स्वाध्याय स्वमन्थन स्वावलम्बन

UTTAR PRADESH RAJARSHI TANDON OPEN UNIVERSITY

सरस्वती नः सुभगा मयस्करत्।।

UGPHS – 01
MACHANICS

SECOND BLOCK

SYSTEMS OF PARTICLES

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UGPHS - 01
Mechanics

Block

2

SYSTEMS OF PARTICLES

BLOCK INTRODUCTION

In Block 1 you have studied the basic concepts of mechanics. You have learnt the language for describing motion. You have applied Newton's laws of motion to a variety of systems executing linear as well as angular motion. In addition you have studied the concepts of work, energy and gravitation. In the process you have also learnt to apply the principles of conservation of linear momentum, angular momentum, and energy.

In almost all the applications you must have noticed one common feature. We have represented each object by a single particle, be it a cricket ball, a car or the moon. We have seen that the single-particle models were good enough. However, there are many situations in which we need to deal with systems of many particles. For example, the Solar System comprising the sun, the planets, their satellites, comets and asteroids is a many-particle system. A cylinder containing gas, and a rigid body are other examples. We need to extend the concepts of Block 1 to study the motion of such systems.

We shall begin by discussing the motion of objects under central conservative forces in Unit 6. We will mainly deal with the inverse square forces. This will enable you to understand the motion of planets, their satellites and comets. In Unit 7 we shall first consider two-body systems. You will learn to express their motion in terms of the centre-of-mass and relative coordinates. We shall then extend these concepts to analyse the motion of a three-body and N-body systems.

In Unit 8 we shall study the phenomenon of scattering of two or more particles. You will learn some new concepts, such as the scattering cross-sections and impact parameters. You will study the motion of rigid bodies in Unit 9. We shall chiefly concentrate on their rotational motion.

In Block 1 you have studied the motion of objects from the point of view of an inertial observer. The same will be the case upto Unit 9 of this block. However, there are many phenomena which are easier to analyse from the point of view of non-inertial observers. We experience many of them daily. For example, we are pushed sideways in a bus when it takes a turn. Many important natural phenomena, such as cyclones, variation of g with latitude, etc. arise due to the rotation of the earth. Therefore, we shall discuss motion of objects in non-inertial frames of reference in Unit 10.

The study time required for each unit is almost equal which comes to about 4h. It will also depend on how well you have studied Block 1. The tables of constants given in Block 1 are being repeated here for your convenience.

Study Guide

In order to understand this block you will have to keep in mind the suggestions we gave in the study guide to Block 1. Go through it again and follow those suggestions. As in Block 1, here too we are giving the answers to SAQs and terminal questions at the end of each unit. But we hope that you will try to solve them on your own.

You may find some derivations in this block difficult, particularly the ones in Secs. 8.2.4 and 10.3.1. You need not memorise these derivations. The purpose is to demonstrate that there is a logic behind each result. The appendices given at the end of the block are for enrichment only. You will not be examined for that material.

We once again hope that you will enjoy reading these units. Our best wishes are with you.

Acknowledgements

Prof. R.N. Mathur and Dr. S.C. Garg for comments on the units.

UNIT 6 MOTION UNDER CENTRAL CONSERVATIVE FORCES

Structure

6.1 INTRODUCTION

In Block 1 you have studied the basic concepts of mechanics. In Unit 5 of Block 1 we have discussed gravitation. You know that the planets move under the influence of the gravitational field of the sun. How do we solve the equation of motion of a planet ? In this unit we will try to answer this and similar questions.

In fact one of the most important problems of mechanics is to understand the motion of a particle moving under the influence of a force field: The force may be due to another particle or a system of particles, as in the Solar System or a system of fixed charged particles. It could even be due to an electromagnetic field. In this unit we will restrict ourselves to what we call central conservative forces. You will first learn what a central conservative force is. The motion of particles under the influence of such forces has special properties which simplify its description. So you will also study these properties.

There are many examples of such motion. We have mentioned the motion of planets around the sun. Other examples are the motion of satellites around the earth, of spacecrafts sent out to probe the universe and that of two charged particles with respect to each other. The forces associated with these systems, namely the gravitational and electrostatic, obey the inverse square law. We shall see that the inverse square central conservative forces are of special importance. So we shall concentrate chiefly on inverse square central conservative forces. We shall solve the equation of motion of a particle moving under the influence of such forces. We shall then apply the results to determine the possible orbits of a body moving around the sun. This provides the theoretical basis for Kepler's empirical laws. We shall also determine the trajectory of an alpha particle approaching the nucleus. Such a calculation led to the nuclear model of the atom.

So far we have studied single particle motion. In the next unit, we shall turn our attention to many-particle systems. In this unit we shall refer to the contents of Units 3, 4 and 5 of Block 1 very often. So we suggest that you go through these units once again before studying this unit. You may also go through Appendix A on conic sections before studying Sec. 6.3. It is given after Unit 10.

Objectives

After studying this unit you should be able to

- identify a central conservative force
- solve problems by applying the properties of motion under a central conservative force
- determine the possible orbits under a given inverse square central conservative force.

6.2 CENTRAL CONSERVATIVE FORCE

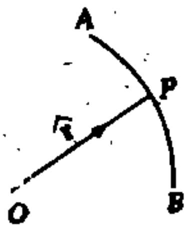


Fig. 6.1 : A particle moving under a central force.
O = Centre of force
P = Particle
APB = Trajectory

In nature we come across many forces which are either directed towards or away from a fixed point. For example, the gravitational force experienced by a mass due to a fixed point mass is directed towards the point mass. Again the force experienced by a positive charge due to another fixed positive charge is directed away from the latter charge. The force on a particle of mass m attached to a string and moving in a circle in a horizontal plane is also directed toward the centre of the circle (see Fig. 4.17 of Unit 4). Such forces are examples of central forces. We define a **central force** as one that is everywhere directed towards or away from a fixed point. This fixed point is called the **centre of force.** Mathematically, we can express a central force acting on a particle as

$$\mathbf{F} = F\,\hat{\mathbf{r}} \qquad (6.1)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from the centre of force to the particle (see Fig. 6.1). For the above mentioned first three examples of central forces, F depends only on the separation between the centre of force and the particle. For such forces Eq. 6.1 can be written as

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}}\,. \qquad (6.2)$$

We can show that the central forces given by Eq. 6.2 are also conservative. For this recall the definition of a conservative force from Sec. 3.3 of Unit 3. Let us compute the work done by the force on particle P as it moves from point A to B (Fig. 6.2).

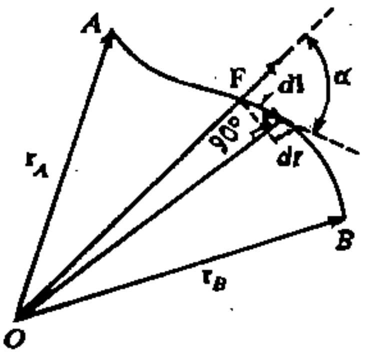


Fig. 6.2 : Work done on a particle moving from A to B. Since $d\mathbf{l}$ is infinitesimal, the angle indicated by double stripes can be considered alternate and hence equal to α.

Let dW be the work done by the central force on the particle as it undergoes a displacement $d\mathbf{l}$ along the path. It is given by

$$dW = \mathbf{F}.\,d\mathbf{l} = f(r)\,\hat{\mathbf{r}}.d\mathbf{l} = f(r)\,dl\cos\alpha, \qquad (6.3)$$

where α is the angle between $\hat{\mathbf{r}}$ and $d\mathbf{l}$. Since $d\mathbf{l}$ is infinitesimal, from Fig. 6.2 you can see that

$$dl\,\cos\alpha = dr\,,$$

where dr is the change in the particle's separation from O, as it undergoes displacement $d\mathbf{l}$. So Eq. 6.3 becomes

$$dW = f(r)\,dr.$$

The work done by the force on the particle as it moves from the point A to B is given as

$$W = \int_{r_A}^{r_B} f(r)dr. \qquad (6.4)$$

Now, the value of this integral depends on its limits only. So the work done depends only on the end points and not on the path being followed by the particle. Thus, the central force given by Eq. 6.2 is conservative. We term the forces represented by Eq. 6.2 as **central conservative forces.**

Have you wondered about the use of the term 'conservative' force field? You know that for a conservative force, the work done in taking a system around a closed path is zero. If this were not so, we could find a closed path, traversing which would yield negative work, i.e. energy to us. Thus we could recover any amount of energy going around the loop. That this does not happen is related to the conservation of energy. Thus path independence of work in a 'conservative' force field is related to 'energy conservation'.

You may like to use this concept to identify some central conservative forces in the following SAQ.

SAQ 1

Which forces among the following are central ? Also identify the central conservative force.

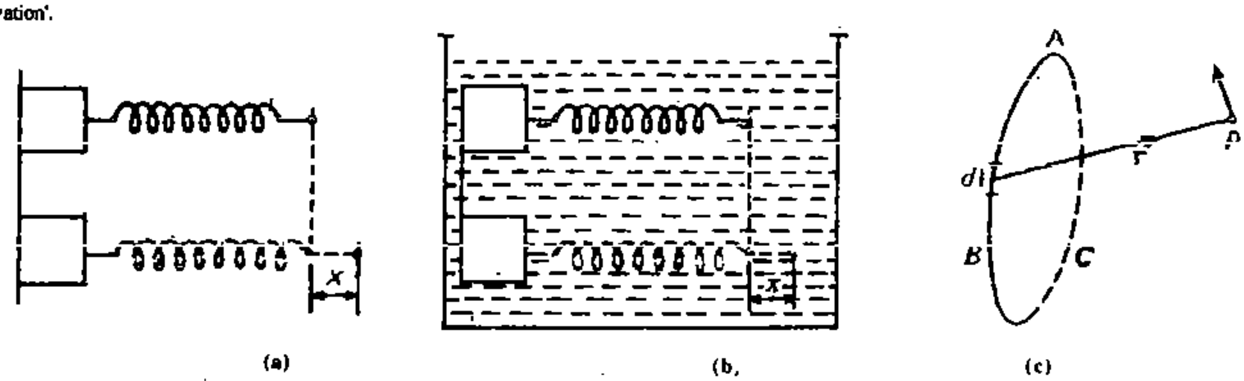


Fig. 6.3 : (a) Ideal spring-mass system; (b) real spring-mass system; (c) a current-carrying conductor

a) The force acting on a particle of mass m in the spring-mass system shown in Fig. 6.3a for which $\mathbf{F} = -k\mathbf{x}$.

b) The force acting on a particle of a real spring-mass system kept inside water (shown in Fig. 6.3b). Its vibration is subject to damping due to water. For such a system

$$\mathbf{F} = -k_2\mathbf{x} - k_3\mathbf{x}$$

where k_2 and k_3 are constants.

c) The force acting on a charge P due to an element $d\mathbf{l}$ of a current-carrying conductor shown in Fig. 6.3c for which

$$\mathbf{F} = \frac{k_1 (\mathbf{v} \cdot \hat{\mathbf{r}}) d\mathbf{l} - (\mathbf{v} \cdot d\mathbf{l})\, \hat{\mathbf{r}}.}{r^2}$$

where $\mathbf{v}$ is the velocity of the charge and k_1 is a constant depending on the magnitude of the current and the nature of the medium.

Now that you know what a central conservative force is, let us find out the equation of motion for a particle of mass m moving under its influence. From Newton's second law it is given as

$$m\mathbf{a} = f(r)\,\hat{\mathbf{r}} \tag{6.5}$$

We find that the study of motion under central conservative forces is much simplified because it has certain general properties. Let us first discuss these properties.

6.2.1 Properties of Motion under Central Conservative Forces

The central force is directed along $\hat{\mathbf{r}}$. So, the torque on the particle about the centre of force is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F\,\hat{\mathbf{r}} = \mathbf{0}.$$

Angular momentum is constant

You know from Unit 4 that $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$. So for zero net torque, $\mathbf{L}$ is a constant. This means that *for motion under central force, the magnitude and direction of angular momentum is constant*. We shall now see that another interesting property arises only from the fact that the direction of angular momentum is constant.

Motion is restricted to a plane

We know from Unit 4 that $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\,\mathbf{v}$. So $\mathbf{L}$ is a vector perpendicular to $\mathbf{r}$. In other words, the vector $\mathbf{r}$ always remains in a plane perpendicular to $\mathbf{L}$. Since the direction of $\mathbf{L}$ is fixed, this plane is also fixed (Fig. 6.4).

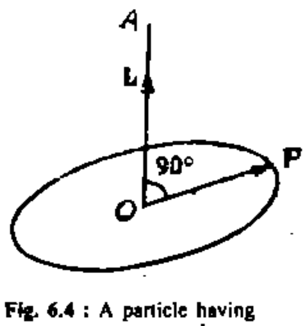


Fig. 6.4 : A particle having constant angular momentum L moves on a fixed plane perpendicular to L.

Since the motion is restricted to a plane, we can use a two-dimensional coordinate system to describe the particle's motion. Since $\mathbf{r}$ will be occurring very often in the mathematical treatment, it will be convenient to use plane polar coordinates which you have studied in Unit 4.

We have already determined the magnitude of $\mathbf{L}$ in Unit 4. From Eq. 4.25

$$L = mr^2\,\dot{\theta} \tag{6.6}$$

which is constant for a central force.

The property that angular momentum is constant for central force motion gives rise to the following law.

Law of equal areas

Refer to Fig. 6.5. Let $\mathbf{r}$ be the radius vector of a particle at a time t, executing central force

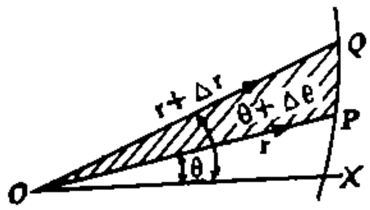


Fig. 6.5 : Area swept out by the radius vector. OX – polar axis, $OP = r$, $OQ = r + \Delta r$.

The area of a triangle can be expressed as half of the product of the length of any two sides and the sine of the angle contained between them.

motion. Let its radius vector be $\mathbf{r} + \Delta\mathbf{r}$ at time $t + \Delta t$. The polar coordinates of the particle at t and $t + \Delta t$ are (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$, respectively. The area ΔA swept out by the radius vector during the time interval Δt is shown shaded in the figure. For small values of $\Delta\theta$, the area ΔA is approximately equal to the area of the triangle OPQ, i.e.

$$\Delta A = \frac{1}{2} r (r + \Delta r) \sin \Delta\theta \simeq \frac{r}{2} (r + \Delta r) \Delta\theta, \; (\because \sin \Delta\theta \simeq \Delta\theta, \text{ for small } \Delta\theta)$$

Ignoring the term $\Delta r \Delta\theta$, we get $\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$.

Therefore, the rate at which area is swept out is given by

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} \right) = \frac{1}{2} r^2 \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \dot{\theta}.$$

From Eq. 6.6, we understand that $r^2 \dot{\theta}$ = a constant for a particle of given mass m. So $\frac{dA}{dt}$ is a constant, which gives the **law of equal areas**. It states that *for any central force the radius vector of a particle sweeps out equal areas in equal times.* Kepler's second law of planetary motion is precisely this law applied to the central force of gravitation. You will understand the physical meaning of this law better after deriving Kepler's first law.

The property that angular momentum is a constant vector holds for all central forces. Motion under central conservative forces has another property that the total mechanical energy is constant.

Total mechanical energy is constant

From Eq. 3.21 of Unit 3, you know that the total mechanical energy E for a conservative force is constant, i.e.,

$$E = \frac{1}{2} m v^2 + U(r) = \text{constant.} \qquad (6.7a)$$

The potential energy $U(r)$ is given by

$$U(r) - U(r_0) = - \int_{r_0}^{r} f(r)\, dr \qquad (6.7b)$$

where r_0 is some arbitrary reference position. Both these equations 6.7a and 6.7b apply to those central forces which are conservative.

Let us now apply the concepts that angular momentum and total mechanical energy of a particle moving under a central conservative force are constants of motion.

Example 1

A spacecraft is launched from the point A of the surface of a spherical planet of mass M having no atmosphere with a speed v_0 at an angle of 30° from the radial direction. It goes into an orbit, where its maximum distance OB from the centre of the planet is twice its radius R. Find v_0 in terms of G, R and M.

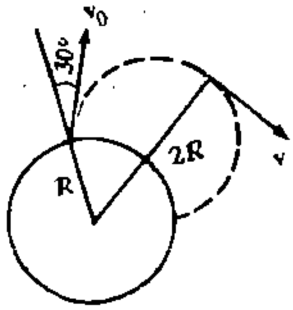


Fig. 6.6

Refer to Fig. 6.6. Let the mass of the spacecraft be m. When the spacecraft is at position $\mathbf{r}$ with respect to the centre of the planet then the force of gravitation on it is $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$.

On comparing with Eq.6.2 we realise that this is a central conservative force. This means that the spacecraft is moving under the influence of a central conservative force. Hence, its angular momentum and the total mechanical energy E are constant. We know that E = K.E + P.E. From Eq. 5.16 we also know that the P.E. of a mass m at a point at a distance r from the centre of a spherical mass M is $-\frac{GMm}{r}$. Therefore, the total mechanical energy of the spacecraft at point A on the surface of the planet is

$$E_A = \frac{1}{2} m v_0^2 - \frac{GMm}{R}.$$

The total mechanical energy of the spacecraft at point B corresponding to the maximum distance $2R$ is

$$E_B = \frac{1}{2} mv^2 - \frac{GMm}{2R}.$$

We know that $E_A = E_B$, since the total mechanical energy is constant. The conservation of angular momentum gives us another relation. Recalling that $\mathbf{L} = m\,\mathbf{r} \times \mathbf{v}$, we get for the magnitudes of angular momentum at points A and B,

$$L_A = mRv_0 \sin 30^\circ = \frac{mRv_0}{2},$$

$$L_B = mv\,(2R)\sin 90^\circ = 2mRv.$$

Since $L_A = L_B$, we get

$$v = \frac{v_0}{4}$$

Setting $E_A = E_B$ and putting $v = \frac{v_0}{4}$ in the equation, we get

$$\frac{1}{2} mv_0^2 - \frac{GMm}{R} = \frac{1}{2} m \left[\frac{v_0}{4}\right]^2 - \frac{GmM}{2R}.$$

After simplification, we get

$$v_0 = 4\sqrt{\frac{GM}{15R}}.$$

So far we have studied some general properties of motion under central conservative forces. We shall now use these properties to determine the path of a particle moving under inverse square central conservative forces. Examples of such forces are the familiar gravitational and electrostatic forces.

6.3 INVERSE SQUARE CENTRAL CONSERVATIVE FORCES

For any general inverse square central conservative force, Eq. 6.2 is expressed as

$$\mathbf{F} = \frac{k}{r^2}\,\hat{\mathbf{r}}. \tag{6.8}$$

If k is positive, then the force is repulsive and if it is negative, the force is attractive. For example, you know that the force between two like charges is repulsive and that between two unlike charges is attractive. Similarly, gravitation is an attractive inverse square force. Let us now solve the equation of motion to determine the orbit of a body moving under the influence of gravitational force of the sun. We will regard the sun to be stationary. In order to determine the orbit, we need to know $r\,(t)$ and $\theta\,(t)$, or r as a function of θ. We will now use a simple method to obtain $r\,(\theta)$.

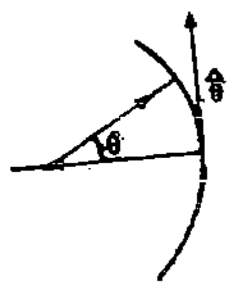


Fig. 6.7 : Motion of a body moving under the gravitational force of the sun (S). P is the position of the body at $t = 0$.

Refer to Fig. 6.7. As has been pointed out in Sec. 6.2.1, we shall be using plane polar coordinates. Let the sun be at the origin located at the centre of force. The equation of motion of the body under the influence of the gravitational attraction of the sun is given by

$$m\frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}}, \tag{6.9a}$$

where m and M are the masses of the body and the sun, respectively.

$$\therefore \quad \frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\,\hat{\mathbf{r}}, \qquad (6.9b)$$

Let us first solve this equation to obtain **v**. Then we will use Eq. 4.13a to obtain r from the expression of **v**.

Since the force is central, we have from Eq. 6.6 that $L = mr^2\,\dot{\theta}$ = a constant.

We also know from Eq. 4.10 that $\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\,\hat{\mathbf{r}}$. Using Eqs. 4.10 and 6.6 we can write Eq. 6.9b as

$$\frac{d\mathbf{v}}{dt} = \frac{GM}{r^2\dot{\theta}}\,\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{GMm}{L}\,\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{A}{L}\,\frac{d\hat{\boldsymbol{\theta}}}{dt},$$

where $A = GMm$ = a constant.

$$\text{or} \quad \frac{L}{A}\,d\mathbf{v} = d\hat{\boldsymbol{\theta}}.$$

On integrating, we get

$$\frac{L}{A}\,\mathbf{v} = \hat{\boldsymbol{\theta}} + \mathbf{C}, \qquad (6.10a)$$

where **C** is a constant vector of integration. We shall use the initial conditions to determine **C**. Let us choose the origin of time ($t = 0$) at the instant when the body is closest to the sun, i.e., r is a minimum. Thus $\frac{dr}{dt} = 0$ at $t = 0$. Again **v** (0) (i.e. **v** at $t = 0$) is in the same direction as $\hat{\boldsymbol{\theta}}$ (0) (i.e. $\hat{\boldsymbol{\theta}}$ at $t = 0$). Let $\hat{\boldsymbol{\theta}}(0) = \hat{\mathbf{n}}$. Hence, from Eq. 6.10a we get

$$\frac{L}{A}\,v(0)\,\hat{\mathbf{n}} = \hat{\mathbf{n}} + \mathbf{C},$$

$$\therefore \quad \mathbf{C} = \left(\frac{L}{A}\,v(0) - 1\right) = \hat{\mathbf{n}} = e\hat{\mathbf{n}}, \text{ say}$$

where
$$e = \frac{l}{A}\,v(0) - 1 = \text{a constant.} \qquad (6.10b)$$

Hence, from Eqs. 6.10a and b, we get

$$\frac{L}{A}\,\mathbf{v} = \hat{\boldsymbol{\theta}} + e\hat{\mathbf{n}}. \qquad (6.10c)$$

Now that we have obtained an expression for **v**, we can find r as a function of θ in a simple manner. Taking the scalar product of Eq.6.10c with $\hat{\boldsymbol{\theta}}$, we get

$$\frac{L}{A}\,\mathbf{v}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} + e\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\theta}} = 1 + e\cos\theta. \qquad (6.11)$$

We know from Eq.4.13a that $\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}$.

Since $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = 0$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = 1$, we get $\mathbf{v}\cdot\hat{\boldsymbol{\theta}} = r\,\dot{\theta} = \frac{1}{r}\,r^2\,\dot{\theta} = \frac{L}{mr}$ from Eq. 6.6.

So we get from Eq. 6.11,

$$\frac{L^2}{Am}\,\frac{1}{r} = 1 + e\cos\theta. \qquad (6.12)$$

Comparing Eq. 6.12 and Eq. A.3 of Appendix A, we get

$$p = \frac{L^2}{Am}. \qquad (6.13)$$

So we can say that the orbit of the body is a conic with its pole inside. e is called the **eccentricity**, of the conic. Now this conic can be either a parabola, a hyperbola or an ellipse depending on whether e is equal to, greater than or less than 1.

In the special case when $e = 0$, the conic is a circle.

The solution that we have obtained for the path of a body moving under the sun's gravitational field is based on some simplifying assumptions. We have assumed that the sun is stationary and that the only force acting on the body is the gravitational attraction of the sun. We know that both these assumptions are not exactly true in the real Solar System. The sun is not stationary and all other members of the Solar System also exert gravitational forces on the body. However, these forces are negligible in comparison with the gravitational attraction of the massive sun. For our Solar System containing one huge sun and a small number of little planets (called a Keplerian system) these assumptions are reasonable.

Let us now analyse what kinds of orbits (elliptical, parabolic or hyperbolic) are followed by the various bodies in the Solar System. For this we shall relate the eccentricity e to the total mechanical energy of the moving body.

Energy and eccentricity

We know that E = K.E. + P.E. (6.14a)

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}.\mathbf{v}$$

To calculate K.E. in polar coordinates we use Eq.6.10c to get

$$\text{K.E.} = \frac{m}{2}\frac{A^2}{L^2}(\hat{\boldsymbol{\theta}} + e\,\hat{\mathbf{n}})\,.\,(\hat{\boldsymbol{\theta}} + e\,\hat{\mathbf{n}}) = \frac{A^2m}{2L^2}\,(1 + 2e\cos\theta + e^2). \quad (6.14b)$$

Similarly, from Eq.5.16 we know that P.E. $= -\dfrac{GMm}{r} = -\dfrac{A}{r}$

From Eq. 6.12, $\text{P.E.} = -\dfrac{A^2m}{L^2}\,(1 + e\cos\theta)$. (6.14c)

From Eqs.6.14a, 6.14b and 6.14c we get

$$E = \frac{A^2m}{2L^2}(e^2 - 1), \quad (6.15a)$$

$$\text{or } e = \sqrt{1 + \frac{2L^2E}{A^2m}} \quad (6.15b)$$

Eqs. 6.13 and 6.15b give the values of p and e, which together determine the orbit of the body moving under the sun's gravitation. These can be calculated if we know the values of E, L and A. Although we have determined the orbit of a body moving under the sun's gravitation, these results can be applied more generally. These equations hold for every particle of mass m moving under the influence of an attractive inverse square force given by

$$\mathbf{F} = -\frac{a}{r^2}\hat{\mathbf{r}}\,.$$

Before proceeding further you may like to solve an SAQ to get some practice on Eqs. 6.12 to 6.15.

SAQ 2

The elliptical orbit of a 2000 kg satellite about the earth is given by the equation

$$r = \frac{8000\text{ km}}{1 + 0.5\cos\theta}$$

Find the (a) eccentricity of the orbit; (b) angular momentum and (c) total mechanical energy of the satellite. (Note that for this problem m and M are the masses of the satellite and the earth, respectively.)

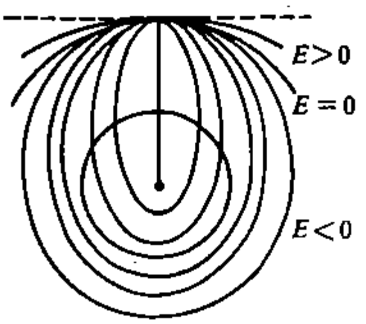


Fig. 6.8: Possible orbits under an inverse square attractive central conservative force.

Let us now consider the various kinds of orbits corresponding to different values of E.

Case 1: $E > 0$. For this $e > 1$ and the orbit is **hyperbolic**. This means that the object starts its motion at an infinite distance from the sun and slowly falls towards the sun. Its loss in P.E. appears as a gain in K.E. It passes by the sun at some minimum separation and goes away along the hyperbola, never to return (see Fig. 6.8). Some comets have been seen with hyperbolic orbits.

Case 2: $E = 0$. For zero energy, $e = 1$ and the object moves along a **parabola**. It too passes the sun once and moves away, without ever returning. Parabolic orbits are highly unlikely because $E = 0$ means that a perfect balance is required between the negative P.E. and the positive K.E.

Case 3: $E < 0$. In addition, we have to put another condition on E, namely $E \geq \frac{-A^2 m}{2L^2}$.

If $E < \frac{-A^2 m}{2L^2}$ the number under the square root in Eq. 6.15b becomes negative and no orbit is possible. For $\frac{-A^2 m}{2L^2} \leq E < 0$, we have $0 \leq e < 1$. For $0 < e < 1$, the orbit is an **ellipse** and for $e = 0$, it is a **circle**. Negative total energy means that the gravitational P.E. is always greater in magnitude than the positive K.E. The object never gains enough K.E. to escape. So it remains bound to the sun or to the centre of force forever in a closed elliptical orbit. Such is the case for all planets and the asteroids of the Solar System. Let us consider these orbits in some detail.

Orbits of planets and comets

You have already read about Kepler's laws of planetary motion in Unit 5 of Block 1. As you know Kepler had arrived at these laws on the basis of the detailed observations made by Tycho Brahe. Here we have applied Newton's laws of motion to show that a planet, comet, meteor or any heavenly body that orbits the sun must move along a conic section given by Eq. 6.12. The shape of the orbit is determined by Eqs. 6.13 and 6.15.

Table 6.1

Planet	e
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0483
Saturn	0.0560
Uranus	0.0461
Neptune	0.0100
Pluto	0.2484

In fact, Case 3 corresponds to Kepler's first law. Let us recall the law of equal areas derived in Sec. 6.2.1. When applied to planetary motion, this is Kepler's second law. Kepler observed that a planet did not orbit the sun with a constant angular speed $\dot{\theta}$. For constant $\dot{\theta}$ the law of equal areas demands that r should remain constant, i.e. the orbit should be circular. Since $\dot{\theta}$ varied, Kepler conjectured that the planetary orbits were not circular but elliptical. This turned out to be consistent with the observations. However, the eccentricities (e) of most of the planetary orbits are very small and they are very nearly circular (Table 6.1). For example, the earth's distance from the sun varies by only 3% throughout the year. From Table 6.1, you can see that the approximation that a planet's orbit is circular, made in Sec. 5.2.1, is quite justifiable.

Let us complete the discussion of planetary orbits by deriving Kepler's third law from his first and second laws.

Using the result $r^2\dot{\theta} = \frac{L}{m}$, Kepler's second law can be written as

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m}. \qquad (6.16)$$

If the time taken to complete one elliptical orbit is T, then integrating Eq. 6.16 from $t = 0$ to $t = T$, we find

$$\int_0^T \frac{dA}{dt}\, dt = \frac{L}{2m} \int_0^T dt = \frac{LT}{2m}$$

The quantity on the left side of the above equation is the area of the region enclosed by the ellipse. Now, the area of an ellipse = πab, where a is the semi-major axis and b the semi-minor axis of the ellipse. So we have,

$$\pi ab = \frac{L}{2m} T$$

$$\text{or } T^2 = \left[\frac{2m\pi}{L}\right]^2 a^2 b^2$$

We know that for an ellipse (see Eqs. A.5 to A.8 of Appendix A),

$b^2 = a^2(1-e^2)$ and $p = a(1-e^2)$. Hence, from Eq. 6.13,

$$a(1-e^2) = \frac{L^2}{A\,m}.$$

So, we get

$$T^2 = \left(\frac{2m\pi}{L}\right)^2 a^4(1-e^2) = \left(\frac{2m\pi}{L}\right)^2 \frac{L^2}{A\,m} a^3$$

$$\text{or } T^2 = \frac{4\pi^2\, ma^3}{A}.$$

Since $A = GMm$, we have Kepler's third law :

$$T^2 = \frac{4\pi^2 a^3}{GM} = ka^3, \qquad (6.17)$$

where $k = \left(\frac{4\pi^2}{GM}\right)$ depends only on the mass of the sun, and is the same for all the planets.

Kepler's third law holds not only for planetary orbits but also for elliptical orbits of the satellites of planets. For the motion of satellites, M, in Eq. 6.17, is the mass of the planet.

So we have studied the laws of planetary motion. We shall now study comets very briefly.

The motion of comets remained an enigma for a long time even after Kepler formulated the three laws. In fact, it was Isaac Newton who observed a comet in 1682 and was the first to explain its trajectory. He could see that the orbit of the comet was governed by the same principles of dynamics that applied to the motion of the planets. He realised that some comets could move past the sun in parabolic and hyperbolic orbits and so would never return. But other comets should move along the elliptical path like the planets. Only the eccentricity would be much higher. Newton's insight revealed that comets are members of the Solar System. You must be familiar with Halley's comet which returns every 76 years It has a highly elliptical orbit with $e = 0.967$.

We have seen earlier that **L** and E are constants of motion. We must also be able to determine these constants provided the geometrical features of the orbits are known. We can use the results $p = a(1-e^2) = \frac{L^2}{A\,m}$ and $A = GMm$ in Eq. 6.15a to obtain the following relation between E and a.

$$E = -\frac{GMm}{2a} \qquad (6.18)$$

You may now like to apply these results (Eq. 6.15 to 6.18) to some actual situations. So how about trying the following SAQ.

SAQ 3

a) Given that $e = 0.0167$ and $a = 1.5 \times 10^8$ km for the earth's orbit, calculate its energy and angular momentum about the sun.

b) The absolute magnitude of energy of a meteor approaching the sun is given by $|E| = \frac{G^2M^2m^3}{L^2}$ where M and m are the masses of the earth and the meteor, respectively. **L** is the angular momentum of the meteor. What are the types of the possible orbits?

So, we have taken care of the factors that determine the orbit of a planet, a comet or a satellite. Let us now try to see how the orbit of a body can be determined if some initial conditions are known.

Example 2: Calculating the orbit from initial conditions

Let us consider the example of a satellite of mass m launched from a space shuttle at a distance r_0 from the centre (C) of the earth (see Fig. 6.9). The initial velocity $\mathbf{v}_0$ of the satellite with respect to the earth and the angle of launch ϕ are given as initial conditions. What will the orbit of the satellite be?

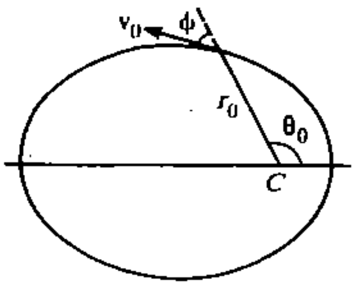


Fig. 6.9: Orbit of a satellite.

Let us first determine the energy of the satellite which is a constant. As you know the value of E will give us the shape of the orbit. It is given by

$$E = \frac{1}{2} mv_0^2 - \frac{GMm}{r_0}$$

where M is the mass of the earth. If the orbit of the satellite is to be a closed one, then $E < 0$, i.e.

$$\frac{1}{2} mv_0^2 < \frac{GMm}{r_0}.$$

$$\text{or } v_0^2 < \frac{2GM}{r_0} \quad \text{or} \quad v_0 < \sqrt{\frac{2GM}{r_0}}$$

Thus, for the satellite to be in an elliptical or a circular orbit v_0 must satisfy the above condition. The size of the orbit will be determined from the length of the major axis which is given from Eq. 6.18 as

$$2a = -\frac{GMm}{E}.$$

Similarly, the angular momentum of the satellite remains a constant equal to its initial value given by

$$L = mv_0 r_0 \sin \phi. \tag{6.19}$$

The point on a planetary orbit where the planet is nearest from the sun is called the *perihelion* and where it is farthest is called the *aphelion*.

The eccentricity of the orbit can then be found from Eq. 6.15b. Then we can find the point where the satellite is nearest from the earth (called the **perigee**) and where it is farthest (called the **apogee**). From Eq. A.6 of the Appendix A, these points are, respectively, given as

$$r_p = a(1-e),\ r_a = a(1+e). \tag{6.20}$$

This is how the shape of the orbit given by 'e', and its size given by 'a' can be found from the initial conditions. To completely specify the satellite's orbit we also need to know its orientation in space. It is specified by the line joining the focus to the perigee. We can find this line by determining the angle θ_0 between this line and the known vector $\mathbf{r}_0$ as shown in Fig. 6.9. The angle θ_0 can be found from the polar equation of the ellipse by putting the values of r_0, e, m and L, i.e.

$$r = \frac{L^2}{Am\,(1 + e\cos\theta)}$$

However, from the equation only $\cos\theta_0$ is determined. It does not tell us the sign of θ_0 which can be positive or negative depending on whether we are moving away from the perigee or approaching it. This information is obtained by considering the angle ϕ in Fig. 6.9. You can see that $\phi < 90^\circ$ when moving away from the perigee and $\phi > 90^\circ$ when approaching it. You can now apply these results to determine the orbit of an actual satellite.

SAQ 4

A satellite of mass 5,000 kg is launched in space with an initial speed of 4,000 m s^{-1} at a distance 3.6×10^7 m from the centre of the earth. It is projected at an angle of 30° with respect to the radial direction. Calculate (a) the lengths of semi-major and semi-minor axes, (b) the angular momentum, and (c) the apogee and perigee distances of the orbit.

So far we have determined the possible orbits of a body moving under an attractive inverse square force. As you know the electrostatic force between two positively charged particles is a repulsive inverse square central conservative force. What will the path of a particle acted upon by such a force be?

Orbits under a repulsive inverse square force

We can follow the same procedure that we adopted for determining the planetary orbits. But we have to replace the right hand side of Eq. 6.9a by $\frac{k}{r^2}\hat{\mathbf{r}}$, where k is a positive constant.

You can now determine such an orbit by solving the following SAQ.

SAQ 5

Show that the path followed by an alpha particle approaching an atomic nucleus is a hyperbola.

Let us now summarise what we have studied in this unit.

6.4 SUMMARY

- A central force is one which, everywhere, is directed towards or away from a fixed point called the centre of force. It is represented as

$$\mathbf{F} = F\,\hat{\mathbf{r}}.$$

- A central force whose magnitude depends only on r is also conservative. A central conservative force can be represented as :

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}}.$$

- For motion under central conservative forces, angular momentum $\mathbf{L}$ and total mechanical energy E are constant. The law of equal areas holds for such a motion which is restricted to a plane.

- The equation of an orbit for an inverse square central conservative force $\mathbf{F} = \pm\frac{k}{r^2}\hat{\mathbf{r}}$ is a conic given by

$$\frac{1}{r} = \frac{\mp 1 + e\cos\theta}{p}$$

For a repulsive force, the orbit will be a hyperbola. For an attractive force, its shape would depend on the value of e. Eccentricity depends on E and is given by

$$e = \sqrt{1 + \frac{2L^2E}{k^2m}}$$

6.5 TERMINAL QUESTIONS

1. Indicate which of the following central force fields are attractive and which are repulsive.

$$\text{(i)}\ \mathbf{F} = -4\,r^3\hat{\mathbf{r}}, \quad \text{(ii)}\ \mathbf{F} = \frac{\hat{\mathbf{r}}}{\sqrt{r}}, \quad \text{(iii)}\ \mathbf{F} = \frac{(r-1)}{r^2+1}\,\hat{\mathbf{r}}.$$

2. a) Had the force of gravitational attraction been inverse cube instead of being inverse square, which one of the three Kepler's laws would still be true?

   b) Justify the statement: The angular speed of a planet in its orbit is minimum at aphelion and maximum at the perihelion.

3. A rocket is fired from Thumba with an initial speed

$$v_0 = \frac{3}{4}\sqrt{\frac{2GM_E}{R_E}},$$

where R_E and M_E are the radius and mass of the earth, respectively.
Ignore air resistance and the earth's rotation. Consider conservation of energy and angular momentum and calculate the farthest distance it reaches from the centre of the earth if it is fired off (a) radially and (b) tangentially.

4. It is given that the eccentricity of the orbit of Halley's comet is 0.967 and the length of its semi-major axis is approximately 2.7×10^{12}m. Calculate the following for the comet :

   a) the perihelion and the aphelion distances.

   b) the period.

6.6 ANSWERS

SAQs

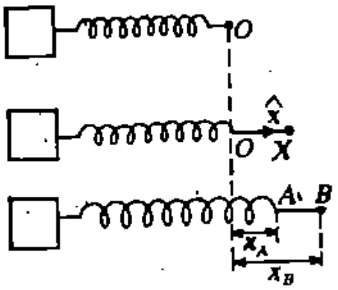


Fig. 6.10: Diagram for SAQ1a.

1. a) Since $\mathbf{F} = -k\mathbf{x}$ where $\mathbf{x} = \mathbf{OX}$ (Fig. 6.10), the force is directed towards the fixed point O. So it is central.

   The work done in stretching it from A $(x = x_A)$ to B $(x = x_B)$ is given by

$$W = \int_A^B \mathbf{F}.d\mathbf{x} = \int_{x_A}^{x_B} -kx\,\hat{\mathbf{x}}.(-dx\,\hat{\mathbf{x}})$$

or $$W = \int_{x_A}^{x_B} kx\,dx = \frac{k}{2}(x_B^2 - x_A^2)$$

   So the work done is dependent only on the initial and final positions. Hence, the force is also conservative.

   b) This force is again directed towards a fixed point and hence it is central. However, from the working of part (a), it is evident that the work done in taking the particle from $x = x_A$ to $x = x_B$ will not only depend on the initial and final positions of the path, it would also depend on the velocity. So the force is not conservative.

   c) The given force can be expressed as $\mathbf{F} = m\,d\mathbf{l} - n\,\hat{\mathbf{r}}$ where m and n are scalars ($\because$ $\mathbf{v}.\mathbf{r}$ and $\mathbf{v}.d\mathbf{l}$ are scalar quantities). It is evident from the expression of $\mathbf{F}$ that it is not along $\hat{\mathbf{r}}$ as $m \neq 0$ in general. Hence, it is not central.

2. On comparing the given equation with Eq. 6.12, we get

   a) $e = 0.5$ and

   b) $\frac{L^2}{Am} = 8000 \times 1000\text{m} = 8 \times 10^6\text{m}$

   But $A = GMm$

$$= (6.673 \times 10^{-11}\text{Nm}^2\,\text{kg}^{-2}) \times (5.97 \times 10^{24}\,\text{kg}) \times (2000\text{kg})$$

$$= 7.97 \times 10^{17}\,\text{Nm}^2$$

   or $L^2 = (7.97 \times 10^{17}\text{Nm}^2) \times (2000\text{kg}) \times (8 \times 10^6\,\text{m})$

$$= 7.97 \times 16 \times 10^{26}\,\text{kg}^2\,\text{m}^4\,\text{s}^{-2}.$$

   or $L = 1.13 \times 10^{14}\,\text{kg m}^2\,\text{s}^{-1}$

   c) From Eq. 6.15 a, $E = \frac{A^2m}{2L^2}(e^2 - 1)$

$$= \frac{(7.97 \times 10^{17})^2\,\text{N}^2\,\text{m}^4 \times (2000\text{kg})}{2 \times (1.13 \times 10^{14})^2\,\text{kg}^2\,\text{m}^4\,\text{s}^{-2}} \times \left(-\frac{3}{4}\right) = -3.73 \times 10^{10}\text{J}.$$

3. a) From Eq. 6.18, $E = -\frac{GMm}{2a}$

$$= -\frac{(6.673 \times 10^{-11}\text{Nm}^2\,\text{kg}^{-2}) \times (1.99 \times 10^{30}\,\text{kg}) \times (5.97 \times 10^{24}\,\text{kg})}{2 \times 1.5 \times 10^{11}\text{m}}$$

$$= -2.6 \times 10^{33}\text{J}$$

Since from Eq. 6.13 $p = \frac{L^2}{Am}$ and also $p = a(1-e^2)$ (See Eq. A.5 of Appendix A.) we get

$$\frac{L^2}{Am} = a(1-e^2). \qquad \text{and as} \quad A = GMm$$

$$L^2 = GMm^2 a (1-e^2).$$

For the earth's orbit $e = 0.0167$

$$\therefore L^2 = (6.673 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}) \times (1.99 \times 10^{30} \text{kg}) \times (5.97 \times 10^{24} \text{ kg})^2$$

$$\times (1.5 \times 10^{11} \text{m}) \times (0.9997) = 7.1 \times 10^{80} \text{ kg}^2 \text{ m}^4 \text{ s}^{-2}$$

$$\therefore L = 2.7 \times 10^{40} \text{ kg m}^2 \text{s}^{-1}$$

b) From Eq.6.15b, $e = \sqrt{1 + \frac{2L^2E}{A^2m}}$.

Since, the absolute magnitude of E is given, it can be positive as well as negative.

If $E > 0$, $e = \sqrt{1 + \frac{2L^2}{A^2m}\left(\frac{G^2M^2m^3}{L^2}\right)} = \sqrt{3}$, so the orbit is a hyperbola.

If $E < 0$, $e = \sqrt{1-2} = \sqrt{-1}$, so orbit is not possible.

Using Eq. 6.18, we get

$$a = -\frac{GMm}{2E} \qquad (6.21)$$

Now, we know that the initial energy is

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{r_0},$$

where $r_0 = 3.6 \times 10^7$m, $v_0 = 4000 \text{ m s}^{-1}$, $m = 5000$ kg and M is the mass of the earth. Putting these values alongwith those of G and M, we get $E = -1.5 \times 10^{10}$J.

Putting this value in Eq. 6.21, we get $a = 6.6 \times 10^7$m.

The angular momentum $L = m v_0 r_0 \sin\phi$ (where $\phi = 30°$). Putting the values of m, v_0, r_0 and ϕ, we get $L = 3.6 \times 10^{14} \text{ kg m}^2 \text{s}^{-1}$.

Once we know both E and L, we can use Eq. 6.15b to calculate e.

$$e = \sqrt{1 + \frac{2L^2E}{A^2m}} \cdot \text{ where } A = GMm.$$

Putting the values of L, E, G, M and m, we get $e = 0.9$. $\therefore b = 2.9 \times 10^7$m

The apogee and perigee distances are given by

$$r_a = a(1+e) = 1.3 \times 10^8 \text{m},$$

$$r_p = a(1-e) = 6.6 \times 10^6 \text{m}.$$

The orbit is shown in Fig. 6.11.

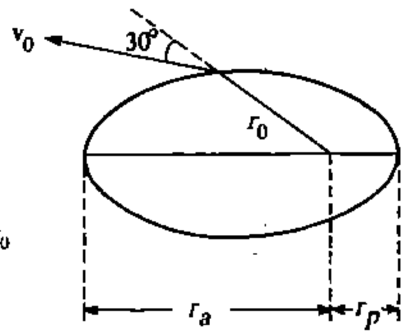


Fig. 6.11: Orbit of the satellite of SAQ 4.

5. Refer to Fig. 6.12. Let the charges on the positively charged nucleus N and the alpha-particle A be q_1 and q_2, respectively. Let the mass of alpha-particle be m. So, as we have obtained Eq. 6.9a, we can get here,

$$m \frac{d\mathbf{v}}{dt} = C \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}.$$ where C is a constant dependent on the nature of the medium.

$$\text{or} \quad \frac{d\mathbf{v}}{dt} = \frac{k}{r^2} \hat{\mathbf{r}}, \qquad (6.22)$$

where $k = \frac{C}{m} q_1 q_2 =$ a positive constant.

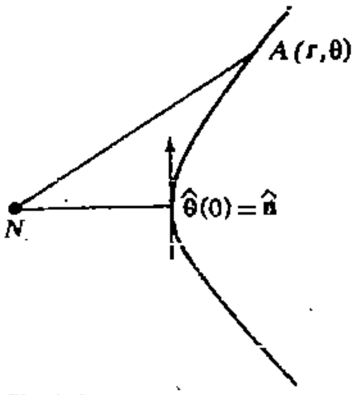


Fig. 6.12 : An alpha particle approaching a nucleus.

You may recall from Eq. 4.10 of Unit 4 that $\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{\mathbf{r}}$, or $\hat{\mathbf{r}} = -\frac{1}{\dot{\theta}}\frac{d\hat{\theta}}{dt}$

Thus, $$\frac{d\mathbf{v}}{dt} = -\frac{k}{r^2\dot{\theta}}\frac{d\hat{\theta}}{dt} = -\frac{km}{L}\frac{d\hat{\theta}}{dt} \quad (\because L = mr^2\dot{\theta})$$

$$\text{or } d\mathbf{v} = -\frac{km}{L}\,d\hat{\theta}$$

$$\mathbf{v} = -\frac{km}{L}\hat{\theta} + \mathbf{C}_1, \qquad (6.23)$$

where $\mathbf{C}_1$ is a constant vector of integration. We shall now determine the constant $\mathbf{C}_1$. For this we shall follow the same procedure as we did for determining **C** in the problem of planetary orbits. So looking back at the few steps worked out after Eq. 6.10a you will realise that $\mathbf{v}(0)$ and $\hat{\theta}(0)$ are in the same direction. Let $\hat{\theta}(0) = \hat{\mathbf{n}}$ then $\mathbf{v}(0) = v(0)\,\hat{\mathbf{n}}$.

Hence, from Eq. 6.23, we get

$$v(0)\,\hat{\mathbf{n}} = -\frac{km}{L}\hat{\mathbf{n}} + \mathbf{C}_1,$$

$$\mathbf{C}_1 = \left[v(0) + \frac{km}{L}\right]\hat{\mathbf{n}} = k_1\,\hat{\mathbf{n}}, \text{ where } k_1 \text{ is a positive constant.}$$

$$\therefore\ \mathbf{v} = -\frac{km}{L}\hat{\theta} + k_1\hat{\mathbf{n}}.$$

Taking dot product with $\hat{\theta}$ on both sides and using $\mathbf{v}\cdot\hat{\theta} = r\dot{\theta}$, $\hat{\theta}\cdot\hat{\theta} = 1$ and $\hat{\mathbf{n}}\cdot\hat{\theta} = \cos\theta$, we get

$$r\dot{\theta} = -\frac{km}{L} + k_1\cos\theta$$

$$\text{or } \frac{L}{mr} = -\frac{km}{L} + k_1\cos\theta \quad [\because L = mr^2\dot{\theta}]$$

$$\text{or } \frac{1}{r} = -\frac{km^2}{L^2} + \frac{mk_1}{L}\cos\theta$$

$$\text{or } \frac{1}{r} = \frac{km^2}{L^2}\left[\frac{Lk_1}{mk}\cos\theta - 1\right]. \qquad (6.24)$$

Eq. 6.24 can be compared with Eq. A.9 of Appendix A

$$\frac{1}{r} = \frac{e\cos\theta - 1}{p}, \text{ where } p = \frac{L^2}{km^2},\ e = \frac{Lk_1}{mk}$$

which is the equation of a conic with pole outside.

Such a conic can only be a hyperbola. Hence, the orbit is hyperbolic, with the nucleus being the pole, i.e. the focus.

Terminal questions

1. (i) The negative sign indicates that the force is directed towards the centre of force and hence it is attractive.

   (ii) The positive sign on the right-hand side indicates that the force is directed away from the centre and hence it is repulsive.

   (iii) The force is attractive for $0 < r < 1$ and repulsive for $r > 1$. It vanishes at $r = 1$.

2. a) So long as the force is central, we have from Sec. 6.2.1 that

$$\tau = 0 \text{ or, } \frac{d\mathbf{L}}{dt} = 0, \text{ i.e. } \mathbf{L} = \text{a constant vector.}$$

   We have seen that the law of equal areas follows from the constancy of angular momentum vector. Kepler's second law is precisely the law of equal areas. So Kepler's second law will still be true.

b) From the law of equal areas we know that $r^2\dot{\theta}$ = a constant for all the planets. At aphelion r is maximum, so $\dot{\theta}$ is minimum. And at perihelion r is minimum, so $\dot{\theta}$ is maximum. So a planet moves faster as it approaches the sun and slower when it moves far away from the sun.

3. (a) Refer to Fig. 6.13. The total mechanical energy of the rocket at point P on the surface of earth is

$$E_P = \frac{1}{2} m v_0^2 - \frac{GM_E m}{R_E} = -\frac{7}{16} \frac{GM_E m}{R_E} .$$

The total mechanical energy of the rocket at point A corresponding to the maximum distance a is

$$E_A = \frac{1}{2} m v_A^2 - \frac{GM_E m}{a} .$$

where v_A = the velocity of the rocket at A.

We know from the principle of conservation of energy that $E_P = E_A$,

$$\text{i.e.} \quad \frac{1}{2} m v_A^2 - \frac{GM_E m}{a} = -\frac{7}{16} \frac{GM_E m}{R_E} . \qquad (6.25)$$

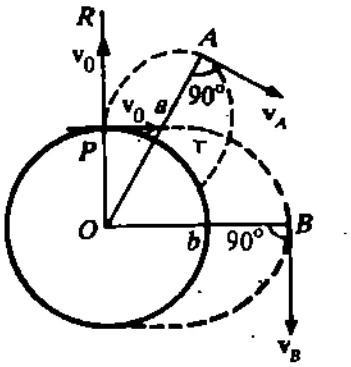


Fig. 6.13: Trajectories of the rocket. PR and PT show, respectively, the radial and tangential directions of firing the rocket.

We know that $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$. At P, $\mathbf{r} = \mathbf{OP}$ which is parallel to $\mathbf{v}_0$. So the magnitude of angular momentum at P is zero. Hence $L_P = 0$. Again $L_A = mav_A$ ($\because \mathbf{v}_A$ is perpendicular to the radial direction **OA**). Now since, $L_P = L_A$, $v_A = 0$.

Hence, we get from Eq. 6.25 that $a = \frac{16}{7} R_E$.

b) The total mechanical energy of the rocket at point B corresponding to the maximum distance b is

$$E_B = \frac{1}{2} m v_B^2 - \frac{GM_E m}{b} ,$$

where $\mathbf{v}_B$ = the velocity of the rocket at B . Now $E_P = E_B$.

$$\frac{1}{2} m v_B^2 - \frac{GM_E m}{b} = -\frac{7}{16} \frac{GM_E m}{R_E} . \qquad (6.26)$$

Now, at P, $\mathbf{r}$ is perpendicular to $\mathbf{v}_0$. $\therefore L_P = mR_E v_0$.

Again $L_B = mbv_B$. Since $L_P = L_B$, we get $v_B = \frac{R_E v_0}{b}$.

$$v_B^2 = \frac{R_E^2 v_0^2}{b^2} = \frac{R_E^2}{b^2} \frac{9}{16} \frac{2GM_E}{R_E} = \frac{9}{8} \frac{GM_E R_E}{b^2} .$$

So, we get from Eq. 6.26, that

$$\frac{9}{16} \frac{GM_E m R_E}{b^2} - \frac{GM_E m}{b} = -\frac{7}{16} \frac{GM_E m}{R_E}$$

or $$\frac{9R_E}{16b^2} - \frac{1}{b} = -\frac{7}{16R_E}$$

or $$9R_E^2 - 16bR_E = -7b^2$$

or $$7b^2 - 16bR_E + 9R_E^2 = 0$$

or $$(7b - 9R_E)(b - R_E) = 0$$

But $b \neq R_E$, so $b = \frac{9R_E}{7}$

4. a) The perihelion and aphelion distances are given by

$$r_p = a(1 - e) \text{ and } r_a = a(1 + e) .$$

For Halley's comet $a = 2.7 \times 10^{12}$m and $e = 0.967$. Hence we get

$$r_p = 8.9 \times 10^{10}\text{m}, r_a = 5.3 \times 10^{12}\text{m}.$$

b) From Eq. 6.17, we get

$$T = 2\pi\sqrt{\frac{a^3}{GM}}\,.$$

where M is the mass of the sun. So on putting the values of a, G and M, we get

$$T = 2\pi\sqrt{\frac{(2.7\times10^{12})^3\ \text{m}^3}{(6.673\times10^{-11}\ \text{Nm}^2\,\text{kg}^{-2})\times(1.99\times10^{30}\ \text{kg})}}$$

$$= 2.42\times10^9\text{s} = 76.7\ \text{years.}$$

UNIT 7 MANY-PARTICLE SYSTEMS

Structure

7.1 INTRODUCTION

So far you have studied the motion of single particles. In Unit 6 we did take up the example of a planet moving in the sun's gravitational field. However, we assumed that the sun was at rest. You may have wondered as to why only the planet moves due to their mutual gravitational attraction. Should not the sun also move ? Indeed, as we shall find in this unit the sun also has a motion. Then why did we neglect it in Unit 6 ? We can answer this question if we analyse the motion of the two-body system of the sun and the planet.

In this unit we shall first study the motion of two bodies moving under the influence of their mutual interaction force. We shall, of course, be applying the basic concepts of mechanics to this system. In addition, you will learn the concepts of the motion of centre-of-mass and the relative coordinates and apply them to two-body systems. We shall then determine the other dynamical variables like the linear and the angular momenta and the K.E. of each system.

We shall next extend these concepts to study the motion of many-particle systems. The Solar System made up of planets and their satellites, asteroids and comets is one such system. Gas filled in a cylinder is also a many-particle system if its molecules can be regarded as point masses in a given problem. Objects such as exploding stars, an acrobat, a javelin thrown in air, a cup of tea, a planet, a car, a ball are all systems composed of many particles. In some systems, e.g. a solid metallic sphere the distances between the particles remain fixed. We shall study the motion of such systems in Unit 9. In other systems the constituent particles move with respect to one another. In this unit you will learn the basic concepts needed to understand these more complex and realistic systems. However, predicting the motion of even more complicated many-particle systems, such as air masses that determine earth's weather, is still very difficult. We need supercomputers to apply these concepts to such systems.

In the next unit we shall use the concepts of mechanics to study the phenomenon of scattering.

Objectives

After studying this unit you should be able to

- define the centre-of-mass and relative coordinates, and reduced mass
- solve problems involving motion of two-body systems
- derive and explain the physical significance of the expressions of linear and angular momenta and K.E. of a many-particle system.

7.2 MOTION OF TWO-BODY SYSTEMS

The motion of a planet around the sun is an example of a two-body motion. In Unit 6 we had approximated this motion as a one-body motion around a stationary sun, for reasons you will study in this section. However, when the masses of the two bodies are comparable, such an approximation cannot be made. Such is the case for the earth-moon system or the system of two charges. For these systems we need to solve the equation of motion of both the bodies moving under each other's influence. In this section we will study a method of solving these equations.

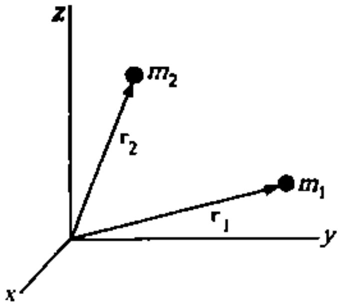


Fig. 7.1: A two-body system

Let us consider the motion of a system of two particles 1 and 2 of masses m_1 and m_2, respectively. Let their position vectors be $\mathbf{r}_1$ and $\mathbf{r}_2$ at time t with respect to an origin O in an inertial frame of reference (Fig. 7.1). We will study the case when no external force acts on the system. The only forces responsible for their motion are the mutual action and reaction forces. For example, planets interact via gravitational attraction and molecules interact via inter-molecular forces. Two charged bodies carrying like charges repel each other. In all these cases no external force acts on the systems.

Let the force on 1 due to 2 be $\mathbf{F}_{21}$, then the force on 2 due to 1 is $\mathbf{F}_{12}$ (= $-\mathbf{F}_{21}$, from Newton's third law of motion). The equations of motion for the two particles are

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21}. \tag{7.1a}$$

$$m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{7.1b}$$

We need to solve these two differential equations in order to determine the path of the two particles. However, we can reduce these two equations to a single differential equation of motion. We will use another set of coordinates, namely the centre-of-mass and relative coordinates to arrive at that single equation of motion.

7.2.1 Equation of Motion in Centre-of-mass and Relative Coordinates

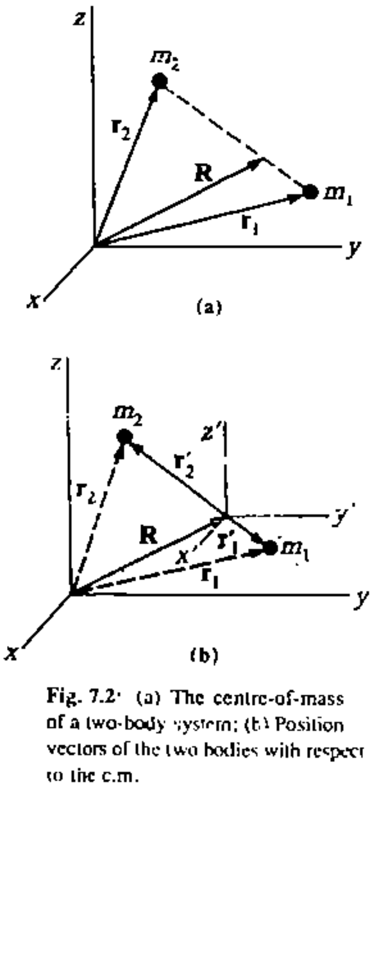


Fig. 7.2: (a) The centre-of-mass of a two-body system; (b) Position vectors of the two bodies with respect to the c.m.

Refer to Fig. 7.2a. We define the position of the centre-of-mass (c.m.) of this system to be

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{7.2}$$

$\mathbf{R}$ is referred to as the **centre-of-mass coordinate.** The relative coordinate of m_1 with respect to m_2 is defined as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{7.3}$$

The position vectors (Fig. 7.2b) of the particles with respect to the c.m. are given by

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\,\mathbf{r}. \tag{7.4a}$$

$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\,\mathbf{r}, \tag{7.4b}$$

where $M = m_1 + m_2$.

Eq. 7.2 defines the centre-of-mass coordinate which together with the relative coordinate of Eq. 7.3 constitutes a new coordinate system to study the two-body motion. Let us now express the equations of motion in terms of these coordinates.

Adding Eqs. 7.1a and 7.1b, we get

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = 0,$$

$$\text{or} \quad \frac{d^2}{dt^2}\,(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) = 0.$$

From Eq. 7.2, we get

$$\frac{d^2}{dt^2}\,\{(m_1 + m_2)\,\mathbf{R}\} = 0,$$

or $$M \frac{d^2\mathbf{R}}{dt^2} = \mathbf{0}. \tag{7.5a}$$

Again from Eqs. 7.1a and 7.1b, we get

$$\ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{21}}{m_1}, \; \ddot{\mathbf{r}}_2 = -\frac{\mathbf{F}_{21}}{m_2},$$

or $$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \mathbf{F}$$

From Eq. 7.3 we can see that $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$.

$$\therefore \quad \ddot{\mathbf{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \mathbf{F}_{21}. \tag{7.5b}$$

Let us now introduce a quantity μ such that

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2},$$

$$\text{i.e.} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{7.6}$$

μ is called the **reduced mass** of the system. So Eq. 7.5b becomes

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21}.$$

The equations of motion for particles 1 and 2 given by Eqs. 7.1a and 7.1b are thus equivalent to

$$M\ddot{\mathbf{R}} = \mathbf{0}, \tag{7.7}$$

$$\text{and} \quad \mu\ddot{\mathbf{r}} = \mathbf{F}_{21}. \tag{7.8}$$

Let us now study the significance of these two equations.

Centre-of-mass motion

Eq. 7.7 describes the motion of the centre-of-mass. This can be integrated to give

$$M\dot{\mathbf{R}} = \text{constant}. \tag{7.9}$$

Since M is a constant, we have $\dot{\mathbf{R}}$ = a constant, i.e. the centre-of-mass moves with constant velocity. Let us now choose an inertial frame of reference which is moving with respect to the present frame with a velocity $\dot{\mathbf{R}}$. Using Eq. 1.37 of Unit 1 we find that the c.m. will be at rest in this new frame.

So we have found an inertial frame in which the c.m. is at rest. Such a frame of reference is called the **centre-of-mass frame of reference**. Its origin lies at the c.m. In this frame we need not solve Eq. 7.7. So it is very convenient to describe the motion in the c.m. frame of reference. The position vectors of 1 and 2 with respect to the c.m., are given by $\mathbf{r}_1'$ and $\mathbf{r}_2'$ as given by Eqs. 7.4a and 7.4b. Now, if we want to arrive at the solution in any other frame of reference we may use Eqs. 7.4a and 7.4b to find $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{r}_1'$, $\mathbf{r}_2'$ and $\mathbf{R}$. These may also be used for determining the velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$.

Relative motion

In the c.m. frame we have to solve only Eq. 7.8. It is the equation of motion for a single fictitious particle of mass μ moving under the force $\mathbf{F}_{21}$. If we solve this differential equation, we get $\mathbf{r}(t)$, which describes the relative motion of particle 1 with respect to the particle 2. We can also determine the paths of the two particles 1 and 2 by solving for $\mathbf{r}_1$ and $\mathbf{r}_2$ using Eqs. 7.4a and 7.4b.

So, by introducing the concept of c.m. we have reduced the task of solving two second order differential equations (7.1a and 7.1b) to solving a single equation 7.8. If we can solve this one-body problem then we can also solve the two-body problem. Thus, the motion of a two-body system is equivalent to a one-body system. All the concepts and laws concerning single particle motion which you studied in Block 1 can now be applied, once the mutual interaction force is known. If it is a central conservative force then the concepts that you have studied in Unit 6 will apply. Note that a mutual force need not always be central as you have worked out in the SAQ 1(c) of Unit 6.

You may how like to work out an SAQ.

SAQ 1

a) Verify the relations (7.4a) and (7.4b).

b) Write down Eqs. 7.1a and 7.1b when an external force alongwith the mutual forces of interaction, acts on the system. Recast these equations using the centre-of-mass and relative coordinates. Does it still reduce to an equivalent one-body problem ?

c) What happens if the external force in (b) is the force of gravity ?

Now that you have solved this SAQ, you must have realised the following fact. The reduction of two-body problem to an equivalent one-body problem is possible if no external force acts on the system. The force of gravity, of course, is an exception.

Let us now consider a system in which the mass of one particle, say m_1, is very large compared with the other, so that $\frac{m_2}{m_1} \ll 1$ as in the case of the earth and the sun. Then

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_2}{1 + \frac{m_2}{m_1}} \simeq m_2, \text{ and} \qquad (7.10\text{ a})$$

$$\mathbf{R} = \frac{\mathbf{r}_1 + \frac{m_2}{m_1}\mathbf{r}_2}{1 + \frac{m_2}{m_1}} \simeq \mathbf{r}_1. \qquad (7.10\text{ b})$$

So, the reduced mass is equal to the smaller mass. And the centre-of-mass is located almost at the position of the greater mass, which can then be regarded as fixed. The motion of the two-body system is thus equivalent to the motion of the lighter body around the heavier one. Let us consider the example of a planet orbiting the sun. In Unit 6, we should have, in principle, determined the planet's orbit by solving Eq. 7.8. Instead we regarded the sun as fixed and solved the equation of motion of the planet with respect to the sun. Can that method be criticised ? We know that the sun is much more massive than any other planet, the ratio $\frac{m_2}{m_1}$ being 2.5×10^{-4} for the most massive planet Jupiter. So you can apply Eqs. 7.10a and 7.10b, and see that the approximate method which we adopted in Unit 6 is quite valid.

However, even when one particle is very heavy, its motion should be considered and we should use Eq. 7.8. Note that if m_1 and m_2 occur in the expression of the force $\mathbf{F}_{21}$, then they should be retained as such and no replacement with μ is to be made ! Let us now work out an example to have a comparative study between the use of Eq. 7.8 and the method adopted in Unit 6 towards the analysis of the planetary motion problem.

Example 1

Write down Eq. 7.8 for the case of a two-body system comprising a planet of mass m and the sun of mass M. Hence explain how Eq. 6.17 (Kepler's third law) of Unit 6 will be modified.

Let the relative coordinate of the planet with respect to the sun be $\mathbf{r}$. Then Eq. 7.8 takes the form

$$\mu \ddot{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}},$$

where $\mu = \frac{Mm}{M+m}$.

So we get,

$$\mu \ddot{\mathbf{r}} = -\frac{GM_0\mu}{r^2}\hat{\mathbf{r}},$$ where $M_0 = M + m$ = the sum of the masses of the planet and the sun.

$$\therefore \quad \ddot{\mathbf{r}} = -\frac{GM_0}{r^2}\hat{\mathbf{r}}.$$

This is similar to Eq. 6.9b of Unit 6 with M replaced by M_0. So we can solve this equation in the same way as we did in Sec. 6.3.

Then in place of Eq. 6.17 we shall obtain the following :

$$T^2 = \frac{4\pi^2 a_0^3}{GM_0},$$

where a_0 is the semi-major axis of the relative orbit (Fig. 7.3) and $M_0 = M + m$.

