its angular momentum, i.e.,

$$\boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = I\frac{d^2\boldsymbol{\theta}}{dt^2} \left[\because \mathbf{L} = I\boldsymbol{\omega} = I\frac{d\boldsymbol{\theta}}{dt} \right] \tag{4.14a}$$

The net external torque on the inertial element is

$$\boldsymbol{\tau}_{ext} = \boldsymbol{\tau} + \boldsymbol{\tau}_K + \boldsymbol{\tau}_B \tag{4.14b}$$

where $\boldsymbol{\tau}_K$ is the torque applied by the spring and $\boldsymbol{\tau}_B$ is the torque due to the damper D. These are, respectively, given as

$$\boldsymbol{\tau}_K = -K\boldsymbol{\theta} \tag{4.14c}$$

where K is the stiffness constant of the spring

and $$\boldsymbol{\tau}_B = -B\boldsymbol{\omega} = -B\frac{d\boldsymbol{\theta}}{dt} \tag{4.14d}$$

Then, we have

$$I\frac{d^2\boldsymbol{\theta}}{dt^2} = \boldsymbol{\tau} - B\frac{d\boldsymbol{\theta}}{dt} - K\boldsymbol{\theta} \tag{4.15a}$$

Since $\boldsymbol{\theta} = \theta\,\hat{\mathbf{AB}}$ and $\hat{\mathbf{AB}}$ is a constant vector in the direction of the axis of rotation,

therefore, we can write Eq. (4.15a) as

$$I\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = \tau \tag{4.15b}$$

This is a nonhomogeneous linear second order ODE. The mathematical model of the system is complete if we specify two initial conditions like the following:

$$\theta(t_0) = C_0, \quad \theta'(t_0) = C_1$$

You may like to solve this equation under given conditions. Try the following SAQ.

SAQ 4

Spend 10 min

Solve Eq. (4.15b) given that $\tau = \tau_0 \cos\omega t$, $\theta(0) = 0$, $\theta'(0) = 0$. If $B = 0$, what is the resonance frequency ω_0 and the solution $\theta(t)$ for the system? [Hint: Use the method discussed in Sec. 2.4.1 of Unit 2.]

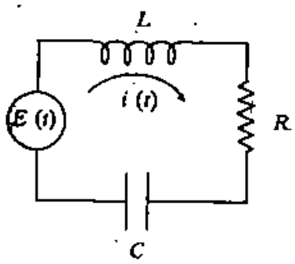


Fig. 4.9: A signal $E(t)$ is imposed on an LCR circuit, which is an electrical circuit having an inductor (L), resistor (R) and capacitor (C) as its elements. These elements are in series in the above circuit.

Now that you have solved SAQ 4, you can draw an analogy with the LCR circuit on which a sinusoidal ac signal has been impressed (Fig. 4.9). Using Kirchoff's voltage law, you can model the circuit with the following nonhomogeneous second order ODE with constant coefficients:

$$\frac{q}{C} + Ri + L\frac{di}{dt} = E(t) = E_0\cos\omega t$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0\cos\omega t, \left[\because i(t) = \frac{dq(t)}{dt}\right] \tag{4.16}$$

You can see that its general solution will be analogous to the general solution of Eq. (4.15b) obtained in SAQ 4. Let us now consider another application of second order ODEs in mechanics.

4.4.2 Planetary Orbits

Let us consider the motion of a planet of mass m in the gravitational field of the sun of mass M. Let us assume the sun to be stationary and neglect the effect of the gravitational field of other planets on this planet. You have solved this problem in Unit 6 of PHE-01 using a different method. And from Unit 7 of PHE-01, you know that these assumptions are valid. We can model this system by Newton's second law from which we have

We would advise you to refer to Units 4,6 and 7 of the Physics course "Elementary Mechanics" before studying this section.

$$\mathbf{F} = m\mathbf{a}$$

or
$$-\frac{GMm}{r^2}\hat{\mathbf{r}} = m\mathbf{a}. \quad (4.17a)$$

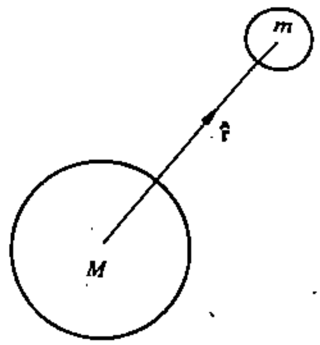


Fig. 4.10 : Force of gravitation on the earth due to the sun is $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$

where $\hat{\mathbf{r}}$ is the unit vector pointing from the sun to the planet (Fig. 4.10). Since $\mathbf{F}$ is a central force, the planet's angular momentum is constant. Therefore, its motion is restricted to a plane and it is convenient to use plane polar coordinates to solve this equation (refer to Unit 3 of PHE-04). In the plane polar coordinate system, acceleration $\mathbf{a}$ is given as (refer to Unit 4, PHE-01 or Unit 3, PHE-04):

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

On substituting $\mathbf{a}$ in Eq. (4.17a), we get two differential equations

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} \quad (4.17b)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \quad (4.17c)$$

We can multiply Eq. (4.17c) by r and write it as

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

which gives on integration

$$mr^2\dot{\theta} = \text{constant}$$

The term $mr^2\dot{\theta}$ is nothing but the magnitude of the angular momentum of the planet about the sun. You know that $\mathbf{L} = \mathbf{r} \times \mathbf{p} = mr\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}\,\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector normal to both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Thus, $mr^2\dot{\theta} = L$.

We now have to solve Eq. (4.17b) to determine the path of the planet in space, i.e., the shape of its orbit. Thus, we have to find r as a function of θ. To do this, we first have to eliminate t from Eqs. (4.17b) and (4.17c). Let us make the substitutions

$$r = \frac{1}{u} \text{ and } \frac{d\theta}{dt} = \frac{L}{mr^2} = \frac{L}{m}u^2$$

in Eqs. (4.17b) and (4.17c). Then, differentiating r with respect to t, we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = -\frac{L}{m}\left(\frac{du}{d\theta}\right)$$

Differentiating again, we get

$$\frac{d^2r}{dt^2} = -\frac{L}{m}\left(\frac{d^2u}{d\theta^2}\right)\left(\frac{d\theta}{dt}\right) = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2}$$

Thus, Eq. (4.17b) becomes

$$-\frac{L^2}{m}u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{GMm}{r^2} = -GMm\,u^2$$

or
$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} = A \quad (4.18a)$$

This is again a nonhomogeneous linear second order ODE. But we can remove the nonhomogeneity in this equation by substituting $u' = u - A$. Eq. (4.18a) then becomes

$$\frac{d^2u'}{d\theta^2} + u' = 0 \tag{4.18b}$$

Its general solution is

$$u' = B\cos(\theta - \theta_0)$$

or $$u = A + B\cos(\theta - \theta_0)$$

Here θ_0 signifies a phase lag. So by suitably shifting the origin of θ-axis, we can put $\theta_0 = 0$. Thus, we get

$$u = A + B\cos\theta, \text{ or } \frac{1}{r} = A + B\cos\theta \tag{4.19}$$

This is the equation of a conic section (ellipse, parabola or hyperbola) with focus at $r = 0$ (Fig. 4.11). The shape of the orbit is determined by the relation between A and B. For

$A > B$, orbit is an ellipse
$B = A$, orbit is a parabola
$0 < A < B$, orbit is a hyperbola

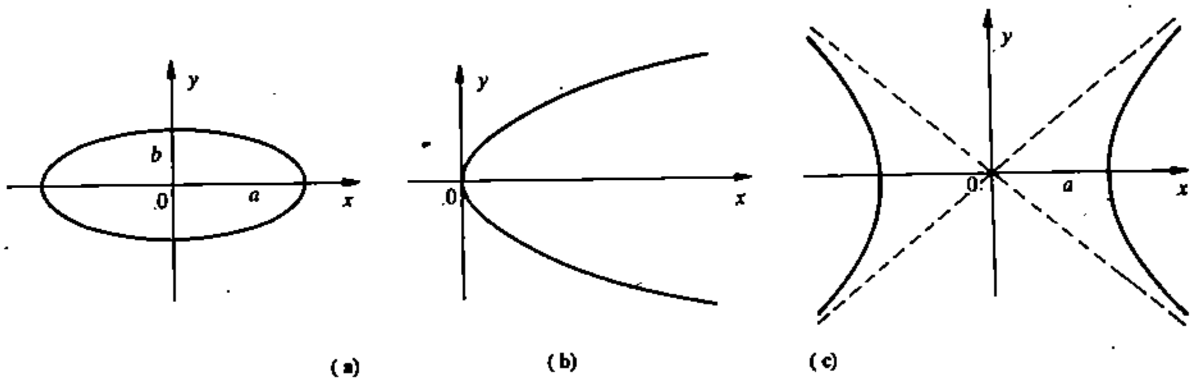


Fig. 4.11 (a) Ellipse; (b) parabola; (c) hyperbola.

The constant B can be determined for planetary orbits using energy considerations. It has been determined in Unit 6 of PHE-01, though in a different form. It is given by

$$B^2 = \frac{m^2(GmM)^2}{L^4} + \frac{2mE}{L^2}$$

So far we have considered initial value problems involving second order ODEs, i.e., the differential equations are solved by specifying the initial conditions. In certain physical situations, we come across boundary value problems. You may like to work out such a problem yourself based on what you have studied so far.

Spend 5 min

SAQ 5

A beam of length L is supported at its ends and weighs w kg/ unit length. The ODE governing the deflection of the beam is

$$C\frac{d^2y}{dx^2} = w\left(\frac{x^2}{2} - \frac{Lx}{2}\right)$$

where C is a constant which depends on the elasticity of the material of the beam and its geometry.

Solve this equation given that $y = 0$ when $x = 0$ and $y = 0$ when $x = L$.

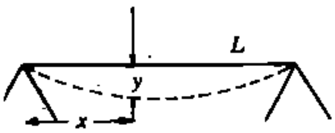


Fig. 4.12 : Deflection of a beam

So far we have taken up applications of ODEs in which a single differential equation describes the system being modelled. Now suppose we wish to study the motion of a mechanical system involving two (or more) objects. In such cases, we will need to solve linked or coupled differential equations in more than one variable. Let us consider some applications of this kind in the final part of the unit.

4.5 COUPLED DIFFERENTIAL EQUATIONS

Many physical situations can be described in terms of coupled differential equations. Let us study a few typical and simple applications. These are related to coupled oscillators in mechanics, coupled electrical circuits, and the motion of electrons in electric and magnetic fields.

4.5.1 Coupled Oscillators

You may have studied about coupled oscillators in the course 'Oscillations and Waves' (PHE-02). Fig. 4.13 shows two identical pendulums, each having a mass m suspended on a rigid massless rod of length L. The masses are connected by a spring of stiffness constant k. Its natural length equals the distance between the masses when neither is displaced from the equilibrium. Such a system can be used to model the vibrations of two atoms set in a crystal lattice. The atoms experience a mutual coupling force.

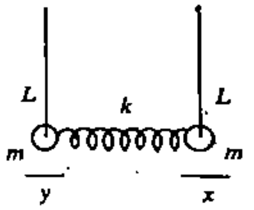


Fig. 4.13 : Two identical pendulums coupled by a spring of stiffness k. Each light rigid rod of length L supports a mass m. The natural length of the spring is equal to the separation of the masses at zero displacement.

Let us assume that the amplitude of oscillations is small and these are restricted in the plane of the paper. If x and y are the displacements of the masses, then the equations of motion are (refer to Block 1, PHE-02):

$$m\ddot{x} = -mg\frac{x}{L} - k(x-y) \qquad (4.20a)$$

and

$$m\ddot{y} = -mg\frac{y}{L} + k(x-y) \qquad (4.20b)$$

These are the differential equations representing the normal simple harmonic motion of each pendulum plus a coupling term $k(x-y)$ from the spring. Writing $\omega_0 = \sqrt{\frac{g}{L}}$, where ω_0 is the natural angular frequency of each pendulum, gives

$$\ddot{x} + \omega_0^2 x = -\frac{k}{m}(x-y) \qquad (4.20c)$$

$$\ddot{y} + \omega_0^2 y = +\frac{k}{m}(x-y) \qquad (4.20d)$$

You can see that these ODEs are coupled together. Each of them involves x and y, and so cannot be solved independently. We can solve these equations by uncoupling them. With the choice of suitable coordinates X and Y, we can obtain two independent equations in X and Y. Let

$$X = x + y, \quad Y = x - y$$

In fact, you can yourself uncouple these equations and solve them.

SAQ 6

Spend 5 min

From Eqs. (4.20c) and (4.20d) obtain the two differential equations in the new variables X and Y, and solve them. (Hint: Add the equations and subtract one from the other).

From your solutions, you can see that if $Y = 0$, $x = y$ at all times,i.e., the pendulums always swing in phase (Fig. 4.14a). Then the motion is completely described by the equation $\ddot{X} + \omega_0^2 X = 0$. The coupling has no effect and the frequency of oscillation is the same as that of either pendulum in isolation.

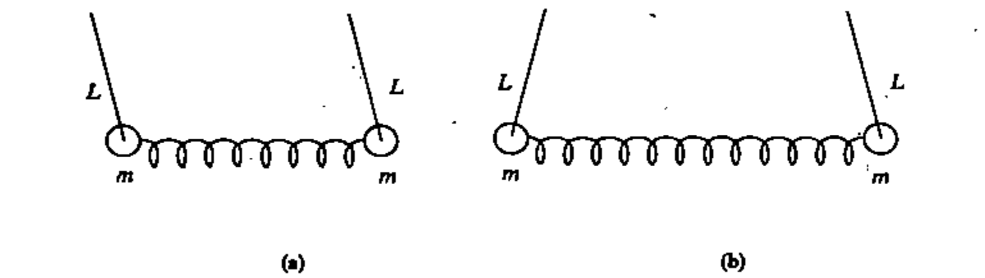


Fig.4.14: (a) The 'in phase' mode of oscillation described by $\ddot{X} + \omega_0^2 X = 0$, where X is the normal co-ordinate $X = x + y$ and $\omega_0^2 = g/L$; (b) 'out of phase' mode of oscillation described by $\ddot{Y} + (\omega_0^2 + 2k/m)\ Y = 0$, where Y is the normal coordinate $Y = x - y$.

If $X = 0, x = -y$, at all times, the pendulums are always out of phase and the equation of motion is $\ddot{Y} + \left(\omega_0^2 + \dfrac{2k}{m}\right) Y = 0$. Now the coupling is effective, the spring is either compressed or extended (Fig. 4.14b). Consequently, the frequency of oscillation is greater than ω_0.

4.5.2 Coupled Electrical Circuits

Fig. 4.15 shows two loops in an electrical circuit joined together by a resistive coupling. Applying Kirchoff's law to the left and right loops, respectively, we get

$$L_1 \frac{dI_1}{dt} + R_1 (I_1 - I_2) = E(t) \tag{4.21a}$$

$$L_2 \frac{dI_2}{dt} + R_2 I_2 + R_1 (I_2 - I_1) = 0 \tag{4.21b}$$

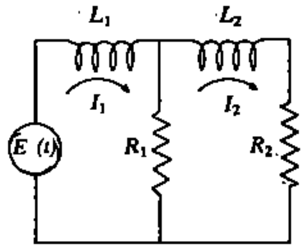


Fig. 4.15 : Resistively coupled *RL* electrical circuits.

Here, we have used the fact that the current in R_1 is $(I_1 - I_2)$ relative to the left loop and $(I_2 - I_1)$ relative to the right loop. These are once again coupled equations. We can eliminate either I_1 or I_2 from these equations. Let us rewrite Eqs. (4.21a) and (4.21b) as follows:

$$L_1 \frac{dI_1}{dt} + R_1 I_1 - R_1 I_2 = E(t) \tag{4.21c}$$

and

$$-R_1 I_1 + L_2 \frac{dI_2}{dt} + (R_1 + R_2) I_2 = 0. \tag{4.21d}$$

Multiplying Eq. (4.21c) by R_1 and operating on Eq. (4.21d) by $\left(L_1 \dfrac{d}{dt} + R_1\right)$, we get

$$R_1 L_1 \frac{dI_1}{dt} + R_1^2 I_1 - R_1^2 I_2 = R_1 E(t)$$

$$-R_1 L_1 \frac{dI_1}{dt} - R_1^2 I_1 + L_1 L_2 \frac{d^2 I_2}{dt^2} + R_1 L_2 \frac{dI_2}{dt}$$

$$+ (R_1 + R_2) L_1 \frac{dI_2}{dt} + R_1 (R_1 + R_2) I_2 = 0.$$

Adding the two equations, we get

$$L_1 L_2 \frac{d^2 I_2}{dt^2} + (R_1 L_2 + R_1 L_1 + R_2 L_1) \frac{dI_2}{dt}$$

$$+ R_1 R_2 I_2 = R_1 E(t) \tag{4.22}$$

Eq. (4.22) is a second order nonhomogeneous linear ODE with constant coefficients. Why don't you solve it for some given values of L_1, L_2, R_1, R_2 and $E(t)$?

Spend 15 min

SAQ 7

Solve Eq. (4.22) for $I_2(t)$ given $L_1 = L_2 = 2\text{H}$, $R_1 = 3\Omega$, $R_2 = 8\Omega$ and $E(t) = 6\text{V}$. Assume the initial current in the circuits to be zero. Determine $I_1(t)$ from Eq. (4.21d).

4.5.3 Charged-Particle Motion in Electric and Magnetic Fields

You must have studied about the Lorentz force in your school physics courses. You know that a particle of charge q moving in an applied electric field **E** and a magnetic field **B** is acted upon by the force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (4.23a)$$

From Newton's second law of motion, its equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (4.23b)$$

We will consider only constant electric and magnetic fields, i.e., the fields which are constant in time and uniform in space. Let us solve this equation of motion (Eq. 4.23b) for a few specific situations.

Uniform electrostatic field

Let us first consider the situation in which the applied magnetic field is zero. Then Eq. (4.23b) simplifies to

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E} \qquad (4.24a)$$

Since **E** is constant, we get upon direct successive integrations

$$\mathbf{v}(t) = \frac{q\mathbf{E}}{m}t + \mathbf{v}_0 \qquad (4.24b)$$

and $$\mathbf{r}(t) = \frac{q\mathbf{E}}{2m}t^2 + \mathbf{v}_0 t + \mathbf{r}_0 \qquad (4.23c)$$

Here $\mathbf{v}_0$ and $\mathbf{r}_0$ are constants of integration which can be determined from the given initial conditions.

Thus, the charged particle moves with a constant acceleration $\frac{q\mathbf{E}}{m}$ in the direction of **E** when $q > 0$ and in the opposite direction when $q > 0$.

Uniform magnetostatic field

When the applied electric field is zero, Eq. (4.23b) becomes

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \qquad (4.25)$$

Let us use the Cartesian coordinate system to solve this equation. Let the z-axis be along **B**, i.e., $\mathbf{B} = B\hat{\mathbf{k}}$. We can simplify Eq. (4.25) as follows:

$$m\frac{d}{dt}(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) = q(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) \times B\hat{\mathbf{k}}$$

$$= -qBv_x\hat{\mathbf{j}} + qBv_y\hat{\mathbf{i}}$$

Taking each component separately, we have

$$m\frac{dv_x}{dt} = qBv_y \qquad (4.26a)$$

$$m\frac{dv_y}{dt} = -qBv_x \qquad (4.26b)$$

$$m\frac{dv_z}{dt} = 0. \qquad (4.26c)$$

Again Eqs. (4.26a) and (4.26b) are coupled together. We can solve them by uncoupling them as follows. Differentiating Eq. (4.26a) and using Eq. (4.26b) we get

$$m\frac{d^2v_x}{dt^2} = qB\frac{dv_y}{dt} = -\frac{q^2B^2}{m}v_x$$

or

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0, \text{ where } \omega_c = \frac{qB}{m}$$

You know the solution of this equation very well. It is

$$v_x = C_1 \sin \omega_c t + C_2 \cos \omega_c t.$$

C_1 and C_2 have the dimensions of speed. For the sake of convenience, we can rewrite v_x as

$$v_x = v_\perp \sin(\omega_c t + \phi),$$

where $C_1 = v_\perp \cos\phi$, $C_2 = v_\perp \sin\phi$. Now from Eq. (4.26a) we have for v_y:

$$v_y = \frac{1}{\omega_c}\frac{dv_x}{dt} = \frac{1}{\omega_c}\omega_c v_\perp \cos(\omega_c t + \phi) = v_\perp \cos(\omega_c t + \phi)$$

Note that $v_x^2 + v_y^2 = v_\perp^2$. v_z is found by simply integrating Eq. (4.26c):

$$v_z = v_\parallel$$

where $v_\parallel$ is an arbitrary constant having the dimension of speed. You can see that it is the component of v parallel to the z-axis. Thus, we have

$$\mathbf{v} = v_\perp \left[\sin(\omega_c t + \phi)\,\hat{\mathbf{i}} + \cos(\omega_c t + \phi)\,\hat{\mathbf{j}} \right] + v_\parallel \hat{\mathbf{k}} \tag{4.27a}$$

Since $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ we can integrate v with respect to t to get

$$\mathbf{r} = \frac{v_\perp}{\omega_c}\left[-\cos(\omega_c t + \phi)\,\hat{\mathbf{i}} + \sin(\omega_c t + \phi)\,\hat{\mathbf{j}} \right] + v_\parallel t\,\hat{\mathbf{k}} + \mathbf{R}_0 \tag{4.27b}$$

where $\mathbf{R}_0$ is a constant vector of integration. This is a general solution of Eq. (4.25). You can quickly work out a particular solution under given initial conditions.

Spend 10 min

SAQ 8

Find the particular solutions for v and r given that $B = 10^{-2}$ tesla, $q = 1.6 \times 10^{-19}$ C, $m = 1.6 \times 10^{-24}$ kg and

(i) $\mathbf{v}(0) = (2000 \text{ m s}^{-1})\,\hat{\mathbf{i}}$, $\mathbf{r}(0) = 2\text{m}\,\hat{\mathbf{j}}$

(ii) $\mathbf{v}(0) = (2000 \text{ m s}^{-1})(\hat{\mathbf{i}} + \hat{\mathbf{k}})$, $\mathbf{r}(0) = 2\text{m}\,\hat{\mathbf{j}}$.

Now that you have worked out the particular solutions, you can interpret them. In case (i), you see that v (0) is perpendicular to **B**, i.e., when the particle enters the magnetic field **B**, it is moving in a direction normal to **B**. Your solution for **r** (t) comes out to be the equation of a particle moving in a circle in the xy plane. Thus, when a charged particle enters the magnetic field in a plane normal to the field's direction, it moves in a circle in that plane (Fig. 4.16).

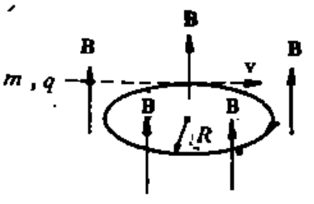


Fig. 4.16 : Motion of a charged particle in a uniform magnetic field B when the particle enters in a plane normal to B. Here $R = \frac{mv_\perp}{qB}$.

In case (ii), v (0) is at an angle to **B**, i.e., when the particle enters the field it is moving at an angle to the field. In the particular solution of v (t), the z component, i.e., the component parallel to **B**, is constant. The component normal to **B** [i.e., $\mathbf{v}_\perp = (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}})$] produces a circular motion of radius $R = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{qB}$. The combined motion is, therefore, a helix around **B**. Having worked out SAQ 8 and studied the interpretation of its results, you can readily appreciate the interpretation of Eqs. (4.27a) and (4.27b). We shall discuss this briefly for enrichment purposes only. **You will not be examined on the following material.**

Let $\quad \mathbf{r} = \mathbf{r}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + z_0\hat{\mathbf{k}} \quad$ at $\quad t = 0$.

Then you can verify that

$$\mathbf{R}_0 = \mathbf{r}_0 + \frac{v_\perp}{\omega_c}\cos\phi\,\hat{\mathbf{i}} - \frac{v_\perp}{\omega_c}\sin\phi\,\hat{\mathbf{j}}$$

$$\text{or } \mathbf{R}_0 = \left(x_0 + \frac{v_\perp}{\omega_c}\cos\phi\right)\hat{\mathbf{i}} + \left(y_0 - \frac{v_\perp}{\omega_c}\sin\phi\right)\hat{\mathbf{j}} + z_0\hat{\mathbf{k}}$$

Thus, the Cartesian components of $\mathbf{r}(t)$ are given as

$$x(t) = -\frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi) + X_0$$

$$y(t) = \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi) + Y_0$$

$$z(t) = v_\parallel t + z_0$$

where $X_0 = x_0 + \frac{v_\perp}{\omega_c}\cos\phi$

and $Y_0 = y_0 - \frac{v_\perp}{\omega_c}\sin\phi$

The equations for $x(t)$ and $y(t)$ can be rewritten as follows:

$$(x - X_0) = -\frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi)$$

$$(y - Y_0) = \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi)$$

Adding and squaring these equations, we get

$$(x - X_0)^2 + (y - Y_0)^2 = \frac{v_\perp^2}{\omega_c^2} = r_c^2$$

Thus, the projection of the charged particle trajectory in the plane normal to **B** is a circle (the shaded circle in Fig. 4.17) with centre at (X_0, Y_0) and radius equal to r_c. The point (X_0, Y_0), i.e., point G at a distance r_c from the particle is called the **guiding centre** (seeFig. 4.17). The radius of the orbit ($r_c = v_\perp/\omega_c = mv_\perp/qB$) is called the **radius of gyration** or the **Larmor radius.** The angular frequency ω_c ($= qB/m$) is termed the **gyrofrequency** or **Larmor frequency.** Since the particle has a component of **v** along **B**, it moves in a helical path.

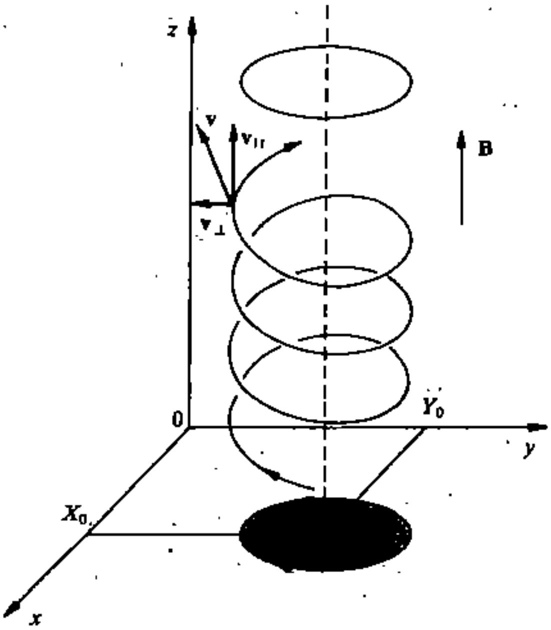


Fig. 4.17: Helical path of a charged particle moving in a uniform magnetic field. The guiding centre moves parallel to the z-axis along the dashed line.

Let us now summarise what you have studied in this unit.

4.6 SUMMARY

- Mathematical modelling is a simplified mathematical representation of reality created to solve a specific problem. The process can be broken down into seven steps, viz., specifying the real problem, setting up the model, formulating the mathematical problem, solving it, interpreting the solution, comparing it with reality and using the result.

- Many physical phenomena can be modelled using first order ODEs. Specific applications of first order ODEs in thermal physics, Newtonian mechanics and electrical circuits have been considered in the unit.

- Second order ODEs have been used to mathematically model physical phenomena in the area of Newtonian mechanics and electrical circuits.

- Coupled differential equations used for modelling physical systems like coupled oscillators, coupled electric circuits, charged particles in electric and magnetic fields have also been solved.

4.7 TERMINAL QUESTIONS

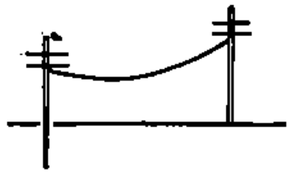


Fig. 4.18 : A telephone wire suspended from two poles. What does the shape of the wire look like to you ? It would be revealing to compare your first impression with your solution.

1) A long suspended telephone wire hangs under its own weight (Fig. 4.18). Determine the differential equation governing the shape that the hanging wire assumes and obtain its general solution. [Hint: Consider a portion of the wire between the lowest point P of the wire and any arbitrary point Q. Write down the equations for forces for PQ in equilibrium and use the fact that $\frac{dy}{dx}$ is the slope of the wire at Q. You may also need to refer to the mathematics course MTE-01 on calculus for solving this problem.]

2) Fig. 4.19 shows a very long strip of thickness D and length L in a furnace attached to a hot wall which is maintained at a temperature of 200°C. Heat is conducted steadily along the strip and is lost from the sides by convection to the surrounding air.

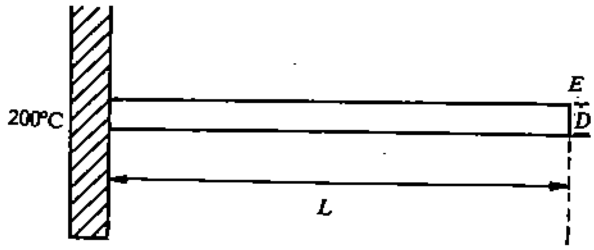


Fig. 4.19

The strip temperature θ, assumed to depend only on the distance x along the strip, is modelled by the differential equation

$$C\frac{d^2\theta}{dx^2} = 2H(\theta - \theta_{air})$$

where C and H are constants, and $\theta_{air} = 70°C$. Assuming that the strip is long enough so that the end E is at the same temperature as the surrounding air, we have the boundary conditions $\theta = 200°C$ when $x = 0$, $\theta = 70°C$ when $x \to \infty$. Solve the equation for the given boundary conditions. (Beware of the boundary condition for $x \to \infty$!)

4.8 SOLUTIONS AND ANSWERS

SAQs (Self-assessment questions)

1) This problem can be mathematically modelled using the law given in the hint. You know from your school courses that the rate at which heat is absorbed by the object is $\frac{dQ}{dt} = ms\frac{dT}{dt}$, where m is the object's mass, and s, its specific heat. If T is the temperature of the object at any instant t, then $\frac{dT}{dt}$ is the time rate of change of temperature. Since the rate at which heat is absorbed by an object is proportional to the temperature difference of the object and its surroundings we have

$$\frac{dQ}{dt} \propto (T_s - T)$$

or $$ms\frac{dT}{dt} = C'(T_s - T)$$

or $$\frac{dT}{dt} = \frac{C'}{ms}(T_s - T) = K(T_s - T),$$

where K^{-1} has the dimensions of time.

Now the difference between the temperature of the oven and the casting is $(200 - T)$ °C. So the first order ODE describing the given system is

Note that in solving this ODE it does not matter whether we use the unit (C) or (K) for the temperature, since the variable T appears throughout.

$$\frac{dT}{dt} = K[200 - T]\,°C^{-}$$

with the boundary conditions that

when (i) $t = 0$ min, $T = 20$ °C and when (ii) $t = 1$ min, $T = 30$ °C.