So the orbital period does not depend only on the semi-major axis. It also depends on the mass of the planet. Hence, Kepler's third law is only approximately true.

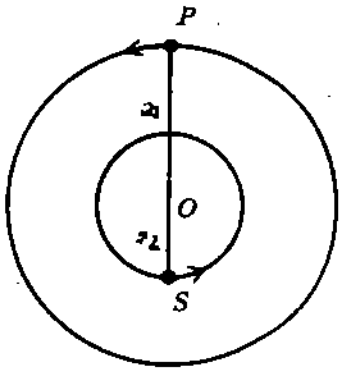


Fig. 7.3: Orbits of a planet and the sun, $OP = a_1$, $OS = a_2$, $SP = a_1 + a_2 = a_0$.

You may now like to work out an SAQ on these concepts.

SAQ 2

One of the most massive stars known at present is a binary or double star, i.e. it consists of two stars bound together by gravitation. It is known from spectroscopic studies that

(a) The period of revolution of the stars about their c.m. is 14.4 days (1.2×10^6s).

(b) Each component has a velocity of about 220 km s^{-1}. Since both components have nearly equal, but opposite velocities we may infer that they are at nearly the same distance from the centre-of-mass, and so their masses are nearly equal.

(c) The orbit is nearly circular.

From this data calculate the reduced mass and the separation of the two components.

We have thus seen that a two-body motion can be reduced to the centre-of-mass and relative motion. In such cases, various kinematical quantities, like linear and angular momenta and kinetic energy of the two bodies can also be expressed in terms of c.m. and relative coordinates. We can also say that these quantities are redistributed in the centre-of-mass and relative motion. Let us see how this is done.

7.2.2 Linear and Angular Momentum and Kinetic Energy

From Eq. 2.20 of Unit 2, the total linear momentum of the two-body system of Fig. 7.1 is given as

$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the linear momenta of 1 and 2,

$$\text{or} \quad \mathbf{p} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2. \qquad (7.11a)$$

Differentiating Eq. 7.2 with respect to time we get

$$(m_1 + m_2)\,\dot{\mathbf{R}} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2,$$

$$\text{or} \quad \mathbf{p} = M\dot{\mathbf{R}}. \qquad (7.11b)$$

According to Eq. 7.9, $M\dot{\mathbf{R}}$, i.e. $\mathbf{p}$ is a constant provided no external force acts on the system Thus, we arrive at the principle of conservation of linear momentum for a two-body system which is as follows:

The total linear momentum of a two-body system remains constant provided no external force acts on it. If the mass of the two-body system remains constant, then it leads to the following statement:

The velocity of the centre-of-mass of a two-body system remains constant provided no external force acts on it.

Let us now work out a simple example on this concept

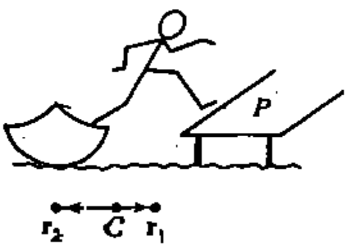


Fig. 7.4

Example 2

A 70 kg man tries to step out of a 35 kg boat, initially at rest, onto a platform P beside a lake (Fig. 7.4). What happens if he tries to step 1m sideways from the boat without holding on to the platform ?

The boat has no keel. So we can assume that the reaction of the water on the boat, sideways to it is negligible for the brief time in which the action takes place. Thus, the net external force on the two-body system (man and boat) is zero and the velocity of c.m. of the system remains constant. Before the man jumped from the boat, the c.m. of the system was at rest. Therefore, it should remain at rest, i.e.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \text{a constant},$$

where m_1, m_2 and $\mathbf{r}_1$, $\mathbf{r}_2$ are the masses and position vectors of the man and the boat, respectively.

If we select the origin of the coordinate system at the position of the c.m., as in Fig. 7.2b, then $\mathbf{R} = \mathbf{0}$, i.e.

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}.$$

Substituting the values of m_1, m_2 and $\mathbf{r}_1$ we get

$(70\text{ kg})(1\text{m})\,\hat{\mathbf{r}}_1 = -(35\text{ kg})\,\mathbf{r}_2$, where $\hat{\mathbf{r}}_1$ is the unit vector in the direction of $\mathbf{r}_1$.

$$\text{or} \quad \mathbf{r}_2 = -2\text{m}\,\hat{\mathbf{r}}_1.$$

The boat will thus move 2 m in a direction opposite to the man. So the man has to hold on to something or bring the boat nearer, otherwise he will be in danger of falling in the lake.

You may now like to work out an SAQ.

SAQ 3

Suppose in a nightmare you find yourself locked in a light cage on rollers on the edge of a cliff (Fig. 7.5) ! Assuming that no external forces act on the system consisting of you and the cage, what could you do to move the cage away from the edge ? What must you avoid doing ? If you weigh 60 kg and the cage weighs 90 kg and is 2m long, how far can you move the cage ?

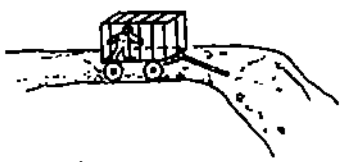


Fig 7.5

So far we have discussed the linear momentum of a two-body system. Let us now find an expression for the angular momentum of a two-body system.

The total angular momentum of the two-body system is the vector sum of the angular momentum of each body.

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$$

$$= \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$$

$$\text{or} \quad \mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2.$$

Substituting $\mathbf{r}_1$ and $\mathbf{r}_2$ from Eqs. 7.4 a and b, we get

$$\mathbf{L} = m_1\left(\mathbf{R} + \frac{m_2}{M}\mathbf{r}\right)\times \mathbf{v}_1 + m_2\left(\mathbf{R} - \frac{m_1}{M}\mathbf{r}\right)\times \mathbf{v}_2$$

$$= \mathbf{R} \times (m_1\mathbf{v}_1 + m_2\mathbf{v}_2) + \frac{m_1 m_2}{M}(\mathbf{r}\times\mathbf{v}_1 - \mathbf{r}\times\mathbf{v}_2).$$

Using Eqs. 7.3, 7.6 and 7.11 we get

$$\mathbf{L} = M(\mathbf{R}\times\dot{\mathbf{R}}) + \mu\,\mathbf{r}\times\mathbf{v},$$

$$\therefore \quad \mathbf{L} = \mathbf{R}\times M\mathbf{V} + \mu\,\mathbf{r}\times\mathbf{v}, \tag{7.12}$$

where $\quad \mathbf{V} = \dot{\mathbf{R}}$ and $\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$.

You may now try an SAQ. The first part is concerned with Eq. 7.12 and the second part is associated with the K.E. of the two-body system.

SAQ 4

a) Use Eq. 7.12 to prove that the angular momentum of a two-body system is conserved provided that no external force acts on it and they move only under their mutual interaction force which is central.

b) Express the K.E. of the two-body system as $T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$ and use the relevant equations to show that

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu v^2. \qquad (7.13)$$

You can see from Eqs. 7.11b, 7.12 and 7.13 that to obtain the values of dynamical variables in any frame from those in c.m. frame, we only need to add the contribution of a particle of mass M located at the c.m. **R**.

So far we have analysed the motion of a two-body system. We saw that the introduction of the centre-of-mass and relative coordinates made it easier to study this system. We could treat the motion of individual bodies as equivalent to the motion of one body relative to the centre-of-mass. Can we extend such an analysis to a many-particle system ? Let us find out.

7.3 DYNAMICS OF MANY-PARTICLE SYSTEMS

To begin with, let us consider a system of three particles A, B, C of masses m_1, m_2 and m_3 and position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, respectively, at a time t with respect to an origin O in a given frame of reference (Fig. 7.6). We will analyse the motion of this system and extend each result to an N-particle system. We define the position vector of the c.m. of the three-body system as

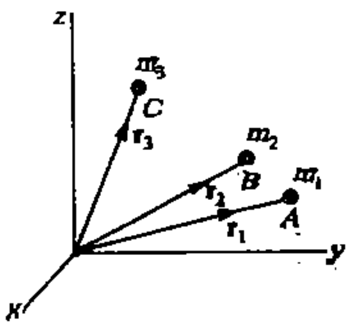


Fig. 7.6: Three-particle system

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} = \frac{\sum_{i=1}^{3} m_i \mathbf{r}_i}{M}, \qquad (7.14a)$$

where Σ, as you know, represents the sum over the three terms in the numerator of Eq. 7.14a and

$$M = m_1 + m_2 + m_3 = \sum_{i=1}^{3} m_i .$$

We also define the relative coordinate of the j^{th} particle with respect to the i^{th} particle as

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = -\mathbf{r}_{ji} . \qquad (7.14b)$$

Let us now write down the equation of motion for particle 1 of the system. In general this particle may be subjected to an external force $\mathbf{F}_{e1}$ and the mutual forces of interaction due to the other two particles in the system (Fig. 7.7).

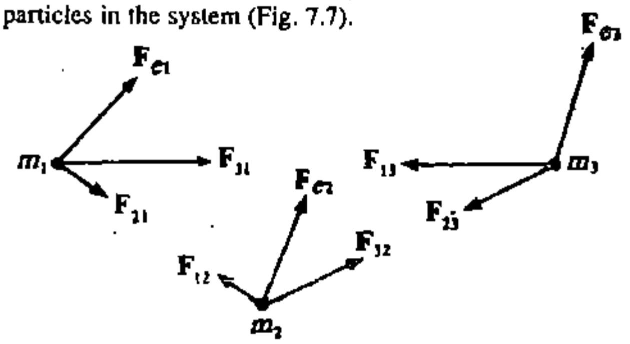


Fig 7.7: Forces acting on a three-particle system

So the net force experienced by particle 1 is

$$\mathbf{F}_1 = \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31}. \qquad (7.15)$$

Since the particle does not exert a force on itself, the term $\mathbf{F}_{11}$ does not appear in Eq. 7.15. The equation of motion for particle 1 then becomes

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31}. \qquad (7.16)$$

You can now work out the following SAQ to obtain the equations of motion for particles 2 and 3.

SAQ 5

a) Write down the relative coordinates of each particle with respect to the other for the three-particle system of Fig. 7.7.

b) Let the external forces acting on the particles 2 and 3 be $\mathbf{F}_{e2}$ and $\mathbf{F}_{e3}$, respectively. Write down the equations of motion for these two particles.

Now that you have written the equations of motion for the other two particles, let us add the equations of motion of all the three particles. This gives

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 = \; & \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31} \\ & + \mathbf{F}_{e2} + \mathbf{F}_{12} + \mathbf{F}_{32} \\ & + \mathbf{F}_{e3} + \mathbf{F}_{13} + \mathbf{F}_{23} \end{aligned}$$

Now the mutual forces of interaction between each pair of particles are equal and opposite, so that

$$\mathbf{F}_{12} = -\mathbf{F}_{21}, \; \mathbf{F}_{13} = -\mathbf{F}_{31}, \; \mathbf{F}_{23} = -\mathbf{F}_{32} \text{ and we get}$$

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3},$$

$$\text{or} \quad \sum_{i=1}^{3} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{3} \mathbf{F}_{ei} = \mathbf{F}_e, \tag{7.17a}$$

where $\mathbf{F}_e$ is the net external force acting on the system. Now if we differentiate Eq. 7.14a twice with respect to time, we obtain $M\ddot{\mathbf{R}} = \sum_{i=1}^{3} m_i \ddot{\mathbf{r}}_i$, provided m_1, m_2, m_3 are constant. Eq. 7.17a then becomes

$$M\ddot{\mathbf{R}} = \mathbf{F}_e. \tag{7.17b}$$

This is the equation of motion of a single particle of mass M situated at $\mathbf{R}$ under an external force $\mathbf{F}_e$. So, the introduction of the centre-of-mass allows us to apply Newton's second law to the entire system rather than to each individual particle. As far as its overall motion is concerned, the system acts as if its entire mass were concentrated at the centre-of-mass. However, using Eqs. 7.17a and b we cannot obtain a general analytical solution for the individual motion of the three bodies.

However, we can use the concept of the centre-of-mass to explain many important features of the motion of a three-body system. We will illustrate this with the three-body system of the earth, moon and the sun.

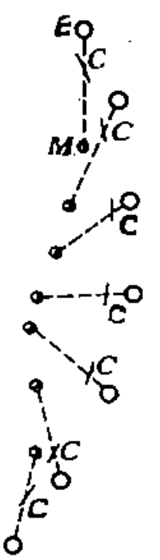


Fig. 7.8: The earth and the moon execute elliptical motion about their c.m.

Example 3: A three-body system—earth, moon and sun

Recall your study of the two-body problem. If we consider the earth-moon (E.M.) system only, then both the bodies would execute elliptical motion about their centre-of-mass (Fig. 7.8). Let us see what happens when we include the sun in the system. The c.m. of the earth-moon-sun system lies at

$$\mathbf{R} = \frac{M_e\mathbf{R}_e + M_m\mathbf{R}_m + M_s\mathbf{R}_s}{M_e + M_m + M_s}, \tag{7.18}$$

where M_e, M_m, M_s are the masses and $\mathbf{R}_e, \mathbf{R}_m, \mathbf{R}_s$ the position vectors of the earth, moon and the sun, respectively. Dividing the numerator and denominator by M_s, we can show that $\mathbf{R} = \mathbf{R}_s$, since M_s is much larger than M_e and M_m. Thus, to a good approximation, the c.m. of this three-body system lies at the centre of the sun (Fig. 7.9a). Now, the external forces due to the gravitational attraction of other celestial bodies are negligible. So, the c.m. moves with a constant velocity. We have seen in Sec. 7.2.1 that the c.m. will be at rest in an inertial frame of reference moving with the same velocity as that of the c.m. Thus, in such an inertial frame, the sun is effectively at rest and we can use a coordinate system with its origin at the centre of the sun, so that $\mathbf{R} = \mathbf{0}$ (Fig. 7.9b). Then we need to consider the motion of the earth and the moon about the sun.

Let $\mathbf{r}_e$ and $\mathbf{r}_m$ be the positions of the earth and moon with respect to the sun. Their c.m. lies at

$$\mathbf{R}_{cm} = \frac{M_e\mathbf{r}_e + M_m\mathbf{r}_m}{M_e + M_m}$$

The external force on the earth-moon system is the gravitational attraction of the sun given as

$$\mathbf{F} = -GM_s\left(\frac{M_e}{r_e^2}\ \hat{\mathbf{r}}_e + \frac{M_m}{r_m^2}\ \hat{\mathbf{r}}_m\right)$$

The equation of motion of the c.m. is

$$(M_e + M_m)\ \ddot{\mathbf{R}}_{cm} = \mathbf{F}.$$

Now you can verify from the table of physical constants that the earth and moon are very close to each other when compared with their distances from the sun. So we can assume to a good approximation that $\mathbf{r}_e \approx \mathbf{r}_m = \mathbf{R}_{cm}$.

With this approximation the equation of motion of c.m. becomes

$$(M_e + M_m)\ \ddot{\mathbf{R}}_{cm} \approx \frac{-GM_s}{R_{cm}^2}\ (M_e\,\hat{\mathbf{r}}_e + M_m\,\hat{\mathbf{r}}_m) = \frac{-GM_s}{R_{cm}^2}\,(M_e + M_m)\hat{\mathbf{R}}_{cm}.$$

So the c.m. of the earth-moon system moves around the sun like a planet of mass $(M_e + M_m)$. Its orbit can be determined to be an ellipse by the method used in Unit 6 (See Fig. 7.10).

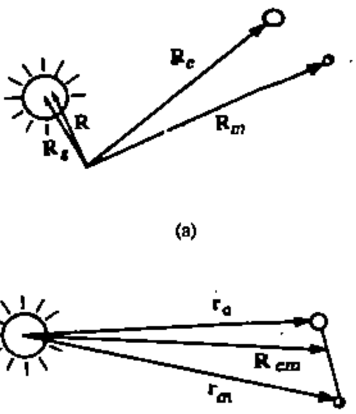


Fig. 7.9: The earth-moon-sun system.

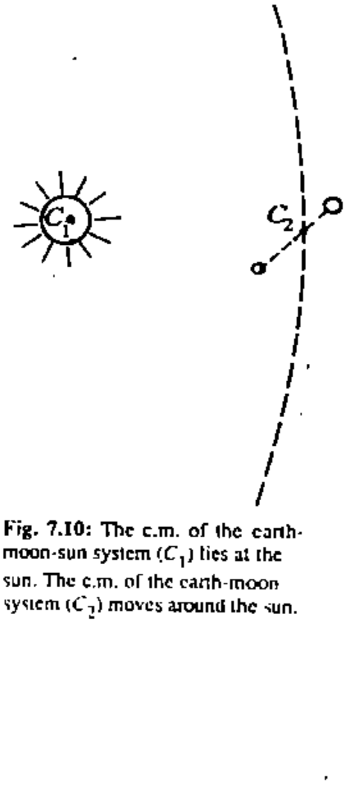


Fig. 7.10: The c.m. of the earth-moon-sun system (C_1) lies at the sun. The c.m. of the earth-moon system (C_2) moves around the sun.

N-particle system

Let us now study the motion of a system of N-particles of masses $m_1, m_2, m_3, \ldots\ldots, m_N$ located at positions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots\ldots, \mathbf{r}_N$ at an instant t with respect to an origin O. The position of the c.m. of this N-particle system is given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_N\mathbf{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i\mathbf{r}_i}{M}, \qquad (7.19)$$

where $M = \sum_{i=1}^{N} m_i$.

Let $\mathbf{F}_{e1}, \mathbf{F}_{e2}, \ldots, \mathbf{F}_{eN}$ be the external forces acting on the particles 1,2, ..., N, respectively. These particles are also subject to the mutual forces of interaction. We can now write the equations of motion for all members of the N - particle system. Each particle is subjected to an external force, and forces of interaction due to the other $(N-1)$ particles. Thus, we have

$$\begin{aligned}
m_1\ddot{\mathbf{r}}_1 &= \mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_{41} + \ldots + \mathbf{F}_{N1} + \mathbf{F}_{e1}, \\
m_2\ddot{\mathbf{r}}_2 &= \mathbf{F}_{12} + \mathbf{F}_{32} + \mathbf{F}_{42} + \ldots + \mathbf{F}_{N2} + \mathbf{F}_{e2}, \\
m_3\ddot{\mathbf{r}}_3 &= \mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{43} + \ldots + \mathbf{F}_{N3} + \mathbf{F}_{e3}, \\
m_4\ddot{\mathbf{r}}_4 &= \mathbf{F}_{14} + \mathbf{F}_{24} + \mathbf{F}_{34} + \ldots + \mathbf{F}_{N4} + \mathbf{F}_{e4}, \qquad (7.20) \\
&\ \ \vdots \\
m_N\ddot{\mathbf{r}}_N &= \mathbf{F}_{1N} + \mathbf{F}_{2N} + \mathbf{F}_{3N} + \mathbf{F}_{4N} + \ldots + \mathbf{F}_{eN}
\end{aligned}$$

Now, from, Newton's third law we have

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \quad \text{for } i = 1, 2, 3, \ldots, N \text{ and } j = 1, 2, 3, \ldots, N.$$

Thus, $\mathbf{F}_{12} = -\mathbf{F}_{21}$, $\mathbf{F}_{13} = -\mathbf{F}_{31}, \ldots, \mathbf{F}_{1N} = -\mathbf{F}_{N1}$ and so on.

Now, if we add all these equations, the terms due to mutual interaction of the particles cancel out and we get

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + \ldots + m_N\ddot{\mathbf{r}}_N = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \ldots + \mathbf{F}_{eN}. \quad (7.21a)$$

Using the summation notation we can write Eq. 7.21a in a compact form as

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_{ei} = \mathbf{F}_e \quad (7.21b)$$

where $\mathbf{F}_e$ is the net external force on the N-particle system.

These equations may appear difficult to you in the first instance. Do *not* feel scared. You don't have to memorise them. Try to understand the reasoning behind them. The following SAQ may also help you in this regard.

SAQ 6

Draw the mutual forces of action and reaction acting on each member of the four-particle system shown in Fig. 7.11. Write down the equation of motion for this system.

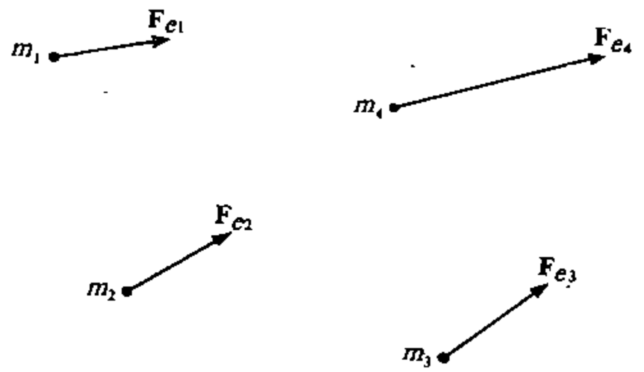


Fig. 7.11: A four-particle system

We can again differentiate Eq. 7.19 twice to obtain

$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i$ so that Eq. 7.21b becomes

$$M\ddot{\mathbf{R}} = \mathbf{F}_e, \quad (7.22)$$

which is the equation of motion of the c.m. of the system. Again, as long as we are interested only in the motion of a body as a whole, we may replace it by a particle of mass M located at the centre-of-mass. In Fig. 7.12, you can see an example of this result in action for the external force of gravity. In this case we can apply the Eq. 7.35 from the solution of SAQ 1c. The solution of this equation tells us that the centre-of-mass of a complex object follows a simple parabolic path under gravity (see Sec. 2.2.2).

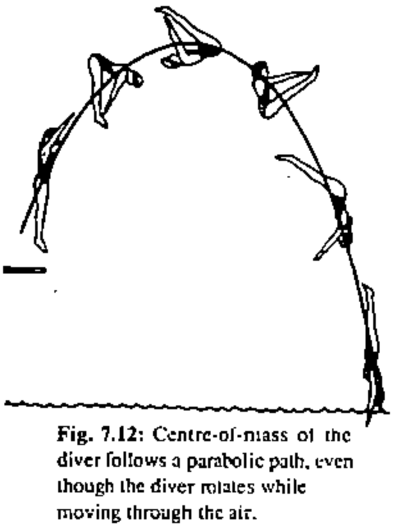


Fig. 7.12: Centre-of-mass of the diver follows a parabolic path, even though the diver rotates while moving through the air.

Let us now determine the expressions of the linear and angular momenta and the kinetic energy of an N-particle system in terms of the c.m. coordinate.

7.3.1 Linear Momentum, Angular Momentum and Kinetic Energy of an N-Particle System

We can extend Eq. 7.11 for a two-body system to express the total linear momentum of an N-particle system as

$$\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + m_3\dot{\mathbf{r}}_3 + \ldots + m_N\dot{\mathbf{r}}_N = \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i \quad (7.23a)$$

Again differentiating Eq. 7.19 with respect to time we get

$$M\dot{\mathbf{R}} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i, \text{ so that } \mathbf{P} = M\dot{\mathbf{R}}. \quad (7.23b)$$

Now, if the net external force acting on the system is zero then from Eq. 7.22 we get

$$\mathbf{P} = M\dot{\mathbf{R}} = \text{constant}. \quad (7.24)$$

This is the principle of conservation of linear momentum which can also be stated as follows:

The velocity of the c.m. of an N-particle system remains constant provided no external forces act on it.

You can now apply these ideas to work out an SAQ.

SAQ 7

Consider a system of three particles, each of mass m, which remain always in the same plane. The particles interact among themselves in a manner consistent with Newton's third law. The three particles A, B, C have positions at various times as given in Table 7.1, i.e. it shows the (x,y) components (in metres) of their position vectors at three instants.

Table 7.1

Time(s)	A	B	C
0	(1,1)	(2,2)	(3,3)
1	(1,0)	(0,1)	(3,3)
2	(0,1)	(1,2)	(2,0)

Determine whether any external forces are acting on the system.

The total angular momentum of the N-particle system about any origin O is the vector sum of the angular momenta of individual particles about that origin, i.e.

$$\mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2 + \ldots + m_N\mathbf{r}_N \times \mathbf{v}_N = \sum_{i=1}^{N} m_i\mathbf{r}_i \times \mathbf{v}_i. \quad (7.25)$$

The value of $\mathbf{L}$ depends on the choice of the origin O, just as it did for a single particle. We can express $\mathbf{L}$ in terms of $\mathbf{R}$ by subtracting and adding the quantity $\sum_i m_i \mathbf{R} \times \mathbf{v}_i$ from Eq. 7.25. Thus,

$$\mathbf{L} = \sum_{i=1}^{N} m_i (\mathbf{r}_i - \mathbf{R}) \times \mathbf{v}_i + \sum_{i=1}^{N} m_i \mathbf{R} \times \mathbf{v}_i$$

$(\mathbf{r}_i - \mathbf{R}) = \mathbf{r}_i'$, say, is the position vector of the i^{th} particle about the c.m. So $m_i\mathbf{r}_i' \times \mathbf{v}_i$ is the angular momentum of the i^{th} particle about the c.m. Thus, the first term is the sum of the angular momenta of the particles about the c.m. It can be denoted by $\mathbf{L}_{cm}$.

Since $\mathbf{R}$ is constant, from Eq. 7.23a the second term can be expressed as

$$\mathbf{R} \times \sum_{i=1}^{N} m_i\mathbf{v}_i = \mathbf{R} \times \mathbf{P}.$$

Therefore, the total angular momentum of the N-particle system can be expressed as

$$\mathbf{L} = \mathbf{L}_{cm} + \mathbf{R} \times \mathbf{P}. \quad (7.26)$$

If no net external force acts on the system, then as we have seen in Example 2, the c.m. can be taken to be at rest. So we can choose the origin of the coordinate system at the c.m., i.e. $\mathbf{R} = \mathbf{0}$. In this case the expression of $\mathbf{L}$ further simplifies to

$$\mathbf{L} = \mathbf{L}_{cm} = \sum_{i=1}^{N} m_i\mathbf{r}_i \times \mathbf{v}_i \quad (7.27)$$

Note that in the Eq. 7.27, $\mathbf{r}_i$ *is the position vector of the* i^{th} *particle with respect to the centre-of-mass.* We can make use of the expression to estimate the angular momentum of the Solar System.

Example 4: Angular momentum of the Solar System

The sun is very massive when compared with the planets. So according to Example 3, the c.m. of the Solar System is very nearly at the position of the sun. Thus, according to Eq. 7.27 the total angular momentum of the Solar System is the sum of the angular momenta of the planets and that of the sun about the centre of the sun. Let us make an estimate of the angular momentum of one of the planets, say Jupiter. Since Jupiter's orbit is very nearly circular, the magnitude of its angular momentum about the centre of the sun is

$$L_j = M_j\, \omega_j r_j^2$$

where M_j, ω_j, r_j are Jupiter's mass, angular speed and mean distance from the sun, respectively. But $\omega_j = 2\pi/T_j$, where T_j is the time period of revolution of Jupiter around the sun. Substituting the numerical values $M_j = 1.90 \times 10^{27}$ kg, $T_j = 11.9$ years, $r_j = 7.78 \times 10^{11}$m, we get

$$L_j = (1.9 \times 10^{27}\,\text{kg}) \times \left(\frac{2\pi}{(11.9)\times(365.25)\times(86400)\text{s}}\right) \times (7.78 \times 10^{11}\text{m})^2$$

$$= 1.92 \times 10^{43}\,\text{kg m}^2\,\text{s}^{-1}.$$

Likewise, the angular momenta of other planets can be estimated by assuming circular orbits.

The angular momentum of the sun about its axis is approximately 6×10^{41}kg m^{2}s^{-1}. Now, all planets move in the same sense around the sun and the sun moves in that sense about its axis. So the directions of the angular momenta of the planets and the sun are the same.

Therefore, the magnitude of the total angular momentum of the solar system about the centre of the sun is obtained by simply adding the magnitude of the planets' and the sun's angular momenta. It is 3.2×10^{43}kg m^{2}s^{-1}, which is a constant. It can be seen that a huge torque (which may act for a small duration of time) will be required to disrupt this system. You can also see that the sun's angular momentum about an axis through its centre is less than 2% of the total angular momentum of the Solar System. A typically hotter star may carry about 100 times as much angular momentum as that of the sun. Thus the process of formation of a planetary system is apparently a mechanism for carrying off angular momentum from a cooling star.

The total kinetic energy of the system of N particles is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2. \qquad (7.28)$$

We can express the total kinetic energy in terms of the c.m. coordinates. In our discussion on angular momentum we have defined the position vector of the i^{th} particle with respect to the c.m. as

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}, \quad i = 1, 2, \ldots\ldots, N. \qquad (7.29)$$

From the definition of the c.m. we get the condition

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = \sum_{i=1}^{N} m_i \mathbf{R},$$

$$\text{or} \quad m_1(\mathbf{r}_1 - \mathbf{R}) + m_2(\mathbf{r}_2 - \mathbf{R}) + \ldots + m_N(\mathbf{r}_N - \mathbf{R}) = 0$$

$$\text{or} \quad \sum_i m_i(\mathbf{r}_i - \mathbf{R}) = \sum_i m_i \mathbf{r}_i' = 0. \qquad (7.30)$$

Differentiating Eqs. 7.29 and 7.30 with respect to time, we also get

$$\mathbf{v}_i' = \mathbf{v}_i - \dot{\mathbf{R}}\ , \qquad (7.31a)$$

$$\text{and} \quad \sum_i m_i \mathbf{v}_i' = 0\ . \qquad (7.31b)$$

Substituting $(\mathbf{v}_i' + \dot{\mathbf{R}})$ for $\mathbf{v}_i$ (from Eq. 7.31a) in Eq. 7.28 we get

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i (\mathbf{v}_i . \mathbf{v}_i) = \frac{1}{2} \sum_{i=1}^{N} m_i [v_i'^2 + \dot{R}^2 + 2\mathbf{v}_i' . \dot{\mathbf{R}}]$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{R}^2 + (\sum_{i=1}^{N} m_i \mathbf{v}_i') . \dot{\mathbf{R}}$$

($\because$ $\dot{\mathbf{R}}$ is a constant independent of i.)

The last term in this expression is zero in view of Eq.7.31b. Again as $\dot{R}$ does not depend on i the second term is simply $\frac{1}{2} M\dot{R}^2$.

Hence we can express total kinetic energy as

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2 \qquad (7.32)$$

The first term in Eq. 7.32 depends on the total mass M and on the motion of the c.m.

The second term depends on the internal coordinates and velocities of the system. Eq. 7.32 implies that a certain amount of K.E. is locked up, as it was in the motion of c.m. In the absence of external forces, $\dot{R}$ remains constant and thus the first term does not change. This means that during the collision of two objects, only a certain fraction of their total K.E. is available for conversion to other purposes. Let us now consider a simple example to explain the above fact.

Example 5

Show that if a moving object of mass m_1 (=2 units) strikes a stationary object of mass m_2 (= 1 unit), then 66.7% of the initial K.E. is locked up in the motion of c.m. and only the remaining is available for the purpose of producing deformations and so on,when the objects collide.

We have from Eq. 7.32 that

$$T = \underline{\frac{1}{2} (m_1 + m_2) \dot{R}^2} + \frac{1}{2} (m_1 v_1'^2 + m_2 v_2'^2) \qquad (7.33)$$

where the underlined part is the contribution of the first term.

Here $m_1 = 2, m_2 = 1, \mathbf{v}_1 = \mathbf{v}$ (say), $\mathbf{v}_2 = 0$.

$\therefore$ $m_1 + m_2 = 3$ and $\dot{R} = \frac{2}{3} v$.

Again from Eq. 7.31a,

$$\mathbf{v}_1' = \mathbf{v}_1 - \frac{2\mathbf{v}_1}{3} = \frac{\mathbf{v}}{3} \text{ and } \mathbf{v}_2' = \mathbf{v}_2 - \frac{2\mathbf{v}_1}{3} = -\frac{2\mathbf{v}}{3}$$

$\therefore$ From Eq. 7.33,

$$T = \frac{3}{2} \times \frac{4v^2}{9} + \frac{1}{2} \left(\frac{2v^2}{9} + \frac{4v^2}{9}\right)$$

$$\text{or } T = \frac{2v^2}{3} + \frac{v^2}{3} = v^2.$$

Hence the first term is equal to 2/3 of the total. Or in other words (2/3) × 100. i.e. 66.7 per cent of the initial K.E. is locked up in the motion of c.m. and the remaining is available for conversion to other purposes.

While studying this unit you must have noticed the remarkable similarities between the results for a single particle and a many-particle system. The analogy is direct for the expression of linear momentum and equations of motion under an applied force. In the expressions for angular momentum and K.E. of a many-particle system we find an additional term for each. We are presenting these similarities in a compact form in Table 7.2.

Table 7.2

Results	Single particle	Many-particle system
Linear momentum	$\mathbf{p} = m\,\dot{\mathbf{r}}$	$\mathbf{P} = M\,\dot{\mathbf{R}}$
Equation of motion	$\dot{\mathbf{p}} = m\,\ddot{\mathbf{r}} = \mathbf{F}_e$	$\dot{\mathbf{P}} = M\,\ddot{\mathbf{R}} = \mathbf{F}_e$
Equation of motion when external force is absent	$\dot{\mathbf{p}} = 0$	$\dot{\mathbf{P}} = 0$
Angular momentum	$\mathbf{l} = \mathbf{r} \times \mathbf{p}$	$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mathbf{L}_{cm}$
Kinetic Energy	K.E. $= \frac{1}{2} m \dot{r}^2$	K.E. $= \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \sum_i m_i v_i'^2$

Let us now summarise what you have learnt in this unit.

7.4 SUMMARY

- For two bodies of masses m_1, m_2 and position vectos $\mathbf{r}_1$, $\mathbf{r}_2$, respectively, the coordinates of c.m. and the relative coordinate of m_1 with respect to m_2 are given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

- The differential equation of motion for each particle in a two-body system under their mutual interaction force can be expressed as

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad m_2\,\ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}$$

  These can be reduced effectively to a single differential equation of motion given by

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21} \quad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

  μ is known as the reduced mass of the system.

- The expressions of linear and angular momenta and K.E. of a two-body system are given by

$$\mathbf{p} = M\mathbf{V}$$

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \mu\mathbf{r} \times \mathbf{v},$$

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\mu v^2,$$

  where $M = m_1 + m_2$,

$$\mathbf{V} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2} = \text{the velocity of c.m.}$$

  $\mathbf{v} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$ = the relative velocity of m_1 with respect to m_2.

- The c.m. coordinate of an N-particle system having masses m_1, m_2, m_3, ... and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, ... is given by

$$\mathbf{R} = \sum_{i=1}^{N} m_i\mathbf{r}_i \Big/ \sum_{i=1}^{N} m_i$$

- The differential equation of motion of an N-particle system under the influence of a total external force $\mathbf{F}_e$ and mutual interaction forces is given as

$$M\ddot{\mathbf{R}} = \mathbf{F}_e \text{ where } M = \sum_{i=1}^{N} m_i$$

This indicates that the motion of the system is equivalent to the motion of its c.m. with mass M under the influence of the external force only.

- The linear and angular momenta and K.E. of an N-particle system are given by

$$\mathbf{P} = M\dot{\mathbf{R}}$$

$$\mathbf{L} = \mathbf{L}_{cm} + \mathbf{R} \times \mathbf{P},$$

where $\mathbf{L}_{cm}$ = Angular momentum of the system about the c.m. and

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2 \text{ where } \mathbf{v}_i' = \mathbf{v}_i - \dot{\mathbf{R}}$$

7.5 TERMINAL QUESTIONS

1. Two particles P and Q of masses 0.1 kg and 0.3 kg, respectively, are initially at rest 1 m apart. They attract each other with a constant force 1 N. No external force acts on the system. Describe the motion of the c.m. At what distance from P's original position do the particles collide?

2. a) Two astronauts (Fig. 7.13) each having a mass of 80 kg are connected by a light rope 8 m long. They are isolated in space, orbiting their c.m. (C) at a speed of 5 m s^{-1}. Treat the astronauts as particles and a) calculate the angular momentum and K.E. of the system.

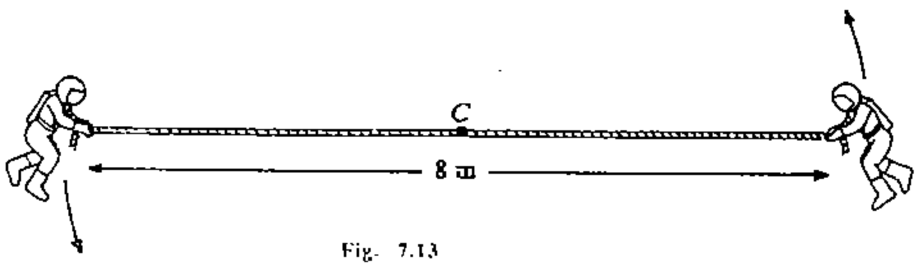


Fig. 7.13

By pulling the rope, the astronauts move closer to each other and their separation becomes 4 m.

b) What is the present angular momentum of the system ?

c) What are their new speeds ?

d) Does the K.E. of the system remain the same as that in case (a) ?

3. Two identical balloons are joined by a thin membrane (Fig. 7.14). Initially one is filled with gas while the other is in a collapsed state. The mass of the material of the balloons is negligible in comparison to the mass of the gas. At a certain instant the membrane ruptures, allowing the gas to fill the balloons equally. Assuming that there is no friction and that only horizontal motion can occur, determine which way (left or right) must the balloons move.

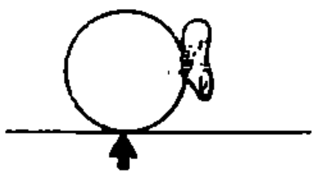


Fig. 7.14

7.6 ANSWERS

SAQs

1. a) Using Eq. 7.2, we get

$$\mathbf{r}_1' = \mathbf{r}_1 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_2(\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2} = \frac{m_2\mathbf{r}}{M}$$

and $$\mathbf{r}_2' = \mathbf{r}_2 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_2(\mathbf{r}_2 - \mathbf{r}_1)}{m_1 + m_2} = -\frac{m_1\mathbf{r}}{M}$$

b) If there is an additional external force then Eqs. 7.1a and 7.1b take the following forms :

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21} + \mathbf{F}_{e1}, \qquad (7.34a)$$

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{12} + \mathbf{F}_{e2}. \qquad (7.34b)$$

Adding Eqs. 7.34a and 7.34b and using $\mathbf{F}_{21} = -\mathbf{F}_{12}$ we get $m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_e$, where $\mathbf{F}_e = \mathbf{F}_{e1} + \mathbf{F}_{e2}$ = the net external force.

Hence using Eq. 7.2, we have

$$M\ddot{\mathbf{R}} = \mathbf{F}_e \qquad (7.35)$$

where $M = m_1 + m_2$.

Again, from Eqs. 7.34a and 7.34b, we get

$$\ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{21}}{m_1} + \frac{\mathbf{F}_{e1}}{m_1}, \quad \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{12}}{m_2} + \frac{\mathbf{F}_{e2}}{m_2}$$

or $$\frac{d^2}{dt^2}(\mathbf{r}_1 - \mathbf{r}_2) = \left[\frac{1}{m_1} + \frac{1}{m_2}\right]\mathbf{F}_{21} + \left[\frac{\mathbf{F}_{e1}}{m_1} - \frac{\mathbf{F}_{e2}}{m_2}\right] \quad (\because\ \mathbf{F}_{21} = -\mathbf{F}_{12})$$

or $$\ddot{\mathbf{r}} = \frac{\mathbf{F}_{21}}{\mu} + \left[\frac{\mathbf{F}_{e1}}{m_1} - \frac{\mathbf{F}_{e2}}{m_2}\right] \qquad (7.36)$$

So, we get two equations (7.35 and 7.36). Thus the case cannot be reduced to an equivalent one-body problem.

c) If the external force is that of gravity then $\mathbf{F}_{e1}/m_1 = \mathbf{F}_{e2}/m_2 = \mathbf{g}$. Hence Eq. 7.36 can be simplified to $\mu\ddot{\mathbf{r}} = \mathbf{F}_{21}$ which is the same as Eq. 7.8. The right-hand side of Eq. 7.35 still remains non-zero. But in this case it reduces to $\ddot{\mathbf{R}} = \mathbf{g}$ whose solution is quite well-known (see Eq. 2.9 of Block 1). It is given by $\mathbf{R} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \mathbf{B}$, where the symbols have their usual meanings. So effectively, this reduces to a one-body problem where we have to solve Eq. 7.36 only.

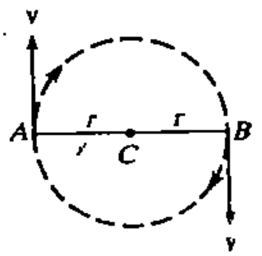


Fig. 7.15

2. Refer to Fig. 7.15. The two stars A and B are moving in uniform circular motion about their common centre-of-mass C. Let their masses be m_1 and m_2. According to condition (b) $m_1 = m_2 = m$ (say), $AC = BC = r$ and the separation between the stars = $2r$.

Now, if T be the time period of rotation of a star then

$$\frac{2\pi r}{T} = v$$

or $$2r = \frac{vT}{\pi} = \frac{(220 \times 1000\,\mathrm{m\,s^{-1}}) \times (1.2 \times 10^6\,\mathrm{s})}{\pi} = 8.4 \times 10^{10}\,\mathrm{m}$$

Our next task is to calculate the reduced mass μ :

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}, \text{ or } m = 2\mu.$$

Again using Eq.7.8 we may write that

$$\mu a = \frac{G\, m_1 m_2}{(2r)^2}, \qquad (7.37)$$

where a = the magnitude of the relative acceleration of A and B. At any instant their accelerations are directed towards C. They are equal in magnitude ($=v^2/r$) and opposite in direction. So using Eq. 1.36 of Unit 1, we understand $a = 2v^2/r$. So on putting $m_1 = m_2 = m = 2\mu$ and $v = 2\pi r / T$, and using Eq. 7.37, we get

$$\frac{G\,4\mu^2}{4r^2} = \frac{2\mu}{r}\frac{4\pi^2 r^2}{T^2}$$

$$\text{or } \mu = \frac{8\pi^2 r^3}{GT^2} = \frac{\pi^2 (2r)^3}{GT^2}$$

$$\therefore \mu = \frac{\pi^2 \times (8.4 \times 10^{10}\,\text{m})^3}{(6.673 \times 10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}) \times (1.2 \times 10^6\,\text{s})^2} = 6.1 \times 10^{31}\,\text{kg}.$$

3. As we have assumed that no external force acts on the system, the c.m. of yourself and the cage will remain at rest. Thus in order to take the cage away from the edge of a cliff you must move towards the edge. You must avoid moving towards the other side as in this case the cage will move towards the edge of the cliff.

For the final part of the problem we shall apply the equation $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \mathbf{0}$, ($\because$ the velocity of the c.m. is zero), where m_1 and m_2 are the masses of the cage and yourself, respectively. Now, your speed $v_2 = x_2/t$, where x_2 is the maximum distance that you can move in the cage, say in a time t, i.e., $x_2 = 2$m. Then $v_1 = x_1/t$, where x_1 is the maximum distance through which the cage can be moved in the opposite direction (as $\mathbf{v}_1$ must be opposite to $\mathbf{v}_2$) during the same time t. Thus

$$x_1 = \frac{m_2 x_2}{m_1} = \frac{(60\text{kg})\,(2\text{m})}{(90\text{kg})} = 1.3\text{ m}.$$

4. a) From Eq.7.12 on applying the result $\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$, we get

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{R}} \times M\mathbf{V} + \mathbf{R} \times M\dot{\mathbf{V}} + \mu\dot{\mathbf{r}} \times \mathbf{v} + \mu\mathbf{r} \times \dot{\mathbf{v}}.$$

Now since $\dot{\mathbf{R}} = \mathbf{V}$ and $\dot{\mathbf{r}} = \mathbf{v}$, the first and the third term vanish.

Again as no internal force acts on the system, the velocity of c.m. is constant and $\dot{\mathbf{V}} = \mathbf{0}$. So the second term also does not survive and we are left with

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mu\dot{\mathbf{v}}.$$

Now $\mu\dot{\mathbf{v}} = \mu\ddot{\mathbf{r}} = \mathbf{F}_{21}$. From Eq. 7.7 it is given that $\mathbf{F}_{21}$ is central. So $\mathbf{F}_{21}$ is either parallel or anti-parallel to $\mathbf{r}$. Hence the cross product of $\mathbf{r}$ with $\mathbf{F}_{21}$ vanishes.

$\therefore \frac{d\mathbf{L}}{dt} = 0$ which means that $\mathbf{L}$ is conserved.

b) $$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

We know from Eqs. 7.4a and 7.4b that

$$\mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\mathbf{r}.$$

$$\text{So } \dot{\mathbf{r}}_1 = \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}} \text{ and } \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}.$$

$$\therefore \dot{r}_1^2 = \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = \dot{R}^2 + \frac{(m_2)^2}{M^2}\dot{r}^2 + \frac{2m_2}{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}$$

$$\text{and } \dot{r}_2^2 = \dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = \dot{R}^2 + \frac{(m_1)^2}{M^2}\dot{r}^2 - \frac{2m_1}{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}.$$

$$\therefore T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{M^2}\dot{r}^2(m_1 + m_2)$$

$$\text{or } T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{M}v^2 \qquad (\because m_1 + m_2 = M,\ \dot{r} = v)$$

Now as $\mu = \frac{m_1 m_2}{M}$, we get $T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu v^2$.

a) From Eq. 7.14, we get $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2$ and $\mathbf{r}_{31} = \mathbf{r}_1 - \mathbf{r}_3$.

b) The equation of motion for particles 2 and 3 will be as follows :

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{e2} + \mathbf{F}_{12} + \mathbf{F}_{32}$$

$$\text{and} \qquad m_3\ddot{\mathbf{r}}_3 = \mathbf{F}_{e3} + \mathbf{F}_{13} + \mathbf{F}_{23}.$$

6.

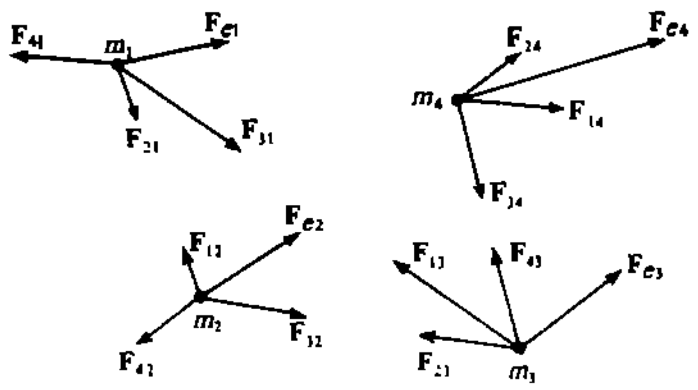


Fig. 7.16: The forces acting on each member of the four particle system

The equation of motion for this system is

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 + m_4\ddot{\mathbf{r}}_4 = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3} + \mathbf{F}_{e4}.$$

7. We shall first construct a Table (7.3) following Table 7.1 to indicate the position vectors $\mathbf{r}_A$, $\mathbf{r}_B$ and $\mathbf{r}_C$ of A, B, and C and the position vector $\mathbf{R}_t$ of the c.m. at t = 0, 1 and 2s, respectively. Remember that $m_A = m_B = m_C = m$.

Table 7.3

t	$\mathbf{r}_A$	$\mathbf{r}_B$	$\mathbf{r}_C$	$\left[\mathbf{R}_t = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C}{m_A + m_B + m_C}\right]$
0	$\hat{i} + \hat{j}$	$2\hat{i} + 2\hat{j}$	$3\hat{i} + 3\hat{j}$	$\mathbf{R}_0 = \frac{m(6\hat{i} + 6\hat{j})}{3m} = 2\hat{i} + 2\hat{j}$
1	$\hat{i}$	$\hat{j}$	$3\hat{i} + 3\hat{j}$	$\mathbf{R}_1 = \frac{m(4\hat{i} + 4\hat{j})}{3m} = \frac{4}{3}\hat{i} + \frac{4}{3}\hat{j}$
2	$\hat{j}$	$\hat{i} + 2\hat{j}$	$2\hat{i}$	$\mathbf{R}_2 = \frac{m(3\hat{i} + 3\hat{j})}{3m} = \hat{i} + \hat{j}$

Now the average velocity of the c.m. during the interval

$$t = 0 \text{ to } 1\text{s} = \frac{\mathbf{R}_1 - \mathbf{R}_0}{1 - 0} = \frac{-2}{3}(\hat{i} + \hat{j})$$

and that during the interval t = 1 to 2s $= \frac{\mathbf{R}_2 - \mathbf{R}_1}{2-1} = \frac{-1}{3}(\hat{i} + \hat{j})$.

This indicates that the velocity of the c.m. has changed. Hence, some external force has acted on the system.

Terminal Questions

1. Since there is no external force, the velocity of the c.m. remains constant. In other words, the c.m. remains at rest as its initial velocity is zero.

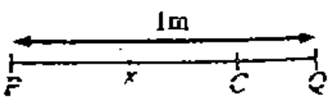


Fig. 7.17

Since the c.m. remains at rest, its position must coincide with that of the particles at the instant of their collision. So they collide at the position of their stationary c.m. C (Fig.7.17). Our task is now to obtain PC.

Let $PC = x$ m.

Then $CQ = (1-x)$ m. As C is the c.m. we have

$m_P(PC) = m_Q(CQ)$

or $(0.1 \text{ kg}) \times x \text{ m} = (0.3 \text{ kg}) \times (1 - x) \text{ m}$

or $4x = 3$.

or $x = \frac{3}{4}$, i.e. $PC = 0.75$ m.

2a) Refer to Fig. 7.13. The angular momentum vectors of the astronauts are parallel (perpendicular to the plane of the paper and pointing towards us) and equal in magnitude. The magnitude of the angular momentum of the system is given by

$$L = L_1 + L_2 = mvr + mvr = 2mvr,$$

where $m = 80$ kg, $v = 5\,\mathrm{m\,s^{-1}}$ and $r = (8/2)\mathrm{m} = 4\,\mathrm{m}$.

$$\therefore\ L = 2\,(80\,\mathrm{kg})\,(5\mathrm{ms^{-1}})\,(4\mathrm{m}) = 3200\,\mathrm{kg\,m^2 s^{-1}}$$

The K.E. of the system $= 2(\frac{1}{2}\,mv^2) = (80\,\mathrm{kg})\,(5\,\mathrm{m\,s^{-1}})^2 = 2000\,\mathrm{J}$.

b) The astronauts move close to each other due to equal and opposite internal forces that act along the line joining them. This means that the mutual force is central. And there is no external force. Hence the angular momentum of the system remains conserved, i.e. $3200\,\mathrm{kg\,m^2\,s^{-1}}$ in the same direction as that stated in (a).

c) Let V and R be the new speed and radius, respectively.

Then we have,

$$2mVR = 3200\ \mathrm{kg\,m^2s^{-1}}, \text{ where } m = 80\ \mathrm{kg} \text{ and } R = 4\mathrm{m}/2 = 2\mathrm{m}.$$

$$\therefore\ V = \frac{3200\ \mathrm{kg\,m^2s^{-1}}}{2(80\mathrm{kg})\,(2\mathrm{m})} = 10\ \mathrm{ms^{-1}}.$$

d) The new total K.E. $= 2(\frac{1}{2}mV^2) = (80\mathrm{kg})\,(10\mathrm{m\,s^{-1}})^2 = 8000\,\mathrm{J}$.

So the new K.E. is greater than that in (a).

3. Refer to Fig.7.18. Here the net external force is zero. So the velocity of c.m. remains invariant. Before the rupture (Fig.7.18a) the c.m. is at rest roughly at the centre of the gas filled balloon. So after the rupture (Fig. 7.18b) the c.m. must remain at the same position. Now, after rupture, the position of c.m. is at the meeting point of the balloons. The arrows in Fig. 7.18 indicate the positions of the c.m. before and after rupture. In order to maintain this fixed position of c.m. the balloons must move to the left.

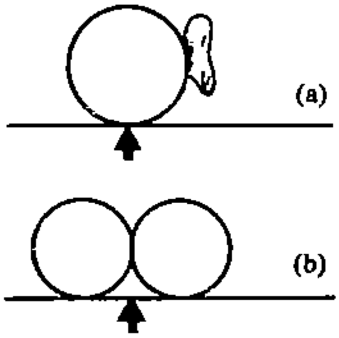


Fig. 7.18

UNIT 8 SCATTERING

Structure

8.1 INTRODUCTION

In Unit 7 you have learnt to apply the concepts of mechanics to many-particle systems. You are familiar with the phenomenon of collisions, which you have studied in Unit 3. It is also called scattering. In this unit we intend to study scattering in more detail. As you know, it involves two or more particles interacting with each other for a brief time. Collisions of particles or scattering of particles is an important feature of our physical universe. On a larger scale, we wonder if the earth's collision with an asteroid led to the extinction of dinosaurs. Galaxies also collide with each other giving rise to new formations. Much of our knowledge of atomic and nuclear structure and elementary particles comes from scattering experiments. These microscopic bodies are bombarded with microscopic particles and the number of particles scattered in various directions is measured. The angular distribution of scattered particles is expressed in terms of scattering cross-sections.

In this unit we shall begin our discussion with scattering cross-sections. The cross-sections are *calculated* in the *centre-of-mass* frame of reference but *experimentally determined* in the laboratory frame of reference. So you will study these two frames of reference and determine the relationship of the relevant physical quantities as observed from each of them. The impact parameter method makes the study of many a scattering phenomenon fairly easy. So you will learn this method and study two of its applications, namely, scattering of two hard spheres and Rutherford scattering. Rutherford scattering is one of the most dramatic scattering experiments. Performed in 1911 by Geiger and Marsden it led to the nuclear model of the atom. In Unit 9 you will learn to apply the concepts of mechanics to the rotational motion of rigid bodies.

Objectives

After studying this unit you should be able to

- distinguish between the c.m. and laboratory frames of reference
- compute differential and total scattering cross-sections in c.m. and laboratory frames of reference
- apply the impact parameter method to solve problems based on elastic scattering of two hard spheres and Rutherford scattering.

8.2 SCATTERING CROSS-SECTIONS

You already know what a collision or scattering of particles is. Recall the collision of two particles with which you are familiar (see Sec. 3.4). We can identify three distinct stages in the entire scattering process. We show these three stages of the collision process in Fig. 8.1. The first stage shown in Fig. 8.1a, corresponds to a time long before the interaction of the colliding particles. At this stage each particle is effectively free, i.e. its energy is positive. As the particles approach each other (Fig. 8.1b), interaction forces much larger than any other force acting on them come into play. Finally, long after the interaction (Fig. 8.1c), the emerging particles are again free and move along straight lines with new velocities in new directions. The emerging particles may or may not be the same as the original particles.

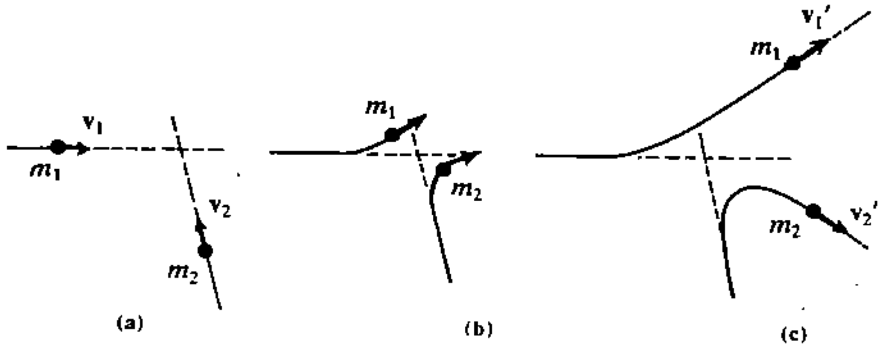


Fig. 8.1: Scattering of two particles

In a typical scattering experiment, a parallel beam of particles, also called projectiles, of given energy and momentum is incident upon a target (Fig. 8.2). The particles interact with the target for a short time, which deflects or scatters them in various directions. Eventually, these particles are detected at large distances from the target. The scattered particles may or may not have the same energies and momenta.

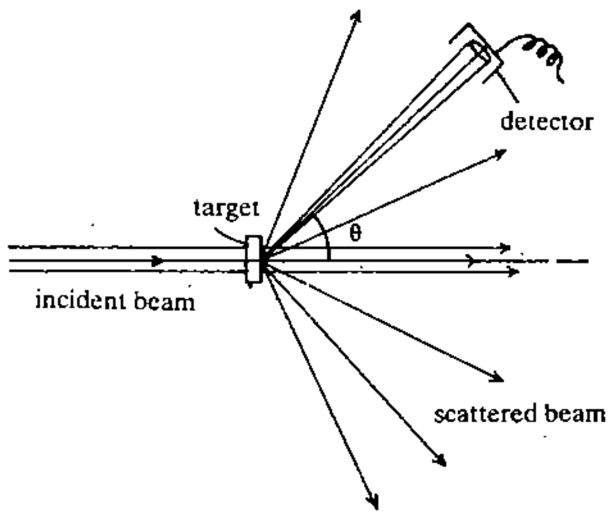


Fig. 8.2: A typical scattering process

An experimenter may be interested in knowing the velocities, linear momenta and energies of the particles before and after scattering. Then the changes brought about in these quantities can be determined. As you have studied in Unit 3, the principles of conservation of linear momentum and total energy allow us to determine these parameters. For example, James Chadwick discovered the neutron by making use of similar information about scattering of these unknown particles. When a beam of these particles was bombarded on the hydrogenous material paraffin, the protons had maximum recoil velocity of 3.3×10^{7} m s^{-1}. When these were bombarded on the nitrogenous material para-cyanogen, the maximum recoil velocity of nitrogen nuclei was 4.7×10^{6} m s^{-1}. Using the methods you have studied in Unit 3, the mass of these particles was calculated and it was found to be a totally different and new particle, the neutron.

There is another aspect of interest in scattering. We may want to know how likely a particle's motion in a given direction is, after its interaction with the target. In other words, we may want to know the probability of scattering in a given direction. This is important because it gives us information about the nature of force between the projectiles and the target, and also their internal structures. For example, the size of the electron was determined by electron-electron scattering experiments. Similarly, electron-atom scattering experiments give us information about the internal structure of the target atom, i.e. their energy levels, configurations etc. The probability of scattering in a given direction is found by determining the **scattering cross-sections**. Let us now define the scattering cross-sections for a typical scattering process.

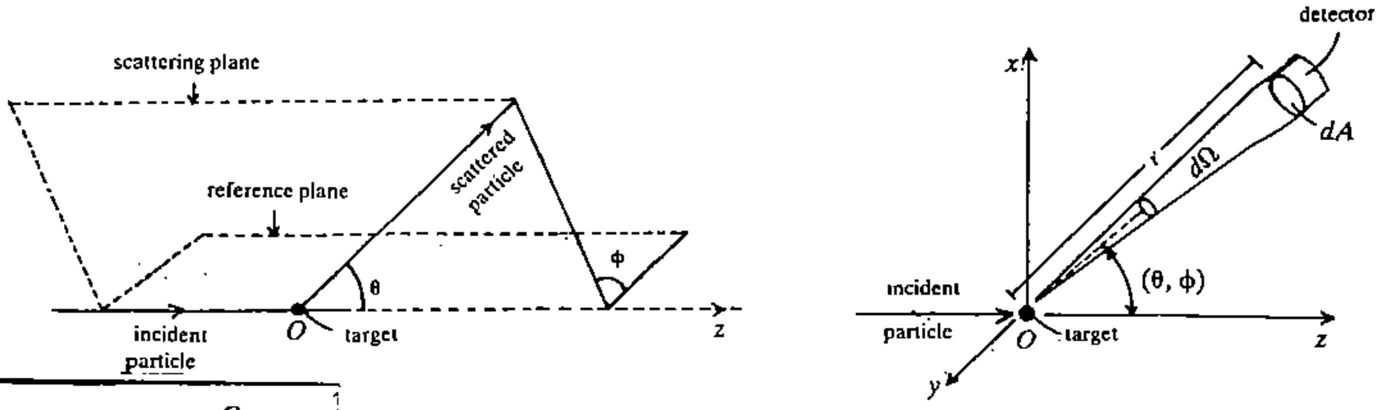


Fig. 8.3: (a) Schematic diagram of a scattering event showing the angles (θ,φ). (b) $d\Omega$ is the solid angle about the angle (θ, φ). The particles scattered into the solid angle $d\Omega$ are received by the detector. dA is the cross-sectional area of the detector.

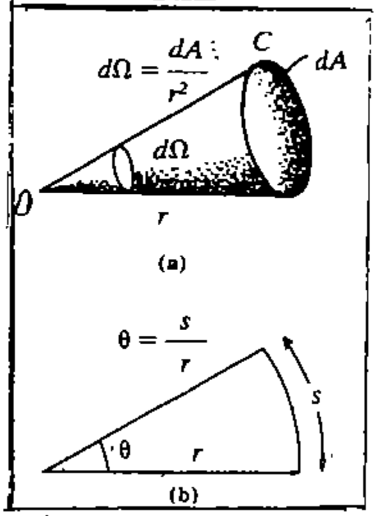


Fig. 8.4:(a) Let the surface of area dA be bounded by a closed curve C shown in the figure. The lines from the point O to the points of C generate a cone. Now visualise the area of a unit sphere about O (or the area of a sphere of radius r about O, divided by r^2). This area intercepted by the cone is called the solid angle subtended at O by the portion dA of the sphere's surface, enclosed by C. You can think of the solid angle as the space enclosed by a cone. The measure of a solid angle is defined as the ratio of the subtended area (dA) to the radius (r) squared, i.e. $d\Omega = \frac{dA}{r^2}$. Its unit is called steradian (sr). You can see that it plays the same role for a sphere as the angle (in radians) for a circle; (b) as you know an angle in the plane is the space between two intersecting lines. The measure of an angle, in radians, is defined as the ratio of the subtended arc length to the radius, i.e. $\theta = \frac{s}{r}$.

Let us suppose that a uniform parallel beam of n particles, all of the same mass and energy, is incident upon a target containing N number of identical particles or scattering centres. Such scattering centres might, for example, be the positive nuclei of atoms in a thin metal foil which could be bombarded by α particles. Let us assume that the particles in the beam do not interact with each other and the scattering centres in the target are sufficiently far apart. With these assumptions we can regard the incident particles and target particles to be sufficiently far apart. Then we can think of this scattering event as if at a given time only one projectile was being scattered by one target particle, without being affected by the presence of other particles. So, effectively at any instant we deal with a two-body collision process. For convenience we choose the origin of the coordinate system at the position of the target and one of the axes, say z-axis, in the direction of the incident beam.

The direction of scattering is given by the angles (θ,φ) as shown in Fig. 8.3a. The angle θ, called the *angle of scattering*, is the angle between the scattered and the incident directions. These two directions define the plane of scattering. The angle φ specifies the orientation of this plane with respect to some reference plane containing the z-axis. The shaded plane in Fig. 8.3a is a reference plane. The probability of the scattering of a particle in a given direction (θ,φ) is measured in terms of the differential cross-section. So let us understand what it is.

8.2.1 Differential Cross-Section

Let F be the number of projectiles incident per unit area per unit time on the target. F represents the incident flux. Let Δn be the number of particles scattered into a small solid angle $d\Omega$ about the angle (θ,φ) in time Δt(Fig. 8.3b). Study Fig.8.4a and read its caption carefully to understand what a solid angle is. Then the number of scattered particles by a single target particle in time Δt, must be proportional to the incident flux F, the duration Δt and also the solid angle in which they are scattered, i.e.

$$\Delta n \propto F\,(d\Omega)(\Delta t). \qquad (8.1a)$$

The constant of proportionality is defined as the differential scattering cross-section and is denoted by the symbol $\frac{d\sigma}{d\Omega}$. We also write it as dcs. in short. So

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right) F\,(d\Omega)\,(\Delta t), \qquad \text{or} \qquad \frac{d\sigma}{d\Omega} = \frac{\Delta n}{F\Delta t\ d\Omega} \tag{8.1b}$$

Thus, we can also express the differential cross-section as the following ratio :

$$\frac{d\sigma}{d\Omega} = \frac{\text{The number of particles scattered per unit time in a solid angle } d\Omega \text{ in the direction } (\theta,\phi)}{\text{Incident flux, i.e. the number of particles incident on the target per unit area per unit time.}}$$

Cross-section literally means the surface formed by cutting through something, especially at right angles. Areas, as you know, are associated with surfaces.

You can see that defined as a ratio like this, the differential cross-section (dcs) gives a probability. In fact, it is a measure of the probability that an incident particle will be scattered in solid angle $d\Omega$ in the direction (θ,ϕ). You can also see that $\frac{d\sigma}{d\Omega}$ has the dimension of area. This explains the use of the term 'cross-section'. Therefore, it can also be thought of as the 'effective' area offered by the scatterer to the incident particle. More precisely, $\frac{d\sigma}{d\Omega}$ is equal to the cross-sectional area of the incident beam that contains the number of particles scattered into the solid angle $d\Omega$ by a single target particle. The unit of dcs is $m^2\,sr^{-1}$. The dcs depends only on the parameters of the incident particle, nature of the target and the nature of the interaction between the two.

So far we have discussed the scattering of particles from a single scattering centre in the target. For the N scattering centres the number of particles scattered will be just N times the number scattered by a single scattering centre. Thus for N scattering centres, the number of particles scattered is

$$\Delta n' = \frac{d\sigma}{d\Omega} NF\, d\Omega\, \Delta t. \tag{8.1c}$$

Of course, Eq. 8.1c is valid only when the target scattering centres are far enough apart so that the same particle is not scattered by two of them. Having defined the differential cross-section we will introduce you to the total scattering cross-section.

8.2.2 Total Cross-Section

Let us place the detector at all possible values of (θ,ϕ) and count the total number of scattered particles entering all the corresponding solid angles. Then we will get the **total scattering cross-section** (tcs, in short). It is denoted by σ. It can also be calculated from the differential scattering cross-sections by integrating over all possible values of $d\Omega$. Thus,

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega. \tag{8.2a}$$

So the tcs represents the number of particles scattered in all directions per unit flux of incident particles. It has the dimension of area. So its unit is m^2. Now, we also define the solid angle subtended by an area to be $d\Omega = \sin\theta\, d\theta\, d\phi$, where the limits of θ and ϕ are 0 to π and 0 to 2π, respectively. You will learn about these relations in the course on Mathematical Methods in Physics-I. If you wish to understand their proofs now, you may read the last book given in the references. Using these relations we get,

$$\sigma = \int_0^{\pi} \int_0^{2\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta\, d\theta\, d\phi\, . \tag{8.2b}$$

We can show that for the cases in which the force is central and its magnitude depends only on r, $\frac{d\sigma}{d\Omega}$ is independent of ϕ. We will not prove this result here. In such cases, we can integrate over ϕ so that

$$\sigma = 2\pi \int_0^{\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta\, d\theta \cdot \tag{8.2c}$$

In the discussion that follows, we shall limit ourselves to the cases in which $\frac{d\sigma}{d\Omega}$ does not depend on ϕ, i.e. it is the same for all values of ϕ. We will now work out an example based on these concepts. Then you may like to work out an SAQ to concretise the concepts you have just studied.

Example 1

A beam of α-particles with a flux of 3×10^8 m^{-2} s^{-1} strikes a thin foil of aluminium, which contains 10^{21} atoms. A detector of cross-sectional area 400 mm^2 is placed 0.6 m from the target in a direction at right angles to the direction of the incident beam. If the rate of detection of α-particles is 8.1×10^3 s^{-1}, compute the dcs.