The solution of this equation is obtained as follows:

$$\int \frac{dT}{200 - T} = K\int dt + C$$

$$-\ln|200 - T| = Kt + C$$

or $$200 - T = C_1 \exp[-Kt]$$

or $$T\,°C = 200\,°C - C_1 \exp[-Kt]$$

Checking the solution: R.H.S. $= K(200 - T) = KC_1 e^{-Kt}$

$$\text{L.H.S.} = \frac{dT}{dt} = KC_1 e^{-Kt}$$

So the solution is correct as L.H.S = R.H.S.

C_1 is a constant of integration having the dimensions of temperature. Using the boundary conditions we get

(i) $20°C = 200°C - C_1$, at $t = 0$ min,

$$\therefore C_1 = 180°C$$

(ii) $30°C = 200°C - (180°C)\exp[-K\,\text{min}^{-1} \times 1\,\text{min}]$, at $t = 1$ min

or $$e^{-K} = \frac{170}{180} = \frac{17}{18}, \therefore K = -\ln\left|\frac{17}{18}\right| = 0.057$$

Thus the particular solution is

$$T\,°C = 200°C - 180°C \exp[(-0.057)t]$$

To find the time t at which $T = 190°C$ we have to solve the following equation:

$$190°C = 200°C - 180°C \exp[-0.57t]$$

or $$\exp[-.057t] = \frac{1}{18} \text{ or } t = \frac{\ln|1/18|}{-0.057} = 5\text{ min.}$$

2) Using the method of separation of variables, we have that

$$\int \frac{dv}{g - \frac{k}{m}v} = \int dt + C$$

Letting $w = g - \frac{k}{m}v$, $dw = -\frac{k}{m}dv$, we get

or $$\int \frac{dw}{w} = -\frac{k}{m}\int dt + C'$$

or $$w = C_1 \exp\left(-\frac{k}{m}t\right)$$

$$\therefore\ C_1 = \frac{-i\,\omega_0\,\tau_0}{(K-\omega^2 I)(i\,\omega_0 + i\,\omega_0)} = -\frac{\tau_0}{2(K-\omega^2 I)} = -\frac{\tau_0}{2I(\omega_0^2-\omega^2)}$$

and

$$C_2 = \frac{i\,\omega_0\,\tau_0}{(K-\omega^2 I)(-2i\,\omega_0)} = -\frac{\tau_0}{2I(\omega_0^2-\omega^2)}$$

Thus

$$\theta(t) = -\frac{\tau_0}{2I(\omega_0^2-\omega^2)}\left[e^{i\omega_0 t}+e^{-i\omega_0 t}\right]+\frac{\tau_0\cos\omega t}{I(\omega_0^2-\omega^2)}$$

$$= \frac{\tau_0}{I(\omega_0^2-\omega^2)}\left[\cos\omega t-\cos\omega_0 t\right]$$

Thus, ω_0 is the resonance frequency of the system given by

$$\omega_0 = \sqrt{K/I}$$

5) This equation can be solved by simple integration

$$C\frac{dy}{dx} = w\left[\frac{x^3}{2.3}-\frac{Lx^2}{2.2}\right]+C_1$$

and

$$C\,y = w\left[\frac{x^4}{2.3.4}-\frac{Lx^3}{2.2.3}\right]+C_1 x + C_2$$

Applying the boundary conditions we get that

$$C_2 = 0$$

and $$w\left[\frac{L^4}{24}-\frac{L^4}{12}\right]+C_1 L = 0$$

or $$C_1 = -\frac{1}{L}\left(-\frac{wL^4}{24}\right) = \frac{wL^3}{24}$$

Thus $$Cy = w\left[\frac{x^4}{24}-\frac{Lx^3}{12}\right]+\frac{wL^3}{24}x.$$

6) Adding Eqs. (4.20c) and (4.20d) we get

$$(\ddot{x}+\ddot{y})+\omega_0^2(x+y) = 0$$

Since $X = x+y, \dot{X} = \dot{x}+\dot{y}$ and $\ddot{X} = \ddot{x}+\ddot{y}$

Thus, the above differential equation becomes

$$\ddot{X}+\omega_0^2 X = 0$$

which has the well known solution of the form

$$X = A\cos\omega_0 t + B\sin\omega_0 t,\ \text{where}\ \omega_0^2 = g/L.$$

A and B can be determined from given initial conditions. Subtracting Eq. (4.20d) from (4.20c) we have

$$(\ddot{x}-\ddot{y}) = -\omega_0^2(x-y)-\frac{2k}{m}(x-y)$$

Again since $Y = x-y, \dot{Y} = \dot{x}-\dot{y}$ and $\ddot{Y} = \ddot{x}-\ddot{y}$, we get

$$\ddot{Y} = -\omega_0^2 Y - \frac{2kY}{m}$$

or $$\ddot{Y}+\left(\omega_0^2+\frac{2k}{m}\right)Y = 0.$$

The solution of this equation is

$$Y = C\cos\omega t + D\sin\omega t$$

with $$\omega^2 = \omega_0^2 + \frac{2k}{m}$$

7) Let us first plug in the values of L_1, L_2 etc. in Eq. (4.22). It then becomes

$$\frac{4\,d^2I_2}{dt^2} + [\,6 + 6 + 16\,]\frac{dI_2}{dt} + 24\,I_2 = 18$$

or $$\frac{d^2I_2}{dt} + \frac{7\,dI_2}{dt} + 6I_2 = \frac{9}{2}$$

Here we have not written the units explicitly in the ODE. We can remove the nonhomogeneity of this ODE by substituting $I' = I_2 - \frac{3}{4}$ in it. We thus have

$$\frac{d^2I'}{dt^2} + \frac{7\,dI'}{dt} + 6I' = 0$$

The characteristic equation for this homogeneous second order ODE is

$$\lambda^2 + 7\lambda + 6 = 0$$

which has the roots $\lambda_1 = -6$, $\lambda_2 = -1$.

Hence, the solution is

$$I' = C_1 e^{-6t} + C_2 e^{-t}$$

or $$I_2 = \frac{3}{4} + C_1 e^{-6t} + C_2 e^{-t}$$

We can get I_1 from Eq. (4.21d):

$$-3I_1 + \frac{2dI_2}{dt} + 11\,I_2 = 0$$

Substituting for I_2, we have

$$-3I_1 + 2\,(-6C_1 e^{-6t} - C_2 e^{-t}) + 11\left(\frac{3}{4} + C_1 e^{-6t} + C_2 e^{-t}\right) = 0$$

or $$I_1 = \frac{11}{4} - \frac{C_1}{3}e^{-6t} + 3C_2 e^{-t}$$

Let us quickly check the general solutions for I_1 and I_2:

L.H.S. of Eq. (4.21c) is

$$\frac{2dI_1}{dt} + 3I_1 - 3I_2 = 2\,[\,2C_1 e^{-6t} - 3C_2 e^{-t}\,]$$

$$+ \frac{33}{4} - C_1 e^{-6t} + 9C_2 e^{-t} - \frac{9}{4} - 3C_1 e^{-6t} - 3C_2 e^{-t}$$

$$= 6 + 4C_1 e^{-6t} - 6C_2 e^{-t} - 4C_1 e^{-6t} + 6C_2 e^{-t} = 6 = \text{R.H.S.}$$

Similarly,

L.H.S. of Eq. (4.21d) is

$$-3I_1 + \frac{2dI_2}{dt} + 11\,I_2 = \frac{-33}{4} + C_1 e^{-6t} - 9C_2 e^{-t} - 12C_1 e^{-6t} - 2C_2 e^{-t}$$

$$+ \frac{33}{4} + 11C_1 e^{-6t} + 11C_2 e^{-t}$$

$$= 0 = \text{RHS}$$

Now it is given that at $t = 0$, $I_1(t) = 0$, $I_2(t) = 0$.

These initial conditions give us two equations for C_1 and C_2:

$$C_1 + C_2 + \frac{3}{4} = 0$$

or $$\frac{dy}{dx} = \sinh(Kx + C_1)$$

Thus $$y = \int \sinh(Kx + C_1)\,dx + C_2$$

or $$y = K\cosh(Kx + C_1) + C_2$$

Did you find the problem interesting? In fact, we had valid reasons for posing it. For one, it illustrates that, in mathematically modelling physical systems, you often need to collect and relate different kinds of information from various fields. There is another interesting aspect to it. Did you try to guess the shape of the telephone wire before solving the problem? We would not be surprised if you thought it was a parabola.

But from your solution, you know that a wire hanging between two poles under its own weight takes the shape of a hyperbolic cosine. The graph of the hyperbolic cosine is called a catenary after the Latin word '**catena**' meaning 'chain'. So what is the moral of the story? It is: *beware, appearances are deceptive!*

2) This is a nonhomogeneous second order ODE. We can remove its nonhomogeneity. By making the substitution $\phi = \theta - \theta_{air}$, we get

$$\frac{d^2\phi}{dx^2} - k^2\phi = 0, \quad k^2 = \frac{2H}{C}$$

We can write down the solution of this equation as

$$\phi = C_1 e^{kx} + C_2 e^{-kx}$$

or $$\theta = \theta_{air} + C_1 e^{kx} + C_2 e^{-kx} \qquad \text{(i)}$$

Let us now apply the boundary conditions. At $x = 0$, $\theta = 200°\text{C}$.

$$C_1 + C_2 = 200°\text{C} - 70°\text{C} = 130°\text{C}$$

As $$x \to \infty, \theta = 70°\text{C}$$

Now, as $x \to \infty$, $e^{kx} \to \infty$, i.e., the solution for θ tends to infinity, as $x \to \infty$. But the temperature of the thin strip tends to a finite value (70°C) as $x \to \infty$. Therefore, the term containing e^{kx} in the general solution for θ is physically unacceptable. Hence, we put $C_1 = 0$ in (i). Thus,

$$C_1 = 0, \quad C_2 = 130°\text{C}$$

and the particular solution is

$$\theta = 70°\text{C} + 130°\text{C}\, e^{-kx}$$

UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGPHS-10

Mathematical Methods in Physics-II

Block

2

PARTIAL DIFFERENTIAL EQUATIONS

BLOCK 2 PARTIAL DIFFERENTIAL EQUATIONS

Introduction

In Block 1, you have used ordinary differential equations to model physical phenomena or systems involving quantities which depend on one variable only. However, in physics, we often need to model the spatial and time variations of quantities which depend on more than one variable, for example, the electrostatic, gravitational and magnetic fields. Such problems can be formulated only in terms of **partial differential equations (PDEs)**,i.e., differential equations involving functions of more than one variable.

Partial differential equations arise in such diverse areas as wave motion, heat conduction, electrostatics, magnetism, hydrodynamics, aerodynamics, nuclear physics, to mention a few. The solution of partial differential equations with given boundary conditions can be truly described as one of the 'gems' of late eighteenth and early nineteenth century mathematics. This subject arose when a number of great mathematicians—Euler, Laplace, Lagrange, Poisson, Cauchy, the Bernoullis, Fourier among others—turned their minds to search for answers to a host of interesting questions pertaining to a variety of physical phenomena.

One of the first phenomenon to be modelled with a PDE was the wave motion—since it occurs in a wide variety of natural phenomena. These include vibrating strings (sitar, guitar, piano, etc.), vibrating membranes (drum heads), waves travelling through a solid media (earth quakes), water waves, vibrating shafts (machines), electromagnetic waves (radio, TV, radar) etc. Whatever the nature of wave phenomenon—whether it is the displacement of a tightly stretched string, deflection of a stretched membrane, propagation of electromagnetic waves in free space or propagation of currents along a telephone or power transmission line—these entities are governed by a single PDE, **the wave equation.**

Another class of important physical phenomena is related to the process of diffusion which may be likened to spreading, flow, or mixing. One of the most common diffusion processes encountered is the transfer of energy in the form of heat. For example, heat flow in fuel rods of a nuclear reactor. The study of diffusion of particles from a region of high concentration to that of low concentration finds applications in industrial and chemical processes, viz., evaporation, distillation, acid and sugar concentration, industrial drying of products, etc. The **diffusion equation** is used to model such phenomena.

Laplace's equation is another interesting PDE in physics having wide applications. It can be applied to obtain the gravitational (electrostatic) potential in free space devoid of matter (charge), to study the steady (time independent) flow of heat across various bodies, to model surface waves on a fluid or to describe the irrotational motion of an incompressible fluid. Indeed, these three PDEs (the wave equation, diffusion equation and Laplace's equation) are so widely applicable that they are often called 'differential equations of physics'. True, there are other important PDEs apart from these, such as the wave equation in quantum mechanics, Dirac's equation in relativistic quantum mechanics, the Klein-Gordon equation in quantum field theory, etc. But, even for these more complex PDEs, the study of these three PDEs is a necessary introduction.

It is chiefly for these reasons that, after a brief introduction to PDEs in Unit 5, we have mainly concentrated on the method of solving physical problems modelled by these equations in Unit 6. A major breakthrough in solving such problems came with Fourier's work on representing an 'arbitrary' function as the sum of a trignometric series. Therefore, in Unit 7, we have introduced Fourier's method involving the use of Fourier series. In Unit 8, we have taken up its applications for solving PDEs of physical interest. Once again we will discuss only the relevant version of a very useful body of work in the area of PDEs. The PDEs considered in this block are applicable to a wide variety of physical phenomena and should make an interesting study.

One last word before you start studying the block. You should go through the Study Guide of Block 1 again to be able to get the best out of this block. The time you would need to spend on each unit could vary between 5 and 6 hours giving a total of a maximum of 24 h of study time for this block.

We hope that you will enjoy studying and working through the block. Wish you good luck!

UNIT 5 AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

Structure

5.1 INTRODUCTION

In Block 1, you have studied ordinary differential equations (ODEs) and learnt various methods of solving them. You have also obtained some understanding of the process of mathematical modelling and used it to solve some simple real-life problems pertaining to physics.

However, the real world around us throws up an astounding variety of problems which cannot be solved with the knowledge of ODEs alone. For example, suppose a sitar string is plucked at some point. What is the ensuing sound ? Any sitarist will tell you that the sound depends (among other things) upon where the string is plucked. Now if you want to model the motion of the sitar string, you cannot use the techniques you have studied in Block 1. Similarly, if you heat a casting in a furnace and want to know its temperature distribution at a given time, you need to look for new methods.

In order to solve such real-world problems, we need to study **partial differential equations.** This unit being the first in our study of PDEs, we shall discuss some basic concepts related to them.

As you have studied in Unit 1 of Block 1, PDEs arise in connection with various physical problems when the functions involved depend on two or more variables. Recall that, in the study of ODEs, we had asked you to go through calculus. So while modelling physical systems with ODEs, you were in a position to verify that the function of one variable occurring in any ODE was continuous and differentiable in the domain under consideration.

You ought to know similar concepts about functions of more than one variable before you go on to study PDEs. Therefore, we begin this unit by defining a function of more than one variable, and explaining the concepts of limits, continuity and differentiability for such functions. You will also learn about partial differentiation in this connection. Then, if you wish to go into greater details about these concepts, you may refer to Units 1 to 8 of Blocks 1 and 2 of MTE-07, the mathematics course entitled 'Advanced Calculus'.

Once you have learnt these basic concepts, we will introduce you to PDEs. You will see how PDEs arise in physical problems, and learn to classify them in various ways as you did for ODEs. You will also learn what is meant by the solution of a PDE.

Having become familiar with these concepts, you will be able to solve PDEs arising in problems of physical interest. This forms the subject of Unit 6.

Objectives

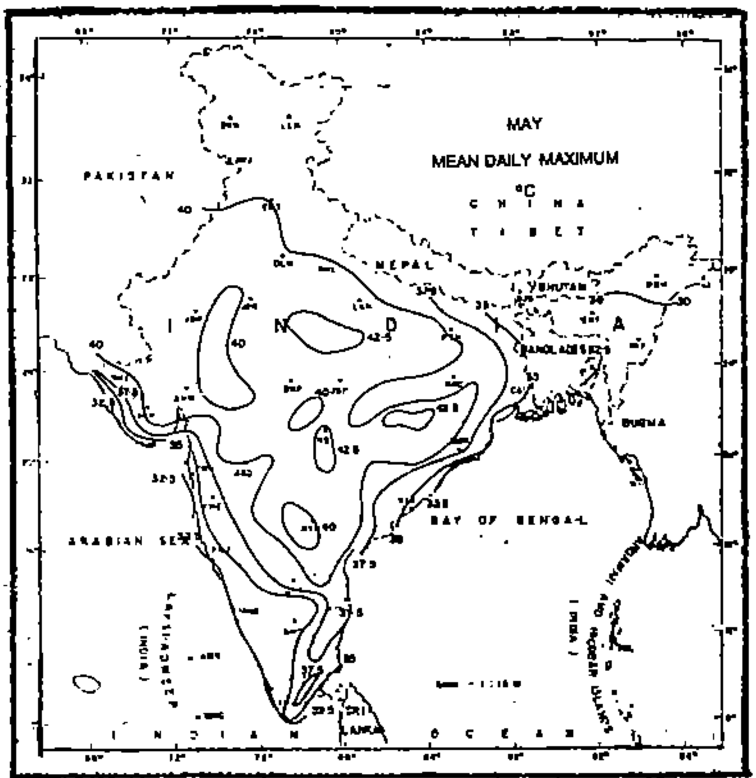
After studying this unit, you should be able to:

- verify that a function of more than one variable is continuous and differentiable
- compute the first and higher order partial derivatives of a function of several variables
- classify partial differential equations by way of order and degree, linearity/nonlinearity, homogeneity/nonhomogeneity
- verify that a function is a solution of a given PDE.

5.2 FUNCTIONS OF MORE THAN ONE VARIABLE

So far in this course you have studied differential equations involving functions of one variable. However, many physical quantities depend on several variables. For example, Fig. 5.1 shows the temperature distribution over India on a particular day of summer. The solid lines join up places where the surface temperature was the same at that time. These lines are called isotherms. You must have seen such pictures shown sometimes in the weather report televised every night in the national newscast. Now suppose we set up a coordinate system with New Delhi as the origin and take the x, y-axes in the East and the North directions, respectively. Then each place in India can be represented by its coordinates (x, y). So the variable T representing the temperature at any place at a given instant is a function of two variables x and y, i.e., $T \equiv T(x, y)$.

Remember that the surface of the earth is not planar, and so this example holds only for a small area on the surface of the earth. We cannot represent the temperature distribution over large areas (for instance, over the area of China or USA) by such a function $T(x,y)$.



Based upon Survey of India map with the permission of the Surveyor General of India.
The territorial waters of India extend into the sea to a distance of twelve nautical miles measured from the appropriate base line.

Fig. 5.1 : Temperature distribution at the surface of India on a hot day at a given time

Fig. 5.1 has been reproduced from Agroclimatic Atlas of India (1986), courtesy India Meteorological Department.

Now suppose we have a record of temperature distribution over India at different hours of a day. On how many variables would T depend? In this case T will be a function of x, y and t, i.e., $T \equiv T(x, y, t)$ where t represents the variable time. Once again you can represent $T(x, y, t)$ with the help of isotherms. But now the isotherms will keep changing with time; in this case, the solid lines of Fig. 5.1 would keep wiggling.

Can you now think of some more examples of functions of more than one variable? Remember, you have read about such functions in Unit 2 (Sec. 2.3) of the course Mathematical Methods in Physics-I (PHE-04). You may like to jot down in the margin, some more examples of such functions before you study further.

Our basic purpose in this block is to set up and solve differential equations involving one or more derivatives of functions of several variables, which are continuous and differentiable over a given domain. Therefore, before studying such DEs you should know certain mathematical concepts, such as the limits and continuity, partial derivatives and differentiability of such functions.

5.2.1 Limits and Continuity

In the calculus course you have studied the concepts of limit and continuity of a real-valued function of one variable. Let us now extend these concepts to functions of more than one variable. We will first consider a function of two variables and understand what is meant by its limit.

Recall that a function of one variable is defined as follows:

If A and B are two sets, a function f from A to B is a rule which connects every member of A to a **unique** member of B. The set A is called the **domain** and B, the **co-domain** of f. $f(x)$ denotes that unique element of B which is associated with the element x to A.

If A and B are both subsets of the set of real numbers R, then $f(x)$ is called a **real-valued function**.

Suppose $f(x, y)$ is a real single valued function of x and y. L is said to be the limit of $f(x, y)$ as the point (x, y) approaches (x_0, y_0), if $f(x, y)$ approaches the value L, as (x, y) approaches (x_0, y_0). It is written as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \qquad (5.1)$$

Now (x, y) can approach (x_0, y_0) along any one of an infinite number of curves passing through (x_0, y_0). The limit (L) of a function $f(x, y)$ is said to **exist**, only if the function always approaches the value L, irrespective of the curve along which (x, y) approaches (x_0, y_0). Thus intuitively, we can say that L is the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0), if $f(x, y)$ is as close to L as we wish whenever (x, y) is close enough to (x_0, y_0). You may like to study Fig. 5.2 which shows the geometric interpretation of this limit.

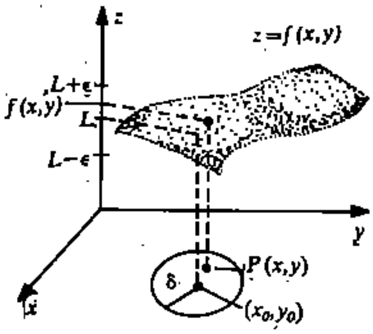


Fig. 5.2

This concept can be extended to functions of three or more variables. For instance, intuitively L is the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) if $f(x, y, z)$ is as close to L as we wish whenever (x, y, z) is close enough to (x_0, y_0, z_0). It is not possible to represent this limit pictorially because that would require four dimensions.

A natural question follows: When can we say that the limit of a function f does **not** exist? Let us find the answer. Let f be a function of two variables and let $\lim\limits_{(x, y) \to (x_0, y_0)} f(x, y) = L$. Then the concept of the limit as explained above implies that $f(x, y)$ must approach L as (x, y) approaches (x_0, y_0) along **each** line (or curve) through (x_0, y_0). Thus, to show that $\lim\limits_{(x, y) \to (x_0, y_0)} f(x, y) = L$ **does not exist**, it is enough to show that $f(x, y)$ approaches different numbers as (x, y) approaches (x_0, y_0) along different lines (curves) through (x_0, y_0). This idea can be extended to functions of more than two variables.

Note that the limits

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right)$$

and

$$\lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right)$$

termed the **repeated limits** are not the same as $\lim\limits_{(x, y) \to (x_0, y_0)} f(x, y)$ which is called the **simultaneous limit**. For a more detailed study in this regard you may like to go through Block 2 of MTE-07 (Advanced Calculus).

A rigorous mathematical treatment of these concepts is given in an appendix to this unit. You should go through its contents to understand the mathematical basis of these ideas. However, **you will not be examined on the material presented in the appendix.**

Let us now consider an example to illustrate the concepts you have studied so far.

Example 1

a) Evaluate $\lim\limits_{(x, y) \to (-1, 2)} \dfrac{x^3 + y^3}{x^2 + y^2}$

Solution

Using the results given in Eq. (A.1) of the appendix, we have

$$\lim_{(x, y) \to (-1, 2)} x = -1 \text{ and } \lim_{(x, y) \to (-1, 2)} y = 2$$

Using the product formula [(Eq. (A. 3) of the appendix)], we get

$$\lim_{(x, y) \to (-1, 2)} x^3 = -1 \text{ and } \lim_{(x, y) \to (-1, 2)} y^3 = 8$$

$$\lim_{(x, y) \to (-1, 2)} x^2 = 1 \text{ and } \lim_{(x, y) \to (-1, 2)} y^2 = 4$$

Combining the sum and quotient formulas [Eqs.(A. 2 and 4)] we get

$$\lim_{(x,y)\to(-1,2)} \frac{x^3+y^3}{x^2+y^2} = \frac{\lim\limits_{(x,y)\to(-1,2)} x^3 + \lim\limits_{(x,y)\to(-1,2)} y^3}{\lim\limits_{(x,y)\to(-1,2)} x^2 + \lim\limits_{(x,y)\to(-1,2)} y^2} = \frac{-1+8}{1+4} = \frac{7}{5}$$

b) Evaluate $\lim\limits_{(x,y,z)\to(2,1,-1)} \dfrac{2x^2y - xz^2}{y^2 - xz}$

Solution

You can see that

$$\lim_{(x,y,z)\to(2,1,-1)} x = 2, \quad \lim_{(x,y,z)\to(2,1,-1)} y = 1, \quad \lim_{(x,y,z)\to(2,1,-1)} z = -1$$

Using the sum, product and quotient formulas we get

$$\lim_{(x,y,z)\to(2,1,-1)} \frac{2x^2y - xz^2}{y^2 - xz} = \frac{2.4.1 - 2(-1)^2}{1^2 + 2.1} = \frac{8-2}{3} = 2.$$

You can see that we can obtain the limits of these functions at a point (x_0, y_0) or (x_0, y_0, z_0) effectively by evaluating the value of the function at these points. The only exceptions to this practice will be those functions whose limit does **not** exist at a point. Let us consider one such function and learn how to find out whether the limit of a function at a point exists or not.

c) Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist for $f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

Solution

Let us set $y = mx$. Then

$$\frac{y^2 - x^2}{y^2 + x^2} = \frac{m^2x^2 - x^2}{m^2x^2 + x^2} = \frac{m^2 - 1}{m^2 + 1}$$

The value of $\dfrac{m^2-1}{m^2+1}$ will be different for different values of m. This means that $f(x, y)$ approaches different values along the lines corresponding to different values of m as (x, y) approaches (0, 0). Hence the $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

You should now work out an SAQ to concretise these ideas.

Spend 5 minutes

SAQ 1

a) Show that $\lim\limits_{(x,y)\to(-1,1)} \dfrac{x^2 + 2xy^2 + y^4}{1+y^2} = 0$

b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{(x^2+y^2)}$ does not exist.

Now that you have understood the concept of limits of functions of several variables, we will define the continuity of such functions.

- A function f of two variables is continuous at (x_0, y_0) if

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$$

- A function f of three variables is continuous at (x_0, y_0, z_0) if

$$\lim_{(x, y, z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

- A function of several variables is continuous if it is continuous at each point in its domain.
- The sums, products, quotients of continuous functions are continuous.
- The composite of continuous functions is continuous.

In addition to the definitions and rules you have studied so far, you should also know the substitution rule for such functions.

Consider a pair of functions f and g. Let the co- domain of f be the domain of g, i.e.,

$f : X \to Y$ and
$g : Y \to Z$

The function $h : X \to Z$ defined by setting

$h(x) = g(f(x))$

is called the **composite** of f and g. The function

$g(f(x, y))$

is also a composite of the functions f and g of two variables.

Substitution rule
For the two variable case let us suppose that

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

Let g be a function of a single variable t and let g be continuous at $t = L$. Then

$$\lim_{(x, y) \to (x_0, y_0)} g(f(x, y)) = g(L)$$

Let us illustrate these concepts through an example.

Example 2

a) Show that the function $\ln(x/y)$ is continuous at $(e, 1)$.

Solution
In effect, here we have to show that

$$\lim_{(x, y) \to (e, 1)} \ln(x/y) = \ln(e/1) = 1.$$

Let $f(x, y) = x/y$ and $g(t) = \ln t$.

Using the quotient formula for limits

$$\lim_{(x, y) \to (e, 1)} (x/y) = e.$$

Since $\lim_{t \to e} g(t) = \ln(e) = 1 = g(e)$, therefore $g(t)$ is continuous at $t = e$. Thus it follows from the substitution rule that

$$\lim_{(x, y) \to (e, 1)} \ln(x/y) = g(e) = 1$$

Hence the function $\ln(x/y)$ is continuous at $(e, 1)$.

b) Let us consider an example from physics. The electric field at a point P due to a point charge q is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q\hat{\mathbf{r}}}{r^2}$$

E for a point charge is shown in Fig. 5.3a. You can see that as $r \to 0$, the magnitude of the electric field tends to infinity. Thus **E** for a point charge is not continuous at the point $r = 0$, i.e., at the location of the charge. This is the reason why we do not talk about the electric field at the point at which the charge is located.

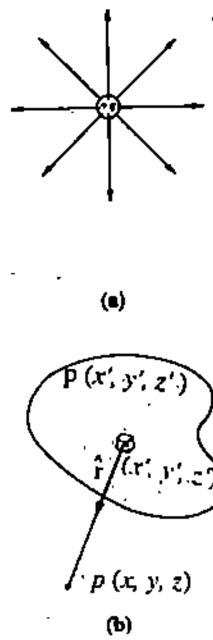


Fig. 5.3 : (a) E for a point charge is not continuous at the point where the charge is located; (b) Each element of the charge distribution $\rho(x', y', z')$ makes a contribution to the field E at the point (x, y, z). The total field at this point is the sum of all such contributions.

However, the electric field due to a finite continuous charge distribution is continuous. The field at a point $P(x, y, z)$ due to a continuous charge distribution $\rho(x', y', z')$ (Fig. 5.3b) is given as

$$\mathbf{E}(x, y, z) = \int_V \frac{\rho(x', y', z')\hat{\mathbf{r}}dV}{r^2}$$

Here $\hat{\mathbf{r}}$ points from (x', y', z') to (x, y, z). Now you know from Unit 3 of PHE-04 that, in spherical polar coordinates, dV is given as

$$dV = r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\therefore \; \mathbf{E}(r, \theta, \phi) = \int_V \rho(r', \theta', \phi') \, \hat{\mathbf{r}} dr \sin\theta \, d\theta \, d\phi$$

This integral is finite at every point in space and in the limit as r tends to any point in space, the integral tends to the value of $\mathbf{E}$ at that point. Therefore, so long as ρ remains finite, $\mathbf{E}$ is continuous everywhere, even in the interior or on the boundary of a charge distribution.

You may now try the following SAQ.

Spend 5 minutes

SAQ 2

Show that $f(x, y) = \sin \dfrac{xy}{1 + x^2 + y^2}$ is a continuous function.

So far you have studied the basic concepts of limits and continuity of functions of several variables. We are now ready to consider the following questions: How do we differentiate such functions? As an example, consider a vibrating guitar string of length L at time t (Fig. 5.4). The string is fixed at points A and B, and vibrates in the xy plane in such a way that each point on it moves in a direction perpendicular to the x-axis (transverse vibrations). Suppose $u(x, t)$ is the vertical displacement of a point P on the string, measured from the x-axis at time $t > 0$. We may like to determine how fast the point P is moving, i.e., the velocity of the string along the vertical line with abscissa x. We may also want to find the slope of the curve in Fig. 5.4 at P. In the former case, we keep x fixed and differentiate u with respect to t. In the latter case, we differentiate u with respect to x, keeping t fixed. Thus, a function of several variables can be differentiated with respect to one variable at a time keeping other variables fixed.

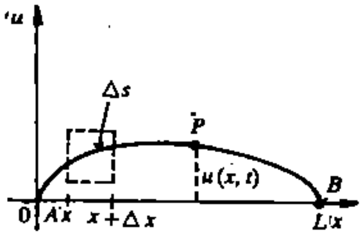


Fig.5.4 : The displacement $u(x, t)$ of a vibrating guitar string at an instant $t > 0$

This example gives us an intuitive idea that the rate of change of a function of several variables is not just a single function. This is because the independent variables may vary in different ways. All the rates of change for a function of n variables are described by n functions, called its **partial derivatives.** Let us learn about partial derivatives in some detail.

5.2.2 Partial Differentiation

In this section we will define partial derivatives and practice computing them. Consider a function of two variables $f(x, y)$ and let (x_0, y_0) be in the domain of f. The first order partial derivatives of f with respect to x at (x_0, y_0) is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y) - f(x_0, y_0)}{\Delta x} \qquad (5.2a)$$

provided that this limit exists. Similarly, the first order partial derivative of f with respect to y at (x_0, y_0) is defined by

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \qquad (5.2b)$$

provided that this limit exists.

So if these limits do not exist at any given point for a function, its partial derivatives also do not exist. The functions f_x and f_y that arise through partial differentiation and are defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \qquad (5.3a)$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \qquad (5.3b)$$

are called partial derivatives of f with respect to x and y, respectively. These are also denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. First order partial derivatives of functions of three or more variables are defined in the same way.

We can think of $f_x(x_0, y_0)$ as the rate of change of $f(x, y)$ at (x_0, y_0) with respect to x, when y is held constant. For example, let $T(x, y)$ be the temperature at any point (x, y) on a flat metal plate lying in the xy plane. Then $T_x(x_0, y_0)$ is the rate at which temperature changes at (x_0, y_0) along the line $y = y_0$ (Fig. 5.5). Similarly, the partial derivative $T_y(x_0, y_0)$ is the rate at which the temperature changes at (x_0, y_0) along the line $x = x_0$.

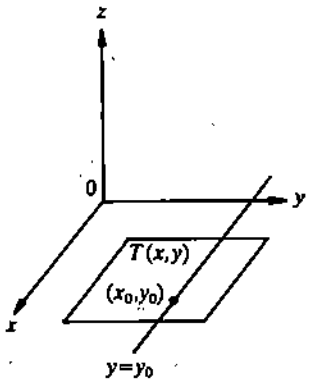


Fig. 5.5

A notation for partial derivatives which is frequently used in applications (particularly in thermodynamics) is $(\partial z/\partial x)_y$. It represents the partial derivative of $z(x, y)$ w.r.t. x when y is held constant. For example, in thermodynamics we use the notation

$$\left(\frac{\partial T}{\partial p}\right)_V, \ \left(\frac{\partial T}{\partial p}\right)_U$$

etc., where T, p, V and U are the thermodynamical variables, temperature, pressure, volume and internal energy, respectively. You can see that these two partial derivatives are different.

Thus, computing partial derivatives is no more difficult than finding derivatives of functions of a single variable. The following rule will help you compute partial derivatives:

> To calculate the partial derivative of a function f of several variables with respect to a certain variable
>
> - treat the remaining variables as constants
> - differentiate f as usual by using the rules of one variable calculus.

The **sum, product,** and **quotient rules** for ordinary derivatives have counterparts for partial derivatives. Thus, if $f(x, y)$ and $g(x,y)$ have partial derivatives then

$$\frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}(f \pm g) = \frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y} \qquad (5.4a)$$

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y} g + f\frac{\partial g}{\partial y} \qquad (5.4b)$$

$$\frac{\partial}{\partial x}\frac{f}{g} = \frac{\frac{\partial f}{\partial x} g - f\frac{\partial g}{\partial x}}{g^2} \quad \text{and} \quad \frac{\partial}{\partial y}\frac{f}{g} = \frac{\frac{\partial f}{\partial y} g - f\frac{\partial g}{\partial y}}{g^2} \qquad (5.4c)$$

As an example, consider $f(x, y) = x^2y^3 - x^3y^2$. We hold y constant and differentiate f with respect to x to get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} x^2y^3 - \frac{\partial}{\partial x} x^3y^2 = 2xy^3 - 3x^2y^2$$

Similarly, we hold x constant and differentiate with respect to y and get

$$\frac{\partial f}{\partial y} = x^2 \frac{\partial}{\partial y} y^3 - x^3 \frac{\partial}{\partial y} y^2 = 3x^2y^2 - 2x^3y$$

Let us consider an example from physics to illustrate these concepts.

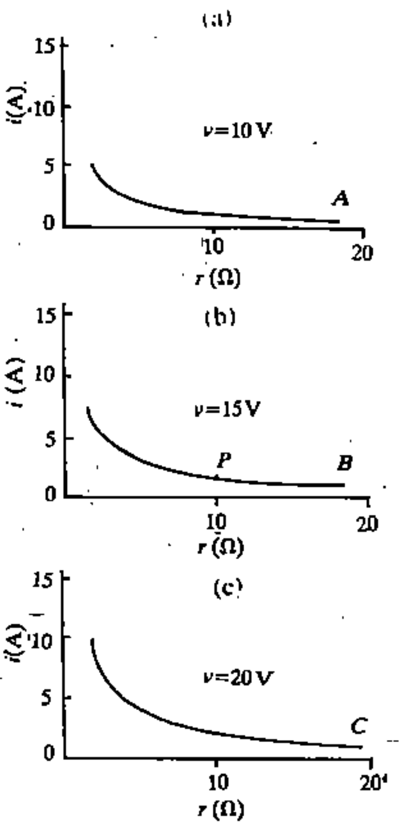


Fig. 5.6 : The variation of current in a resistive circuit element with the resistance for different values of applied voltage

Example 3

Consider the variation of current i in a circuit as we change the resistance r for different values of the applied voltage v (Fig. 5.6). The relation between these quantities is given by the familiar Ohm's law $i = \frac{v}{r}$.

Now suppose we are asked to find the slope at point P on the curve B (Fig. 5.6 b). Treating v as constant we get

$$\frac{\partial i}{\partial r} = -\frac{v}{r^2}$$

For the curve B, $v = 15$V and at point P, $r = 10\Omega$.

Thus $\left.\frac{\partial i}{\partial r}\right|_P = -\frac{15}{100} \text{A}\,\Omega^{-1} = -0.15\ \text{A}\,\Omega^{-1}$

We say that the current varies with resistance at a negative rate of 0.15 ampere per ohm, **other things being equal.**

You will find several other applications of partial derivatives in physics. For example, in the physics course PHE-06 entitled 'Thermodynamics and Statistical Mechanics' you will study about thermodynamic potentials (which involve an extensive use of partial derivatives).

However, you must keep in mind that it is not always possible to compute partial derivatives of functions in this manner. In some exceptional cases we have to use the limiting process. We shall deal with such cases as and when we come across them. In the study of PDEs, you will also come across higher order partial derivatives and you should know about them too.

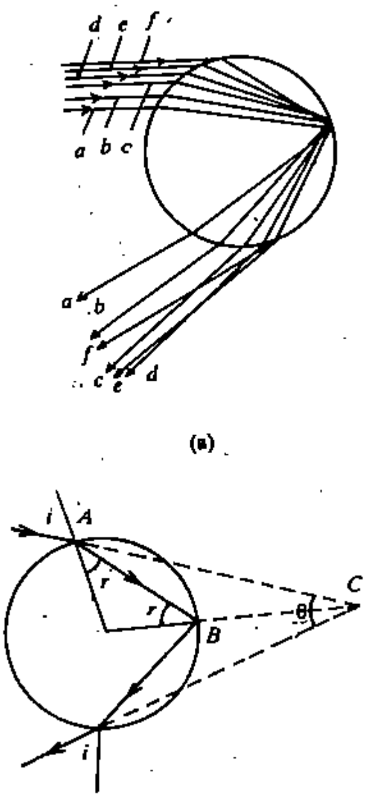


Fig. 5.7 : (a) The main rainbow you see in the sky is created by various monochromatic rays that have been refracted, reflected and finally refracted again by water droplets (b) the angle θ formed by the path of a monochromatic ray before it enters the droplet and the path after it leaves the droplet depends on both μ and i

Higher order partial derivatives

Since the partial derivatives are themselves functions, we can take their partial derivatives to obtain higher order partial derivatives. There are four ways to take a second derivative of $f(x, y)$. We may compute

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy}(x, y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yx}(x, y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \qquad (5.5)$$

f_{xy} and f_{yx} are called mixed partial derivatives or mixed partials.

If $f(x, y)$ has continuous second partial derivatives then the mixed partial derivatives are equal, i.e.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{yx} = f_{xy} \qquad (5.6)$$

You should now compute some first and second partials.

Spend 15 minutes

SAQ 3

a) Find all the first-order partial derivatives of $f(x, y, z) = x^4 - 2x^2y^2z^2 + 3yz^4$ and $h(x, y, t) = xe^t - y^2e^{2t}$. What are the values of $\frac{\partial f}{\partial y}(1, 1, 1)$ and $\frac{\partial h}{\partial t}(4, 1, 0)$?

b) Show that the function $z = \ln(x^2 + y^2)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

c) On a rainy day, you may have observed a rainbow in the sky. A rainbow is formed due to the refraction, reflection and another refraction of various monochromatic rays in the sunlight by water droplets suspended in air (see Fig. 5.7a). The angle θ shown in Fig. 5.7b for one monochromatic ray is given as

$$\theta(\mu, i) = 4\sin^{-1}\left(\frac{\sin i}{\mu}\right) - 2i$$

where μ is the index of refraction of water for the ray and i, its angle of incidence. For any given μ, find the angle i_μ for which

$$\frac{\partial \theta}{\partial i}(\mu, i_\mu) = 0$$

d) The entropy S of a gas is given by

$$S = C_v \ln P + C_p \ln V + A \quad \text{(i)}$$

where C_p, C_v and A are constants. We can substitute for V from the ideal gas law $PV = RT$ to obtain

$$S = (C_v - C_p) \ln P + C_p \ln T + B \quad \text{(ii)}$$

where B is a constant. Compute $\partial S/\partial P$ from (i) and (ii). Why do the two expressions differ ?

Certain results follow logically from the discussion so far. We state them without proof. *The mere existence of partial derivatives does not imply the continuity of a function of several variables. Also a function of several variables which is continuous at a point need not have any of the partial derivatives at the point.* You can read about these ideas in detail in Block 2 of the mathematics course MTE-07. With this background we would now like to consider these questions : When can we say that a function of several variables is 'differentiable'? Is a **continuous** function $f(x, y)$ **differentiable** at a point ? Or is $f(x, y)$ **differentiable at a point provided its partial derivatives exist?** Let us answer these questions very briefly.

5.2.3 Differentiability

Recall that a real-valued **continuous** function of one variable need **not be differentiable.** The same applies to functions of several variables. Similarly, since the existence of partial derivatives does not even guarantee continuity, it cannot guarantee differentiability. So we need additional conditions. We will not go into a formal mathematical definition of a differentiable function. Instead we state here the sufficient conditions which, if satisfied, ensure that a function of several variables is differentiable.

> If $f(x, y)$ has partial derivatives on a disc centred at (x_0, y_0), and if f_x and f_y are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

However, these conditions are not necessary. Thus, a function can be differentiable at a point even when none of its partial derivatives is continuous at that point.

Having studied these concepts, you should know what the term 'a real-valued, continuous differentiable function of several variables' means, whenever you come across it. With this mathematical background, we are ready to discuss partial differential equations, i.e., equations involving the partial derivatives of functions of several variables.

5.3 PARTIAL DIFFERENTIAL EQUATIONS

Let us begin by considering examples of how some special partial differential equations (PDEs) arise in physical situations. We will then classify PDEs and understand what is meant by their solutions. Consider a steadily flowing stream of water with velocity field $\mathbf{v} = v_1\,\hat{\mathbf{i}} + v_2\,\hat{\mathbf{j}} + v_3\,\hat{\mathbf{k}}$. Let the velocity field be incompressible and irrotational. Then you know from Unit 2 (Sec. 2.4) of the course Mathematical Methods in Physics-I (PHE-04) that for this velocity field $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$, i.e. $\mathbf{v} = \nabla\phi$ where ϕ is a scalar field. Now you can use the definitions of divergence and gradient to express these relations in a Cartesian coordinate system in their differential form

$$\nabla \cdot \mathbf{v} = \ldots\ldots\ldots\ldots\ldots\ldots = 0 \quad (5.7a)$$

$$\mathbf{v} = \nabla\phi, \text{ i.e.,} \quad v_1 = \ldots\ldots, \quad v_2 = \ldots\ldots, \quad v_3 = \ldots\ldots \quad (5.7b)$$

Then, substitute Eq. (5.7b) in Eq. (5.7a) and you will get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (5.8)$$

Thus, you have set up a well known PDE in physics termed the **Laplace equation.** This equation is satisfied by the velocity potential function of any incompressible and irrotational flow. It also finds applications in such diverse fields as gravitation, electrostatics, elasticity and steady state heat conduction. Let us take up another example.

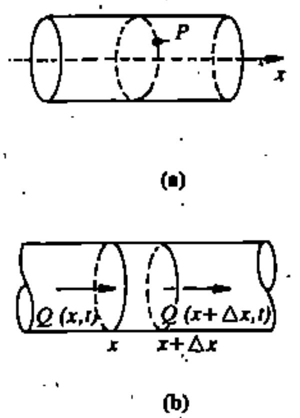


Fig. 5.8 : Flow of heat in a cylindrical metal rod

Consider the flow of heat along a cylindrical metal rod (Fig. 5.8a). We choose the x-axis to be along the axis of the cylinder. Let us assume that heat can flow only in a direction parallel to the x-axis. This means that at any instant t, the temperature T is the same at all points of a cross-section x = constant (see Fig. 5.8a). Then $T(x, t)$ describes the temperature of any point $P(x)$ in the rod at time t. We also assume that no heat is generated within the rod. We can model heat flow in the rod according to two experimentally verified laws: Fourier's law and the principle of conservation of heat. Fourier's law states that the rate of heat flow per unit area, $Q(x, t)$ perpendicular to the flow is proportional to the temperature gradient. Would you like to express this law mathematically ? Give it a try.

Since the flow is one-dimensional, you should get a relation of the following kind.

$$Q(x, t) = -AK \frac{\partial T}{\partial x}(x, t), \qquad (5.9)$$

where $Q(x, t)$ is the rate of heat flow in the positive x direction across the section x = constant at time t. Here K is the thermal conductivity of the metal and A is the rod's cross-sectional area. The minus sign appears because heat flows from hotter to colder regions, so that Q is positive where the temperature gradient is negative, and vice versa.

The principle of conservation of heat states that the rate at which heat accumulates in a region containing no heat sources is equal to the net rate at which heat enters that region through its boundaries. Let us apply this principle to a small portion of the rod between x and $x + \Delta x$ (See Fig. 5.8b). The rate at which heat accumulates in this portion at a time t is

$$Q(x, t) - Q(x + \Delta x, t)$$

Now you can write the expression for the heat accumulated in this portion in the short time interval between t and $t + \Delta t$ in the space below :

.... (5.10a)

You know that the heat energy required to raise the average temperature of a body of mass m and specific heat s, from T_1 to T_2 is $ms\,(T_2 - T_1)$. For the small portion of the rod, we have $m = \rho A \Delta x$, where ρ is the density of the metal. Let $T(x, t)$ and $T(x, t + \Delta t)$ be the average temperatures of the portion at time t and $(t + \Delta t)$, respectively and let s be the metal's specific heat. Then you can write down the heat energy added to the portion between these times:

.... (5.10b)

The principle of conservation of heat demands that the expressions contained in Eqs. (5.10a) and Eq. (5.10b) should be equal. Thus, we have

$$[Q(x, t) - Q(x + \Delta x, t)]\,\Delta t = (\rho A \Delta x)s\,[T(x, t + \Delta t) - T(x, t)] \qquad (5.10c)$$

Now divide both sides by $(\Delta x\ \Delta t)$ and take the simultaneous limit as Δx and Δt tend to zero. What do you get ? Write down the result :

.... (5.10d)

Substituting Eq. (5.9) in Eq. (5.10d) you should get

$$\frac{\partial}{\partial x}\left[AK\frac{\partial T}{\partial x}(x, t)\right] = \rho A s \frac{\partial T}{\partial t}(x, t) \qquad (5.10e)$$

On simplifying the equation further we get

$$\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0, \qquad (5.11)$$

where $k = \dfrac{K}{\rho s}$ is called the **thermal diffusivity** of the material of which the rod is made. Remember that in writing Eq. (5.11) we have assumed K to be a constant, which need not be true always.

Eq. (5.11) is termed the **one-dimensional diffusion equation.** It is so called because it models the 'diffusion' or gradual change of various physical quantities that are continuous functions of time and space coordinates in one dimension. This equation is also used to describe the diffusion of liquid or gas concentrations. A slight variation of Eq. (5.11) describes the diffusion of neutrons in a nuclear reactor. In the situations when Eq. (5.11) models heat flow in a one-dimensional object as in the example considered above it is also termed the **one- dimensional heat flow equation.**

Eqs. (5.8) and (5.11) are two PDEs which occur quite often in physics. You will come across other PDEs in Unit 6 of this course and other physics courses. Our main aim, of course, is to learn the methods of solving the PDEs which occur in physics. However, before you learn these methods you should know how to classify PDEs. You must also understand what constitutes the solution of a PDE. These will be our concerns in the next two sub-sections.

5.3.1 Classification of PDEs

We classify PDEs in much the same way as ODEs, i.e., in terms of their order, degree and linearity/nonlinearity. Linear PDEs are further classified as homogeneous/nonhomogeneous, and as elliptic, parabolic and hyperbolic PDEs. Let us see what these terms mean.

Order and Degree

Just as in the case of ODEs, the **order** of a PDE is the order of the highest derivative occurring in the equation.

For example, the equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0 \qquad (5.12)$$

is a **first order** PDE. A first order PDE for a function $f(x, y)$ contains at least one of the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ but no partial derivative of order higher than one. **A second order** PDE for $f(x, y)$ contains at least one of the partial derivatives f_{xx}, f_{yy}, f_{xy} or f_{yx} but no partial derivatives of order higher than two. Eqs. (5.8) and (5.11) are second order PDEs. Second order PDEs may also contain first order terms like $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, as in Eq. (5.11).

The **degree** of a PDE is the degree of the highest derivative in the equation.

For example, Eqs. (5.8), (5.11) and (5.12) are PDEs of degree one. The PDE

$$\left(\frac{\partial f}{\partial x}\right)^3+\frac{\partial f}{\partial t}=0 \qquad (5.13)$$

is a first order PDE of degree 3. You may turn to SAQ 4 and write down the order of the PDEs listed there right away if you so wish.

Linear and nonlinear PDEs

Just as in the case of ODEs, we say that a PDE is **linear** if (i) it is of the first degree in the unknown function (the dependent variable) and its partial derivatives, (ii) it does not contain the products of the unknown functions and either of its partial derivatives and (iii) it does not contain any transcendental functions. Otherwise it is nonlinear. For example, the PDEs given by Eqs. (5.8), (5.11), (5.12) and the following PDEs are all linear :

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial y^2} \qquad (5.14)$$

and

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x, y) \qquad (5.15)$$

The PDE given by Eq. (5.13) is nonlinear as it is of degree 3.

A linear PDE can further be classified as **homogeneous, or nonhomogeneous.**

Homogeneous and nonhomogeneous linear PDEs
If each term of a PDE contains either the unknown function or one of its partial derivatives, it is said to be **homogeneous**: otherwise it is **nonhomogeneous**. Which of the Eqs. (5.8), (5.11), (5.12), (5.14) and (5.15) are homogeneous and which ones nonhomogeneous? You are right. All the equations except Eq. (5.15) are homogeneous.

You can practise classifying PDEs further by working out SAQ 4.

SAQ 4

Spend 5 minutes

Write down the order and degree of each of the PDEs listed below. Determine which of the PDEs are linear, nonlinear. Classify the linear PDEs as homogeneous, nonhomogeneous.

i) $x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} = 0$

ii) $xy \frac{\partial^2 f}{\partial x^2} + x \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} = x^2 + y^2$

iii) $\left(\frac{\partial y}{\partial x}\right)^3 + \frac{\partial y}{\partial t} = 0$

iv) $x^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = e^{xy}$

v) $\frac{\partial^3 u}{\partial x^3} + 2 \frac{\partial^3 u}{\partial x \partial y^2} - 6 \left(\frac{\partial u}{\partial y}\right)^4 = 0$

In this course we shall restrict ourselves to linear **second order partial differential equations** because these occur most frequently in physics. The most general form of such an equation, for a function $u(x, y)$ is

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} + fu = g(x, y) \tag{5.16}$$

where a, b, c, d, e and f are functions of (x, y). If the coefficients a, b, c, d, e, f are constants, Eq. (5.16) is termed a **linear, second order, constant coefficient PDE**. Equations of the form (5.16) with constant coefficients, are further classified as **elliptic, hyperbolic** and **parabolic**, depending on the relationship between the second- order coefficients a, b, c:

if $ac - b^2 > 0$, the equation is elliptic,

if $ac - b^2 < 0$, the equation is hyperbolic,

if $ac - b^2 = 0$, the equation is parabolic.

You can verify that the **Laplace equation** Eq. (5.8) and Eq. (5.15) known as **Poisson's equation** are **elliptic**. The **diffusion equation** Eq. (5.11) is **parabolic** and Eq. (5.14), known as the **wave equation** is **hyperbolic**. Check these results before studying further. We have introduced you to the PDEs (Eqs. 5.8, 5.11, 5.14, 5.15) that occur most frequently in physics. You will learn the methods of solving these PDEs under specified boundary and initial conditions in Unit 6. But before that you must know what is meant by the solution of a PDE and some properties of the solutions. This is the subject of Sec. 5.3.2.

5.3.2 What is a Solution of a PDE ?

In part (b) of SAQ 3 you have verified that the function $u = \ln(x^2 + y^2)$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.17}$$

The function $\ln(x^2 + y^2)$ is a **solution** of this PDE. The process of solving a PDE involves finding all the functions which satisfy it. A more formal definition of a solution is as follows.

> A **solution** of a PDE in some region R of the space of independent variables is a function, all of whose partial derivatives appearing in the equation exist in some domain containing R and which satisfies the equation everywhere in R.

In general, there can be a large number of solutions of a PDE. For example, the functions $u = x^2 - y^2$ and $u = e^x \cos y$ are also solutions of Eq. (5.17). A unique solution of a PDE corresponding to a given physical problem is obtained by applying appropriate initial conditions and boundary conditions.

Let us now ask: What is involved in solving a PDE? Consider the following rather simple PDE

$$\frac{\partial u(x, t)}{\partial x} = 1 \qquad (5.18a)$$

To find $u(x, t)$ we may keep t constant and integrate with respect to x. We then obtain

$$u(x, t) = x + C \qquad (5.18b)$$

However, C is a constant only if t is kept fixed. For different values of t, C will be different, i.e., C is a function of t. Thus, the most general solution of Eq. (5.18a) is

$$u(x, t) = x + f(t) \qquad (5.18c)$$

where f is an arbitrary function of t. You can verify that $u(x, t)$ of Eq. (5.18c) does satisfy Eq. (5.18a).

So you see that while the solution of an ODE involves arbitrary constants, **the solution of a PDE involves arbitrary functions**. As we increase the order of the partial derivatives in a PDE, we introduce more arbitrary functions.

Recall that a linear combination of the linearly independent solutions of an ODE is also its solution. The same principle applies to the solutions of a linear homogeneous partial differential equation. Thus, if u_1 and u_2 are any linearly independent solutions of a linear homogeneous PDE in some region then

$$u = C_1 u_1 + C_2 u_2$$

where C_1 and C_2 are arbitrary constants, is also a solution of that equation in that region. This principle is called the **principle of superposition** and it can be extended to the case where n solutions of a PDE exist. To sum up, in this section we have studied partial differential equations, their classification and the meaning of their solutions. We would like to end this section with an exercise for you.

SAQ 5

Spend 10 minutes

a) Verify that $u_1 = \cos x \cos cy$ and $u_2 = \sin x \sin cy$ are both solutions of the PDE

$$\frac{\partial^2 u}{\partial y^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Show that $\cos(x + cy)$ and $\cos(x - cy)$ are also solutions of this PDE.

b) Show that for each integer n, the function

$$u_n = e^{-kn^2y} \sin nx$$

is a solution of the PDE

$$\frac{\partial u}{\partial y} - k \frac{\partial^2 u}{\partial x^2} = 0$$

Deduce that for any positive integer N and real numbers, $a_1, a_2, \ldots, a_N$, the function

$$\sum_{n=1}^{N} a_n e^{-k n^2 y} \sin nx$$

is also a solution of the PDE.

Let us now sum up what you have studied in this unit.

5.4 SUMMARY

We summarise below the concepts for a function of two variables. These can be extended to functions of more than two variables.

- A function $f(x, y)$ of two variables x and y is one whose value is determined by the values of x and y. We call x and y the *independent variables*: a variable equal to $f(x, y)$ is called the *dependent variable*.
- The **limit** $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

  if $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ then $| f(x, y) - L | < \varepsilon$
- A function f of two variables is **continuous** at (x_0, y_0) if

  $$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$$
- The limits

  $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

  if they exist, are called the first order partial derivatives of f with respect to x and y, respectively..
- The **partial derivative** of a function of several variables w.r.t. a variable is calculated by differentiating the function w.r.t. that variable alone, treating other variables as constant.
- A function $f(x, y)$ is said to be **differentiable** at (x_0, y_0) if it has partial derivatives on a disc centred at (x_0, y_0) and if the partial derivatives are continuous at (x_0, y_0).
- Differential equations involving functions of more than one variable are termed **partial differential equations** (PDEs).
- PDEs occur quite often in physics. In this unit, we have discussed the setting up of **Laplace's equation** and **one-dimensional diffusion equation.**
- Like ODEs, PDEs are also classified by way of **order** and **degree, linearity** and **nonlinearity**. Linear PDEs are further classified as **homogeneous** and **nonhomogeneous.**
- A linear, second order, constant coefficient PDE can also be classified as **elliptic, hyperbolic** and **parabolic** depending on the relationship between the coefficients of second order partial derivatives occurring in the PDE.
- A **solution** of a PDE in some region R of the space of independent variables is a function, which satisfies the PDE everywhere in R.
- A **linear combination** of the **linearly independent solutions** of a PDE is also a **solution** of the PDE.

5.5 TERMINAL QUESTIONS

Spend 15 minutes

1) According to Newton's law of gravitation the magnitude of the force of attraction between two particles of mass m is

$$F = -\frac{Gm^2}{r^2}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is the distance between the two particles. Determine whether F is continuous and differentiable at all points in space. The potential of this gravitational force field is given as

$$f(x, y, z) = -\frac{Gm^2}{r} \quad (r > 0)$$

Show that f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

2) Given below are some PDEs that appear in physics. Classify them by way of order and degree, linearity (L)/nonlinearity(NL), homogeneity(H)/nonhomogeneity(NH).

i) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial u}{\partial t} + \alpha u = 0$ (the telegraph equation)

ii) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ (the wave equation)

iii) $-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(r)\psi = i\hbar \frac{\partial \psi}{\partial t}$ (Schrödinger's time-dependent equation)

iv) $\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\right) = 0$ (the continuity equation)

v) $\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$ (the two- dimensional diffusion equation)

vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\varepsilon_0}\rho(x, y, z)$ (Poisson's equation)

5.6 SOLUTIONS AND ANSWERS

1) a) $\lim\limits_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^2}$

Using Eq. (A.1) of the Appendix and the sum, product, quotient rules we get

$$\lim_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^2}$$

$$= \frac{\lim\limits_{(x,y)\to(-1,1)} x^2 + 2 \lim\limits_{(x,y)\to(-1,1)} x \lim\limits_{(x,y)\to(-1,1)} y^2 + \lim\limits_{(x,y)\to(-1,1)} y^4}{1 + \lim\limits_{(x,y)\to(-1,1)} y^2}$$

$$= \frac{(-1)^2 + 2[-1]\,[1]^2 + 1^4}{1 + 1^2} = \frac{1 - 2 + 1}{2} = 0$$

b) Let $y = mx$, then $f(x, y) = \frac{xy}{x^2 + y^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}$

This will have different values for different values of m. This means that $f(x, y)$ approaches different values along the lines corresponding to different values of m as (x, y) approaches $(0, 0)$. Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

2) Let $g(x, y) = \left(\frac{xy}{1 + x^2 + y^2}\right)$ and $u(t) = \sin t$

Then $f(x, y) = u(g)$, i.e., $f(x, y)$ is a composite of $g(x, y)$ and u. We have shown in Eq. (A.1) of the Appendix that $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} y = y_0$. Since x_0 and y_0 can be any points in the domain of x and y, $p(x, y) = x$ and $q(x, y) = y$, are continuous. The sums, products and quotients of continuous functions are continuous. Hence $g(x, y)$ is continuous. Similarly, you can verify that $u(t)$ is continuous. Therefore, their composite $f(x, y)$ is also continuous.

3) a) $f(x, y, z) = x^4 - 2x^2y^2z^2 + 3yz^4$

$$\frac{\partial f}{\partial x} = 4x^3 - 4xy^2z^2$$

$$\frac{\partial f}{\partial y} = -4x^2yz^2 + 3z^4 \ ; \ \frac{\partial f}{\partial y}(1, 1, 1) = -4 + 3 = -1$$

$$\frac{\partial f}{\partial z} = -4x^2y^2z + 12yz^3$$

$$h(x, y, t) = xe^t - y^2e^{2t}$$

$$\frac{\partial h}{\partial x} = e^t$$

$$\frac{\partial h}{\partial y} = -2ye^{2t}$$

$$\frac{\partial h}{\partial t} = xe^t - 2y^2e^{2t}$$

$$\frac{\partial h}{\partial t}(4,1,0) = 4.1 - 2.1^2.1 = 2$$

b) $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}, \ \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$$

Thus, $z = \ln(x^2 + y^2)$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

c) $\theta(\mu, i) = 4\sin^{-1}\left(\frac{\sin i}{\mu}\right) - 2i$

$$\frac{\partial \theta}{\partial i} = \frac{4}{\sqrt{1 - \frac{\sin^2 i}{\mu^2}}} \frac{\cos i}{\mu} - 2$$

$$= \frac{4\mu}{\sqrt{\mu^2 - \sin^2 i}} \cdot \frac{\cos i}{\mu} - 2$$

$$= \frac{4\cos i}{\sqrt{\mu^2 - \sin^2 i}} - 2$$

$$\frac{\partial \theta}{\partial i} = 0 \text{ for } i = i_\mu$$

or $\frac{4\cos i_\mu}{\sqrt{\mu^2 - \sin^2 i_\mu}} = 2$

$4\cos^2 i_\mu = \mu^2 - \sin^2 i_\mu$

or $4\cos^2 i_\mu + 1 - \cos^2 i_\mu = \mu^2$

or $3\cos^2 i_\mu = \mu^2 - 1$

or $\cos i_\mu = \sqrt{\frac{\mu^2 - 1}{3}}$

$\therefore\ i_\mu = \cos^{-1}\left(\sqrt{\frac{\mu^2-1}{3}}\right)$

d) From (i)

$$\frac{\partial S}{\partial P} = \frac{C_v}{P}$$

and from (ii)

$$\frac{\partial S}{\partial P} = \frac{C_v - C_p}{P}$$

In finding $\frac{\partial S}{\partial P}$ from (i), we keep V constant, whereas in computing $\frac{\partial S}{\partial P}$ from (ii), we keep T constant. Therefore, these two partial derivatives are different.