Here we shall use Eq. 8.1c to compute the dcs. It is given that the flux $F = 3\times10^8$m^{-2}s^{-1}, $\theta = 90°$, the rate of detection of α-particles is $\frac{\Delta n}{\Delta t} = 8.1 \times 10^3$ s^{-1} and the number of target atoms, $N = 10^{21}$. From Eq. 8.1 c

$$\frac{d\sigma}{d\Omega} = \left(\frac{\Delta n}{\Delta t}\right)\left(\frac{1}{NF}\right)\left(\frac{1}{d\Omega}\right)$$

In this case, $d\Omega$ is the solid angle subtended by the detector at the target for $\theta = 90°$. You know from Fig. 8.4a that

$$d\Omega = \frac{dA}{L^2}.$$

where dA is the area of the detector and L, its distance from the target.

$$\text{Thus } d\Omega = \frac{(400 \times 10^{-6})\text{m}^2}{(0.6\text{m})^2} = 1.1 \times 10^{-3} \text{ sr.}$$

$$\text{Therefore } \frac{d\sigma}{d\Omega} = \frac{8.1 \times 10^3\text{s}^{-1}}{10^{21} \times (3\times10^8\text{m}^{-2}\text{s}^{-1}) \times 1.1\times10^{-3}\text{sr}} = 2.4\times10^{-23} \text{ m}^2\,\text{sr}^{-1}$$

SAQ 1

A beam of neutrons is passed through paraffin. Its incident flux is 5×10^{10} m^{-2} s^{-1}. The dcs is measured to be 1.5×10^{-26} m^{2}sr^{-1} at an angle 60°. Compute the number of particles scattered per unit time by (i) a single paraffin molecule and (ii) 10^{22} paraffin molecules, into a solid angle 10^{-3} sr.

So far we have defined the dcs and tcs. We would next like to find out how these can be determined for various scattering processes. To do this we need some additional information about the cross-sections. Let us see what it is !

Experimentalists measure these cross-sections in laboratory experiments. Theoretical physicists make models of the forces of interaction and calculate these cross-sections. If the calculated values agree well with the experimental values then those models are held to be valid.

When a scattering experiment is performed in the laboratory, the target is taken to be at rest. But for calculating the cross-sections it is easier to use the frame of reference in which the c.m. is at rest because then the two-body problem can be reduced to a one-body problem (recall Sec. 7.2.1). Then we have to deal with only the relative motion of the target and the projectile. So the first question is how to compare the measured cross-sections with the calculated ones ? For this we need to define these frames of reference and determine the relationship of the cross-sections as observed or calculated in them.

8.2.3 Laboratory and Centre-of-mass Frames of Reference

In the laboratory frame of reference (Fig. 8.5a), the target particle of mass m_2 is taken to be at rest before the collision. It is taken to be situated at O, the origin of the coordinate system. Let the projectile of mass m_1 approach the target with velocity $\mathbf{u}_1$. After collision, let the two particles have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$ with respect to O at any instant t. From Eq. 7.2 the position and velocity vectors of the c.m. in the laboratory frame of reference after collision, are given by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \quad (8.3a)$$

$$\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} = \frac{m_1 \mathbf{u}_1}{m_1 + m_2}. \quad (8.3b)$$

since from conservation of linear momentum, $m_1 \mathbf{u}_1 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$.

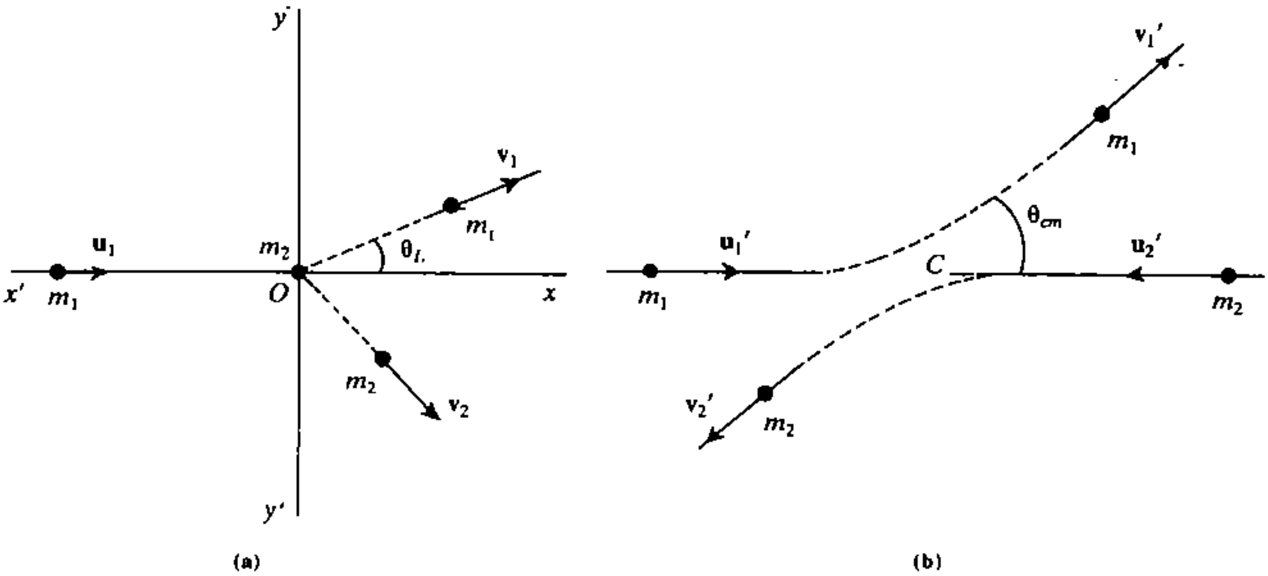


Fig. 8.5: (a) The laboratory frame of reference in which the target particle of mass m_2 is initially at rest; (b) centre-of-mass frame of reference in which the c.m. (C) is initially and always at rest.

It is convenient to study collisions using the c.m. frame of reference. As you know, in this frame, the c.m. is initially and always taken to be at rest (Fig. 8.5b). The origin of the coordinate system is located at the c.m. Since the c.m. is at rest always, its linear momentum and so the linear momentum of the entire system is zero before and after collision. Therefore, the c.m. frame of reference is also known as the **zero momentum frame of reference.** For the two particle system, let the velocities of the particles in the c.m. frame of reference be $\mathbf{u}'_1$ and $\mathbf{u}'_2$ before collision. Let their velocities after collision be $\mathbf{v}'_1$ and $\mathbf{v}'_2$. Then putting the velocity of c.m. equal to zero, we get

$$m_1 \mathbf{u}_1' + m_2 \mathbf{u}_2' = 0 = m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2', \quad (8.4a)$$

or

$$-\frac{\mathbf{u}_2'}{\mathbf{u}_1'} = \frac{m_1}{m_2} = -\frac{\mathbf{v}_2'}{\mathbf{v}_1'} \quad (8.4b)$$

Thus, the colliding particles have equal and opposite momenta before and after collision in the c.m. frame of reference. You can see from Eqs. 8.4a and b that for elastic collisions, the magnitudes of the particles' velocities will remain the same after scattering. In fact, you can work out this result yourself in the following SAQ.

SAQ 2

Show that for elastic collisions $u_1' = v_1'$, $u_2' = v_2'$ in the c.m. frame of reference. (Hint : Recall the definition of an elastic collision from Sec. 3.4 of Block 1 and use the condition of the conservation of kinetic energy alongwith Eqs. 8.4a and b).

We have specified the laboratory and c.m. frames of reference. We would now like to determine the relationship between the angles and the differential scattering cross-sections in the two frames of reference. For this, let us look at the relation between the position and velocity vectors of the particles after scattering in these two frames of reference.

Recall that we have chosen the incident direction along the z-axis. The coordinates of m_1 and m_2 as measured from the origin O of the lab system after collision are $\mathbf{r}_1$ and $\mathbf{r}_2$. The c.m. (C) has the coordinate $\mathbf{R}$ with respect to O. Let $\mathbf{r}_1'$ and $\mathbf{r}_2'$ be the coordinates of m_1 and m_2 with respect to C after collision. As you know from its definition, the c.m. lies on the line joining m_1 and m_2. Thus $\mathbf{r}_1'$ and $\mathbf{r}_2'$ lie along the same line. So, we can relate the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{R}$, $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in a vector diagram as shown in Fig. 8.6a.

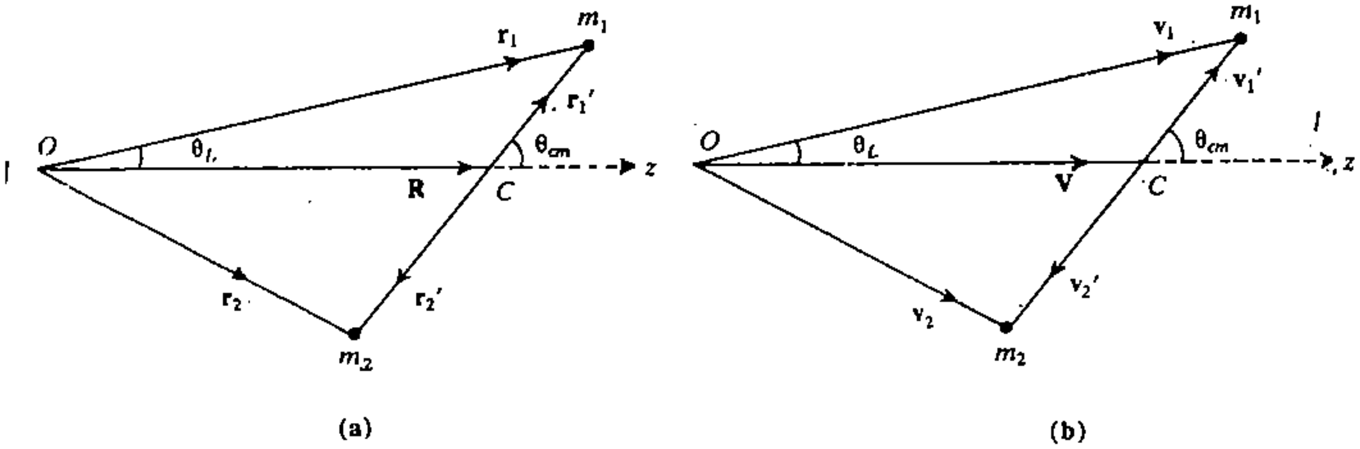


Fig. 8.6: Relation between (a) the position vectors and (b) velocities of the colliding particles in lab and c.m. frames of reference, after collision.

From Fig. 8.6a we have

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1' \quad , \quad \mathbf{r}_2 = \mathbf{R} + \mathbf{r}_2' \tag{8.5a}$$

The relative coordinate of particle 1 with respect to particle 2 is $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$. From Eq. 8.5a you can see that

$$\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2' \equiv \mathbf{r}, \text{ say.} \tag{8.5b}$$

Thus, the separation of the two particles is the same in both frames of reference. Using Eqs. 8.3a, 8.5a and b, we can write $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in terms of $\mathbf{r}$:

$$\mathbf{r}_1' = \frac{m_2}{m_1 + m_2}\mathbf{r} \quad , \quad \mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{8.6}$$

Differentiating Eq. 8.5a we can relate the velocity vectors in both frames of reference after scattering :

$$\mathbf{v}_1 = \mathbf{V} + \mathbf{v}_1', \tag{8.7a}$$

$$\mathbf{v}_2 = \mathbf{V} + \mathbf{v}_2'. \tag{8.7b}$$

Similar relations can be derived for the position and velocity vectors of particles 1 and 2 before scattering, so that

$$\mathbf{u}_1 = \mathbf{V} + \mathbf{u}_1' \tag{8.7c}$$

$$\mathbf{u}_2 = \mathbf{V} + \mathbf{u}_2'. \tag{8.7d}$$

Since the particle 2 is initially at rest in the lab system, $\mathbf{u}_2 = 0$ and we have

$$\mathbf{u}_2' = -\mathbf{V}. \tag{8.7e}$$

Using Eqs. 8.3 to 8.7 we can determine the relations between the angles of scattering and scattering cross-sections in the laboratory and c.m. frames of reference.

8.2.4 Relations Between Angles and Scattering Cross-Sections in the Lab and C.M. Frames of Reference

Let θ_L and θ_{cm} be the angles of scattering in the laboratory and c.m. frames of reference, respectively (see Fig. 8.6). Resolving Eq. 8.7a into its components along the initial z-direction and perpendicular to it (see Fig. 8.6b), we get

$$v_1 \cos\theta_L = v_1' \cos\theta_{cm} + V. \tag{8.8a}$$

$$v_1 \sin\theta_L = v_1' \sin\theta_{cm} \qquad (8.8b)$$

Dividing Eq. 8.8b by Eq. 8.8a gives

$$\tan\theta_L = \frac{v_1' \sin\theta_{cm}}{v_1' \cos\theta_{cm} + V} = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \frac{V}{v_1'}}$$

$$\text{or} \quad \tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \gamma} \quad \text{with} \quad \gamma = \frac{V}{v_1'} . \qquad (8.9)$$

You can see that γ is the ratio of the speed of the c.m. in the laboratory system to the speed of the observed particle in the c.m. system. The value of γ can be determined for both elastic and inelastic scattering. We shall limit ourselves to the case of elastic scattering. Let us find $\frac{V}{v_1'}$ for elastic scattering.

γ for elastic scattering

You have already shown in SAQ 2 that $v_1' = u_1'$. We can obtain u_1' in terms of V from Eqs. 8.7c and 8.3b as follows:

$$u_1' = u_1 - V = \frac{(m_1 + m_2)}{m_1} V - V = \frac{m_2}{m_1} V ,$$

$$\text{or} \quad u_1' = \frac{m_2}{m_1} V \qquad (8.10a)$$

Thus, for elastic scattering

$$\gamma = \frac{V}{v_1'} = \frac{V}{u_1'} = \frac{m_1}{m_2} \qquad (8.10b)$$

You may now like to apply these relations to solve a problem.

SAQ 3

An experiment is to be designed to measure the differential scattering cross-section for elastic pion-proton scattering. In the c.m. frame, the scattering angle is 70° and kinetic energy of the pion is 490 keV. (The eV is the atomic unit of energy.) Find the corresponding angle in the lab at which the scattered pions should be detected and the required lab kinetic energy in eV of the pion beam. The ratio of pion to proton mass is 1/7.

1eV = 1.6×10^{-19} J

Let us now determine the relation between the differential scattering cross-sections in the lab and c.m. frames of reference. The incident flux F and the number of particles (Δn) scattered per unit time in the solid angle $d\Omega$, will be the same in the laboratory and the c.m. systems. So Eq. 8.1a gives us the condition that

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)_{lab} F\,(d\Omega)_{lab}\,(\Delta t) = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F\,(d\Omega)_{cm}\,(\Delta t),$$

$$\text{or} \quad \left(\frac{d\sigma}{d\Omega}\right)_{lab} (d\Omega)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} (d\Omega)_{cm} \qquad (8.11a)$$

We know from Eqs. 8.2a and 8.2b that $d\Omega = \sin\theta\, d\theta\, d\phi$. Since we are dealing with situations in which the cross-sections are independent of ϕ, we can write $d\phi_{lab} = d\phi_{cm}$ so that

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \frac{\sin\theta_{cm}\, d\theta_{cm}}{\sin\theta_L\, d\theta_L}$$

$$\text{or} \quad \left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \frac{d(\cos\theta_{cm})}{d(\cos\theta_L)}, \quad (\because d(\cos\theta) = -\sin\theta\, d\theta) . \qquad (8.11b)$$

We can use Eq. 8.9 to simplify Eq. 8.11b further as follows:

$$\text{Since } \tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \gamma}$$

you can verify that

$$\cos\theta_L = \frac{\cos\theta_{cm} + \gamma}{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{1/2}}$$

and
$$\frac{d(\cos\theta_L)}{d(\cos\theta_{cm})} = \frac{(1 + \gamma\cos\theta_{cm})}{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{3/2}}$$

Thus, we get the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{3/2}}{(1 + \gamma\cos\theta_{cm})}\left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.11c)$$

It is $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ which is obtained from theory. Eq. 8.11c tells us how to transform it to the laboratory system to compare with experimental data.

For elastic scattering, $\gamma = \frac{m_1}{m_2}$ and we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta_{cm}\right)^{3/2}}{\left(1 + \frac{m_1}{m_2}\cos\theta_{cm}\right)}\left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.12)$$

If the masses of the target and projectile are equal, i.e. $m_2 = m_1$, then Eq. 8.12 reduces to

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = 4\cos\frac{\theta_{cm}}{2} \cdot \left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.13)$$

The total scattering cross-sections will be the same in both the frames of reference.

You will get some practice on these equations if you work out the following SAQ.

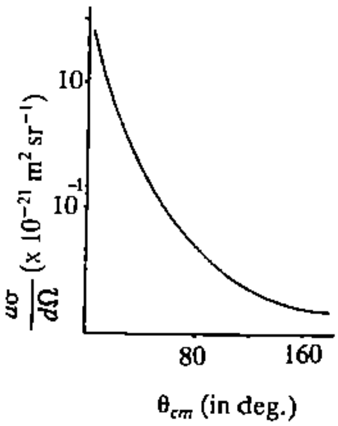


Fig. 8.7

SAQ 4

a) The differential scattering cross-sections in a proton-proton elastic scattering experiment are measured to be 2.3×10^{-27} $m^2 sr^{-1}$ and 2.6×10^{-28} $m^2 sr^{-1}$ at the scattering angles 30° and 60°. Find the corresponding quantities in the c.m. frame of reference.

b) Fig. 8.7 shows the variation of the differential cross-section (dcs) with the angle of scattering for the elastic scattering of electrons by lithium atoms in the c.m. frame of reference. What is the corresponding curve in the lab system?

So far we have defined scattering cross-sections and established the relations between the scattering angles and cross-sections in the laboratory and c.m. frames of reference. Let us now determine the cross-sections for a few scattering processes. One of the methods commonly used for this purpose is the method involving impact parameters, which we shall now study.

8.3 IMPACT PARAMETERS

Let us suppose that the projectile does not make a head-on collision with the target. Instead, it travels along a path, which if continued in a straight line, would pass the target at a distance b (Fig. 8.8a). This, indeed, is the case most of the times. The distance b is known as the **impact parameter.** You can see that b is the perpendicular distance between the projectile's initial path and the target.

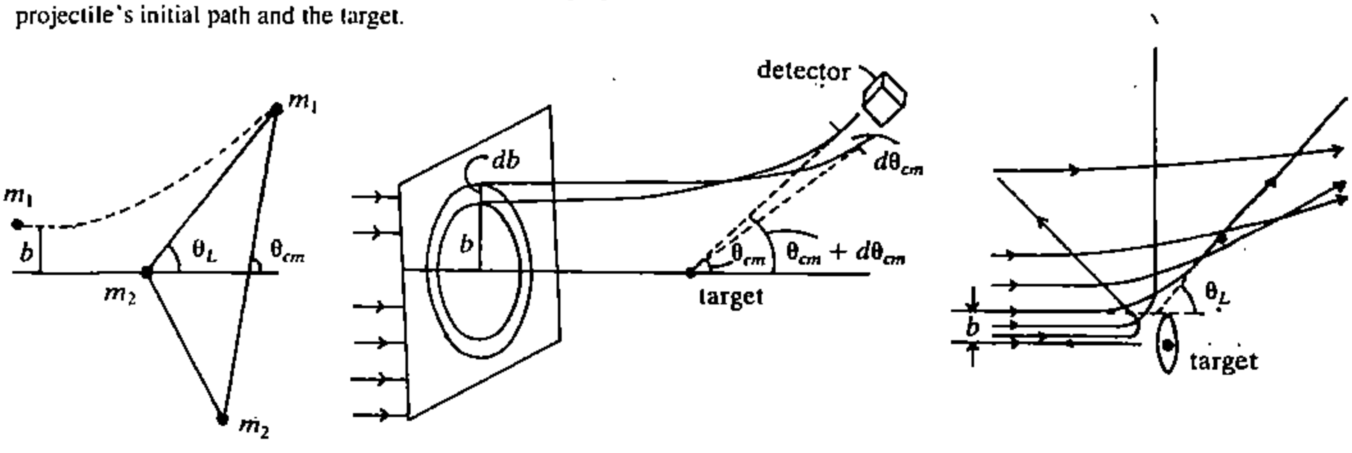


Fig. 8.8: (a) The impact parameter b; (b) the particles having impact parameters between b and $b + db$ are scattered into angles between θ_{cm} and $\theta_{cm} + d\theta_{cm}$; (c) the scattering angle decreases with increasing impact parameter.

Let us now express the differential scattering cross-sections in terms of the impact parameter. We will study the scattering process in the c.m. frame of reference with θ_{cm} as the angle of scattering (Fig. 8.8b). What is the number of particles incident on the target during time Δt having impact parameters between b and $b + db$? Let us consider a circular ring having radii between b and $b + db$. The area of the ring is $2\pi b db$ for infinitesimal values of db. If the incident flux is F then,

The number of incident particles having an impact parameter between b and $(b + db)$

$$= F(\Delta t)\,(2\pi\, b\, db). \qquad (8.14)$$

Let us suppose that these particles are scattered into angles between θ_{cm} and $\theta_{cm} + d\theta_{cm}$. The particles with larger b will be scattered through smaller angles as shown in Fig. 8.8c. This happens because larger b means lesser interaction, i.e. less scattering. For very large b, scattering will be minimal and the particles will go almost undeflected in a straight line. Now in the c.m. frame of reference the number of particles scattered in the solid angle $d\Omega$ in time Δt is given from Eq. 8.1a as

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F\,(d\Omega)_{cm}\,\Delta t\,.$$

This is the same as the number of incident particles in time Δt having impact parameters between b and $b + db$, given by Eq. 8.14, i.e.

$$F\,(\Delta t)\,2\pi b db = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F(d\Omega)_{cm}\,\Delta t,$$

$$\text{or} \quad 2\pi b db = -\left(\frac{d\sigma}{d\Omega}\right)_{cm} 2\pi \sin\theta_{cm}\, d\theta_{cm}. \qquad (8.15a)$$

Here we have assumed that $\left(\frac{d\sigma}{d\Omega}\right)$ is independent of ϕ. Taking into account all values of ϕ in $d\Omega$, we have $d\Omega = 2\pi \sin\theta\, d\theta$. The negative sign expresses the fact that as b increases, θ_{cm} decreases, i.e. db and $d\theta_{cm}$ have opposite signs. From Eq. 8.15a we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}} \left|\frac{db}{d\theta_{cm}}\right|. \qquad (8.15b)$$

We have not written the negative sign in Eq. 8.15b because $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ has the dimension of area and its magnitude has to be positive. So, if we know b as a function of scattering angle θ_{cm}, we can calculate the differential scattering cross-section using Eq. 8.15b

How do we determine b as a function of θ_{cm}? We will not study any general method for finding $b(\theta_{cm})$. Instead, we will study two specific cases, namely, the hard sphere scattering and Rutherford scattering as applications of Eq. 8.15b.

8.3.1 Elastic Scattering of Two Hard Spheres

Let us consider the elastic scattering of a sphere of mass m_1 and radius R by a target sphere of mass m_2 and radius R_s (Fig. 8.9a). Let the distance between the centres of the two spheres at any instant be r.

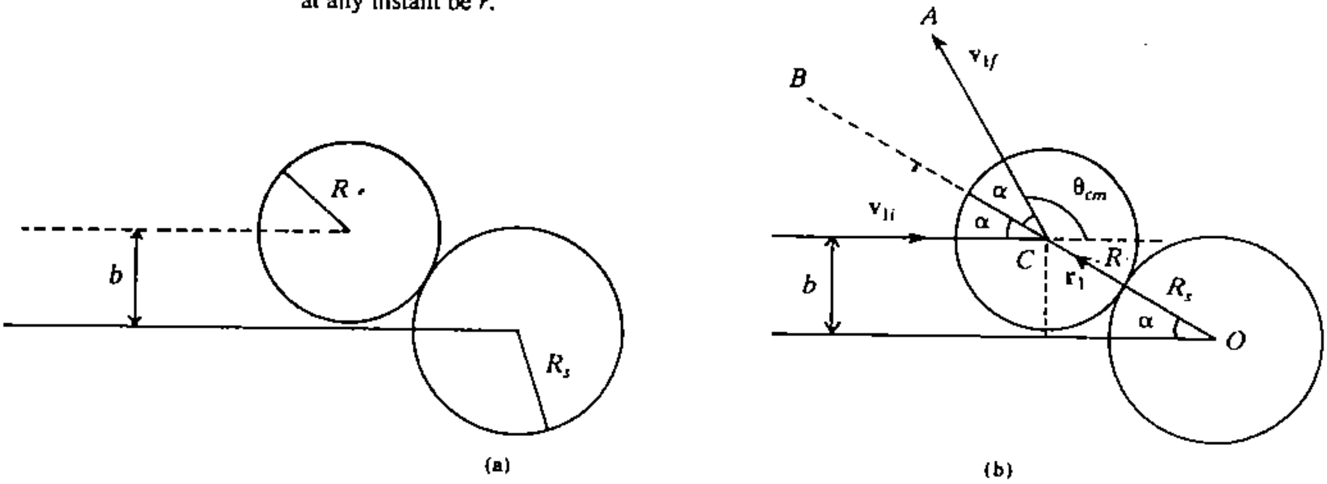


Fig. 8.9: (a) Scattering of two hard spheres; (b) the incident hard sphere rebounds at the same angle as the incident angle after scattering from the target sphere.

The incident hard sphere will get scattered after rebounding from the target hard sphere. What do we mean by the term 'hard sphere'? This means that the spheres cannot penetrate a distance smaller than $R + R_s$. So we can say that the force or potential is infinite for $r < (R + R_s)$. For a distance $r > (R + R_s)$, the spheres are free to move both before and after the collision, i.e. there is no force between them. Mathematically we can express such a situation in terms of a potential $V(r)$ such that

$$V(r) = \infty \quad \text{for} \quad r < (R + R_s). \tag{8.16}$$
$$= 0 \quad \text{for} \quad r > (R + R_s).$$

You know that $F = -\frac{dV}{dr}$. So you can see that the force on the spheres corresponding to such a potential is infinite for $r < (R + R_s)$ and zero for $r > (R + R_s)$. This means that the torque is zero for $r > (R + R_s)$. Since the torque $= \frac{d\mathbf{L}}{dt}$, the total angular momentum will remain constant before and after the collision.

Let us now find out the relation between b and θ_{cm}. Refer to Fig. 8.9b. For an elastic collision, K.E. is conserved. You have already worked out in SAQ 2 that for elastic scattering the target and projectile velocities remain the same before and after collision. Let α be the angle between the direction of the initial velocity $\mathbf{v}_{1i}$ of m_1 and the line joining the centres of the two spheres at the time of impact as shown in Fig. 8.9b. Let $\mathbf{r}_1$ be the position vector of the centre of sphere of radius R with respect to the centre of sphere of radius R_s. The magnitude of the angular momentum of m_1 with respect to the centre of m_2 just before the impact is

$$L_1 = m_1 |(\mathbf{v}_{1i} \times \mathbf{r}_1)| = m_1 v_{1i} r_1 \sin(\pi - \alpha) = m_1 v_{1i} r_1 \sin\alpha .$$

Just after the impact it is

$$L_1 = m_1 |(\mathbf{v}_{1f} \times \mathbf{r}_1)| = m_1 v_{1f} r_1 \sin \angle ACB.$$

Since from SAQ 2, $v_{1i} = v_{1f}$ for elastic scattering we have that
$m_1 v_{1i} \sin\alpha = m_1 v_{1f} \sin \angle ACB$, i.e. $\angle ACB = \alpha$.

Thus, the sphere m_1 will bounce off the sphere m_2 at an angle to the normal, equal to the incident angle α. So from Fig. 8.9b you can see that

$$\theta_{cm} = \pi - 2\alpha . \tag{8.17}$$

Now, we can relate the impact parameter b to α using Fig. 8.9b as follows:

$$b = r_1 \sin\alpha = (R + R_s)\sin\alpha$$

$$= (R + R_s)\ \sin\frac{\pi - \theta_{cm}}{2}, \text{ using Eq. 8.17,}$$

or $$b = (R + R_s)\cos\frac{\theta_{cm}}{2}. \quad (8.18a)$$

$$\therefore \frac{db}{d\theta_{cm}} = -\frac{R + R_s}{2}\sin\frac{\theta_{cm}}{2}. \quad (8.18b)$$

Then from Eq. 8.15b we get the differential scattering cross-section as

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}}\ \frac{R + R_s}{2}\ \sin\frac{\theta_{cm}}{2}$$

$$= \left(\frac{b}{2\cos\frac{\theta_{cm}}{2}}\right)\left(\frac{R + R_s}{2}\right).$$

Using $\dfrac{b}{\cos\frac{\theta_{cm}}{2}} = R + R_s$ from Eq. 8.18a, we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_s)^2}{4} \quad (8.19)$$

The total scattering cross-section is

$$\sigma = 2\pi \int_0^{\pi} \left(\frac{d\sigma}{d\Omega}\right)_{cm} \sin\theta_{cm}\, d\theta_{cm}$$

$$= 2\pi \int_0^{\pi} \frac{(R + R_s)^2}{4}\ \sin\theta_{cm}\, d\theta_{cm}$$

or $$\sigma = 2\pi\ \frac{(R + R_s)^2}{4} \times 2 = \pi\,(R + R_s)^2. \quad (8.20)$$

If the projectile is a point particle instead of a sphere, then the total scattering cross-section is πR_s^2 which is the cross-sectional area of the target sphere. You may like to work out an SAQ applying the ideas of this section.

SAQ 5

A beam of point particles strikes a wall. Each atom in the wall behaves like a sphere of radius 3×10^{-15}m. The mass of each particle is much less than that of an atom. What is the dcs, tcs and the impact parameter of the particles entering a detector placed at an angle of 60° to the direction of the beam?

Let us now study another application of Eq. 8.15b, namely the Rutherford scattering.

8.3.2 Rutherford Scattering

The Rutherford scattering experiment was an important milestone in understanding the structure of the atom. Until the early twentieth century Thomson's plum pudding model of the atom was believed to be valid. J.J. Thomson had proposed, in 1898, that atoms were uniform spheres of positively charged matter in which electrons were embedded (Fig. 8.10a). It was almost 13 years later that a definite experimental test of this model was made. Now, the most direct way to find out what is inside a plum pudding is to plunge a finger into it! A similar technique was used in the classic experiment performed in 1911, by Geiger and Marsden who were working with Lord Rutherford. They bombarded thin foils of various materials with α-particles (helium nuclei) and recorded the angular distribution of the scattered α- particles (see Fig. 8.11).

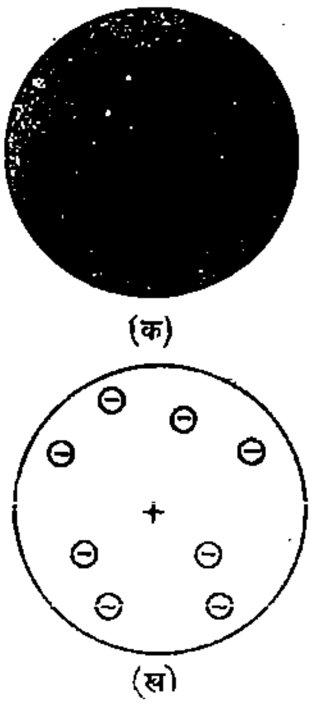


Fig. 8.10: (a) Thomson's plum pudding model of the atom; (b) Rutherford's nuclear model.

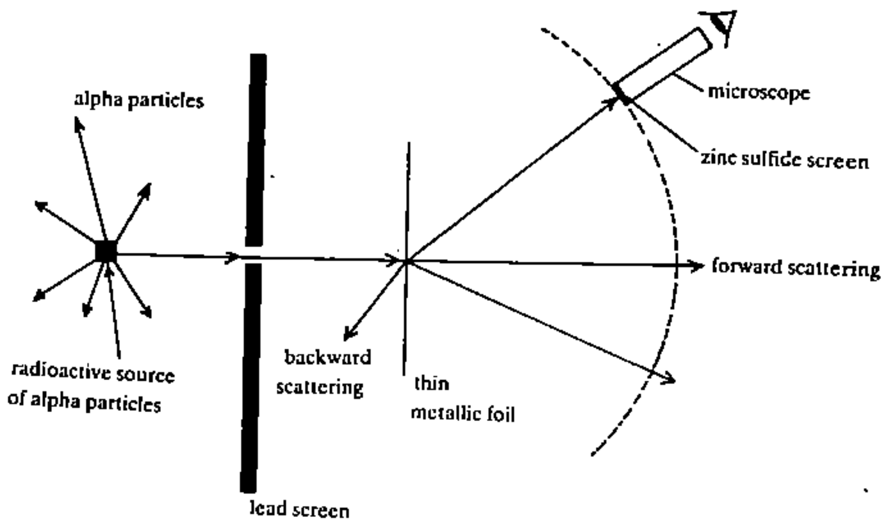


Fig. 8.11 Rutherford scattering experiment. A source of α-particles is placed behind a lead screen with a small hole, so that a narrow beam is directed at a thin metallic foil. A movable zinc sulphide screen is placed at the other side of the foil. When an α-particle strikes the screen it gives off a flash of light.

The thickness of the foils used by Geiger and Marsden was of the order of 10^{-7}m. Compare this with the human hair which is about 10^{-4}m in diameter.

On seeing these results, Rutherford remarked, "It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

It was found that most of the α-particles pass through the foil (i.e. scattering angle $\theta < 90°$). However, about 1 in 6.17×10^6 alpha particles was scattered backward, i.e. deflected through an angle greater than 90°. This result was unexpected according to Thomson's model. It was anticipated that the alpha particles would go right through the foil with only slight deflections. This follows from the Thomson model. If this model were correct, only weak electric forces would be exerted on alpha particles passing through a thin metal foil. In such a case their initial momenta should be enough to make them go through with only slight deflections. It would indeed need strong forces to cause such considerable deflections in α-particles as were observed.

In order to explain these results Rutherford proposed a nuclear model of the atom. Using this model he calculated the dcs. In doing so, he reasoned that the backward scattering could not be caused by electrons in the atom. The alpha particles are so much more massive than electrons that they would hardly be scattered by them. He assumed that the positive charge in the atom was concentrated in a very small volume, which he termed the nucleus, rather than being spread out over the volume of the atom. So the scattering of alpha particles was due to the atomic nucleus. As you know the force of interaction between the α - particles and the nucleus is simply the repulsive inverse square electrostatic force. On the basis of this model, Rutherford calculated the differential cross-sections.

There was a striking agreement between the calculated and observed cross-sections. This established the nuclear model of the atom, i.e. the positive charge of the atom is concentrated in the nucleus, which is surrounded by electrons (Fig. 8.10b).

Let us consider the scattering of a particle carrying charge q by the atomic nuclei having charge q'. For this scattering process Rutherford derived the relation between the impact parameter b and the angle of scattering θ_{cm} to be

$$b = \frac{r_0}{2} \cot \frac{\theta_{cm}}{2}, \qquad (8.21)$$

where $r_0 = \dfrac{qq'}{4\pi\,\varepsilon_0\, E_{cm}}$. E_{cm} is the total mechanical energy of the projectile and the target in the c.m. system.

ε_0 is known as the permittivity of free space. Its value is $8.8 \times 10^{-12}\ C^2 N^{-1} m^{-2}$.

The differential scattering cross-section in the c.m. system for Rutherford scattering is then given from Eq. 8.15b as

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}}\left(\frac{db}{d\theta_{cm}}\right)$$

$$= \frac{r_o \cot\frac{\theta_{cm}}{2}}{2\sin\theta_{cm}}\frac{r_o}{2}\cdot\frac{1}{2}\operatorname{cosec}^2\frac{\theta_{cm}}{2}$$

$$= \frac{r_o^2\cot\frac{\theta_{cm}}{2}}{16\sin\frac{\theta_{cm}}{2}\cos\frac{\theta_{cm}}{2}}\operatorname{cosec}^2\frac{\theta_{cm}}{2}.$$

or $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\operatorname{cosec}^4\frac{\theta_{cm}}{2}, \text{ where } r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}} \qquad (8.22)$$

This is the Rutherford scattering cross-section. For scattering of an α-particle by a nucleus of atomic number Z, $qq' = (2e)(Ze) = 2Ze^2$, where e is the electronic charge. You can see that the Rutherford scattering cross-section is strongly dependent on both the energy of the incoming particle and the scattering angle. Also we expect the number of particles scattered to increase as Z^2 with increasing atomic number. Let us now apply the ideas discussed in this section to a concrete situation.

Example 2

In one of their experiments on scattering of α-particles, Geiger and Marsden bombarded 7.7 MeV α-particles on a gold target, for which $Z = 79$. Its atomic weight is 197 amu. Find the impact parameters and differential scattering cross-sections of the α-particles which are scattered elastically through angles equal to (i) 10°, (ii) 90° and (iii) 150°.

It is given here that the K.E. of the incident α-particles in the laboratory system is 7.7 MeV, i.e. 1.2×10^{-12}J. The angles of scattering in the lab system are (i) $\theta_L = 10°$, (ii) $\theta_L = 90°$ and (iii) $\theta_L = 150°$. In order to apply Eqs. 8.21 and 8.22 we must determine the scattering angle θ_{cm} and total mechanical energy E_{cm} in the c.m. frame of reference. We have also to find out r_0.

1 MeV = 10^6 eV

The total mechanical energy E_L in the lab system is simply the initial K.E. of the α-particles, since the target is initially at rest and the two particles are free. This is given to be 1.2×10^{-12}J. We have to determine E_{cm} in terms of E_L. As you know the total mechanical energy in the c.m. frame before scattering is

$$E_{cm} = \frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2$$

Now from Eqs. 8.10a, 8.7e and 8.3b, $u_1' = \frac{m_2}{m_1}V$, $u_2' = -V$ and

$$V = \frac{m_1 u_1}{m_1 + m_2}, \text{ so that}$$

$$E_{cm} = \frac{1}{2}m_1\frac{m_2^2}{m_1^2}\cdot\frac{m_1^2}{(m_1+m_2)^2}u_1^2 + \frac{1}{2}m_2\frac{m_1^2 u_1^2}{(m_1+m_2)^2}$$

or $$E_{cm} = \frac{1}{2}m_1 u_1^2\left[\frac{m_2^2}{(m_1+m_2)^2} + \frac{m_2 m_1}{(m_1+m_2)^2}\right] = E_L\frac{m_2}{m_1+m_2} = \frac{E_L}{\left(1+\frac{m_1}{m_2}\right)}. \qquad (8.23)$$

For α–particle scattering by gold atoms, we have $m_1 = 4$ amu and $m_2 = 197$ amu.

1amu = 1.67×10^{-27}kg

$$\therefore\ E_{cm} = \frac{1.2\times10^{-12}\text{J}}{\left(1+\frac{4}{197}\right)} = 1.2\times10^{-12}\text{ J}$$

From Eq. 8.22, $$r_o = \frac{qq'}{4\pi\varepsilon_o E_{cm}} = \frac{2Ze^2}{4\pi\varepsilon_o E_{cm}}$$

$$= \frac{2 \times 79 \times (1.6 \times 10^{-19} C)^2}{(4\pi) \times (8.8 \times 10^{-12}\ C^2N^{-1}m^{-2}) \times (1.2 \times 10^{-12}\ J)}$$

or $\qquad r_0 = 3.0 \times 10^{-14}$ m.

Let us now use Eqs. 8.9, 8.21 and 8.22 to calculate θ_{cm}, b and $\left(\frac{d\sigma}{d\Omega}\right)$ for (*i*) $\theta_L = 10°$, (ii) $\theta_L = 90°$, (iii) $\theta_L = 150°$, respectively.

Since $\frac{m_1}{m_2} \doteq .02 \ll 1$, we can neglect it, so that $\theta_{cm} \simeq \theta_L$. In fact, you can verify this yourself by calculating the exact value of θ_{cm} using Eq. 8.9.

(i) For $\theta_{cm} = 10°$, $b = \frac{r_0}{2}\cot 5° = \frac{(3.0 \times 10^{-14}\ m) \times 11.4}{2} = 1.7 \times 10^{-13}$ m.

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(3.0 \times 10^{-14}\ m)^2}{16} \operatorname{cosec}^4 (5°) = \frac{(3.0 \times 10^{-14}\ m)^2}{16} \times (11.5)^4$$

$$= 9.8 \times 10^{-25}\ m^2\ sr^{-1}$$

(ii) For $\theta_{cm} = 90°$, $\quad b = 1.5 \times 10^{-14}$ m, $\quad \left(\frac{d\sigma}{d\Omega}\right)_{cm} = 2.2 \times 10^{-28}\ m^2\ sr^{-1}$

(iii) For $\theta_{cm} = 150°$, $\quad b = 4 \times 10^{-15}$ m, $\quad \left(\frac{d\sigma}{d\Omega}\right)_{cm} = 6.5 \times 10^{-29}\ m^2\ sr^{-1}$

Let us again understand the physical significance of the Rutherford scattering cross-section in the light of what we have studied so far. The distance of closest approach between the alpha particle and the nucleus is given by $r_{min} = \frac{r_0 + \sqrt{r_0^2 + 4b^2}}{2}$, where

$$r_0 = \frac{q\,q'}{4\pi\,\varepsilon_0\,E_{cm}}.$$

So to investigate the structure of the atom at small distances, E_{cm} should be large because only for those values of E_{cm}, r_{min} would be sufficiently small. Thus we should bombard the target with high energy particles and examine large angle scattering for which b is small.

You can see from Example 2 that the cross-section is large for small values of scattering angles. But physically we are interested in large angle scattering. This is because of the fact that only very strong forces acting at very short distances can give rise to scattering at such large angles. On what basis can we say this? Let us find out.

If the positive nuclear charge were spread out over a larger volume as proposed by Thomson, the force would be inverse-square law force only down to a distance equal to the radius of the charge distribution. Beyond this point it would decrease as we go to even smaller distances. (Recall the Example 2 in Sec. 5.4 of Unit 5, Block 1. A force law with a similar r-dependence would hold for a charge placed inside a spherical charge distribution. The constants would, of course, change). As a result, charged particles which penetrate inside the charge distribution would experience a weaker force than the inverse square force. Thus, particles with smaller b and smaller r_{min} would be scattered through smaller angles. But this does not turn out to be true, experimentally.

This was why Rutherford assumed the nuclear charge to be concentrated in a very small volume. Only in such a case the strong inverse square force would act at very small distances of the order of r_{min}, giving rise to large deflections. The agreement of theory and experiment vindicated Rutherford's nuclear model. Thus, Rutherford is credited with the 'discovery' of atomic nucleus. In fact, if we neglect electrons completely in Thomson's model, the electric field intensity at the atom's surface is calculated to be about 10^{13}V m^{-1}. On the other hand using Rutherford's model, the electric field intensity at the surface of the nucleus exceeds 10^{21}V m^{-1}. This is greater by a factor of 10^8, enough to reverse the direction of alpha particles.

An interesting aspect of this scattering experiment is that it determines an upper limit to the dimensions of atomic nuclei. This is none else than the parameter r_0, since for $b = 0$,

$r_{min} = r_o$. For the typical α-particle scattering discussed in Example 2, $r_o = 3.0 \times 10^{-14}$m. The radius of gold nucleus is, therefore, less than 3.0×10^{-14}m. In recent years, however, α-particles of higher energies have been used to determine nuclear dimensions. It has been found that the Rutherford scattering formula does fail to agree with experiment. From these experiments the radius of gold nucleus comes out to be 1/6 of the values of r_o found in Example 2.

Another interesting feature of the dcs of Eq. 8.22 is that the corresponding total cross-section is infinite. This is because of the infinite range of the Coulomb force. Even if a particle is very far away from the nucleus, it experiences some force and is scattered through a non-zero (though small) angle. So the total number of particles scattered is indeed infinite.

From these applications you must have realised that scattering is an important tool for investigating the microscopic structure of matter. Let us now summarise what we have studied in this unit.

8.4 SUMMARY

- When a beam of particles strikes a target, the angular distribution of scattered particles for different values of (θ, ϕ) may be found from the differential scattering cross-section $\frac{d\sigma}{d\Omega}$. The total scattering cross-section is obtained by integrating the dcs over all values of θ and ϕ.
- The dcs are measured in the laboratory frame of reference but calculated in the c.m. frame of reference. For elastic scattering the relations between the scattering angle and the dcs in the lab and c.m. frames are

$$\tan \theta_L = \frac{\sin \theta_{cm}}{\cos\theta_{cm} + \frac{m_1}{m_2}}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta_{cm}\right)^{3/2}}{\left(1 + \frac{m_1}{m_2}\cos\theta_{cm}\right)} \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

- If we know the impact parameter b as a function of θ_{cm} we can calculate the dcs for any given scattering process using the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}} \left| \frac{db}{d\theta_{cm}} \right|$$

- For the elastic scattering of two hard spheres

$$b = (R + R_s) \cos \frac{\theta_{cm}}{2}, \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_s)^2}{4} \text{ and } \sigma = \pi (R + R_s)^2$$

- For the scattering of a point charge q from another point charge q',

$$b = \frac{r_o}{2} \cot \frac{\theta_{cm}}{2}, \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_o^2}{16} \operatorname{cosec}^4 \frac{\theta_{cm}}{2}, \text{ where } r_o = \frac{qq'}{4\pi\varepsilon_o E_{cm}}$$

  This is known as the Rutherford scattering cross-section. The tcs is infinite due to the infinite range of Coulomb forces.

8.5 TERMINAL QUESTIONS

1. Show that for Rutherford scattering the total cross-section for particles scattered through any angle θ' greater than a lower limit θ_o is

$$\sigma(\theta' > \theta_o) = \frac{\pi r_o^2}{4} \cot^2 \frac{\theta'}{2}.$$

2. At low energies neutrons and protons behave roughly like hard spheres of radius about 1.3×10^{-12} cm. A parallel beam of neutrons with a flux of 3×10^6 neutrons cm^{-2} s^{-1}

strikes a target containing 4×10^{22} protons. A circular detector of radius 2 cm is placed 70 cm away from the target. Calculate the rate of detection, i.e. $\Delta n/\Delta t$ of neutrons for a scattering angle $\theta_L = 30°$.

3. Find the dcs of 7 MeV α-particles scattered from a lead target ($Z = 82$, atomic weight = 207 amu) for $\theta_L = 30°$, given that 1 amu = 1.67×10^{-27} kg.

8.6 ANSWERS

SAQs

1. (i) Here we will use Eq. 8.1b. We have to calculate $\Delta n/\Delta t$ given $F = 5 \times 10^{10} \text{m}^{-2}\text{s}^{-1}$,

$$\frac{d\sigma}{d\Omega} = 1.5 \times 10^{-26}\, \text{m}^2\, \text{sr}^{-1} \text{ and } d\,\Omega = 10^{-3}\text{sr}.$$

From Eq. 8.1b

$$\frac{\Delta n}{\Delta t} = \left[\frac{d\sigma}{d\Omega}\right](F)(d\,\Omega)$$

$$= (1.5 \times 10^{-26}\, \text{m}^2\text{sr}^{-1})(5 \times 10^{10}\, \text{m}^{-2}\text{s}^{-1})(10^{-3}\text{sr})$$

$$= 7.5 \times 10^{-19}\, \text{s}^{-1}$$

(ii) For $N = 10^{22}$, we have

$$\frac{\Delta n}{\Delta t} = \left[\frac{d\sigma}{d\Omega}\right](NF)(d\Omega) = (7.5 \times 10^{-19}\, \text{s}^{-1}) \times 10^{22} = 7.5 \times 10^{3}\, \text{s}^{-1}.$$

2. For elastic collisions, the total kinetic energy of the system remains constant. Its value for the entire system is the same before and after collision. Thus, we have that

$$\frac{1}{2}\, m_1 u_1'^2 + \frac{1}{2}\, m_2\, u_1'^2 = \frac{1}{2} m_1\, v_1'^2 + \frac{1}{2} m_2 v_2'^2.$$

Substituting for u_2' and v_2' in terms of u_1' and v_1' from Eq. 8.4b we have

$$\frac{1}{2} m_1\, u_1'^2 + \frac{1}{2}\, m_2 \left[\frac{m_1^2}{m_2^2} u_1'^2\right] = \frac{1}{2}\, m_1\, v_1'^2 + \frac{1}{2}\, m_2 \left[\frac{m_1^2}{m_2^2}\, v_1'^2\right]$$

or $$\frac{1}{2}\, u_1'^2 \left[m_1 + \frac{m_1^2}{m_2}\right] = \frac{1}{2}\, v_1'^2 \left[m_1 + \frac{m_1^2}{m_2}\right],$$

or $u_1' = v_1'$.

Similarly you can show that $u_2' = v_2'$.

3. Here $\gamma = 1/7$, $\theta_{cm} = 70°$, E_{cm} for pion = 490 keV. From Eq. 8.9

$$\tan\theta_L = \frac{\sin 70°}{\cos 70° + 1/7} = 1.94 \text{ or } \theta_L = 62.7°.$$

The pion K.E.s in laboratory and c.m. frames of reference are $E_L = \frac{1}{2} m_1 u_1^2$ and

$E_{cm} = \frac{1}{2} m_1 u_1'^2$, respectively.

From Eqs. 8.10a and 8.3b we have

$$E_{cm} = \frac{1}{2} m_1 \frac{m_2^2}{m_1^2}\left[\frac{m}{m_1 + m_2}\right]^2 u_1^2 = \frac{1}{2}\, m_1\, u_1^2 \frac{m_2^2}{(m_1 + m_2)^2} = \frac{E_L}{\left[1 + \frac{m_1}{m_2}\right]^2} \quad (8.24)$$

Thus $E_L = (1 + \frac{1}{7})^2 \times 490$ keV = 640 keV.

4. (a) Here we have to apply Eqs. 8.9 and 8.13. For elastic scattering, $\gamma = m_1/m_2$ and in this case $m_1 = m_2$. So $\gamma = 1$.

Let us first determine the angles of scattering in c.m. frame of reference for (i) $\theta_L = 30°$ and (ii) $\theta_L = 60°$.

For $\gamma = 1$, $\tan\theta_L = \dfrac{\sin\theta_{cm}}{\cos\theta_{cm} + 1} = \dfrac{2\sin\dfrac{\theta_{cm}}{2}\cos\dfrac{\theta_{cm}}{2}}{2\cos^2\dfrac{\theta_{cm}}{2}} = \tan\dfrac{\theta_{cm}}{2}$

or $\theta_{cm} = 2\theta_L$.

So for $\theta_L = 30°$, $\theta_{cm} = 60°$ and for $\theta_L = 60°$, $\theta_{cm} = 120°$

We can now use Eq. 8.13 to obtain dcs in c.m. frame

(i) $\left(\dfrac{d\sigma}{d\Omega}\right)_{lab} = 2.3 \times 10^{-27}\ \text{m}^2\ \text{sr}^{-1}$ for $\theta_L = 30°$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4\cos 30°} \times (2.3 \times 10^{-27}\text{m}^2\ \text{sr}^{-1}) = 6.60 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}$$

(ii) $\left(\dfrac{d\sigma}{d\Omega}\right)_{lab} = 2.6 \times 10^{-28}\text{m}^2\ \text{sr}^{-1}$ for $\theta_L = 60°$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4\cos 60°} \times (2.6 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}) = 1.3 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}$$

b. In this case $m_1 \ll m_2$, since m_1 = mass of the electron = 9.31×10^{-31} kg and m_2 = mass of the Li atom = 1.5×10^{-26} kg.

So $\gamma = \dfrac{m_1}{m_2} = 8.09 \times 10^{-5}$, i.e. $\dfrac{m_1}{m_2} \ll 1$.

Thus from Eq. 8.9 $\tan\theta_L \approx \tan\theta_{cm}$, or $\theta_L \approx \theta_{cm}$.

Again from Eq. 8.12, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left[\frac{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{3/2}}{(1 + \gamma\cos\theta_{cm})}\right]\left(\frac{d\sigma}{d\Omega}\right)_{cm} \approx \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

since $\gamma \ll 1$ and $\gamma\cos\theta_{cm} \ll 1$ for all θ_{cm}.

Therefore, the dcs vs. θ curve in the lab system will be the same as in Fig. 8.7.

5. Since $m_1 \ll m_2$, we have $\theta_L \approx \theta_{cm}$.

Again since the incident particle is a point mass, we put $R = 0$ in Eqs. 8.19, 8.20 and 8.18a and get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{R_s^2}{4} = \frac{(3 \times 10^{-15}\,\text{m})^2}{4}\ \text{sr}^{-1} = 2.2 \times 10^{-30}\text{m}^2\ \text{sr}^{-1}$$

$\sigma_{cm} = \pi R_s^2 = \pi \times (3 \times 10^{-15}\text{m})^2 = 2.8 \times 10^{-29}\ \text{m}^2$.

For $\theta_{cm} = 60°$

$b = R_s \cos\dfrac{\theta_{cm}}{2} = (3 \times 10^{-15}\text{m}) \times 0.87 = 2.6 \times 10^{-15}\text{m}$.

Terminal Questions

1. The dcs for Rutherford scattering is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\,\text{cosec}^4\frac{\theta_{cm}}{2},\ \text{where}\ r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}}.$$

We can use Eq. 8.2c to determine σ, since the dcs does not depend on ϕ. Now instead of zero, the lower limit for integration over θ is any angle θ' greater than θ_0 in this question. Thus we have

$$\sigma = 2\pi \int_{\theta}^{\pi} \left(\frac{d\sigma}{d\Omega}\right)_{cm} \sin\theta_{cm}\, d\theta_{cm}$$

$$= 2\pi \int_{\theta'}^{\pi} \frac{r_o^2}{16} \operatorname{cosec}^4 \left[\frac{\theta_{cm}}{2} \right] \sin\theta_{cm}\, d\theta_{cm}$$

Now $\operatorname{cosec}^4\left(\frac{\theta_{cm}}{2}\right) = \frac{1}{\left(\sin^2\frac{\theta_{cm}}{2}\right)^2} = \frac{1}{\left(\frac{1-\cos\theta_{cm}}{2}\right)^2}$, $[\because \cos 2\theta = 1 - 2\sin^2\theta]$.

$$\therefore \quad \sigma = \frac{\pi r_o^2}{8} \int_{\theta'}^{\pi} \frac{4\sin\theta_{cm}\, d\theta_{cm}}{(1-\cos\theta_{cm})^2}.$$

Putting $\cos\theta_{cm} = t$ we get

$$\sigma = \frac{\pi r_o^2}{2} \int_{-1}^{\cos\theta'} \frac{dt}{(1-t)^2} = \frac{\pi r_o^2}{2}\left[+\frac{1}{1-t}\right]_{-1}^{\cos\theta'}$$

$$= \frac{\pi r_o^2}{2}\left[\frac{1}{1-\cos\theta'} - \frac{1}{2}\right]$$

$$= \frac{\pi r_o^2}{4}\left[\frac{1+\cos\theta'}{1-\cos\theta'}\right] = \frac{\pi r_o^2}{4}\,\frac{2\cos^2\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}$$

or $\sigma(\theta') = \frac{\pi r_o^2}{4}\cot^2\frac{\theta'}{2}$.

2. From Eq. 8.1c, the rate of detection of the scattered neutrons is

$$\frac{\Delta n}{\Delta t} = \left(\frac{d\sigma}{d\Omega}\right) N F\, d\Omega = \left(\frac{d\sigma}{d\Omega}\right) N F \frac{dA}{L^2}$$

where dA is the cross-sectional area of the detector placed at a distance L from the target. We have been given the following data :

Incident flux $F = 3 \times 10^6\ \text{cm}^{-2}\,\text{s}^{-1} = 3 \times 10^{10}\text{m}^{-2}\text{s}^{-1}$

Number of target scattering centres $N = 4 \times 10^{22}$.

Cross-sectional area of the detector, $dA = \pi r^2 = \pi\,(0.02\text{m})^2$

$$= 1.2 \times 10^{-3}\ \text{m}^2$$

Distance between the detector and the target, $L = 70$ cm $= 0.7$m Let us calculate the dcs using Eq. 8.19 for the elastic scattering of two hard spheres. In the c.m. frame it is

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_t)^2}{4} = \frac{1}{4}(1.3 \times 10^{-14}\text{m} + 1.3 \times 10^{-14}\ \text{m})^2\text{sr}^{-1}$$

$$= 1.7 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}$$

We have to find out $\left(\frac{d\sigma}{d\Omega}\right)_{lab}$ for which we will use Eq. 8.13, wherein we also need θ_{cm}.

From Eq. 8.9, for $m_1 = m_2$ we have $\theta_L = \frac{\theta_{cm}}{2}$.

For $\theta_L = 30°$, $\theta_{cm} = 60°$, so that

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = 4\cos 30° \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

$$= 2\sqrt{3} \times 1.7 \times 10^{-28}\text{m}^2\ \text{sr}^{-1}$$

$$= 5.9 \times 10^{-28}\text{m}^2\text{sr}^{-1}$$

Therefore, the rate of detection of neutrons is

$$\frac{\Delta n}{\Delta t} = (5.9 \times 10^{-28}\mathrm{m^2 sr^{-1}}) \times (4 \times 10^{22}) \times (3 \times 10^{10}\mathrm{m^{-2}s^{-1}}) \times \left[\frac{(1.2 \times 10^{-3}\mathrm{m^2})}{(0.7\ \mathrm{m})^2}\ \mathrm{sr}\right]$$

$$= 1.7 \times 10^3\ \mathrm{s^{-1}}$$

3. Here we have to essentially follow the method used in Example 2. It is given that

$$E_L = 7\ \mathrm{MeV} = 7 \times 1.6 \times 10^{-13}\mathrm{J}$$

$$= 1.1 \times 10^{-12}\mathrm{J}$$

Putting $m_1 = 4$ a.m.u. and $m_2 = 207$ a.m.u. in Eq. 8.23, we get

$$E_{cm} = \frac{E_L}{1 + \frac{m_1}{m_2}} = \frac{1.1 \times 10^{-12}\mathrm{J}}{\left(1 + \frac{4}{207}\right)} = 1.1 \times 10^{-12}\ \mathrm{J}.$$

From Eq. 8.22, $r_o = \frac{2Z e^2}{4\pi\varepsilon_o E_{cm}} = \frac{2 \times 82 \times (1.6 \times 10^{-19}\mathrm{C})^2}{(4\pi) \times (8.8 \times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}}) \times (1.1 \times 10^{-12}\mathrm{J})}$

$$= 3.4 \times 10^{-14}\mathrm{m}$$

Since $\frac{m_1}{m_2} = 0.019 \ll 1$, $\theta_L = \theta_{cm}$.

For $\theta_L = 30°$, $\theta_{cm} = 30°$ and

$$\left(\frac{d\sigma}{d\Omega}\right)_m = \frac{r_o^2}{16}\,\mathrm{cosec}^4\,\frac{\theta_{cm}}{2}$$

$$= \frac{(3.4 \times 10^{-14}\mathrm{m})^2}{16}\ \mathrm{sr^{-1}}\ \mathrm{cosec}^4\,(15°)$$

$$= 1.6 \times 10^{-26}\mathrm{m^2\ sr^{-1}}$$

UNIT 9 RIGID BODY DYNAMICS

Structure

9.1 INTRODUCTION

In the previous unit you have studied the phenomenon of scattering. We had treated the projectile there as a point mass. In Units 6 and 7 you have studied about the motion of planets around sun by treating them as point masses. As a matter of fact so far in this course, we have been concerned primarily with the motion of point masses. In nature, however, we hardly come across an ideal point mass. We have to deal with motion of bodies which have finite dimensions. So we need to develop a technique for studying the motion of such bodies.

A special class of such bodies is known as *rigid bodies*. In this unit you will first learn what a rigid body is. You will see that the definition of a rigid body provides a model for studying the motion of various kinds of physical bodies. You will then study about the different kinds of motion of a rigid body. A rigid body can execute both translational and rotational motion. We shall see that the general motion of a rigid body is a combination of both translation and rotation.

You will find that the translational motion of a rigid body can be described in terms of the motion of its centre-of-mass. So, we shall be able to apply the dynamics of point masses for description of translational motion. Hence, our chief concern will be the study of dynamics of rotational motion of rigid bodies.

In Unit 4 of Block 1 you have studied the dynamics of rotational motion of a particle. You already know the concepts of angular displacement, angular velocity, angular acceleration, moment of inertia, kinetic energy, torque and angular momentum for a particle. In this unit we shall extend these concepts to the case of rigid bodies. This will enable us to study about a variety of applications such as the rotation of flywheels, despinning of satellites, motion of rolling objects and so on.

Finally, in this unit we shall revisit the important principle of conservation of angular momentum. We shall see that the principle holds for rigid and other extended bodies. We shall apply the principle to explain the acrobatics performed by a diver or a ballerina. Finally we shall discuss very briefly about precessional motion.

In this unit we shall very often refer to the contents of Unit 4 of Block 1. So it is suggested that you go through that unit once again before you start this unit.

In the next unit we shall aim to study the analysis of motion from the point of view of a non-inertial observer.

Objectives

After studying this unit you should be able to

- identify a rigid body
- distinguish between the features of translational and rotational motion of a rigid body
- outline the features of the general motion of a rigid body
- explain the significance of moment of inertia of a rigid body about a certain axis
- solve problems based on the concept of rotational dynamics of rigid bodies.

9.2 A RIGID BODY AND ITS MOTION

Let us consider the motion of a Yo-Yo (Fig.9.1). It runs up and down as the spool winds and unwinds. The Yo-Yo rotates about an axis passing through its centre and perpendicular to the plane of this paper. You can see that this axis does not remain fixed in space. It moves vertically downward or upward with the Yo-Yo. In principle we can use Newton's laws of motion to analyse such a motion as each particle of the Yo-Yo obeys them. But obtaining a description on a particle-by-particle basis will be an uphill task as the number of particles is very large. So we would like to find a simple method for analysing the general motion of an extended body like a Yo-Yo. We can find such a method by using the model of a rigid-body. So let us first learn what a rigid body is.

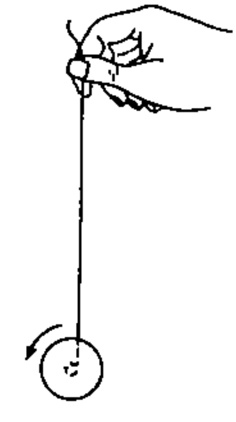


Fig. 9.1: A Yo-Yo

9.2.1 What is a Rigid Body ?

You must have seen a wheel rotating about its axle. Let us consider any two points on the wheel. We find that the relative separation between them does not change when the wheel is in motion. But if we take the example of the diver of Fig. 7.12 we find that the relative separation between two different parts of her body does change. The former is an example of a rigid body whereas the latter is not.

Technically speaking, a rigid body is defined as an aggregate of point masses such that the relative separation between any two of these always remains invariant, i.e. for any position of the body r_{ik} = a constant (Fig. 9.2). So a rigid body is one which has a definite shape. It does not change even when a deforming force is applied. In nature there is no perfectly rigid body as all real bodies experience some deformation when forces are exerted. So a perfectly rigid body can only be idealised. But we shall see that this model is quite useful in cases where such deformations can be ignored. For example, the deformation of a cricket ball as it bounces off the ground can be ignored. You know that if a heavy block is dragged along a plane, frictional force acts on it (see Sec. 2.2.2 of Block 1). But its deformation due to the frictional force can be neglected. However, you cannot neglect the deformation of a railway track due to the weight of the train. Likewise, the deformation of the fibre glass pole used by a pole-vaulter can also not be neglected. So in the last two cases we cannot apply the rigid body model.

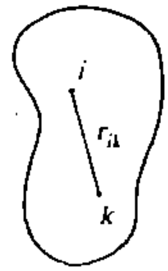


Fig. 9.2: For any position of a rigid body, r_{ik} = a constant

You may now like to identify the objects that can be approximated by the rigid body model.

SAQ 1

Which of the following can be considered as rigid bodies ?

a) A top b) A rubber band c) A bullet d) A balloon e) The earth.

Let us now study the motion of a rigid body. A rigid body can execute both translational and rotational motion. Let us discuss their basic features.

9.2.2 Translational Motion of a Rigid Body

Suppose you are travelling in a train. Then during a certain interval of time your displacement will be exactly equal to that of your co-passenger provided both of you do not move with respect to the train. This will also be true for any two objects attached to the body of the train, say a bulb and a switch. This is the characteristic of translational motion. A rigid body is said to execute pure translational motion if each particle in it undergoes the same displacement as every other particle in any given interval of time. Translational motion of a rigid body is shown schematically in Fig. 9.3.

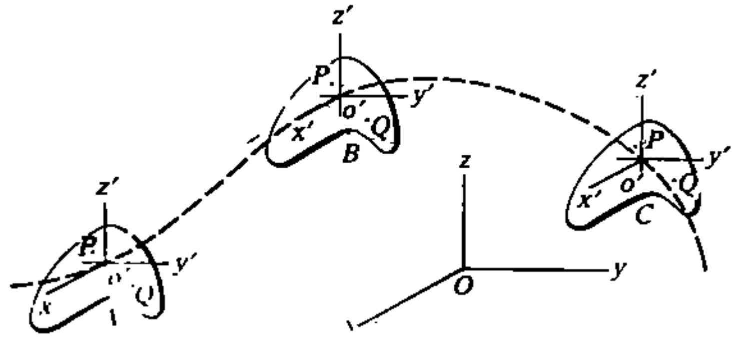


Fig. 9.3: Translational motion of a rigid body

You must have noted that the path taken is not necessarily a straight line. Now let us measure the magnitudes of the displacements of the points P, O', Q as the body moves from the position A to B. Each is equal to 3.9 cm and the lines joining these positions are parallel to each other. So they undergo the same displacement. You may verify the same for the motion of the body between positions B and C.

SAQ 2

(a) Measure the magnitudes of the displacements of P, O' and Q between positions B and C and verify that they are equal.

(b) Give two examples of a pure translational motion.

Now that you have worked out SAQ 2, you can see that if we are able to describe the motion of a single particle in the body, we can describe the motion of the body as a whole. We have done this exercise a number of times before. However, you may like to consolidate your understanding by working out the following SAQ.

SAQ 3

A rigid body of mass M is executing a translational motion under the influence of an external force $\mathbf{F}_e$. Suggest a suitable differential equation of motion of the body.

What does the answer to SAQ 3 signify? We know that the relative separation between any two points of a rigid body does not change, i.e.

$$\frac{dr_{ik}}{dt} = 0. \qquad (9.1)$$

So all the points follow the same trajectory as the c.m. Hence, for studying translational motion, the body may be treated as a particle of mass M located at its c.m. You may recall that we had treated the sun and a planet as particles in Unit 6. They were treated as particles as their sizes are negligible compared to the distances between them and also because the shapes of these bodies were insignificant. But here we are considering a rigid body as a particle for another reason as explained above.

Thus we can represent the translational motion of a rigid body as a whole in terms of the motion of its c.m. It becomes easier to describe the translational motion in this way. In the previous units we have dealt with cases like a body falling down an inclined plane, a cricket ball hit by a bat, etc. There we had applied the above idea. So before we go over to the next sub-section it would be worthwhile to know about the position of c.m. of a rigid body.

The problem of locating the c.m. of a rigid body is complicated when its shape is asymmetrical. However, we shall deal mostly with bodies having a symmetrical shape. Positions of c.m.s (c) of several symmetrical bodies have been shown in Fig. 9.4.