4) We have used the notations L for linear, NL for nonlinear, H for homogeneous and NH for nonhomogeneous PDEs in the answer. The first term gives the order and the second, the degree of each PDE.

i) 2, 1, L, H

ii) 2, 1, L, NH

iii) 1, 3, NL

iv) 2, 1, L, NH

v) 3, 1, NL

5) a) The partial derivatives of $u_1 = \cos x \cos cy$ and $u_2 = \sin x \sin cy$ are

$$\frac{\partial u_1}{\partial x} = -\sin x \cos cy, \quad \frac{\partial^2 u_1}{\partial x^2} = -\cos x \cos cy$$

$$\frac{\partial u_1}{\partial y} = -c \cos x \sin cy, \quad \frac{\partial^2 u_1}{\partial y^2} = -c^2 \cos x \cos cy$$

and $\frac{\partial u_2}{\partial x} = \cos x \sin cy, \quad \frac{\partial^2 u_2}{\partial x^2} = -\sin x \sin cy$

$$\frac{\partial u_2}{\partial y} = c \sin x \cos cy, \quad \frac{\partial^2 u_2}{\partial y^2} = -c^2 \sin x \sin cy$$

Substituting the relevant partial derivatives of u_1 and u_2 in the given PDE we get two identities implying that both u_1 and u_2 are its solutions. Now from the principle of superposition a linear combination of linearly independent solutions of a PDE is also its solution. Since u_1 and u_2 are linearly independent we get that

$\cos(x + cy) = \cos x \cos cy - \sin x \sin cy = u_1 - u_2$

and $\cos(x - cy) = \cos x \cos cy + \sin x \sin cy = u_1 + u_2$

are also solutions of the PDE.

b) Let us first compute the partial derivatives of u_n :

$$\frac{\partial u_n}{\partial y} = -kn^2 e^{-kn^2 y} \sin nx$$

$$\frac{\partial u_n}{\partial x} = n\, e^{-kn^2 y} \cos nx$$

$$\frac{\partial^2 u_n}{\partial x^2} = -n^2 e^{-kn^2 y} \sin nx$$

Substituting $\frac{\partial u_n}{\partial y}$ and $\frac{\partial^2 u_n}{\partial x^2}$ in the PDE gives us an identity. Therefore, u_n is a solution of the PDE. Again, since $u_1, u_2, u_3, \ldots, u_n$ are linearly independent functions, we get from the principle of superposition that their linear combination, i.e. $u = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ is also a solution of the PDE.

In concise form we may write

$$u = \sum_{n=1}^{N} a_n u_n = \sum_{n=1}^{N} a_n e^{-kn^2 y} \sin nx$$

Terminal Questions

1) F is not continuous and therefore not differentiable at the point $r = 0$ for reasons explained in part (b) of Example 2.

$$\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{Gm^2}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}\right]$$

$$= \frac{1}{2}\frac{2(x-x_0)\, Gm^2}{\left\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right\}^{3/2}} = \frac{(x-x_0)Gm^2}{r^3}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{3}{4}\frac{4(x-x_0)^2 Gm^2}{\left\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right\}^{5/2}} + \frac{Gm^2}{r^3} = -\frac{3(x-x_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$$

Similarly $\frac{\partial^2 f}{\partial y^2} = -\frac{3(y-y_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$ and $\frac{\partial^2 f}{\partial z^2} = -\frac{3(z-z_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$

Thus $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{3Gm^2}{r^5}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right] + \frac{3Gm^2}{r^3}$

$$= -\frac{3Gm^2}{r^5} r^2 + \frac{3Gm^2}{r^3} \equiv 0$$

Therefore, $f = -\frac{Gm^2}{r}$ $(r > 0)$ satisfies the given PDE.

2) i) 2, 1, L, H

ii) 2, 1, L, H

iii) 2, 1, L, H

iv) 1, 1, L, H

v) 2, 1, L, H

vi) 2, 1, L, NH

APPENDIX A LIMITS OF A FUNCTION OF MORE THAN ONE VARIABLE

We will give here the formal mathematical definitions of the limits of a function of two or more variables and state the rules for evaluating limits of such functions.

Recall that the distance between two points (x, y) and (x_0, y_0) in the plane containing these points is less than δ if

$$\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

Let $f(x, y)$ be a real-valued function defined throughout a set containing a disc centred at (x_0, y_0) except possibly at (x_0, y_0) itself (see Fig. 5.2). Let L be a real number. Then L is the limit of f at (x_0, y_0) if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta, \text{ then } |f(x, y)-L|<\varepsilon$$

Then we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)=L$$

and say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.

Fig. 5.2 shows the geometric interpretation of the limit.

We can extend the definition to a function of three variables.

Once again, recall that the distance between the points (x, y, z) and (x_0, y_0, z_0) is less than δ if

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta$$

The formal definition of the limit of $f(x, y, z)$ is then given as follows:

Let f be defined throughout a set containing a ball centred at (x_0, y_0, z_0) except possibly at (x_0, y_0, z_0) itself. Then L is the limit of f at (x_0, y_0, z_0) if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta, \text{ then } |f(x, y, z)-L|<\varepsilon$$

In this case we write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)=L$$

and say that $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$ exists.

Let us consider a simple example to evaluate the limits using these basic definitions. Let us show that

$$\lim_{(x,y)\to(x_0,y_0)} x=x_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} y=y_0 \qquad \text{(A.1)}$$

Let $\varepsilon > 0$. Now $(x-x_0)^2 \le (x-x_0)^2+(y-y_0)^2$

Therefore, if we let $\delta=\varepsilon$, it follows that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta, \text{ then } |f(x, y)-L|=|x-x_0|=\sqrt{(x-x_0)^2}<\varepsilon\ (\because\ \delta=\varepsilon).$$

This proves that $\lim\limits_{(x,y)\to(x_0,y_0)} x=x_0$. You can prove the second limit in a similar way.

The result of Eq. (A.1) and the following limit formulas for the sum, products and quotients of functions of several variables will enable you to determine the limits of a variety of functions. We state the formulas for functions of two variables. Similar formulas would apply to functions of three and more variables.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y)$ exist, then

i) $$\lim_{(x,y)\to(x_0,y_0)} (af+bg)(x,y) = a \lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm b \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.2)}$$

where a and b are constants.

ii) $$\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.3)}$$

iii) $$\lim_{(x,y)\to(x_0,y_0)} (f/g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \Big/ \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.4)}$$

In Example 1, we have determined the limits of some functions using these formulas.

UNIT 6 PARTIAL DIFFERENTIAL EQUATIONS IN PHYSICS

Structure

6.1 INTRODUCTION

In Unit 5 you have learnt the basic concepts of order, degree, linearity, and type of partial differential equations (PDEs). Such equations arise for systems whose behaviour is governed by more than one independent variable. We come across PDEs in such diverse fields as meteorology, structural engineering, fluid mechanics, elasticity, heat flow, pollutant and neutron diffusion, wave propagation, aerodynamics, electromagnetics and nuclear physics. Most applied problems in physics are formulated in terms of second-order PDEs. From PHE-02 course on Oscillations and waves, you are familiar with the wave equation which governs wave propagation—a phenomenon responsible for hearing, seeing, music and our communication with the world at large. In your course on electric and magnetic phenomena, you would have come across Laplace's and Poisson's equations. These equations can also be used to determine gravitational potential, steady- state temperature etc.

A particularly useful method employed frequently to solve several second-order partial differential equations is the method of **separation of variables**. Depending on the number of independent variables, this method facilitates to reduce a linear PDE to two or more ordinary differential equations, which you already know to solve. This method is illustrated in Sec. 6.2. Boundary value problems in physics invariably exhibit rectangular, spherical or cylindrical symmetry in one or more dimensions. In Sec. 6.3 we illustrate the above said method to obtain a unique solution, subject to the given initial and boundary conditions. Since the same PDE may apply to many problems, the method discussed here can be used to solve many more problems than are illustrated here.

The term separation of variables was used in Unit 1 of this course in a completely different context.

Objectives

After studying this unit you should be able to

- solve a given PDE using the method of separation of variables
- obtain a unique solution to a given physical problem.

6.2 THE METHOD OF SEPARATION OF VARIABLES

Linear second order PDEs form the backbone of theoretical physics, Apart from Laplace's equation and Poisson's equation, the most important of these are the Helmholtz equation, Telegraph equation, wave equation, Klein-Gordon equation, Schrödinger equation and Dirac's equation.

Nonlinear PDEs are encountered in the study of shock wave phenomenon, atmospheric physics and turbulence. Higher order PDEs occur in the study of viscous fluids and elasticity.

The first question that should logically come to your mind is: How to solve a PDE? As a first strategy, we would like to reduce the given PDE to simpler differential equations containing fewer variables. (The process may be continued until a set of ordinary differential equations is obtained). Next, we put the ODEs so obtained in easily solvable form using methods discussed in Block 1 of this course. The simplest and most widely used method for reducing common and physically important PDEs is the **method of separation of variables**. Let us now learn how it works.

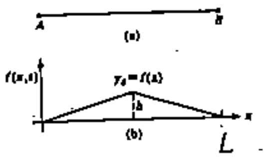


Fig. 6.1 : Vibrations of a string fixed at both ends

You will learn other analytical methods for solving PDEs, such as Green's function technique or numerical methods in later courses.

To illustrate the method of separation of variables, we consider a finite string AB of length L fixed at both ends, as shown in Fig. 6.1(a). Suppose that the string is plucked (initial displacement $h(x)$) and then released from rest, as shown in Fig. 6.1(b). If we choose x-axis along the length of the string, you may recall from Unit 5 of PHE-02 course Oscillations and Waves that the motion of the string is described by the 1-D wave equation:

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2} \tag{6.1}$$

You would note that there is no term containing mixed partials like $\frac{\partial^2 f}{\partial t \partial x}$ or $\frac{\partial f}{\partial t}\frac{\partial f}{\partial x}$ in Eq. (6.1). This is because this equation is obtained under the assumption that the string is displaced only slightly from its equilibrium position.

We now assume that the solution of Eq. (6.1) can be written in the form of a product as

$$f(x, t) = X(x)\ T(t) \tag{6.2}$$

Physically, it means that the dependence of the unknown function on one variable is in no way affected by its dependence on other variable. Does this imply that there is no connection at all between X and T? No, it only means that the function X does not depend on t and the function T does not depend upon x. For instance, the function

$$f(x, t) = x \sin \omega t \tag{6.3a}$$

is completely separable in x and t. On the other hand, the function

$$f(x, t) = x + t \tag{6.3b}$$

is inseparable in that the function cannot be written as a product of two functions.

To illustrate the method, we differentiate Eq. (6.2) twice with respect to x. This gives

$$\frac{\partial f}{\partial x} = X'T$$

and

$$\frac{\partial^2 f}{\partial x^2} = X''T \tag{6.4}$$

where prime(s) denote ordinary differentiation with respect to x. This emphasises the fact that the derivative is the total derivative and the function X has only one independent variable. Similarly, if we differentiate Eq. (6.2) with respect to t, we obtain

$$\frac{\partial f}{\partial t} = X\dot{T}$$

and

$$\frac{\partial^2 f}{\partial t^2} = X\ddot{T} \tag{6.5}$$

where dot(s) denote ordinary differentiation with respect to t. We have used primes and dots just to distinguish the independent variables with respect to which differentiation has been carried out.

By inserting results contained in Eqs. (6.4) and (6.5) into Eq. (6.1), you would obtain

$$X(x)\ddot{T}(t) = v^2 X''(x) T(t)$$

Dividing throughout by $v^2 X(x) T(t)$, we get

$$\frac{\ddot{T}(t)}{v^2 T(t)} = \frac{X''(x)}{X(x)} \tag{6.6}$$

The left hand side of this equation involves functions which depend only on t whereas the expression on right-hand side is a function of x only. Thus, if we vary t and keep x fixed, the right-hand side cannot change. This means that $\ddot{T}(t)/v^2 T(t)$ must remain constant for all t. Similarly, if we vary x holding t fixed, the left-hand side must not change. That is, the quantity $X''(x)/X(x)$ must be the same for all x. Mathematically, we express this fact by saying that both sides must be equal to a constant, k say. Is this argument sound? To discover the answer to this question, let us write k to represent either side of Eq. (6.6), i.e.,

$$\frac{\ddot{T}(t)}{v^2 T(t)} = k = \frac{X''(x)}{X(x)} \qquad (6.7)$$

This is really the key to the process of separation of variables.

Then from the right-hand side, we have

$$\frac{\partial}{\partial t}(k) = \frac{\partial}{\partial t}\left[\frac{X''(x)}{X(x)}\right] = 0$$

and from the LHS, we have

$$\frac{\partial}{\partial x}(k) = \frac{\partial}{\partial x}\left[\frac{\ddot{T}(t)}{v^2 T(t)}\right] = 0$$

Since the first order partial derivative of k with respect to t or x is zero, k must be a constant. It is called the **separation constant**. It means that if $y = a_0 \sin \omega t$ is a solution of the ODE

$$\ddot{y} + \omega^2 y = 0$$

we will get an identity, for all values of t, on substituting the assumed form of the solution in the given equation.

Thus, you can now rewrite the given equation as two ordinary differential equations :

$$X''(x) - k\,X(x) = 0 \qquad (6.8a)$$

and

$$\ddot{T}(t) - k\,v^2\,T(t) = 0 \qquad (6.8b)$$

That is, by assuming a separable solution, we have reduced a partial differential equation in two variables into two equivalent ordinary differential equations.

SAQ 1

Spend 15 minutes

Use the method of separation of variables to reduce the following PDEs to a set of ODEs :

i) $$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

ii) $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r,\theta)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

iii) $$\frac{\partial^2 \psi}{\partial x^2} + \alpha \frac{\partial \psi(x,t)}{\partial t} = 0$$

The first of these equations describes the steady-state temperature distribution in a cylindrical body, like control/fuel rods in the reactor core. The second PDE describes the potential in the region on either side of a spherical surface. The third PDE is the one-dimensional Schrödinger wave equation.

You can now solve these equations without much difficulty using the methods developed in Unit 2, Block 1 of this course. For instance, for a nonzero value of k, the solutions of Eqs. (6.8a) and (6.8b) are of the form $\exp(mx)$ and $\exp(nt)$ respectively. The characteristic equations are

$$m^2 - k = 0 \qquad (6.9a)$$

and

$$n^2 - k\,v^2 = 0 \qquad (6.9b)$$

which have roots

$$m_1 = \sqrt{k} = \mu, \; m_2 = -\sqrt{k} = -\mu \qquad (6.10a)$$

and

$$n_1 = v\sqrt{k}, = v\mu,\ n_2 = -v\sqrt{k} = -\mu v \qquad (6.10b)$$

The resulting solutions, therefore, are

$$X(x) = A\exp(\mu x) + B\exp(-\mu x) \qquad (6.11a)$$

and

$$T(t) = C\exp(\mu vt) + D\exp(-\mu vt) \qquad (6.11b)$$

which are sums of growing and decaying exponentials. If you calculate time derivative of $T(t)$, you will obtain velocity, which too will increase or decrease with respect to time. This means that the kinetic energy of an element of the string will increase and decrease with time simultaneously, which is physically unacceptable.

You can now write the general solution as

$$f(x,t) = X(x)\,T(t) = [A\exp(\mu x) + B\exp(-\mu x)]\,[C\exp(\mu vt) + D\exp(-\mu vt)] \qquad (6.12)$$

However, in view of the argument given before Eq. (6.12), this solution does not give the desired wave motion. So k cannot have positive values. Similarly the value $k = 0$ leads to a **trivial** solution and is not acceptable. However, for $k < 0$, $\sqrt{k}$ will be imaginary. Therefore, we can write

$$\sqrt{k} = i\beta$$

where β is a real number and $i = \sqrt{-1}$. Then, Eq. (6.11a) becomes

$$X(x) = A\exp(i\beta x) + B\exp(-i\beta x) \qquad (6.13a)$$

and Eq. (6.11b) takes the form

$$T(t) = C\exp(i\beta vt) + D\exp(-i\beta vt) \qquad (6.13b)$$

$\exp(i\theta) = \cos\theta + i\sin\theta$

Using the Euler's relation, you can rewrite Eqs. (6.13a) and (6.13b) as

$$X(x) = A_1\sin\beta x + A_2\cos\beta x \qquad (6.14a)$$

and

$$T(t) = G_1\sin\beta vt + G_2\cos\beta vt \qquad (6.14b)$$

where A_1, A_2, G_1 and G_2 are new constants. You can easily verify that $A_1 = i(A - B)$, $A_2 = A + B$, $G_1 = i(C - D)$ and $G_2 = C + D$. The solutions given by Eqs. (6.14a, b) are periodic in space and time. You can now write the general solution of 1 – D wave equation as

$$f(x,t) = X(x)\,T(t) = (A_1\sin\beta x + A_2\cos\beta x)\,(G_1\sin\beta vt + G_2\cos\beta vt) \qquad (6.15)$$

In the above example we have illustrated the method of separation of variables by considering PDEs in two variables (x, t). Can you think of a physical situation where the PDE of interest involves more thantwo variables? The music produced by a drum used in folk dances involves the vibrations of a circular membrane. The wave motion is two dimensional and the PDE involves three variables (r, θ, t). Similarly, in the heat flow in a rectangular plate, the number of independent variables is three : (x, y, t). This is of particular interest to a reactor physicist since plate type fuel elements may be used in the reactor core. It is, therefore, important for us to extend the method of separation of variables to three (or more) variables. For simplicity, let us first consider a rectangular membrane whose edges are fixed at $x = 0$, $x = a$, $y = 0$ and $y = b$, as shown in Fig. 6.2.

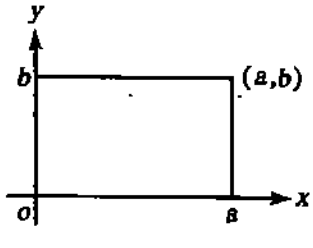


Fig. 6.2 : A rectangular membrane fixed at edges

Rectangular Membrane

The function $f(x, y, t)$ satisfies the wave equation

$$\frac{\partial^2 f(x,y,t)}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y,t) \qquad (6.16)$$

While solving this equation by the method of separation of variables, we expect to reduce it to three second order ODEs, which possess periodic solutions in space and time. You can do so in two ways :

i) By partial separation of Eq. (6.16) in space variables (x, y) taken together and the time variable by writing

$$f(x, y, t) = F(x, y)\, T(t) \tag{6.17a}$$

where $F(x, y)$ is a function of space and T depends only on time.

This will result in an ODE in time and a PDE in space variables, which may then be further split to arrive at ODEs in x and y. This **two-stage process** is worthwhile to attempt as it invariably facilitates mathematical steps.

ii) Separate all the three variables by writing

$$F(x, y, t) = X(x)\, Y(y)\, T(t) \tag{6.17b}$$

How do we know that this is valid? The answer is simple. We do not say that it is valid; We only wish to discover if it works. But we expect that both substitutions should lead us to the same result. Why? Because a tool (mathematical technique) cannot influence physics. We now illustrate this by solving Eq. (6.16) using both substitutions.

As before, let us assume a separable solution

$$f(x, y, t) = F(x, y)\, T(t)$$

where $F(x, y)$ is a function of space only and $T(t)$ is a function of time only. Substituting it in Eq. (6.16) we find that

$$F\ddot{T} = v^2 T\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y)$$

Dividing throughout by v^2FT, we get

$$\frac{\ddot{T}}{v^2T} = \frac{1}{F}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y) \tag{6.18}$$

By comparing it with Eq. (6.6) you can say that the expression on the left-hand side depends only on t, whereas the expression on the right-hand side depends only on space variables. Following the arguments used for wave equation in two variables, we can say that both sides must be equal to a constant. We now know that only negative values of this constant will lead to a nontrivial solution. If we denote this constant by $-p^2$, we have

$$\frac{\ddot{T}}{v^2T} = \frac{1}{F}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y) = -p^2 \tag{6.19}$$

This yields two differential equations :

$$\ddot{T} + p^2v^2T = 0 \tag{6.20}$$

and

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + p^2F = 0 \tag{6.21}$$

You will note that whereas Eq. (6.20) is an ordinary differential equation, Eq. (6.21) still contains partial derivatives in x and y. That is, although we have separated the space and time variables, we have to separate space dependences. To do so, we assume that

$$F(x, y) = X(x)\, Y(y) \tag{6.22}$$

Substituting it in Eq. (6.21), we obtain

$$\frac{d^2X}{dx^2}Y = -X\left(\frac{d^2Y}{dy^2} + p^2Y\right)$$

On dividing both sides by XY, we find that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\left(\frac{d^2Y}{dy^2} + p^2Y\right) \tag{6.23}$$

Note that the expression on LHS depends only on x, whereas the expression on RHS depends only on y. Therefore, both sides must be equal to a constant, which we take $-q^2$:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\left(\frac{d^2Y}{dy^2} + p^2Y\right) = -q^2 \tag{6.24}$$

This immediately leads to two ordinary equations :

$$\frac{d^2X}{dx^2} + q^2X = 0 \tag{6.25}$$

and

$$\frac{d^2Y}{dy^2} + \alpha^2Y = 0 \tag{6.26}$$

where $\alpha^2 = p^2 - q^2$.

We thus find that Eq. (6.16) which contained derivatives with respect to three independent variables has been reduced to three separate second-order ODEs (Eqs. (6.20), (6.25) and (6.26). Thus in the two-stage process of separation of variables, we separated the time dependence from the space dependence by clubbing them in one function, $F(x, y)$, which is subsequently separated.

Let us now split Eq. (6.16) by taking $f(x, y, t)$ as a product of three functions as in Eq. (6.17b). Then we can write

$$\frac{\partial^2 f}{\partial t^2} = X(x)\,Y(y)\,\ddot{T}(t)$$

$$\frac{\partial^2 f}{\partial x^2} = X''(x)\,Y(y)\,T(t)$$

and

$$\frac{\partial^2 f}{\partial y^2} = X(x)\,Y''(y)\,T(t)$$

On substituting these in Eq. (6.16) you will obtain

$$X(x)\,Y(y)\,\ddot{T}(t) = v^2\,[Y(y)\,T(t)\,X'' + X(x)\,Y''(y)\,T(t)]$$

On dividing throughout by $T(t)\,X(x)\,Y(y)$, this equation simplifies to

$$\frac{\ddot{T}(t)}{v^2T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} \tag{6.27}$$

The left hand side of this identity is a function only of time and the right hand is a function only of the space variables. Therefore, we can write

$$\frac{1}{v^2}\frac{\ddot{T}}{T} = -k^2$$

or

$$\ddot{T} + k^2v^2T = 0 \tag{6.28a}$$

and

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2$$

or

$$\frac{1}{X}\frac{d^2X}{dt^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} \tag{6.28b}$$

Here we have a function of x equated to a function of y. As before, we equate each side to another constant, $-m^2$. So we can split Eq. (6.28b) into two ODEs:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -m^2 \tag{6.29a}$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + m^2 = -n^2 \tag{6.29b}$$

where we have introduced a new constant by $k^2 = m^2 + n^2$ to produce a symmetric set of equations. Thus we find that Eq. (6.16) has been replaced by three ODEs (Eqs. (6.28a), (6.29a) and (6.29b)).

If you identify p with k, m with q and n with α, Eqs. (6.28a), (6.29a) and (6.29b) will become identical to Eqs. (6.20) (6.25) and (6.26), respectively. We hope that now you have understood both the processes. To get a better grasp of these concepts you may like to work out an SAQ.

SAQ 2

Spend 15 minutes

The Helmholtz equation in Cartesian coordinates can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2 f(x, y, z) = 0$$

Reduce it to three ODEs using one step process.

So far we have considered a rectangular membrane fixed at its edges. The space variables in the PDE describing its vibrations were taken to be Cartesian. But you will readily recognise that for musical instruments like beating-drum and cymbal, use of spherical polar coordinates is desirable. You can mathematically model wave propagation in these instruments by considering the vibrations of a circular membrane. Let us now learn to separate wave equation in spherical coordinates.

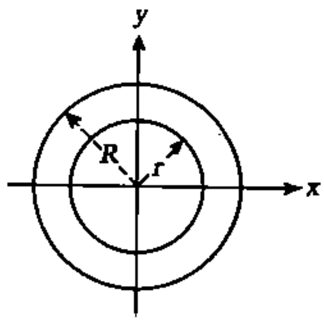


Fig. 6.3 : A circular membrane fixed at the perimeter

Circular Membrane

For a circular membrane held fixed at the perimeter, as shown in Fig. 6.3, the wave equation takes the form

$$\frac{\partial^2 f}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f(r, \theta, t) \tag{6.30}$$

By assuming a solution in the separable form as

$$f(r, \theta, t) = F(r, \theta)\, T(t) \tag{6.31}$$

you can readily show that Eq. (6.30) reduces to

$$\ddot{T} + \lambda^2 T = 0 \tag{6.32a}$$

and

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F(r, \theta) = 0 \tag{6.32b}$$

You would recall from PHE-02 course that the three dimensional wave equation is

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where ∇^2 is the Laplacian. You have studied its form in different coordinate systems in Unit 3 of the PHE-04 course.

where $\lambda = vk$; k being the separation constant. You will note that Eq. (6.32b) still contains two variables. We separate these as well and write

$$F(r, \theta) = R(r)\, \Theta(\theta) \tag{6.33}$$

Substituting in Eq. (6.32b), we obtain

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + r^2k^2R\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = p^2$$

so that Eq. (6.32b) reduces to two ODEs; one involving R and the other one for Θ

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (r^2k^2 - p^2)\,R = 0 \qquad (6.34a)$$

and

$$\frac{d^2\Theta}{d\theta^2} + p^2\,\Theta = 0 \qquad (6.34b)$$

Spend 10 minutes

SAQ 3

By substituting

$$f(r, \theta, t) = R(r)\,\Theta\,(\theta)\,T(t)$$

in Eq. (6.30), show that it can be split into three ODEs.

To put Eq. (6.34a) in a more familiar form, let us introduce a change of variable by defining

$$s = kr$$

Then

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = k\frac{dR}{ds}$$

and

$$\frac{d^2R}{dr^2} = k^2\frac{d^2R}{ds^2}$$

Substituting these in Eq. (6.34a) you will get

$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + (s^2 - p^2)\,R = 0 \qquad (6.35)$$

which is Bessel's equation of order p.

So far we have familiarised you with the basic technique of separating variables for reducing a PDE to a set of ODEs. You can solve these using the methods described in Block 1. However, in physical problems, we have to usually obtain unique solutions of PDEs, which correspond to certain initial and boundary conditions.

Before proceeding further to solve initial and boundary value problems in PDEs, let us stop for a while and summarise what we know about the method of separation of variables.

1) First of all, the unknown function of two (or more) variables is expressed as a product of two (or more) functions so that the dependence of one on an independent variable is in no way affected by the dependence of the other variable(s).

2) The assumed form of solution is inserted in the given differential equation. A second-order PDE in two variables splits into two ODEs. When the number of independent variables is more than two, we get ODEs equal in number to the independent variables.

3) You can solve the ODEs so obtained using methods known from Block-1. The solutions may be exponential functions, trigonometric functions, or power series.

4) The general solution of the given PDE is obtained by taking the product of the solutions of ODEs.

We hope that you can now use the method of separation of variables to reduce a PDE to a set of ODEs. (The number of ODEs equals the number of independent variables in the given PDE.) For the remainder of this unit, we shall confine ourselves to finding product solutions of wave equation, heat equation and Laplace's equation for different physical situations, under specific initial and boundary conditions.

6.3 SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS IN PHYSICS

When we solve a PDE, the number of solutions is, in general, very large. For example, if you consider Laplace's equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

you can readily verify that each of the functions

$$f=x^2-y^2, f=e^{x} \cos y \text{ and } f=\ln(x^2+y^2)$$

satisfies the given equation. However, all these are completely different from each other. There are many other functions which would satisfy the above-said equation. Does this mean that we cannot obtain a general solution of a PDE? In physical problems, a general solution is seldom sought. Even if we can obtain a general solution, it involves too much arbitrariness. That is, it is not unique. You may ask: Why is it so? This is because a PDE with independent variables in space (x) and time (t), which is of second order in each of these variables, requires two conditions at some x and two conditions at some t. (If the condition on x is specified at a boundary, we say that we are specifying **boundary conditions.** Usually the conditions on time are given at the instant we start making observations. These are referred to as **initial conditions**). So you may conclude that

Refer to IVP and BVP defined in Block 1.

> To obtain a unique solution to a given PDE, we have to specify initial conditions (ICs) and boundary conditions (BCs) which correspond to the particular physical problem.

Let us consider certain IVPs and BVPs in PDEs that arise in physics.

One-dimensional wave equation

Let us consider a wave propagating on a string. We put equidistant marks to identify particles of the string. We wish to determine instantaneous displacement of a particle at any of these marked positions. Mathematically speaking, we wish to determine a function $f(x, t)$, which depends on two independent variables. We have to supplement the PDE describing this phenomenon by BCs. The boundary conditions will involve f, or some of its derivatives, or both, on the curve (boundary) enclosing the region (of independent variables) over which a solution is being sought.

Proceeding further, we note that we have to solve a one- dimensional equation for a string of length L such as a guitar, an Ektara or a violin string illustrated in Fig. 6.1 :

$$\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 f(x,t)}{\partial x^2}$$

where v is wave speed.