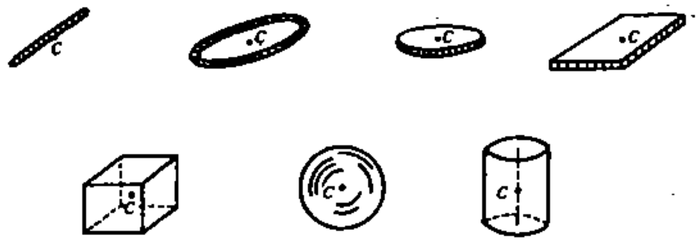


Fig. 9.4: Centres-of-mass of symmetrical rigid bodies

Let us now discuss the rotational motion of a rigid body.

9.2.3 Rotational Motion of a Rigid Body

Let us consider the motion of the earth. Every point on it moves in a circle (the corresponding latitude), the centres of which lie on the polar axis. Such a motion is an example of a rotational motion. A rigid body is said to execute rotational motion if all the particles in it move in circles, the centres of which lie on a straight line called the axis of rotation. Fig. 9.5 shows the rotational motion of a rigid body about the z-axis. When a rigid body rotates about an axis every particle in it remains at a fixed distance from the axis. So each point in the body, such as P, describes a circle about this axis. You must have realised that perpendiculars drawn from any point in the body to the axis will sweep through the same angle as any other such line in any given interval of time.

Fig. 9.5: An example of rotational motion of a rigid body.

We shall now study about the general motion of a rigid body.

9.2.4 General Motion of a Rigid Body

The general motion of a rigid body is a combination of translation and rotation. This can be understood by considering a simple example shown in Fig. 9.6.

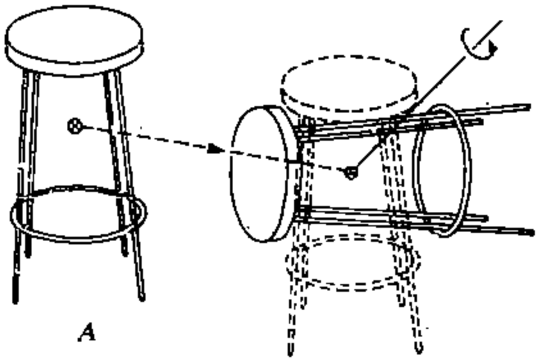


Fig. 9.6: To bring the body from position A to some new position B, first translate it so that the centre-of-mass coincides with the new centre-of-mass, and then rotate it around the appropriate axis through the centre-of-mass until the body is in the desired position.

You may now perform an activity for the sake of better understanding of the general motion of a rigid body.

Activity

Take any book lying on your table and keep it in the bookshelf in its erect posture.

Here you first shift the c.m. of the book to a new position. Then you turn the book about a suitable axis through the c.m. to make it stand erect on the shelf. So you can see that the above motion of the book is a combination of translation and rotation. Now, study the following figure carefully.

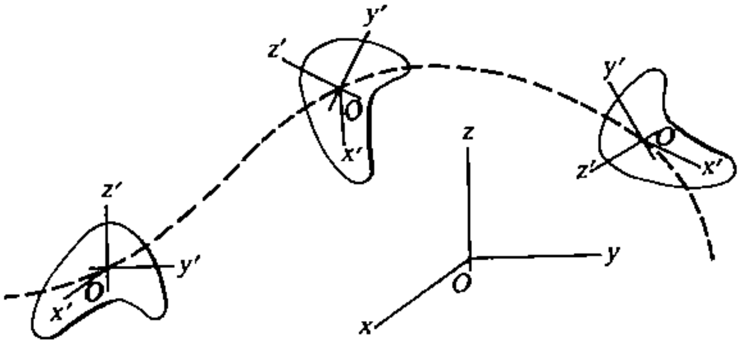


Fig. 9.7: A rigid body moving in combined translational and rotational motion as seen from reference frame (x, y, z). Notice that the reference frame fixed on the body (x', y', z') changes its orientation with respect to (x, y, z) as the motion proceeds.

Fig. 9.7 shows a case of combined translational and rotational motion of a rigid body. It can be considered as a schematic extension of Fig. 7.12. Study Fig. 9.7 and work out the following SAQ.

SAQ 4

Compare Figs. 9.3 and 9.7. Mention very briefly the distinctive features in respect of the observer's reference axes (x, y, z) and the body-fixed axes (x', y', z').

Now that you have worked out SAQ 4 you can realise that determining the location O in Fig. 9.7 is the good old problem of the motion of c.m. which we have studied in detail. As stated earlier our chief concern in this unit is to suitably study the rotational aspect. For this we have to develop a formalism to analyse rotational motion of a rigid body. Now, in Unit 4 of Block 1, you have already studied the dynamics of angular motion of a particle. We shall only make an extension of that study here.

Recall from Sec. 4.3.3 of Block 1 that a particle executing rotational motion possesses a moment of inertia (denoted by I). For rotational motion I plays the same role as the mass of the particle plays for translational motion. It is very important to understand the meaning of moment of inertia of a rigid body for its rotational motion. So let us now learn about the '*moment of inertia*' of a rigid body. We shall start by determining the angular momentum of a rotating rigid body about the axis of rotation.

9.3 MOMENT OF INERTIA

We know that the earth rotates about the line joining the poles which passes through the centre of earth. How can we calculate its angular momentum about the axis of rotation? We know that when the earth rotates about its axis, every point on it executes a uniform circular motion about this axis. The radius of this circle decreases with the latitude of the point. The circle along which New Delhi moves has a smaller radius than that along which Trivandrum moves. So the linear velocity of each point is in general different. In order to determine the angular momentum of a body we shall first have to determine the angular momentum of each particle in it. And as angular momentum is a vector quantity we shall add vectorially the individual angular momenta to get the angular momentum of the body. Let us now consider a general situation.

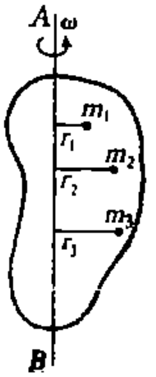


Fig. 9.8: A rigid body rotating about an axis AB.

Refer to Fig. 9.8. A rigid body is rotating about an axis AB fixed in an inertial frame with a uniform angular speed ω. Three point masses m_1, m_2, m_3 at distances r_1, r_2, r_3, respectively, from AB have been shown. m_1 moves along a circle of radius r_1 and let its velocity be $\mathbf{v}_1$. Using Eq. 4.23 of Unit 4, Block 1, we may say that the angular momentum $\mathbf{L}_1$, of m_1 is given by

$$\mathbf{L}_1 = m_1 \mathbf{r}_1 \times \mathbf{v}_1$$

Now the mass m_1 is rotating along a circle of radius r_1 whose plane is perpendicular to AB. In fact every point mass is moving along a circle whose plane is perpendicular to AB. Using Eq. 4.13a of Unit 4, Block 1, we get

$$\mathbf{v}_1 = \dot{r}_1 \hat{\mathbf{r}}_1 + r_1 \dot{\theta}_1 \hat{\boldsymbol{\theta}}_1,$$

where $\hat{\mathbf{r}}_1$ is the unit vector along $\mathbf{r}_1$ and $\hat{\boldsymbol{\theta}}_1$ is perpendicular to $\hat{\mathbf{r}}_1$ in the sense of increasing angle θ_1. You may recall that the directions of $\hat{\mathbf{r}}_1$ and $\hat{\boldsymbol{\theta}}_1$ change with time. Again $\dot{\theta}_1 = \omega$, which is same for all the point masses. So we get,

$$\mathbf{L}_1 = m_1 r_1 \hat{\mathbf{r}}_1 \times (\dot{r}_1 \hat{\mathbf{r}}_1 + r_1 \dot{\theta}_1 \hat{\boldsymbol{\theta}}_1)$$

Now, $\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_1 = 0$ and $\hat{\mathbf{r}}_1 \times \hat{\boldsymbol{\theta}}_1 = \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector along BA (See SAQ 3c of Unit 4 of Block 1).

$$\therefore \; \mathbf{L}_1 = m_1 r_1^2 \omega \hat{\mathbf{n}} = m_1 r_1^2 \boldsymbol{\omega}.$$

Similarly $\mathbf{L}_2 = m_2 r_2^2 \boldsymbol{\omega}$, $\mathbf{L}_3 = m_3 r_3^2 \boldsymbol{\omega}$ and so on.

So the angular momentum of the body is given by

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \ldots\ldots \\ &= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots.) \boldsymbol{\omega} \\ &= I\boldsymbol{\omega}, \end{aligned} \tag{9.2}$$

$$\text{where } I = \sum_i m_i r_i^2 \tag{9.3}$$

is called the **moment of inertia** of the body about the given axis of rotation. Here the summation extends over all the point masses that constitute the body. The SI unit of moment of inertia is kg m^2.

As stated in Sec. 9.2.3 the general motion of a rigid body can be considered as a translational motion of its c.m. and a rotational motion about its c.m. Hence, the considerations of this unit apply also to rotations about an axis that is *not* fixed in an inertial frame, provided the axis passes through the c.m. and the moving axis always has the same direction in space.

In situations involving asymmetric objects, $\mathbf{L}$ and $\boldsymbol{\omega}$ may be in different directions. In that case $\mathbf{I}$ cannot be expressed as a single number but in a more complicated mathematical form called tensor.

If the mass of the body be M then we can express I as $I = Mk^2$, (9.4)

where k is a quantity having the dimension of length. This quantity is called the *radius of gyration*. If we compare Eq. 9.4 with Eq. 4.21b of Unit 4, we find that k is equivalent to the distance from the axis of rotation of the point where the entire mass of the body can be considered to be concentrated. In other words it is the distance between the axis of rotation and the c.m. of the body.

Did you notice the similarity between the expression (9.2) and the same for linear momentum (i.e. $M\mathbf{v}$). Since $\boldsymbol{\omega}$ is analogous to $\mathbf{v}$ (see Table 4.1), I must be analogous to M. In other words, I is the rotational analogue of mass about which you have read in Sec. 4.3.3. This analogy also becomes evident from the expression of K.E. of rotation K which you may work out in the following SAQ.

SAQ 5

Show that for the body in Fig. 9.8 the K.E. of rotation is given by

$$K = \frac{1}{2} I \omega^2 \tag{9.5}$$

Compare the expression for K with that of K.E. of linear motion and find the rotational analogue of mass.

So far in this section we have considered a case where the axis of rotation lies within the body. The above analysis also holds if the axis lies outside the body; e.g. the bob of a conical pendulum (Fig. 9.9).

Fig. 9.9: B is the bob of a conical pendulum and OA, the axis of rotation.

Now that you have understood the meaning of the term 'moment of inertia' we may proceed to study the method for its determination.

9.3.1 Determination of Moment of Inertia of a Rigid Body

We shall now put Eq. 9.3 to use. To start with let us try to determine the moment of inertia of a dumb-bell (Fig. 9.10a). We shall assume that the thin rod joining the masses m_1 and m_2 is of negligible mass. We shall also consider m_1 and m_2 as point masses. These assumptions may appear oversimplifying. But this model finds many applications in molecular spectroscopy as this can represent a diatomic molecule. Let us first work out the following example related to the determination of moment of inertia of the dumb-bell. Then we shall study an application of this model.

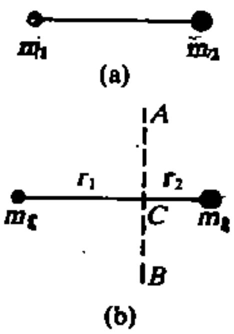


Fig. 9.10: (a) A dumb-bell having masses m_1 and m_2 at its ends; (b) the determination of moment of inertia of the dumb-bell in (a) about an axis passing through the c.m. of m_1 and m_2 and perpendicular to the line joining them.

Example 1

Refer to Fig.9.10b. *AB* is perpendicular to the line joining the masses m_1 and m_2 and it passes through *C*, the c.m. Using the assumptions stated above show that the moment of inertia of the system described in Fig. 9.10b is μr^2, where μ is the reduced mass of the system and r is the distance between the masses.

For the given system the summation of Eq. 9.3 will have two terms. i.e.

$$I = m_1 r_1^2 + m_2 r_2^2 .$$

Since C is the c.m. we have $m_1 r_1 = m_2 r_2$

$$\text{or} \quad \frac{r_1}{m_2} = \frac{r_2}{m_1} = \frac{r_1 + r_2}{m_2 + m_1} \quad \text{(by addendo)}$$

$$\therefore \quad r_1 = \frac{m_2 r}{m_1 + m_2}, \; r_2 = \frac{m_1 r}{m_1 + m_2} \quad (\because r = r_1 + r_2)$$

$$I = m_1 \left(\frac{m_2}{m_1 + m_2} r\right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} r\right)^2 = \frac{m_1 m_2}{m_1 + m_2} r^2.$$

Hence, using Eq. 7.6 we get,

$$I = \mu r^2.$$

You may now like to study an application of Example 2 by working out the following SAQ.

SAQ 6

The atoms in the oxygen molecule (O_2) may be considered to be point masses separated by a distance of 1.2°A. The molecular speed of an oxygen molecule at s.t.p. is 460m s^{-1}. It is known that the rotational K.E. of the molecule is 2/3 of its translational K.E. Calculate its angular velocity at s.t.p. assuming that molecular rotation takes place about an axis through the c.m. of, and perpendicular to the line joining the atoms.

We have just now applied Eq. 9.3 to determine the moment of inertia of a system made up of discrete particles. In each of the systems (dumb-bell and diatomic molecule) the total mass is distributed among particles which are not attached to one another, i.e. the particles that comprise the system can be enumerated. We shall now take up the case of systems where there is a continuous distribution of matter. Here the particles cannot be enumerated. For example, we have bodies like a uniform rod, a sphere, a cylinder and so on. For that we shall modify Eq. 9.3 in the following manner

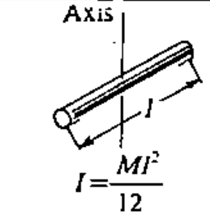
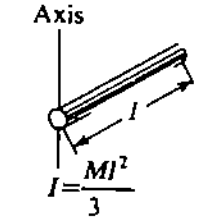
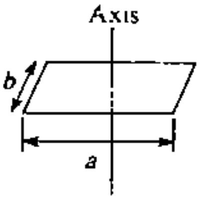
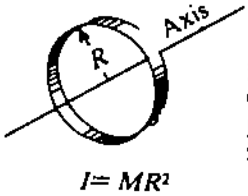
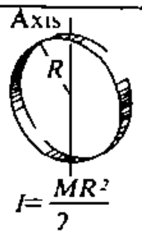
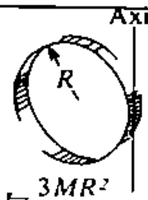
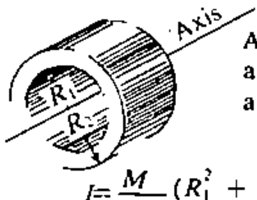
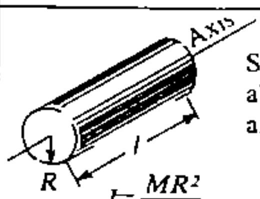
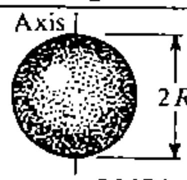
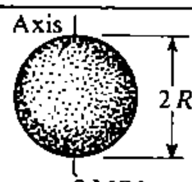
Let r be the perpendicular distance of an infinitesimal mass Δm of the body from the axis. Then from Eq. 9.3, we get

$$I = \lim_{\Delta m \to 0} \Sigma r^2 \Delta m = \int r^2 dm \qquad (9.6)$$

where Δm gets replaced by dm, the differential of mass and the summation by integral. The integral is a definite one extending over the entire body. Using Eq. 9.6 the moments of inertia of symmetrical bodies about certain axes can be determined.

The moments of inertia about certain axes of a few common symmetrical bodies have been given in Table 9.1 (In all cases M represents the mass of the body in the diagram). We have derived these results in Appendix B.

Table 9.1

 Thin rod about axis through centre perpendicular to length $I = \frac{Ml^2}{12}$ (a)	 Thin rod about axis through one end perpendicular to length $I = \frac{Ml^2}{3}$ (b)
 Rectangular plate about axis through centre and perpendicular to its plane $I = \frac{M}{12}(a^2 + b^2)$ (c)	 Ring about axis passing through centre and perpendicular to its plane $I = MR^2$ (d)
 Ring about any diameter $I = \frac{MR^2}{2}$ (e)	 Ring about any tangent line $I = \frac{3MR^2}{2}$ (f)
 Annular cylinder about cylinder axis $I = \frac{M}{2}(R_1^2 + R_2^2)$ (g)	 Solid cylinder about cylinder axis $I = \frac{MR^2}{2}$ (h)
 Solid sphere about any diameter $I = \frac{2MR^2}{5}$ (i)	 Thin spherical shell about any diameter $I = \frac{2MR^2}{3}$ (j)

So you have understood the meaning of the term 'moment of inertia'. You have also come to know the value of moments of inertia of several bodies about certain axes. So we may proceed to study the dynamics of rotational motion.

9.4 ROTATIONAL DYNAMICS OF A RIGID BODY

You know that dynamics is the study of accelerated motion and its causes. For translational motion it is governed by Newton's second law, i.e.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The rotational analogue of Newton's second law of motion, as you know (see Eq. 4.24 of Unit 4, Block 1) is given by

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

where $\boldsymbol{\tau}$ is the torque acting on the particle and $\mathbf{L}$ its angular momentum and I the moment of inertia about the axis of rotation. You may recall that you have studied the dynamics of angular motion of a particle in Sec. 4.3 of Block 1. We shall now apply the concepts you have studied in Unit 4 of Block 1 mostly to a rigid body. You have studied the necessary principles and laws there. We shall now list them in the Table 9.2. Here we have shown the equivalent aspects of translational and rotational motion. A few spaces have been left blank which you may fill in.

Table 9.2

S.No.	Translational Motion	Rotational Motion
i)	Position, $\mathbf{r}$	Angular position, θ
ii)	Velocity, $\mathbf{v} = \frac{d\mathbf{r}}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
iii)	Acceleration, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$	Angular acceleration, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
iv)	Mass, m	Moment of inertia, I
v)	Linear momentum, $\mathbf{p} = m\mathbf{v}$	Angular momentum, $\mathbf{L} = I\omega$
vi)	Force, $\mathbf{F}$	Torque, τ
vii)	Newton's second law $\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$	Analogue of second law $\tau = \frac{d\mathbf{L}}{dt}$ = ...
viii)	Work done $= \int \mathbf{F}.d\mathbf{r}$	Work done $= \int \tau.d\theta$
ix)	K.E. $= \frac{1}{2}mv^2$	K.E. = ...
x)	Principle of conservation of linear momentum: When the net external force acting on a body is zero the linear momentum of its c.m. remains constant.	....
xi)	Impulse $= \int_{t_1}^{t_2} \mathbf{F}(t)dt = \mathbf{p}(t_2) - \mathbf{p}(t_1)$	Angular Impulse = ...

SAQ 7

Fill in the blank spaces of Table 9.2.

Now that you have studied Table 9.2 and worked out SAQ 7, we can discuss some applications of the principles of rotational dynamics. We shall start with the rotational analogue of Newton's second law.

9.4.1 Rotational Analogue of Newton's Second Law

We have used the equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ to describe the dynamics of linear motion of a body. For a system having constant mass this equation becomes $\mathbf{F} = m\mathbf{a}$. To study the rotational dynamics of a body we first need to know its moment of inertia I about the axis of rotation. Then we shall use the rotational analogue of the above equation, i.e.

$$\tau = \frac{d\mathbf{L}}{dt}$$

Now, we know from Eq. 9.2 that, $\mathbf{L} = I\omega$. $\therefore \tau = \frac{d}{dt}(I\omega)$.

For a system having constant I, we get

$$\tau = I\frac{d\omega}{dt} = I\alpha \quad (9.7)$$

τ in Eq. 9.7 is the net torque acting on the body. So we must take care to determine all torques that act on the body and take their vector sum to obtain the net torque.

We have studied about the linear motion of a many-particle system in Sec. 7.3. There we found that only external forces matter. The internal forces cancel in pairs according to Newton's third law. Now, let us see what happens in the case of internal torques. Refer to Fig. 9.11. It shows two particles 1 and 2 of a rigid body. The internal force on 1 due to 2 is $\mathbf{F}_{21}$ and that on 2 due to 1 is $\mathbf{F}_{12}$. Let us find out the total internal torque about a point O due to these forces. You may recall from Eq. 1.16 of Block 1 that this total internal torque is given by,

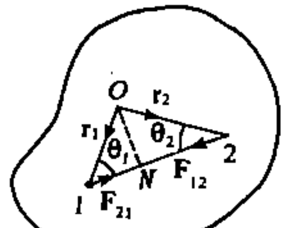


Fig. 9.11: Internal forces on two particles of a rigid body.

$$\boldsymbol{\tau}_{int} = \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12}$$

Now, $\mathbf{r}_1 \times \mathbf{F}_{21} = r_1 F_{21} \sin(\pi - \theta_1)\, \hat{\mathbf{n}} = F_{21} r_1 \sin \theta_1 \hat{\mathbf{n}}$,

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane of this page and pointing towards you. And,

$$\mathbf{r}_2 \times \mathbf{F}_{12} = r_2 F_{12} \sin(\pi - \theta_2)\,(-\hat{\mathbf{n}}) = -F_{12} r_2 \sin \theta_2\, \hat{\mathbf{n}}$$

$$\therefore \quad \boldsymbol{\tau}_{int} = (F_{21} r_1 \sin \theta_1 - F_{12} r_2 \sin \theta_2)\, \hat{\mathbf{n}}. \qquad (9.8)$$

We know from Newton's third law that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal and opposite. So $F_{12} = F_{21}$. Again, we can see from Fig. 9.11 that,

$$r_1 \sin \theta_1 = r_2 \sin \theta_2 = ON,$$

where ON is the length of the perpendicular drawn from O on the line joining the points 1 and 2. Hence, from Eq. 9.8, we get

$$\boldsymbol{\tau}_{int} = \mathbf{0}$$

So we see that internal torques cancel in pairs. Thus, the torque in Eq. 9.7 is the net external torque.

Let us now work out an example to illustrate Eq. 9.7. You will find that the situation is analogous to the case of accelerated linear motion as the applied torque and the angular velocity of the rotating body are in the same direction.

Example 2

A solid cylinder of mass M is mounted on a horizontal axle over a well (Fig. 9.12a). A rope is wrapped around the cylinder and a bucket of mass m is suspended from the rope. Find in terms of m, M and g an expression for the acceleration of the bucket as it falls down. Neglect the mass of the rope and any friction between the axle and the cylinder. Assume that the rope does not slip over the cylinder as it unwinds.

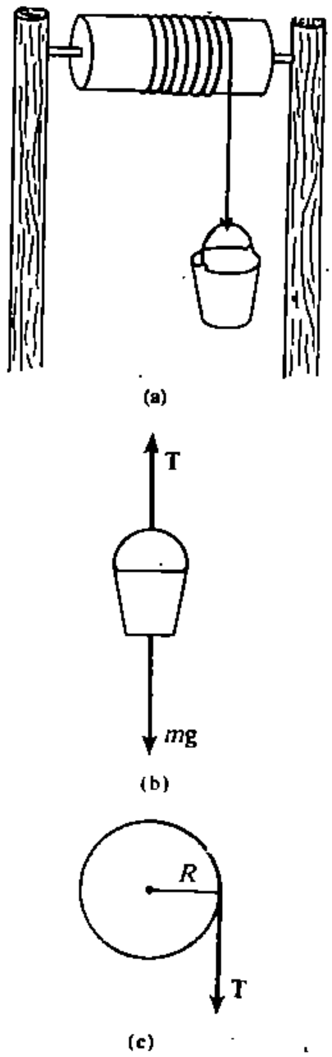


Fig. 9.12: Diagram for Example 2

If the bucket were not connected to the cylinder it would have accelerated downward at the rate g. But now there is an upward tension T on the bucket due to the rope. It reduces the net downward force on the bucket. It also exerts a torque on the cylinder. The magnitude of the downward force on the bucket (Fig. 9.12b) is given by

$$F = mg - T.$$

But $F = ma$, where a is the linear acceleration of the bucket.

$$\therefore\ ma = mg - T. \qquad (9.9)$$

If we take the end view of the cylinder (Fig 9.12c), we see that the rope exerts a torque of magnitude $\tau (= RT)$ on the cylinder. This gives rise to an angular acceleration α given by Eq. 9.7 as

$$\alpha = \frac{\tau}{I} = \frac{RT}{I}, \qquad (9.10)$$

where I is the moment of inertia of the cylinder about the axis.

Since the rope unwinds without slipping, a is related to α. Using Eq. 4.11a of Block 1 we get from Eq. 9.10 that,

$$a = \alpha R = \frac{R^2 T}{I} \qquad (9.11)$$

Hence, from Eqs. 9.9 and 9.11 we get

$$ma = mg - \frac{Ia}{R^2}$$

$$\therefore \quad \left[m + \frac{I}{R^2}\right]a = mg$$

$$\text{or} \qquad a = \frac{mg}{m + \dfrac{I}{R^2}} \qquad (9.12a)$$

We know from result (h) of Table 9.1 that for the cylinder $I = \frac{1}{2}MR^2$. So, we can rewrite Eq. 9.12a as

$$a = \frac{mg}{m + \dfrac{M}{2}} \qquad (9.12b)$$

Eq. 9.12b indicates that if $M << m$, then $a \approx g$. In other words if the mass of the cylinder is very small compared to that of the bucket then the rotation of the cylinder does not matter. The acceleration of the bucket is simply equal to g.

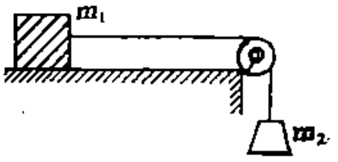


Fig. 9.13: The falling mass m_2 can provide horizontal acceleration to m_1.

However, in general we can say that the gravitational force on the bucket does not only provide its linear acceleration, it also gives rise to the angular acceleration of the cylinder. As a result, the linear acceleration of the bucket decreases. A falling mass can provide horizontal acceleration to another mass (Fig. 9.13). From Example 2, we have just now seen that a falling mass can also generate angular acceleration in another body.

So we have learnt to apply the rotational analogue of Newton's second law of motion. This study throws some light on the concept of equilibrium of a body. You may recall that we have studied about equilibrium of forces in Sec 2.2.2. of Block 1. A body had been said to be in equilibrium if the vector sum of all the forces acting on it is zero. This is equivalent to saying that the linear acceleration of the c.m. of the body is zero. But we know that the general motion of a rigid body is a combination of the translational motion of the c.m. and a rotational motion about an axis passing through the c.m. So we can say that our study in Sec. 2.2.2 of Block I was restricted to the case of translational equilibrium only. The general condition of equilibrium of a body must include the rotational aspect too. We shall study briefly about this now.

Equilibrium of a Rigid Body

A rigid body is said to be in mechanical equilibrium if with respect to an inertial frame

(i) the linear acceleration $\mathbf{a}_{cm}$ of its c.m. is zero and (ii) its angular acceleration α about any axis fixed in this frame is zero.

The above conditions do not imply that the body must be at rest with respect to the frame. It should only be unaccelerated. Its c.m., for example, may be moving with a constant velocity $\mathbf{v}_{cm}$ and the body may be rotating about a fixed axis with a constant angular velocity.

The translational motion of the body, as you know is governed by the equation.

$$\mathbf{F}_e = M\,\mathbf{a}_{cm},$$

where $\mathbf{F}_e$ is the net external force acting on the body of mass M. So condition (i) may be expressed as follows : **The vector sum of all the external forces acting on the body is zero.** In other words, if a rigid body is in translational equilibrium under the action of several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and so on, we may write the above condition as

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0 \qquad (9.13a)$$

The other conditiopn is given by $\alpha = \mathbf{0}$ for any axis. We know that the angular acceleration of a rigid body is related to the net external torque τ as

$$\tau = I\alpha$$

where I is the moment of inertia of the body about the axis of rotation. So condition (ii) may be expressed as follows : **The vector sum of all the external torques acting on the body is zero.** In other words if a rigid body is in rotational equilibrium under the action of several torques τ_1, τ_2, τ_3 and so on we may express this condition as

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 + ... = 0 \qquad (9.13b)$$

Hence, a rigid body is said to be in mechanical equilibrium if both the conditions 9.13 a and b hold.

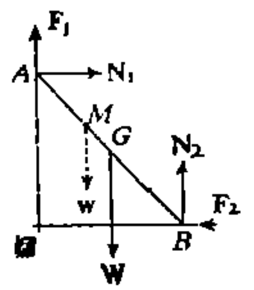


Fig. 9.14: A ladder in equilibrium

Let us take the example of a man standing on a ladder (Fig. 9.14). Suppose that the entire system is in equilibrium when the man is at the point M of the ladder AB. We shall first find out what are the forces acting on the system. The weight of the man acting vertically downwards through M is $\mathbf{w}$. The weight $\mathbf{W}$ of the ladder acts vertically downward through its mid point G. $\mathbf{N}_1$ and $\mathbf{N}_2$ are the normal reactions at the points of contact A and B of the ladder with the vertical and horizontal surfaces, respectively. Since the point A has a tendency to slip towards O, the force of friction $\mathbf{F}_1$ at A acts along OA. Again B has a tendency to slip along OB. So the force of friction $\mathbf{F}_2$ at B is along BO. So condition (9.13a) demands that

$$\mathbf{w} + \mathbf{W} + \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F}_1 + \mathbf{F}_2 = 0.$$

Now let us define the Cartesian x and y-axes along OB and OA, respectively. Then the above condition may be written as

$$-w\,\hat{\mathbf{j}} - W\,\hat{\mathbf{j}} + N_1\,\hat{\mathbf{i}} + N_2\hat{\mathbf{j}} + F_1\,\hat{\mathbf{j}} - F_2\hat{\mathbf{i}} = 0$$

$$\text{or} \quad (N_1 - F_2)\,\hat{\mathbf{i}} + (N_2 + F_1 - w - W\,)\,\hat{\mathbf{j}} = 0.$$

Hence, we get

$$N_1 - F_2 = 0, \; N_2 + F_1 - w - W = 0$$

$$\text{i.e.} \;\; N_1 = F_2 \text{ and } N_2 + F_1 = w + W. \qquad (9.14a)$$

Now, we shall take care of the condition (9.13b). For this we have to determine the total torque acting on the system about any point. The choice of this point is quite important. A proper choice helps us in getting the final condition in a simple form. Let us see how. If we select the point A, the torques of $\mathbf{F}_1$ and $\mathbf{N}_1$ vanish. Similarly for the point B, the torques of $\mathbf{F}_2$ and $\mathbf{N}_2$ vanish. So if we select any one of these two points, we may get rid of the expressions of torques of a pair of forces while writing the condition (9.13b). This considerably simplifies the final condition. However, the meaning of the condition is independent of the choice of the point about which the torques are being determined.

So, let us now write (9.13b) with reference to the point B. We have,

$$\mathbf{AB} \times \mathbf{F}_1 + \mathbf{AB} \times \mathbf{N}_1 + \mathbf{MB} \times \mathbf{w} + \mathbf{GB} \times \mathbf{W} = 0$$

Now, let $AB = 2l$, $BM = a$ and $\angle OBA = \theta$ (Fig. 9.14). So we get

$$2l\,F_1 \sin(90° + \theta)\,\hat{\mathbf{k}} + 2lN_1 \sin\theta\,\hat{\mathbf{k}} - aw\sin(90° - \theta)\hat{\mathbf{k}} - lW\sin(90° - \theta)\hat{\mathbf{k}} = 0,$$

where $\hat{\mathbf{k}}$ is the unit vector perpendicular to the xy - plane and pointing towards you.

$$\text{or } 2lF_1 \cos\theta + 2lN_1 \sin\theta - aw\cos\theta - lW\cos\theta = 0,$$

$$\text{or } \cot\theta = \frac{2lN_1}{aw + lW - 2lF_1} \qquad (9.14b)$$

So for the equilibrium of the system (ladder and man) both the equations (9.14a and 9.14b) should hold good.

So far we have studied how to apply the rotational analogue of Newton's second law of motion. In Unit 3 of Block 1 you have read about 'Work and Energy', as applied to linear motion. We shall now study about these quantities with reference to rotational motion of a rigid body.

9.4.2 Work and Energy in Rotational Motion

In general work done by a force **F** during linear motion is given by

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

where $d\mathbf{r}$ is an infinitesimal displacement. $\boldsymbol{\tau}$ is the rotational analogue of **F**. The angular displacement $\boldsymbol{\theta}$ is the analogue of **r**. So work done in rotational motion by a torque can be obtained by replacing **F** with $\boldsymbol{\tau}$ and **r** with $\boldsymbol{\theta}$ in the above expression of W. It is given by

$$W_{rot} = \int \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \qquad (9.15a)$$

For a constant torque acting in the direction of the angular displacement, we get

$$W_{rot} = \tau\, \Delta\theta \qquad (9.15b)$$

where $\Delta\theta$ is the overall angular displacement.

Let us now apply Eq. 9.15b to a simple example.

Example 3

An automobile engine develops 72kW of power when rotating at a rate of 1800 r.p.m. What torque does it deliver?

Power is the rate of doing work. Now if the work W_{rot} $(= \tau\Delta\theta)$ is done in a time Δt, then the power will be given by

$$P = \frac{\tau\, \Delta\theta}{\Delta t}$$

where $\dfrac{\Delta\theta}{\Delta t} = \omega$ = the angular speed,

$$\text{or} \quad \tau = \frac{P}{\omega}$$

For this example, $P = 72 \times 10^3\,\mathrm{W} = 72 \times 10^3\,\mathrm{kg\,m^2\,s^{-3}}$

and $\omega = 2\pi \times \dfrac{1800}{60}\,\mathrm{rad\,s^{-1}} = 60\pi\,\mathrm{rad\,s^{-1}}$

$$\therefore\ \tau = \frac{72 \times 10^3\,\mathrm{kg\,m^2\,s^{-3}}}{60\pi\,\mathrm{rad\,s^{-1}}} = 382\,\mathrm{Nm}.$$

You may recall that here Nm is not equivalent to joule.

Let us now discuss the K.E. of rotation. We have derived the expression for the K.E. of a rotating body in Sec. 9.3. It is given by

$$K_{rot} = \frac{1}{2} I\omega^2, \qquad (9.16)$$

where I is the moment of inertia of the body about the axis of rotation and ω is its angular speed. We shall now apply Eq. 9.16 to discuss briefly about the motion of rolling objects.

Rolling Objects

A rolling object exhibits both rotational and translational motion. As the object moves forward, it rotates about a point that is itself moving along a straight line. How do we express the total K.E. of such a rolling object? The expression must contain both the translational and rotational K.E. So the total K.E. is given by

$$K = K_{trans} + K_{rot}$$

$$K_{trans} = \frac{1}{2} M v_{cm}^2 \quad , \quad K_{rot} = \frac{1}{2} I_{cm}\omega^2,$$

where M is the mass of the object, v_{cm} is the speed of the c.m., I_{cm} is the moment of inertia of the object about an axis passing through the c.m. and ω the angular speed.

Thus. $K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm}\omega^2$.

Now, if the object has a radius R and it is **rolling without slipping,** then $\omega = v_{cm}/R$. Hence for an object which is rolling without slipping,

$$K = \frac{1}{2} \left(M + \frac{I_{cm}}{R^2}\right) v_{cm}^2. \tag{9.17}$$

Let us apply Eq. 9.17 to work out the following example.

Example 4

A solid cylinder and a solid sphere, each of the same mass M and radius R, start from rest and roll without slipping down an inclined plane (Fig. 9.15). Which one reaches the bottom of the incline first?

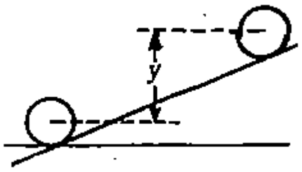


Fig. 9.15: Diagram for Example 4

Let the finishing line be at a vertical distance y below the starting line. The object whose c.m. finishes with greater speed reaches first. Using Eq. 9.17 and applying the principle of conservation of energy, we get

$$Mgy = \frac{1}{2}\left[M + \frac{I_{cm}}{R^2}\right] v_{cm}^2$$

or
$$v_{cm}^2 = \frac{2gy}{1 + \frac{I_{cm}}{MR^2}}. \tag{9.18}$$

For the solid cylinder, $I_{cm} = \frac{1}{2} MR^2$ or $v_{cm}^2 = \frac{4}{3} gy$

and for the solid sphere $I_{cm} = \frac{2}{5} MR^2$ or $v_{cm}^2 = \frac{10}{7} gy$.

Since $(10/7) > (4/3)$, we find that the sphere reaches first. You may like to work out an SAQ based on the above concept.

SAQ 8

A spherical ball rolls without slipping down a slope of vertical height 35 cm, and reaches the bottom moving at $2 ms^{-1}$. Is the ball hollow or solid?

So far you have studied some applications of the principles of rotational dynamics. You may recall from Sec. 4.4.2 of Block 1 that the principle of conservation of angular momentum is used widely in physics. We have already studied some applications of this principle in Unit 4 of Block 1. The law of equal areas which you have read in Unit 6 is also an application of this principle. We shall now review the principle of conservation of angular momentum and study some other of its applications.

9.4.3 Conservation of Angular Momentum and its Applications

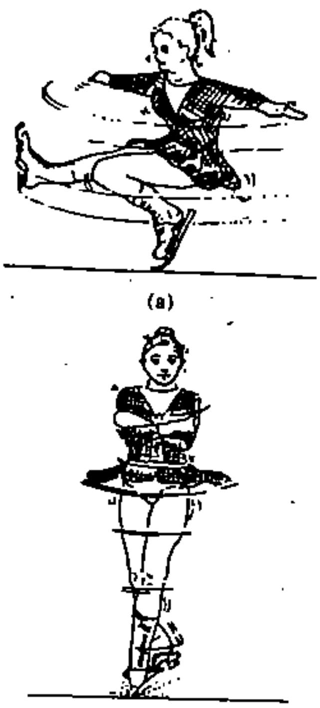
Now, you are quite familiar with the relation

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

You may recall that we have proved this result for a single particle right at the beginning of Sec. 4.4 of Block 1. For a many-particle system $\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$ and $\mathbf{L} = \sum_i \mathbf{L}_i$, where $\boldsymbol{\tau}_i$ and $\mathbf{L}_i$ are the torque experienced and the angular momentum, respectively, of the ith particle. Now, we know that,

$$\boldsymbol{\tau}_i = \frac{d\mathbf{L}_i}{dt}$$

Again $\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum_i \mathbf{L}_i\right) = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \boldsymbol{\tau}_i =$ the sum of torques acting on the particles.



(a)

(b)

Fig. 9.16: Motion of a figure skater. (a) The I of the skater is large and ω is small (b) I is small and ω is large

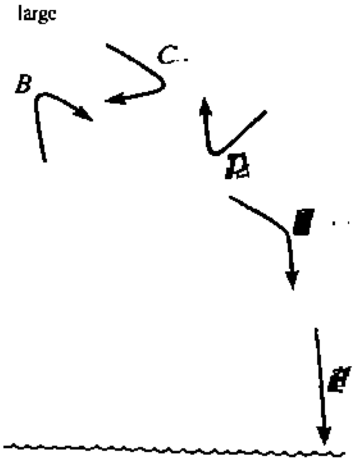


Fig. 9.17: Different stages of the motion of a diver

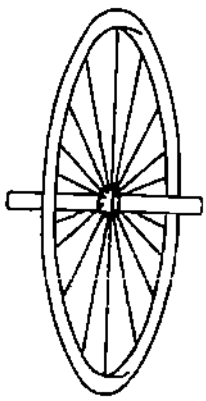


Fig. 9.18: A bicycle wheel

But we have seen in Sec. 9.4.1 that internal torques cancel in pairs. So the sum of the torques is equal to the net external torque.

$$\therefore \quad \frac{d\mathbf{L}}{dt} = \tau_e. \tag{9.19}$$

where τ_e is the net external torque.

When there is no external torque on a system, Eq. 9.19 tells us that $\frac{d\mathbf{L}}{dt} = \mathbf{0}$, or the angular momentum is constant. This is the principle of conservation of angular momentum. It implies that the angular momentum of an isolated system cannot change. We shall now study some applications of this principle.

Did you notice that while deriving Eq. 9.19, we did not require that the system in question be a rigid body? So conservation of angular momentum also applies to systems that undergo changes in configuration, and hence in moment of inertia. A common example is that of a figure skater, who starts spinning relatively slowly with her arms extended (Fig. 9.16a) and then pulls her arms in to spin much more rapidly (Fig. 9.16b). Let us find out why this happens. As her arms move in, her mass gets concentrated more towards the axis of rotation. In other words in the expression $\sum mr^2$ of I, r's become small. So I decreases. But the angular momentum $I\omega$ is conserved. Hence ω increases. The principle also applies to the case of the diver in Fig. 7.12. A schematic representation of Fig. 7.12 is shown in Fig. 9.17. At the positions A, E and F the value of I is high and so ω is low, whereas at the positions B, C and D, I is low and ω is high. So the diver utilises the principle of conservation of angular momentum to do somersaults in mid-air and enter the pool with head and hands down.

You may now like to work out an SAQ on the above concept.

SAQ 9

The earth is suddenly condensed so that its radius becomes half of its usual value without its mass being changed. How will the period of daily rotation change ?

We have studied the application of the principle of conservation of angular momentum. We know that the angular momentum vector changes when an external torque is applied to the system. The change in the angular momentum vector when the applied torque is perpendicular to the direction of the angular momentum presents an interesting situation. The resulting motion is called 'precession' about which we shall study now.

9.4.4 Precession

At some time you must have played with a top. You must have seen that the axis of rotation of a spinning top slowly rotates about the vertical. This means that the direction of **L** of the top (which lies along its axis of rotation) changes. This must be due to a torque acting on the top. You can also observe this effect if you carry out the following activity.

Activity

Turn a bicycle upside down and make it stand on its seat and handle. Rotate its front wheel. When the wheel is rotating reasonably fast, lift it upwards by applying force at the tip of the axle (Fig. 9.18). What happens if you do this?

In doing this activity you must have seen that when you applied the force, the wheel turned, i.e. its axis of rotation changed. Why does this happen? To understand this, study Fig. 9.19.

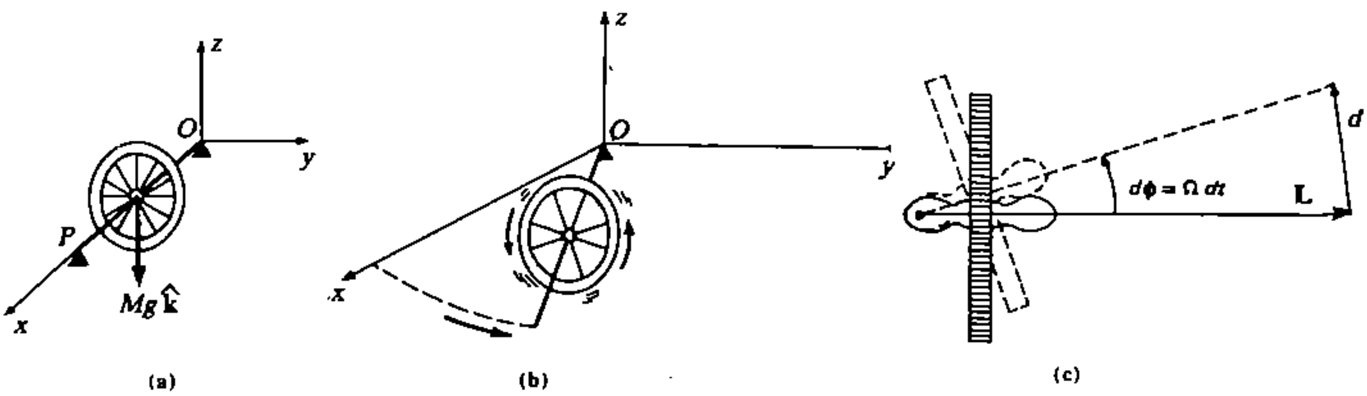


Fig. 9.19 : a) Axle of the wheel is supported at both ends; b) A rapidly spinning wheel does not fall on removing the support at P, but exhibits precession; c) top view of the precessing wheel.

Fig. 9.19 shows a free wheel with an axle. Initially the axle is supported at both end points O and P (Fig. 9.19a). If the support at P is removed, the torque due to force of gravity mg causes the wheel to fall. Now suppose you rotate this wheel anticlockwise and remove the support at P. What happens in this case? This time the wheel does not fall. Instead the axle remains almost horizontal and begins to revolve about the z-axis (Fig. 9.19b). Why does this happen?

This happens because the torque due to gravity acts on the wheel and changes its angular momentum ($\because \boldsymbol{\tau} = d\mathbf{L}/dt$). Since $\mathbf{L}$ is along the axis of rotation, the axis of rotation also turns. We can calculate the angular velocity $\boldsymbol{\Omega}$ at which the axis of rotation moves using the relation $\boldsymbol{\tau} = d\mathbf{L}/dt$. Let the axis of rotation turn by an angle $d\phi$ during time interval dt, then

$$\Omega = \frac{d\phi}{dt}$$

Let the angular speed of the wheel (ω) be constant. Then since $\mathbf{L} = I\boldsymbol{\omega}$, the magnitude of $\mathbf{L}$ is constant and only its direction changes. From Fig. 9.19c we have

$$d\phi = \frac{dL}{L}, \quad \Omega = \frac{d\phi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{\tau}{L}. \tag{9.20}$$

The direction in which the axis of rotation turns will be along $d\mathbf{L}$, i.e. along the torque's direction. Now if r be the distance of the point of support to the centre of the wheel then

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (r\,\hat{\mathbf{i}}) \times (-Mg\,\hat{\mathbf{k}}) = rMg\,(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) = rMg\,\hat{\mathbf{j}}\,.$$

Substituting $L = I\,\omega$ and $\tau = rMg$ in Eq. 9.20 we get

$$\Omega = \frac{rMg}{I\omega}. \tag{9.21}$$

Eq. 9.21 indicates that Ω increases as ω decreases. As rotational energy is lost due to friction, ω will decrease and the wheel's axis of rotation will change faster.

Such a motion in which the axis of rotation changes is called *precession*. $\boldsymbol{\Omega}$ is termed as the **angular velocity of precession**, i.e. the velocity at which the axis of rotation precesses.

SAQ 10

Perform the activity suggested in this section once again. In the light of what we have discussed in this section attempt the following question giving reasons for your answers.

a) In which direction will the wheel turn when you apply an upward force at P, if as seen from P, it were rotating (i) clockwise and (ii) anticlockwise?

b) If you applied upward forces at both P and Q, would the wheel's axis of rotation change?

Let us now summarise what we have studied in this unit.

9.5 SUMMARY

- A rigid body is one in which the relative separation between any two of its constituent particles always remains constant.
- A rigid body is said to execute pure translational motion if each particle in it undergoes the same displacement as every other particle in any given interval of time.
- A rigid body is said to execute rotational motion if each particle in it moves in a circle, the centres of which lie on a straight line called the axis of rotation.
- The general motion of a rigid body is a combined effect of the translation of its c.m. and a rotation about an axis passing through the c.m.
- The rotational analogue of mass is moment of inertia. It measures the resistance of a body to changes in rotational motion. It depends on the mass of a body and on the distribution of mass about the axis of rotation. It is given by

  $$I = \sum_i m_i r_i^2$$

  for a body consisting of discrete masses, and by

  $$I = \int r^2 dm$$

  for a continuous distribution of matter.
- Torque is the rotational analogue of force. Torque, moment of inertia and angular acceleration are related by the rotational analogue of Newton's second law

  $$\tau = I\alpha$$
- A rigid body is said to be in mechanical equilibrium if

  $$\Sigma \mathbf{F} = 0, \ \Sigma \tau = 0$$
- The work done during a rotational motion by a torque is given by

  $$W_{rot} = \int \tau \, . \, d\theta$$
- The expression for K.E. of rotation is similar to that of K.E. for linear motion with mass replaced by I and linear speed by angular speed. It is given by

  $$K_{rot} = \frac{1}{2} I\omega^2$$
- The total K.E. of a rolling object may be written as the sum of the translational K.E. of its c.m. and its rotational K.E. about an axis through its c.m.
- The expression for angular momentum of a rigid rotating object is given by

  $$\mathbf{L} = I\omega$$
- The rotational analogue of Newton's second law may be written in terms of angular momentum as

  $$\tau = \frac{d\mathbf{L}}{dt}$$
- In the absence of external torques, the angular momentum of a system is conserved.
- When a torque is applied perpendicular to the angular momentum vector, then the axis of rotation exhibits a precessional motion.

9.6 TERMINAL QUESTIONS

1. a) Explain with reasons whether the mass of a body can be considered as concentrated at its c.m. for the purpose of computing its moment of inertia ?

   b) Two circular discs of the same mass and thickness are made from metals having different densities. Which disc will have the larger moment of inertia about its central axis ?

c) Comment on the following statement : "The melting of polar icecaps is a possible cause of the variation in the time period of rotation of earth."

2. Refer to Fig. 9.20. It shows a satellite of mass 960 kg. Assume that it is in the form of a solid cylinder of 1.6m diameter and that the total mass is uniformly distributed throughout its volume. Now, suppose that the satellite is spinning at 10 r.p.m. about its axis and it has to be stopped so that a space shuttle crew can make necessary repairs. Two small gas jets are mounted diametrically opposite on the satellite as shown in Fig. 9.20. The jets aim tangentially to the surface of the satellite and each of them produces a thrust of 20N. How long must the jets be fired in order to stop the rotation of the satellite?

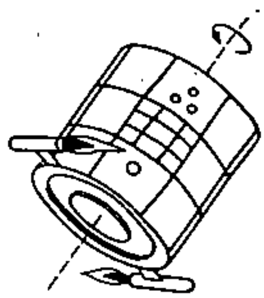


Fig. 9.20 : A spinning satellite

3. The rotational energy of the earth is decreasing steadily because of tidal friction. Estimate the change in the rotational energy of the earth in a day. It is given that the rotational period of the earth decreases by about 10 microseconds in a year. Assume the earth to be a solid sphere.

9.7 ANSWERS

SAQs

1. (a), (c), (e).

2. a) Each has a magnitude of 4.2 cm.

   b) i) A stone falling freely under gravity.

   ii) The motion of a block on a table when it is given a push.

3. The required differential equation (see Eq. 7.22) would be $M\ddot{\mathbf{R}} = \mathbf{F}_e$, where $\mathbf{R}$ is the position vector of the c.m. of the body and $\ddot{\mathbf{R}}$ is its acceleration.

4. In Fig. 9.3 the x', y', z' - axes are always parallel to the x, y, z - axes, whereas in Fig. 9.7 the former continually changes its orientation with respect to the latter. In case of Fig 9.3 the location of the body can be obtained only by locating O', the c.m. of the body while in Fig. 9.7 one has to know in addition the orientation of x', y', z' - axes with respect of the x, y, z - axes.

5. From Sec. 4.3.4 of Block 1 we may say that the K.E. of rotation K_1 of the point mass m_1 is given by

$$K_1 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly the K.E.s of m_2 and m_3 are $K_2 = \frac{1}{2} m_2 r_2^2 \omega^2$,

$K_3 = \frac{1}{2} m_3 r_3^2 \omega^2$. So the K.E. of rotation of the body is given by

$$K = K_1 + K_2 + K_3 + \ldots\ldots$$

$$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots.)\omega^2 = \frac{1}{2} I\omega^2.$$

The expression for the K.E. of linear motion is $\frac{1}{2} Mv^2$ and since ω is analogous to v, I must be the rotational analogue of M.

6. Let the mass in kg of each atom be m. Then from Eq. 7.5 we get $\mu = m/2$. Here $r = 1.2 \times 10^{-10}$ m. If the required angular speed be ω, then from Eq. 9.2 and Example 1, the rotational K.E. is given by

$$E_R = \frac{1}{2}\left[\frac{m}{2}\ \mathrm{kg}\right](1.2 \times 10^{-10}\ \mathrm{m})^2 \omega^2.$$

The translational K.E. is given by

$$E_T = 2 \times \frac{1}{2}\ mv^2 = mv^2, \text{ where } v = 460\ \mathrm{ms}^{-1}.$$

It is given that $E_K = \frac{2}{3} E_T$.

$$m\,(0.36 \times 10^{-20}\ \mathrm{kg\,m^2})\,\omega^2 = \frac{2}{3} m\,(460)^2\ \mathrm{kg\,m^2\,s^{-2}}.$$

$$\text{or } \omega = 6.3 \times 10^{12}\ \mathrm{rad\,s^{-1}}$$

7. vii) $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I\boldsymbol{\alpha}$

ix) K.E. $= \frac{1}{2} I\omega^2$

x) Principle of conservation of angular momentum : When the net torque acting on a body is zero, its angular momentum remains conserved.

xi) Angular impulse $= \int_{t_1}^{t_2} \boldsymbol{\tau}(t)\,dt = \mathbf{L}(t_2) - \mathbf{L}(t_1)$.

8. For (a) a hollow ball, $I_{cm} = \frac{2}{3} MR^2$,

and (b) a solid ball, $I_{cm} = \frac{2}{5} MR^2$.

Now, from Eq. 9.18, we get for (a), $(v_{cm}^2)_a = \frac{6}{5} gy$

and for (b) $(v_{cm}^2)_b = \frac{10}{7} gy$.

For our problem $y = 0.35\,\mathrm{m}$ and we put $g = 9.8\,\mathrm{m\,s^{-2}}$.

So $(v_{cm}^2)_a = 4.1\ \mathrm{m^2\,s^{-2}}$, $(v_{cm}^2)_b = 4.9\ \mathrm{m^2\,s^{-2}}$. The observed value of $v_{cm}^2 = 4\mathrm{m^2 s^{-2}}$ which agrees more closely with (a). Hence the ball is hollow.

9. From the principle of conservation of angular momentum, we get,

$$I_1\omega_1 = I_2\omega_2$$

Here $I_1 = \frac{2}{5} MR_1^2$, $I_2 = \frac{2}{5} MR_2^2$ and $R_2 = \frac{R_1}{2}$

$$\therefore \quad \frac{2}{5} MR_1^2\,\omega_1 = \frac{2}{5} M\,\frac{R_1^2}{4}\,\omega_2$$

$$\text{or} \quad \omega_2 = 4\omega_1$$

But $\omega_1 = \frac{2\pi}{T_1}$ and $\omega_2 = \frac{2\pi}{T_2}$ where T_1 and T_2 are the usual and changed time periods of daily rotation of earth.

$$\therefore \quad T_2 = \frac{T_1}{4} = \frac{24}{4}\ \mathrm{h} = 6\mathrm{h}.$$

So the time period of daily rotation will become 6h.

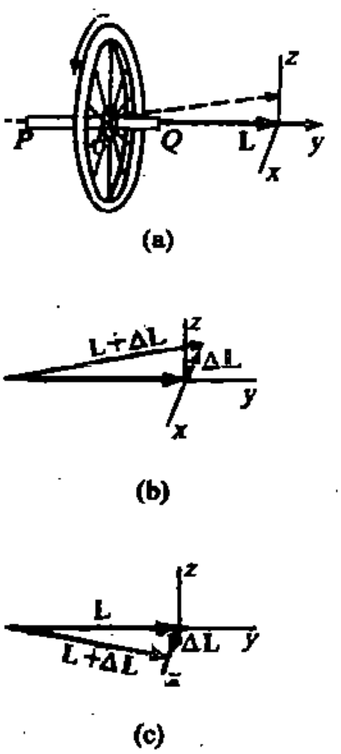


Fig. 9.21: (a) If a rotating bicycle wheel is lifted vertically, it swerves to the side; (b) the change in angular momentum vector for (i); (c) the change in angular momentum vector for (ii).

10. a) (i) Refer to Fig. 9.21a. The direction of **L** is along the positive direction of y-axis. A vertically upward (i.e. along the positive direction of z-axis) force **F** is applied at P. The resulting torque ($\mathbf{r} \times \mathbf{F}$) about O is along the negative direction of x-axis. So the change $\Delta\mathbf{L}$ in the angular momentum vector is along that direction (Fig. 9.21b). Accordingly the new direction will be along $\mathbf{L} + \Delta\mathbf{L}$. So the wheel will swerve so that the axle moves in the xy-plane in the sense $+x$ to $+y$ axis.

(ii) Following similar argument as in (i), we can draw the angular momentum vector **L**, its change $\Delta\mathbf{L}$, and the resulting vector $\mathbf{L} + \Delta\mathbf{L}$ as shown in Fig. 9.21c. So the wheel will again swerve in the xy-plane in the sense $+x$ to $-y$ axis.

b) If upward forces are applied at both points P and Q, then the torques due to them about O will be equal and opposite. So the resulting torque is zero. Hence there would be no change in **L**. So the axis of rotation of the wheel will not turn.

Terminal Questions

1. a) $I = \Sigma m_i r_i^2$ and r_i is not same for all i. So the mass of a body cannot be considered as concentrated at its c.m. for the purpose of computing its moment of inertia.

b) A disc of thickness t, radius R and mass M is essentially a right circular cylinder of the same radius and of length t.

$$\therefore \qquad I = \frac{1}{2} MR^2.$$

But $M = \pi R^2 t \rho$, where ρ = the density of the metal of which the disc is made.

$$\therefore \qquad I = \frac{\pi\rho t R^4}{2} = \frac{\pi\rho t}{2}\left[\frac{M}{\pi\rho t}\right]^2 = \frac{M^2}{2\pi\rho t}.$$

So we see that for same mass and thickness, I is inversely proportional to ρ. Hence the disc made of the metal having lower density will have larger moment of inertia.

c) When the polar icecap melts the water flows towards the equator. This leads to a redistribution of matter over the globe as a result of which I for the earth changes. But as the angular momentum of the earth remains constant its angular speed changes. But, $\omega = 2\pi/T$, where T is the time period of rotation. So T also changes.

2. The satellite's angular speed has to change by $\Delta\omega$ = 10r.p.m. If the angular acceleration α is constant then the time taken for the change is given by

$$\Delta t = \frac{\Delta\omega}{\alpha} = \frac{\Delta\omega I}{\tau} \qquad (\because I\alpha = \tau)$$

Since, the satellite is cylindrical, $I = \frac{1}{2}MR^2$, where M is the mass of the satellite and R its radius. The torque is exerted by two jets, each at a distance R from the rotational axis and directed perpendicular to the radius (Fig. 9.22). If F is the thrust of each jet we get, $\tau = 2RF$.

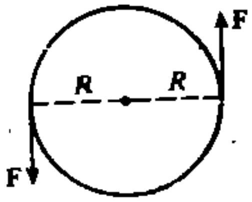


Fig. 9.22

$$\therefore \qquad \Delta t = \frac{(\Delta\omega)\left(\frac{1}{2}MR^2\right)}{2RF} = \frac{\Delta\omega MR}{4F}$$

$$\therefore \qquad \Delta t = \left[\frac{10}{60} \times 2\pi \text{ rads}^{-1}\right] \times \frac{(960\text{ kg}) \times (0.8\text{m})}{4 \times 20\text{N}} = 10\text{s}.$$

3. The moment of inertia of the earth about its axis of rotation is given by

$$I = \frac{2}{5}MR^2, \text{ where } M = 5.97 \times 10^{24}\text{kg}, R = 6.37 \times 10^6\text{m}.$$

$$\therefore \qquad I = 9.7 \times 10^{37}\text{ kgm}^2.$$

The daily rotational period of earth is T = 24h = 86400s. Now the rotational K.E. is given by

$$E = \frac{1}{2} I\omega^2 = \frac{2\pi^2 I}{T^2} \qquad (\because \omega = \frac{2\pi}{T})$$

Now the relative changes in E and T are small in comparison to E and T themselves. So we can treat the changes as differentials dE and dT. We have,

$$dE = 2\pi^2 I\ (-2T^{-3}dT) = -\frac{4\pi^2 I dT}{T^3}$$

The change in T in one year ($\simeq 365$ days) is 10×10^{-6} s, i.e. 10^{-5} s.

$\therefore$ The change in a day is $dT = \dfrac{10^{-5}\text{s}}{365} = 2.7 \times 10^{-8}\text{s}.$

Hence, the change in rotational K.E. will be

$$dE = -\frac{4\pi^2 \times (9.7 \times 10^{37}\,\text{kgm}^2) \times (2.7 \times 10^{-8}\text{s})}{(86400\text{s})^3}$$

$$= -1.6 \times 10^{17}\,\text{kg m}^2\,\text{s}^{-2}$$

So the rotational energy decreases by 1.6×10^{17} J per day.

UNIT 10 MOTION IN NON-INERTIAL FRAMES OF REFERENCE

Structure

10.1 INTRODUCTION

In the previous unit you have read about rigid body dynamics. The present unit will be the final one of our Elementary Mechanics course. We had introduced the concept of frame of reference in the very first unit of Block 1. In Unit 2 of Block 1 we introduced the idea of inertial and non-inertial observers. So far we have explained motion from the point of view of inertial observers. But as a matter of fact we live on a frame of reference (the earth) which is non-inertial. Moreover, we shall see that certain problems can be answered quite elegantly if we take the point of view of a non-inertial observer. So in this unit we shall study the description of motion relative to a non-inertial frame of reference. First we shall study what is meant by a non-inertial frame of reference.

You must have had the following experiences while travelling in a bus. You fall backward when the bus suddenly accelerates and forward when it decelerates. When the bus takes a turn you have sensation of an outward force. We shall explain these features by introducing the concept of inertial forces. Thereby we shall see how Newton's second law of motion gets modified in a non- inertial frame. This will be used to develop the concept of weightlessness.

Frames attached with rotating bodies like a merry-go-round, the earth and so on form the most interesting examples of non- inertial frames of reference. We shall derive the equation of motion of a body in such a frame of reference. Thereby we shall come across two inertial forces, namely, the centrifugal force and the Coriolis force. The former can be used to explain the action of a centrifuge. We will study a variety of applications of these forces in connection with the earth as a non-inertial frame of reference. Centrifugal force finds application in studying the variation of g with the latitude of a place.

Several natural phenomena like erosion of the banks of rivers, cyclones etc. can be explained using the concept of Coriolis force. Finally we shall study about Foucault's Pendulum experiment with a view to establishing the fact that the earth rotates about an axis passing through the poles.

Objectives

After studying this unit you should be able to

- distinguish between an inertial and a non-inertial frame of reference
- write down the equation of motion of a body in a non-inertial frame of reference
- identify the inertial forces appearing in any non-inertial frame of reference
- solve problems on motion from the point of view of a non-inertial frame of reference.

10.2 NON-INERTIAL FRAME OF REFERENCE

In Sec. 2.2.1 of Block 1 we have discussed about inertial and non-inertial *observers*. You may recall that a car moving with a constant velocity and a man standing on the road are inertial with respect to each other. Let us now specify inertial and non-inertial *frames of reference*. Refer to Fig. 10.1.

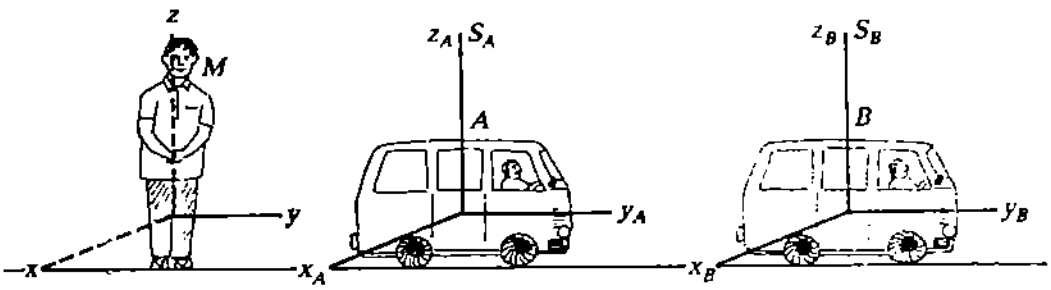


Fig. 10.1: S and S_A are inertial with respect to each other. S and S_B are non-inertial with respect to each other.

M is a person standing on the road. We take some point on the person of M as origin and define a three-dimensional Cartesian coordinate system S. Let Car A move with a uniform velocity and Car B accelerate with respect to S. Let us now choose a point on each of Car A and B as origin and define the coordinate systems S_A and S_B.

The person will locate any object with reference to the coordinate system S. The drivers in the cars will locate objects with respect to S_A and S_B. They may choose a common zero on the time scale. Then you may recall from Sec. 1.2 of Block 1 that S, S_A and S_B are frames of reference. S and S_A are two inertial frames of reference with respect to each other. And S and S_B are two non-inertial frames of reference with respect to each other. In other words, *the frames of reference moving with uniform velocity with respect to each other are inertial and those accelerating with respect to each other are called non-inertial*. For the sake of convenience, from now onward we shall mostly use the word 'frame' in place of the phrase "frame of reference". Let us now discuss some examples of inertial and non-inertial frames of reference.

Consider a child sitting on a revolving merry-go-round in a park. A frame attached to a fixed structure S in the park and the child are non-inertial with respect to each other because the merry-go-round has an acceleration due to rotation. Likewise the frame attached to a ball thrown up in the air by a child and S are non-inertial with respect to each other as the ball has an acceleration equal to **g**. The frame attached to some bench in the park and S are inertial with respect to each other as the bench is at rest with respect to the fixed structure. Similarly, the frame attached with a child walking leisurely (i.e. with a low uniform speed) and S are inertial with respect to each other.

You may now like to work out a simple SAQ to determine the nature of a frame, i.e. whether a frame is inertial or non-inertial with respect to any given frame.

SAQ 1

State giving reasons the nature of the frame attached

i) to a car moving along a curved path with a uniform speed with respect to a frame attached to a man standing on the road,

ii) to a falling rain drop during a drizzle (when it has attained a terminal velocity) with respect to a frame attached to the ground,

iii) to an electron moving in a uniform magnetic field produced by an electromagnet, with respect to a frame attached with a pole piece of the magnet.

So you have learnt how to identify inertial and non-inertial frames. Recall from what you have studied in Sec. 2.2.1 of Block 1 that for many purposes a frame fixed on the surface of earth can be considered as inertial. In all our previous units we had been analysing motion from the point of view of an inertial frame.

We shall see that certain problems of rotational dynamics become simpler when analysed from the point of view of a non-inertial frame. You may recall from Sec. 2.2.1 that Newton's first law of motion holds only in an inertial frame. You also know that the first law can be obtained from the second law. So we can say that the second law also holds only in an inertial frame. Let us now see how the second law will be modified for a non-inertial observer.

10.2.1 Motion Observed from a Non-Inertial Frame

Let us take a simple example. Suppose you are standing on a road and observe a car about to start. We know that in order to start, a car has to accelerate. You would see that a person sitting inside the car gets pressed back against the seat by the acceleration. How would you explain this? Since you are an inertial observer with respect to another inertial observer, you will explain this as follows: This happens due to inertia of rest. The hips and the waist form part of the body of the man that is in direct contact with the seat of the car. The head and the torso are not in direct contact. This portion has a tendency to remain at rest. So as long as the car accelerates, the torso and the head tend to remain behind the waist and the hips. Thus, the person in the car gets pressed back against the seat.

Now, let us try to visualise the situation in a frame S' attached to the car. Due to the acceleration of the car, S' is non-inertial with respect to the person at rest. With respect to S' the portion of the person's body that is in direct contact with the seat of the car is at rest. The other portion falls back. How can this behaviour be explained from S'? We can say that in S' some force acts on the person in a direction opposite to the acceleration of the car. This force neutralises the accelerating force on the waist and hips and causes the other part to fall back.

But where does this force arise from? We have seen in Sec. 5.5 of Block 1 that forces occur either by way of contact (e.g. push, pull, friction) or due to some action at a distance (e.g. gravitational or electromagnetic field). But the force here does not have either of these as its origin. Moreover, such a force does not exist from the point of view of an inertial observer. However, this force is very much real from the point of view of S'. This is called the **inertial force**. From the example we have just now considered you can understand that the magnitude of this force is equal to the accelerating force and it is directed opposite to it. However, we shall quantify this force very soon in this section.

Continuing with the example, we find that in S' the man is held at rest by a force exerted on him by the back of the seat. If you were to remain at rest or in uniform motion with respect to an inertial frame of reference, no force would be needed. But in order to be at rest in a non-inertial frame of reference like that of the accelerating car, some force is required. This implies that the second law of motion will take a different form in a non-inertial frame. We shall now study that. In the process, we shall be able to quantify 'inertial force'.

10.2.2 Newton's Second Law and Inertial Forces

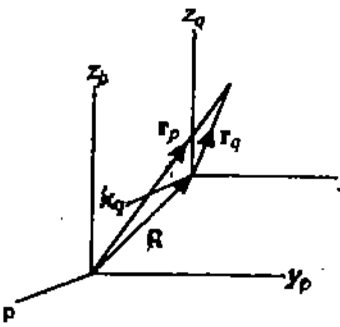


Fig. 10.2: The frames of reference of P and Q

Suppose that two scientists P and Q decide to observe a series of events such as the position of a body of mass m as a function of time. Each has his own set of measuring devices and each works in his own laboratory. Let us suppose that P has confirmed by performing some experiments in his laboratory that the second law of motion holds there precisely. His frame of reference is, therefore, inertial. How can P find out whether Q's frame is inertial or not?

As per convention let the frames be defined by two Cartesian coordinate systems (Fig. 10.2) with identical scale units. In general, the coordinate systems do not coincide. We shall assume that none of the frames is executing a rotation and that they are executing relative motion with their corresponding axes always parallel to each other. Let the position vectors of m be $\mathbf{r}_p$ and $\mathbf{r}_q$ with respect to P and Q, respectively. If the origins of the two frames are displaced by a vector $\mathbf{R}$, then we have from Fig. 10.2

$$\mathbf{r}_q = \mathbf{r}_p - \mathbf{R}. \tag{10.1}$$

The process of obtaining accelerations from the position vectors involves differentiation with respect to time. Incidentally, the time intervals are, strictly speaking, not the same in the two frames of P and Q. However the mathematical treatment corresponding to unequal time intervals will be very complicated. This issue will be resolved for two inertial and non-inertial frames by studying, respectively, the special and general theories of relativity. For the sake of simplicity here we shall assume the time intervals to be equal.

If P sees m accelerating at a rate $\mathbf{a}_p = \ddot{\mathbf{r}}_p$ he concludes from the second law that there is a force on m given by

$$\mathbf{F}_p = m\mathbf{a}_p.$$

Q observes m to be accelerating at a rate $\mathbf{a}_q = \ddot{\mathbf{r}}_q$ as if it were experiencing a force

$$\mathbf{F}_q = m\mathbf{a}_q$$

Let us now find out how $\mathbf{F}_q$ is related to the force $\mathbf{F}_p$. We know from Sec. 1.5 of Block 1 that if Q be moving with a uniform velocity relative to P, i.e. if Q is also inertial, then $\mathbf{a}_q = \mathbf{a}_p$ and

$$\mathbf{F}_q = m\mathbf{a}_q = m\mathbf{a}_p = \mathbf{F}_p.$$

So we find that the force is same in both the frames. In other words, the equations of motion have the same form in both the frames. So all inertial frames are equivalent. *There is no dynamical experiment that leads us to prefer one inertial frame from another.*

Let us now see what happens if Q were accelerating with respect to P. How about working out the relation between $\mathbf{F}_p$ and $\mathbf{F}_q$ in this case?

SAQ 2

Find the relation between $\mathbf{F}_p$ and $\mathbf{F}_q$ when the acceleration of Q with respect to P is $\mathbf{a}$?

Now that you have solved SAQ 2, we can express the relation between $\mathbf{F}_q$ and $\mathbf{F}_p$ as

$$\mathbf{F}_q = \mathbf{F}_p + \mathbf{F}' = m\mathbf{a}_q \tag{10.2a}$$

where $$\mathbf{F}' = -m\mathbf{a}. \tag{10.2b}$$

So we are able to preserve the relationship between the net force on the object and its acceleration. But the net force in the Q-frame is now made up of two parts: a force $\mathbf{F}_p$ and another force $\mathbf{F}'$ equal to $-m\mathbf{a}$. The latter originates from the fact that the frame Q has an acceleration $\mathbf{a}$ with respect to P. This force $\mathbf{F}'$ is called the **inertial force**. Its expression is given by Eq. 10.2b. Its magnitude is equal to the product of the mass of the body and the acceleration of the non-inertial frame. It is directed opposite to the acceleration of the frames. An important special case of Eq.10.2a is that in which the force $\mathbf{F}_p$ is zero. In such a case the body as observed in Q, moves under the action of the inertial force alone. The situation of the torso and the head of the man in the car is very much like that. Let us now work out an example to understand the meaning of inertial force better.

Example 1

A small ball of mass m hangs from a string in a car (Fig. 10.3a) which accelerates at a rate a. What angle does the string make with the vertical and what is the value of tension in it?

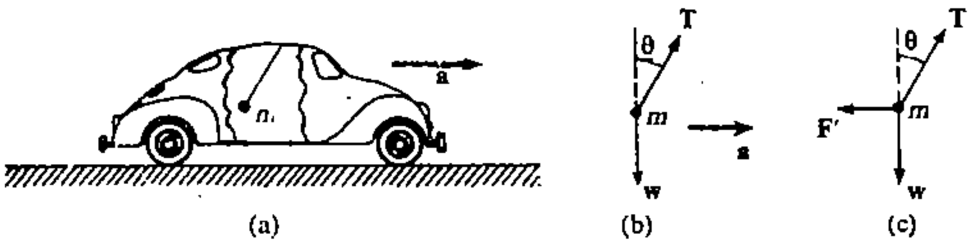


Fig. 10.3: (a) A car accelerating at the rate a; (b) Force diagram with respect to an inertial frame; (c) force diagram with respect to a frame accelerating with the Car.

We shall analyse the problem both with respect to an inertial frame and in a frame accelerating with the car. Let the tension in the string be T and let it make an angle θ with the vertical.

Motion in inertial frame

Refer to Fig. 10.3b. With respect to an inertial frame the mass moves in the direction of motion of the car with an acceleration a ($\mathbf{a} = a\,\hat{\mathbf{i}}$). This is caused by the tension T and the weight mg ($\mathbf{g} = -g\,\hat{\mathbf{j}}$). There is no motion in the y-direction.

$$\therefore \quad T\cos\theta\,\hat{\mathbf{j}} + mg\,(-\hat{\mathbf{j}}) = 0 \qquad \text{or} \qquad T\cos\theta = mg. \tag{10.3a}$$

Equation of motion in the x-direction is given by

$$T\sin\theta\,\hat{\mathbf{i}} = ma\,\hat{\mathbf{i}} \text{ or } T\sin\theta = ma\,. \tag{10.3b}$$

From Eqs. 10.3a and b, we get

$$\tan\theta = \frac{a}{g} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a}{g}\right). \tag{10.3c}$$

$$\text{and } T = \sqrt{(T\cos\theta)^2 + (T\sin\theta)^2}$$

$$\therefore \quad T = m\sqrt{g^2 + a^2}\,. \tag{10.3d}$$

Motion in the frame accelerating with the car

Refer to Fig 10.3c. In this frame apart from the forces **T** and mg there is an inertial force **F'** arising out of the acceleration of the frame. With respect to this frame the mass is at rest, i.e. it is in equilibrium under the influence of **T**, $m\mathbf{g}$ and **F'**:

$$\therefore \quad \mathbf{T} + m\mathbf{g} + \mathbf{F}' = 0$$

$$\text{or} \quad T\cos\theta\,\hat{\mathbf{j}} + T\sin\theta\,\hat{\mathbf{i}} + mg\,(-\hat{\mathbf{j}}) + F'\,(-\hat{\mathbf{i}}) = 0$$

$$\text{or} \quad (T\cos\theta - mg)\,\hat{\mathbf{j}} + (T\sin\theta - F')\,\hat{\mathbf{i}} = 0$$

$$\therefore \quad T\cos\theta - mg = 0, \quad \text{i.e. } T\cos\theta = mg \tag{10.3a'}$$

and $$T\sin\theta - F' = 0 \quad \text{or} \quad T\sin\theta = F'.$$

F' is the magnitude of **F'** and it is equal to ma. So we get

$$T\sin\theta = ma. \tag{10.3b'}$$

From Eqs. 10.3a and 10.3b' we get as in the previous case

$$\theta = \tan^{-1}\left(\frac{a}{g}\right) \tag{10.3c'}$$

and $$T = m\sqrt{g^2 + a^2}\,. \tag{10.3d'}$$

which are identical with the values of θ and T obtained in Eqs. 10.3c and 10.3d. In fact Eq. 10.3a′ is identical with Eq. 10.3a and 10.3b′ is same as 10.3b. But there is an element of difference. Eqs. 10.3a and 10.3a′ both occur as conditions of equilibrium. But 10.3b occurs as an equation of motion whereas 10.3b′ arises out of a condition of equilibrium.

Sometimes the inertial force is called 'fictitious force' or 'pseudo-force' (pseudo means false) as it does not arise from any basic interaction. But these names are misleading as the force actually exists for a non-inertial observer.

Moreover, you must remember that the inertial force does not exist for the inertial observer. This is because inertial forces experienced in an accelerating frame of reference do not arise from physical interactions. They originate in the acceleration of the frame of reference. So for a non-inertial observer such forces are present. For example, suppose we wish to keep an object at rest in a non-inertial frame by tying it down with springs. Then these springs would be observed to elongate or contract in such a way as to provide an opposing force to balance the inertial force.

You may now like to work out an SAQ on the above concept.

SAQ 3

a) A glass half filled with water is kept on a horizontal table in a train. Will the free surface of water remain horizontal as the train starts?

b) A man of mass m is standing in a lift which is accelerating upwards at a rate f. Write down the expression for the inertial force acting on the man. Hence prove that he feels heavier than usual.

Now that you have worked out SAQ 3(b), you will be able to appreciate the concept of weightlessness.

10.2.3 Weightlessness

Suppose that the lift was accelerating downwards at the rate **f** (Fig. 10.4a). Then the net force acting on the man in the frame attached with the lift is given by

$$\mathbf{F} = m\mathbf{g} - m\,\mathbf{f}$$
$$= m\,(g - f)\,\hat{j},$$ where $\hat{j}$ is the unit vector in the vertically downward direction.

Now if the lift were falling freely, i.e. $f = g$, then $\mathbf{F} = \mathbf{0}$. Thus, the force acting on the man is zero. You know that the weight of an object is defined as the force needed to keep it at rest. So in the lift's frame, the reaction of **F** is the weight of the man, since it is the force required to keep the man at rest. Since **F** is zero in a freely falling lift, the man feels weightless. Likewise, every freely falling object is weightless in a frame attached with itself.

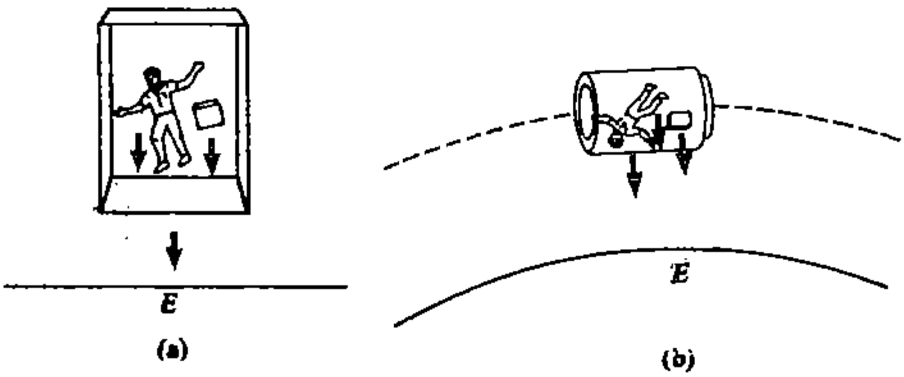


Fig. 10.4: Objects feel weightless in a freely falling frame of reference as they experience the same acceleration as the frame: a) A freely falling elevator near the earth's surface; b) a spacecraft orbiting the earth E. The person, book and the elevator or spaceship all have the same acceleration towards the earth.

You may have seen Squadron Leader Rakesh Sharma floating in the spaceship. In fact, he could lift his fellow astronaut on the tip of his finger. How could this happen?

This is because weightlessness occurs in any orbiting spaceship (Fig. 10.4b), as it is always in a state of free fall. You must remember that weight depends on the frame of reference. The astronaut is weightless only in the freely falling frame of the spaceship. So weightlessness does not imply absence of gravitational force.

Let us now consider the same situation in a frame at rest with respect to the earth. In this frame the net force acting on the astronaut is mg. Therefore, both the spaceship and the astronaut have weight with respect to this frame. The astronaut can float because he is falling towards the earth at the same rate as that of the spaceship.

So far we have not considered the rotation of frames with respect to one another. We know that a rotating body has an acceleration. So a frame attached with such a body rotates and is non-inertial. Our interest in rotating frames of references arises mainly because we live on one such frame, the earth. Another example of a rotating frame is the one attached to a merry-go-round. We shall be able to explain several natural phenomena by considering rotating frames. For example, the occurrence of weather disturbances, the variation of g with latitude and many other phenomena can be explained if we regard the earth as a rotating frame. So let us now analyse motion from the point of view of a rotating frame of reference.

10.3 ROTATING FRAME OF REFERENCE

In Sec. 10.2.2 we have seen how the second law of motion transforms from an inertial frame to a translating non-inertial frame. We shall now see how the second law transforms when one goes from an inertial frame to a rotating frame of reference. As in the previous case the transformed version of the second law will contain the inertial force. We shall see that in a rotating frame more than one inertial force will occur. Our aim will be to determine these inertial forces.

Let us consider a particle of mass m which is accelerating at a rate $\mathbf{a}_{in}$ with respect to an inertial frame. Then its equation of motion in that frame is

$$\mathbf{F} = m\mathbf{a}_{in}.$$

Again let its acceleration with respect to a rotating frame be $\mathbf{a}_{rot}$.

Then its equation of motion in that frame would be

$$\mathbf{F}_{rot} = m\mathbf{a}_{rot}.$$

Let the relative acceleration of the inertial frame with respect to the rotating frame be $\mathbf{a}$. Then we have

$$\mathbf{a}_{in} = \mathbf{a}_{rot} + \mathbf{a}'$$

$$\text{or} \quad \mathbf{F}_{rot} = m\,(\mathbf{a}_{in} - \mathbf{a}') = \mathbf{F} + \mathbf{F}', \qquad (10.4)$$

where $\mathbf{F}'$ is the inertial force given by $\mathbf{F}' = -m\,\mathbf{a}'$. Our task now is to determine $\mathbf{a}'$ for a rotating frame. We know that acceleration is the time-derivative of velocity which again is the time-derivative of displacement. So we shall first relate the infinitesimal displacements of a particle as measured from an inertial and a rotating frame of reference. We shall take the time-derivative of this relation to obtain the relation between the velocities of the particle measured in these frames. Then the time derivative of the relation between the velocities will give the desired expression of the accelerations. So effectively, we shall now study the relations between the time-derivatives of different kinematical variables in inertial and rotating frames of reference.

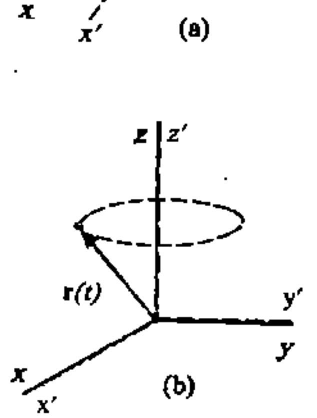


Fig. 10.5: (a) The inertial frame (x, y, z) and the rotating frame (x', y', z') (b) A position vector $\mathbf{r}(t)$ in the xz (and $x'z'$) plane.

10.3.1 Time Derivatives in Inertial and Rotating Frames

Let the motion of a particle of mass m be observed by an inertial and a rotating observer. Let the inertial observer O have a Cartesian coordinate system (x, y, z) as its frame of reference (Fig. 10.5a). The frame of reference of another observer O', who is rotating, is given by another Cartesian coordinate system (x', y', z'). In practice we will be dealing with situations where a frame rotates uniformly about an inertial frame. So here we shall assume that the set of axes (x', y', z') rotates about (x, y, z) with a uniform angular velocity. We are interested in pure rotation, i.e. O' has no translational motion with respect to O. So we have taken the origin of the coordinate systems to be coincident. Also let us suppose that the (x', y', z') system is so rotating that the z and z'-axes always coincide. Thus, the constant angular velocity ω of the rotating system, lies along the z-axis. Further, let the x and x'-axes coincide at an instant of time t.

Imagine now that the particle has a position vector $\mathbf{r}(t)$ in the xz-plane (and $x'z'$-plane) at time t (Fig.10.5b). At time $t+\Delta t$, the position vector is $\mathbf{r}(t+\Delta t)$ and from Fig. 10.6a the displacement of the particle of the particle in the inertial frame is given by

$$\Delta \mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t). \tag{10.5a}$$

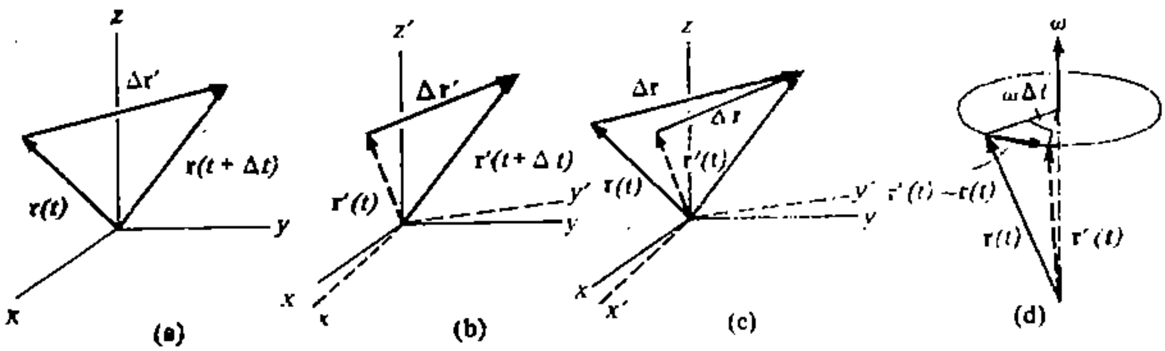


Fig. 10.6: (a) The change $\Delta\mathbf{r}$ in the position vector in the inertial frame; (b) the change $\Delta\mathbf{r}'$ in the position vector in the rotating frame; (c) illustrating that $\Delta\mathbf{r}$ and $\Delta\mathbf{r}'$ are not the same; (d) diagram for obtaining the relation between $|\mathbf{r}'(t) - \mathbf{r}(t)|$ and $\boldsymbol{\omega}$.

The situation is different for the rotating observer. He also notes the same final position vector $\mathbf{r}(t+\Delta t)$ but in obtaining the displacement he ensures that the initial position vector $\mathbf{r}'(t)$ in his coordinate system (Fig. 10.6b) was in the $x'z'$-plane. So he measures the displacement as

$$\Delta \mathbf{r}' = \mathbf{r}'(t+\Delta t) - \mathbf{r}'(t). \tag{10.5b}$$

It can be seen from Fig. 10.6c that the $x'z'$- plane is now rotated away from its previous position. So $\Delta\mathbf{r}$ and $\Delta\mathbf{r}'$ are not the same. From Eqs. 10.5a and b we get

$$\Delta \mathbf{r} = \Delta \mathbf{r}' + \mathbf{r}'(t) - \mathbf{r}(t). \tag{10.6}$$

We shall now express $\{\mathbf{r}'(t) - \mathbf{r}(t)\}$ in terms of $\boldsymbol{\omega}$ and Δt. For this let us refer to Fig. 10.6d. It can be seen that

$$|\mathbf{r}'(t) - \mathbf{r}(t)| = (r \sin\theta)(\omega\,\Delta t)$$

$$= \omega r \sin\theta\, \Delta t = |\boldsymbol{\omega} \times \mathbf{r}|\, \Delta t$$

where r and $\mathbf{r}$ stand for $r(t)$ and $\mathbf{r}(t)$, respectively. Again from the right hand rule for determining the direction of vector product, we see that $\{\mathbf{r}'(t) - \mathbf{r}(t)\}$ is along $(\boldsymbol{\omega} \times \mathbf{r})$. So the vector quantity $(\boldsymbol{\omega} \times \mathbf{r})\,\Delta t$ represents $\{\mathbf{r}'(t) - \mathbf{r}(t)\}$ in magnitude as well as direction. Thus

$$\mathbf{r}'(t) - \mathbf{r}(t) = (\boldsymbol{\omega} \times \mathbf{r})\,\Delta t.$$

Hence, from Eq. 10.6, we get

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\Delta \mathbf{r}'}{\Delta t} + \boldsymbol{\omega} \times \mathbf{r}.$$

Now taking limits on both sides of above as $\Delta t \to 0$ we get

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \boldsymbol{\omega} \times \mathbf{r}.$$

Now $\dfrac{d\mathbf{r}}{dt} = \mathbf{v}_{in}$ = velocity of particle in the inertial frame.

and $\dfrac{d\mathbf{r}'}{dt} = \mathbf{v}_{rot}$ = velocity of the particle in the rotating frame. Thus

$$\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r}. \tag{10.7}$$

You must have noted that in the above proof we did not use the special arrangement of axes of our choice. So the result given by Eq. 10.7 is a general one.

An alternative way of expressing Eq. 10.7 is as follows.

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.8)$$

For obtaining Eq. 10.8 we have only used the geometric properties of **r**. So it can be generalised for any vector **A** Thus we have the general result

$$\left(\frac{d\mathbf{A}}{dt}\right)_{in} = \left(\frac{d\mathbf{A}}{dt}\right)_{rot} + (\boldsymbol{\omega} \times \mathbf{A}). \qquad (10.9)$$

We shall now use Eq. 10.8 to determine $\mathbf{a}'$ ($= \mathbf{a}_{in} - \mathbf{a}_{rot}$).

We know that $\mathbf{a}_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in}$, i.e. the time derivative of $\mathbf{v}_{in}$ in the inertial frame.

and $\mathbf{a}_{rot} = \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot}$, i.e. the time derivative of $\mathbf{v}_{rot}$ in the rotating frame.

On applying Eq. 10.9 for $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\mathbf{a}_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{in}$$

On using Eq. 10.7 we get

$$\mathbf{a}_{in} = \frac{d}{dt}(\mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r})_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Since $\boldsymbol{\omega}$ is constant, we get

$$\mathbf{a}_{in} = \mathbf{a}_{rot} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

or $\mathbf{a}_{in} = \mathbf{a}_{rot} + 2\boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

$$\therefore \quad \mathbf{a}' = 2\boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (10.10)$$

Thus the inertial force is given by

$$\mathbf{F}' = -m\mathbf{a}' = -2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.11)$$

In Eq. 10.11 we have written $\mathbf{v}'$ in place of $\mathbf{v}_{rot}$ for the sake of convenience.

Hence, from Eq. 10.4 we can write

$$\mathbf{F}_{rot} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}') - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.12)$$

Eq. 10.12 shows that the dynamics of motion as observed from a uniformly rotating frame of reference may be analysed in terms of the following three categories of forces:

i) **F**: This is the sum of all forces on the particle, arising out of physical interactions or due to contact. They may be tensions in strings and forces due to fundamental interactions. Only these forces are present in an inertial frame.

ii) $-2m(\boldsymbol{\omega} \times \mathbf{v}')$: This is called the **Coriolis force**. It acts at right angles to the plane containing $\boldsymbol{\omega}$ and $\mathbf{v}'$ and points in the direction of advancement of the screwhead when the screw is rotated from $\mathbf{v}'$ towards $\boldsymbol{\omega}$. This force is absent when the particle has no velocity with respect to the rotating frame.

Coriolis force is named after the French engineer and mathematician Gustave Gaspard Coriolis (1792-1843). He was the first man to provide a description of the force. The term centrifugal comes from 'centre' and 'fugal'. The latter means to fly off.

iii) $-m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$: This is called the **Centrifugal force**. It always acts radially outward. The two observers in the inertial and rotating frame do agree on the position vector of a particle at a given instant. Hence **r** may be replaced by $\mathbf{r}'$, provided their origins coincide.

We shall now study some examples of these forces. Let us begin with the centrifugal force.

10.3.2 Centrifugal Force

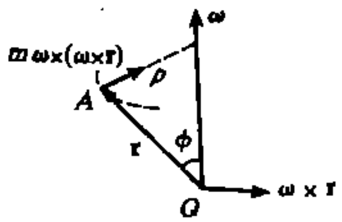


Fig. 10.7: The centrifugal force. Mathematically $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ acts at O. But physically $-m\boldsymbol{\omega}(\boldsymbol{\omega} \times \mathbf{r})$ is a force acting on the body. Thus $\mathbf{F}_{cent}$ acts at A and is a vector antiparallel to $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

Let us first determine the magnitude and direction of the centrifugal force $\mathbf{F}_{cent} = -m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Fig. 10.7. $\boldsymbol{\omega} \times \mathbf{r}$ is perpendicular to the plane of $\boldsymbol{\omega}$ and $\mathbf{r}$. Let the angle between $\boldsymbol{\omega}$ and $\mathbf{r}$ be ϕ. Then the magnitude of $\boldsymbol{\omega} \times \mathbf{r}$ is $\omega r \sin\phi = \omega\rho$, where $\rho = r \sin\phi$ is the perpendicular distance from the axis of rotation to the head of $\mathbf{r}$. Hence $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector with magnitude $\omega^2 \rho$, since the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \mathbf{r}$ is 90°. From the right-hand rule this vector is directed radially inward towards the axis of rotation. Therefore, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector of magnitude $m\omega^2$. It points radially outward from the axis of rotation to the head of $\mathbf{r}$. So we can also write

$$\mathbf{F}_{cent} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\omega^2 \rho\, \hat{\boldsymbol{\rho}} = m\,\omega^2 r \sin\phi\, \hat{\boldsymbol{\rho}}. \qquad (10.13a)$$

where $\hat{\boldsymbol{\rho}}$ is the unit vector along the direction from the axis of rotation to the head of $\mathbf{r}$. If the body's position vector $\mathbf{r}$ were measured from the centre of the circle in which it is rotating, then $\phi = 90°$ and

$$\mathbf{F}_{cent} = m\omega^2 r\, \hat{\mathbf{r}}. \qquad (10.13b)$$

The centrifugal force is familiar to us in our daily life. If we tie an object to a string and whirl it around it seems to pull on us This effect can be explained in terms of the centrifugal force. Let's see how.

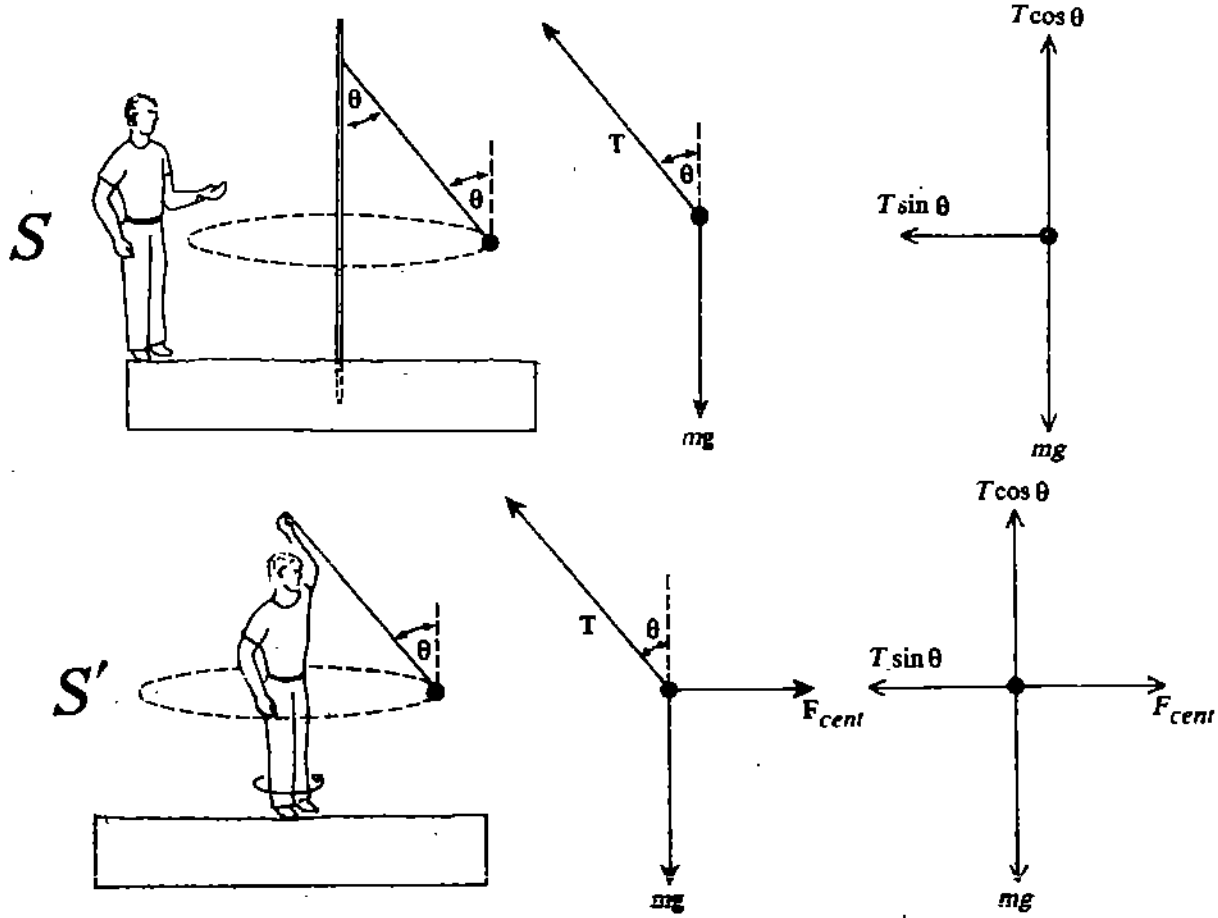


Fig. 10.8: In the frame S the forces acting are the tension in the string and the weight of the object. $T\cos\theta$ balances mg and $T\sin\theta$ provides the necessary centripetal force. In the frame S' apart from the tension and weight we have the centrifugal force. These forces are in equilibrium.

Suppose that a ball is being whirled around in horizontal circular motion (Fig. 10.8) with constant angular speed ω. Let us analyse the motion of the ball from two frames of reference. A stationary (inertial) frame S, and a rotating (non-inertial) frame S' that rotates with the same angular speed as the ball. So the angular speed of S' with respect to S is also ω. Look at the force diagrams in the S frame and S' frame.

In the S frame the ball has a centripetal acceleration ($-\omega^2 r\,\hat{\mathbf{r}}$). The force responsible for this acceleration is provided by the tension in the cord. On resolving the force **T** into its components we get

$$T\cos\theta = mg,$$

$$T\sin\theta = m\omega^2 r$$

In the S' frame, the ball is at rest. This is because in this frame alongwith **T** and $m\mathbf{g}$ a centrifugal force $\mathbf{F}_{cent}$ also acts on it. Resolution of forces gives

$$T\cos\theta = mg,$$

$$F_{cent} = T\sin\theta = m\omega^2 r.$$

We have taken this example also to caution you against the misuse of the term centrifugal force. Sometimes you may come across statements like 'The Moon does not fall down as it moves around the Earth because the centrifugal force balances the force of gravitation and hence there is no net force to make it fall.'

Any such statement goes against Newton's first law. Why? Because if no net force were acting on a body, it would *move in a straight line*. Any body moving on a curved path must have an unbalanced force on it. Now in the inertial frame the moon (or the ball) is seen to *move in a circular path*. Thus, an unbalanced centripetal force given by the force of gravitation (or the tension in the string) acts on the moon or the ball.

However, in the rotating frame of reference moving at the same angular speed, these objects would be seen to be at *rest*. Only in such frames would the centrifugal force balance the gravitational force on the moon (or the horizontal component of the tension in the string). So remember centrifugal forces arise only in rotating frames of reference. If we analyse a rotating object's motion from a non-rotating frame there is no such thing as centrifugal force. Of course, either frame is valid for analysing the problems. But never use inertial forces in inertial frames. They arise only in non-inertial frames.

Let us round off this section with an example of centrifugal force.

Example 2: Centrifuge

An interesting application of the centrifugal force is a device called a centrifuge. It has uses, such as for separating heavy particles suspended in a liquid, for separating chemicals etc. You may like to know how it works.

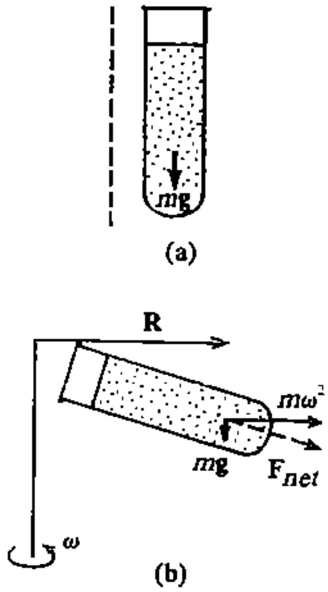


Fig. 10.9: (a) A test-tube in a centrifuge. The dotted line is the axis of the centrifuge; (b) when the centrifuge rotates, the centrifugal force makes the free end of the test tube swing out.

Suppose we have a test tube containing small particles suspended in a liquid. If the particles are heavier than the liquid, they will settle to the bottom, but if the particles are extremely small, this will take a long time. To speed up the process, we attach the test tube to a centrifuge. It is a mechanical device whose operation depends on centrifugal force.

For a rigorous analysis of the situation we need to account for the buoyant forces on the suspended particles and the viscous force acting on the mobile particles. Since these forces are small compared to the force of gravity and the centrifugal force, we shall ignore them.

Initially the tube hangs vertically, as in Fig. 10.9a. The centrifuge is carefully balanced with other tubes (not shown in the figure). When the centrifuge is spun about its central vertical axis, the tubes feel a centrifugal force (in the frame rotating with the centrifuge) pointing in the horizontal direction. The resultant of the force of gravity and centrifugal force acts like an *effective force of gravity*. At high values of angular speed $\mathbf{F}_{cent}$ is much greater than $m\mathbf{g}$. So this effective force is much stronger and points almost horizontally (Fig. 10.9b). The tube rises until it is oriented along the direction of the net force $\mathbf{F}_{net}$ on it. The surface of the liquid orients itself normal to the net force it feels. A particle suspended in the liquid moves in the direction of the net force it feels. This is essentially towards the bottom of the tube. Since $\mathbf{F}_{net}$ is much greater than $m\mathbf{g}$ for high values of ω, the suspended particles settle to the bottom of the tube much more rapidly than they would otherwise.

You may now like to work out an SAQ to consolidate your understanding of centrifugal forces.

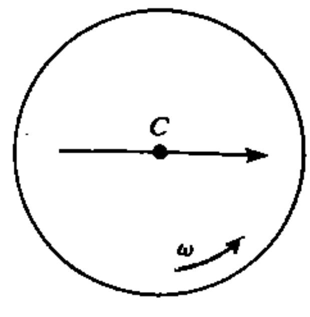
SAQ 4

(a) When we drive a car too fast around a curve, it skids outward. To us it seems as if it is pushed by a centrifugal force. If you were standing by the roadside watching this happen, how would you explain the car's motion?

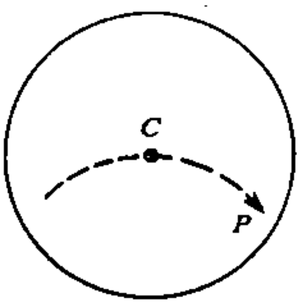
(b) A tiny virus particle of mass 6×10^{-19} kg is in a water suspension in an ultracentrifuge which is essentially a centrifuge where extremely high angular speed can be generated. It is 4 cm from the vertical axis of rotation. The angular speed of rotation is $2\pi \times 10^3$ rad s^{-1}.

(i) What is the effective value of 'g' relative to the frame rotating with the ultracentrifuge?

(ii) What is the net centrifugal force acting on the particle?



(a)



(b)

Fig. 10.10: Motion of a frictionless ball passing over the rotation axis at C, as seen from above in (a) an inertial frame (solid line) and (b) the rotating frame (dashed line).

10.3.3 Coriolis Force

Let us consider a particle which moves with a velocity $\mathbf{v}_{rot}$ with respect to a rotating frame. The effect of Coriolis force is relatively easy to visualize at the axis of rotation, where the centrifugal force is negligible. So let us begin with that case.

A rotating horizontal disc is shown in Fig. 10.10a. The axis of rotation is perpendicular to the plane of this paper at point C which is the centre of the disc. Let us now consider a ball passing through C. If friction can be ignored, the ball is free of horizontal forces. Therefore, it moves in a straight line (the solid line of Fig. 10.10a) with constant velocity $\mathbf{v}$ relative to the inertial frame. As seen from this frame, the rotating disc turns, say, counterclockwise with angular speed ω. But as seen from a frame fixed in the disc, it is the inertial frame that rotates, with the same angular speed in the opposite sense, clockwise. So in the rotating frame the ball's trajectory also turns clockwise, following the curved path indicated by the dashed line in Fig. 10.10b. Thus, there must be an inertial force in the rotating frame to provide the curvature that was not present in the inertial frame. It is indeed the Coriolis force.

The magnitude of the Coriolis force can be appreciable on a turntable or merry-go-round. For example, if ω is 1 rad s^{-1} and v_{rot} is 5 m s^{-1} the Coriolis acceleration $2\omega\, v_{rot}$ is 10 m s^{-2}, equal to the acceleration due to gravity.

The Coriolis force associated with the earth's rotation is much weaker than the effect considered above because the earth rotates only once per day, corresponding to an angular speed $\omega \simeq 2\pi \times 10^{-5}$ rad s^{-1}. Even at projectile velocities of 10^3 m s^{-1}, the Coriolis acceleration $2\omega\, v_{rot}$ is only of the order of 10^{-2} m s^{-2} which is far less than g. That is why the Coriolis force is not intuitively familiar. However, when the Coriolis force associated with the earth's rotation acts over a sufficient period of time, say for several days, it can have striking effects. The centrifugal and Coriolis forces associated with the earth's rotation are responsible for many a natural phenomena. For example, the variation in g with latitude, the deflection of a moving body, wind patterns in the two hemispheres can be explained using the concepts of centrifugal or Coriolis force arising on a rotating earth. So let us now study the earth as a rotating frame.

10.4 THE EARTH AS A ROTATING FRAME OF REFERENCE

A number of important phenomena are driven by the inertial forces acting in a rotating frame of reference attached to the earth's surface. Let us study some of these phenomena.

10.4.1 The Variation of g with Latitude

You may know that a person weighs more at the poles than at the equator. This effect arises due to the rotation of the earth. In fact we have already stated this result giving the variation of g with latitude (recall Eq. 5.44 of Unit 5, Block 1). Let us now prove the result.

Let a particle P be at rest with respect to the earth at latitude λ near the earth's surface. Then in the earth's frame it is subjected to the force of gravity $\mathbf{F}_g$ $(= m\mathbf{g})$ and the centrifugal force

$\mathbf{F}_{cent}$ shown in Fig. 10.11a. The Coriolis force is zero for this particle, since it is at rest in the rotating frame. The magnitude of $\mathbf{F}_{cent}$ is given from Eq. 10.13 as

$$F_{cent} = m\omega^2 R \sin\phi = m\omega^2 R \cos\lambda, \; [\because \lambda = \frac{\pi}{2} - \phi]$$

where R is the earth's radius. Let the resultant of $\mathbf{F}_g$ and $\mathbf{F}_{cent}$ be $\mathbf{F}_g^*$. Let us resolve these three forces along the radial and transverse directions. Note that on the earth, the radial direction corresponds to the vertical (opposite to $\mathbf{F}_g$) and the transverse to the horizontal. Let g_v^* and g_h^* represent the vertical and horizontal components of $\mathbf{g}^*$, respectively (Fig. 10.11b). So we have

$$m\, g_v^* = F_g - F_{cent}\cos\lambda = mg - m\,\omega^2 R\cos^2\lambda$$

$$\text{or}\;\; g_v^* = g - \omega^2 R\cos^2\lambda \qquad (10.14a)$$

$$\text{and}\;\; mg_h^* = F_{cent}\sin\lambda = m\,\omega^2 R\cos\lambda\,\sin\lambda$$

$$\text{or}\;\; g_h^* = \omega^2 R\cos\lambda\sin\lambda \qquad (10.14b)$$

Now, the maximum magnitude of the centrifugal acceleration, (F_{cent}/m), is $\omega^2 R$. Let us calculate its value.

$$\omega^2 R = \left(\frac{2\pi\ \text{rad}}{24\times 60\times 60\,\text{s}}\right)^2 \times (6.37\times 10^6\,\text{m}) = 3.4\times 10^{-2}\ \text{ms}^{-2}.$$

Thus $\omega^2 R \ll g$ and $g_v^* \approx g$, i.e. the angle between $\mathbf{g}_v^*$ (the apparent vertical) and $\mathbf{g}$ (the real vertical) is very small. Let us compute its value. From Fig. 10.11b

$$\tan\alpha \approx \alpha = \frac{g_h^*}{g_v^*} = \frac{\omega^2 R\cos\lambda\sin\lambda}{g} = \frac{\omega^2 R\sin 2\lambda}{2g}$$

It has a maximum value at $\lambda = 45^\circ$ which is

Fig. 10.11 Variation of g with λ. (a) Resultant of $\mathbf{F}_g$ and $\mathbf{F}_{cent}$. The dotted line E represents the equator. PV is the vertical direction at P (b) g^*, g_h^* and g_v^*.

$$\alpha_{max} = \frac{(3.4\times 10^{-2}\,\text{ms}^{-2})}{2\times 9.8\ \text{ms}^{-2}} = 0.0017\ \text{rad} = 0^\circ\,6'.$$

So effectively $g_h^* = 0$ and $g_v^* = g^*$.

From Eq. 10.14a, we get

$$g^* = g - \omega^2 R\cos^2\lambda. \qquad (10.15)$$

At the poles $\lambda = 90^\circ$ and $g_v^* = g$, i.e. $g_{pole} = g$.

At the equator $\lambda = 0$, so that

$$g_h^* = 0,\;\; g_v^* = g - \omega^2 R.$$

$$\therefore\;\; g_e = g - \omega^2 R, \text{ where } g_e \text{ in the value of } g \text{ at equator.}$$

Now using Eq. 10.15, we may write,

$$g^* = g - \omega^2 R\,(1 - \sin^2\lambda) = (g - \omega^2 R) + \omega^2 R\sin^2\lambda$$

$$\text{or}\;\; g^* = g_e + \omega^2 R\sin^2\lambda,$$

which is same as Eq. 5.44 of Block 1.

So the value of acceleration due to gravity at the poles will be greater by $3.4\times 10^{-2}\ \text{ms}^{-2}$, than its value at the equator if we take earth's rotation into account. However, the measured difference is $5.2\times 10^{-2}\ \text{ms}^{-2}$. This discrepancy arises because the earth is not a perfect sphere. It is flattened at the poles and bulging at the equator. Due to the centrifugal force arising from earth's rotation a plumb line does not point exactly towards the centre of the earth. Instead it swings through a small angle. You may now like to work out an SAQ on the above concept.

SAQ 5

a) What must be the angular speed of the earth so that the centrifugal force makes objects fly off its surface? (Take $g = 10\ \mathrm{m\,s^{-2}}$).

b) If the angular speed is just enough to make this happen, from which part of the earth would the objects fly off?

In the above discussion we have considered the body to be at rest with respect to the earth. What can you say about a body moving with respect to the earth's surface? We will now have to take into account the Coriolis force also. Let us analyse this motion.

10.4.2 Motion on the Rotating Earth

Let us consider a particle of mass m moving with velocity $\mathbf{v}$ at latitude λ on the surface of the spherical earth. So $\mathbf{v}$ is tangential to the sphere. Let the earth's angular velocity be $\boldsymbol{\omega}$. Then in the earth's frame of reference, the force on m is given from Eq. 10.12 as

$$\mathbf{F} = m\mathbf{g} - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Let us analyse the additional term due to Coriolis force. Refer to Fig. 10.12a. Let us decompose $\boldsymbol{\omega}$ into a vertical part $\boldsymbol{\omega}_V$ and horizontal part $\boldsymbol{\omega}_H$. Then the Coriolis force is given by

$$\mathbf{F}_{cor} = -2m\,(\boldsymbol{\omega} \times \mathbf{v})$$

$$= -2m\,(\boldsymbol{\omega}_V \times \mathbf{v}) - 2m\,(\boldsymbol{\omega}_H \times \mathbf{v}).$$

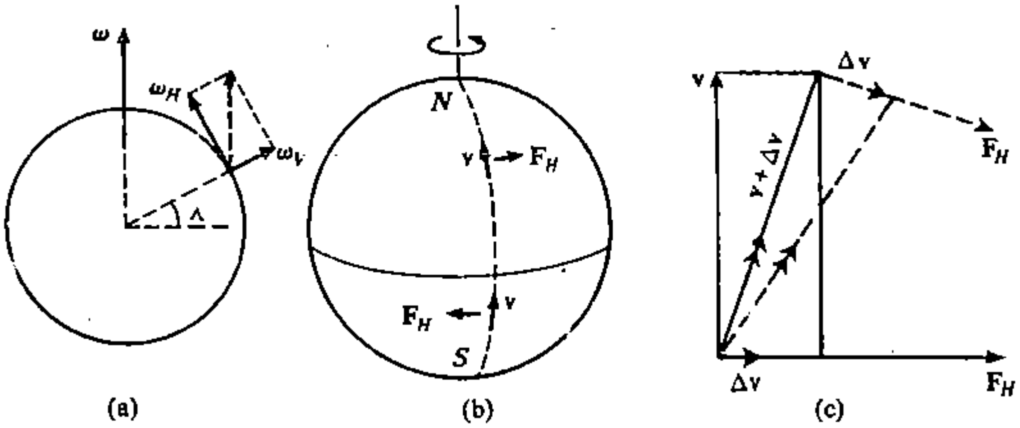


Fig.10.12: Deflection of a moving particle due to Coriolis force. (a) Breaking $\boldsymbol{\omega}$ into components $\boldsymbol{\omega}_H$ and $\boldsymbol{\omega}_V$; (b) directions of $\mathbf{F}_H$ in N and S-hemisphere; (c) clockwise turning of $\mathbf{v}$ in N-hemisphere

Now $\boldsymbol{\omega}_H$ and $\mathbf{v}$ are horizontal, so $\boldsymbol{\omega}_H \times \mathbf{v}$ is vertical. And $\boldsymbol{\omega}_V \times \mathbf{v}$ alone gives rise to the horizontal component $\mathbf{F}_H$ of the Coriolis force. $\boldsymbol{\omega}_V$ is perpendicular to $\mathbf{v}$. So $\boldsymbol{\omega}_V \times \mathbf{v}$ has magnitude $\omega_V\, v$. Now let $\hat{\mathbf{r}}$ be a vector perpendicular to the surface at latitude λ, i.e. $\hat{\mathbf{r}}$ is along $\boldsymbol{\omega}_V$. Then we have that

$$\boldsymbol{\omega}_V = \omega_V\,\hat{\mathbf{r}} = \omega \cos\left(\frac{\pi}{2} - \lambda\right)\hat{\mathbf{r}} = \omega \sin\lambda\,\hat{\mathbf{r}}$$

and $\mathbf{F}_H = -2m\,(\boldsymbol{\omega}_V \times \mathbf{v}) = -2m\,\omega \sin\lambda\,(\hat{\mathbf{r}} \times \mathbf{v})$

The magnitude of $\mathbf{F}_H$ is $2mv\omega \sin\lambda$. $\mathbf{F}_H$ is a force perpendicular to $\mathbf{v}$ (Fig. 10.12b). So its effect is to produce circular motion. Let us see how.

The effect of $\mathbf{F}_H$ will be to produce a deflection towards the right in the northern hemisphere. $\mathbf{F}_H$ produces a change in the direction of $\mathbf{v}$. Let the change in $\mathbf{v}$ be $\Delta\mathbf{v}$ in an infinitesimal interval of time Δt. From Fig. 10.12c you can see that the resultant velocity vector moves towards the right. $\mathbf{F}_H$ is now perpendicular to $\mathbf{v} + \Delta\mathbf{v}$. So in the next such time interval Δt, the velocity vector will further turn towards right. So the effect of $\mathbf{F}_H$ in the northern hemisphere is to produce a clockwise rotation of the velocity vector. In the southern hemisphere this will be anticlockwise.

So you can see that this effect of Coriolis force is that it turns straight line motion into circular motion. This result has a number of interesting consequences. For example, rivers flowing in the northern hemisphere wash out their right banks, and those in the southern hemisphere their left banks. Again in the northern hemisphere the right hand rails of the rail tracks are worn out faster if it is a double-track railway. This is because on each track the train always goes in one direction. Due to $\mathbf{F}_{//}$ its motion has a component to the right from the direction of motion. Similarly, the left hand rail is worn out faster in the southern hemisphere.

Air flow patterns in the atmosphere can also be explained by this result. Imagine that temperature difference in the various layers of air has given rise to a low pressure region in the atmosphere (Fig. 10.13a). The closed curves in the figure represent lines of constant pressure, called isobars. The pressure gradient gives rise to a force on each element of air. In the absence of other forces winds would blow inward and the pressure in the region would become uniform.

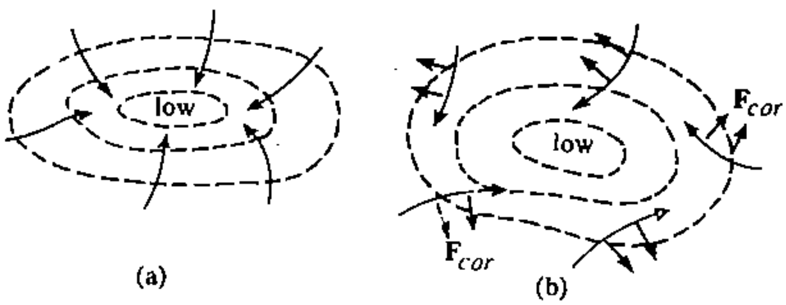


Fig. 10.13: Air-flow patterns: (a) Dotted lines represent the isobars, (b) right deflection of the air particles.

However, the pressure of Coriolis force considerably changes the air flow pattern. Let us consider this event in the northern hemisphere. As the air flows inward towards the low pressure region it is deflected toward the right as shown in Fig. 10.13b. The result is that wind rotates anticlockwise about the regions of low pressure. This effect causes most cyclones to be anticlockwise in the northern hemisphere and clockwise in the southern hemisphere. This effect can be seen quite clearly in the INSAT pictures of clouds taken during a cyclonic storm.

So far we have discussed some natural phenomena which arise due to rotation of the earth. We can also demonstrate rotation of the earth in a laboratory using the Foucault's pendulum.

10.4.3 Foucault's Pendulum

In 1851, J.B.L. Foucault for the first time demonstrated the rotation of the earth. He suspended a heavy metal sphere of 28 kg on a wire almost 70 m long. The suspension point of the pendulum was free to rotate in any direction. The motion of the pendulum was observed from a point above. With successive swings of the pendulum it seemed that the plane of its motion rotated. In 1 h the plane of the swing changed by 11°. A full circuit was completed in about 32 h.

Why does the plane of motion of the pendulum rotate?

To understand this, we shall visualise this experiment at the North Pole (Fig. 10.14a). In an inertial frame the only forces acting on the pendulum are the force of gravity and the tension of the wire. Both these forces act in the plane of oscillation. So they cannot rotate it. Therefore, with respect to an inertial frame the plane of the oscillation of the pendulum would remain fixed. The earth would, of course, rotate from west to east under the pendulum once in every 24 h. The rotation of the earth is anticlockwise as seen from the North Pole. So to an observer standing at the North Pole, the plane of the oscillation would seem to rotate clockwise (east to west) (Fig. 10.14b). It can also be explained for other latitudes but we are not going into those details here.

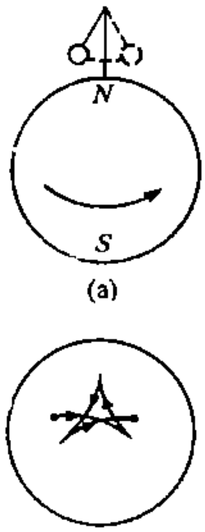


Fig. 10.14: Foucault's pendulum. (a) The pendulum on the N Pole. The arrow indicates the direction of rotation of the earth. (b) rotation of the plane of oscillation

Let us now summarise what we have studied in this unit.

10.5 SUMMARY

- The frames of reference accelerating with respect to each other are called non-inertial frames.
- The net force acting on any object in the non-inertial frame S' having an acceleration **a** with respect to an inertial frame S is made up of two parts: the force $\mathbf{F}_s$ acting on the object in the S frame and an inertial force equal to $-m\,\mathbf{a}$. Inertial forces arise only in non-inertial frames.
- The equation of motion of an object in a rotating frame of reference is given as

$$\mathbf{F}_{rot} = \mathbf{F} - 2m\,(\boldsymbol{\omega} \times \mathbf{v}') - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

  where **F** is the sum of all forces acting on the object as seen from the inertial frame. The second and the third terms are the Coriolis and the centrifugal forces, respectively.
- Any frame of reference attached to the earth is a non-inertial frame of reference. Rotation of the earth is responsible for many a natural phenomena, such as variation of g with latitude, deflection of moving bodies, etc. The earth's rotation can be demonstrated with the help of Foucault's pendulum.

10.6 TERMINAL QUESTIONS

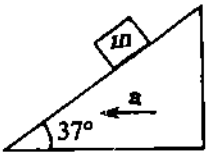


Fig. 10.15: Diagram for TQ 1

1. An inclined plane (Fig. 10.15) is accelerated horizontally to the left. The magnitude of the acceleration is gradually increased until a block of mass m, originally at rest with respect to the plane, just starts to slip up the plane. The coefficient of static friction between the plane and the block is 0.8. (It is given that $\sin 37^\circ = 3/5$, $g = 10\ \mathrm{m\,s^{-2}}$).

   a) Draw diagrams showing the forces acting on the block, just before it slips (i) in an inertial frame fixed to the floor and (ii) in the non-inertial frame moving along with the block.

   b) Find the acceleration at which the block begins to slip using both the force diagrams (i) and (ii) of part (a).

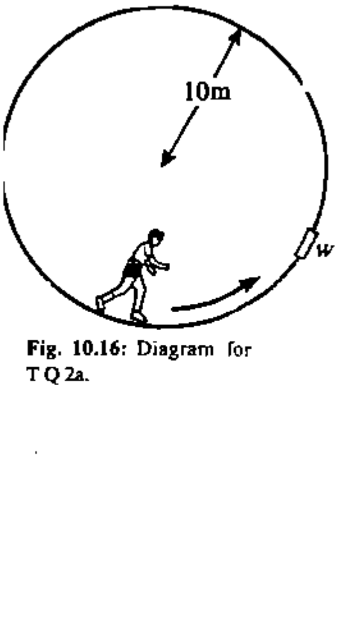


Fig. 10.16: Diagram for TQ 2a.

2. a) A space station of radius 10m spins so that a person inside it (Fig. 10.16) has a sensation of 'artificial gravity' when afloat in space. The rate of spin is chosen to attain $g = 10\ \mathrm{ms^{-2}}$. Find the length of the 'day' as seen in the spacecraft through a window W.

   b) A 1.0 × 10^5 kg train runs south at a speed of 30 $\mathrm{ms^{-1}}$ at a latitude of 60° N. What is the horizontal force on the track? What is the direction of this force?

3. Your weight is measured to be equal to W when you are at rest with respect to the earth. Will your weight be different from W when you are in motion with respect to the earth ?

10.7 ANSWERS

SAQs

1. i) Since the car is moving along a curved path its velocity vector is continually changing its direction. So it has a non-zero acceleration with respect to the man standing on the road. So the frame attached to it is non-inertial with respect to the man.

   ii) Since the raindrop has attained a terminal velocity it is falling with a constant velocity with respect to the ground. So the frame attached to it is inertial with respect to the ground.

   iii) An electron moving in a uniform magnetic field experiences a force. So it will be accelerating with respect to a pole piece. Hence, the frame attached to the electron is non-inertial with respect to the pole piece.

2. Differentiating Eq. 10.1 twice with respect to time, we get

$$\ddot{\mathbf{r}}_q = \ddot{\mathbf{r}}_p - \ddot{\mathbf{R}}$$

or $$\mathbf{a}_q = \mathbf{a}_p - \mathbf{a}$$

$\therefore$ $$\mathbf{F}_q = m\mathbf{a}_q = m\mathbf{a}_p - m\mathbf{a}$$

or $$\mathbf{F}_q = \mathbf{F}_p - m\mathbf{a}.$$

3. a) In order to start, the train has to accelerate. Let this acceleration be $\mathbf{a}$, and directed along the x-axis. Now, following Eqs. 10.2a and 10.2b, we can write the total force acting on the water in the frame of reference of the train as (see Fig. 10.17)

$$\mathbf{F}_Q = m\mathbf{g} + (-m\mathbf{a}),$$

where m is the total mass of the water and the glass.

$$\mathbf{g} = -g\,\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{a} = a\,\hat{\mathbf{i}}.$$

The surface of water takes up a position normal to the force $\mathbf{F}_Q$ as shown in Fig. 10.17.

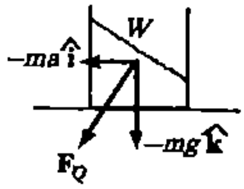


Fig. 10.17

b) Let the lift be accelerating in the z-direction (Fig. 10.18). The inertial force acting on the man is given by

$$\mathbf{F}' = -m\mathbf{f},$$

where m is the mass of the man. So the total force on the man is given by

$$\mathbf{F} = m\mathbf{g} + \mathbf{F}' = m\mathbf{g} - m\mathbf{f}.$$

But $\mathbf{g} = -g\hat{\mathbf{k}}$, $\mathbf{f} = f\hat{\mathbf{k}}$.

Hence, $\mathbf{F} = -m\,(g + f)\,\hat{\mathbf{k}}$.

So the magnitude of the force on the man is greater than mg. Hence, he feels heavier than usual.

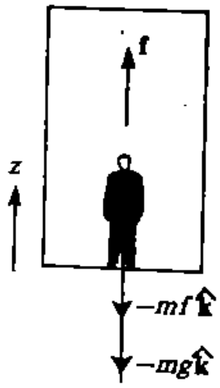


Fig. 10.18

a) The observer on the roadside will analyse the situation as follows: A centripetal force ($= mv^2/r$) where m is the mass of the car, v its speed and r the radius of curvature of the bend, is required by the car to move along the curve. You may recall from Sec. 4.3.1 of Block 1 that this is normally provided by way of the banking on the road and the friction between the tyres and the road. Let the contribution due to banking and friction be F_1, F_2, respectively. Then the equation of motion of the car will be

$$F_1 + F_2 = mv^2/r \quad \text{or} \quad \frac{F_1 + F_2}{m} = \frac{v^2}{r}$$

Now, the left hand side is a fixed quantity depending on m. So if v is large, r should be large in order to make the above equation hold. In other words, the car has to move more outward to have a large r, when it is moving very fast.

b. i) For this problem

$$\omega^2 r = (2\pi \times 10^3\ \text{s}^{-1})^2 \times (0.04\text{m}) = 1.6 \times 10^6\ \text{m s}^{-2}.$$

Since this is much larger than the usual value of 'g' the effective value of 'g' can be considered to be equal to $1.6 \times 10^6\ \text{m s}^{-2}$.

ii) The net centrifugal force $= m\omega^2 r$, where $m = 6 \times 10^{-19}$ kg. So its value is $(6 \times 10^{-19}\ \text{kg}) \times (1.6 \times 10^6\ \text{m s}^{-2}) = 9.6 \times 10^{-13}$ N.

5. a) The required angular speed will correspond to $g^* = 0$. We know from Eq. 10.15 that $g^* = g - \omega^2 R \cos^2\lambda$. So the required condition is

$$\omega = \sqrt{\frac{g}{R\cos^2\lambda}}.$$

So the minimum value of ω corresponds to the maximum value of $\cos^2\lambda$, i.e. 1 for $\lambda = 0^\circ$. This happens at the equator. And the required angular speed of earth is given

$$\omega_{min} = \sqrt{\frac{g}{R}}\,,$$

where R is the equatorial radius of the earth = 6.37×10^6 m.

$$\therefore\ \omega_{min} = \sqrt{\frac{10\ \text{ms}^{-2}}{6.37 \times 10^6\ \text{m}}} = 1.3 \times 10^{-3}\ \text{rad s}^{-1}.$$

b) At equator as explained in the answer to part (a).

Terminal Questions

1. a) Refer to Figs. 10.19 (a and b) for parts (i) and (ii), respectively.

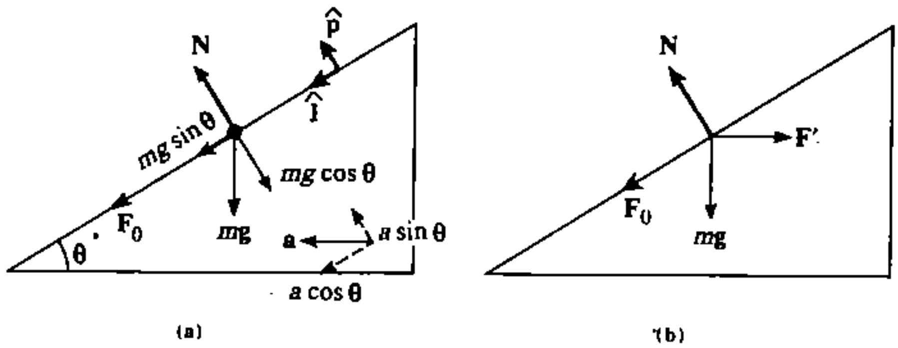


Fig. 10.19: F_0 is the force of friction. N is the normal reaction and mg is the weight of the block. a) The resultant of three forces F_0, N and mg is equal to ma. Components of a along $\hat{p}$ and $\hat{i}$ have also been shown; (b) In addition to F_0, N and mg, we have F', the inertial force (= $-m$a). The forces F_0, N, mg and F' are in equilibrium.

b) Using the force diagram for part (i), i.e. Fig. 10.19a, we have the equation of motion

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_o = m\mathbf{a}. \qquad (10.16)$$

Now, let the unit vectors along $\mathbf{F}_o$ and $\mathbf{N}$ be $\hat{\mathbf{i}}$ and $\hat{\mathbf{p}}$, respectively. So we have

$$mg\cos\theta\,(-\hat{\mathbf{p}}) + mg\sin\theta\,(\hat{\mathbf{i}}) + N\,(\hat{\mathbf{p}}) + F_o\,(\hat{\mathbf{i}}) = ma\cos\theta\,(\hat{\mathbf{i}}) + ma\sin\theta\,(\hat{\mathbf{p}}).$$

Thus,

$$(F_o + mg\sin\theta - ma\cos\theta)\ \hat{\mathbf{i}} + (N - mg\cos\theta - ma\sin\theta)\ \hat{\mathbf{p}} = \mathbf{0}.$$

$$\left.\begin{aligned}\therefore\ F_o + mg\sin\theta - ma\cos\theta &= 0\\ \text{and } N - mg\cos\theta - ma\sin\theta &= 0\end{aligned}\right\} \qquad (10.17)$$

Now if a be the magnitude of acceleration at which the block just begins to slip up we have $F_o = \mu N$ where $\mu = 0.8$.

So from Eqs. 10.17 we get

$$\mu N = m(a\cos\theta - g\sin\theta)$$

$$\text{or } \mu m(g\cos\theta + a\sin\theta) = m(a\cos\theta - g\sin\theta)$$

$$\therefore\ g(\mu\cos\theta + \sin\theta) = a(\cos\theta - \mu\sin\theta)$$

$$\text{or } a = g\left(\frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}\right)$$

Since, $\sin\theta = 0.6$, $\cos\theta = 0.8$

$$\therefore\ a = (10\ \text{ms}^{-2})\left(\frac{0.8 \times 0.8 + 0.6}{0.8 - 0.8 \times 0.6}\right) = 39\ \text{ms}^{-2}.$$

Using the force diagram for part (ii), i.e. Fig. 10.19b, we have,

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_o + \mathbf{F}' = \mathbf{0}.$$

Since $\mathbf{F}' = -m\mathbf{a}$, we get

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_o = m\mathbf{a}.$$

This is same as Eq. 10.16. So the succeeding analysis will follow as in the previous case and we shall get $a = 39\ \text{ms}^{-2}$. You must have noted that we come across an equation of motion in the inertial frame, but a condition of equilibrium in the non- inertial frame.

2. a) Let the required rate of spin be ω. Then the corresponding length of day is given by

$$T = \frac{2\pi}{\omega}.$$

Since the person inside has a sensation of artificial gravity, we have

$$\omega^2 r = g, \text{ where } r = 10\,\text{m}.$$

$$\therefore \frac{4\pi^2}{T^2} r = g$$

$$\text{or } T = 2\pi\sqrt{\frac{r}{g}}$$

$$= 2\pi\sqrt{\frac{10\text{m}}{10\ \text{ms}^{-2}}} = 6.3\,\text{s}.$$

b) Refer to Fig. 10.20. *NPM* and *QPR* are, respectively, the longitude and latitude through *P*, the position of the train. *AB* is the equator. The horizontal force is due to the Coriolis force given by

$$\mathbf{F}_{cor} = -2m\,(\boldsymbol{\omega} \times \mathbf{v}).$$

Since the angle between $\boldsymbol{\omega}$ and $\mathbf{v}$ is $(180° - \lambda)$ (see figure caption), the magnitude of the horizontal force is $2\,mv\omega \sin\lambda$.

where $m = 4.0 \times 10^5$ kg, $v = 30\ \text{ms}^{-1}$, $\omega = \dfrac{2\pi}{24 \times 60 \times 60}$ rad s^{-1} and $\lambda = 60°$.

So the magnitude of the horizontal force on the tracks is

$$2 \times (4 \times 10^5\ \text{kg}) \times (30\ \text{ms}^{-1}) \times \left(\frac{2\pi}{24 \times 60 \times 60}\ \text{s}^{-1}\right) \times \sin 60° = 1.5 \times 10^2\ \text{N}.$$

The direction is opposite to $(\boldsymbol{\omega} \times \mathbf{v})$. Now, $(\boldsymbol{\omega} \times \mathbf{v})$ points tangentially to the latitude *QPR* in the sense *Q* to *P*. So $\mathbf{F}_{cor}$ will be tangential to *QPR* in the sense *P* to *Q*, i.e. towards west.