Since the string is fixed at $x=0$ and $x=L$ for all times, the boundary conditions may be written as

$$f(0,t)=0$$

and

$$f(L,t)=0 \text{ for all } t>0 \tag{6.36}$$

Since the solution of the wave equation depends on t as well, we must also know as to what happens at $t=0$. That is, we have to specify initial conditions on displacement and velocity. Since the string is released from rest, the initial velocity is zero. In mathematical terms, we seek a function $f(x, t)$ which satisfies the initial conditions

$$f(x,0)=h(x) \qquad 0<x<L$$

and

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0 \qquad (6.37)$$

Separating variables in wave equation, you will obtain

$$X'' + \mu^2 X(x) = 0$$

and

$$\ddot{T}(t) + \mu^2 v^2 T(t) = 0.$$

whose solutions are given by Eq. (6.11):

$$X(x) = A\cos\mu x + B\sin\mu x$$

and

$$T(t) = C\cos\mu vt + D\sin\mu vt$$

Now since $f(0, t) = X(0)\,T(t) = 0$ and $f(L, t) = X(L)\,T(t) = 0$, we must have $X(0) = 0$ and $X(L) = 0$. Using the first of these conditions, we find that $A=0$. Therefore

$$X(x) = B\sin\mu x$$

The second condition now implies that

$$X(L) = B\sin\mu L = 0$$

This equality will be satisfied if $B = 0$ or $\sin\mu L = 0$. If $B = 0$, then $X = 0$ so that $f = 0$, which is a trivial solution. Hence, we must have $B \neq 0$ and the only option is $\sin\mu L = 0$. This implies that $\mu L = n\pi$ or $\mu = n\pi/L$ for $n = 0, 1, 2, 3, \ldots$. The solution for $n = 0$ is a trivial solution. For any arbitrary value of B, we obtain infinite solutions of the form

$$X(x) \equiv X_n(x) = B_n \sin\left(\frac{n\pi}{L}\right)x \qquad n = 1, 2, 3, \ldots \qquad (6.38)$$

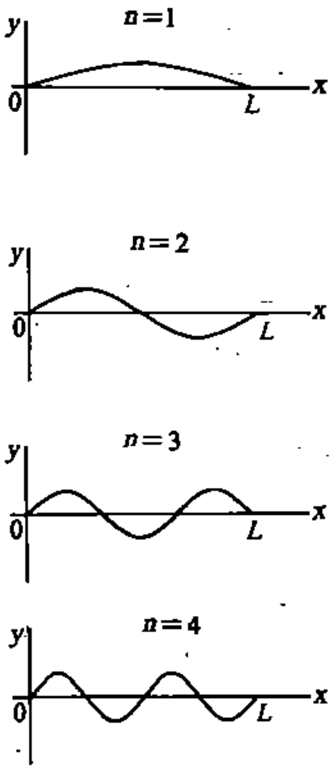


Fig. 6.4 : A plot of Eq. (6.38) for the first few modes

The values of $\mu = n\pi/L$ for $n = 1, 2, 3, \ldots$ are called **eigenvalues** of Eq. (6.1). With $B = 1$, Eq. (6.38) is depicted in Fig. 6.4 for $n = 1, 2, 3$ and 4. Hence, the solution of Eq. (6.2) which satisfies the given boundary conditions can now be written as

$$f_n(x,t) = \left[C\cos\left(\frac{n\pi vt}{L}\right) + D\sin\left(\frac{n\pi vt}{L}\right)\right]B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \left[a_n\cos\left(\frac{n\pi vt}{L}\right) + b_n\sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad (6.39)$$

where we have put $CB_n = a_n$ and $DB_n = b_n$ since each value of n may require different constants. You would note that the subscript n has been added to $f(x, t)$. Do you know why? This is just to allow for a different function for each value of n. In the present case, each value of n defines harmonic motion of the string with frequency $(n\,v/2L)$ Hz. Whereas $n = 1$ defines the **fundamental mode**, $n > 1$ characterises **overtones**.

You would agree that $f_n(x, t)$ is not a solution of the given problem since initial conditions have not yet been imposed. Moreover, since the wave equation is linear and homogeneous, we expect that the most general solution, which satisfies the given boundary conditions, is given by the superposition principle:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi vt}{L}\right) + b_n\sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad (6.40)$$

To match the initial conditions, we set $t=0$ in the above equation. This gives

$$f(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = h(x) \qquad (6.41)$$

Now the question arises: How to evaluate a_n? To determine the constants a_n, we must know the form of the function $h(x)$. Let us take $h(x) = \xi_0 \sin \frac{\pi x}{L}$. Then by comparison, we have

$$a_1 = \xi_0$$

and

$$a_2 = a_3 = \ldots = 0 \tag{6.42}$$

For any general form of $h(x)$ you would require Fourier series, which you will learn in the next two units.

To determine b_n, we first differentiate Eq. (6.40) with respect to t and then set $t = 0$. The result is

$$\frac{\partial f}{\partial t} = \sum_{n=1}^{\infty} \left[-a_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi vt}{L}\right) + b_n \left(\frac{n\pi v}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

so that

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = 0 = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

You will readily conclude by looking at this expression that

$$b_n = 0\,;\; n = 1, 2, \ldots \tag{6.43}$$

Hence the unique solution of the one-dimensional wave equation on a string tied at both ends corresponding to the given initial and boundary conditions is given by

$$f(x, t) = \xi_0 \cos \omega t \sin\left(\frac{\pi x}{L}\right) \tag{6.44}$$

where $\omega = \frac{\pi v}{L}$ is angular frequency and ξ_0 is amplitude.

SAQ 4

Spend 2 minutes

Determine the constants a_n's occurring in Eq. (6.40) when

$$h(x) = \xi_0 \left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

The boundary value problems considered in SAQ 4 refer to Cartesian geometry. You know of many physical problems which involve spherical and cylindrical coordinates. You have studied these in Unit 3 of Block 1, PHE-04 course entitled Mathematical Methods in Physics-I. In particular, we may mention wave propagation on the membrane of a tabla or a beating drum, electric field around a long current carry wire, energy produced in a nuclear reactor, etc. In the following examples, we have considered physical problems involving spherical polar and cylinderical coordinates.

Example 1 : Circular Membrane

The radial part of wave equation for a circular membrane of radius r_0 fixed at its circumference is

$$\frac{\partial^2 f}{\partial t^2} = v^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right)$$

Specify the boundary conditions and obtain a unique solution.

$$J_0(2\lambda) = 0 \qquad \text{(iv)}$$

This equation will hold for $\lambda_1 = \alpha_1/2,\ \lambda_2 = \alpha_2/2, \ldots, \lambda_n = \alpha_n/2$ where $\alpha_1, \alpha_2, \ldots . \alpha_n$ are zeros of the Bessel function. Lastly $Z(0) = 0$ implies $c_3 = 0$. Hence we have $R = c_1 J_0(\lambda_n \rho)$, $Z = c_4 \sinh \lambda_n z$, and

$$u_n = A_n \sinh \lambda_n z\ J_0(\lambda_n \rho)$$

The general solution is therefore of the form

$$u(r, z) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n z\ J_0(\lambda_n \rho)$$

Example 3

Find the steady-state temperature $T(r, \theta)$ in the semi-circular plate shown in Fig. 6.7.

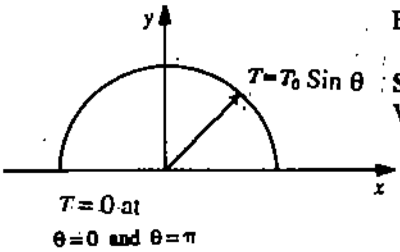


Fig. 6.7 : A semi circular plate

Solution

We must solve

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad 0 < \theta < \pi;\ 0 < r < a_0$$

subject to the boundary conditions

$$T(a_0, \theta) = T_0 \sin\theta \qquad 0 < \theta < \pi$$

$$T(r, 0) = 0,\ T(r, \pi) = 0 \qquad 0 < r < a_0$$

If we define $T = R(r)\,\Theta(\theta)$, then separation of variables gives

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

so that original PDE reduces to

$$r^2 R'' + rR' - \lambda^2 R = 0 \qquad \text{(i)}$$

and $$\Theta'' + \lambda^2 \Theta = 0 \qquad \text{(ii)}$$

Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi) = 0$ to the solution $\Theta = c_1 \cos\lambda\theta + c_2 \sin\lambda\theta$ of (ii) yields $c_1 = 0$ and $\lambda = n$; $n = 1, 2, 3, \ldots$. Hence $\Theta = c_2 \sin n\theta$. For $\lambda = n$, (i) is Cauchy-Euler equation. You can solve it using methods discussed in Unit 3. It has the solution

$$R = c_3 r^n + c_4 r^{-n}$$

In order that the solution $T(r, \theta)$ is finite as $r \to 0$ we must demand that $c_4 = 0$. Therefore

$$T_n = A_n r^n \sin n\theta$$

and $$T(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

The boundary condition at $r = a_0$ gives

$$T_0 \sin\theta = \sum_{n=1}^{\infty} A_n a_0^n \sin n\theta$$

so that $A_1 = \dfrac{T_0}{a_0}$ and $A_2 = A_3 = 0 = \ldots\ldots = A_n$

Hence

$$T(r, \theta) = \frac{T_0}{a_0} r \sin\theta$$

Let us now sum up the unit.

6.4 SUMMARY

- A linear partial differential equation in two variables can be solved by assuming a solution in the form of a product $f = XY$ where X is a function of x only, and Y is a function of y only. This **method of separation of variables** leads to two ordinary differential equations.
- A **boundary value problem** consists of finding a function that satisfies a partial differential equation as well as conditions consisting of boundary conditions and initial conditions.
- The method of solving a boundary value problem using separation of variables consists of the following basic steps.
  - i) Write the function f as a product of two (or more) functions involving independent variables, i.e. $f(x, y) = X(x)\,Y(y)$ and insert it in the given PDE. You will obtain a set of ODEs. A PDE in two variables splits into two ODEs. If the number of independent variables is more, we get ODEs whose number is equal to the number of variables.
  - ii) Solve the separated ordinary differential equations. The solutions may be exponential functions, trignometric functions or a power series.
  - iii) Substitute the solutions so obtained in the above product.
  - iv) Use boundary and/or the initial condition(s), and solve for the coefficients in the series.

6.5 TERMINAL QUESTIONS

1. i) The electrostatic potential in the exterior and interior of a spherical shell is calculated by using Laplace's equation : $\nabla^2 f = 0$. Use the method of separation of variables to split it into three ODEs.
(Hint : Express ∇^2 in spherical polar coordinates.)

   ii) The wave propagation in space is described by 3-D equation

$$\nabla^2 f(\mathbf{r}, t) = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}(\mathbf{r}, t)$$

Working in Cartesian coordinates show that it can be reduced to four ODEs:

$$\ddot{T} + \omega^2 T = 0$$

$$X'' + l^2 X = 0$$

$$Y'' + m^2 Y = 0$$

and

$$Z'' + n^2 Z = 0$$

where $\omega = vk$ and l, m, n are separation constants.

2. The one dimensional wave equation for em wave propagation in free space is given by (for $\mathbf{E} \,||\, \hat{\mathbf{y}}$)

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0$$

Solve this equation and obtain the eigen frequencies of the cavity if $E_y = 0$ at $x = 0$ and $x = L$.

3. Consider a rod whose ends are kept at a constant temperature and the lateral surface is insulated. The heat flow is described by one-dimensional heat equation subject to the conditions

$$f(0, t) = 0, f(L, t) = 0 \qquad \text{for} \quad t > 0$$

and

$$f(x, t) = f(x) \qquad \text{for} \quad 0 < x < L$$

Obtain a unique solution.

6.6 SOLUTIONS AND ANSWERS

SAQs

1. i) Let us take

$$T(r, z) = R(r)\, Z(z) \tag{i}$$

Then,

$$\frac{\partial T}{\partial r} = \frac{dR}{dr} Z$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{d^2 R}{dr^2} Z$$

and

$$\frac{\partial^2 T}{\partial z^2} = R \frac{d^2 Z}{dz^2} \tag{ii}$$

Substituting these in the given PDE, we obtain

$$Z \frac{d^2 R}{dr^2} + \frac{Z}{r}\frac{dR}{dr} + R \frac{d^2 Z}{dz^2} = 0$$

On dividing throughout by ZR, we get

$$\frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] = -\frac{1}{Z}\frac{d^2 Z}{dz^2} \tag{iii}$$

The LHS of this equality involves functions which depend only on r, whereas the expression on RHS is a function of z only. So both sides must be equal to a constant, k. Hence the given equation splits into the following two ODEs:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - kR = 0 \tag{iv}$$

and

$$\frac{d^2 Z}{dz^2} + kZ = 0 \tag{v}$$

ii) The given PDE can be rewritten as

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r} + \cot\theta \frac{\partial V}{\partial \theta} + \frac{\partial^2 V}{\partial \theta^2} = 0 \tag{i}$$

Let us now write

$$V(r, \theta) = R(r)\, \Theta(\theta) \tag{ii}$$

Then, differentiation with respect to r gives

$$\frac{\partial V}{\partial r} = \frac{dR}{dr}\Theta$$

and

$$\frac{\partial^2 V}{\partial r^2} = \frac{d^2 R}{dr^2}\Theta \qquad \text{(iii)}$$

Similarly, differentiation with respect to θ gives

$$\frac{\partial V}{\partial \theta} = R\frac{d\Theta}{d\theta}$$

and

$$\frac{\partial^2 V}{\partial \theta^2} = R\frac{d^2\Theta}{d\theta^2} \qquad \text{(iv)}$$

Substituting these results in the given equation, we obtain

$$r^2\Theta\frac{d^2R}{dr^2} + 2r\Theta\frac{dR}{dr} + \cot\theta\, R\frac{d\Theta}{d\theta} + R\frac{d^2\Theta}{d\theta^2} = 0$$

As before, on dividing throughout by $R\,\Theta$, we get

$$\frac{1}{R}\left[r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right] = -\frac{1}{\Theta}\left[\cot\theta\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2}\right] \qquad \text{(v)}$$

The LHS is a function of r only whereas RHS is a function of θ only. Hence, putting them equal to a constant, k, we get

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0$$

or

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - kR = 0 \qquad \text{(vi)}$$

and

$$\cot\theta\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2} + k\Theta = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + k\Theta = 0 \qquad \text{(vii)}$$

iii) The given equation is

$$\frac{\partial^2\psi}{\partial x^2} + \alpha\frac{\partial\psi}{\partial t} = 0$$

Express $\psi(x, t)$ as a product of two separable functions:

$$\psi(x, t) = X(x)\,T(t)$$

Substituting it in the given PDE, you will obtain

$$X''(x)\,T(t) + \alpha X(x)\,\dot{T}(t) = 0$$

Dividing throughout by $X(x)\,T(t)$, we find that

$$\frac{X''(x)}{X(x)} = -\alpha\frac{\dot{T}(t)}{T(t)} = -k^2$$

so that the Schrödinger equation splits into following equations:

Terminal Questions

1. i) In spherical polar coordinates, the Laplace's equation can be written as

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right]=0$$

In the method of separation of variables, we write

$$f(r,\theta,\phi)=R(r)\,\Theta(\theta)\,\Phi(\phi)$$

By substituting back into the given equation and dividing by $R\,\Theta\,\Phi$, we have

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\, r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

On multiplying throughout by $r^2\sin^2\theta$, we can isolate ϕ dependent term:

$$-\frac{1}{R}\sin^2\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

As before, you will note that this equation relates a function of ϕ alone to a function of r and θ. Since r, θ, ϕ are independent variables, we can equate each side to a constant. Let us choose it to be $-m^2$. Then

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2 \qquad \text{(i)}$$

and

$$\frac{1}{R}\sin^2\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=m^2$$

or

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{m^2}{\sin^2\theta}-\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

Again equating each side to a constant, we get the required result:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-nR=0 \qquad \text{(ii)}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\Theta+n\,\Theta=0 \qquad \text{(iii)}$$

where n is separation constant.

ii) The 3-D wave equation is

$$\nabla^2 f(\mathbf{r},t)=\frac{1}{v^2}\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} \qquad \text{(i)}$$

In Cartesian coordinates, the Laplacian can be written as

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad \text{(ii)}$$

so that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(x,y,z,t)=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \text{(iii)}$$

The function f depends on four variables. Let us write it as

$$f(x, y, z, t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

On substituting back in the given equation and dividing by $XYZT$, we find that

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} \qquad \text{(iv)}$$

The LHS is a function of space variables whereas RHS is a function of time alone. Let us therefore choose

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2$$

so that

$$\frac{d^2T}{dt^2} + \omega_0^2 T = 0 \qquad \text{(v)}$$

where $\omega_0 = kv$.

Then, (iv) reduces to

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad \text{(vi)}$$

Again LHS is a function of x only, whereas the RHS depends only on y and z. We, therefore, choose

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2$$

so that

$$\frac{d^2X}{dx^2} + l^2X = 0 \qquad \text{(vii)}$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = l^2 - k^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}$$

Proceeding along the same lines you can show that

$$\frac{d^2Y}{dy^2} + m^2Y = 0$$

and

$$\frac{d^2Z}{dz^2} + n^2Z = 0$$

where $n^2 = k^2 - l^2 - m^2$.

2. The given equation describes e.m. wave propagation in free space:

$$\frac{\partial^2E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2E_y}{\partial t^2} = 0$$

Let us make the substitution

$$E_y = X(x)\,T(t)$$

so that

$$X''\,T(t) - \frac{1}{c^2}X(x)\,\ddot{T} = 0$$

UNIT 7 FOURIER SERIES

Structure

7.1 INTRODUCTION

In Unit 6 you have learnt to solve second order PDEs which arise in the study of physical phenomena. While solving a BVP in Sec. 6.4 you came to know that you would need to learn a new method to be able to completely solve many BVPs in physics. In this unit you will learn this method based on **Fourier Series.** To begin with, in Sec. 7.2 we will again examine why such a method is needed to solve BVPs. In this context, let us briefly recount the historical development of this method.

Following the invention of calculus by Newton and Leibnitz, a great burst of activity was seen in mathematical physics. Problems that particularly attracted the attention of scientists of that period pertained to vibrations in instruments. These were modelled by boundary value problems related to vibrations of strings, elastic bars and columns of air. By the 1750s, d'Alembert, Bernoulli and Euler had established the PDE for a vibrating string, found its general solution (as in Eq. 6.15 of Unit 6) and determined the solution for a given BVP for strings. You can see that the solution given by Eq. (6.40) of Unit 6 is the sum of a series of trigonometric functions. This further led these mathematicians to the problem of representing arbitrary functions by trigonometric series. Later on, Euler found expressions for the coefficients in those series. However, the question of the validity of representing arbitrary functions by such series was not settled at that time.

Later, in his work on BVPs in heat conduction, the French scientist Jean Baptiste Fourier (1768-1830) presented many examples of representing arbitrary functions by the sum of infinite series containing sine and cosine terms. Since it was Fourier's work that aroused major interest in representing any function by such a series, this particular series has been named after him, as **Fourier Series.** In Sec. 7.3 to 7.5 of this unit you will learn the technique of representing a function by the Fourier series and apply it to a wide variety of functions. A relevant question in this context is : How accurately does the Fourier series of a function represent it? In Sec. 7.6 you will learn the conditions under which the Fourier series of a function gives the same value at a point as does the function itself.

In Unit 8 you will study the applications of Fourier series in solving certain important BVPs in physics.

Objectives

After studying this unit you should be able to:

- compute the coefficients of Fourier series for a function defined on an interval $(-L, L)$
- obtain the half-range expansions of functions defined on $(0, L)$
- ascertain whether or not the Fourier series representation of a function is valid on the given interval.

7.2 THE NEED FOR FOURIER SERIES

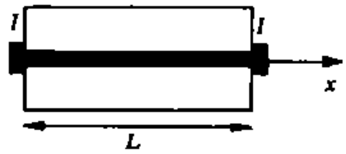


Fig. 7.1 : An insulated rod (shown shaded) of length L with both ends immersed in ice (I). The x-axis is chosen along the length of the rod with the origin at the left end

Let us consider a typical physics problem from Unit 6 involving a PDE. Consider the flow of heat in a uniform rod, which is insulated along its length (Fig. 7.1). Both ends of the rod are immersed in ice. We would like to determine the temperature distribution along the rod at time t. You know that heat flow along the rod can be modelled by the heat flow equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{7.1a}$$

where $T(x, t)$ is the temperature of the rod (measured in K) at a distance x from the left end at time t. Here $k = \frac{s\rho}{K}$, where ρ is the density of the material of the rod, s its specific heat and K, its thermal conductivity. For this specific problem we have $0 < x < L$, for $t > 0$. Since both ends of the rod are maintained at 0°C (i.e. 273K), we get the following boundary conditions

$$T(0, t) = T(L, t) = 273\text{K, for } t \geq 0 \tag{7.1b}$$

Let the initial temperature of the rod be given by

$$T(x, 0) = (60\text{K})\left(\sin\frac{2\pi x}{L} + \sin\frac{3\pi x}{L}\right) \tag{7.1c}$$

What is the solution of Eq. (7.1a) given the boundary and initial conditions specified by Eqs. (7.1b) and (7.1c)?

Recall that you have solved a similar problem in Unit 6. You can verify that the function

$$T(x, t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\sin\frac{n\pi x}{L} \tag{7.1d}$$

satisfies the heat flow equation as well as these boundary conditions. Here the coefficients b_n have the dimensions of temperature. Applying the initial condition to Eq. (7.1d) we get

$$T(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Substituting for $T(x, 0)$ from Eq. (7.1c) we have

$$b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + b_3 \sin\frac{3\pi x}{L} + b_4 \sin\frac{4\pi x}{L} + \ldots = 60\text{K}\left(\sin\frac{2\pi x}{L} + \sin\frac{3\pi x}{L}\right) \tag{7.1e}$$

This equation is satisfied only if $b_1 = 0$, $b_2 = 60\text{K}$, $b_3 = 60\text{K}$, $b_4 = b_5 = b_6 = \ldots = 0$. Thus, the solution is

$$T(x, t) = (60\text{K}) \exp\left(-\frac{4\pi^2 kt}{L^2}\right)\sin\frac{2\pi x}{L}$$
$$+ (60\text{K}) \exp\left(-\frac{9\pi^2 kt}{L^2}\right)\sin\frac{3\pi x}{L} \tag{7.1f}$$

You may recall that in Unit 3 you have used the power series in x to represent a continuous function. Both power series method and Fourier's method arise from the idea that a continuous function can be represented by an infinite series of functions. The added advantage of Fourier series is that it can be used to validly represent even those functions which have serveral points of discontinuity. You will read more about this in Sec. 7.6.

This was a fairly straight-forward problem, wasn't it? Now suppose $T(x, 0)$ were an arbitrary function of x, say, $T(x, 0) = \frac{100}{L}x$. How would we determine the coefficients b_n? In other

Let us put all these results for a_n and b_n together. These are termed **Euler formulas.**

i)	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$		(7.5a)
ii)	$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$	$n = 1, 2, 3, \ldots$	(7.5b)
iii)	$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,$	$n = 1, 2, 3, \ldots$	(7.5c)

The numbers a_0, a_n and b_n (for $n \geq 1$) given by Eqs. (7.5a, b, c) are called the **Fourier coefficients** of $f(x)$. The series

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

You can glance at Sec. 7.6 to know what a piecewise continuous function means.

with coefficients given by Eqs. (7.5 a, b, c) is called the **Fourier series** of $f(x)$ on the interval $-L < x < L$. Note that in order to obtain a_0, a_n, b_n, etc. the integrals of Eqs. (7.5a, b, c) must exist. From the definition of a definite integral, you know that if $f(x)$ is *continuous* or merely *piecewise continuous* on this interval, the integrals in (7.5a, b, c) exist. So if $f(x)$ is a continuous or piecewise continuous function on a given interval, we can compute its Fourier coefficients using these equations.

Recall from Sec. 7.2 that the idea of representing a function by a trigonometric series arose in connection with BVPs relating to vibrating systems, which mostly have periodic solutions. Therefore, in many a text-book you will come across discussions that begin with representing arbitrary *periodic* functions by Fourier series. However, Fourier series can be used to represent a much larger class of functions that arise in physics problems. Of course, it is valid only under certain conditions. We shall discuss these conditions for the validity of Fourier series for a given function later in the unit (in Sec. 7.6), only after you have computed the coefficients and found Fourier series for a variety of functions.

A function that is not periodic is termed as non-periodic or aperiodic.

Let us now consider an example to illustrate these ideas by determining the Fourier series for both non-periodic and periodic functions.

Example 1 : Fourier series for exponential function and the square wave

a) Determine the Fourier series for the function $e^{\alpha x}$ on the interval $-1 < x < 1$.

Solution

In this case $L = 1$. Using Eq. (7.5a) we get

$$a_0 = \frac{1}{2}\int_{-1}^{1} e^{\alpha x}\,dx = \frac{1}{2\alpha}(e^{\alpha} - e^{-\alpha})$$

$$a_n = \int_{-1}^{1} e^{\alpha x}\cos n\pi x\,dx$$

Here we have used the result $\cos n\pi = (-1)^n$

Integrating by parts you can verify that

$$a_n = \frac{\alpha(e^{\alpha} - e^{-\alpha})(-1)^n}{\alpha^2 + n^2\pi^2}, \qquad n = 1, 2, 3, \ldots$$

You can evaluate b_n in the same way.

$$b_n = \int_{-1}^{1} e^{\alpha x}\sin n\pi x\,dx = \frac{-n\pi(e^{\alpha} - e^{-\alpha})(-1)^n}{\alpha^2 + n^2\pi^2}$$

We have deliberately chosen a rather involved example (in terms of integration) here in order to bring out the fact that the computation of Fourier coefficients is just an exercise in integration. So if you can evaluate the integrals you should be able to determine the Fourier series for any function on the interval $-L < x < L$. Thus, the Fourier series for the function $e^{\alpha x}$ on the interval $-1 < x < 1$ is

$$\frac{1}{2\alpha}(e^{\alpha} - e^{-\alpha}) + \sum_{n=1}^{\infty} \frac{\alpha(e^{\alpha} - e^{-\alpha})(-1)^n}{\alpha^2 + n^2\pi^2} \cos n\pi x - \sum_{n=1}^{\infty} n\pi \frac{(e^{\alpha} - e^{-\alpha})}{\alpha^2 + n^2\pi^2} (-1)^n \sin n\pi x$$

$$= (e^{\alpha} - e^{-\alpha}) \left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2 + n^2\pi^2} (\alpha \cos n\pi x - n\pi \sin n\pi x) \right]$$

b) Find the Fourier series for the function $f(x)$ representing a periodic square wave (Fig. 7.2) of period 2π, defined as

$$E(t) = \begin{cases} 0 & \text{if } -\pi < x < -\pi/2 \\ E & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < \pi \end{cases} \qquad \text{(i)}$$

Functions of this type represent voltages impressed upon electrical circuits.

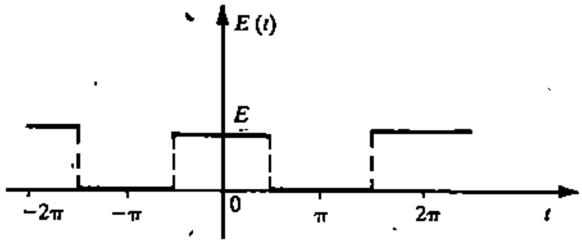


Fig. 7.2 : Periodic square wave of period 2π

Solution

Here $L = \pi$. Therefore, from Eq. (7.5a) we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} E(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/2} (0)\, dt + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} E\, dt + \frac{1}{2\pi} \int_{\pi/2}^{\pi} (0)\, dt = \frac{E}{2\pi}\pi = \frac{E}{2}$$

From Eq. (7.5b)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} E(t) \cos nt\, dt = \frac{E}{\pi} \int_{-\pi/2}^{\pi/2} \cos nt\, dt$$

$$= \frac{E}{n\pi} [\sin nt]_{-\pi/2}^{\pi/2} = \frac{2E}{n\pi} \sin \frac{n\pi}{2}$$

Thus $a_n = 0$ if n is even

$$a_n = \frac{2E}{n\pi}, \qquad \text{if } n = 1, 5, 9, \ldots$$

and $$a_n = -\frac{2E}{n\pi} \qquad \text{if } n = 3, 7, 11, \ldots$$

Similarly, from Eq. (7.5c) we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} E(t) \sin nt\, dt = \frac{E}{\pi} \int_{-\pi/2}^{\pi/2} \sin nt\, dt$$

$$= \frac{E}{\pi}\left[-\frac{\cos n\pi}{n}\right]_{-\pi/2}^{\pi/2} = -\frac{E}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right] = 0.$$

Thus $b_n = 0$ for all n.

Hence, the Fourier series for the periodic square wave of period 2π, represented by $E(t)$ of (i) is

$$E(t) = \frac{E}{2} + \frac{2E}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x + \ldots\right]$$

Let us now summarise the technique used to determine the Fourier series for any function $f(x)$ defined on an interval $-L < x < L$.

Finding Fourier Series

1) Write down the Fourier series for a function $f(x)$ defined on the interval $-L < x < L$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

2) Evaluate

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

3) Evaluate

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \ldots.$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \ldots.$$

You will be doing the exercise of numerical computation of Fourier Series in Table-Top Experiment–I of the physics laboratory Course PHE–12 (L).

As you have seen, a Fourier series has an infinite number of terms. Nowadays, IVPs and BVPs in PDEs are solved numerically through computers. Clearly, then, in the numerical calculations we can cope up with only a finite number of terms. The question therefore arises: How many terms do we need to get a reasonably good approximation to the original function? Let us briefly consider this question here to sensitise you about this topic. We will not go into too many details here.

7.3.2 The Use of Fourier Series as an Approximation

Let us get a 'feel' for this question by looking at the Fourier series of the function e^x (which can be obtained by putting $\alpha = 1$ in Example 1a). Let us compare the graph of e^x with the graphs of functions obtained by adding an increasing number of terms in the Fourier series. From Example 1(a), the Fourier series for e^x on the given interval is

$$(e - e^{-1})\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2\pi^2}(\cos n\pi x - n\pi \sin n\pi x)\right]$$

Let us consider the sum of the first N terms of this series, which is called the Nth **partial sum** S_N:

$$S_N = (e - e^{-1})\left[\frac{1}{2} + \sum_{n=1}^{N} \frac{(-1)^n}{1 + n^2\pi^2}(\cos n\pi x - n\pi \sin n\pi x)\right]$$

In Fig. 7.3, we compare the graph of e^x with the graphs of the partial sums S_2, S_5 and S_{10}, i.e., the sum of the first two, the first five and the first ten terms, respectively.