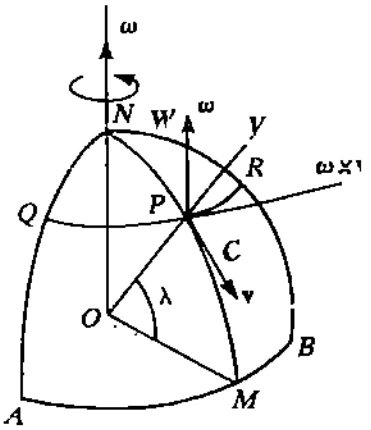


Fig 10.20 : Diagram for terminal question 2b. You must note that $\angle NOP = 90° - \lambda$. It is equal to the corresponding angle ($\angle WPY$). And $\angle YPC = 90°$. So $\angle WPC = 90° - \lambda + 90° = 180° - \lambda$

3. The weight of your body is given by

$$\mathbf{F} = m\mathbf{g} - \mathbf{F}_{cent} - \mathbf{F}_{cor},$$

where m is your mass. If you are at rest with respect to the earth $\mathbf{F}_{cor} = \mathbf{0}$. But if you are moving $\mathbf{F}_{cor} \neq \mathbf{0}$. So your weight will be different from W when you are in motion with respect to the earth.

FURTHER READING

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APPENDIX A

CONIC SECTIONS

The curves obtained by slicing a cone with a plane not passing through its vertex are called *conic sections* or simply *conics*. If the cutting plane is parallel to the side of the cone, as in Fig. A.1a, the conic is a *parabola*. Otherwise the intersection is called an *ellipse* or a *hyperbola*, according as the plane cuts just one or both nappes (portion of the cone) as shown in Figs. A.1b and A.1c. *Circle* is the special case of ellipse when the intersecting plane is parallel to the base of the cone (Fig. A.1d).

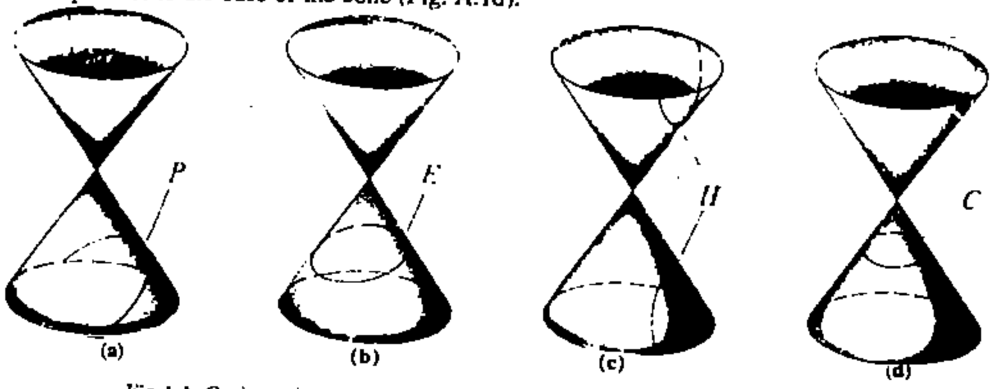


Fig.A.1: Conic sections: P - Parabola, E - Ellipse, H - Hyperbola, C - Circle

We shall now present a unified treatment for all conics. For this we shall define a term called 'eccentricity'.

A.1 Eccentricity and Polar Equation of a Conic

Refer to Fig. A.2. A conic section may be defined as a curve traced out by a point moving in a plane such that the ratio of its distance from a fixed point F (a focus) and a fixed line AB (a directrix) is constant. This constant ratio is called the *eccentricity*. It is denoted by e.

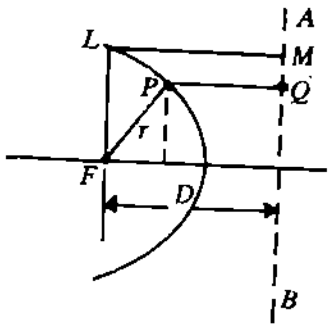


Fig. A.2: Polar equation of a conic

If $0 < e < 1$, the conic is an *ellipse*. If $e = 1$ it is a *parabola* and if $e > 1$, it is a *hyperbola*.

In the Fig. A.2, let P be any point on the conic. PQ is perpendicular on AB from P. Then according to the definition,

$$e = \frac{FP}{PQ}. \tag{A.1}$$

Using Eq. A.1 we shall obtain the polar equation of a conic, when the pole (i.e. the origin of the plane polar coordinates) is inside the curve. Let the pole be at F. The polar axis Fx is so chosen that it is perpendicular to the directrix. L is a point on the conic such that FL is perpendicular to Fx. FL is called the *semi-latus rectum* of the conic. Let $FL = p$. LM is again the perpendicular from L on AB. Let $LM = D$.

Now, we have

$$FP = r \quad \text{and} \quad QP = D - r\cos\theta.$$

So from Eq. A.1, we get

$$r = e\,(D - r\cos\theta)$$

$$\text{or} \quad r = \frac{eD}{1 + e\cos\theta}. \tag{A.2}$$

But from the definition of e, we find that

$$\frac{FL}{LM} = e, \quad \text{i.e.} \ \frac{p}{D} = e \quad \text{or} \quad p = eD.$$

So, we get from Eq. A.2 that

$$r = \frac{p}{1 + e\cos\theta}. \tag{A.3}$$

Eq. A.3 is the *polar equation of a conic with pole inside*.

The three types of conics have been shown in Fig. A.3. Because $\cos(-\theta) = \cos\theta$, all the three conics are symmetrical about the polar axis

The ellipse ($0 < e < 1$) is shown in Fig. A.3a. Here FP is always less than PQ. The curve crosses the polar axis at two points corresponding to $\theta = 0$, and $\theta = \pi$. Because of symmetry, the curve is closed.

The parabola ($e = 1$) is shown in Fig. A.3b. Here FP is always equal to PQ. When $\theta = 0$, $r = \frac{p}{2}$ · But as $\theta \to \pi$, $1 + \cos\theta \to 0$ and $r \to \infty$. So, the curve crosses the axis only once and spreads out to arbitrarily large distances from the axis as θ goes towards π.

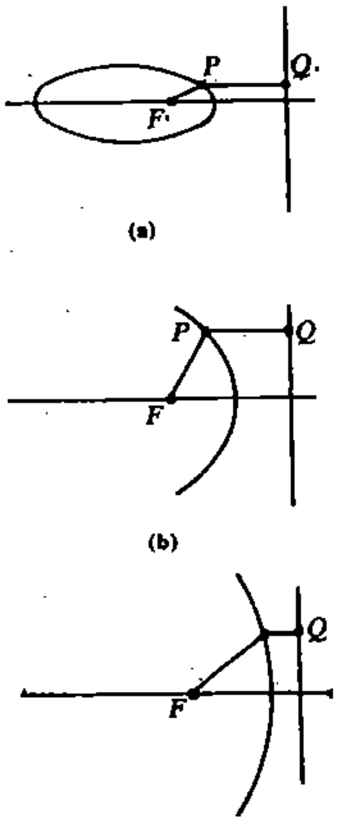


Fig. A.3: (a) Ellipse; (b) Parabola; (c) Hyperbola

The hyperbola ($e > 1$) is shown in Fig. A.3c. Here FP is always greater than PQ. When $\theta = 0$, $r = \frac{p}{1+e}$. There exists a value α of θ in the range $\frac{\pi}{2} < \theta < \pi$, such that $\cos\alpha = -\frac{1}{e}$. Using this we may write Eq. A.3 as

$$r = \frac{p}{e\,(\cos\theta - \cos\alpha)}.$$

As $\theta \to \alpha$, $r \to \infty$ and so the curve is open-ended but it spreads out in a manner different from that of the parabola. As r increases θ also increases but it never reaches the value α.

Circle is a special case of ellipse. It can be obtained by letting $e \to 0$ and $D \to \infty$ in Eq. A.2 so that eD, i.e. p tends to a finite quantity, say a. The limiting form of Eq. A.2 is then $r = a$. The parabola may be considered as the intermediate stage between the ellipse and the hyperbola. In other words it marks the transition from a closed to an open curve.

Among the three conics we have mostly dealt with the ellipse. So we shall study the properties of an ellipse in a little detailed manner.

A.2 Properties of the Ellipse

Refer to Fig. A.4. An ellipse has two foci F and F' and two directrices MN and $M'N'$. Both the points F and F' lie inside the curve.

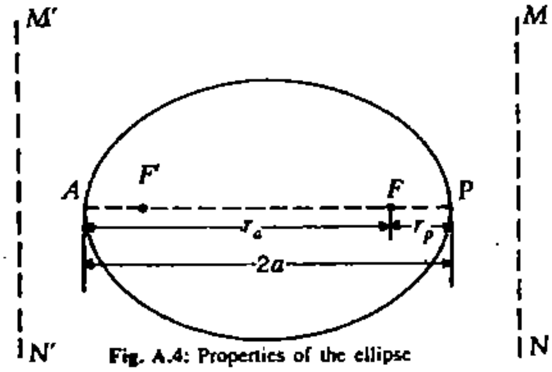


Fig. A.4: Properties of the ellipse

P and A are fixed points on the ellipse that are, respectively, nearest and farthest from F. Let $FP = r_p$ and $FA = r_a$. The axis of symmetry (shown dotted) is called the *major axis* of the ellipse. The sum $(r_p + r_a)$ is the length of the segment of the major axis intercepted by the ellipse. We write

$$r_p + r_a = 2a, \qquad \text{(A.4)}$$

where a is called the *semi-major axis* of the ellipse. We shall now express r_p and r_a in terms of a and e.

Here, r_p and r_a are, respectively, the minimum and maximum values of r. We can see from Eq. A.3 that when $\cos\theta$ is maximum (=1), i.e. $\theta=0$, then r is minimum and when $\cos\theta$ is minimum (= –1), i.e. $\theta = \pi$, then r is maximum.

$$\therefore r_p = \frac{p}{1+e}, \quad r_a = \frac{p}{1-e}$$

From Eq. A.4, we get

$$p\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = 2a$$

or $p = a(1-e^2)$ (A. 5)

$r_p = a\,(1-e\,); \quad r_a = a\,(1+e\,).$ (A. 6)

Using Eqs. A.3 and A.5, we may write the polar equation of the ellipse as

$$r = \frac{a\,(\,1 - e^2\,)}{1 + e\,\cos\theta}\,. \qquad \text{(A. 7)}$$

Eq. A. 7 tells us that the shape of the ellipse is governed by the value of e. In Fig. A. 5 several ellipses have been shown with the same focus F and the same value of a, but with different eccentricities. From Eq. A. 7 we see that $r \to a$ as $e \to 0$, which tells us that the smaller the eccentricity the rounder is the shape of the ellipse.

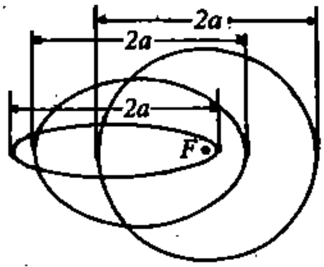


Fig.A.5: Ellipses of different shapes

Refer to Fig. A. 6. The dotted line represents a directrix of the ellipse.

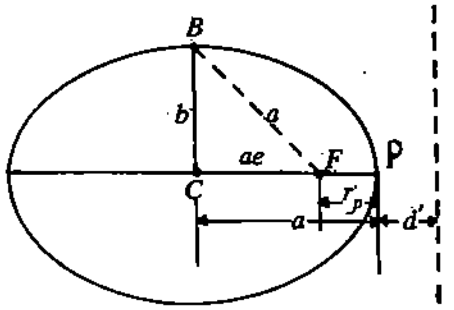


Fig. A.6: Relation between *a*, *b* and *e*

Let C be the midpoint of the major axis. It is called the *centre* of the ellipse. The chord of the ellipse through C and perpendicular to the major axis is called the *minor axis*. Let B be the point of the minor axis directly above C. Then BC is the semi-minor axis of the ellipse. Let $BC = b$. We shall now express b in terms of a and e.

We have $CP = a$ and $FP = a(1-e)$,

$$CF = CP - FP = ae.$$

Now the point B is at a distance $(a + d')$ from the directrix. From the definition of eccentricity, we have

$$\frac{FP}{d'} = e \quad \text{or} \quad d' = \frac{FP}{e} = \frac{a}{e}\,(1 - e)$$

$$\therefore \quad a + d' = a + \frac{a}{e} - a = \frac{a}{e}\,.$$

Since B is a point on the ellipse, we have

$$\frac{FB}{a+d'} = e$$

$$\text{or} \quad \frac{FB}{a/e} = e. \quad \therefore \; FB = a.$$

So in the right angled triangle BCF, we have

[illegible], $CF = ae$ and $FB = a$

$$\therefore \; a^2 = (ae)^2 + b^2$$

$$\text{or} \quad b^2 = a^2\,(1-e^2). \qquad \text{(A. 8)}$$

We saw that the ellipse has two foci each lying inside the conic. Like an ellipse a hyperbola has two foci and two directrices. We shall see that since hyperbola is open-ended it would have two open segments. For each open segment one focus is inside and another outside.

With $e > 1$, Eq. A.3 represents the equation of the segment of a hyperbola with focus (i.e. pole) inside. The polar equation of the other segment will also be of some interest. So let us take up the discussion on hyperbola with a view to studying the polar equation of a conic with pole outside.

A.3 Polar Equation of a Conic with Pole Outside

As we have seen just now the hyperbola is the only possible conic with focus outside.

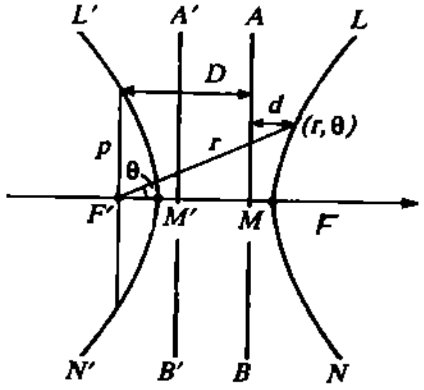


Fig. A.7: Polar equation of a conic with pole outside

Fig. A. 7 shows a hyperbola. F and F' are two foci. AB and $A'B'$ are the directrices. With $e > 1$, Eq. A. 3 represents the equation of the segment $L'M'N'$ with origin at F'. We shall now determine the equation of the segment LMN with respect to F' as origin.

As we had seen while deriving Eq. A. 3, we have,

$$e = \frac{r}{d} = \frac{p}{D} \qquad \text{or } D = \frac{p}{e}$$

Again $D = r\cos\theta - d$

$$\therefore \ \frac{p}{e} = r\cos\theta - \frac{r}{e}$$

$$\text{or } \ p = r\ (e\cos\theta - 1)$$

$$\text{or } \ r = \frac{p}{-1 + e\cos\theta} \qquad \text{(A.9)}$$

So Eq. A. 9 is the *equation of a conic with pole outside.*

Studying Eqs. A.3 and A.9 it can be said that the polar equation of a conic with pole inside or outside is given by

$$r = \frac{p}{\varepsilon + e\cos\theta} \qquad \text{(A. 10)}$$

according as ε is equal to +1 or −1.

APPENDIX B
METHODS OF DETERMINATION OF MOMENT OF INERTIA

In Sec. 9.3 of Unit 9 we have discussed about the determination of moment of inertia of bodies made up of continuous matter. The moment of inertia can be expressed by Eq. 9.6 as

$$I = \int r^2 \, dm$$

where the integral is a definite one and it extends over the entire body.

We shall see that it will be possible to express dm in terms of position and angular variables that describe the body. Then the limits of integration will correspond to these position and angular variables. They have to be chosen in a manner so that the integration extends over the whole body. Let us see how we do that. Since density (ρ) is mass per unit volume, $\rho = \frac{dm}{dV}$, where dV is the infinitesimal volume occupied by dm, we get from Eq. 9.6 that

$$I = \int r^2 \rho \, dV,$$ which for a body having uniform density reduces to

$$I = \rho \int r^2 dV. \tag{B.1}$$

For a two-dimensional body like a circular lamina, we define a quantity σ called the mass per unit area, i.e. $\sigma = \frac{dm}{dA}$, where dA is the infinitesimal area occupied by dm. So we get from Eq. 9.6 for a body having uniform σ, that

$$I = \sigma \int r^2 \, dA. \tag{B.2}$$

Similarly, for a one-dimensional body like a thin uniform rod, we define a quantity λ called the mass per unit length, i.e. $\lambda = \frac{dm}{dl}$, where dl is the infinitesimal length of the mass dm. So, we get from Eq. 9.6 for a body having uniform λ, that

$$I = \lambda \int r^2 \, dl. \tag{B.3}$$

We shall now put these methods to use. In the process we shall derive all the results of Table 9.1. First we shall verify the results (a), (b) and (g) of Table 9.1.

Verifying (a) and (b)

Since the rod is thin we consider it to be equivalent to a straight line AB lying along the x-axis (Fig. B.1a). The y-axis is then the axis about which the moment of inertia has to be determined. We take an infinitesimal element dx included between the points P and Q. The rod is an aggregate of such elements and x ranges from $-L/2$ to $+L/2$. On applying Eq. B.3, where we put $\lambda = \frac{M}{L}$, we get

$$I = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx = \frac{2M}{L} \int_{0}^{L/2} x^2 dx = \frac{ML^2}{12}.$$

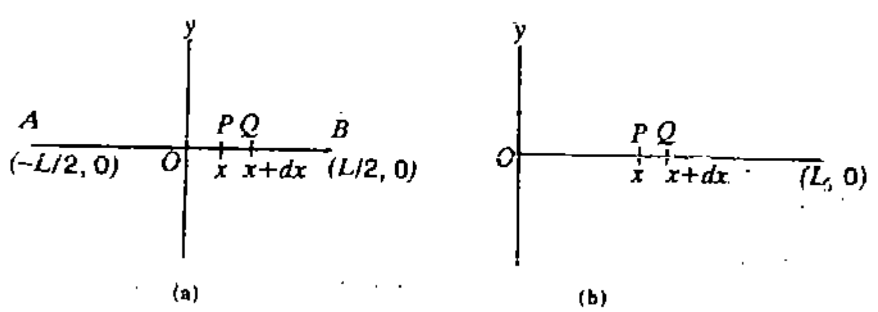


Fig. B.1: Moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through (a) its centre and (b) one end.

For (b), we shall adopt the same method. The only difference here is that the limits of integration are 0 and L (Fig. B.1b).

$$\therefore \quad I = \frac{M}{L}\int_0^L x^2 dx = \frac{ML^2}{3}$$

Verifying (g)

For this we shall first obtain the moment of inertia of an annular circular lamina about an axis perpendicular to its plane and passing through its centre. Its inne: ɟ outer radii are R_1 and R_2, respectively. It is shown in Fig. B.2a.

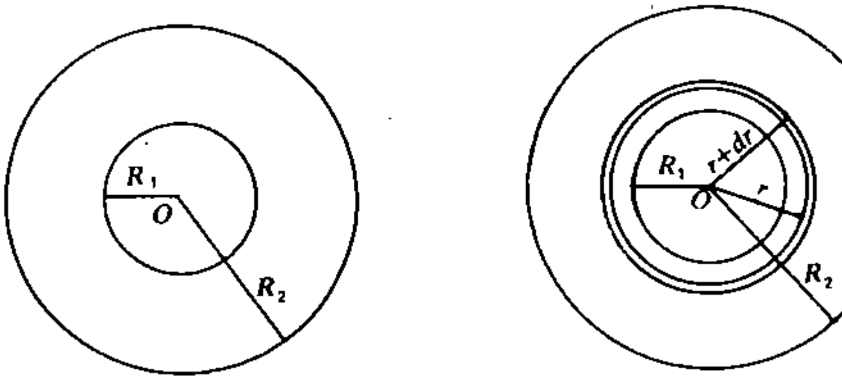


Fig. B.2: Moment of inertia of an annular circular lamina about its axis

Let the mass of this lamina be m. The area of the annular circular lamina is $\pi\,(R_2^2 - R_1^2)$. So,

$$\sigma = \frac{m}{\pi(R_2^2 - R_1^2)}.$$

Now we shall apply Eq. B.2. For that we have taken an element of a circular strip included between the radii r and $r+dr$ (Fig. B.2b). The area of this element $dA = 2\pi r dr$. The annular circular lamina is an aggregate of such strips with r ranging from R_1 to R_2. So from Eq. B.2, we get

$$I_s = \frac{m}{\pi(R_2^2 - R_1^2)}\int_{R_1}^{R_2} r^2\, 2\pi r dr,$$ where I_s is the moment of inertia of the lamina.

$$\therefore \quad I_s = \frac{2m}{(R_2^2 - R_1^2)}\int_{R_1}^{R_2} r^3\, dr = \frac{m\,(R_2^4 - R_1^4)}{2(R_2^2 - R_1^2)} = \frac{m}{2}\,(R_1^2 + R_2^2).$$

Fig. B.3: Annular cylinder

Now, suppose we stack such exactly similar laminas, one above the other, what do we get? We get an annular cylinder of same inner and outer radii (Fig. B.3) as that of the lamina. If the height of the stack is much smaller than R_2 then we get an annular disc. Let the moment of inertia of the annular cylinder (or disc) be I. As the axis remains unchanged we have

$I = \Sigma I_s$. Since R_1, R_2 are constants, we have

$$I = (\Sigma m)\,\frac{(R_1^2 + R_2^2)}{2}.$$

But $\Sigma m = M$ = the mass of the annular cylinder (or disc).

$$\therefore \quad I = \frac{M}{2}\,(R_1^2 + R_2^2)$$

Now the result (h) of Table 9.1 can be treated as a special case of (g) by putting $R_1 = 0$ and $R_2 = R$. So we get

$$I = \frac{1}{2}MR^2.$$

We shall now use (h) to obtain (i).

Verifying (i)

Fig. B.4: Diagram for verifying (i)

Refer to Fig. B.4. Moment of inertia of the sphere has to be determined about a diameter. Let the y-axis be along the diameter. We now consider an infinitesimal portion of the sphere included between two planes perpendicular to the y-axis. These are at distances y and $y + dy$, respectively, from the centre of sphere. This portion is a circular disc of infinitesimal thickness dy with A as centre and radius $\sqrt{R^2 - y^2}$ where $OA = y$. So its mass is equal to

$\pi\rho\,(R^2 - y^2)\,dy$, where ρ is the density of the material of the sphere. Hence, using the result (h), we get that the moment of inertia of the disc is given by

$$dI = \frac{1}{2}\pi\,\rho\,(R^2 - y^2)\,dy\,\left(\sqrt{R^2 - y^2}\,\right)^2$$

and the moment of inertia I of the sphere can be obtained by integrating the expression for dI over the entire range of y.

$$\therefore\quad I = \frac{\pi\rho}{2}\int_{-R}^{+R}(R^2 - y^2)^2\,dy = \pi\,\rho\int_0^R (R^4 - 2R^2y^2 + y^4)\,dy = \frac{8\pi\rho}{15}R^5,$$

$$\text{or}\quad I = \frac{8\pi}{15}\,\frac{M}{\frac{4}{3}\pi R^3}\,R^5 = \frac{2}{5}MR^2.$$

We shall next verify the results (d), (e) and (f).

Verifying (d)

A ring may be considered as a circle, i.e. its thickness may be neglected. We consider an element of arc ds of the ring. Every point of this element is at a perpendicular distance R from the axis AOB (Fig. B.5). So, on using Eq. 9.6, we get

$$I = R^2\int dm = MR^2\ \left(\because \int dm = M\right).$$

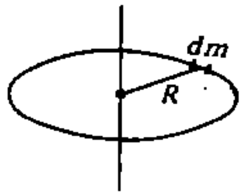


Fig. B.5: Moment of inertia of a ring

We shall now derive (e) and (f). For these you will find that it would be more convenient to use the plane polar coordinates.

Verifying (e)

Refer to Fig. B.6. We have identified the y-axis with the axis about which moment of inertia has to be determined. The x-axis is taken perpendicular to it through the centre of the ring. We consider an element PQ of the ring included between the angles θ and $\theta + d\theta$. Its distance from the axis is $R\cos\theta$. We shall now use Eq. B.3. Here dl = the length of the element of $PQ = Rd\theta$ and $r = R\cos\theta$. Note that the integration has to be performed over the entire ring, and the variable is θ. So θ ranges from 0 to 2π. Thus, we get

$$I = \frac{M}{2\pi R}\int_0^{2\pi} R^2\cos^2\theta\,R\,d\theta = \frac{MR^2}{2\pi}\int_0^{2\pi}\cos^2\theta\,d\theta.$$

The definite integral is a standard one and its value is π. So, $I = \frac{1}{2}MR^2$. Now we shall verify (f).

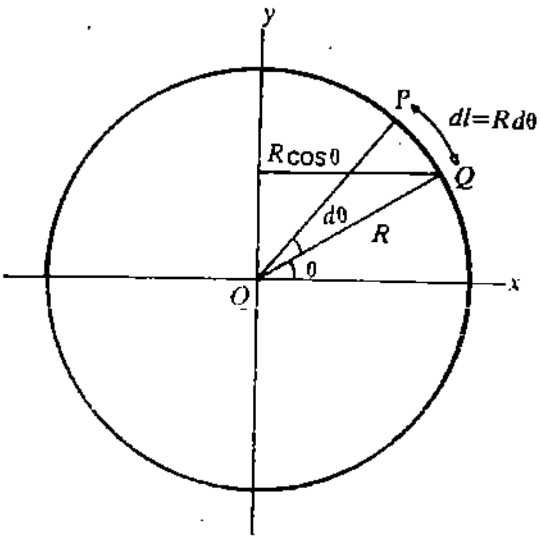


Fig. B.6: Diagram for verifying (e)

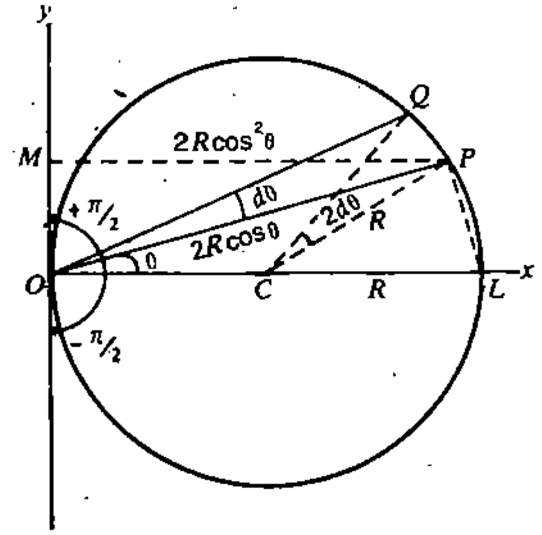


Fig. B.7: Diagram for verifying (f)

Verifying (f)

Refer to Fig. B.7. We identify the y-axis with the axis about which moment of inertia has to be determined. The x-axis is taken perpendicular to it through the point of contact of y-axis with the ring. Thus x-axis is along a diameter. We consider an element PQ of the ring included between the angles θ and $\theta + d\theta$. As the angle subtended at the centre of a circle by an arc is twice of that which it subtends at the circumference we have $\angle PCQ = 2\angle POQ = 2d\theta$. $\therefore PQ = R(2d\theta) = 2Rd\theta$. Since $\angle OPL = 90°$, $OP = 2R\cos\theta$, hence, the perpendicular distance of the element of PQ from the y-axis = $PM = OP\cos\theta = 2R\cos^2\theta$. We shall now use Eq. B.3 where $\lambda = \dfrac{M}{2\pi R} \cdot r = 2R\cos^2\theta$ and θ ranges from $-\pi/2$ (lower semicircle) to $+\pi/2$ (upper semicircle).

If $J_n = \int_0^{\pi/2} \cos^n\theta\, d\theta$, then

$J_n = \frac{n-1}{n} J_{n-2}$

$\therefore \int_0^{\pi/2} \cos^4\theta\, d\theta = J_4 = \frac{3}{4} J_2$

$J_2 = \frac{1}{2} J_0 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$

$\therefore \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}$

$$\therefore \quad I = \frac{M}{2\pi R} \int_{-\pi/2}^{+\pi/2} (2R\cos^2\theta)^2\, 2R\, d\theta = \frac{8MR^2}{\pi} \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$I = \frac{8MR^2}{\pi} \frac{3\pi}{16} = \frac{3}{2} MR^2.$$

We had to use quite a bit of geometry and trigonometry for verifying (c) and (f). In fact this can be done much more elegantly by applying the theorems of parallel and perpendicular axes. We shall now learn these theorems.

Parallel Axes Theorem

Refer to Fig. B.8. C is the c.m. of the body of mass M and AB is an axis passing through it. $A'B'$ is an axis parallel to AB. Now, we want to determine the moment of inertia of the body about $A'B'$ (I) if the same about AB (I_{cm}) is known. Let the perpendicular distance between the axes be L. We consider a point mass m_i at P. Let the position vectors of the points O and P with respect to C as origin be $\mathbf{h}$ and $\mathbf{R}_i$ respectively, and let $\mathbf{OP} = \mathbf{r}_i$. We know that, $\mathbf{OP} = \mathbf{OC} + \mathbf{CP} = \mathbf{CP} - \mathbf{CO}$,

$$\text{or} \quad \mathbf{r}_i = \mathbf{R}_i - \mathbf{h}.$$

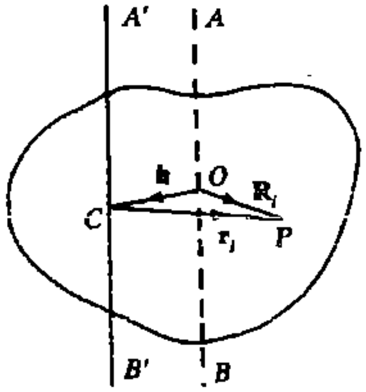


Fig. B.8: Parallel-axes theorem

Now $I = \sum_i m_i r_i^2 = \sum_i m_i \mathbf{r}_i \cdot \mathbf{r}_i$, where the summation is over all the points that comprise the body.

$$\therefore I = \sum_i m_i (\mathbf{R}_i - \mathbf{h}) \cdot (\mathbf{R}_i - \mathbf{h}) = \sum_i m_i (R_i^2 - 2\mathbf{R}_i \cdot \mathbf{h} + h^2).$$

Now, the relative position of AB and $A'B'$ is independent of the position of P. In other words, $\mathbf{h}$ is independent of i.

$$\therefore\ I = \sum_i m_i R_i^2 - 2\,(\sum_i m_i \mathbf{R}_i)\,.\,\mathbf{h} + (\sum_i m_i)\,h^2.$$

The first term is the moment of inertia of the body about the c.m., i.e. I_{cm}. Again

$\dfrac{\sum_i m_i \mathbf{R}_i}{\sum_i m_i}$ = the position vector of the c.m. = $\mathbf{0}$, as it is at the origin. So the second term is zero. And $\sum_i m_i = M$. So the third term is Mh^2. Hence, we get

$$I = I_{cm} + Mh^2. \qquad \text{(B.4)}$$

Eq. B.4 is the mathematical form of the *parallel axes theorem*. According to it, the moment of inertia of a body about an axis is equal to the sum of the inertia of the body about the axis parallel to the given axis and passing through its c.m. and the product of the mass of the body and the square of the distance between the axes.

Theorem of Perpendicular Axes (For a Laminar Body)

Refer to Fig. B.9. Suppose we know the moments of inertia of the body about two mutually perpendicular axes lying on the plane of the lamina. Then we shall see that with the help of this theorem we shall be able to know the moment of inertia of the body about a third axis which is perpendicular to the above pair of axes at their point of intersection.

Let us consider a point mass m_i within the body. On the plane of the lamina, we take two axes Ox, Oy with respect to which the coordinates of m_i are (x_i, y_i). Oz is perpendicular to Ox and Oy. The perpendicular distance of m_i from the z-axis is r_i and is given by

$$r_i^2 = x_i^2 + y_i^2.$$

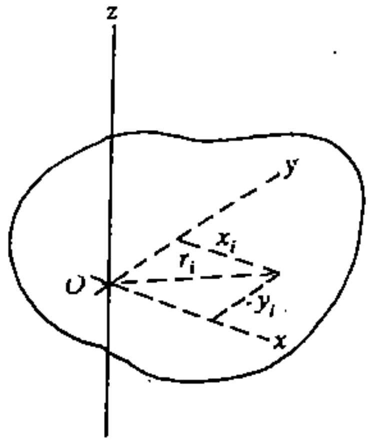


Fig B.9: Theorem of perpendicular axes for a two-dimensional body

Let the moments of inertia of the body about the three mutually perpendicular axes Ox, Oy and Oz be I_x, I_y and I_z, respectively. Let us now express I_z in terms of I_x and I_y.

Now,

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i\,(x_i^2 + y_i^2)$$

$$\text{or } I_z = \sum_i m_i x_i^2 + \sum_i m_i y_i^2$$

$$\therefore\ I_z = I_x + I_y. \qquad \text{(B.5)}$$

Eq. B.5 is the mathematical form of the theorem of perpendicular axes for a laminar body, according to which the sum of the moments of inertia of a laminar body about two mutually perpendicular axes on its plane is equal to the moment of inertia of the body about a third axis perpendicular to the above pair of axes, passing through their point of intersection.

We shall now apply these theorems to verify (e) and (f).

Refer to Fig. B.10a.

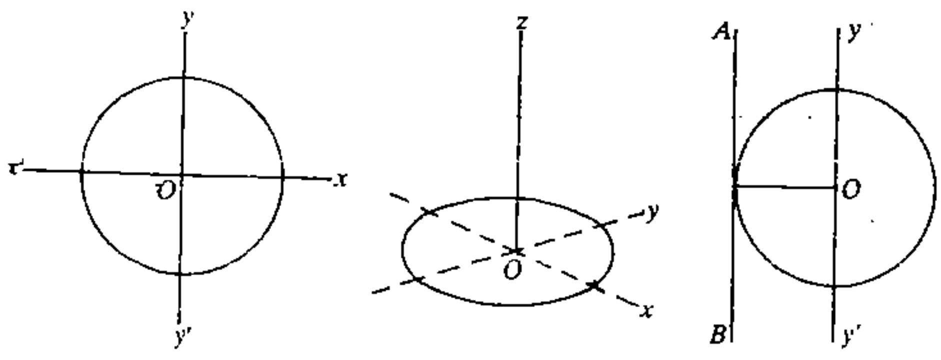


Fig. B.10: Determination of moment of inertia of a ring.

We have to determine the moment of inertia of the body about a diameter. xx' and yy' are two perpendicular axes along its diameters. Let the moments of inertia about these axes be I and I_y. If the ring is now turned through 90° about an axis perpendicular to its plane and passing through O, the x-axis takes the position of y-axis and vice-versa. But the ring looks the same and it coincides exactly with its previous position. This means that $I_x = I_y$. Let us now consider Fig. B.10b where the x and y-axes have been shown along with the z-axis. By applying the theorem of perpendicular axes (Eq. B.5) we get

$$I_z = I_x + I_y = 2I_y$$

$$\text{or } I_y = \frac{1}{2} I_z.$$

But from the result (d) of Table 9.1 we know that, $I_z = MR^2$.

Thus, $I_y = \frac{1}{2} MR^2$, and hence (e) is verified.

We shall now verify (f). The axis AB about which moment of inertia is required is parallel to yy', which passes through the c.m. (Fig. B.10c). Now on applying the theorem of parallel axes (Eq. B.4), we get

$$I_{AB} = I_{cm} + MR^2.$$

But $I_{cm} = I_y = \frac{1}{2} MR^2$ from the result (e).

$$\therefore \quad I_{AB} = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2, \text{ and hence (f) is verified.}$$

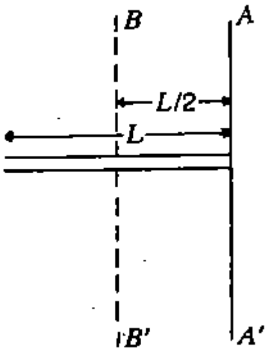


Fig. B.11:

So, we see that these two theorems provide us with an elegant method for determining moments of inertia. The methods adopted earlier were more complicated. Moreover, if the moment of inertia of a body about an axis passing through its c.m. is known then we can obtain the moment of inertia of the body about any parallel axis by using the parallel-axes theorem. The theorem of perpendicular axes can be found handy in determining I_z for a laminar body if I_x and I_y are known. For symmetric bodies I_x and I_y can be determined if I_z is known. We may now verify the result (b) of Table 9.1 using parallel-axes theorem. The distance between the given axis AA' and the axis BB' parallel to it and passing through the c.m. is $\frac{L}{2}$ (Fig. B.11). So on applying the result (a) and Eq. B.1, we get

$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = ML^2\left(\frac{1}{12} + \frac{1}{4}\right) = \frac{ML^2}{3}$$

We shall now verify (c) using the perpendicular - axes theorem.

Refer to Fig. B.12

Let us consider an infinitesimal strip of width dx at a distance x from the y-axis. The area of the strip is bdx and every point on it is at a distance x from the y-axis. So its moment of inertia about the y-axis is given by Eq. 9.6c ($\sigma = M/ab$) as

$$I_y = \frac{M}{ab} \int_{-a/2}^{+a/2} x^2 \, b dx$$

$$= \frac{2M}{a} \int_0^{a/2} x^2 dx = \frac{2M}{a} \frac{a^3}{24} = \frac{Ma^2}{12}$$

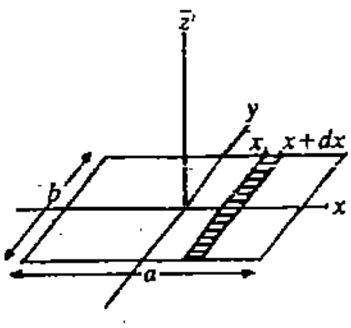


Fig. B.12: Diagram for verifying (c)

Similarly, the moment of inertia about x-axis can be determined. It is given by

$$I_x = \frac{M}{12} b^2.$$

From Eq. B.5 we get $I_z = I_x + I_y = \frac{M}{12} (a^2 + b^2)$.

Let us now consider three mutually perpendicular axes (x, y and z) through the centre of a solid sphere of mass M and radius R. As these axes are along the diameters we can say from result (i) of Table 9.1 that $I_x = I_y = I_z = \frac{2}{5}MR^2$. So $I_z \neq I_x + I_y$. Does this violate the theorem of perpendicular axes? It does not because a sphere is a three-dimensional body and Eq. B.5 is valid for a two-dimensional body only. However, there is a theorem of perpendicular axes for a three-dimensional body also. We shall study that now.

Theorem of Perpendicular Axes for a Three-dimensional Body

Refer to Fig. B.13. It shows a three-dimensional body. P is the position of a point mass m_i within the body and r_i is its distance from the origin O of a three-dimensional rectangular coordinate system. The coordinates of the point P are (x_i, y_i, z_i). So, the perpendicular distances of the point P from the x, y and z-axes are $\sqrt{y_i^2 + z_i^2}$, $\sqrt{z_i^2 + x_i^2}$ and $\sqrt{x_i^2 + y_i^2}$, respectively. Hence, the moment of inertia of m_i about the x-axis is given by

$$I_{xi} = m_i (y_i^2 + z_i^2).$$

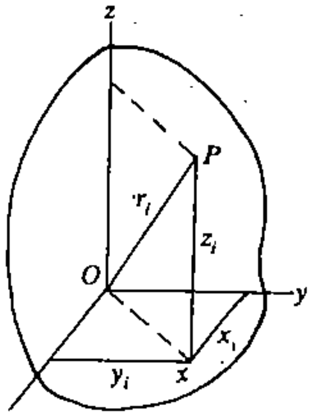


Fig. B.13: Theorem of perpendicular axes for a three-dimensional body

Now the moment of inertia of the whole body about the x-axis will be the sum of I_{xi} taken over all mass points that make up the body. It is given by

$$I_x = \sum_i I_{xi} = \sum_i m_i (y_i^2 + z_i^2).$$

Similarly,

$$I_y = \sum_i m_i (z_i^2 + x_i^2).$$

$$\text{and } I_z = \sum_i m_i (x_i^2 + y_i^2).$$

$$\therefore \quad I_x + I_y + I_z = 2\sum_i m_i (x_i^2 + y_i^2 + z_i^2).$$

$$\text{But } r_i^2 = x_i^2 + y_i^2 + z_i^2.$$

$$\therefore \quad I_x + I_y + I_z = 2\sum_i m_i r_i^2 . \qquad \text{(B.6)}$$

Eq. B.6 is the mathematical form of the theorem of perpendicular axes for a three-dimensional body. We shall now verify the result (j) of Table 9.1 using this theorem.

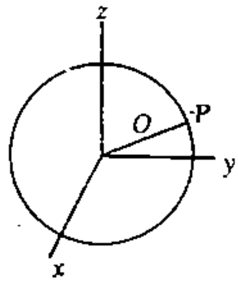


Fig. B.14: Diagram for verifying (j)

Refer to Fig. B.14. Ox, Oy and Oz are the three rectangular coordinate axes, where O is the centre of the spherical shell. So Ox, Oy, Oz are along three diameters of the shell. Since the shell is symmetrical, $I_x = I_y = I_z = I$, where I is the moment of inertia of the shell about any diameter. So the left hand side of Eq. B.6 is $3I$. P is any point of the shell. Let its mass be m_i and its distance from O, r_i. Now, for all points of the shell r_i is equal to the radius of the shell, R. So from Eq. B.3, we get

$$3I = 2\sum_i m_i R^2$$

or $$3I = 2R^2 \left(\sum_i m_i\right) = 2MR^2 \left(\because \sum_i m_i = \text{the mass of the shell} = M\right).$$

or $$I = \frac{2}{3}MR^2.$$

This result could have been obtained without using Eq. B.3. But the process would have been lengthy.

A list of commonly occurring quantities in the block along with their unit symbols, special names (if any) and dimensions is given below. Dimensions are given in terms of length [L], mass [M], time [T], temperature [K], and charge [Q].

Quantity	SI UNIT		Dimensions
	Special names (if any)	Symbol	
Displacement		m	[L]
Velocity		$m\ s^{-1}$	$[LT^{-1}]$
Acceleration		$m\ s^{-2}$	$[LT^{-2}]$
Angular displacement	radian	rad	—
Angular velocity		$rad\ s^{-1}$	$[T^{-1}]$
Angular acceleration		$rad\ s^{-2}$	$[T^{-2}]$
Angular momentum		$kg\ m^2 s^{-1}$	$[ML^2T^{-1}]$
Force	newton	N	$[MLT^{-2}]$
Work, Energy	joule	J	$[ML^2T^{-2}]$
Power	watt	W	$[ML^2T^{-3}]$
Gravitational potential		$J\ kg^{-1}$	$[L^2T^{-2}]$
Gravitational Intensity		$N\ kg^{-1}$	$[LT^{-2}]$
Momentum, Impulse		$kg\ m\ s^{-1}$	$[MLT^{-1}]$
Period		s	[T]
Moment of inertia		$kg\ m^2$	$[ML^2]$
Area		m^2	$[L^2]$
Volume		m^3	$[L^3]$
Density		$kg\ m^{-3}$	$[ML^{-3}]$
Torque		Nm	$[ML^2T^{-2}]$
Temperature	kelvin	K	[K]
Electric charge	coulomb	C	[Q]
Electric current	ampere	A	$[T^{-1}Q]$

Table of Constants

Physical Constants

Symbol	Quantity	Value
c	speed of light in vacuum	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
μ_0	permeability of free space	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
ϵ_0	permittivity of free space	$8.854 \times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}$
$1/4\,\pi\epsilon_0$		$8.988 \times 10^{9}\,\mathrm{N\,m^2C^{-2}}$
e	charge of the proton	$1.602 \times 10^{-19}\,\mathrm{C}$
$-e$	charge of the electron	$-1.602 \times 10^{-19}\,\mathrm{C}$
h	Planck's constant	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
$\hbar$	$h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
m_e	electron rest mass	$9.109 \times 10^{-31}\,\mathrm{kg}$
$-e/m_e$	electron charge to mass ratio	$-1.759 \times 10^{11}\,\mathrm{C\,kg^{-1}}$
m_p	proton rest mass	$1.673 \times 10^{-27}\,\mathrm{kg}$
m_n	neutron rest mass	$1.675 \times 10^{-27}\,\mathrm{kg}$
R	Rydberg constant	$1.097 \times 10^{7}\,\mathrm{m^{-1}}$
a_o	Bohr radius	$5.292 \times 10^{-11}\,\mathrm{m}$
N_A	Avogadro constant	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
R	Universal gas constant	$8.314\,\mathrm{J\,K^{-1}mol^{-1}}$
k_B	Boltzmann constant	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
G	Universal gravitational constant	$6.673 \times 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$

Astrophysical Data

Celestial Body	Mass (kg)	Mean Radius(m)	Mean distance from the centre of Earth (m)
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.85×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

NOTES

NOTES

NOTES

Uttar Pradesh
Rajarshi Tandon Open University

UGPHS - 01
Mechanics

Block

1

CONCEPTS IN MECHANICS

MECHANICS

In our everyday life we come across a wide variety of objects in motion. The branch of physics dealing with the motion of bodies and bodies at rest in equilibrium is called *mechanics.* You use the laws of mechanics when you ride a bicycle, lift heavy loads, play football or build a house. Many fascinating developments of the space age, such as launching of space probes and artificial satellites are direct applications of laws of mechanics.

Today, mechanics is regarded as the most fundamental area of physics. In order to study other areas of physics, such as electromagnetism, thermal physics, vibrations and waves you need to have a sound knowledge of mechanics. Quantum mechanics and relativity are the two gateways to modern physics. A profound concept of Newtonian Mechanics is required to understand these subjects.

The development of mechanics started when attempts were made to understand the motion of bodies in everyday life on one hand and heavenly bodies, such as planets on the other. As you know, the three laws of motion discovered by Newton form the basis of mechanics. These laws appear simple to us, the way they are taught at the school. But it took more than two thousand years, from the Aristotelian views of fourth century BC to the seventeenth century AD when the Newtonian laws were put forward, to arrive at the right conclusions. Aristotle's description of motion was based on the idea of a "natural place". Recall from Unit 3 of FST 1 that in his times it was believed that all matter was made up of one or more of the four elements — earth, fire, water and air. In his view earth had the lowest place. So if you dropped a heavy object made of earth, it fell. Heavy bodies, containing more earth, fell faster than light ones. Similarly, smoke rose because it was made up of fire and the natural place of fire is in the air. Heavenly bodies, such as the Sun, the Moon and the planets moved around the Earth in circles because a circle was regarded as the most perfect shape.

These Aristotelian ideas about motion held sway for about 2,000 years till the emergence of modern science. Observation, experiment and careful measurement are the hallmarks of modern science. It was in all these respects that Galileo broke away from the ancient tradition. Whereas others before Galileo speculated on properties of matter in motion, he based his arguments on detailed observations. His experiments with balls rolling down smooth inclined planes led to the law of falling bodies and law of inertia. Galileo's work set the tone and the seventeenth century saw a rapid development in mechanics. Tycho Brahe's detailed observations of planetary motion enabled Johannes Kepler to arrive at his laws of planetary motion. Many a brilliant mind of the times, especially Robert Hooke, tried to bring about a synthesis of the laws governing the motion of heavenly bodies and bodies on the earth. Eventually, it was Isaac Newton who presented mechanics as a scientific theory. Newton's laws of motion alongwith his law of universal gravitation provide a complete description of motion of all material bodies in the universe. The only systems known to us today to which Newtonian Mechanics does not apply are the subatomic particles and the bodies moving at velocities close to the velocity of light.

In this course, we shall first develop the basic concepts of mechanics and apply them to simple physical situations in Block 1. We shall also study three important conservation laws of linear momentum, angular momentum and energy. In Block 2, we shall extend these concepts to the study of more complex situations, such as planetary motion, many-particle systems and rigid body dynamics.

BLOCK 1 CONCEPTS IN MECHANICS

In this block you will study the basic concepts of mechanics. You are already familiar with these concepts. So essentially it will be a recapitulation of what you have studied in your school science courses. The only difference is that we shall use three-dimensional vectors extensively as we develop these concepts. In Unit 1 you will learn the language for describing *how* things move, which is known technically as *kinematics*. We will discuss the important kinematical concepts of displacement, velocity and acceleration. Using them we will describe uniform circular motion.

In Unit 2 we go on to the causes of motion, technically known as *dynamics*. We shall discuss the concepts of force and linear momentum which are bound together in Newton's laws of motion. You will study the equilibrium of forces, which finds many applications in mechanical devices. You will also study the principle of conservation of linear momentum. This principle makes it easier to study complex mechanical phenomena where direct application of Newton's laws is difficult. In Unit 3 we shall take up other dynamical concepts, such as work and energy. Interestingly, the notion of energy is one of the few elements of mechanics not handed down to us by Sir Isaac Newton. This idea was clearly grasped only by the middle of nineteenth century. The concept of energy is useful in studying diverse phenomena, such as the evolution of the universe, properties of elementary particles, biochemical reactions in living systems, design of machines etc. The law of conservation of energy is particularly important. This law along with the law of conservation of linear momentum will be used to study collisions.

In Unit 4, you will study the kinematics and dynamics of angular motion. We shall discuss the concepts of angular displacement, angular velocity and acceleration, torque and angular momentum and apply them to the angular motion of a particle. The law of conservation of angular momentum and its applications will also be discussed. Finally, in Unit 5 we shall undertake the study of Newton's law of universal gravitation which leads to an understanding of the motion of all bodies in this universe, be they planets, satellites or objects on the earth. In this context we shall also discuss the force of gravity which plays such an important role in our everyday lives.

As far as possible, we have tried to illustrate all these concepts by application to problems of real physical interest. We have also provided numerous solved examples throughout the text. Our emphasis all along is not only on understanding these concepts but also on applying them to a wide variety of physical phenomena.

We have supplemented this block with two audio-vision productions. You will get access to these programmes at your Study Centres. Through these programmes we have tried to develop problem solving ability in mechanics.

The units are not of equal length. On an average, we can suggest the following estimate for the study time required for each unit: $3\frac{1}{2}$ h for Unit 1, 4 h for Unit 2, 4 h for Unit 3, 5 h for Unit 4, $3\frac{1}{2}$ h for Unit 5 giving a total of 20 h for working through the text and solving the SAQs and Terminal Questions. Your actual study time will, of course, depend on your background. For example, if you have done your +2 or twelfth standard recently, the mathematics, especially calculus, used in these units will be easy to follow. Since you have done +2, we are, in fact, assuming that you know the calculus used here. However, if you have done your twelfth a few years ago, you may like to brush up your knowledge of calculus. For this you could either study the IGNOU course MTE-01 on calculus or the twelfth standard NCERT book on mathematics. It will be available at your study centre

We will use the SI units throughout. A table of the units and their symbols is given in the block. A table of the physical constants is also provided on the same page. You will have to refer to this table for solving a number of problems. You will also find a list of suggested books for further reading at the end of the block.

Some of the abbreviations used in the text are Sec. for Section, Fig. for Figure and Eq. for Equation. Fig. X.Y refers to the Yth Figure of Unit X, i.e., Fig. 1.10 is the tenth figure in Unit 1. Similarly, Sec. 2.4 is the fourth section in Unit 2 and Eq. 3.9 is the ninth equation in Unit 3.

Study Guide

Physics, as you know, cannot be learnt passively. This is equally true for this course. You will have to work through the derivations and the solved examples given in the text yourself. So, always keep a pen or a pencil and paper with you while studying. You will also need a ruler and a protractor. For performing calculations in course of working numerical problems you will have to use either a calculator or the standard booklet containing logarithmic and trigonometrical tables. So, keep either of these aids handy. The idea is *not* to memorise, but to *understand* concepts. You will be able to acquire a better understanding of the concepts of mechanics only if you apply them to problems. We have tried to select a representative set of problems. Some are straightforward and intended for practice, but many of them require some effort. There are a few challenging problems as well, in the form of Terminal Questions. We advise you to make an honest attempt at solving the Self Assessment Questions (SAQs) and the Terminal Questions. Do *not* immediately turn to the answers given at the end of each unit if you cannot solve a problem in the first instance. In many problems values of fundamental constants like the Universal Gravitational constant G or the charge of electron e may be required. You will not find these values in the text of the problem as they are given in the table of physical constants. The same also holds for certain astrophysical data like the radius of earth, the mean distance between the moon and the earth and so on.

You must also take sufficient care to see that the answers to numerical problems are expressed in proper units. For this we advise you to write the necessary units at every step while performing the calculations as we have done in the worked examples. While arriving at the answers to the numerical problems we have adopted the conventional method of rounding off to the desired number of significant digits. By the way, you will have to work out these questions on separate papers, as no space has been provided in the text for this purpose.

We hope you will enjoy studying the material and wish you success.

Acknowledgments

Prof. R.N. Mathur for comments on the units

UNIT 1 MOTION

Structure

1.1 INTRODUCTION

Nothing characterises our daily lives more than motion itself. A game of cricket or football, the graceful movements of a dancer, falling leaves, rising and setting sun are all examples of matter in motion. You have studied about motion in your school science courses. However, your study was limited to motion along a straight line and in a two-dimensional plane. Among the prominent examples of two-dimensional motion that you have studied are circular motion and projectile motion.

But you know that our world is three-dimensional in space. Therefore, we shall begin by studying motion in three dimensions. We shall first understand what we mean when we say that an object is moving. Since vectors will be used extensively, we shall quickly go through relevant vector algebra. Using vectors, we shall develop a language for describing *how* things move. For this we shall recall the basic concepts of displacement, velocity and acceleration. Finally, we shall use these concepts to study uniform circular motion and relative motion.

The concept of acceleration is related to the causes of motion, which we shall study in Unit 2. There we shall also study equilibrium of forces and conservation of linear momentum principle.

Objectives

After studying this unit you should be able to:

- specify an appropriate frame of reference for a given physical situation
- express one-dimensional, two-dimensional and three-dimensional vectors using unit vectors
- compute the sum, difference, scalar and vector products of two vectors
- determine the displacement, velocity and acceleration of a particle in a given frame of reference
- distinguish between average and instantaneous velocity, and average and instantaneous acceleration
- determine relative velocity and acceleration of one particle with respect to another
- solve problems concerning relative motion and uniform circular motion.

1.2 WHAT IS MOTION?

Can you imagine what your life would be like if you were confined to some place, unable to move from one position to another as time passed? Read this sentence again and you will see that it suggests an answer to the question: What is motion? We say that an object is moving if it occupies different positions at different instants of time. The study of motion thus deals with the questions: where? and when?

Frame of reference

The actual motion of an object can be determined by measuring the changes of position during measured intervals of time. And to determine the position of an object at a given instant of time, or the changes in its position with time, we need a **frame of reference.** To understand this, let us take the example of a moving train. The change in the train's position in a given time interval has one value if measured by an observer standing on the ground. It has a different value if measured by an observer moving in a car. It will have the value zero if measured by an observer sitting in the train itself. And each of these values is equally correct from the point of view of the observer making the measurement.

In general, the measured value of any physical quantity depends on the reference frame of the observer who is making the measurement. *To specify a physical quantity, each observer may choose a zero of the time scale, an origin in space and an appropriate coordinate system. We shall refer to these collectively as a frame of reference.* Since the space of our experience has three dimensions, we must in general specify three coordinates to fix uniquely the position of an object. The Cartesian coordinates x, y, z are commonly used in mechanics. Thus, the position and time of any event may be specified with respect to the frame of reference by three Cartesian coordinates x, y, z and the time t. We may suppose, for example, that the observer is located on a solid body, such as the earth. He or she chooses some point of this body as the origin and takes the axes to be rigidly fixed to it. Now, wouldn't you like to select a frame of reference yourself ? See Fig. 1.1 and answer SAQ 1.

Fig. 1.1: A familiar athletic event, the 200m race in which the athlete A is about to overtake B.

SAQ 1

(a) Select the frames of reference to describe the motion of A, from the points of view of the observers B, A and S in Fig. 1.1.

(b) Which one of these observers B, A and S measures the velocity of A correctly?

Having specified the concepts of motion and the frame of reference, we would next like to describe how objects move in a given frame of reference. The language for describing motion is known, technically, as **kinematics.** The easiest to describe is the motion of a particle or a point. You may ask what a particle is. If in the study of a given phenomenon, the dimensions, shape and internal structures of an object are of no consequence, we can represent it by a point or call it a particle. For example, with respect to the distance between the earth and the sun, both these objects can usually be considered as particles. Let us now learn how to describe the motion of a particle.

1.3 THE LANGUAGE FOR DESCRIBING MOTION

The most important concepts used for describing motion are displacement, velocity and acceleration. In order to develop the language for describing motion, we need to explain and express these concepts. As you know, this is best done with the help of mathematics. Therefore, we will first develop some necessary mathematical tools, such as vectors by considering the example of displacement. We will then use these tools to understand other kinematical concepts, such as velocity and acceleration.

1.3.1 Vectors

We are all familiar with the meaning of the word 'vector'. Our basic motivation for using vectors is that it enables us to express physical concepts in compact and simple forms. We shall first learn the use of vectors in one and two dimensions and then extend the ideas to three dimensions.

But first, a word about the notation that we will adopt in our discussion of vectors. This will be as follows: In print, vectors will be distinguished from scalars through the use of boldface type; e.g. **A** is a vector. In written work we denote a vector **A** by putting a wavy line under the letter A (A̰) or an arrow over it ($\overrightarrow{A}$). The magnitude of **A** is printed as *A* or as |**A**|. In diagrams, vectors are shown by putting arrows on the lines representing them, as in Figs. 1.2a and 1.2b.

Vectors are quantities that (1) have both magnitude and direction independent of the choice of a coordinate system and (2) combine according to the following law of addition: **A** + **B** = **B** + **A** (Commutative law)

Vectors in one and two dimensions

Suppose you start cycling from a point *O* and travel upto a point *P* along a straight path (Fig. 1.2 a). What is your displacement relative to *O*? As you know, it is **OP** in the direction from *O* to *P*. Note that we have specified both the magnitude and direction of the displacement, i.e. it is a vector quantity. *O* is the **tail** and *P* the **head** of the vector **OP**.

Now, how do we get the length of the segment *OP*, i.e. the magnitude of **OP**? We can fix *O* as the origin, so that the coordinate of *P* gives the magnitude of **OP**. And if we want to express **OP** in terms of its magnitude we can use the concept of a unit vector. First let us give a name, say **A**, to the vector **OP**. We select a length *OL* of unit magnitude (Fig. 1.2b) so that *OP* is *A* times the length *OL*. Let **OL** denote a vector in the same direction as **OP**. We can then write

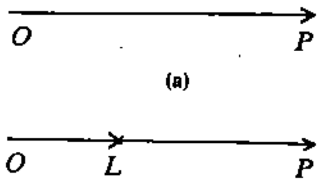


Fig. 1.2: (a) Displacement OP; (b) unit vector OL.

$$A\,(\mathbf{OL}) = \mathbf{OP} = \mathbf{A}. \tag{1.1}$$

OL is called a **unit vector.** By definition, *a unit vector has length equal to one unit.* The vector of unit length parallel to **A** is denoted by $\hat{\mathbf{A}}$ (pronounced A cap). By convention, it is chosen to be dimensionless. Thus, from Eq. 1.1 we have

$$\mathbf{OL} = \hat{\mathbf{A}} = \mathbf{A}\,/A, \tag{1.2a}$$

$$\text{and } \mathbf{A} = A\,\hat{\mathbf{A}}. \tag{1.2b}$$

Note that **A** and *A* have the same dimension, as $\hat{\mathbf{A}}$ is dimensionless.

We may also choose **OP** to be in the same direction as the *x*-axis. By convention, the unit vector along *x*-axis is represented by $\hat{\mathbf{i}}$. Then we can rewrite Eq. 1.2 b as

$$\mathbf{A} = A\hat{\mathbf{i}} = \hat{\mathbf{i}}A. \tag{1.2c}$$

Now suppose the path you are travelling along lies in a playground and you do not move in a straight line (Fig. 1.3a).

Can you specify your displacement **PQ** by one number along any one axis as for Fig. 1.2b? You could still choose that axis along **PQ**. But then using one coordinate, how would you specify a displacement **OP** in a different direction? Clearly, one axis along **PQ** alone will not be sufficient to specify displacements in other directions.

In such a case, we would need to know two lengths measured along the two axes *x*–*y*, perpendicular to each other and drawn from the origin *O* (Fig. 1.3 b). As you know, lengths x_1 and y_1 are equal to the coordinates of the point *P*. These coordinates completely specify the displacement **OP**. Now, how do we specify **PQ** or a vector in any other direction?

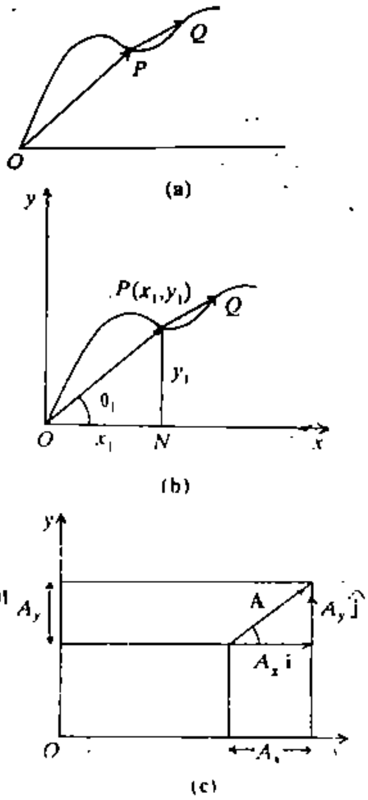


Fig. 1.3: (a) Motion in two dimensions; (b) $ON = x_1$, $NP = y_1$; (c) the x and y-components of A, which is a sum of vectors $A_x\hat{\mathbf{i}}$ and $A_y\hat{\mathbf{j}}$.

For this, let us consider a vector **A** in the *xy*-plane. The projections of **A** along the coordinate axes *x* and *y* are called the *x* and *y*-components of **A** (Fig. 1.3c). We may denote them by A_x and A_y, respectively. By convention, the unit vectors along *x* and *y*-axes are denoted by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, respectively. Then from the law of vector addition, **A** will be the sum of vectors $A_x\,\hat{\mathbf{i}}$ and $A_y\,\hat{\mathbf{j}}$.

$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}. \tag{1.3a}$$

Using Pythagoras' theorem, the magnitude of **A** is given by

$$A = |\,\mathbf{A}\,| = \sqrt{(A_x^2 + A_y^2)} \tag{1.3b}$$

and its direction, represented by the angle θ which **A** makes with the x-axis, is given by

$$\tan\theta = \frac{A_y}{A_x} \qquad (1.3c)$$

We can choose a set of axes inclined at any angle to each other. However, it is convenient to choose a set of axes perpendicular to each other because the mathematics is simplified.

In a given coordinate system **A** is uniquely expressed in terms of A_x and A_y. So if **A** is given, then A_x and A_y have unique values given by

$$A_x = A\cos\theta, \quad A_y = A\sin\theta. \qquad (1.3\ d))$$

Notice that when any displacement is referred to the origin, e.g. **OP** in Fig. 1.3 b, the coordinates of the point P are also the components of **OP**, i.e.

$$\mathbf{OP} = x_1\,\hat{\mathbf{i}} + y_1\,\hat{\mathbf{j}}, \qquad (1.4\ a)$$

$$\tan\theta_1 = \frac{y_1}{x_1} \qquad (1.4\ b)$$

Let us now recall briefly certain concepts related to vectors, which you already know from your school courses.

Equality of vectors

Two vectors **A** and **B** are equal iff their corresponding components are equal, i.e. given

'iff' stands for the phrase 'if and only if'.

$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} \text{ and } \mathbf{B} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}, \qquad (1.5)$$

$$\mathbf{A} = \mathbf{B} \text{ iff } A_x = B_x \text{ and } A_y = B_y. \qquad (1.6)$$

Multiplication of a vector by a scalar

If we multiply **A** by a positive scalar m, the result is a new vector $\mathbf{C} = m\mathbf{A}$. **C** is parallel to **A** and its length is m times greater (Fig. 1.4 a).

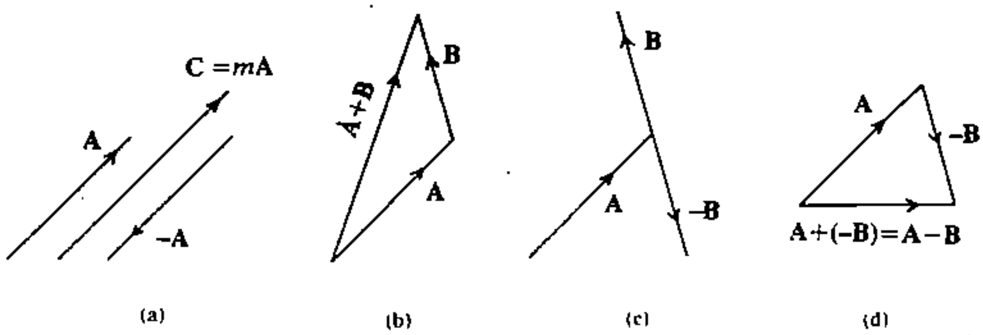


Fig. 1.4: (a) Multiplication of a vector by a scalar; (b) addition of vectors; (c) and (d) subtraction of vector **B** from **A**.

The result of multiplying a vector by –1 is a new vector equal in magnitude but opposite in direction or antiparallel to the original vector.

Addition and subtraction of vectors

The sum of two vectors can be obtained by the triangle law of addition (Fig. 1.4 b). Thus, to add **B** to **A**, place the tail of **B** at the head of **A**. The sum is a vector from the tail of **A** to the head of **B**.

Since $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, in order to subtract **B** from **A**, we can simply multiply it by –1 and then add as shown in Figs. 1.4 c and 1.4 d.

Associative and distributive laws

Vectors obey the associative laws of addition, and multiplication by scalars:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \qquad (1.7\ a).$$

$$m(n\mathbf{A}) = n(m\mathbf{A}) = mn\mathbf{A}. \qquad (1.7\ b)$$

They also obey the distributive law with respect to multiplication by scalar:

$$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}. \qquad (1.7\ c)$$

Let us express some of the above relations in terms of the vector components. These will be useful in solving problems. Given Eq. 1.5, we may write

$$\mathbf{C} = m\mathbf{A} = mA_x\,\hat{\mathbf{i}} + mA_y\,\hat{\mathbf{j}}, \tag{1.8 a}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} \tag{1.8 b}$$

We will now solve an example using the above concepts.

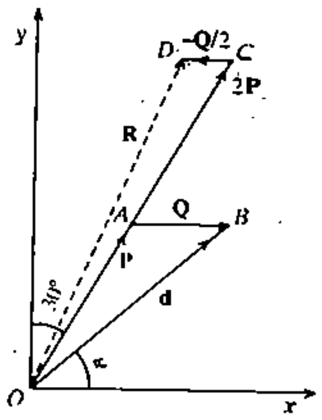


Fig. 1.5

Example 1

A ship travels a distance of 8 km from a point O along a direction 30° East of North upto A and then moves along the East for 4 km upto B. Let **OA = P, AB = Q.** Draw the resultant displacement vector **d** of the ship and find

i) the components of vectors **P** and **Q**. Express **P** and **Q** in terms of unit vectors.

ii) the components, magnitude and direction of **d**.

iii) **R** in terms of the unit vectors where $\mathbf{R} = 2\mathbf{P} - \frac{1}{2}\mathbf{Q}$ and draw **R**.

Let us draw x- and y-axes to represent the direction of East and North, respectively (Fig. 1.5). Then **P** is a vector of magnitude 8 km at an angle $\theta = 30°$ from the y-axis. **Q** is a vector of magnitude 4 km parallel to the x-axis. **OB** is the resultant displacement **d**.

i) The components of **P**:

Along x-axis $= OA \cos 60° = 8 \text{ km} \times \frac{1}{2} = 4 \text{ km}$, and .

along y-axis $= OA \sin 60° = 8 \text{ km} \times \frac{\sqrt{3}}{2} = 4\sqrt{3} \text{ km}$

Thus, $\mathbf{P} = \mathbf{OA} = (4\hat{\mathbf{i}} + 4\sqrt{3}\,\hat{\mathbf{j}})$ km

Components of **Q**:

Along x-axis $= 4 \text{ km} \times \cos 0° = 4 \text{ km} \times 1 = 4$ km, and

along y-axis $= 4 \text{ km} \times \sin 0° = 4 \text{ km} \times 0 = 0$ km

Thus $\mathbf{Q} = \mathbf{AB} = 4\,\hat{\mathbf{i}}$ km.

ii) $\mathbf{d} = \mathbf{P} + \mathbf{Q}$

$= 4\hat{\mathbf{i}} + 4\sqrt{3}\,\hat{\mathbf{j}} + 4\hat{\mathbf{i}} \text{ km} = [(4+4)\hat{\mathbf{i}} + 4\sqrt{3}\hat{\mathbf{j}}] \text{ km}$

$= (8\hat{\mathbf{i}} + 4\sqrt{3}\hat{\mathbf{j}})$ km

$\therefore d = \sqrt{64 + 48} \text{ km} = \sqrt{112} \text{ km} = 4\sqrt{7} \text{km}$

$\alpha = \tan^{-1}\left(\frac{4\sqrt{3}}{8}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

iii) $\mathbf{R} = 2\mathbf{P} - \frac{1}{2}\mathbf{Q}$

$= \{2\,(4\hat{\mathbf{i}} + 4\sqrt{3}\hat{\mathbf{j}}) - \frac{1}{2}\,(4\hat{\mathbf{i}})\}$ km.

Using the distributive and associative properties of vectors

$\mathbf{R} = (8\,\hat{\mathbf{i}} + 8\sqrt{3}\hat{\mathbf{j}} - 2\,\hat{\mathbf{i}})$ km

$= (6\,\hat{\mathbf{i}} + 8\sqrt{3}\,\hat{\mathbf{i}})$ km.

For drawing **R**, we extend OA to twice its length up to C, which gives the vector **2P**. Then we draw the vector $-\frac{1}{2}\mathbf{Q}$ at the head of **2P**. The resultant **R** can then be drawn from O to D.

You may now like to work out an SAQ based on what you have studied so far.

SAQ 2

The trajectory of a ball is shown in Fig. 1.6.

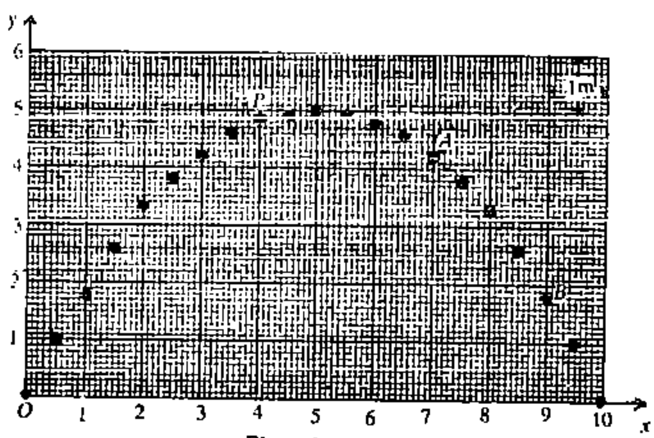


Fig. 1.6

a) Sketch the displacement **p** with respect to O and **q** with respect to A, given that i) $p_x = 4$ m, $p_y = 4.9$ m, ii) $q_x = -3$ m, $q_y = 0.7$ m.

b) Express **OA** (= **r**, say) in terms of the numerical values of components r_x and r_y.

c) Find $(q_x + r_x)$, $(q_y + r_y)$ and compare their values with p_x and p_y, respectively, and give a comment.

d) Measure the length OB with a ruler and the angle θ made by it with the positive direction of x-axis with a protractor.

Let **OB** = **s**. Now find s_x and s_y graphically and also using Eq. 1.3d. Compare these values.

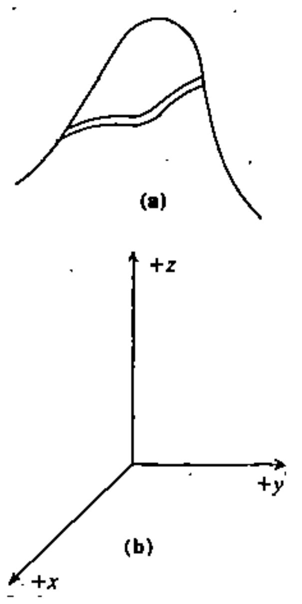


Fig. 1.7: (a) A path uphill; (b) three-dimensional Cartesian coordinate system.

So far we have quickly recalled the concepts of vectors in one and two dimensions with which you are familiar. The only new idea introduced here was to express vectors in terms of components and unit vectors. Let us now extend these concepts to three dimensions.

Vectors in three dimensions

Suppose we were travelling uphill along a path as shown in Fig. 1.7 a. Then we would need to know a third number to specify our displacement. In other words, we will need to use vectors in three dimensions. For this, we need a three-dimensional coordinate system. You are familiar with the three-dimensional Cartesian coordinate system shown in Fig. 1.7 b. It has three axes perpendicular to each other, passing through the point of reference O. The point O divides each line into halves, one of which is taken to be positive. Three unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ point along the positive x-, y- and z-axes, respectively.

Once the positive x- and y- axes have been chosen, a restriction is imposed on the choice of positive z-axis. Clearly there are two ways of choosing the positive z-direction as shown in Figs. 1.8 a and 1.8 b. By convention, the choice of $+z$-axis is made in the following manner: a 90° - rotation of $+x$-axis towards $+y$-axis appears anticlockwise when seen from any point on the positive z-axis. A coordinate system defined in this way is called a **right-handed** system, as shown in Fig. 1.8 a. Two alternative definitions of positive z-direction are shown in Figs. 1.9 a and 1.9 b. Read the captions carefully and try the following SAQ.

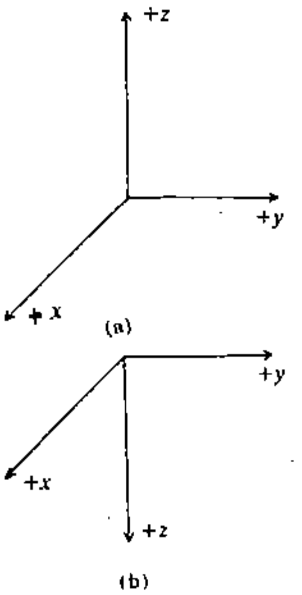


Fig. 1.8: The two ways (a) and (b) of choosing +z-axis.

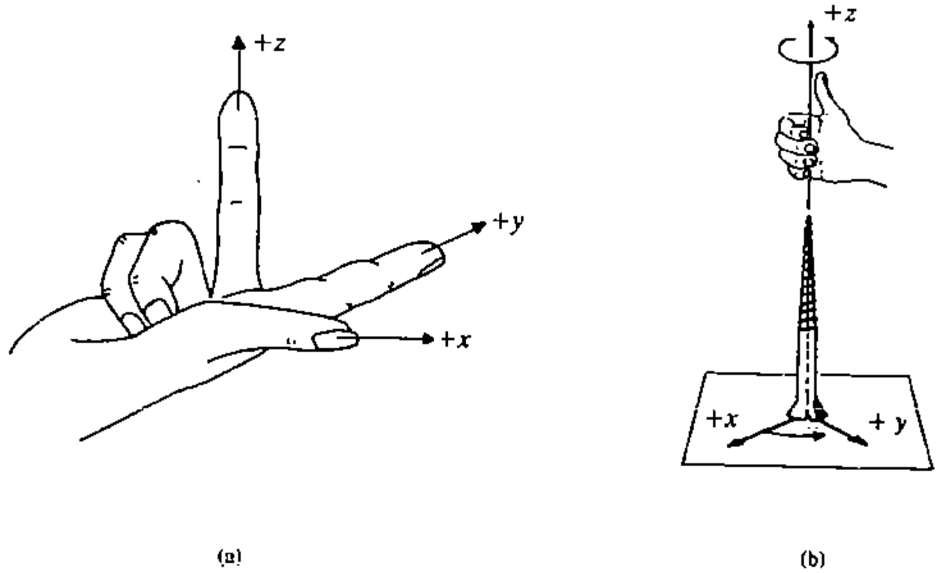
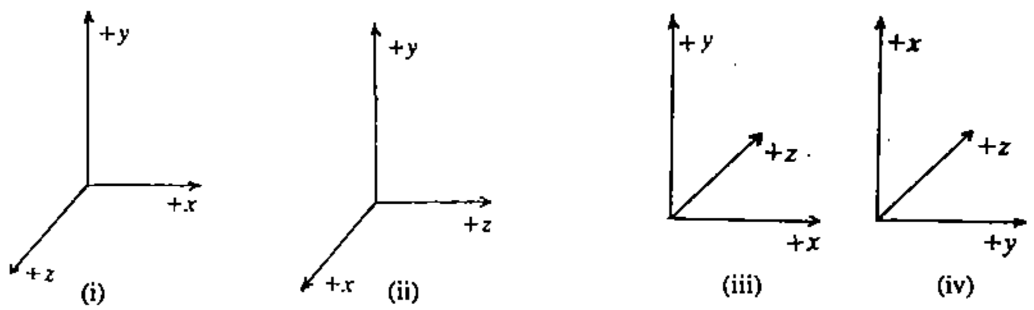


Fig. 1.9: (a) The right hand is extended so that the middle finger, first finger and thumb are placed perpendicular to each other, and the other two fingers closed. If the thumb and the first finger represent the $+x$ and $+y$-axes, respectively, then the middle finger represents the $+z$-axis; (b) curl the fingers of your right hand around the line perpendicular to the xy-plane. Let the orientation of the knuckles be such that the fingertips point in the direction of 90° rotation of x-axis towards y-axis. Then the thumb gives the direction of positive z-axis. It is also the direction in which the screwhead advances when a right-handed screw is rotated from $+x$-axis to $+y$-axis.

SAQ 3

Determine which of the following sets of perpendicular axes define right-handed systems?



Let us now use this system to specify the displacement **OP** with respect to the origin O (Fig. 1.10). To reach the point P we can walk a distance x along the x-axis, turn to our left, walk a distance y parallel to the y-axis and then travel a distance z parallel to the z-axis. If we know the distances x, y and z, we can always specify the position of the point P with respect to O by the three Cartesian coordinates (x, y, z). For example, the point (2,3,4) is shown in Fig. 1.10 a. Thus, the displacement $\widehat{\mathbf{OP}}$ which we can denote by **r** can be expressed as a sum of three vectors $x\hat{\mathbf{i}}$, $y\hat{\mathbf{j}}$ and $z\hat{\mathbf{k}}$:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1.9}$$

The displacement vector drawn from the origin O to the position of point **P** *is known as its position vector.* You can see that the components of the position vector of point P are the same as its coordinates.

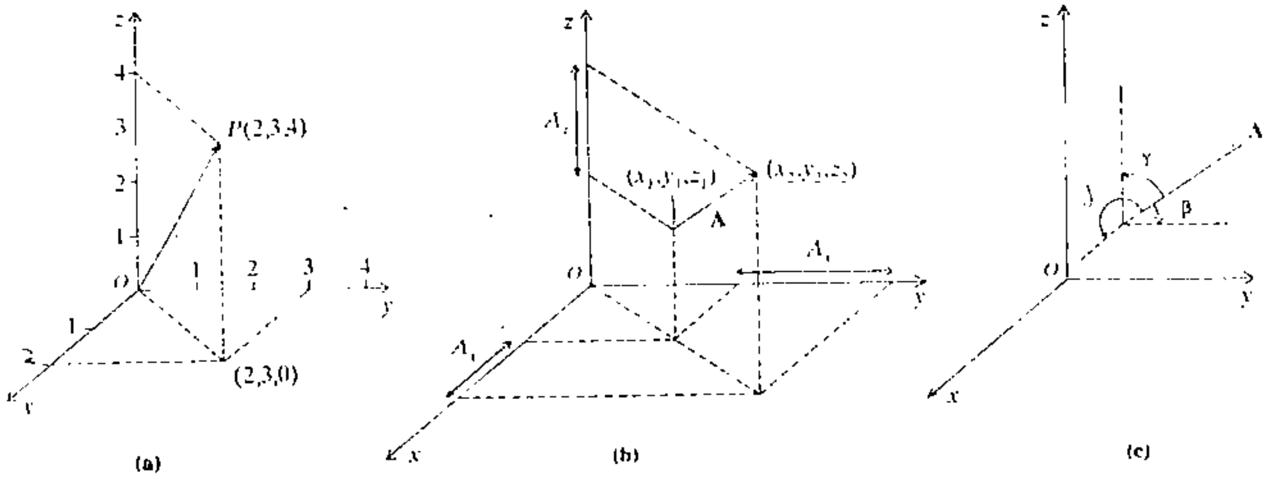


Fig. 1.10: (a) The position vector of a particle at P; (b) Cartesian components of a vector $\mathbf{A}$: $A_x = x_2 - x_1$, $A_y = y_2 - y_1$, $A_z = z_2 - z_1$; (c) angles α, β, γ.

We can now extend the above concept to any vector **A** (see Fig. 1.10 b). If A_x, A_y, A_z are the projections of **A** on the x, y and z-axes, respectively, then **A** can be expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}. \tag{1.10 a}$$

The magnitude of **A** is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.10 b}$$

The direction of **A** can be specified in terms of the angles between **A** and the three coordinate axes. These angles are labelled as α, β, γ. See Fig. 1.10 c. The dotted lines through the tail of **A** are parallel to the *x*, *y* and *z*-axes. In actual practice we use the cosines of these angles and they are called **direction cosines.**

$$l = \cos\alpha = \frac{A_x}{A},\ m = \cos\beta = \frac{A_y}{A},\ n = \cos\gamma = \frac{A_z}{A} \qquad (1.10\text{ c})$$

You can see that

$$l^2 + m^2 + n^2 = 1. \qquad (1.10\text{ d})$$

Example 2

Calculate the magnitude and direction cosines of $\mathbf{A} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

Here, $A_x = 2, A_y = 1, A_z = -2$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = 3$$

$$l = \cos\alpha = A_x/A = 2/3,\ m = A_y/A = \cos\beta = 1/3,\ n = \cos\gamma = A_z/A = -2/3$$

SAQ 4

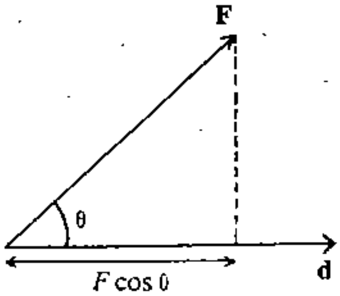
Note that position vector is a physical quantity and an appropriate unit of length should be attached to it. However, since we do not always write units with the coordinates of a point, we have omitted the same for position vectors.

a) Draw the position vector **r** of a point Q (2,4,4) in Fig. 1.10a. Express it in terms of unit vectors and calculate its magnitude and direction cosines.

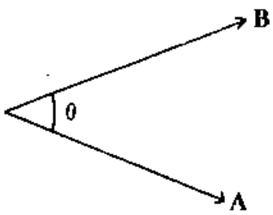
b) $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the points R and S, given by $\mathbf{r}_1 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{r}_2 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.
Indicate the points R and S on Fig. 1.10 a. Draw the vector **d** (=**RS**). Write down its components.

So far you have learnt to express vectors in terms of their components in one, two and three dimensions. Though we have taken only the example of displacement to explain these concepts, they can be extended to any other kind of vector, such as velocity, acceleration, force etc. We have also learnt how to add vectors and multiply them by scalars. The next question is how do we multiply two vectors? Is the product a vector, a scalar or some other quantity. The choice is up to us, and we shall define two types of products which are useful in applications to physics.

1.3.2 Products of Vectors



(a)



(b)

Fig. 1.11: The scalar product of **A** and **B**.

Let us apply a force **F** at an angle θ to the direction in which the displacement **d** takes place (Fig. 1.11 a). You may already know that the work done by force **F** on an object is the magnitude of the displacement **d** of the object multiplied by the component of **F** along the direction of **d**, i.e.

$$W = (F\cos\theta)d. \qquad (1.11\text{a})$$

As you know, work is a scalar quantity. We also express work as a product of the two vectors **F** and **d** as:

$$W = \mathbf{F}.\ \mathbf{d}. \qquad (1.11\text{b})$$

This type of product of two vectors **A** and **B** which yields a scalar quantity is called the **scalar product**. It is denoted by **A . B** (pronounced A dot B) and is also called the dot product. The **scalar product** is defined as

$$\mathbf{A}.\ \mathbf{B} = AB\cos\theta, \qquad (1.12)$$

where θ is the angle between **A** and **B** when they are drawn tail to tail (Fig. 1.11 b). It is conventional to take θ as the angle smaller than or equal to π.

Example 3

Find the scalar product of a unit coordinate vector with itself.

$$\hat{\mathbf{i}}.\ \hat{\mathbf{i}} = |\hat{\mathbf{i}}|\ |\hat{\mathbf{i}}| \cos 0^\circ = 1$$

Similarly $\hat{\mathbf{j}}.\hat{\mathbf{j}} = 1$,

and $\hat{\mathbf{k}}\ .\ \hat{\mathbf{k}} = 1. \qquad (1.13\text{ a})$

SAQ 5

a) In the given dotted spaces, write the scalar product of two different unit coordinate vectors:

(i) $\hat{\mathbf{i}}.\hat{\mathbf{j}}$ = ...

(ii) $\hat{\mathbf{j}}.\hat{\mathbf{k}}$ = ...

(iii) $\hat{\mathbf{k}}.\hat{\mathbf{i}}$ = ... (1.13 b)

b) Fill in the dotted spaces given a vector $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$:

(i) $\mathbf{A}.\hat{\mathbf{i}}$ = ...

(ii) $\mathbf{A}.\hat{\mathbf{j}}$ = ...

(iii) $\mathbf{A}.\hat{\mathbf{k}}$ = ... (1.13 c)

In your calculation of SAQ 5(b), did you notice that the *x, y, z* components of a vector can be expressed as its scalar products with the respective unit coordinate vectors?

We can also express the scalar product in terms of the components of the vectors.

$$\text{Let } \mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}},$$
$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}.$$

Using the distributive property of scalar product of vectors, given as

$$\mathbf{P}.(\mathbf{Q} + \mathbf{R}) = \mathbf{P}.\mathbf{Q} + \mathbf{P}.\mathbf{R}, \quad (1.14)$$

we get

$$\begin{aligned}\mathbf{A}.\mathbf{B} &= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}).(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})\\ &= A_xB_x\hat{\mathbf{i}}.\hat{\mathbf{i}} + A_xB_y\hat{\mathbf{i}}.\hat{\mathbf{j}} + A_xB_z\hat{\mathbf{i}}.\hat{\mathbf{k}}\\ &+ A_yB_x\hat{\mathbf{j}}.\hat{\mathbf{i}} + A_yB_y\hat{\mathbf{j}}.\hat{\mathbf{j}} + A_yB_z\hat{\mathbf{j}}.\hat{\mathbf{k}}\\ &+ A_zB_x\hat{\mathbf{k}}.\hat{\mathbf{i}} + A_zB_y\hat{\mathbf{k}}.\hat{\mathbf{j}} + A_zB_z\hat{\mathbf{k}}.\hat{\mathbf{k}}.\end{aligned}$$

Using Eqs. 1.13 a and b we get,

$$\mathbf{A}.\mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \quad (1.15\text{ a})$$

Thus, the scalar product of two vectors is the sum of the product of their components along each of the coordinate axes. Since, the components are scalars, we get

$$\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A} \quad (1.15\text{ b})$$

You can quickly try another simple SAQ on scalar product.

SAQ 6

Show that $\mathbf{A}.\mathbf{A} = A^2$ (1.15 c)

Apart from the scalar product, there is another product of two vectors which is defined to be a vector, whose direction is perpendicular to the two given vectors. For example, if a force **F** is applied at a point *A* (Fig. 1.12 a), the couple $\boldsymbol{\tau}$ exerted by the force about *O* has a magnitude $(F \sin\theta)r$ and acts about an axis perpendicular to the plane containing **F** and **r**. As you know, torque is a vector quantity. We also express torque as the product of **r** and **F**:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \quad (1.16)$$

Such a product of two vectors A and **B** which yields a vector is called the **vector product** or the **cross product**. It is denoted by **A × B** (pronounced A cross B). It is defined to be the vector

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\theta\,\hat{\mathbf{C}}, \quad (1.17)$$

where θ is the angle between **A** and **B** when they are drawn tail to tail (Fig. 1.12 b). By convention, θ is always taken as the angle smaller than or equal to π. The magnitude of **C** is given by $AB\sin\theta$ and its direction is defined by $\hat{\mathbf{C}}$, which is a unit vector whose direction is perpendicular to **A** and **B**. The sense of $\hat{\mathbf{C}}$ is determined as a matter of convention by the right-hand-rule: First, place together the tails of vectors **A** and **B**; this defines a plane. Rotate **A** into **B** through the lesser of the two angles and curl the fingers of the right-hand in the direction in which **A** is rotated. The thumb will point in the direction of **C = A × B**.

Thus, **B × A** is a vector opposite to **A × B** (Fig. 1.12 d), i.e.

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}, \quad (1.18)$$

We also see that if **A = B**, then

$$\mathbf{C} = \mathbf{A} \times \mathbf{A} = |\mathbf{A}|^2 \sin(0^\circ)\,\hat{\mathbf{C}} = \mathbf{0}.$$

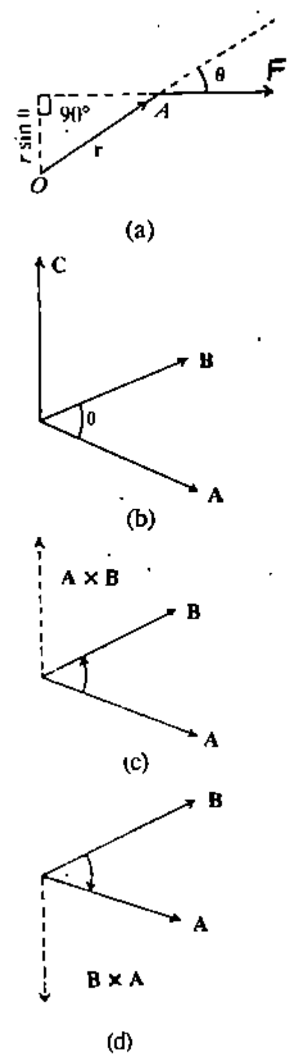


Fig. 1.12: (a) Couple exerted by the force about the origin *O* is $\tau = rF\sin\theta$; (b) **C = A × B**; (c) the direction of **A × B** is determined by the right-hand-rule; (d) **B × A** is a vector equal and opposite to **A × B**.

Thus, the vector product of any vector with itself is a **zero vector**, i.e. a vector having zero magnitude.

Example 4

Find the vector product of two different unit coordinate vectors.

$$|\hat{\mathbf{i}} \times \hat{\mathbf{j}}| = 1.1 \sin 90^{\circ} = 1$$

The direction is given by the right-hand rule, so that

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}.$$

Similarly, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$,

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}. \qquad (1.19\text{ a})$$

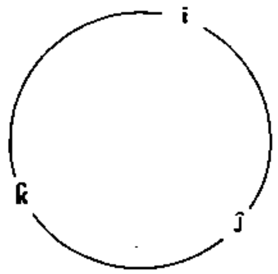


Fig. 1.13 : If we go around the circle clockwise, the cross products are positive; If we move anticlockwise the cross products are negative.

SAQ 7

Using the right-hand rule and the definition of a cross product, fill in the blank spaces below:

i) $\hat{\mathbf{j}} \times \hat{\mathbf{i}} =$ ....

ii) $\hat{\mathbf{k}} \times \hat{\mathbf{i}} =$ ....

iii) $\hat{\mathbf{i}} \times \hat{\mathbf{k}} =$ ....

iv) $\hat{\mathbf{i}} \times \hat{\mathbf{i}} =$ .... (1.19b)

v) $\hat{\mathbf{j}} \times \hat{\mathbf{j}} =$ ....

vi) $\hat{\mathbf{k}} \times \hat{\mathbf{k}} =$ ....

You can see that there is a cyclic pattern in the products $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}}$, which is shown in Fig. 1.13. For example, *if we go around the circle clockwise, the cross products are positive:* $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$.

If we go in an anticlockwise direction, the cross products are negative. $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$.

Like the scalar product the vector product is distributive over vector addition, i.e.

$$\mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R}. \qquad (1.20)$$

Therefore, for any two vectors

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \text{ and } \mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}},$$

we can write

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \\ &= A_xB_x\,(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_xB_y\,(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + (A_xB_z)\,(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \\ &\quad + A_yB_x\,(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_yB_y\,(\hat{\mathbf{j}} \times \hat{\mathbf{j}}) + A_yB_z\,(\hat{\mathbf{j}} \times \hat{\mathbf{k}}) \\ &\quad + A_zB_x\,(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) + A_zB_y\,(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) + A_zB_z\,(\hat{\mathbf{k}} \times \hat{\mathbf{k}}).\end{aligned}$$

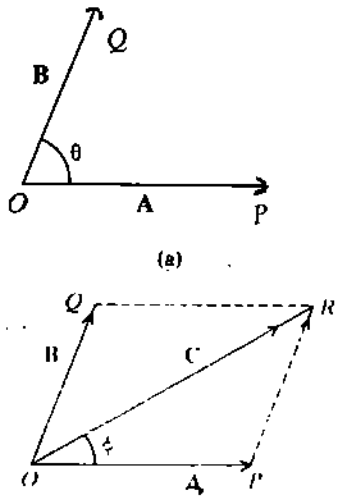


Fig. 1.14: Parallelogram law of vector addition: The resultant of two vectors A and B having a common tail is given by the diagonal through the common tail O of the parallelogram having A and B as adjacent sides.

Using Eq. 1.19, we get

$$\mathbf{A} \times \mathbf{B} = A_xB_y\,\hat{\mathbf{k}} - A_xB_z\,\hat{\mathbf{j}} - A_yB_x\,\hat{\mathbf{k}} + A_yB_z\,\hat{\mathbf{i}} + A_zB_x\hat{\mathbf{j}} - A_zB_y\hat{\mathbf{i}},$$

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{i}}(A_yB_z - A_zB_y) + \hat{\mathbf{j}}\,(A_zB_x - A_xB_z) + \hat{\mathbf{k}}\,(A_xB_y - A_yB_x). \qquad (1.21\text{ a})$$

An alternative way of expressing $\mathbf{A} \times \mathbf{B}$ is in the form of a determinant. Since you already know about determinants you can verify that

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (1.21\text{ b})$$

This is equivalent to Eq. 1.21 a and it is easier to remember.

We can use the scalar and vector products of two vectors **A** and **B** to determine the magnitude and direction of their resultant **A** + **B**. Let **A** and **B** have a common tail as shown in Fig. 1.14a. Can we use the triangle law of addition? Yes, we can translate the vector **B** from the position OQ to PR as shown in Fig. 1.14b. Then **OR** = C is the resultant. Incidentally, OR is the diagonal of the parallelogram $OPRQ$ through the point O. This is the **parallelogram law of vector addition.**

The magnitude of **C** is obtained from the scalar product of **C** with itself as follows:

$$C^2 = \mathbf{C}.\mathbf{C} = (\mathbf{A} + \mathbf{B}).(\mathbf{A} + \mathbf{B})$$
$$= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B}$$
$$= A^2 + 2\mathbf{A}\cdot\mathbf{B} + B^2 \quad (\because \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A})$$

or $C^2 = A^2 + B^2 + 2AB\cos\theta$, where θ is the angle between **A** and **B**.

or $C = \sqrt{A^2 + B^2 + 2AB\cos\theta}$. (1.22 a)

The vector and scalar products of **A** with **C** give the angle ϕ which **C** makes with **A**, i.e. the direction of **C**.

$$\mathbf{A}\times\mathbf{C} = \mathbf{A}\times(\mathbf{A}+\mathbf{B}) = \mathbf{A}\times\mathbf{B}, \quad (\because \mathbf{A}\times\mathbf{A} = 0)$$

or $|\mathbf{A}\times\mathbf{C}| = |\mathbf{A}\times\mathbf{B}|$,

or $AC\sin\phi = AB\sin\theta$

i.e. $C\sin\phi = B\sin\theta$. $(\because A \neq 0)$ (1.22 b)

$$\mathbf{A}\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{B},$$

or $AC\cos\phi = A^2 + AB\cos\theta$,

$\therefore C\cos\phi = A + B\cos\theta$. (1.22 c)

From Eqs. 1.22 b and 1.22 c, we get

$$\tan\phi = \frac{B\sin\theta}{A + B\cos\theta} \qquad (1.22\text{ d})$$

Let us now consider a numerical example on scalar and vector products.

Example 5

Given two vectors $\mathbf{A} = 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$ and $\mathbf{B} = 2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, find **A.B**, **A** × **B** and the angle between **A** and **B**.

Here $A_x = 4, A_y = 4, A_z = -7$; $B_x = 2, B_y = 6, B_z = 3$.

i) $\mathbf{A}.\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$
$= 4.2 + 4.6 + (-7)3$
$= 8 + 24 - 21 = 11.$

ii) $\mathbf{A}\times\mathbf{B} = \hat{\mathbf{i}}(A_yB_z - A_zB_y) + \hat{\mathbf{j}}(A_zB_x - A_xB_z) + \hat{\mathbf{k}}(A_xB_y - A_yB_x)$
$= \hat{\mathbf{i}}(4.3 - 6.(-7)) + \hat{\mathbf{j}}(-7.2 - 4.3) + \hat{\mathbf{k}}(4.6 - 4.2)$
$= 54\hat{\mathbf{i}} - 26\hat{\mathbf{j}} + 16\hat{\mathbf{k}}.$

iii) We know that $\mathbf{A}.\mathbf{B} = AB\cos\theta$

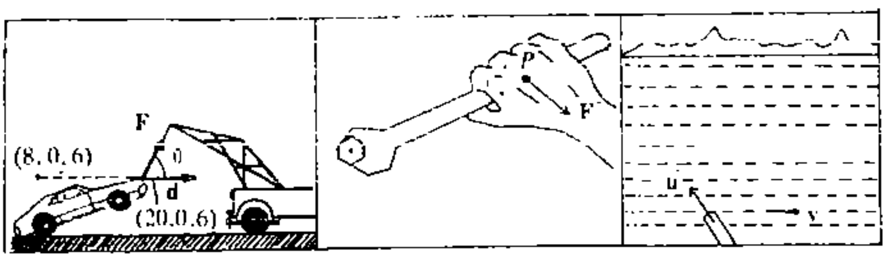
$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = 9, \quad B = \sqrt{B_x^2 + B_y^2 + B_z^2} = 7$

$$\cos\theta = \frac{\mathbf{A}.\mathbf{B}}{AB} = \frac{11}{9\times7} = \frac{11}{63}$$

or $\theta = \cos^{-1}\frac{11}{63}$.

You can now try a little lengthy SAQ. Parts (a) and (b) are on scalar and vector products. Part (c) is based on Eqs. 1.22a and 1.22d.

SAQ 8



(a) (b) (c)

Fig. 1.15

a) A constant force $\mathbf{F} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ acts on a particle (Fig. 1.15 a) resulting in its displacement from point (8, 0, 6) to point (20, 0, 6). Find the work done. Consider the force to be in newton and the displacement in metres:

b) A force $\mathbf{F} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ is applied at the point P (1, −1, 1) (Fig. 1.15 b). Find the torque $\boldsymbol{\tau}$ about the origin. The force is in newton and the displacement is in metres.

c) A man wishes to cross a river along the shortest possible path in a speed boat. In still water, the boat's speed is u. The speed of the river is v ($<u$) (Fig. 1.15c). Show that the boat's resultant speed will be $\sqrt{u^2-v^2}$ and find the direction in which the boat should be steered. Assume that the banks are parallel.

So far we have gone through the fundamentals of vector operations. We shall now use vectors to discuss kinematical quantities like displacement, velocity and acceleration.

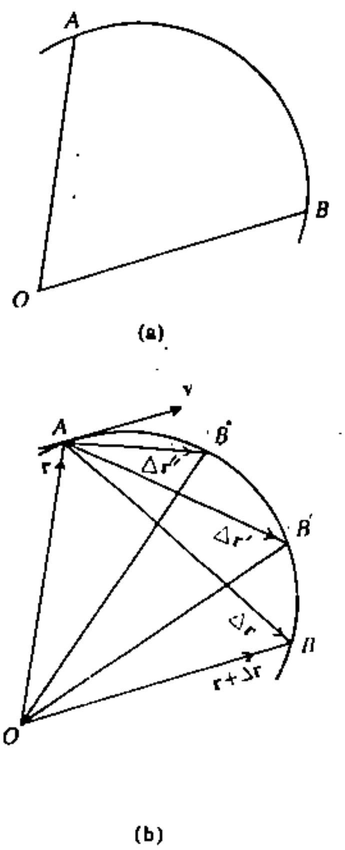


Fig. 1.16

1.3.3 Displacement, Velocity and Acceleration

Let us consider a particle's motion in space (Fig. 1.16a). Let it be at the position A at the instant of time t and at B at the instant of time $t + \Delta t$. As discussed in Sec. 1.3.1, the position of a particle in a particular frame of reference is given by a position vector drawn from the origin of the coordinate system in that frame to the position of the particle. Let the position vectors of A and B with respect to O be $\mathbf{r}$ and $\mathbf{r} + \Delta\mathbf{r}$, respectively. The **displacement** of the particle in the time Δt is equal to $\Delta\mathbf{r}$ in the direction AB.

The average velocity of the particle during the time Δt is given by

$$\mathbf{v}_{av} = \frac{\Delta\mathbf{r}}{\Delta t}. \tag{1.23}$$

Since Δt is a scalar quantity the direction of $\mathbf{v}_{av}$ is the same as that of $\Delta\mathbf{r}$. $\mathbf{v}_{av}$ is the velocity at which the particle would have travelled the distance AB in uniform and rectilinear motion during the interval of time Δt.

SAQ 9

The displacement vs. time equation of a particle falling freely from rest is given by

$$x = (4.9\,\text{ms}^{-2})t^2$$

where x is in metres, t is in seconds. Calculate the average velocity of the particle between $t_1 = 1$ s and $t_2 = 2$ s and also between $t_3 = 3$ s and $t_4 = 4$ s.

On solving SAQ 9, you have found that the values of average velocities during the two time intervals are not the same. Such a motion is called **non-uniform** motion. When a bus leaves a bus-stop and travels up to the next one, it executes a non-uniform motion. In such a case, we may also like to know the velocity of the particle at any given instant of time.

The velocity of a particle may vary by way of change in magnitude, change in direction, or both. In Fig. 1.16b, the average velocity during the interval Δt is directed along the chord AB but the motion has taken place along the arc. The average velocities during the intervals $\Delta t'$ (A to B') and $\Delta t''$ (A to B'') are different both in magnitude and direction. The time interval $\Delta t''$ is smaller than $\Delta t'$, which is smaller than Δt. As we decrease the interval of time, the point B approaches point A, i.e. the chord approximates the actual motion of the particle better. These points finally merge and the direction of $\Delta\mathbf{r}$ coincides with the tangent to the curve at the point of merger.

As Δt decreases, the ratio $\frac{\Delta\mathbf{r}}{\Delta t}$ approaches a limit. The vector $\mathbf{v}$, having the magnitude equal to the limit of the ratio $\frac{\Delta\mathbf{r}}{\Delta t}$ as $\Delta t \to 0$, is called the **instantaneous velocity** of the particle at time t. It is in the direction of the tangent to the curve at the given moment of motion. Thus,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

In other words, the instantaneous velocity is the derivative of $\mathbf{r}$ with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{1, 24 a}$$

It follows from Eq. 1.24a that if $\mathbf{r}$ has components x, y, z, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\,(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= x\frac{d\hat{\mathbf{i}}}{dt} + \hat{\mathbf{i}}\frac{dx}{dt} + y\frac{d\hat{\mathbf{j}}}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + z\frac{d\hat{\mathbf{k}}}{dt} + \hat{\mathbf{k}}\frac{dz}{dt}$$

$$= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dx}{dt}\hat{\mathbf{k}},$$ since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are independent of time.

$$= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}},$$ where

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt} \qquad (1.24\text{ b})$$

As $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are constant in magnitude and direction, their time derivatives are zero.

If we were using coordinates only to write the equations for velocity, we would have to write three equations, as in Eq. 1.24b. The use of vectors enables us to write a single equation 1.24 a.

Let us now represent the instantaneous velocities of the particle in passing through the points A and B of its path (Fig. 1.17). We can see that the velocity at B is different from that at A, i.e. velocity is changing in magnitude and direction. Thus, the particle experiences an acceleration. Just as we defined average and instantaneous velocity, we will now define average and instantaneous acceleration.

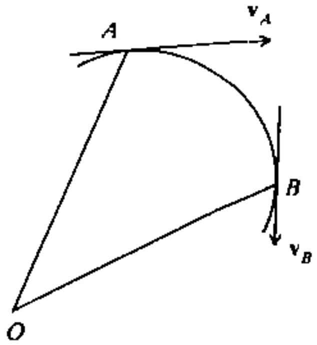


Fig. 1.17

If the velocity of the particle changes from $\mathbf{v}$ to $\mathbf{v} + \Delta\mathbf{v}$ within the time interval from t to $t + \Delta t$, then the **average acceleration** $\mathbf{a}_{av}$ during this interval of time is given by

$$\mathbf{a}_{av} = \frac{\Delta\mathbf{v}}{\Delta t} \qquad (1.25)$$

Once again as Δt is a scalar quantity the direction of $\mathbf{a}_{av}$ is along $\Delta\mathbf{v}$. When the interval of time Δt decreases, the ratio $\frac{\Delta\mathbf{v}}{\Delta t}$ approaches a limit. We define **instantaneous acceleration** of a particle at a given instant of motion as,

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \qquad (1.26\text{ a})$$

So, acceleration is the derivative of $\mathbf{v}$ with respect to time, i.e.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}, \text{ and}$$

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}, \qquad a_z = \frac{dv_z}{dt} \qquad (1.26\text{ b})$$

Example 6

A wire helix of radius R is oriented vertically along the z-axis. A frictionless bead slides down along the wire (Fig. 1.18). Its position vector varies with time as

$\mathbf{r}(t) = (R\cos bt^2)\,\hat{\mathbf{i}} + (R\sin bt^2)\,\hat{\mathbf{j}} - \frac{1}{2}ct^2\,\hat{\mathbf{k}}$, where b and c are constants.

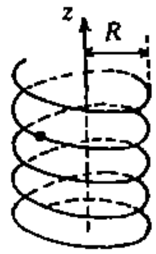


Fig. 1.18

Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$, where $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are the velocity and acceleration expressed as functions of t.

Here $x = R\cos bt^2$, $y = R\sin bt^2$, $z = -\frac{1}{2}ct^2$

We know that

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$$= (-2tbR\sin bt^2)\,\hat{\mathbf{i}} + (2tbR\cos bt^2)\,\hat{\mathbf{j}} - (ct)\hat{\mathbf{k}}$$

and $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (-4t^2b^2R\cos bt^2 - 2Rb\sin bt^2)\;\hat{\mathbf{i}} + (-4t^2Rb^2\sin bt^2 + 2Rb\cos bt^2)\;\hat{\mathbf{j}} - c\hat{\mathbf{k}}$

SAQ 10

A particle moves along the curve $y = Ax^2$ such that $x = Bt$, A and B are constants.

a) Express the position vector of the particle in the form $\mathbf{r}(t) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

b) Calculate the speed $[v = |\frac{d\mathbf{r}}{dt}|]$ of the particle along this path at any instant t.

Let us apply the concepts that we have developed so far to the case of uniform circular motion which plays an important role in physics. Uniform circular motion provides a good approximation to many diverse phenomena, such as artificial satellites in circular orbits, designing of roads, motion of electrons in a magnetic field, etc.

1.4 UNIFORM CIRCULAR MOTION

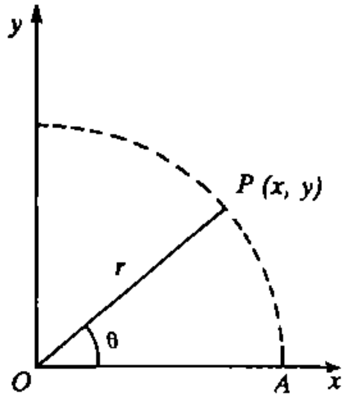


Fig. 1.19: Uniform circular motion.

Let us consider a particle which moves so that it always maintains a constant distance r from a certain point O and turns through a constant angle in a fixed time. A portion of its trajectory is shown dotted in Fig. 1.19. A on x-axis is the position of the particle at time $t = 0$. t seconds later it is at P after describing an angle θ (= $\angle AOP$). Through O we draw y-axis perpendicular to x-axis. Let the coordinates of P with respect to the mutually perpendicular x and y-axes be (x, y). From trigonometry, we know that:

$$x = r\cos\theta,$$
$$y = r\sin\theta. \qquad (1.27\text{ a})$$

If the angle described per second by the particle be a constant equal to ω radians, then $\theta = \omega t$ and Eq. 1.27 a can be written as

$$x = r\cos\omega t,$$
$$y = r\sin\omega t. \qquad (1.27\text{ b})$$

ω is also known as the **angular speed** of the particle.

The position vector of the particle at P is given by $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, (1.28)

$$\text{or } \mathbf{r} = r\cos\omega t\,\hat{\mathbf{i}} + r\sin\omega t\,\hat{\mathbf{j}},$$

$$\text{and } \mathbf{v} = \frac{d\mathbf{r}}{dt} = -r\,\omega\sin\omega t\,\hat{\mathbf{i}} + r\,\omega\cos\omega t\,\hat{\mathbf{j}}, \qquad (1.29\text{ a})$$
$$= v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}},$$

$$\text{where } v_x = -r\,\omega\sin\omega t,\ v_y = r\,\omega\cos\omega t. \qquad (1.29\text{ b})$$

The magnitude of velocity is, therefore,

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{r^2\,\omega^2\,(\sin^2\omega t + \cos^2\omega t)} = \sqrt{r^2\,\omega^2} = r\omega. \qquad (1.29\text{ c})$$

Now, what about its direction? For that, let us calculate $\mathbf{v}.\,\mathbf{r}$.

$$\mathbf{v}.\,\mathbf{r} = (-r\,\omega\sin\omega t\,\hat{\mathbf{i}} + r\,\omega\cos\omega t\,\hat{\mathbf{j}})\,.\,(r\cos\omega t\,\hat{\mathbf{i}} + r\sin\omega t\,\hat{\mathbf{j}})$$
$$= -r^2\,\omega\sin\omega t\cos\omega t + r^2\omega\cos\omega t\sin\omega t = 0$$

Since $\mathbf{v}.\mathbf{r} = 0$, $\mathbf{v}$ is always perpendicular to $\mathbf{r}$. Hence, $\mathbf{v}$ is always along the tangent to the circular path. From Eq. 1.29c we understand that $\mathbf{v}$ has a constant magnitude. But it constantly changes direction as it is along the tangent at any point. So the velocity vector is not constant, i.e. the particle has an acceleration, which we denote by $\mathbf{a}_R$. Since,

acceleration $= \frac{d\mathbf{v}}{dt}$, we have from Eq. 1.29 b,

$$\mathbf{a}_R = -r\,\omega^2\cos\omega t\,\hat{\mathbf{i}} - r\,\omega^2\sin\omega t\,\hat{\mathbf{j}}$$
$$= -\omega^2\,(r\cos\omega t\,\hat{\mathbf{i}} + r\sin\omega t\,\hat{\mathbf{j}}), \qquad (1.30\text{ a})$$

$$\therefore\ \mathbf{a}_R = -\omega^2\,\mathbf{r}. \qquad (1.30\text{ b})$$

Since $v = \omega r$ from Eq. 1.29c, we get

$$a_R = \omega^2 r = \frac{v^2}{r^2}\,r = \frac{v^2}{r} \qquad (1.30\text{ c})$$

Note that the negative sign on the right hand side of Eq. 1.30b indicates that the acceleration is opposite to $\mathbf{r}$, i.e. towards the centre. So a particle moving with uniform angular speed in a circle, experiences an acceleration directed towards the centre. It is known as the **centripetal acceleration**.

Example 7 : Satellite in a circular equatorial orbit

Let us calculate the period of revolution of a satellite moving around the earth in a circular equatorial orbit (Fig. 1.20). Let the velocity of the satellite in the orbit be $\mathbf{v}$, and the radius of the orbit be r. Like any free object near the earth's surface the satellite has an acceleration towards the centre of earth (= g', say), which is the centripetal acceleration. It is this acceleration that causes it to follow the circular path. Hence from Eq. 1.30c, we have

$$g' = \frac{v^2}{r},$$
$$\text{or } v^2 = g'r. \quad (1.31)$$

If the angular speed of the satellite is ω, we get from Eq. 1.29c,

$$\omega^2 r^2 = g'r$$
$$\text{or } \omega^2 = \frac{g'}{r}. \quad (1.32)$$

Again the time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g'}}$

or $$T = 2\pi\sqrt{\frac{R+h}{g'}} \quad (1.33)$$

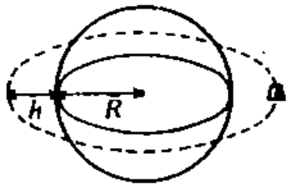


Fig. 1.20

where R = the radius of the earth, and

h = the height of the satellite above the surface of the earth.

The orbit of the first artificial earth satellite Sputnik, was almost circular at a mean height of 1.7×10^5 m above the surface of the earth, where the value of acceleration due to gravity is $9.26\ \text{ms}^{-2}$. Thus, the time the satellite took to complete one orbit around the earth was,

$$T = 2\pi\sqrt{\frac{(6.37 \times 10^6 + 0.17 \times 10^6)\text{m}}{(9.26)\text{ms}^{-2}}} = 5.28 \times 10^3\ \text{s} = 1\ \text{h}\ 28\ \text{min}$$

SAQ 11

A flat horizontal road is being designed for a 60 km h^{-1} speed limit. If the maximum acceleration of a car travelling on this road is to be 1.5 m s^{-2} at the above speed limit, what must be the minimum radius of curvature for curves in the road?

You are now familiar with the language for describing motion. You have learnt about displacement, velocity and acceleration using vectors.

We have already pointed out that the position, velocity and acceleration of a particle can only be defined with respect to some reference frame. When we are travelling in the same bus, both of us are at rest with respect to each other. But with respect to someone standing on the road both of us are in motion. Again, our velocity measured by the person on the road and the one measured by a cyclist on the road will be different. Similarly, when we say that a car is moving at 60 km h^{-1}, usually it means 60 km h^{-1} relative to the earth. But the earth is moving at 30 km s^{-1} relative to the sun. Thus, the car's speed relative to the sun is much greater than 60 km h^{-1}. These examples show that all motion is relative. Therefore, often we have to determine the relative position, velocity and acceleration of a particle with respect to another particle. Let us see how to find out these quantities.

1.5 RELATIVE MOTION

Let $\mathbf{r}_P$ and $\mathbf{r}_Q$ be the position vectors of particles P and Q, respectively, at any instant of time, with respect to a fixed origin O (Fig. 1.21). $\mathbf{r}_{QP}$ is the position of P with respect to Q. We know from the law of addition of vectors, that

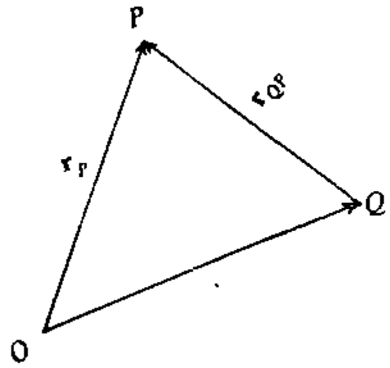


Fig. 1.21 : $\mathbf{r}_{QP} = \mathbf{r}_P - \mathbf{r}_Q$

$$\mathbf{r}_Q + \mathbf{r}_{QP} = \mathbf{r}_P,$$

or $$\mathbf{r}_{QP} = \mathbf{r}_P - \mathbf{r}_Q. \quad (1.34)$$

The relative velocity $\mathbf{v}_{QP}$ of P with respect to Q is obtained by differentiating $\mathbf{r}_{QP}$ with respect to time. Thus,

$$\mathbf{v}_{QP} = \frac{d}{dt}(\mathbf{r}_{QP}) = \frac{d\mathbf{r}_P}{dt} - \frac{d\mathbf{r}_Q}{dt},$$

or $$\mathbf{v}_{QP} = \mathbf{v}_P - \mathbf{v}_Q \quad (1.35)$$

Similarly, relative acceleration $\mathbf{a}_{QP}$ of P with respect to Q is given by,

$$\mathbf{a}_{QP} = \frac{d}{dt}(\mathbf{v}_{QP}) = \frac{d\mathbf{v}_P}{dt} - \frac{d\mathbf{v}_Q}{dt},$$

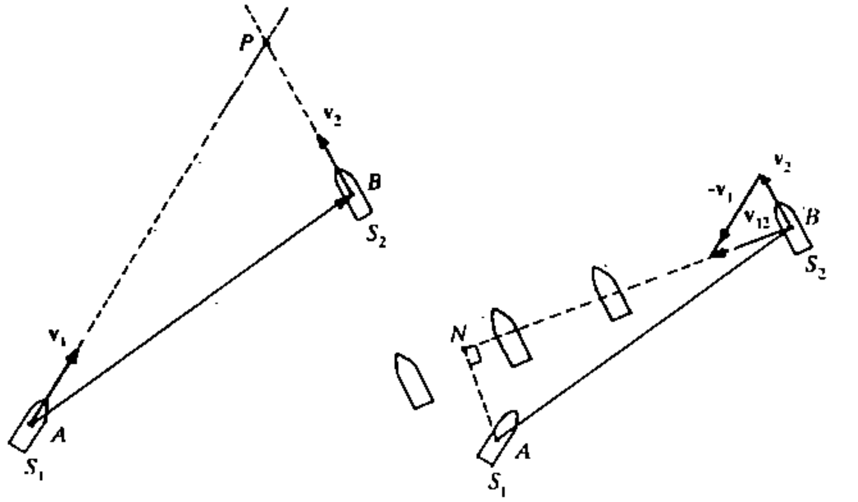
or $$\mathbf{a}_{QP} = \mathbf{a}_P - \mathbf{a}_Q \quad (1.36)$$

If $\mathbf{v}_Q$ is constant then $\mathbf{a}_Q = 0$ and $\mathbf{a}_{QP} = \mathbf{a}_P$.

This means that the relative acceleration of P with respect to Q is the same as the acceleration of P with respect to O, provided Q has a constant velocity with respect to O.

Example 8

Let us consider the practical problem of navigation and avoiding collisions at sea. Imagine that two ships S_1 and S_2 moving with constant velocities are at the positions A and B shown in Fig. 1.22a at some instant of time. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ represent their velocities with respect to the sea. The paths of the ships extended along their directions of motion from the initial points A and B intersect at point P. Will the ships collide, or will they pass one another at a distance?



(a) (b)

Fig. 1.22: (a) Path of two ships moving at constant velocity along courses that intersect; (b) path of S_2 relative to S_1 showing that they do not collide even though their paths cross.

The relative velocity of ship S_2 with respect to ship S_1 is given from Eq. 1.35 as

$$\mathbf{v}_{12} = \mathbf{v}_2 - \mathbf{v}_1$$

$\mathbf{v}_{12}$ is shown in Fig 1.22b. With respect to ship S_1, ship S_2 follows the straight line along $\mathbf{v}_{12}$. It will miss S_1 by the distance AN. If you have travelled in a ship and experienced an event of this sort on an open sea with no landmarks in sight, you will know that it is a curious experience. The observed motion of the other ship seems to be unrelated to the direction in which it is pointing.

Fig. 1.23

We will now state the results of Eqs. 1.35 and 1.36 more generally. Let an object move at velocity $\mathbf{v}$ relative to a frame of reference S'. If another frame of reference S' moves with velocity $\mathbf{V}$ relative to S (Fig. 1.23), then the velocity $\mathbf{v}'$ of the object with respect to the frame S' is

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}. \qquad (1.37)$$

If $\mathbf{V}$ is constant, then

$$\mathbf{a}' = \mathbf{a}. \qquad (1.38)$$

Thus, the acceleration of an object is the same in all frames of reference moving at constant velocity with respect to one another.

From the foregoing discussion we have realised that absolute motion is trivial. One has to always study the motion of one object with respect to another. Let us now summarise what we have learnt in this unit.

1.6 SUMMARY

- A body is said to be in motion if its position changes with time. A frame of reference is required to determine any kind of variation of position with time.
- Any vector $\mathbf{A}$ can be expressed as

  $$\mathbf{A} = A\,\hat{\mathbf{A}},$$

  where $\hat{\mathbf{A}}$, is the unit vector in the direction of $\mathbf{A}$. The magnitude of unit vector is always unity.
- A one-dimensional vector $\mathbf{A}$ along x-axis can be expressed as

  $$\mathbf{A} = A_x\,\hat{\mathbf{i}};$$

  where $\hat{\mathbf{i}}$ is the unit vector along the positive direction of x-axis and A_x is the x-component of $\mathbf{A}$.

- A two-dimensional vector **B** can be expressed with reference to mutually perpendicular x and y-axes as

  $$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}},$$

  where B_x and B_y are the x and y-components of **B**.
  $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along positive x and y-directions. If the angle between **B** and x-axis is θ then

  $$B_x = B\cos\theta,\quad B_y = B\sin\theta,\ B = \sqrt{B_x^2 + B_y^2},\quad \tan\theta = \frac{B_y}{B_x}.$$

- A three-dimensional vector **C** can be expressed in terms of components as

  $$\mathbf{C} = C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}},$$

  where $C = \sqrt{C_x^2 + C_y^2 + C_z^2}$.

  The direction of **C** can be specified in terms of the angles α, β, γ between **C** and the x-, y- and z-axes as,

  $$C_x = C\cos\alpha,\ C_y = C\cos\beta,\ C_z = C\cos\gamma.$$

- The scalar product of two vectors **A** and **B** is defined as $\mathbf{A}.\mathbf{B} = AB\cos\theta$, where θ is angle between **A** and **B**.
- The vector product of two vectors **A** and **B** is defined to be a vector **C** such that

  $$\mathbf{C} = \mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{C}}.$$

  The direction of $\hat{\mathbf{C}}$ is obtained from the right-hand rule.
- The position vector of a particle having coordinates (x, y, z) in a given frame of reference is

  $$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

  The instantaneous velocity **v** and instantaneous acceleration **a** of the particle are

  $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}},$$

  where $\quad v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt}$, and

  $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}},$$

  where $\quad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},\quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},\quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$

- For a particle executing uniform circular motion, the instantaneous velocity is always directed along the tangent and has magnitude

  $$v = r\omega,$$

  where r is the radius of the circle and ω is the angular speed of the particle. The instantaneous acceleration is directed towards the centre, and has the magnitude

  $$a_R = \frac{v^2}{r} = \omega^2 r.$$

- Motion is relative. The relative position and velocity of a particle P with respect to a particle Q are given as

  $$\mathbf{r}_{QP} = \mathbf{r}_P - \mathbf{r}_Q.$$

  $$\mathbf{v}_{QP} = \mathbf{v}_P - \mathbf{v}_Q.$$

  where $\mathbf{r}_P$ and $\mathbf{r}_Q$ are the position vectors of P and Q in a given frame of reference. $\mathbf{v}_P$ and $\mathbf{v}_Q$ are the velocities of P and Q in this frame.

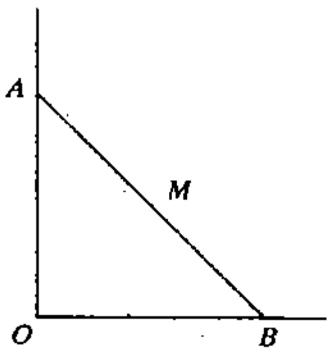


Fig. 1.24

1.7 TERMINAL QUESTIONS

1. Why is the statement "I am moving" meaningless ?
2. A ladder AB of length L rests against a vertical wall OA (Fig. 1.24). The foot B of the ladder is pulled away with constant speed v_o. (a) Obtain the position vector of the midpoint M of the ladder. Show that it describes an arc of a circle of radius $L/2$ with centre at O. (b) Find the velocity and speed of this point at the instant when B is at a distance b ($< L$) from the wall.
3. Refer to Fig. 1.25. A uniform magnetic field **B** exists in a region. Two electrons 1 and 2 moving with uniform velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ enter into this region: 1 parallel

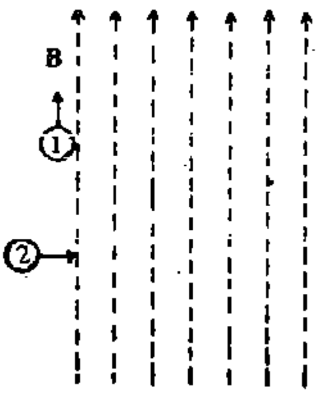


Fig. 1.25

and 2 perpendicular to this field. Write down their position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ at any time t, measured from the instant of their entering the field. Hence, determine the relative velocity and acceleration of 2 with respect to 1. (Hint: The velocity of the electron moving parallel to the field remains unaffected. The electron moving perpendicular to the field executes uniform circular motion in a plane perpendicular to the field.)

1.8 ANSWERS

SAQs

1. a) The frames of reference can be selected as three-dimensional rectangular Cartesian coordinate systems with origins at B, A and S together with a zero on the time scale in each case. The zeros on the time scale could correspond to the instants at which they hear the report of the starter's gun.

   b) All of them are correct in their respective frames of reference.

2. a) See Fig. 1.26.

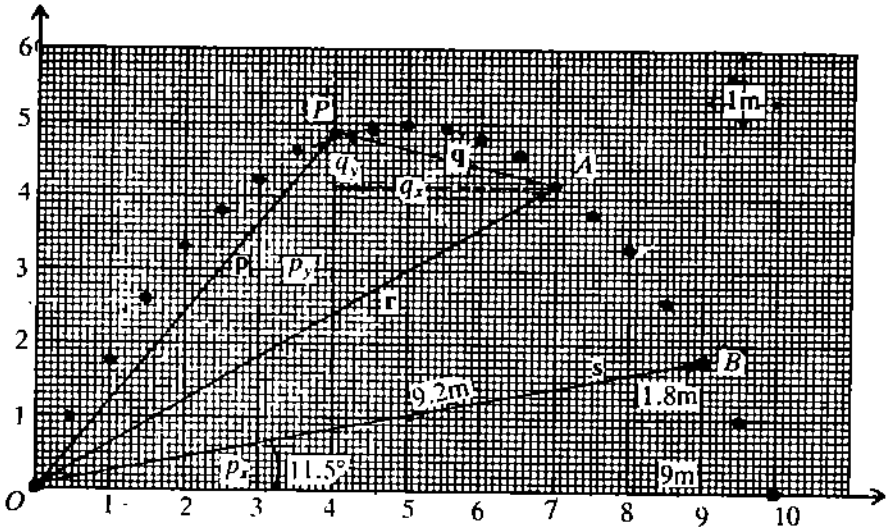


Fig. 1.26

   b) $\mathbf{r} = (7.0\,\hat{\mathbf{i}} + 4.2\hat{\mathbf{j}})$ m. You must have noted that we have taken 4.15m ≃ 4.2m

   c) $q_x + r_x = (-3.0 + 7.0)$ m $= 4.0$ m, $q_y + r_y = (0.7 + 4.2)$m $= 4.9$m

   Thus, $p_x = q_x + r_x$, $p_y = q_y + r_y$.

   Comment : $\mathbf{p} = \mathbf{q} + \mathbf{r}$ from Eq. 1.8b (i.e. $\mathbf{OA} = \mathbf{OP} + \mathbf{PA}$, expressed geometrically). This agrees with the law of vector addition.

   d) $s = 9.2$m, $\theta \simeq 11.5°$, $s_x = 9$m, $s_y = 1.8$m. From Eq. 1.3d, $s_x = s\cos\theta = 9.0$m, $s_y = s\sin\theta = 1.8$m.

   Hence, they are in agreement.

3. (i), (iv).

4. a) See Fig. 1.27a.

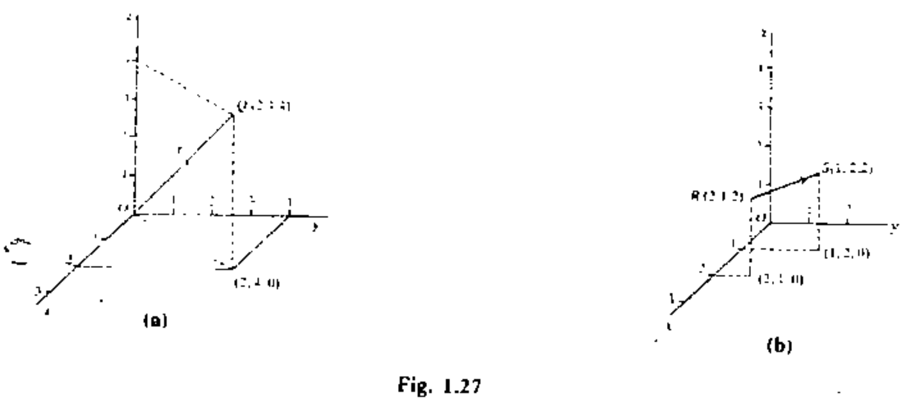


Fig. 1.27

$\mathbf{r} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $r = \sqrt{2^2 + 4^2 + 4^2} = 6.$

$l = \frac{2}{6} = \frac{1}{3}$, $m = n = \frac{4}{6} = \frac{2}{3}$.

b) See Fig. 1.27b. **RS** = **d**, $d_x = -1, d_y = 1, d_z = 0$

5. a) $\hat{\mathbf{i}}.\hat{\mathbf{j}} = 1.1 \cos 90° = 0$. Similarly, $\hat{\mathbf{j}}.\hat{\mathbf{k}} = \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0$

b) $\mathbf{A} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$

or $\mathbf{A}.\hat{\mathbf{i}} = A_x \hat{\mathbf{i}}.\hat{\mathbf{i}} + A_y \hat{\mathbf{j}}.\hat{\mathbf{i}} + A_z \hat{\mathbf{k}}.\hat{\mathbf{i}}$

$= A_x$ ($\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = 1, \hat{\mathbf{j}}.\hat{\mathbf{i}} = \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0$)

Similarly, $\mathbf{A}.\hat{\mathbf{j}} = A_y$, $\mathbf{A}.\hat{\mathbf{k}} = A_z$

6. $\mathbf{A}.\mathbf{A} = A.A \cos 0° = A^2$.

7. $|\hat{\mathbf{j}} \times \hat{\mathbf{i}}| = 1.1 \sin 90° = 1$. The direction of $\hat{\mathbf{j}} \times \hat{\mathbf{i}}$ is along the direction of advancement of screw-head when a right-handed screw rotates from $\hat{\mathbf{j}}$ towards i.e.$-\hat{\mathbf{k}}$. Hence, $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$. Similarly,
$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}, \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$.

8. a) $W = \mathbf{F}.\mathbf{d}$, $\mathbf{d} = [(20\hat{\mathbf{i}} + 6\hat{\mathbf{k}}) - (8\hat{\mathbf{i}} + 6\hat{\mathbf{k}})]\text{m} = 12\hat{\mathbf{i}}\,\text{m}$

$W = [(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}).\ 12\hat{\mathbf{i}}]$ Nm = 12 Nm = 12 joules

b) Torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$m

$\boldsymbol{\tau} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ Nm

$= (3\hat{\mathbf{k}} + 2\hat{\mathbf{k}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{i}})$ Nm

$= (-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ Nm

Note that the unit newton metre (Nm) for torque should be left as such and not written as joules.

c) Refer to Fig. 1.28. **OA** represents the velocity of the river. The shortest possible path is perpendicular to the bank. In order that the boat moves along that path the resultant velocity should be along OC. So the direction of **u** will be such that the resultant of **u** and **v** is along OC. Therefore, OC must be the diagonal of the parallelogram with **u** and **v** as adjacent sides. Now, we have to calculate w and α in terms of u and v.

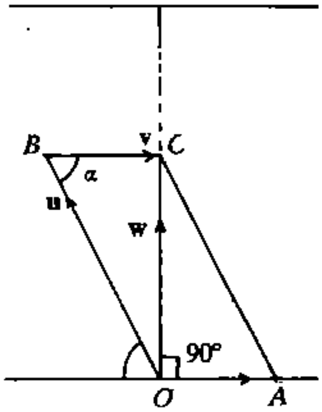


Fig. 1.28

Let $\angle AOB = \beta$. Then from Eq.1.22 d, we get

$\tan \angle AOC = \frac{u \sin\beta}{v + u \cos\beta}$

Since $\angle AOC = 90°$, the denominator of above is zero. Or $\cos \beta = -\frac{v}{u}$.

But $\alpha + \beta = 180°$.

$\therefore \cos \alpha = \cos(180° - \beta) = -\cos\beta = \frac{v}{u}$, or $\alpha = \cos^{-1}(\frac{v}{u})$

Again, from Eq. 1.22 a,

$w = \sqrt{u^2 + v^2 + 2uv \cos\beta}$

$= \sqrt{u^2 + v^2 + 2uv(-\frac{v}{u})}$

$= \sqrt{u^2 - v^2}$

An alternative way of working out the problem will be to apply Pythagoras' theorem in the right-angled triangle OBC, i.e.

$w = OC = \sqrt{OB^2 - BC^2} = \sqrt{u^2 - v^2}$

and $\cos \alpha = \frac{BC}{OB} = \frac{v}{u}$ or $\alpha = \cos^{-1}(\frac{v}{u})$

9. Average velocity between $t_1 = 1$s and $t_2 = 2$ s is $\frac{x_2 - x_1}{t_2 - t_1}$

$= \frac{(4.9 \times 4 - 4.9 \times 1)\text{m}}{(2 - 1)\text{ s}} = 14.7 \text{ m s}^{-1}$ and that between

$t_3 = 3$s and $t_4 = 4$s is $\frac{x_4 - x_3}{t_4 - t_3}$

$= 34.3 \text{ m s}^{-1}$

10. a) $\mathbf{r}(t) = Bt\,\hat{\mathbf{i}} + AB^2t^2\,\hat{\mathbf{j}}$

b) $\mathbf{v} = \frac{d}{dt}\{\mathbf{r}(t)\} = B\hat{\mathbf{i}} + 2AB^2t\,\hat{\mathbf{j}}$,

$\therefore v = \sqrt{B^2 + 4A^2t^2B^4} = B\sqrt{1 + 4A^2t^2B^2}$

11. $\frac{v^2}{r} \not> a$ or $\frac{v^2}{a} \not> r$

or $r \not< \frac{v^2}{a}$, i.e. $r_{min} = \frac{v^2}{a}$

Since $v = 60$ km h^{-1}, $a = 1.5$ m s^{-2},

$\therefore r_{min} = 1.8 \times 10^2$ m

Terminal Questions

1. We have learnt from Sec. 1.5 that the terms 'rest' and 'motion' are relative. So, whenever I say that 'I am moving' or 'I am at rest', I am supposed to mention about the observer with respect to whom I am talking about my state. That is why the statement "I am moving" is meaningless.