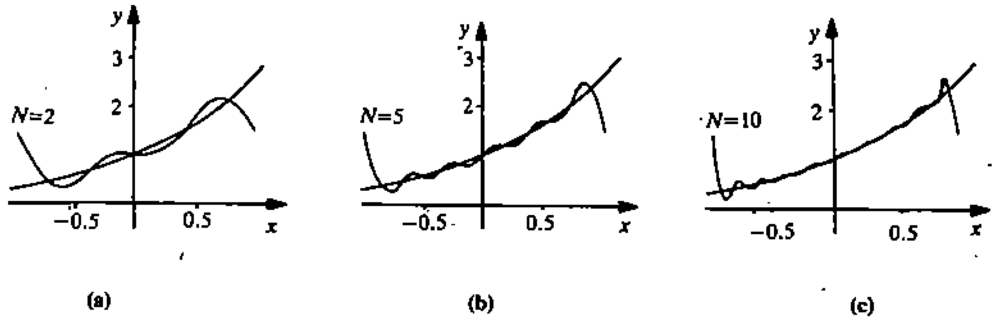


Fig. 7.3: A comparison of e^x with the finite partial sums of its Fourier series for (a) N=2 (b) N=5, (c) N=10.

Do you notice that with $N = 10$, we obtain better approximations to e^x at all points, than for $N = 2$ or $N = 5$? Of course, the end points $x = \pm 1$ are an exception. We are not worried about this at the moment because these points are not included in the interval $-1 < x < 1$ for which we have computed the Fourier series representing e^x.

You can see that the partial sums for $N \geq 2$ in Fig. 7.3 have pronounced spikes near the points of discontinuity. Even if we take N to be large, these spikes near a point of discontinuity remain; they do not smooth out. This behaviour of a Fourier series near a point at which a function is discontinuous is known as the **Gibbs phenomenon** after its discoverer, the American physicist J.W. Gibbs (1839-1903).

We say that as N increases, the Fourier series approximation of the original function **converges** to e^x at all values of x such that $-1 < x < 1$. We shall discuss the convergence of Fourier series in more detail in Sec. 7.6. Then you will also know precisely which functions may be approximated by Fourier series. Evidently the number of terms upto which we need to sum the Fourier series, would depend on how good an approximation we want to our original function. And that would depend on what we want to use the series for.

You should now practise finding Fourier series by solving the following SAQ.

SAQ 3

Spend 15 minutes

a) Obtain the Fourier series expansion of the function

$$T(x, 0) = \frac{100}{L}x, \text{ on the interval } -L < x < L. \text{ See Fig. 7.4a.}$$

b) Find the Fourier series of the periodic function

$$E(t) = \begin{cases} 0 & \text{if } -T/2 < t < 0 \\ E \sin \omega t & \text{if } 0 < t < T/2 \end{cases}, \quad T = \frac{2\pi}{\omega}$$

which represents the output of a half-wave rectifier (Fig. 7.4b).

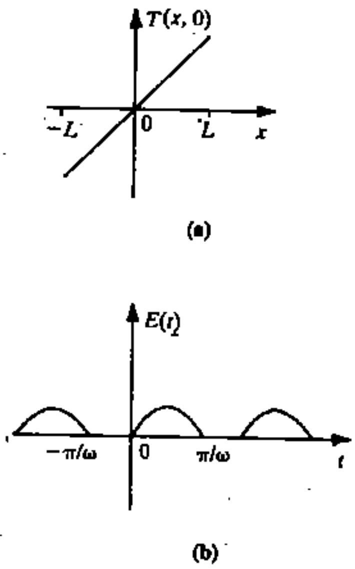


Fig 7.4

In part (b) of Example 1, you have seen that all the sine coefficients ($b_0, b_1, b_2, \ldots$) of the Fourier series for a square wave are zero. Similarly, in part (a) to SAQ 3 you would have found that a_0 and all cosine coefficients ($a_1, a_2, \ldots$) are zero. A natural question is : Could we have avoided the calculation of these coefficients, which turned out to be zero?

If we could find some conditions under which such a thing happens, we could save a great deal of work and also avoid errors ! This is what we are going to do in the next section.

7.4 FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

Study Figs. 7.2 and 7.4a representing the periodic square wave and the function $T(x, 0)$, respectively. Do you observe any symmetry in the graphs of these functions about the origin? Notice that the graph of the square wave is symmetric with respect to vertical axis, Similarly, the graph of $T(x, 0)$ is symmetric with respect to the origin. Recall that the Fourier series of the square wave contained no sine terms, and that of $T(x, 0)$ no cosine terms. So, is there some link between the symmetry of these functions and the form of their Fourier series? Further, can we generalise this notion to any function $f(x)$ possessing some kind of symmetry about the origin?

Indeed, there is a connection between the symmetry of those functions which can be categorised as 'even' or 'odd' functions, and the form of their Fourier series. In fact, when we began the study of Fourier series, we chose the interval $-L < x < L$ which is symmetric about the origin. There is an advantage in choosing such an interval : Such a choice enables us to make use of the fact that some functions are odd and some are even about the origin. So let us first understand what is meant by even and odd functions.

7.4.1 Even and Odd Functions

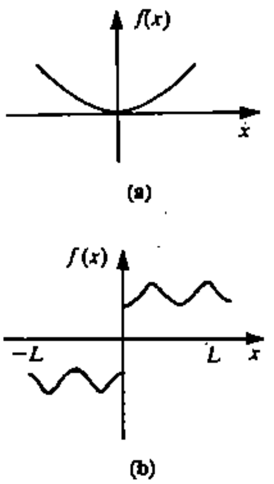


Fig. 7.5 : (a) An even and (b) an odd function

A function $f(x)$ defined on an interval $-L \le x \le L$ is said to be **even** if

$$f(-x) = f(x) \qquad \text{for all } x \in [-L, L]$$

and **odd** if

$$f(-x) = -f(x) \qquad \text{for all } x \in [-L, L]$$

You can see that the square wave of Fig. 7.2 is an even function, whereas $T(x, 0)$ of Fig. 7.4a is an odd function. You can also verify that the functions x^2, $\cos nx$ are even, while the functions x and $\sin nx$, are odd. You will notice that **the graph of an even function can be obtained from its graph to the right of the vertical axis by reflection in that axis. Similarly, the graph of an odd function is obtained by first reflecting its right half in the vertical axis, and then reflecting it in the horizontal axis.** You should verify these ideas using Figs. 7.2, 7.4a and 7.5 before studying further.

Any function can be written as the sum of an even function and an odd function, as follows :

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

The first part on the RHS is even and the second part is odd. For example

$$e^x = \frac{1}{2}[e^x + e^{-x}] + \frac{1}{2}[e^x - e^{-x}]$$

$$= \cosh x + \sinh x$$

where $\cosh x$ is even and $\sinh x$ is odd.

Whether a function is odd or even, or neither odd nor even may depend merely on the choice of the origin and the coordinate axes. For example, the square wave shown in Fig. 7.6a is neither odd nor an even function of t. However, if we shift the t-axis upwards by half the wave's amplitude, we get an odd function as in Fig. 7.6b.

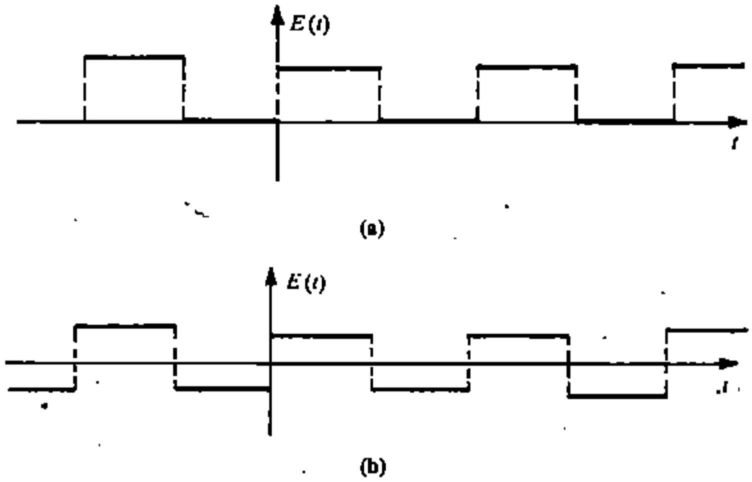


Fig. 7.6: (a) This periodic square wave is neither odd nor even; (b) It becomes an odd function by a shift of the t-axis

Having defined even and odd functions, we would like to find the Fourier series of such functions. Now, there are certain properties of even and odd functions that would be useful when you evaluate the Fourier coefficients of such functions. We state these properties here without proof. The proofs are fairly straight-forward and you can do them yourself.

Properties of Even and Odd Functions

1) If $f(x)$ and $g(x)$ are even (odd) functions then

   a) $f(x) + g(x)$ is an even (odd) function

   b) $f(x) - g(x)$ is an even (odd) function.

2) If $f(x)$ and $g(x)$ are both even functions or both odd functions, then $f(x)\,g(x)$ is an even function.

3) If $f(x)$ is an even function and $g(x)$ is an odd function, then $f(x)\,g(x)$ is an odd function.

4) If $f(x)$ is an even function then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx \qquad (7.6a)$$

5) If $f(x)$ is an odd function then

$$\int_{-L}^{L} f(x)\,dx = 0 \qquad (7.6b)$$

Fig. 7.7 illustrates the properties (4) and (5) for a given function $f(x)$. You may now like to try an exercise quickly, to get familiar with the concept of even and odd functions.

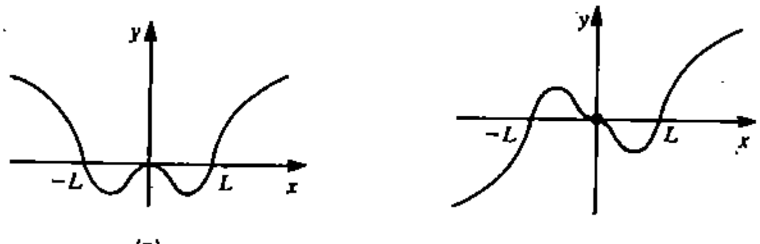


Fig. 7.7 : You can see that the area under the curve in (b) is zero whereas in (a) it is twice that of the area under the curve from 0 to L.

SAQ 4

a) An ac signal in the shape of a triangular waveform (Fig. 7.8) is applied to an electrical circuit. Is it an odd or even function?

b) Is the waveform in Fig. 7.8 odd or even, when it is moved

   i) one unit vertically downwards, and

   ii) one unit vertically downwards and one unit to the left?

c) Are the following functions even, odd or neither odd nor even?

   (i) $|x|$, (ii) $x \sin x$, (iii) e^x, (iv) x^{2n+1}, (v) $\sin nx + \cos nx$, (vi) $(\cos x)/x$

d) Express each of the following functions as the sum of an even and an odd function

   (i) $x e^x$, (ii) $(1 + x)(\sin x + \cos x)$

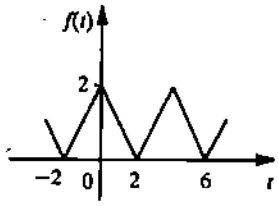


Fig. 7.8 : Triangular wave form

Using these ideas about even and odd functions, we can generalise the results we obtained for the square wave and the function $T(x, 0)$. In this way, we get Fourier cosine series representations for even functions, and Fourier sine series representations for odd functions.

7.4.2 Fourier Sine and Cosine Series

Suppose $f(x)$ is an even function. Then the product $f(x)\,\cos\frac{n\pi x}{L}$ is an even function, and the product $f(x)\ \sin\frac{n\pi x}{L}$ is an odd function. Using property 4 (Eq. 7.6a) of the even functions we get

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

and $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

What would the value of b_n be? Use property 5 (Eq. 7.6b) and determine b_n in the following step:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, the **Fourier series for an even function contains only the constant term and the cosine terms.** It is called the **Fourier cosine series.** What result do you expect if $f(x)$ is an odd function? Use properties 4 and 5 (Eqs. 7.6a and b) once again and solve the following integrals:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

So, you have found that the **Fourier series for an odd function contains only sine terms.** It is called the **Fourier sine series.** Let us sum up these results.

Fourier sine and cosine series

The Fourier series for an **even function** $f(x)$ on the interval $-L < x < L$ is a **Fourier cosine series:**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad (f \text{ even}) \qquad (7.7a)$$

with coefficients

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (7.7b)$$

The Fourier series for an **odd function** on the interval $-L < x < L$ is a **Fourier sine series:**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (f \text{ odd}) \qquad (7.7c)$$

with coefficients

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (7.7d)$$

Henceforth, whenever you have to determine the Fourier series for a function, the first thing you should do is to find out if the function is even or odd. If it is either of these then you can accordingly determine either Fourier cosine series or Fourier sine series. In this manner you can reduce your work considerably. Let us consider an example to illustrate these ideas and then you can work out a problem.

Example 2 : Fourier Series for a Saw-Tooth Wave

Find the Fourier series for the saw-tooth wave shown in Fig. 7.9.

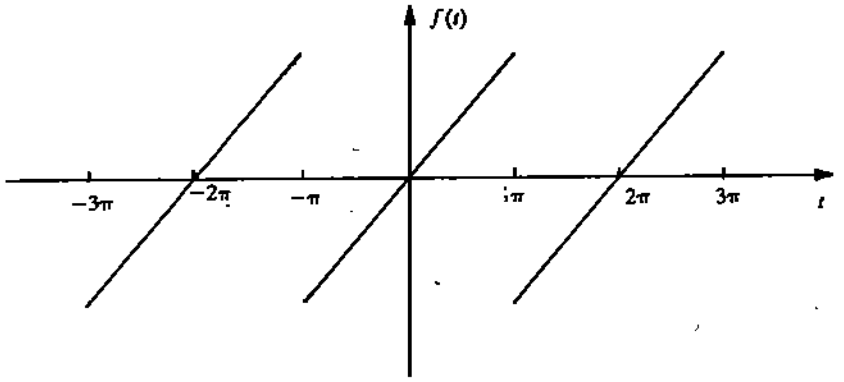


Fig. 7.9 : Certain signals in ac circuits are of the form of a saw-tooth function. Note that this function is not defined at $t = \ldots, -\pi, \pi, 3\pi, \ldots$ Among many applications, such signals are used in an oscilloscope to synchronise signals.

Solution

The function is algebraically expressed as :

$$f(t) = \frac{t}{\pi} \qquad -\pi < t < \pi$$

and

$$f(t + 2\pi) = f(t)$$

Thus, this saw-tooth function is odd and periodic with period 2π. It can be represented by a Fourier sine series.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{\pi} = \sum_{n=1}^{\infty} b_n \sin nt$$

where $$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^{\pi} \frac{t}{\pi} \sin nt \, dt$$

Integrating by parts you can verify that

$$b_n = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

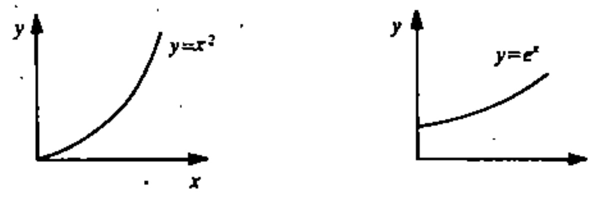
Thus $$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$$

$$= \frac{2}{\pi}\left[\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \ldots\right]$$

SAQ 5

Spend 10 minutes

You know that when a sound wave passes through the air and you hear it, the air pressure around you varies with time. Suppose the excess pressure above (and below) the atmospheric pressure in a sound wave is given by the graph in Fig. 7.10. Represent this function in the form of a Fourier series and thus determine the frequencies you hear when you listen to this sound?



(a) (b)

Fig.7.12

Let us now work out an example to illustrate the ideas of this section.

Example 3

$$\text{Represent } f(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x < 1 \end{cases}$$

shown in Fig. 7.13a in (a) Fourier sine series and (b) Fourier cosine series.

Solution

a) In the first case we need an odd extension $g(x)$ of $f(x)$ over the interval $-1 < x < 1$. Thus, we can define

$$g(x) = \begin{cases} 0 & -1 < x < -\frac{1}{2} \\ -1 & -\frac{1}{2} < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x < 1 \end{cases}$$

The function $g(x)$ is shown in Fig. 7.13(b). Since it is odd, only the coefficients b_n will survive. Using Eq. (7.8b) we get

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2} \sin n\pi x\,dx = \frac{2}{n\pi}[\cos n\pi x]_0^{1/2}$$

$$= -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right)$$

or $b_1 = \frac{2}{\pi}$, $b_2 = \frac{4}{2\pi}$, $b_3 = \frac{2}{3\pi}$, $b_4 = 0, \ldots$

The Fourier sine series for $f(x)$ is

$$f(x) = \frac{2}{\pi}\left[\sin \pi x + \frac{2\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{2\sin 6\pi x}{6} + \ldots\right]$$

b) The even extension $h(x)$ of $f(x)$ over the interval $-1 \le x \le 1$ as shown in Fig. 7.13(c) is

$$h(x) = \begin{cases} 0 & -1 \le x < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases}$$

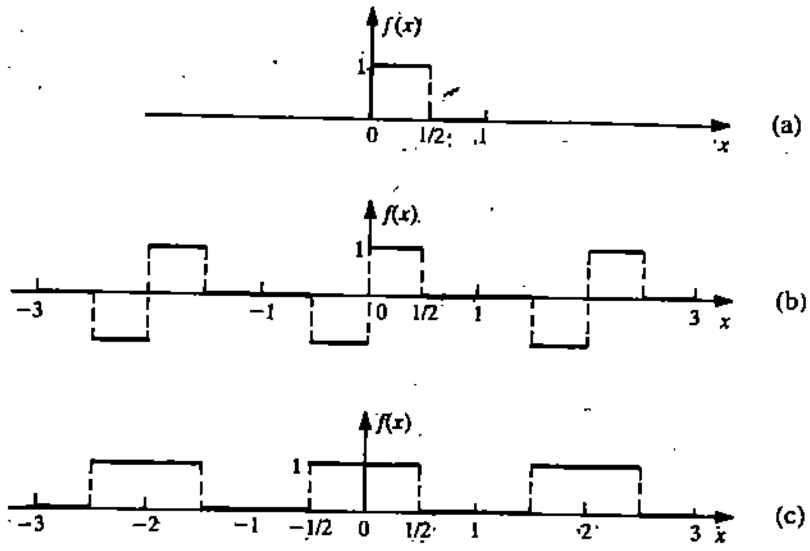


Fig. 7.13 : Parts (b) and (c) show the odd and even extensions, respectively, of $f(x)$ in (a)

In this case, using Eqs. (7.8d) and (7.8e) we get the Fourier cosine series with coefficients :

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = 2\int_0^{1/2}\cos n\pi x\,dx = \frac{2}{n\pi}\left[\sin n\pi x\right]_0^{1/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

Thus $$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\pi x - \frac{1}{3}\cos 3\pi x + \frac{1}{5}\cos 5\pi x - \ldots\right]$$

Note : We could also extend the given function in such a way as to get the Fourier series containing both sine and cosine terms. It may seem strange to you that a function can be represented by several different trigonometric series. What you should understand is that even though all these series represent the given function in the given interval, they represent **different** functions on the extended interval.

Let us now sum up the procedure for finding the Fourier series for a function defined on a finite interval:

> We can represent a function $f(x)$ defined over a finite interval (say $0 < x < L$) by a
>
> Fourier series, if we derive the Fourier series of another function $g(x)$ defined over $-L < x < L$ which is either an **odd** or an **even extension** of $f(x)$. Finally, in writing the Fourier series for $f(x)$, we restrict the values of the independent variable to the original interval.

You should now work out an exercise to find out whether you have grasped the ideas of this section.

SAQ 7

Spend 10 minutes

Determine the Fourier sine series for e^x on the interval $0 \le x < 1$. How does the value of the series at $x = 0$ compare with the value of e^x at $x = 0$?

In SAQ 7, you have discovered an anomaly in the evaluation of the Fourier sine series for e^x at $x = 0$. You have found that the series does not converge to the value of the function it is intended to represent, at the point $x = 0$. However, remember that it converges to the value of the extended function at that point. Recall from Sec. 7.3.2 that the Fourier series of e^x on the interval $-L < x < L$ does not converge to $e^{\pm L}$ at the end points $x = \pm L$. This leads us to the question: How valid is our representation of a given function by Fourier series? If the Fourier series of a function $f(x)$ converges at all points on that interval to its value of those points, can we say that it is a **valid representation** of $f(x)$? This leads us to the question of the *convergence of Fourier series.*

7.6 THE CONVERGENCE OF FOURIER SERIES

Before we can study the convergence of Fourier series, we must define continuous and piece-wise continuous functions. You already know that a function $f(x)$ is **continuous on the interval** $a < x < b$, if, for each point x_0 in the interval (a, b), $f(x)$ tends to a finite limit $f(x_0)$, as x tends to x_0, i.e.,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The limit $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$ where h is a positive number, is termed the right-hand limit of $f(x)$ at $x = a$. It is the limit of $f(x)$ as you approach the point $x = a$ from its right. Similarly

$\lim_{x \to b} f(x) = \lim_{h \to 0} (b-h)$

where h is a positive number is termed the left hand limit of $f(x)$ at $x = b$. It is the limit of $f(x)$ as you approach the point $x = b$ from its left.

(If this equality holds for a point x_0, we say that $f(x)$ is **continuous at the point** x_0). The graph of $f(x)$ is, obviously, an unbroken curve. Moreover, if the function is defined on the interval $a \le x \le b$, we say that it is continuous on the interval $a \le x \le b$ if it is continuous on the interval $a < x < b$, and if $f(x)$ tends to finite limits $f(a)$ and $f(b)$ as x tends to the end points a and b, respectively, from within the interval. We express this as

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

For example, the function $f(x) = x$ or any polynomial function is continuous on any interval. But, the function $f(x) = 1/x$ and $f(x) = \ln|x|$ are not continuous on the interval $0 \le x < 1$, because neither function is defined at $x = 0$. Further, neither $\lim_{x \to 0} \frac{1}{x}$ nor $\lim_{x \to 0} \ln x$ exists. The function (shown in Fig. 7.14) represented algebraically as

$$f(x) = \begin{cases} 1 & -\frac{\pi}{2} < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < \frac{\pi}{2} \end{cases} \tag{7.9}$$

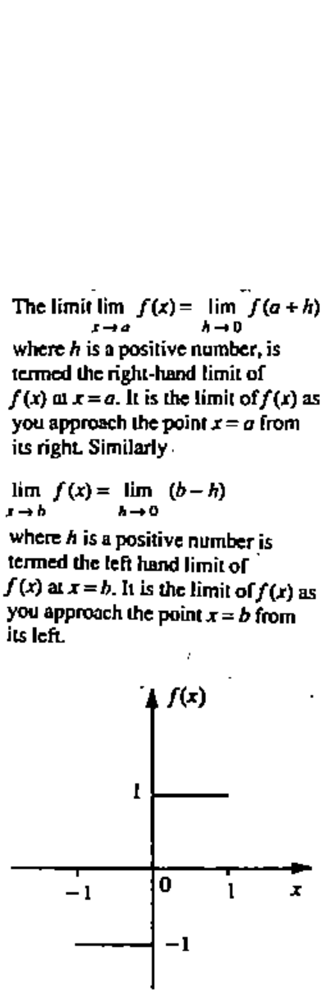


Fig. 7.14 : This function is called the step function. It may represent the output signal of an electronic switch

is not continuous at $x = 0$ even though it is defined at $x = 0$, because the limits $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$ are not the same. Obviously, the $\lim_{x \to 0} f(x)$ does not exist. This example brings us to the concept of piecewise continuous functions.

Piecewise continuous functions

It does seem obvious from the term that a function can be called piecewise-continuous if its graph consists of a **finite** number of continuous pieces (see Figs. 7.14 and 7.15). We say that a function is piecewise continuous on the interval $a \le x \le b$, if there are a finite number of points

$$a = x_0 < x_1 < x_2 \ldots < x_n = b$$

For a point x_0 in the interval and $h > 0$, $f(x_0^+) = \lim_{h \to 0} f(x_0 + h)$

$f(x_0^-) = \lim_{h \to 0} f(x_0 - h)$

such that :

i) f is continuous on each subinterval

$$x_0 < x < x_1,\ x_1 < x < x_2,\ \ldots,\ x_{j-1}\ x < x_j,\ \ldots,\ x_{n-1} < x < x_n$$

ii) f has finite limits as x approaches the end-points of each subinterval from within the subinterval.

Fig. 7.15 shows the graph of a typical piecewise continuous function, f. The dots on the graph represent the value of the function at each of the breakpoints.

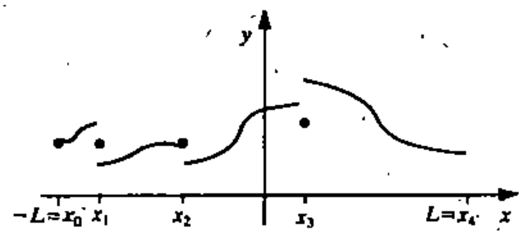


Fig. 7.15 : The graph of a typical piecewise continuous function f on the interval $(-L, L)$

Note that at each of the break-points, the value of the function f may or may not equal its left or right-hand limit. For example, in Fig. 7.15 you can see that

$$f(x_1^-) \neq f(x_1) \neq f(x_1^+), \text{ whereas } f(x_2^-) = f(x_2) \neq f(x_2^+)$$

We can now discuss the convergence of Fourier series. Let us consider the sum of the first N terms of the series for a function $f(x)$ on the interval $-L < x < L$ given by

$$S_N = a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad (7.10)$$

What happens when N tends to infinity? **By definition, the Fourier series converges to f at a point $x = x_0$ if the partial sum given in Eq. (7.10), with $x = x_0$, tends to a finite limit $f(x_0)$, in the limit as** $N \to \infty$, i.e.

$$\lim_{N \to \infty} S_N(x_0) = f(x_0)$$

Having defined the convergence of Fourier series we must know the conditions under which a Fourier series converges to its function. We state below (without proof), a theorem giving these conditions. These conditions are known as **Dirichlet conditions**.

Let the function f and its derivative f' be piecewise continuous on the interval $-L < x < L$. Then the Fourier series for f converges to f at all points in $(-L, L)$ where f is continuous, and to the mean value

$$\frac{f(x_0^+) + f(x_0^-)}{2}$$

at all points x_0 in $(-L, L)$ where f is not continuous. At the end points $-L$ and L, the Fourier series converges to

$$\frac{[f(-L^+) + f(L^-)]}{2}.$$

This theorem tells us that the Fourier series for a function f converges to f at all points where f is continuous. The series converges to the average value of the left- and right-hand limits at each break-point. At the end- points, the Fourier series converges to $\frac{[f(-L^+) + f(L^-)]}{2}$, which is the average of the two end-point limits.

The behaviour of the series at the end points as defined by this theorem may seem to be puzzling (recall Fig. 7.3). To understand why this should happen, consider what the Fourier series represents outside the interval $-L < x < L$. Remember that the Fourier series is the sum of periodic functions $\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$ $(n = 1, 2, 3......)$ which have a period $T = 2L$. Therefore, the Fourier series itself must have period $2L$.

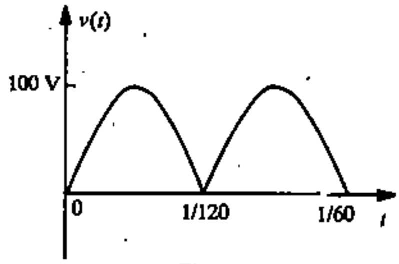


Fig. 7.18

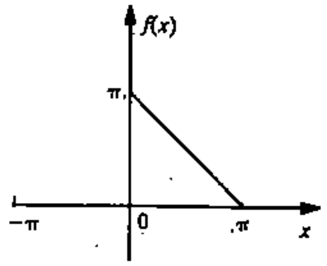


Fig. 7.19

$$f(t) = \begin{cases} \dfrac{2k}{L}\, t & \text{when } 0 < t < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L-t) & \text{when } \dfrac{L}{2} < t < L \end{cases}$$

2) Obtain the Fourier series expansion of the output of the full-wave rectifier shown in Fig. 7.18. The shape of the curve is the absolute value of a sine function. The maximum voltage is 100V.

3) Expand

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x. & 0 < x < \pi \end{cases}$$

in a Fourier series (see Fig. 7.19).

Verify whether the Fourier series representation of this function is valid.

7.9 SOLUTIONS AND ANSWERS

SAQs (Self-assessment questions)

1) Using Eq. (7.3d) we find that all the terms in the second and third series in the RHS of Eq. (7.4) are zero, since

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\, dx = \int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx = 0, \qquad n = 1, 2, 3, \ldots.$$

Thus, we get

$$\int_{-L}^{L} a_0\, dx = a_0 \int_{-L}^{L} dx = 2L\, a_0 = \int_{-L}^{L} f(x)\, dx$$

or $$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

2) Multiplying Eq. (7.2) by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, we get

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\, dx = a_0 \int_{-L}^{L} \cos\frac{m\pi x}{L}\, dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$$

Using Eqs. (7.3d), (7.3b), (7.3c) and (7.3e) we get

Since the remaining terms, for which $n \neq m$, in the first series of RHS are zero by virtue of Eq. (7.3c) and all the terms in the second series on RHS are zero in view of Eq. (7.3a).

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\, dx = 0 + a_m \int_{-L}^{L} \cos^2\frac{m\pi x}{L}\, dx + 0,$$

$$= a_m L$$

or $$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\, dx$$

3) The Fourier series representation of $T(x, 0) = \dfrac{100x}{L}$ is given by

$$T(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} T(x,0)\,dx = \frac{1}{2L}\int_{-L}^{L}\frac{100x}{L} = \frac{100}{2L^2}\left[\frac{x^2}{2}\right]_{-L}^{L} = 0,$

$$a_n = \frac{1}{L}\int_{-L}^{L} T(x,0)\cos\frac{n\pi x}{L}\,dx = \frac{100}{L^2}\int_{-L}^{L} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{100}{L^2}\left(\left[x\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{100}{L^2}\,[0-0] = 0$$

and $b_n = \frac{1}{L}\int_{-L}^{L} T(x,0)\sin\frac{n\pi x}{L}\,dx = \frac{100}{L^2}\int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx$

$$= \frac{100}{L^2}\left(\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx\right)$$

$$= -\frac{100}{L^2}\cdot\frac{L}{n\pi}\;[2L\cos n\pi] + 0 \qquad \text{[Using Eq. 7.3d]}$$

$$= -\frac{200}{n\pi}(-1)^n, \qquad n = 1, 2, \ldots\ldots (\because \cos n\pi = (-1)^n)$$

$$= \frac{200}{n\pi}(-1)^{n+1}$$

Thus, the Fourier series for $T(x, 0)$ is

$$T(x,0) = \sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$$

$$= \frac{200}{\pi}\left[\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \sin\frac{4\pi x}{L} + \ldots\right]$$

Notice that the higher the frequency of a term, the lower is its amplitude.

b) The Fourier series for $E(t)$ is

$$E(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n\sin\frac{2\pi nt}{T} (\because L = T/2)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega t + \sum_{n=1}^{\infty} b_n\sin n\omega t \left(\because \omega = \frac{2\pi}{T}\right)$$

Here $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} E(t)\,dt = \frac{E}{T}\int_{0}^{T/2}\sin\omega t\,dt \quad \left(\because E(t) = 0 \text{ for } \frac{-T}{2} < t < 0\right)$

$$= \frac{E}{T}\left[-\frac{\cos\omega T}{\omega}\right]_0^{T/2} = \frac{E}{T\omega}\left(-\cos\frac{\omega T}{2} + \cos 0\right)$$

$$= \frac{E}{2\pi}(2) = \frac{E}{\pi}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} E(t)\cos n\omega t\,dt$$

Do *not* confuse between the methods of evaluation of a_n here and b_n in SAQ 2 where we have used Eq. 7.3a to put $\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$.

Notice that the intervals of integration in both the cases are different. Therefore, we get different results.

$$= \frac{2E}{T} \int_0^{T/2} \sin \omega t \cos n\omega t \, dt$$

$$= \frac{2E}{2T} \int_0^{T/2} [\sin (1+n)\omega t + \sin (1-n)\omega t] \, dt$$

$$\left(\because \sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)] \right).$$

We evaluate a_1, separately. For $n = 1$, the integral is zero.

$\therefore a_1 = 0.$ For $n = 2, 3, \ldots.$

$$a_n = \frac{E}{T}\left[-\frac{\cos (1+n)\omega t}{(1+n)\omega} - \frac{\cos (1-n)\omega t}{(1-n)\omega} \right]_0^{T/2}$$

$$= \frac{E}{\omega T}\left[-\frac{\cos (1+n)\pi}{(1+n)} - \frac{\cos (1-n)\pi}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n} \right]$$

$$= \frac{E}{2\pi}\left[\frac{-(-1)^{n+1}+1}{n+1} + \frac{-(-1)^{1-n}+1}{1-n} \right].$$

Thus

$a_n = 0,$ for $n = 3, 5, 7, 9, \ldots..$

and $a_n = \frac{E}{2\pi}\left[\frac{2}{n+1} + \frac{2}{1-n} \right] = \frac{2E}{(n+1)(1-n)\pi},$ $n = 2, 4, 6, \ldots..$

Similarly,

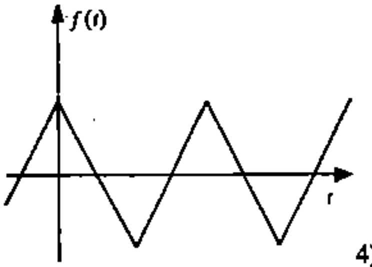
$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} E(t) \sin n\omega t \, dt = \frac{2E}{T} \int_{-T/2}^{T/2} \sin \omega t \sin n\omega t \, dt$$

From Eqs. (7.3b) and (7.3c) only b_1 is non-zero. Thus, we get

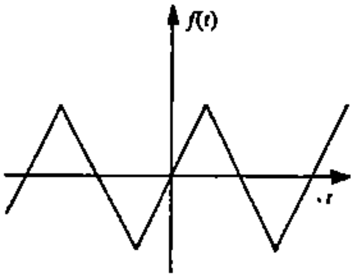
$b_1 = E/2,\ b_n = 0$ for $n = 2, 3, 4, \ldots..$

Therefore, the Fourier series representation of $E(t)$ is

$$E(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi}\left(\frac{1}{1.3} \cos 2\omega t + \frac{1}{3.5} \cos 4\omega t + \ldots \right)$$



(a)



(b)

Fig. 7.20

4) a) It is even, since $f(t) = f(-t)$

b) (i) See Fig. 7.20 a. This is an even function. (ii) See Fig. 7.20 b. This is an odd function.

c) (i) even (ii) even (iii) neither odd nor even (iv) odd (v) neither odd nor even (vi) odd

d) (i) $\frac{xe^x - xe^{-x}}{2} + \frac{1}{2}[xe^x + xe^{-x}] = \frac{x}{2}[\sinh x + \cosh x]$

(ii) $\frac{1}{2}[(1+x)(\sin x + \cos x) + (1-x)(-\sin x + \cos x)]$

$+ \frac{1}{2}[(1+x)(\sin x + \cos x) - (1-x)(-\sin x + \cos x)]$

5) The function shown in Fig. 7.10 is an odd function, with $L = 1/300$. So we get only a Fourier sine series with coefficients

$$b_n = 2(300) \int_0^{1/300} p(t) \sin 300\, n\pi t \, dt$$

$$= 600 \int_0^{1/600} \sin 300\, n\pi t\, dt - \frac{5}{6}(600) \int_{1/600}^{1/300} \sin 300\, n\pi t\, dt$$

$$\left[\therefore p(t) = \begin{cases} 1, & 0 < t < \dfrac{1}{600} \\ \dfrac{5}{6}, & \dfrac{1}{600} < t < \dfrac{1}{300} \end{cases} \right]$$

$$= 600 \left(- \frac{\cos \frac{n\pi}{2} - 1}{300\, n\pi} + \frac{5}{6} \frac{\cos n\pi - \cos \frac{n\pi}{2}}{300\, n\pi} \right) = \frac{2}{n\pi} \left(- \frac{11}{6} \cos \frac{n\pi}{2} + 1 + \frac{5}{6} \cos n\pi \right)$$

This gives $b_1 = \frac{2}{\pi}\left(1 - \frac{5}{6}\right) = \frac{1}{\pi} \cdot \frac{1}{3}$, $\quad b_5 = \frac{2}{5\pi} \cdot \frac{1}{3}$

$b_2 = \frac{2}{2\pi}\left(\frac{11}{6} + 1 + \frac{5}{6}\right) = \frac{1}{2\pi} \cdot \frac{22}{3}$, $\quad b_6 = \frac{1}{6\pi} \cdot \frac{22}{3}$

$b_3 = \frac{2}{3\pi}\left(1 - \frac{5}{6}\right) = \frac{1}{3\pi} \cdot \frac{1}{3}$, $\quad b_7 = \frac{1}{7\pi} \cdot \frac{1}{3}$

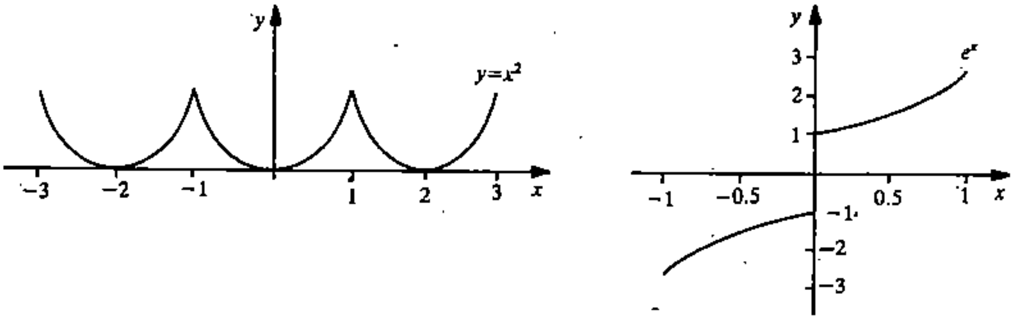
$b_4 = \frac{2}{4\pi}\left(-\frac{11}{6} + 1 + \frac{5}{6}\right) = 0$, $\quad b_8 = 0, \ldots$ etc.

Thus, we have

$$p(t) = \frac{1}{3\pi} \left(\frac{\sin 300\,\pi t}{1} + \frac{22 \sin 600\,\pi t}{2} + \frac{\sin 900\,\pi t}{3} + \frac{\sin 1500\,\pi t}{5} + \frac{22 \sin 1800\,\pi t}{6} + \frac{\sin 2100\,\pi t}{7} + \ldots \right)$$

You can see that the second harmonic at a frequency of 300 cps has the largest amplitude. Since the intensity is proportional to the square of the amplitude of a wave we would principally hear the second harmonic.

6) See Figs. 7.21 a and 7.21 b.



(a) (b)

Fig. 7.21

7) Since we have to determine the Fourier sine series of e^x on the interval $0 < x < 1$, we need an odd extension of e^x (see Fig. 7.21b). Note that $g(0) = 0$ by definition. Then the coefficients b_n are given by

$$b_n = 2 \int_0^1 e^x \sin n\pi x\, dx$$

$$= 2 \left[\frac{-e^x \cos n\pi x}{n\pi} \right]_0^1 + \frac{2}{n\pi} \int_0^1 e^x \cos n\pi x\, dx$$

$$= \frac{-2e \cos n\pi + 2}{n\pi} + \frac{2}{n\pi} \left[\frac{e^x \sin n\pi x}{n\pi} \right]_0^1 - \frac{2}{n^2\pi^2} \int_0^1 e^x \sin n\pi x\, dx$$

$$= \frac{-2e\,(-1)^n + 2}{n\pi} + 0 - \frac{b_n}{n^2\pi^2}$$

$$\text{or } b_n \left(1 + \frac{1}{n^2\pi^2}\right) = \frac{2 - 2e(-1)^n}{n\pi}$$

$$\text{or } b_n = \frac{2n\pi\,(1 - e(-1)^n)}{1 + n^2\pi^2}$$

Hence, the Fourier series for e^x on the interval $0 < x < 1$ is

$$2\pi \sum_{n=1}^{\infty} \frac{n\,[1 - (-1)^n e]}{1 + n^2\pi^2} \sin n\pi x$$

At $x = 0$, the value of Fourier sine series is zero. But $e^0 = 1$. Thus, the Fourier sine series for e^x does not give the value of the function e^x, at $x = 0$. However, it does give the value of the odd extension of e^x, at $x = 0$

8) a) In SAQ 7, the Fourier sine series for e^x on the interval $0 < x < 1$ converged to zero at $x = 0$. This is consistent with the convergence theorem since the sine series is the Fourier series of the odd extension $g(x)$ of e^x and

$$g(0^+) = +1,\ g(0^-) = -1,$$

i.e., at the discontinuity $x = 0$, the series converges to the value 0.

b) According to the convergence theorem, the Fourier series for e^x on the interval $-1 < x < 1$ converges to

$$\frac{e^1 + e^{-1}}{2} \cong 1.5$$

at the points $x = \pm 1$. As you can see this value does not agree with the actual values of e^x at $x = \pm 1$ (see Fig. 7.3).

Terminal Questions

1) The even extension of the triangular pulse is shown in Fig. 7.22

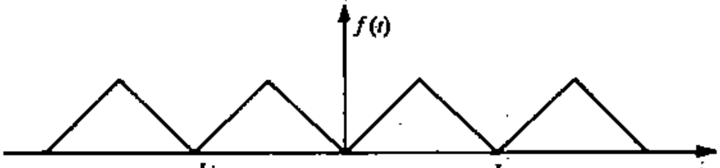


Fig. 7.22

From Eqs. (7.8b) and (7.8e), the coefficients of the Fourier cosine series representation of the given function are

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} t\,dt + \frac{2k}{L}\int_{L/2}^{L} (L - t)\,dt\right] = \frac{1}{L}\cdot\frac{2k}{L}\left[\frac{t^2}{2}\right]_0^{L/2} + \left[Lt - \frac{t^2}{2}\right]_{L/2}^{L}$$

$$= \frac{2k}{L^2}\left(\frac{L^2}{8} + L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{L^2}{8}\right) = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} t\cos\frac{n\pi t}{L}\,dt + \frac{2k}{L}\int_{L/2}^{L} (L - t)\cos\frac{n\pi t}{L}\,dt\right]$$

$$= \frac{2}{L}\cdot\frac{2k}{L}\left(\left[\frac{Lt}{n\pi}\sin\frac{n\pi t}{L}\right]_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi t}{L}\,dt + \right.$$

$$+\left[(L-t)\frac{L}{n\pi}\sin\frac{n\pi t}{L}\right]_{L/2}^{L}+\frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi t}{L}\,dt\Bigg)$$

$$=\frac{4k}{L^2}\left(\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\left[\cos\frac{n\pi}{2}-1\right]-\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}-\frac{L^2}{n^2\pi^2}\left[\cos n\pi-\cos\frac{n\pi}{2}\right]\right)$$

$$=\frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)$$

Thus

$$a_1=0,\ a_2=-\frac{16k}{2^2\pi^2},\ a_3=0,\ a_4=0,\ a_5=0,$$

$$a_6=-\frac{16k}{6^2\pi^2},\ a_7=0,\ a_8=0,\ a_9=0,\ a_{10}=-\frac{16k}{10^2\pi^2},\ldots.$$

Thus $a_n=0$, when $n\neq 2, 6, 10, 14, \ldots$. The desired half-range expansion of the triangular pulse is

$$f(t)=\frac{k}{2}-\frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi t}{L}+\frac{1}{6^2}\cos\frac{6\pi t}{L}+\ldots\right)$$

2) The output of a full-wave rectifier (Fig. 7.18) can be expressed as

$$v(t)=\begin{cases}100\sin\omega t & 0<t<\pi\\ -100\sin\omega t & -\pi<t<0\end{cases}$$

Since $v(t)$ is even, we can represent it by a Fourier cosine series. Here $L=\pi/\omega$. Therefore, from Eq. (7.7b)

$$a_0=\frac{100\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\,dt=-\frac{100}{\pi}\left[\cos\omega t\right]_0^{\pi/\omega}=\frac{200}{\pi}$$

$$a_n=\frac{200}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos n\omega t\,dt$$

You have solved a similar integral in SAQ 3(b). We can use those results and write

$$a_n=-\frac{400}{(n-1)(n+1)\pi},\qquad n=2,4,6,\ldots.$$

Thus

$$v(t)=\frac{200}{\pi}-\frac{400}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\omega t}{4m^2-1},\text{ where we have put } n=2m.$$

You can see that the original frequency ω has been eliminated. The lowest surviving harmonic has frequency 2ω and amplitude $400/3\pi$. The amplitudes of higher harmonics (of frequencies $4\omega, 6\omega, \ldots 2m\omega, \ldots$) fall off as $1/m^2$. Thus the full-wave rectifier does a fairly good job of approximating direct current.

3) Here $L=\pi$ and the coefficients of Fourier series are

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_0^{\pi}(\pi-x)\,dx=\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}=\frac{\pi}{4}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}(\pi-x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} - \frac{1}{n^2\pi}\left[\cos nx\right]_0^{\pi}$$

$$= -\frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{n}$$

Therefore $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}$

This series converges to the periodic extension of $f(x)$ onto the entire x-axis (Fig. 7.23). At the points of discontinuity ($x = 0, \pm 2\pi, \pm 4\pi \ldots$) the series converges to the value

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi}{2}$$

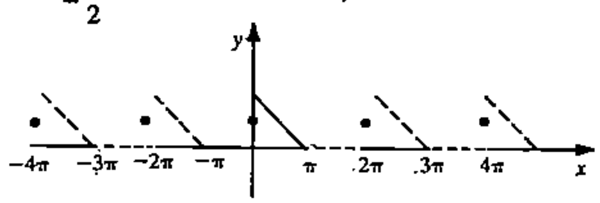


Fig. 7.23

These are shown by the solid dots in the figure. At $n = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the series will converge to the value

$$\frac{f(\pi^-) + f(-\pi^+)}{2} = 0$$

which is the value of the function at these points.

UNIT 8 APPLICATIONS OF FOURIER SERIES TO PDEs

Structure

8.1 INTRODUCTION

In Unit 7 you have studied the technique of expanding an arbitrary function in terms of Fourier series. In this unit we will use this technique to solve some important BVPs in physics. Specifically, we will illustrate the application of Fourier series to solve BVPs involving the diffusion equation, wave equation and Laplace equation. For example, we will study heat conduction along a cylindrical rod as well as diffusion of particles using the diffusion equation. Such BVPs arise in engineering and industrial applications, viz. modelling heat flow in the fuel rods in a nuclear reactor, evaporation of water, drying of granular products, etc.

Using the wave equation we shall solve the 'plucked string' problem which models the motion of a string in a variety of musical instruments like the sitar, guitar, violin etc. We shall also study torsional vibrations which arise in several mechanical systems having a rotating shaft such as the axle in a car, propeller in a ship, drill pipe in an oil well, etc.

We will solve Laplace's equation for steady state heat flow in a rectangular plate (which can be used to model heat flow across refrigerator doors). Finally we will solve Laplace's equation for determining the potential at a point due to a circular disc. This problem will demonstrate the fact that Fourier series can be used to solve problems involving non-Cartesian geometry. We hope that after studying this unit you will be able to appreciate the fact that Fourier series can be used for solving a wide variety of real-world problems.

Objectives
After studying this unit you should be able to apply the Fourier series to

- solve the diffusion equation, wave equation and Laplace's equation for a given BVP
- solve similar BVPs involving other PDEs

8.2 DIFFUSION EQUATION

You have studied the one-dimensional diffusion equation (for heat flow) in Unit 5. You have solved it under specific initial and boundary conditions for a given physical problem in Unit 6. You have also solved the two-dimensional heat flow equation in Unit 6.

In its most general form, the diffusion equation is expressed as

$$\nabla^2 u + G(x, y, z, t) = \frac{1}{k}\frac{\partial u}{\partial t} \quad (8.1a)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian and G is an arbitrary function of x, y, z and t. As you know the function $u(x, y, z, t)$ could represent temperature in a body so that Eq. (8.1a) models heat flow in that body. For example, the temperature T of a current-carrying metallic wire can be modelled, by adding another term in Eq. (5.10c) which accounts for the heat generated due to current conduction. If I is the current in the wire and R its resistance, an additional amount of heat ($\equiv I^2 R\Delta x$) will be accumulated in the portion of the wire between x and $x + \Delta x$. Thus, you can add the term $I^2 R\Delta x$ in Eq. (5.10c) and repeat the remaining steps to obtain the following heat flow equation for a current-carrying wire:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t} - \frac{I^2 R}{KA}, \quad (8.1b)$$

For the two-dimensional diffusion equation, we have to represent the arbitrary function $u(x, y)$ by Fourier series in two variables. It is of the form:

$$u(x, y, 0) = f(x, y)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi}{b} x \sin\frac{n\pi}{c} y$$

This is beyond the scope of this course.

where k is the thermal diffusivity and K, the thermal conductivity of the wire. When Eq. (8.1a) is used to model the diffusion of dissolved substances in a solution, $u(x, y, z, t)$ represents the concentration of the liquid. For example, the PDE

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u \quad (8.1c)$$

can be used to model the loss (diffusion) of moisture from a porous object through its surface. Here γ is constant and u represents the moisture concentration. You can see that Eq. (8.1) is a nonhomogeneous PDE. Now, in Unit 6, you have learnt to solve only homogeneous PDEs. Further in Unit 7, you have learnt to use the Fourier series of a single variable only (read the margin remark). Therefore, in this section we shall restrict the application of Fourier series to one-dimensional homogeneous diffusion equation. Let us consider two specific applications of this equation: in heat conduction and in diffusion of particles.

8.2.1 Heat Conduction

In Unit 7, we introduced the idea of using Fourier series in the solution of a one-dimensional diffusion equation. You had also completed the solution for a specified boundary-value problem. Let us consider another example of heat flow, where the Fourier series can be applied. This is a slightly different application.

Example 1

Consider the flow of heat in a uniform bar of length L, insulated along its length. As you know the temperature of the bar is modelled by the diffusion equation

$$\frac{\partial T(x, t)}{\partial t} = k\frac{\partial^2 T(x, t)}{\partial x^2}, \quad (0 < x < L,\ t > 0) \quad (8.2a)$$

One end of the block is immersed in a block of ice, maintained at 0°C, while the other end is insulated (Fig. 8.1a). This gives rise to the boundary conditions

$$T(0, t) = 0 \ \text{ and } \ \frac{\partial T(L, t)}{\partial x} = 0, \ t \geq 0 \quad (8.2b)$$

If the initial temperature distribution is given by

$$T(x, 0) = \frac{x}{2}(2L - x) \qquad (0 < x < L) \quad (8.2c)$$

(see Fig. 8.1b), then solve the heat equation (8.2a). (Note that the initial condition is physically consistent with the boundary conditions at $x = 0$ and $x = L$).

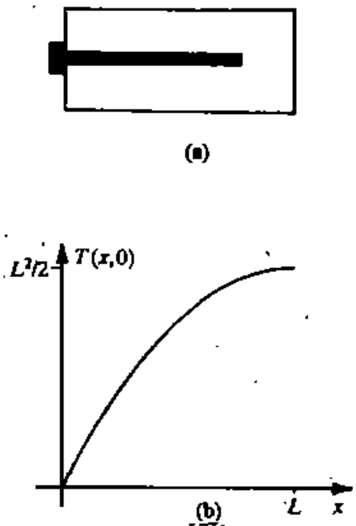


Fig. 8.1: (a) An insulated bar with its left end immersed in ice (b) the initial temperature distribution of the bar

Solution

Using the method of separation of variables we write $T(x, t)$ as a product of two terms: $T(x, t) = X(x)\,Y(t)$. Taking $-\lambda^2$ as the separation constant, we get

$$\frac{X''}{X} = \frac{Y'}{kY} = -\lambda^2 \qquad \text{(i)}$$

or

$$X'' + \lambda^2 X = 0 \qquad \text{(ii)}$$

and $$Y' + k\lambda^2 Y = 0 \qquad \text{(iii)}$$

The solutions of (ii) and (iii) for $X(x)$ and $Y(t)$ are well known

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \qquad \text{(iv)}$$

and $$Y(t) = C_3 \, e^{-k\lambda^2 t} \qquad \text{(v)}$$

From the boundary conditions for $T(x, t)$ we have

$$T(0, t) = X(0)\, Y(t) = 0$$

and $$\frac{\partial T}{\partial x}(L, t) = \left[\frac{dX(L)}{dx} \right] Y(t) = 0$$

Since $Y(t) \neq 0$, X must satisfy the conditions

$$X(0) = X'(L) = 0$$

Application of the first of these condition gives us $C_1 = 0$. Thus

$$X(x) = C_2 \sin \lambda x$$

The second boundary condition gives us

$$X'(L) = C_2 \cos \lambda' L = 0$$

For a non-trivial solution, for which $C_2 \neq 0$, we have

$$\cos \lambda L = 0$$

or $$\lambda L = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots.$$

We call these values of λ as λ_n. The solutions can thus be written as

$$X_n(x) = C_{2n} \sin\left[\frac{(2n+1)\pi}{2L} x\right], \qquad n = 0, 1, 2, 3, \ldots.$$

From (v) we have

$$Y_n(t) = C_{3n} \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right]$$

Thus

$$T_n(x, t) = X_n(x)\, Y_n(t) = b_n \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right] \sin\left[\frac{(2n+1)\pi x}{2L}\right], \qquad n = 0, 1, 2, 3, \ldots.$$

where we have put $b_n = C_{2n}\, C_{3n}$. From the principle of superposition, the most general solution is

$$T(x, t) = \sum_{n=0}^{\infty} b_n \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right) \sin\left[\frac{(2n+1)\pi x}{2L}\right] \qquad (8.3)$$

Notice that $f(x)$ in Eq. (8.4a) cannot be expanded in the Fourier series we have introduced in Unit 7 because the argument of the sine function is different. However, we can use the same technique to expand a given function in terms of any sinusoidal series provided its terms satisfy the orthogonality condition.

Applying the initial condition (8.2c) we have

$$T(x, 0) = \sum_{n=0}^{\infty} b_n \sin\left[\frac{(2n+1)\pi x}{2L}\right] = f(x), \quad 0 < x < L \qquad (8.4a)$$

where $f(x) = \frac{x}{2}(2L - x)$.

We can determine b_n in Eq. (8.4a) using the half-range expansion of $f(x)$. Since $f(x)$ is defined on $0 < x < L$, and $T(x, 0)$ is the sum of a sine series, we can take $g(x)$ to be the odd extension of $f(x)$. Multiplying the LHS of Eq. (8.4a) by $\sin\left[\frac{(2m+1)\pi x}{2L}\right]$ and integrating from $-L$ to L, we have

$$\sum_{n=0}^{\infty} b_n \int_{-L}^{L} \sin\left[\frac{(2m+1)\pi x}{2L}\right] \sin\left[\frac{(2n+1)\pi x}{2L}\right] dx = \int_{-L}^{L} g(x) \sin\left[\frac{(2m+1)\pi x}{2L}\right] dx$$

Applying the technique of Sec. 7.2 we get

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left[\frac{(2n+1)\pi x}{2L}\right] dx$$

$$= \frac{2}{L} \int_0^L \frac{x}{2}(2L - x) \sin\left[\frac{(2n+1)\pi x}{2L}\right] dx$$

You can integrate by parts twice and show that

$$b_n = \frac{16L^2}{(2n+1)^3 \pi^3}$$

Hence the solution is given by

$$T(x, t) = \frac{16L^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right) \sin\left[\frac{(2n+1)\pi x}{2L}\right] \quad (8.4b)$$

You may now like to work out an SAQ yourself by applying Fourier series to the homogeneous heat equation.

Spend 5 minutes

SAQ 1

In Example 1, let the initial temperature of the bar be a constant T_0 °C. Then solve the diffusion equation (8.2a),

with $T(0, t) = \frac{\partial T}{\partial x}(L, t) = 0$, $(t \geq 0)$

and $T(x, 0) = T_0$, $(0 < x < L)$

Determine the expression for $T(x, t)$ and discuss its behaviour at large values of times.

Note that in this case the initial condition does not match the boundary conditions at $x = 0$ and $x = L$. In reality, when the ends of the bar are put into the ice, it would melt to match the temperature of the ends of the bar, which cool rapidly. Only later would we have $T(0, t) = 0$.

Let us consider another application of Fourier Series for solving the diffusion equation.

8.2.2 Diffusion of Particles

Many of our day-to-day experiences involve diffusion of particles. For example, when sugar is added to a cup of tea, it dissolves and then diffuses throughout the tea. Water evaporates from ponds and increases the humidity of the passing air stream. Diffusion of particles plays an important role in many industrial applications. Some examples are: the removal of pollutants from plant discharge streams, the stripping of gases from waste water, acid concentration, salt-production and sugar solution concentration through continuous evaporation, drying of industrial products, such as concrete slabs, wood, etc. Here we will apply the diffusion equation to a typical example of drying of a porous material.

A porous rod containing moisture with one of its ends (for which $x = 0$) sealed is left to be dried. The other end of the rod is in contact with a dry medium, and it loses moisture through its surface to dry air. The concentration of moisture, $u(x, t)$, satisfies the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u, \qquad 0 < x < L, \quad t > 0 \tag{8.5a}$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(L, t) = 0, \qquad t > 0 \tag{8.5b}$$

$$u(x, 0) = u_0, \qquad 0 < x < L \tag{8.5c}$$

Let us find $u(x, t)$ and determine the concentration at $x = 0$, i.e., $u(0, t)$ explicitly.

Using the method of separation of variables, we seek a solution of the form $u(x, t) = X(x)\,T(t)$. The PDE becomes

$$X''(x)\,T(t) = \frac{1}{k}X(x)\,T'(t) + \gamma^2 X(x)\,T(t)$$

Dividing by $X(x)T(t)$ we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t) + k\gamma^2 T(t)}{k\,T(t)} = -\lambda^2$$

Thus, we get two ODEs

$$X''(x) + \lambda^2 X(x) = 0, \qquad 0 < x < L \tag{i}$$

$$T'(t) + k(\gamma^2 + \lambda^2)\,T = 0, \qquad t > 0 \tag{ii}$$

The solution of (i) is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

Applying the boundary conditions, we have

$$X'(0) = 0; \quad X(L) = 0$$

$$X'(0) = \lambda C_2 = 0, \text{ i.e., } C_2 = 0$$

and $\quad X(L) = C_1 \cos \lambda L = 0$

Since $C_1 \neq 0$, this gives

$$\cos \lambda L = 0$$

or

$$\lambda_n L = \frac{(2n+1)\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

or $\quad \lambda_n = \dfrac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$

Thus $X_n(x) = C_{1n} \cos \lambda_n x$, where $\lambda_n = \dfrac{(2n+1)\pi}{2L}$, $\qquad n = 0, 1, 2, \ldots$

The solution of (ii) is

$$T_n(t) = C_{3n} \exp\left[-k(\gamma^2 + \lambda_n^2)t\right] = C_{3n}\, e^{-\gamma^2 kt}\, e^{-\lambda_n^2 kt}$$

Therefore, the general solution of Eq. (8.5a) is

$$u(x, t) = e^{-\gamma^2 kt} \sum_{n=0}^{\infty} a_n \cos \lambda_n x e^{-\lambda_n^2 kt}$$

where we have put $a_n = C_{1n} C_{3n}$. To determine these unknown constants, we note from the initial condition that

$$\sum_{n=0}^{\infty} a_n \cos \lambda_n x = u_0, \qquad 0 < x < L$$

Again we can use the even extension of u_0 (Sec. 7.5.1) to obtain its Fourier series expansion, i.e., the coefficients a_n.

$$a_n = \frac{2}{L} \int_0^L u_0 \cos \lambda_n x \, dx$$

$$= \frac{2u_0}{L}\left(\frac{\sin \lambda_n L}{\lambda_n}\right) = \frac{4u_0 \sin \lambda_n L}{(2n+1)\pi}, \qquad n = 0, 1, 2, \ldots$$

Thus,

$$u(x, t) = \frac{4u_0}{\pi} e^{-\gamma^2 kt} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left[(2n+1)\frac{\pi}{2}\right] \cos\left[\frac{(2n+1)\pi x}{2L}\right] e^{-(2n+1)^2 \pi^2 kt/4L^2}$$

We can find the value of $u(0, t)$ by putting $x = 0$ in this solution

$$u(0, t) = \frac{4u_0}{\pi} e^{-\gamma^2 kt} \sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)\frac{\pi}{2}\right]}{(2n+1)} e^{-(2n+1)^2 \pi^2 kt/4L^2}$$

$$= \frac{4u_0}{\pi} e^{-\gamma^2 kt}\left[e^{-\tau} - \frac{e^{-9\tau}}{3} + \frac{e^{-25\tau}}{5} - \ldots + \ldots\right]$$

where $\tau = \pi^2 kt/4L^2$.

You should now work out an SAQ to know whether you have grasped the application of Fourier series to the one-dimensional diffusion equation.

Spend 10 minutes

SAQ 2

Solve the heat/diffusion problem stated below in terms of dimensionless variables

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad \frac{\partial u}{\partial x}(1, t) = 0, \qquad t > 0$$

$$u(x, 0) = 1 + 2x, \qquad 0 < x < 1.$$

Let us now consider some applications of Fourier series to the wave equation.

8.3 THE WAVE EQUATION

For simplicity we shall restrict our discussion to the solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L, \qquad t > 0 \qquad (8.6)$$

with given initial and boundary conditions. Let us solve this equation by applying Fourier series to two specific categories of physical problems related to (i) vibrating strings, and (ii) torsional vibrations.

8.3.1 Vibrating Strings

When a sitarist plucks the sitar string, several other tones called overtones or harmonics, are generated along with the fundamental frequency (Recall Eq. (6.39) of Unit 6). The richness of musical sound is related to the number of harmonics that can be detected by the human ear. The larger the amplitude of each harmonic, the more likely it is to be detected. The amplitude of each harmonic depends, in turn, on where exactly the string is plucked. So if we know the point at which a sitar string is plucked, we can get a fair idea of the richness of the sound produced. To mathematically model this physical situation, we have to determine a unique solution of Eq. (8.6) for the "plucked string" problem which we consider in the following example.

Example 2 : The 'plucked string' problem

A string is plucked at its mid-point and then released from rest from this position (Fig. 8.2). The resulting vibrations are modelled by Eq. (8.6) along with the following boundary and initial conditions.

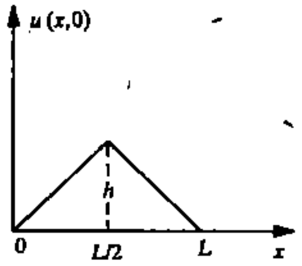


Fig. 8.2

$$u(0, t) = 0, \quad u(L, t) = 0$$

$$u(x, 0) = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

where h is a positive constant which is small compared to L.

These conditions correspond to an initial **triangular deflection** and zero initial velocity.

In Unit 6 you have already obtained the general solution of the wave equation for given boundary conditions. The general solution given by Eq. (6.40) is

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi vt}{L} + b_n \sin \frac{n\pi vt}{L} \right) \sin \frac{n\pi x}{L} \qquad \text{(i)}$$

Let us now apply the initial conditions to (i) :

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases} \qquad \text{(ii)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} \left(-a_n \frac{n\pi v}{L} \sin \frac{n\pi vt}{L} + b_n \frac{n\pi v}{L} \cos \frac{n\pi vt}{L} \right) \sin \frac{n\pi x}{L} \right]_{t=0}$$

$$= \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L} \sin \frac{n\pi x}{L} = 0 \qquad \text{(iii)}$$

Eq. (iii) will be satisfied only if $b_n = 0$ for all n, as you have obtained in Eq. (6.43). So now you have to determine a_n, i.e., you have to expand $u(x, 0)$ in a Fourier sine series. In effect, you have to obtain the odd periodic extension of $u(x,0)$ and hence its half-range expansion in a Fourier sine series. Recall that you have worked out a problem for an even periodic extension of the same function in the terminal question 1 of Unit 7.

So you may like to solve this part of the problem yourself.

Spend 10 minutes

SAQ 3

Show that the solution of the "plucked string" problem specified by Eq. (8.6) is

$$u(x, t) = \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi v t}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi v t}{L} + \ldots\right]$$

Another way to start any string vibrating is to strike it (a piano string, for example). In this case, the initial conditions would be $u(x, t) = 0$ at $t = 0$ and the velocity $\frac{\partial u}{\partial t}$ will be given as a function of x (i.e., the velocity of each point of the string is given at $t = 0$). Now that you have practised determining Fourier series you should feel confident enough to be able to solve any such BVP related to vibrating strings.

Another interesting application of the wave equation is in torsional vibrations. Such vibrations can result from unbalanced torques on shafts in a wide variety of machinery in cars, aircraft, turbines, railway engines, etc. You may know that a shaft is a bar that is usually cylindrical and solid. It is used to support rotating pieces in machines or to transmit power or motion by rotation. Some common examples of shafts are axles connecting the wheels of a car, spindles on a spinning wheel, propeller shafts used for ship propulsion and shafts in belt and pulley arrangements. So let us now consider a typical problem involving torsional vibrations of a shaft.

8.3.2 Torsional Vibrations

Consider a uniform, undamped torsionally vibrating shaft of finite length, subject to given initial conditions of angular displacement and angular velocity (Fig. 8.3). This means that we have to find solutions of the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\theta}{\partial t^2} \qquad (8.7)$$

where θ is the angle of twist of the shaft and $v^2 = E_s/\rho$: Here E_s is the modulus of elasticity in shear, and ρ, the mass per unit volume of the shaft. Once again we use the method of separation of variables to express $\theta(x, t)$ as

$$\theta(x, t) = X(x)\,T(t)$$

Just as in the case of a vibrating string (Eq. 6.15 of Unit 6) we reduce Eq. (8.7) to a set of two ODEs:

$$T'' = \frac{\lambda^2}{v^2}T, \qquad X'' = -\lambda^2 X$$

where $(-\lambda^2)$ is the separation constant. The solutions of these ODEs are

$$T = A\cos\lambda vt + B\sin\lambda vt$$

and $$X = C\cos\lambda x + D\sin\lambda x$$

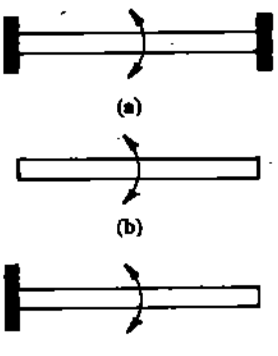


Fig. 8.3: A torsionally vibrating shaft with (a) both ends fixed (b) both ends free (c) one end fixed and one end free

Thus the solution is

$$\theta(x, t) = X(x)\,T(t) = (C\cos\lambda x + D\sin\lambda x)(A\cos\lambda vt + B\sin\lambda vt) \qquad \text{(i)}$$

You can see that the solution is periodic, repeating itself for every increase in time t by $\frac{2\pi}{\lambda v}$. In other words, $\theta(x, t)$ represents a torsional motion of period $\frac{2\pi}{\lambda v}$ or frequency $\lambda v/2\pi$.

It remains now to find the values of λ, A, B, C and D. The values of λ are determined by the given boundary conditions which define how the shaft is constrained at its ends. There are three cases which occur most often in physical systems:

1) Both ends of the shaft are fixed so that no twisting can take place (Fig. 8.3a)
2) Both ends of the shaft are free to twist (Fig. 8.3b)
3) One end of the shaft is fixed, while the other is free to twist (Fig. 8.3c).

Case 1

The boundary conditions in this case are

$$\theta(0, t) = \theta(L, t) = 0, \qquad t > 0$$

Applying these conditions you get the general solution which is the familiar result obtained for the vibrating string (Eq. 6.40 of Unit 6):

$$\theta(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi vt}{L} + b_n \sin \frac{n\pi vt}{L} \right) \sin \frac{n\pi x}{L}$$

This solution has to satisfy the given initial conditions on angular displacement and angular velocity. If we set $t = 0$ in the equation for $\theta(x, t)$ and its derivative we get

$$\theta(x, 0) \equiv f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

and
$$\left. \frac{\partial \theta}{\partial t} \right|_{t=0} \equiv g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi v}{L} b_n \right) \sin \frac{n\pi x}{L}$$

where $f(x)$ and $g(x)$ are some functions of x, representing the initial angular displacement and angular velocity of the shaft. We can then use the half-range sine expansions of $f(x)$ and $g(x)$. This gives

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

and

$$b_n = \frac{L}{n\pi v} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx = \frac{2}{n\pi v} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

Thus, a uniform shaft with both ends restrained against turning vibrates torsionally at any one of the infinite number of natural frequencies

$$f_n = \frac{nv}{2L} \text{ cycles per unit time,} \qquad n = 1, 2, 3, \ldots.$$

Case 2

When both ends of the shaft are free, no torque acts through the end section (i.e., at $x = 0$ and at $x = L$) since there is no shaft material beyond these points. Thus, the torque transmitted through these ends is zero, i.e.,

$$\tau = E_s I \left. \frac{\partial \theta}{\partial x} \right|_{\text{end points}} = 0$$

where I is the moment of inertia of the rod, and E_s is the shear modulus of elasticity. Both E_s and I are non-zero. Thus for a free-end, the boundary conditions are

$$\frac{\partial \theta}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

The subsequent solution proceeds on familiar lines. In fact, you may like to complete the solution for a specific problem. Try the following SAQ.

The torque transmitted through any cross-section of a twisted shaft is proportional to the twist per unit length, i.e., the slope of the (θ, x) curve, at that cross-section.

Thus $\tau \propto \frac{\partial \theta}{\partial x}$

For solid shafts the proportionality constant is equal to $E_s I$ where E_s is the shear modulus of elasticity and I the moment of inertia of the shaft.

SAQ 4

Spend 10 minutes

A uniform shaft free at each end is twisted so that it rotates through an angle proportional to $(2x - L)/2$. If the shaft is released from rest in this position, what will its subsequent angular displacement as a function of x and t be?

Have you noted that the natural frequencies of the vibrating shafts in Cases (1) and (2) are the same? However, the amplitudes of vibration are not the same. For the shaft fixed at both ends, the amplitudes along the shaft are proportional to $\sin \frac{n\pi x}{L}$, whereas for the shaft free at the ends, the amplitudes are proportional to $\cos \frac{n\pi x}{L}$

Case 3

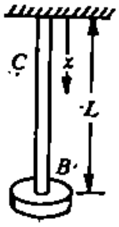


Fig. 8.4

A typical example of a shaft fixed at one end and free at the other is the drill pipe used in oil wells. A drill collar (C) containing the cutting bit (B) is attached to the lower end of the pipe (Fig. 8.4). The boundary conditions for such a shaft are

$$\theta(0, t) = 0 \text{ and } \left.\frac{\partial \theta}{\partial x}\right|_{L, t} = 0, \qquad t > 0$$

When we impose these conditions on Eq. (i) we get

$$C = 0, \cos \lambda L = 0, \text{ which gives}$$

$$\lambda_n L = (2n-1)\frac{\pi}{2}, \qquad n = 1, 2 \ldots$$

and $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2 \ldots$$

The general solution is, therefore,

$$\theta(x, t) = \sum_{n=1}^{\infty} (\sin \lambda_n x)(A_n \cos \lambda_n vt + B_n \sin \lambda_n vt)$$

Now suppose we have the initial conditions that

$$\theta(x, 0) = f(x) \quad \text{and} \quad \left.\frac{\partial \theta}{\partial t}\right|_{x, 0} = g(x)$$

These initial conditions yield

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

and

$$g(x) = \sum_{n=1}^{\infty} \left[\frac{(2n-1)\pi v}{2L} B_n\right] \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

Recall that in Example 1, we have written $\lambda_n L = (2n+1)\frac{\pi}{2}$ but n takes the values of 0, 1, 2 . . . . In this case $\lambda_n L = (2n-1)\frac{\pi}{2}$, where $n = 1, 2 \ldots$ So the expression for λ_n is the same in both the cases.

Recall that we have obtained the coefficients A_n and B_n in the heat conduction problem. You can easily verify that the half-range sine expansions of $f(x)$ and $g(x)$ yield

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left[\frac{(2n-1)\pi x}{2L}\right] dx$$

and

$$B_n = \frac{4}{(2n-1)\pi v} \int_0^L g(x) \sin\left[\frac{(2n-1)\pi x}{2L}\right] dx$$

These coefficients can be obtained for any function $f(x)$ and $g(x)$ integrable on the interval $0 < x < L$.

Finally, we will take up the application of Fourier series to Laplace's equation.

8.4 LAPLACE'S EQUATION

You know the three dimensional Laplace's equation : $\nabla^2 u = 0$. The function u may repsresent many physical qualities: it may be the gravitational potential in a region containing no matter or the electrostatic potential in a charge- free region. The steady-state temperature (i.e., temperature not changing with time) in a region containing no source of heat also satisfies Laplace's equation. In Unit 5, we have derived this equation for the velocity potential for an incompressible and irrotational fluid. Of all these diverse areas where Laplace's equation applies, we have selected two to illustrate the applications of Fourier series. These are the steady-state heat flow and the potential problem. You can extend the procedure explained here to other specific problems.

8.4.1 Steady-state Heat Flow

In Unit 6 you have solved Laplace's equation for determining the steady-state temperature of a circular cylinder (Example 2) and a semi-circular plate (Example 3). However, in both these examples we need not use Fourier series to determine the particular solution. So let us consider a specific BVP for Laplace's equation which involves the use of Fourier series. A modified version of this problem can be used to model the flow of heat across a refrigerator door.

Example 3 : Steady-state temperature of a rectangular metal plate

A thin rectangular metal plate is sandwiched between sheets of insulation (Fig. 8.5a). Since the plate is very thin and insulated at two of its surfaces, one may assume that the temperature does not vary in the z-direction. In the steady state, the temperature of the plate obeys the two-dimensional Laplace equation:

$$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < B \tag{8.8}$$

where L is the length and B the width of the plate.

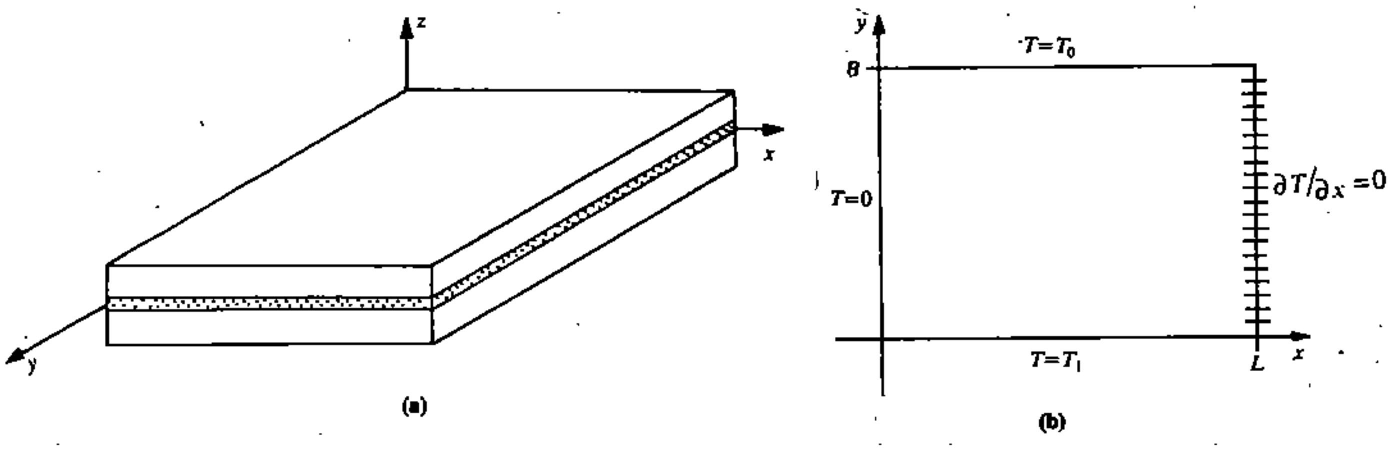


Fig. 8.5: (a) A thin plate between sheets of insulation (b) the boundary conditions for $T(x, y)$

Suppose that the temperature of the plate is held at T_0 at its top edge, T_1 at its bottom edge and 0°C on the left edge. The plate is insulated on the right edge, so that no heat flows in that direction, and the partial derivative of T in the x – direction is zero (See Fig. 8.5b). Can you write down these boundary conditions mathematically? These are

i) $T(0, y) = 0, \quad \dfrac{\partial T(L, y)}{\partial x} = 0, \quad 0 < y < B$

ii) $T(x, 0) = T_1 \quad T(x, B) = T_0, \quad 0 < x < L$

We wish to determine $T(x, y)$ by solving Laplace's equation subject to these boundary conditions.

Solution

For a non-trivial solution we write

$$T(x, y) = X(x)\,Y(y)$$

and use the method of separation of variables to get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

Since $X(x)$ vanishes at the boundaries the ratio $\frac{X'(x)}{X(x)}$ cannot be positive. Hence, we get the two ODEs :

$$X'' + \lambda^2 X = 0 \qquad 0 < x < L$$

and $$Y'' - \lambda^2 Y = 0 \qquad 0 < y < B$$

The solutions are

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

and $$Y(y) = C\cosh\lambda y + D\sinh\lambda y$$

Applying the boundary conditions (i) and (ii) we get

$$A = 0, \quad \cos\lambda L = 0$$

or $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2 \ldots$$

which gives $X_n(x) = B'_n \sin\lambda_n x$

and $$Y_n(y) = C'_n\cosh\lambda_n y + D'_n\sinh\lambda_n y$$

The general solution for $T(x, y)$ is

$$T(x, y) = \sum_{n=1}^{\infty} (C_n\cosh\lambda_n y + D_n\sinh\lambda_n y)\sin\lambda_n x$$

with $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2, \ldots$$

and $$C_n = C'_n B_n, \quad D_n = D'_n B_n.$$

The coefficients C_n and D_n are determined by applying the boundary condition (ii). At $y = 0$

$$T(x, 0) = C_n\sin\lambda_n x = T_1, \qquad 0 < x < L$$

from which you can determine C_n to be

$$C_n = \frac{2}{L}\int_0^L T_1\sin\lambda_n x\,dx = \frac{4T_1}{(2n-1)\pi}$$

At $y = B$,

$$T(x, B) = \sum_{n=1}^{\infty} (C_n\cosh\lambda_n B + D_n\sinh\lambda_n B)\sin\lambda_n x = T_0, \qquad 0 < x < L$$

Now we have to choose D_n so that the quantity within brackets is the Fourier sine coefficient of the function representing the given boundary value (T_0 in this case). Let us put

$$C_n\cosh\lambda_n B + D_n\sinh\lambda_n B = G_n$$

Then, the coefficients G_n are given by the relation

$$G_n = \frac{2}{L}\int_0^L T_0 \sin\lambda_n x\, dx = \frac{4T_0}{(2n-1)\pi}$$

This gives the coefficients D_n in terms of the known coefficients C_n and G_n :

$$D_n = \frac{G_n - C_n \cosh\lambda_n B}{\sinh\lambda_n B} = \frac{4}{(2n-1)\pi}\frac{T_0 - T_1\cosh\lambda_n B}{\sinh\lambda_n B}$$

Thus, the unique solution of this problem is

$$T(x, y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{T_1\cosh\lambda_n y}{2n-1} + \frac{T_0 - T_1\cosh\lambda_n B}{(2n-1)\sinh\lambda_n B}\sinh\lambda_n y\right)\sin\lambda_n x$$

The solution for the case $B = 2L$, $T_1 = 10°C$, $T_0 = 20°C$ is shown in Fig. 8.6. The curves shown are the isotherms $T(x, y) = T_c$ for various values of T.

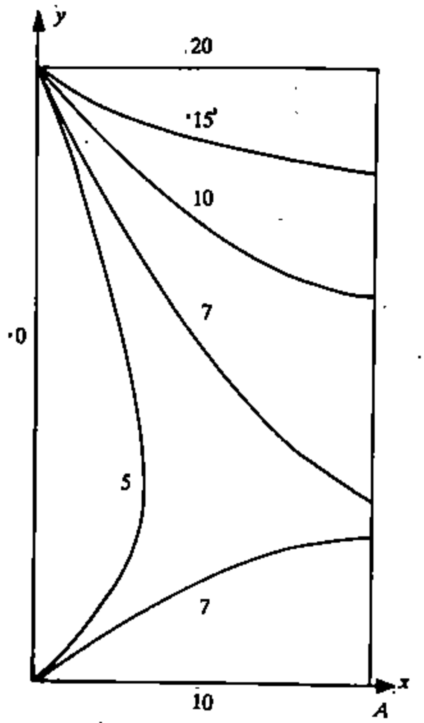


Fig. 8.6: The isotherms $T(x, y) = T_c$ for various values of T when $B = 2L$, $T_1 = 10°C$, $T_0 = 20°C$

You could have solved this problem even if the boundary conditions for $T(x, y)$ on the top and bottom edge had been any piecewise continuous functions, instead of constants, or if the boundary conditions on the left and right edges had been different. Why don't you work out such a problem?

SAQ 5

Spend 15 minutes

Obtain the steady-state temperature for the rectangular plate of Fig. 8.5a given the following boundary conditions :

$$u(0, y) = \frac{U_0 y}{B}, \quad \frac{\partial u}{\partial x}(L, y) = -S, \qquad 0 < y < B$$

$$u(x, 0) = 0, \quad u(x, B) = 0, \qquad 0 < x < L$$

In a certain class of problems involving conductors, all the charge is found on their surfaces. The potential at all points outside the conductor satisfies Laplace's equation. Let us now solve Laplace's equation for the electrostatic potential of a conductor.

So far we have considered problems which require Cartesian coordinate system. In the final section of this unit we are considering an application of Fourier series to a potential problem in a non-Cartesian geometry.

8.4.2 The Potential at a Point due to a Circular Disc

Let us solve Laplace's equation for the potential on a circular metallic disc. For a circular disc it is natural to use plane-polar coordinates (r, θ). The problem is as follows :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2}\right) = 0, \qquad 0 \le r < L, \; -\pi < \theta \le \pi \qquad (8.9a)$$

$$u(L, \theta) = f(\theta), \qquad -\pi < \theta \le \pi \qquad (8.9b)$$

There are two special features of this problem:

(i) The points $\theta = -\pi$ and $\theta = \pi$ coincide. Therefore, the value of u and its angular derivative should match there:

$$u(r, -\pi) = u(r, \pi), \quad \frac{\partial u}{\partial\theta}(r, -\pi) = \frac{\partial u}{\partial\theta}(r, \pi), \qquad 0 \le r < L$$

ii) The point $r = 0$ is singular: the coefficient of $\frac{\partial^2 u}{\partial r^2}$ in Eq. (8.9a) is 1, while the coefficients of other terms are $1/r$ and $1/r^2$ We must, therefore, enforce a condition of boundedness :

$u(r, \theta)$ tends to a finite value, i.e., it is bounded, as $r \to 0$.

Keeping these special features in mind, we can solve the potential problem using the method of separation of variables.

Let $u(r, \theta) = R(r)\Theta(\theta)$

Substituting $u(r, \theta)$ in Eq. (8.9a) and taking into account the special continuity conditions, we get

$$\frac{1}{r}[rR'(r)]'\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0, \qquad 0 \le r < L, \ -\pi < \theta \le \pi \qquad \text{(i)}$$

and $$R(r)\,\Theta(-\pi) = R(r)\,\Theta(\pi), \quad R(r)\,\Theta'(-\pi) = R(r)\,\Theta'(\pi) \quad 0 \le r < L \qquad \text{(ii)}$$

Multiplying Eq. (i) by r^2, dividing it by $R(r)\,\Theta(\theta)$, and eliminating $R(r)$ in (ii) we get

$$\frac{r[rR'(r)]'}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \qquad 0 \le r < L, \qquad -\pi < \theta \le \pi$$

$$\Theta(-\pi) = \Theta(\pi) \text{ and } \Theta'(-\pi) = \Theta'(\pi), \qquad \text{(ii)}$$

Thus

$$\Theta'' + \lambda^2\Theta = 0$$

which gives $\Theta = A\cos\lambda\theta + B\sin\lambda\theta$

The continuity conditions (ii) for Θ give us

$$A\cos\lambda\pi - B\sin\lambda\pi = A\cos\lambda\pi + B\sin\lambda\pi$$

$$A\lambda\sin\lambda\pi + B\lambda\cos\lambda\pi = -A\lambda\sin\lambda\pi + B\lambda\cos\lambda\pi$$

or $$2B\sin\lambda\pi = 0$$

and $$2\lambda A\sin\lambda\pi = 0$$

which gives $\lambda_n = n, \quad n = 0, 1, 2, 3, \ldots$

Thus, we have

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \qquad n = 0, 1, 2, 3, \ldots$$

The ODE for $R(r)$ is

$$\frac{r(rR'_n)'}{R_n} = \lambda_n^2 \quad \text{or} \quad r^2R''_n + rR'_n - \lambda_n^2R_n = 0$$

This is an Euler-Cauchy equation with linearly independent solutions (see Example 3 of Unit 6):

$$R_n(r) = r^n \quad \text{and} \quad R_n(r) = r^{-n}$$

The second of these is physically unacceptable as it tends to ∞ in the limit as $r \to 0$. Thus, we have the general solution for $u(r, \theta)$:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

The boundary condition on $r = L$ yields

$$A_0 + \sum_{n=1}^{\infty} L^n(A_n\cos n\theta + B_n\sin n\theta) = f(\theta) \qquad -\pi < \theta \le \pi$$

This is a Fourier series problem and the coefficients in the series are

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$L^n A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta$$

$$L^n B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

You can solve these three integrals for any given form of $f(\theta)$ provided they exist.

In this unit, we have considered certain specific applications of Fourier series to some PDEs of special interest in physical problems, viz. the diffusion equation, the wave equation and Laplace's equation. The applications discussed here are, by no means, exhaustive, but only illustrative of this powerful method based on the Fourier series. This method applies to a much wider variety of problems which, of course, cannot all be discussed here for want of time. However, we are sure that you have been able to develop an appreciation of the usefulness of the Fourier method based on Fourier series, from what you have studied in Units 7 and 8.

But we would certainly not like to end this unit on the note that the Fourier method is the ultimate method of solving linear BVPs in PDEs. There are other important methods of solving these problems, such as methods based on Laplace transforms, Fourier transforms and other integral transforms, methods that use the Green's functions and numerical methods. In fact, the development of new methods for solving BVPs is an active area of present-day mathematical research. Most of these methods are usually discussed in advanced level courses on Mathematical Methods in Physics or Differential Equations, at the undergraduate as well as postgraduate level.

We will now summarise what you have studied in this Unit.

8.5 SUMMARY

In this unit you have learnt to apply the Fourier series to solve various boundary-value problems related to:

- the diffusion equation, e.g., the problems of heat conduction and diffusion of liquids in porous solids
- the wave equation, e.g., the problems of vibrating strings and torsional vibrations
- Laplace's equation, e.g., the steady-state heat flow and the potential problem.

8.6 TERMINAL QUESTIONS

1. A cylindrical elastic bar (e.g., steel bar) of natural length L is initially stretched by an amount cL and is at rest. The initial longitudinal displacement of any section of the bar is proportional to the distance from the fixed end $x = 0$. At the instant $t = 0$, both ends are released and left free. The longitudinal displacement $y(x, t)$ of the bar satisfies the following BVP

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

where $v^2 = E/\rho$, E is the modulus of elasticity and ρ is the density of the material of the bar. Since the ends are free, the force per unit area on the ends of the bar is zero and we get

$$\frac{\partial y}{\partial x}(0, t) = 0, \qquad \frac{\partial y}{\partial x}(L, t) = 0$$

Further $y(x, 0) = cx$, $\quad \frac{\partial y}{\partial t}(x, 0) = 0$

Solve the BVP and obtain $y(x, t)$.

2. The flow of electric current in a pair of telephone wires or power transmission lines

The boundary conditions for a shaft with both ends free,

$$\frac{\partial \theta}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

imply that for all $t > 0$

$$\frac{\partial X}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

This gives

$$D = 0 \quad \text{and} \quad C \sin \lambda L = 0$$

whence

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots.$$

Thus, the general solution for $\theta(x, t)$ is

$$\theta(x, t) = \sum_{n=1}^{\infty} (a_n \cos \lambda_n vt + b_n \sin \lambda_n vt) \cos \lambda_n x$$

where $a_n = A_n C_n$ and $b_n \equiv B_n C_n$.

At $t = 0$, $\theta(x, 0)$ is proportional to $(2x - L)/2$.

$$\therefore\ \theta(x, 0) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x = k\frac{(2x-L)}{2}, \qquad 0 < x < L$$

Using the half-range expansion technique we get

$$a_n = \frac{2k}{L}\int_0^L \left(x - \frac{L}{2}\right)\cos\frac{n\pi x}{L}\,dx = \frac{2k}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx - k\int_0^L \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2k}{L}\left(\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\right]_0^L + \frac{L^2}{n^2\pi^2}\left[\cos\frac{n\pi x}{L}\right]_0^L\right) - \frac{Lk}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2k}{L}\frac{L^2}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2Lk}{n^2\pi^2}(\cos n\pi - 1)$$

Since the shaft starts vibrating from rest, its initial velocity is zero giving the condition

$$\frac{\partial \theta}{\partial t}(x; 0) = 0$$

or $\displaystyle\sum_{n=1}^{\infty} b_n \lambda_n v \cos \lambda_n x = 0$

This will be satisfied only if $b_n = 0$ for all n. Thus, the solution for the given BVP is

$$\theta(x, t) = \frac{2Lk}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\cos n\pi - 1)\cos\frac{n\pi vt}{L}\cos\frac{n\pi x}{L}$$

5) We seek non-trivial solutions in the product form $u(x, y) = X(x)\,Y(y)$. Applying the method of separation of variables we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \quad 0 < x < L, \quad 0 < y < B$$

with the boundary conditions

$$X(x)\,Y(0) = 0, \qquad X(x)\,Y(B) = 0, \qquad 0 < x < L$$

or $Y(0) = 0, \; Y(B) = 0$

Since Y has to vanish at the boundaries $y = 0$ and $y = B$, the ratio $\frac{Y''}{Y}$ cannot be positive. Thus, we get the ODEs:

$$X'' - \lambda^2 X = 0, \quad Y'' + \lambda^2 Y = 0$$

whence $X(x) = A \cosh \lambda x + B \sinh \lambda x$

$$Y(y) = C \cos \lambda y + D \sin \lambda y$$

The boundary conditions on Y yield the following values of C and λ:

$$C = 0, \quad \lambda_n = \frac{n\pi}{B}, \qquad n = 1, 2, 3, \ldots.$$

Thus $Y_n(y) = D_n \sin \frac{n\pi y}{B}, \quad n = 1, 2, 3, \ldots.$

Thus, the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} (a_n \cosh \lambda_n x + b_n \sinh \lambda_n x) \sin \lambda_n y$$

where $a_n = A_n D_n, b_n = B_n D_n$.

Applying the remaining boundary conditions we get

At $x = 0, \; \sum_{n=1}^{\infty} a_n \sin \lambda_n y = \frac{U_0 y}{B}, \qquad 0 < y < B$

Using the half-range expansion technique, we get

$$a_n = \frac{2}{B} \int_0^B \frac{U_0 y}{B} \sin \lambda_n y \, dy$$

$$= \frac{2U_0}{B^2} \left(\left[-\frac{y}{\lambda_n} \cos \lambda_n y \right]_0^B + \frac{1}{\lambda_n} \left[\frac{\sin \lambda_n y}{\lambda_n} \right]_0^B \right)$$

$$= \frac{2U_0}{B^2} \left(-\frac{B^2}{n\pi} \cos n\pi + 0 \right)$$

$$= -\frac{2U_0 \cos n\pi}{n\pi}$$

At $\quad x = L, \; \frac{\partial u}{\partial x}(L, y) = -S, \qquad 0 < y < B$

Differentiating the series for $u(x,y)$ term by term and applying the given boundary conditions we get

$$\frac{\partial u}{\partial x}(L, y) = \sum_{n=1}^{\infty} \lambda_n (a_n \sinh \lambda_n L + b_n \cosh \lambda_n L) \sin \lambda_n y = -S, \qquad 0 < y < B$$

So we must choose b_n such that the coefficient of $\sin \lambda_n y$ will be

$$C_n = \lambda_n (a_n \sinh \lambda_n L + b_n \cosh \lambda_n L)$$

NOTES