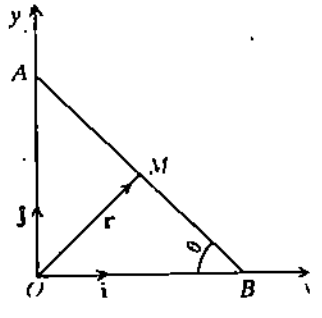


Fig. 1.29

2. Refer to Fig. 1.29. This is a modified form of Fig. 1.24 where the Cartesian x and y-axes are along OB and OA, respectively. (a) Let $\mathbf{r}$ be the position vector of the midpoint M of AB. Let the coordinates of B and A be $(x, 0)$ and $(0, y)$, respectively, at any time t.

$\mathbf{OB} = x\hat{\mathbf{i}}$, $\mathbf{OA} = y\hat{\mathbf{j}}$. Now, $\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$, or $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$

and $\mathbf{r} = \mathbf{OM} = \mathbf{OA} + \mathbf{AM} = \mathbf{OA} + \frac{1}{2}\mathbf{AB} = y\hat{\mathbf{j}} + \frac{1}{2}(x\hat{\mathbf{i}} - y\hat{\mathbf{j}}) = \frac{1}{2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$.

The position vector of M is $\mathbf{r} = \frac{1}{2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$, where $x^2 + y^2 = L^2$ = a constant equal to the square of the length of the ladder.

Now, using Eq. 1.3b, we get $r = L/2$ ($\because x^2 + y^2 = L^2$). This means that the point M is always at a distance $L/2$ from O. In other words, it describes a circle of radius $L/2$ with O as centre.

(b) The velocity of $M = \frac{d\mathbf{r}}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}\right)$.

Now $\frac{dx}{dt}$ = a constant = v_0 (given). Again as $x^2 + y^2 = L^2$, we have

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \text{ or } \frac{dy}{dt} = -\frac{x}{y}v_0 .$$

When B is at a distance b from O, we get $x = OB = b$ and $y = OA = \sqrt{L^2 - b^2}$.

Correspondingly, $\frac{dy}{dt} = -\frac{bv_0}{\sqrt{L^2-b^2}}$

So, the velocity of M at this instant is given by

$$\mathbf{v}_M = \frac{1}{2}v_0\hat{\mathbf{i}} - \frac{1}{2}\frac{bv_0}{\sqrt{L^2-b^2}}\hat{\mathbf{j}} = \frac{v_0}{2}\left[\hat{\mathbf{i}} - \frac{b}{\sqrt{L^2-b^2}}\hat{\mathbf{j}}\right]$$

And the speed $= v_M = \frac{v_0}{2}\left[1 + \frac{b^2}{L^2-b^2}\right]^{1/2} = \frac{Lv_0}{2\sqrt{L^2-b^2}}$

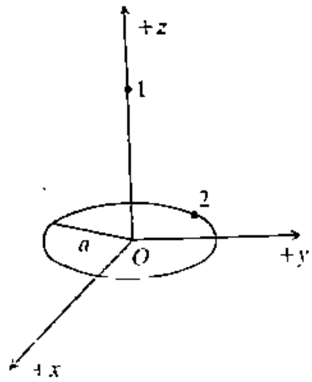


Fig. 1.30

3. Refer to Fig. 1.30. We first select a three-dimensional Cartesian coordinate system. Its origin is at the centre of the circle along which electron 2 executes uniform circular motion. Its z-axis is taken along the direction of the magnetic field, such that electron 1 moves along it. Let the position vectors of 1 and 2 at any time t be $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Let the uniform angular speed of 2 be ω and the radius of the circle a. Then,

$$\mathbf{r}_1 = v_1 t\hat{\mathbf{k}}, \mathbf{r}_2 = a(\cos\omega t\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}})$$

The relative velocity of 2 with respect to 1 is given by

$$\mathbf{v}_{12} = \frac{d}{dt}(\mathbf{r}_2 - \mathbf{r}_1) = \frac{d}{dt}\{a(\cos\omega t\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}}) - v_1 t\hat{\mathbf{k}}\}$$

or $\mathbf{v}_{12} = -a\omega(\sin\omega t\,\hat{\mathbf{i}} - \cos\omega t\,\hat{\mathbf{j}}) - v_1\hat{\mathbf{k}}$, and the acceleration of 2 with respect to 1 is given by $\mathbf{a}_{12} = \frac{d\mathbf{v}_{12}}{dt} = -a\omega^2(\cos\omega t\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}})$

UNIT 2 FORCE AND MOMENTUM

Structure

2.1 INTRODUCTION

In Unit 1, we learnt how to describe the motion of a particle in terms of displacement, velocity and acceleration. We did not ask what *caused* the motion. In this unit we shall study the factors affecting motion. For this we shall recall Newton's laws of motion and apply them to a variety of situations. Using Newton's laws we shall also establish the condition for a particle's equilibrium, when it is acted on by several coplanar forces.

We will use the familiar concept of linear momentum to study the motion of systems having more than one particle. In this process we shall establish the principle of conservation of linear momentum and apply it to solve problems in which a knowledge of the forces acting on the system is not needed. Finally, we shall recall the concept of impulse and use it to study the motion of variable mass systems. Any change of motion of an object is accompanied by performance of work and expenditure of energy. Therefore, in the next unit we shall study the concepts of work and energy.

Objectives

After studying this unit you should be able to:

- apply Newton's laws of motion
- solve problems using conditions for equilibrium of forces
- apply the law of conservation of linear momentum
- solve problems concerning impulse and variable mass systems.

2.2 CAUSES OF MOTION

What makes things move? An answer to this question was suggested by Aristotle, way back in the fourth century B.C. For nearly 2,000 years following the work of Aristotle, most people believed in his answer, that a force— a push or a pull — was needed to keep something moving. And the motion ceased when the force was removed. This idea made a lot of common sense. When an ox stopped pulling an ox-cart, the cart quickly came to a stop.

But these ideas were first critically examined by Galileo who carried out a series of experiments to show that no cause or force is needed to maintain the motion of an object. Study Fig. 2.1 carefully to understand this.

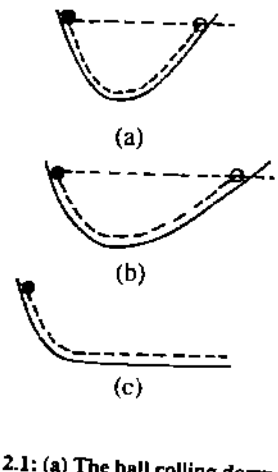


Fig. 2.1: (a) The ball rolling down a frictionless incline will rise approximately to its starting height on a second frictionless incline; (b) making the second incline more gradual would result in the ball's travelling further in the horizontal direction to attain the same height; (c) what happens in this case?

What do you think actually happens in the case of (c)?

The ball does stop on the flat surface after some time. But it is seen that the smoother the surface, the longer it takes for the ball to come to rest. Moreover, if the surface is reasonably smooth and flat, the ball moves more or less in a straight line. So if the element of friction can be completely removed, the ball would move indefinitely with a constant velocity as it would never be able to reach the starting height. Galileo concluded that *any object in motion, if not obstructed will continue to move with a constant speed along a horizontal line*. So, there would be no change in the motion of an object, unless an external agent acted on it to cause the change.

That was Galileo's version of inertia. Inertia resists changes, not only from the state of rest, but also from motion with a constant speed along a straight line. So the interest shifted from the *causes of motion* to the *causes for changes in motion.*

Galileo's work set the stage for centuries of progress in mechanics, beginning with the achievements of Isaac Newton. Newton's laws of motion are the basis of mechanics. We will now briefly discuss these laws.

2.2.1 Newton's Laws of Motion

Galileo's version of inertia was formalised by Newton in a form that has come to be known as Newton's first law of motion.

Newton's first law of motion
Stated in Newton's words, the first law of motion is:

> *"Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it."*

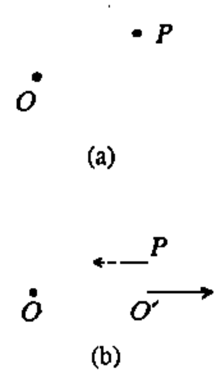


Fig. 2.2: (a) The observer O and the object P are at rest with respect to each other; (b) O' is accelerating with respect to O.

Newton's first law is also known as *law of inertia* and the motion of a body not subject to the action of other forces is said to be **inertial motion.** With the help of this law *we can define force as an external cause which changes or tends to change the state of rest or of uniform motion of a body.*

Have you noted that the first law does not tell you anything about the observer? But we know from Sec. 1.5 that the description of motion depends very much on the observer. So it would be worthwhile to know: For what kind of observer does Newton's first law of motion hold?

Suppose that an object P is at rest with respect to an observer O who is also at rest (Fig. 2.2a). Let another observer O' be accelerating with respect to O. P will appear to O' to be accelerating in a direction opposite to the acceleration of O' (Fig. 2.2b). According to Newton's first law the cause of the acceleration is some force. So O' will infer that P is being acted upon by a force. But O knows that no force is acting on P. It only appears to be accelerated to O'. Hence, the first law does not hold good for O'. It holds good for O.

An observer like O who is at rest or is moving with a constant velocity is called an **inertial** observer and the one like O', a **non-inertial** observer.

But how do we know whether an observer is inertial or not. For this, we need to measure the observer's velocity with respect to some standard. It is a common practice to consider the earth as a standard. Now the place where one is performing one's experiment has an acceleration (as discussed in Sec. 1.5) towards the polar axis due to the daily rotation of the earth. Again the centre of the earth has an acceleration towards the sun owing to its yearly motion around the sun. The sun also has an acceleration towards the centre of the Galaxy, and so on. Hence, the search for an absolute inertial frame is unending.

So we modify the definition of an inertial observer. We say *two observers are inertial with respect to one another when they are either at rest or in uniform motion with respect to one another. If an observer has an acceleration with respect to another then they are non-inertial with respect to one another.* Thus, a car moving with a constant velocity and a man standing on a road are inertial with respect to one another while a car in the process of gathering speed, and the man, are non-inertial with respect to each other.

The first law tells you how to detect the presence or absence of force on a body. In a sense, it tells you what a force does — it produces acceleration (either positive or negative) in a body. But the first law does not give a quantitative, measurable definition of force. This is what the second law does.

Newton's second law of motion
If you are struck by a very fast moving cricket ball you get injured but if you are hit by a flower moving with the same velocity as that of the ball you do not at all feel perturbed. However, if you are struck by a slower ball the injury is less serious. This indicates that any kind of impact made by an object depends on two things — its mass and velocity. Hence, Newton felt the necessity of defining the product of mass and velocity which later came to be known as **linear momentum.** Mathematically speaking, linear momentum

$$\mathbf{p} = m\mathbf{v}. \tag{2.1}$$

Thus, $\mathbf{p}$ is a vector quantity in the direction of velocity. The introduction of the above quantity paved the way for stating the second law, which in the words of Newton is as follows:

> *'The change of motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed."*

By "change of motion", Newton meant the rate of change of momentum with time. So mathematically we have

$$\mathbf{F} \propto \frac{d}{dt}(\mathbf{p}),$$

$$\text{or } \mathbf{F} = k\frac{d}{dt}(\mathbf{p}), \tag{2.2}$$

where $\mathbf{F}$ is the impressed force and k is a constant of proportionality. The differential operator $\frac{d}{dt}$ indicates the rate of change with time. Now, if the mass of the body remains constant (i.e. neither the body is gaining in mass like a conveyer belt nor is it disintegrating like a rocket), then

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ = the acceleration of the body. Thus, from Eq. 2.2, we get

$$\mathbf{F} = k\, m\, \mathbf{a}, \text{ and} \tag{2.3a}$$

$$F = kma, \tag{2.3b}$$

We had seen earlier that the need for a second law was felt in order to provide a quantitative definition of force. So something must be done with the constant k. We have realised that the task of a force $\mathbf{F}$ acting on a body of mass m is to produce in it an acceleration $\mathbf{a}$. Hence, anything appearing in the expression for force other than m and $\mathbf{a}$ must be a pure number, i.e. k is a pure number. So we can afford to make a choice for its numerical value.

We define unit force as one which produces unit acceleration in its direction when it acts on a unit mass. So, we obtain from Eq. 2.3b that $1 = k.\ 1.1$ or $k = 1$. Thus, Eqs. 2.2 and 2.3 take the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \text{ and} \tag{2.4a}$$

$$\mathbf{F} = m\mathbf{a}, \text{ for constant mass.} \tag{2.4b}$$

Now we know from Sec. 1.3 that if the position vector of a particle is $\mathbf{r}$ at a time t then its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by Eqs. 1.24a and 1.26a. Substituting for $\mathbf{a}$ and $\mathbf{v}$ in Eq. 2.4, we get

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right), \text{ or } \mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}. \tag{2.5}$$

Eq. 2.5 is a second order differential equation in $\mathbf{r}$. If we know the force $\mathbf{F}$ acting on a body of mass m, we can integrate Eq. 2.5 to determine $\mathbf{r}$ as a function of t. The function $\mathbf{r}(t)$ would give us the path of the particle. Since Eq. 2.5 is of second order we shall come across two constants of integration. So we require two initial conditions to work out a solution of this equation. Conversely, if we know the path or trajectory of an accelerating particle, we can use Eq. 2.5 to determine the force acting on the body. Eq. 2.5 also enables us to determine unknown masses from measured forces and accelerations.

So far, we have considered only one force acting on the body. But often several forces act on the same body. For example, the force of gravity, the force of air on the wings and body of the plane and the force associated with engine thrust act on a flying jet (Fig. 2.3).

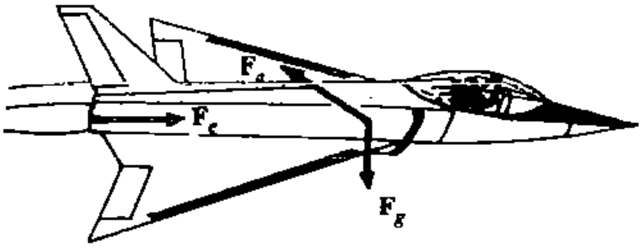


Fig. 2.3: Forces on a jet: $\mathbf{F}_e$, the thrust of the engine, $\mathbf{F}_a$, the force of the air provides both lift and drag, $\mathbf{F}_g$ the force of gravity.

In such cases, we add the individual forces vectorially, to find the *net force* acting on the object. The object's mass and acceleration are related to this net force by Newton's second law. You may now like to apply Newton's second law to a simple situation.

SAQ 1

Astronauts on the Skylab mission of the 1970s found their masses by using a chair on which a known force was exerted by a spring. With an astronaut strapped in the chair, the 15 kg chair underwent an acceleration of 2.04×10^{-2} m s^{-2} when the spring force was 2.07 N. What was the astronaut's mass?

Newton's third law of motion

So far we have been trying to understand how and why a single body moves. We have identified force as the cause of change in the motion of a body. But how does one exert a force on this body? Inevitably, there is an agent that makes this possible. Very often, your hands or feet are the agents. In football, your feet bring the ball into motion. Thus, forces arise from interactions between systems. This fact is made clear in Newton's third law of motion. To put it in his own words:

"To every action there is an equal and opposite reaction."

Here the words 'action' and 'reaction' mean forces as defined by the first and second laws. If a body A exerts a force $\mathbf{F}_{AB}$ on a body B, then the body B in turn exerts a force $\mathbf{F}_{BA}$ on A, such that

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}.$$

So, we have $\mathbf{F}_{AB} + \mathbf{F}_{BA} = \mathbf{0}.$ (2.6)

Notice that Newton's third law deals with *two* forces, each acting on a different body. You may now like to work out an SAQ based on the third law.

SAQ 2

a) When a footballer kicks the ball, the ball and the man experience forces of the same magnitude but in opposite directions according to the third law. The ball moves but the man does not move. Why?

b) The earth attracts an apple with a force of magnitude F. What is the magnitude of the force with which the apple attracts the earth? The apple moves towards the earth. Why does not the reverse happen?

Newton's laws of motion give us the means to understand most aspects of motion. Let us now apply them to a variety of physical situations involving objects in motion.

2.2.2 Applications of Newton's Laws

To apply Newton's laws, we must identify the body whose motion interests us. Then we should identify all the forces acting *on* the body, draw them on a vector diagram and find the *net* force acting *on* the body. Newton's second law can then be used to determine the body's acceleration. We will now use this basic method to solve a few examples.

Example 1: Projectile Motion

The motion of a shot fired by a gun and that of a ball thrown by a fieldsman to another are all examples of projectiles. Let us consider such a projectile of mass m (Fig. 2.4). It is thrown from a point O with a velocity $\mathbf{v}_0$ along OA making an angle θ with the horizontal. Let the particle be at a point P ($\mathbf{OP} = \mathbf{r}$) at time t. If we neglect air resistance, then the only force acting on the particle is a constant force, $\mathbf{F} = m\mathbf{g}$, due to gravity. Let us determine the particle's path. Eq. 2.5 gives

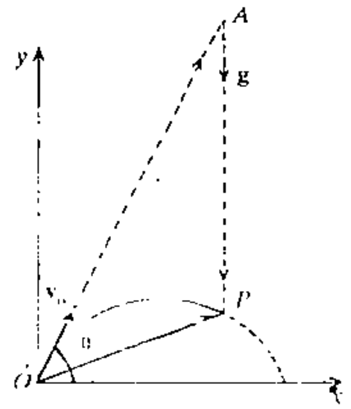


Fig. 2.4: OP = OA + AP

$$m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{g}, \tag{2.7}$$

$$\text{or } \frac{d^2\mathbf{r}}{dt^2} = \mathbf{g},$$

$$\text{or } \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \mathbf{g}.$$

On integrating with respect to t we get

$$\frac{d\mathbf{r}}{dt} = \mathbf{g}t + \mathbf{A}, \tag{2.8a}$$

where **A** is a constant of integration. As the other two factors in Eq. 2.8a are vectors having dimensions of velocity, **A** must also be a vector having the dimension of velocity. To determine **A**, we use the initial condition that velocity $= \frac{d\mathbf{r}}{dt} = \mathbf{v}_0$ when $t = 0$.
So $\mathbf{A} = \mathbf{v}_0$. Hence,

$$\frac{d\mathbf{r}}{dt} = \mathbf{g}t + \mathbf{v}_0. \qquad (2.8b)$$

On integrating with respect to t again, we get

$$\mathbf{r} = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{B}, \qquad (2.9)$$

where **B**, like **A** in Eq. 2.8a, is a constant vector of integration, but it has the dimension of length. To determine **B**, we need another initial condition. Letting $\mathbf{r} = 0$ at $t = 0$, we get $\mathbf{B} = 0$. Hence,

$$\mathbf{r} = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{g}t^2 \qquad (2.10)$$

We have essentially used two initial conditions: $\frac{d\mathbf{r}}{dt} = \mathbf{v}_0$ and $\mathbf{r} = 0$ at $t = 0$. Since $\mathbf{v}_0$ is along *OA* and t is scalar, we understand that $\mathbf{v}_0 t$ is along *OA*. Again **g** is directed vertically downwards and $\frac{1}{2}t^2$ is a scalar, so $\frac{1}{2}\mathbf{g}t^2$ is directed vertically downwards, i.e. along *AP* (Fig. 2.4).

We use the law of vector addition to get

$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}. \qquad (2.11)$$

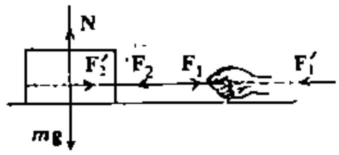
Thus, we get the location of the particle. As time advances *OA* is lengthened and so is *AP*, and we get the location of the particle by adding **OA and AP**.

Example 2: Friction

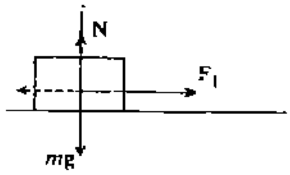
A heavy block is kept on a rough floor. You apply a force by pulling on a rope attached to it, but it still does not move. Is it a contradiction of Newton's laws? Discuss the motion of the block.

Refer to Fig. 2.5a. Let us first find out all the forces that act on the heavy block. There is a force of gravity mg acting downwards. The block exerts this force on the floor. Therefore, the floor exerts an equal and opposite normal force of reaction **N** on the block. **N** is normal to the surface of the floor. The third force results from your pull on the rope. Let $\mathbf{F}_1$ be the force that you exert on the rope. The rope exerts a force of reaction $\mathbf{F}_1'$ on you and a force of action, say $\mathbf{F}_2$ on the block. Let $\mathbf{F}_2'$ be the force that the block exerts on the rope. Then according to Newton's third law of motion

$$\mathbf{F}_1 = -\mathbf{F}_1'; \quad \mathbf{F}_2 = -\mathbf{F}_2'. \qquad (2.12)$$



(a)



(b)

Fig. 2.5

Let us assume that the rope is massless. Then, from Newton's second law, the net force acting on the rope is zero and we have,

$$\mathbf{F}_1 + \mathbf{F}_2' = 0,$$

$$\text{or } \mathbf{F}_2' = -\mathbf{F}_1,$$

$$\text{or } \mathbf{F}_2 = \mathbf{F}_1, \text{ from Eq. 2.12.}$$

So a massless rope transmits the force you exert on it to the block without any change. The three forces $m\mathbf{g}$, **N** and $\mathbf{F}_1$ acting on the block do not add up to zero. **N** and $m\mathbf{g}$ cancel each other, leaving a net force $\mathbf{F}_1$ shown in Fig. 2.5b. Since the block remains at rest, the net force acting on it must be zero according to the first law. There must, therefore, be another force which acts on the block. This force must also be horizontal, directed opposite to $\mathbf{F}_1$ and equal in magnitude. Actually, there is such a force which is the contact force between the floor and the block, known as the force of friction. It is shown in Fig. 2.5b by a dotted line.

Friction is a force that acts between two surfaces to oppose their relative motion (see Fig. 2.6). The **force of static friction** $\mathbf{f}_s$ acts between surfaces at rest with respect to each other. The maximum force of static friction $\mathbf{f}_{sm}$ is the same as the smallest force necessary to start motion. Once motion has started, the force of friction usually decreases, so that a smaller force is required to maintain a uniform motion.

The force acting between surfaces in relative motion is called the **force of kinetic friction** $\mathbf{f}_k$. $\mathbf{f}_k$ is less than $\mathbf{f}_{sm}$. The ratio of the magnitude of maximum force of static friction f_{sm} to the

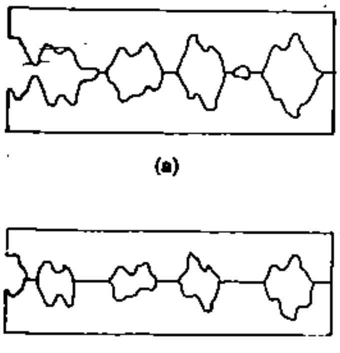


Fig. 2.6: Friction acts between two surfaces to oppose their relative motion. Even the smoothest surface is actually rough on a microscopic scale. (a) When two surfaces are in contact, then irregularities adhere because of electrical forces between the molecules. This gives rise to a force that opposes their relative motion; (b) when the normal force between the surfaces increases, the irregularities are crushed together and the contact area between the surfaces increases. This increases the force of friction.

magnitude of normal force of reaction N between the two surfaces is called the coefficient of static friction μ_s, i.e.

$$f_{sm} = \mu_s N.$$

Similarly,

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction.

The discussion on friction brings us to an important class of problems in which an object undergoes motion against resistive forces. Another example of resistive force is air resistance to projectile motion. The motion of raindrops, or cars is also affected by air resistance. So let us discuss an example on motion where resistive forces are present.

Example 3: Motion against resistive forces

Suppose an object moves under the influence of a constant force $\mathbf{F}_0$, with a resistive force $\mathbf{R}$ opposing its motion. Let $\mathbf{R}$ always act in a direction opposite to the object's instantaneous velocity. In general the resistive force is a function of speed, so that Newton's second law becomes:

$$\mathrm{F}_0 - \mathrm{R}\,(\mathrm{v}) = m\frac{dv}{dt} \qquad (2.13)$$

The resistive force of dry friction (Fig. 2.7a) if almost independent of v, so that

$$R\,(v) = F_f = \text{constant}.$$

In this case Eq. 2.13 reduces to the simple case of acceleration under a constant net force.

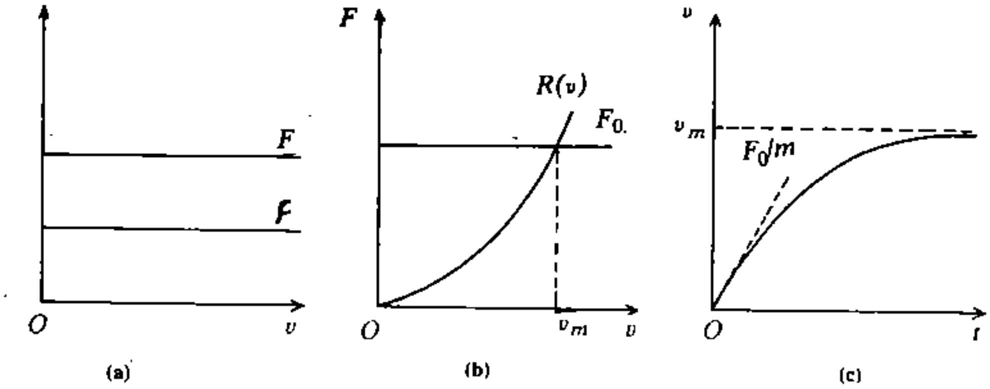


Fig. 2.7: (a) Resistive force for an object resisted by dry friction and (b) by fluid friction; (c) terminal speed v_m of an object in a fluid resistive medium. F_0/m is the slope of the curve at O.

In the case of air resistance or fluid resistance, $R(v)$ increases with v (Fig. 2.7b). It is usually described by the relation

$$R(v) = Av + Bv^2. \qquad (2.14)$$

For the sake of simplicity, let us consider only one-dimensional motion under the resistive force of Eq. 2.14. So we can use the scalar form of Eq. 2.13, which is

$$m\frac{dv}{dt} = F_0 - Av - Bv^2. \qquad (2.15)$$

Eq. 2.15 is not very easy to solve and we do not intend to go into its formal mathematical solution. Let us, however, consider some qualitative features of the possible solution.

Let the object start moving under a constant force F_0. Its initial acceleration will have almost a constant value $\frac{F_0}{m}$, since v is very small. Thus, v will be a linear function of t (Fig. 2.7 c). As v increases, $R\,(v)$ will increase and the net driving force is reduced to a value below F_0, giving a steadily decreasing slope in the graph of $v(t)$. When $R(v)$ approaches F_0, the net force acting on the body tends to be zero. Then the object's velocity acquires a limiting constant magnitude v_m. The value of v_m is the positive solution of the quadratic equation

$$Bv^2 + Av - F_0 = 0.$$

In such a situation, the body moves with zero acceleration under zero net force. ***It is not the unaccelerated motion of objects moving under no force at all.*** So every time we see a car moving along a straight road at a steady speed, a jet plane flying through the air at a constant speed, or raindrops falling with a uniform terminal velocity, we see bodies moving under *zero net force*. Their motion at a constant speed does *not* mean that no force is acting on them. Now you may like to work out an SAQ based on this concept.

SAQ 3

A box of mass m is being pulled across a rough floor by means of a massless rope that makes an angle θ with the horizontal (Fig. 2.8). The coefficient of kinetic friction between the box and the floor is μ_k. What is the tension in the rope when the box moves at a constant velocity?

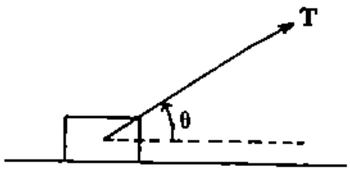


Fig. 2.8

One simple but important application of Newton's laws is the study of bodies in equilibrium. A large number of situations may be reduced to problems concerning the equilibrium of forces on a particle. For example, the construction of buildings and suspension bridges, design of aircrafts and ships, loading or unloading operations, involve forces in equilibrium. So let us now study equilibrium of forces acting on a particle.

2.2.3 Equilibrium of Forces

We say that *a particle is in equilibrium, when the resultant of all the forces acting on it is zero*. It then follows from Newton's first law of motion that a particle in equilibrium is either at rest or is moving in a straight line with constant speed. It is found that for a large number of problems, we have to deal with equilibrium of forces lying in a plane. Therefore, we shall restrict our discussion to the case when a particle is in equilibrium under the influence of a number of coplanar forces, $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$. The required condition is given by

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots = \mathbf{0}. \qquad (2.16a)$$

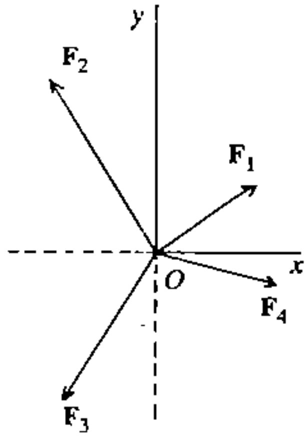


Fig. 2.9

Since the forces are coplanar, we can resolve them along two mutually perpendicular directions of x and y-axes (Fig. 2.9), O being the particle. So Eq. 2.16a can be rewritten as

$$(F_{1x}\hat{\mathbf{i}} + F_{1y}\hat{\mathbf{j}}) + (F_{2x}\hat{\mathbf{i}} + F_{2y}\hat{\mathbf{j}}) + \cdots = 0,$$

$$\text{or } (F_{1x} + F_{2x} + \cdots)\hat{\mathbf{i}} + (F_{1y} + F_{2y} + \cdots)\hat{\mathbf{j}} = \mathbf{0},$$

$$\text{or } \quad F_{1x} + F_{2x} + \ldots = 0,$$

$$\text{and } F_{1y} + F_{2y} + \ldots = 0. \qquad (2.16b)$$

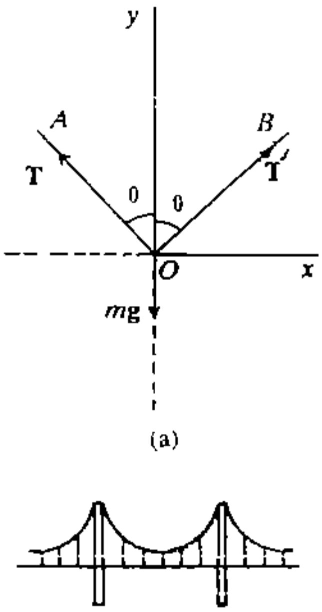
Eqs. 2.16b can be expressed in a concise form as

$$\Sigma F_x = 0, \;\; \Sigma F_y = 0. \qquad (2.16c)$$

where Σ denotes summation of the x- or y-components of the forces. We shall now apply Eq. 2.16c to work out an example.

Example 4

A particle of mass m is hung by two light strings as shown in Fig. 2.10a. The ends A and B are held by hands. The strings OA and OB make angles θ with the vertical. Find the values of T and T' in terms of m and θ.



(a)

(b)

Fig. 2.10

Through O, we consider two mutually perpendicular directions of x and y-axes, the latter being along the vertical.

From Eq. 2.16c, we have

$$-T\cos(90° - \theta) + T'\cos(90° - \theta) = 0, \qquad (2.17)$$

$$\text{and} \quad T\cos\theta + T'\cos\theta - mg = 0 \qquad (2.18)$$

Hence, from Eq. 2.17, we get

$$T' = T. \; (\because \theta \neq 0°).$$

Thus, from Eq. 2.18, we have

$$2T\cos\theta = mg,$$

$$\text{or } T = T' = \frac{mg}{2\cos\theta}. \qquad (2.19)$$

If θ is increased, $\cos\theta$ will decrease, thereby increasing the tension. This may lead to the breaking of the string. It is for this reason that the main cable supporting the suspension of a bridge must be hung with a substantial curvature as shown in Fig. 2.10b. If the cable were stretched straight across, the tension would be so large that it may break.

Now that you have studied equilibrium of forces, you can work out the following SAQ.

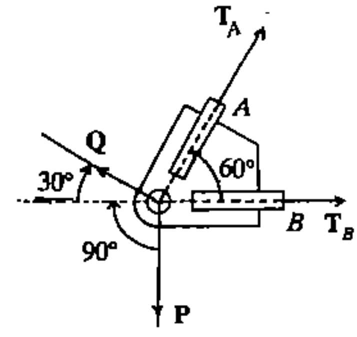


Fig. 2.11

SAQ 4

A connection used for joining different parts of a machine is maintained in equilibrium by applying two forces **P** and **Q** of magnitude $P = 3000$ N and $Q = 4000$ N as shown in Fig. 2.11. Determine the tension in rods A and B.

/e have applied Newton's laws to a single particle or a single body which could be ıs a particle. We will now extend our study to the motion of a system of particles. ımple of such a system is the sun and the planets. These bodies are so far apart ɜd to their diameters that together they can be treated as a system of particles. We t linear momentum (recall Eq. 2.1) plays a vital role in describing the motion of such . It is also significant because of the principle of conservation of linear momentum. s now study linear momentum in some detail.

2.3 LINEAR MOMENTUM

Let us first study a system of two interacting particles '1' and '2' having masses m_1 and m_2 (Fig. 2.12). Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be their linear momenta. The total linear momentum **p** of this system is simply the vector sum of the linear momenta of these two particles.

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2. \qquad (2.20)$$

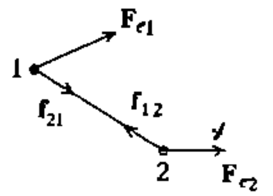


Fig. 2.12

From Newton's second law, the rate of change of $\mathbf{p}_1$ is the vector sum of all the forces acting on 1, i.e. the total external force $\mathbf{F}_{e1}$ on it and the internal force $\mathbf{f}_{21}$ due to 2:

$$\mathbf{F}_{e1} + \mathbf{f}_{21} = \frac{d\mathbf{p}_1}{dt}. \qquad (2.21a)$$

Similarly, for particle 2:

$$\mathbf{F}_{e2} + \mathbf{f}_{12} = \frac{d\mathbf{p}_2}{dt}. \qquad (2.21b)$$

From Newton's third law, we know that $\mathbf{f}_{12} = -\mathbf{f}_{21}$. Therefore, on adding Eqs. 2.21a and 2.21b, we get

$$\mathbf{F}_{e1} + \mathbf{F}_{e2} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt}, \text{ which may be written as}$$

$\mathbf{F}_e = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2)$, where $\mathbf{F}_e$ is the net external force on the system. Therefore, from Eq. 2.20,

$$\mathbf{F}_e = \frac{d\mathbf{p}}{dt}. \qquad (2.22)$$

Thus, in a system of interacting particles, it is the net external force which produces acceleration and not the internal forces. Now, we shall see how Eq. 2.22 leads to the principle of conservation of linear momentum.

2.3.1 Conservation of Linear Momentum

In the special case when the net external force $\mathbf{F}_e$ is zero, Eq. 2.22 gives

$$\frac{d\mathbf{p}}{dt} = 0, \qquad (2.23)$$

so that $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ = a constant vector.

This is the principle of conservation of linear momentum for a two-particle system. It is equally valid for a system of any number of particles. Its formal proof for a many-particle system will be given in Unit 7 of Block 2. It states that:

> *"If the net external force acting on a system is zero, then its total linear momentum is conserved."*

Let us now apply this principle.

Example 5

A vessel at rest explodes, breaking into three pieces. Two pieces having equal mass fly off perpendicular to one another with the same speed of 30 ms^{-1}. Show that immediately after the explosion the third piece moves in the plane of the other two pieces. If the third piece has three times the mass of either of the other piece, what is the magnitude of its velocity immediately after the explosion?

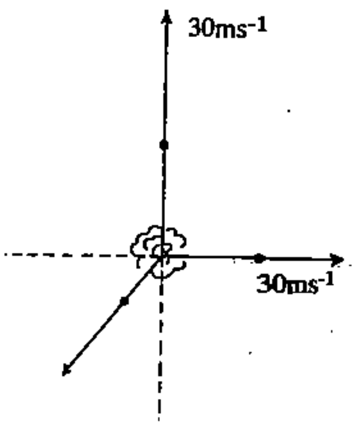


Fig. 2.13

The process is explained in the schematic diagram (Fig. 2.13). The vessel was at rest prior to the explosion. So its linear momentum was zero. Since no net external force acts on the system, its total linear momentum is conserved. Therefore, the final linear momentum is also zero, i.e.

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}, \quad (2.24a)$$

$$\text{or } \mathbf{p}_1 + \mathbf{p}_2 = -\mathbf{p}_3. \quad (2.24b)$$

$(\mathbf{p}_1 + \mathbf{p}_2)$ lies in the plane contained by $\mathbf{p}_1$ and $\mathbf{p}_2$. So in accordance with Eq. 2.24b, $-\mathbf{p}_3$ must also lie in that plane. Hence, $\mathbf{p}_3$ lies in the same plane as $\mathbf{p}_1$ and $\mathbf{p}_2$. Now, from Eq. 2.24b,

$$(\mathbf{p}_1 + \mathbf{p}_2) \cdot (\mathbf{p}_1 + \mathbf{p}_2) = (-\mathbf{p}_3) \cdot (-\mathbf{p}_3),$$

$$\text{or } p_1^2 + p_2^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 = p_3^2$$

But $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$ ($\because$ $\mathbf{p}_1$ is perpendicular to $\mathbf{p}_2$).

$$\text{So, } p_3^2 = p_1^2 + p_2^2. \quad (2.24c)$$

$$\text{or } (3m\,v)^2 = (mu)^2 + (mu)^2,$$

$$\text{or } 9m^2v^2 = 2m^2u^2, \text{ or } v = \frac{\sqrt{2}}{3}u.$$

According to the problem $u = 30$ ms^{-1}. $\therefore v = 10\sqrt{2}$ ms^{-1}

There is another method of finding the magnitude of the velocity. We can express Eq. 2.24b in terms of the components of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ in two mutually perpendicular directions of x and y-axes. Let $\mathbf{p}_1$ be along x-axis, $\mathbf{p}_2$ along y-axis and let $\mathbf{p}_3$ make an angle θ with x-axis. Then Eq. 2.24b gives

$$p_1\hat{\mathbf{i}} + p_2\hat{\mathbf{j}} = -(p_3\cos\theta\hat{\mathbf{i}} + p_3\sin\theta\hat{\mathbf{j}}). \quad (2.25a)$$

This equation is satisfied iff (see Eq. 1.6)

$$-p_3\cos\theta = p_1, \quad -p_3\sin\theta = p_2. \quad (2.25b)$$

$$\text{or } p_3^2 = p_1^2 + p_2^2, \text{ which is the same as Eq. 2.24c.}$$

SAQ 5

Find the direction of v in Example 5

From the above example and the way we obtained the principle of conservation of momentum, it may appear that the principle is limited in its application. This is because we have assumed that no net external force acts on the system of particles. However, the scope of the principle is much broader.

There are many cases in which an external force, such as gravity, is very weak compared to the internal forces. The explosion of a rocket in mid air is an example. Since the explosion lasts for a very brief time, the external force can be neglected in this case. In examples of this type, linear momentum is conserved to a very good approximation.

Again, if a force is applied to a system by an external agent, then the system exerts an equal and opposite force on the agent. Now if we consider the agent and the system to be a part of a new, larger system, then the momentum of this new system is conserved. Since there is no larger system containing the universe, its total linear momentum is conserved.

We have seen that whenever we have a system of particles on which no net external force acts, we can apply the law of conservation of linear momentum to analyse their motion. In fact, the advantage is that this law enables us to describe their motion without knowing the details of the forces involved. Let us now see what happens to each individual particle in the system. In such a case, the particle experiences a net force, and its linear momentum changes. This change depends on the magnitude of the force and also the time for which it acts. Further, the force itself may vary with time. Let us now study the relationship of force, its duration and the resulting change in momentum.

2.3.2 Impulse

In cricket, when a batsman strikes a ball with his bat, the bat is in contact with the ball for a very short but finite time. At the two instants, when the ball is just about to make contact

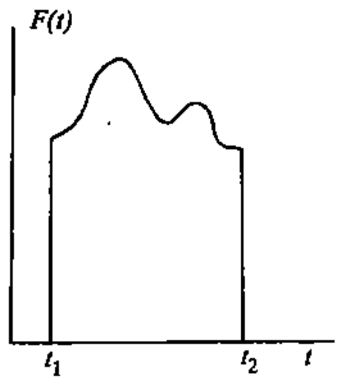


Fig. 2.14

with the bat and when it just leaves the bat, it experiences no force. In between, it experiences a large varying force. The variation of the magnitude of such a force **F**(t) can be as shown in Fig.2.14. We generally assume that the force has a constant direction. We can find the change in the linear momentum of an object on which such a force acts by integrating Eq. 2.4a over the time interval from t_1 to t_2:

$$\int_{t_1}^{t_2} \mathbf{F}(t)dt = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt}dt = \mathbf{p}(t_2) - \mathbf{p}(t_1) \equiv \Delta\mathbf{p}. \qquad (2.26)$$

The integral of force over time is called the **impulse** of the force and is given by

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}(t)dt. \qquad (2.27)$$

Thus, according to Eq.2.26, impulse of a force is equal to the change in linear momentum. If F acts during a time interval Δt but is variable, then to calculate impulse we would need to know the function F(t) explicitly. However, this is usually not known. A way out is to define the **average force** $\overline{\mathbf{F}}$ by the equation

$$\overline{\mathbf{F}} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \mathbf{F}(t)dt, \text{ where } \Delta t = t_2 - t_1. \qquad (2.28)$$

From Eqs. 2.27 and 2.28, we get

$$\mathbf{J} = \overline{\mathbf{F}}\Delta t = \Delta\mathbf{p}. \qquad (2.29)$$

There are many examples which illustrate the relationship between the average force, its duration and change of linear momentum. A tennis player hits the ball while serving with a great force to impart linear momentum to the ball. To impart maximum possible momentum, the player 'follows through' with the serve. This action prolongs the time of contact between the ball and the racquet. Therefore, to bring about the maximum possible change in the linear momentum, we should apply as large a force as possible over as long a time interval as possible. You may now like to apply these ideas to solve a problem.

SAQ 6

a) A ball of mass 0.25 kg moving horizontally with a velocity 20 m s^{-1} is struck by a bat.The duration of contact is 10^{-2} s. After leaving the bat, the speed of the ball is 40 m s^{-1} in a direction opposite to its original direction of motion. Calculate the average force exerted by the bat.

b) Give an example in which a weak force acts for a long time to generate a substantial impulse.

So far we have dealt with examples which do not involve variation of mass of objects in motion. We shall now take up such cases and apply the concepts of impulse and momentum.

2.3.3 Motion with Variable Mass

If the mass of a system varies with time, we can express Newton's second law of motion as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}. \qquad (2.30)$$

Under the special case when v is constant, Eq. 2.30 becomes

$$\mathbf{F} = \mathbf{v}\frac{dm}{dt}. \qquad (2.31)$$

Let us study an example of this special type.

Example 6

Sand falls on to a conveyer belt B (Fig.2.15) at the constant rate of 0.2 kg s^{-1}. Find the force required to maintain a constant velocity of 10 m s^{-1} of the belt.

Here, we shall apply Eq. 2.31, as velocity remains constant. Since the mass is increasing, $\frac{dm}{dt}$ is positive. The direction of F, therefore, is same as that of v, i.e. the direction of motion of the conveyer belt.

Thus, using Eq. 2.31, we get

$$F = (10 \text{ m s}^{-1}) \times (0.2 \text{ kg s}^{-1}) = 2 \text{ kg m s}^{-2} = 2\text{N}.$$

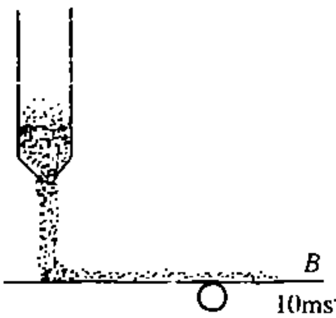


Fig. 2.15

Another example of a varying mass system is the rocket. In a rocket (Fig. 2.16) a stream of gas produced at a very high temperature and pressure escapes at a very high velocity through an exhaust nozzle. Thus, the rocket loses mass and $\frac{dm}{dt}$ is negative. So the main body of the rocket experiences a huge force in a direction opposite to that of the exhaust causing it to move. This is a very simplified way of dealing with the motion of a rocket. We shall next analyse the motion of a rocket with a little more rigour using the idea of impulse.

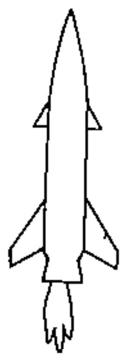


Fig. 2.16

Motion of a rocket

Let us assume that the rocket has a total mass M at a time t. It moves with a velocity $\mathbf{v}$ and ejects a mass ΔM during a time interval Δt. The situation is explained schematically in Figs. 2.17a and 2.17b.

At time t the total initial momentum of the system $= M\mathbf{v}$ (Fig. 2.17a).
At time $t + \Delta t$ the total final momentum of the system $= (M - \Delta M)(\mathbf{v} + \Delta\mathbf{v}) + (\Delta M)\,\mathbf{u}$ (Fig. 2.17b).

Notice that we have used the positive sign for $\mathbf{u}$ because the total final momentum of the system in Fig. 2.17b is a vector sum and not the difference of the momenta of M and $(M - \Delta M)$. Let us now apply Eq. 2.29. If we take the vertically upward direction as positive, the impulse is $-M\mathbf{g}\,\Delta t$ and is equal to the change in linear momentum.

So, $$-M\mathbf{g}\ \Delta t = (M - \Delta M)(\mathbf{v} + \Delta\mathbf{v}) + (\Delta M)\mathbf{u} - M\mathbf{v}$$
$$= M(\Delta\mathbf{v}) + \Delta M(\mathbf{u} - \mathbf{v} - \Delta\mathbf{v})$$

We may use Eq. 1.35 to simplify the above relation:

$$-\mathbf{g} = \frac{\Delta\mathbf{v}}{\Delta t} + \frac{1}{M}\frac{\Delta M}{\Delta t}\mathbf{u}_{rel},$$ where $\mathbf{u}_{rel} = \mathbf{u} - (\mathbf{v} + \Delta\mathbf{v})$ is the relative velocity of the exhaust with respect to the rocket.

Now, in the limit $\Delta t \to 0$, we have

$$-\mathbf{g} = \frac{d\mathbf{v}}{dt} - \frac{1}{M}\frac{dM}{dt}\mathbf{u}_{rel} \qquad (2.32)$$

The negative sign on the right-hand side of Eq. 2.32 appears as

$$\lim_{\Delta t \to 0}\frac{\Delta M}{\Delta t} = -\frac{dM}{dt},$$ because M decreases with t.

So, when we apply Eq. 2.32 in numerical problems we just replace $\frac{dm}{dt}$ *by its magnitude.*

On integrating Eq. 2.32 with respect to t, we get

$$\int_0^t \frac{d\mathbf{v}}{dt}\,dt = -\mathbf{g}t + \mathbf{u}_{rel}\int_{M_0}^{M}\frac{dM}{M},$$

where M_0 is the initial mass of the rocket and M is its mass at time t. Now, if $\mathbf{v}_0$ is the initial velocity, then we get

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}_{rel}\ \ln\frac{M}{M_0} - \mathbf{g}t. \qquad (2.33)$$

We shall illustrate Eq. 2.33 with the help of an example.

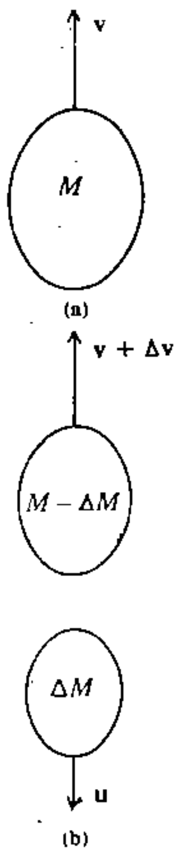


Fig 2.17

Example 7

The stages of a two-stage rocket separately have masses 100 kg and 10 kg and contain 800 kg and 90 kg of fuel, respectively. What is the final velocity that can be achieved with an exhaust velocity of 1.5 km s^{-1} relative to the rocket? (Neglect any effect of gravity).

Since we are neglecting gravity Eq. 2.33 reduces to

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}_{rel}\ \ln\frac{M}{M_0} \qquad (2.34)$$

Now, let the unit vector along the vertically upward direction be $\hat{\mathbf{n}}$. So, Eq. 2.34 can be written as

$$v\hat{\mathbf{n}} - v_0\hat{\mathbf{n}} = -(u_{rel}\,\hat{\mathbf{n}})\ \ln\frac{M}{M_0},$$ where $\mathbf{u}_{rel} = -u_{rel}\,\hat{\mathbf{n}}$, as the relative velocity of the exhaust points vertically downward.

$$\text{or}\quad v - v_0 = -u_{rel}\ln\frac{M}{M_0} \qquad (2.34a)$$

For our problem,

$u_{rel} = 1.5$ km s^{-1}.

For the first stage $v_0 = 0$

$M_0 = (800 + 90 + 100 + 10)$ kg = 1000 kg

$M = (90 + 10 + 100)$ kg = 200 kg, as the 800 kg fuel gets burnt in the first stage.

Hence, from Eq. 2.34a, we get

$$v = -(1.5\,\text{km s}^{-1})\left(\ln\frac{200}{1000}\right)$$

$$= (-1.5\text{ km s}^{-1})(\ln 2 - \ln 10)$$

$$= 1.5 \times 1.6\text{ km s}^{-1}$$

$$= 2.4\text{ km s}^{-1}.$$

Note that the above will be the initial velocity for the second stage. Also note that at the beginning of the second stage there occurs another drop in mass to the extent of the mass of the first stage (i.e. 100 kg).

For the second stage

$v_0 = 2.4$ km s^{-1}

$M_0 = (90 + 10)$ kg = 100 kg, $M = 10$ kg.

$$v = \left(2.4 - 1.5\ \ln\frac{10}{100}\right)\text{km s}^{-1}$$

$$= (2.4 + 1.5 \times 2.3)\text{ km s}^{-1} = 5.85\text{ km s}^{-1} = 5.8\text{ km s}^{-1}.$$

The final result of Example 7 has to be rounded off to two significant digits. Here we have a special case as the digit to be discarded is 5. By convention, we have rounded off to the nearest even number.

Let us now follow up this example with an SAQ.

SAQ 7

Find the final velocity of the rocket in Example 7 taking it to be single-stage, i.e. its mass is 100kg and it carries 890 kg of fuel. Hence comment whether the two-stage rocket has an advantage over single stage or not.

Let us now sum up what we have learnt in this unit.

2.4 SUMMARY

- Newton's first law states that "*Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.*"
- Newton's second law gives a relationship between force and linear momentum and can be expressed as

  $$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

  For a system of constant mass it becomes

  $$\mathbf{F} = m\mathbf{a}.$$

- Newton's third law states that "*To every action there is an equal and opposite reaction.*" Forces of action and reaction act on different bodies.
- A particle is said to be in equilibrium if the net force acting on it is zero. For coplanar forces, this implies that

  $$\Sigma F_x = 0,\ \ \Sigma F_y = 0.$$

- The total linear momentum of a system is conserved if no net external force acts on it.
- The impulse of a force on an object equals the change in its linear momentum and is given by

  $$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}(t)\,dt,$$ where the force acts during the time interval $\Delta t = (t_2 - t_1)$.

- For a variable mass system, Newton's second law of motion is expressed as

  $$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}.$$

- The increase in the velocity of a rocket within a time t of its take off is given by

  $$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}_{rel}\ \ln\frac{M}{M_0} - gt.$$

2.5 TERMINAL QUESTIONS

1. A block of mass 100 kg is placed on an inclined plane of height 6 m and base 8 m (Fig.2.18). The coefficients of static and kinetic frictions are 0.3 and 0.25, respectively. (a) Would the block slide down the plane? (b) What force parallel to the plane must be applied to just support the block on the plane? (c) What force parallel to the inclined plane is required to keep the block moving up the plane at constant speed? (d) If an upward force of 882 N parallel to the plane is applied to the block what will be its acceleration? (e) What will happen if an upward force of 490 N parallel to the plane is applied? (f) What will happen if an upward force of 254.8 N parallel to the plane is applied?

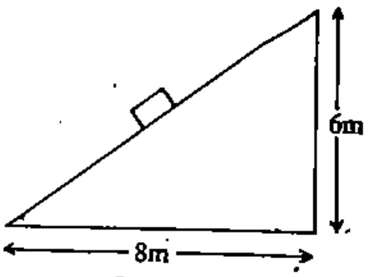


Fig. 2.18

2. A boy of mass 20 kg is standing on a flat boat of mass 30 kg so that he is 3 m from the shore (Fig. 2.19). He walks 1 m on the boat toward the shore and then halts. How far is he from the shore at the end of this time?

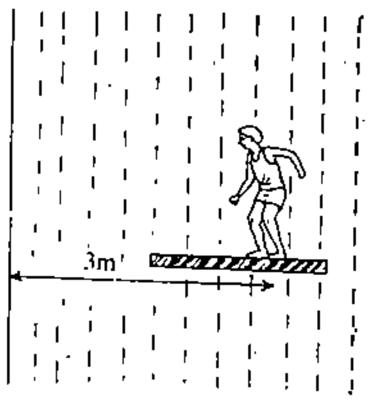


Fig. 2.19

3. Explain why it is less dangerous to fall on a mattress than on a hard floor.
4. A fire-fighter directs a stream of water against the door of a building in flames. The water is delivered by the hose at the rate of 45 kg s^{-1}. Water moving horizontally at 32 ms^{-1} hits the door. After hitting the door, the water drops vertically downward. What is the horizontal force exerted on the door?

2.6 ANSWERS

SAQs

1. Let the mass of the astronaut in kg be m. Then from the given conditions and Eq. 2.4 b, we get

$$[(15+m)\text{ kg}]\,(2.04\times10^{-2}\text{ms}^{-2}) = 2.07\text{ N}$$

$$\text{or } m = \left[\frac{2.07\text{N}}{(2.04\times10^{-2})\text{ms}^{-2}} - 15\right]\text{kg} = 86.5\text{ kg}$$

2. a) The reaction force acts on the man. Due to the large mass (inertia) of the man the force is not able to make him move.

   b) Apple also attracts the earth with a force of magnitude F. The acceleration of the apple and the earth are, respectively, F/m_a and F/m_E, where m_a and m_E are the masses of the apple and the earth, respectively. Since $m_E >> m_a$, $F/m_E << F/m_a$. Hence the earth does not move appreciably.

3. Refer to Fig. 2.20. Let the tension be **T**. The forces are resolved along the directions of x and y-axes. The former is along the floor and the latter is perpendicular to it. **N** is the normal reaction and correspondingly the magnitude of the force of kinetic friction F_k is equal to $\mu_k N$. It is in a direction opposite to the tendency of motion. Since there is no motion in the vertical direction, the resultant of the forces along the y-axis must be zero. Moreover, as the body moves with a uniform velocity, the resultant force along the x-axis is also zero. So we have

$$T\sin\theta + N - mg = 0 \qquad (2.35)$$

$$\mu_k N - T\cos\theta = 0. \qquad (2.36)$$

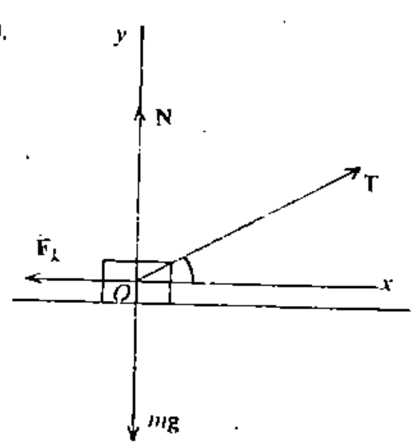


Fig 2.20

To find T we have to eliminate N between Eqs. 2.35 and 2.36. So, we have

$$T\left(\sin\theta + \frac{\cos\theta}{\mu_k}\right) = mg,$$

$$\text{or} \quad T = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

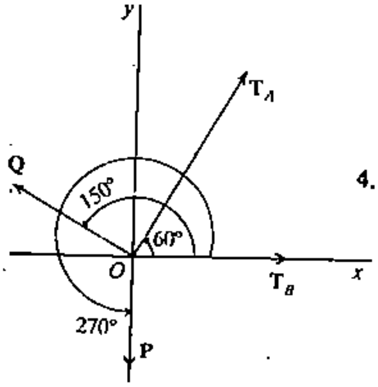


Fig 2.21

4. Refer to Fig. 2.21. We define a two-dimensional rectangular coordinate system with the common point of action of **P**, **Q**, $\mathbf{T}_A$, $\mathbf{T}_B$ as origin. We shall now apply Eq. 2.16c to obtain the condition of equilibrium.

Now, $\Sigma F_x = T_B \cos 0° + T_A \cos 60° + Q \cos 150° + P \cos 270°$

$$= T_B + \frac{T_A}{2} - \frac{\sqrt{3}Q}{2} \text{ and}$$

$$\Sigma F_y = T_B \sin 0° + T_A \sin 60° + Q \sin 150° + P \sin 270°$$

$$= \frac{\sqrt{3}}{2}T_A + \frac{Q}{2} - P.$$

Hence, from Eq. 2.16c, we get

$$T_B + \frac{T_A}{2} - \frac{\sqrt{3}}{2}Q = 0, \qquad (2.37a)$$

$$\frac{\sqrt{3}}{2}T_A + \frac{Q}{2} - P = 0. \qquad (2.37b)$$

From Eq. 2.37b, $T_A = \frac{2}{\sqrt{3}}\left(-\frac{Q}{2} + P\right) = \frac{2,000}{\sqrt{3}}\text{ N} = 1155\text{N}$

And from Eq. 2.37a, $T_B = \frac{\sqrt{3}}{2}Q - \frac{T_A}{2} = \frac{1}{2}(6928 - 1155)\text{N} = 2886\text{N}$

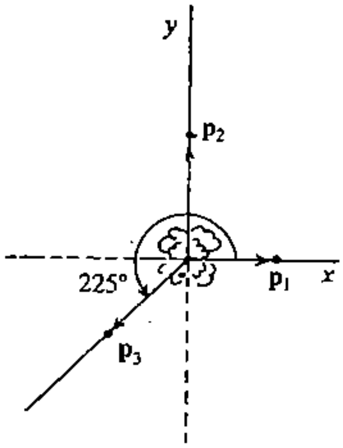


Fig. 2.22

5. As $p_1 = p_2 = mu$ and $p_3 = 3mv$, we get from Eq. 2.25b

$$\cos\theta = \sin\theta = -\frac{u}{3v}. \qquad (2.38)$$

$\therefore$ $\tan\theta = 1$. Again from Eq. 2.38 we understand that $\cos\theta$ and $\sin\theta$ are both negative as $\frac{u}{v}$ is positive. So θ must lie in the third quadrant. Hence, $\theta = 225°$. (see Fig.2.22).

6. a) $J = \Delta p = (0.25\text{ kg}) \times \{40 - (-20)\}\text{ m s}^{-1} = 15\text{ kg m s}^{-1}$

$$\Delta t = 10^{-2}\text{ s}, \bar{F} = \frac{J}{\Delta t} = 1500\text{ N}$$

b) The gravitational force of attraction between sun and earth is very weak but it has been acting since their formation and so it can generate a substantial impulse.

7. Had it been a single stage rocket, then

$$v_0 = 0$$

$$M_0 = (890 + 100)\text{ kg} = 990\text{ kg}. M = 100\text{kg}.$$

$$v = (-1.5\text{ km s}^{-1})\left[\ln\frac{100}{990}\right]$$

$$= (-1.5\text{ km s}^{-1})(\ln 10 - \ln 99)$$

$= 3.4\text{ km s}^{-1}$ which is 41% less than the value of velocity (5.8 km s^{-1}) attained in a double-stage rocket. Hence double-stage has an advantage over the single-stage.

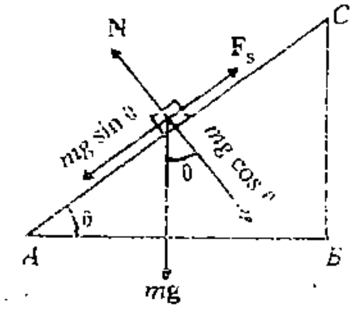


Fig. 2.23

Terminal Questions

1. Refer to Fig. 2.23. $BC = 6\text{m}$, $AB = 8\text{m}$. $AC = \sqrt{6^2 + 8^2}\text{ m} = 10\text{m}$, $\sin\theta = 0.6$, $\cos\theta = 0.8$.

a) We have resolved the forces along and perpendicular to the plane. Since there cannot be any motion perpendicular to the plane, we have $N = mg\cos\theta$, where m = mass of the block. The magnitude of the force of static friction

$$F_s = \mu_s N = \mu_s mg\cos\theta.$$

$$F_s = (0.3)(100\text{ kg})(9.8\text{ m s}^{-2})(0.8) = 235.2\text{ N, and}$$

$mg\sin\theta = (100\text{ kg})(9.8\text{ m s}^{-2})(0.6) = 588\text{ N}$. So $mg\sin\theta > F_s$ and hence the block will slide down the plane.

b) The required magnitude of force parallel to the plane = $mg \sin\theta - F_s$ = 352.8 N.

c) When the block is urged to move up the plane, the force of kinetic friction $\mathbf{F}_k$ of magnitude $\mu_k N$ comes into play and it acts down the plane. So the magnitude of total force down the plane will be ($\mu_k N + mg \sin\theta$). Now, in order that the block moves up at a constant speed, the magnitude of the applied force parallel to the plane must be equal to ($\mu_k N + mg \sin\theta$) = mg ($\mu_k \cos\theta + \sin\theta$)

= (100 kg) (9.8 m s^{-2}) (0.25 × 0.8 + 0.6) = 784 N.

d) The acceleration $= \dfrac{(882-784)\text{N}}{100\text{ kg}} = 0.98\text{ m s}^{-2}$

e) Since 490 < 784, we understand from (c) that the block will not move up. In this case the magnitude of the resultant force down the plane = (588 – 490)N = 98 N, which is less than the magnitude of the maximum force of static friction F_{sm} (=235.2 N). So the force of static friction will adjust itself equal to 98 N and the block remains at rest.

f) Now, the resultant force down the plane has a magnitude (588 – 254.8)N = 333.2 N. This exceeds F_{sm}. So the block will move down the plane. Its motion will be opposed by the force of kinetic friction $\mathbf{F}_k$. Now $F_k = \mu_k\, mg \cos\theta$

= (0.25) × (100 kg) × (9.8 m s^{-2}) × (0.8) = 196 N. So its acceleration down the plane

$= \dfrac{(333.2-196)\text{N}}{100\text{ kg}} = 1.4\text{ m s}^{-2}$

2. Let the masses of the boy and boat be m and M, respectively (Fig.2.24). Let the velocity of the boy relative to the boat be **v** and that of the boat with respect to the shore S be **u**. So the velocity of the boy with respect to the shore is $\mathbf{v} + \mathbf{u} = \mathbf{v}_r$, say. Now, before the boy started walking on the boat, the total linear momentum of the system (boy and boat) with respect to the shore was zero. The motion of the boat due to his walking arises out of the mutual forces of action and reaction. We shall neglect the forces of friction between the boy and boat as well as between the boat and the surface of water. So, no external force acts on the system. Hence, from the principle of conservation of linear momentum, we have

$$m(\mathbf{v}+\mathbf{u}) + M\mathbf{u} = \mathbf{0} \qquad (2.39)$$

It is evident that as the boy walks on the boat towards the shore the boat moves in the opposite direction. Let the unit vector along the direction of motion of the boy be $\hat{\mathbf{n}}$.

So $\mathbf{v} = v\hat{\mathbf{n}}$, $\mathbf{u} = -u\hat{\mathbf{n}}$. Hence, from Eq. 2.39, we get

$$m(v-u)\hat{\mathbf{n}} - Mu\hat{\mathbf{n}} = \mathbf{0}.$$

$$\therefore m(v-u) - Mu = 0, \text{ or } u = \frac{mv}{m+M}$$

or $\dfrac{m}{m+M} = \dfrac{u}{v} = \dfrac{s}{L}$, where s is the distance travelled by the boat and L is the length covered by the boy on the boat. In our problem, m = 20 kg, M = 30 kg, L = 1 m.

∴ s = 0.4 m. So after he halts, the boy is at a distance (3 – 1 + 0.4) m, i.e. 2.4 m from the shore.

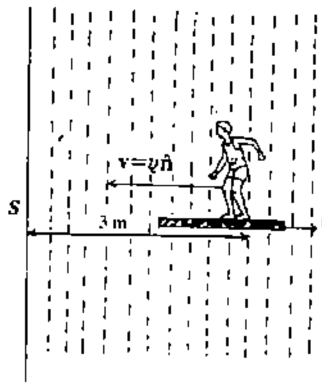


Fig. 2.24

3. In either case the person comes to rest finally and has the same velocity at the point of hitting the mattress or the floor. So the impulse, i.e. change of linear momentum is same. But the mattress being soft, the duration of impact is greater than that in the case of the hard floor. So, from Eq. 2.29 the average force is smaller in the former case. Hence, it is less dangerous to fall on a mattress than on a hard floor.

4. Water drops vertically downward after hitting the door. So the horizontal motion of water stops abruptly at the door. Since water is moving at 32 m s^{-1}, each kg of water loses 32 kg m s^{-1} of linear momentum. But water strikes the door at the rate of 45 kg s^{-1}. So the rate of loss of linear momentum is equal to (45 kg s^{-1}) × (32 kg m s^{-1} kg^{-1}) = 1440 kg m s^{-2} = 1440 N, which is the horizontal force on the door.

UNIT 3 WORK AND ENERGY

Structure

3.1 INTRODUCTION

In the previous unit we have studied the causes behind change of motion. We have also studied the important law of conservation of linear momentum and its applications. In our everyday experience we often feel that when we execute a motion some energy is expended. Some times we say that work is done at the expense of some energy. However, the word 'work' has a special meaning in physics. For instance, if a lecturer stands near a table and delivers a lecture for one hour, then no work is done according to the principles of physics. In this unit you will learn about the work done by various forces and also different kinds of energies. We will go into the details of the very important principle of conservation of energy. This principle has very wide applications and will be used very often in your physics courses. In the next unit we will apply some of the concepts of motion developed in the first three units to the study of angular motion.

Objectives

After studying this unit you should be able to:

- compute work of constant and variable forces
- apply work-energy theorem
- distinguish between conservative and non-conservative forces
- solve problems based on the principle of conservation of energy
- interpret energy diagrams
- solve problems based on elastic and inelastic collisions
- compute power in mechanical systems.

3.2 WORK

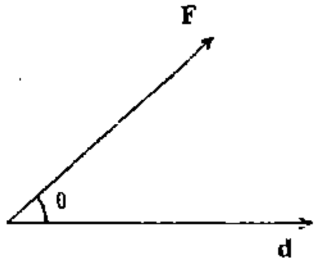


Fig. 3.1

You have studied in Unit 1 that the work done by a force **F** during a displacement **d** of its point of application is given according to Eq.1.11b as **F.d**. So if there is an angle θ between **F** and **d** as shown in Fig. 3.1 then work done is $F\cos\theta d$. The unit of work is newton-metre which is named as joule. If θ is acute ($\cos\theta$ is positive), then work is said to be done by the force and if θ is obtuse ($\cos\theta$ is negative), then work is said to be done **against** the force. If $\theta = 90°$ ($\cos\theta = 0$), then it is a **no-work** force. For example, when a man walks on the ground, the reaction force experienced by him is always normal to his displacement. Hence, reaction is a no-work force.

SAQ 1

Give an example (other than the above) of a no-work force

Now, for a small displacement $\Delta\mathbf{l}$ the work done by a force $\mathbf{F}$ is given by

$$W = \mathbf{F} \cdot \Delta\mathbf{l}. \tag{3.1}$$

3.2.1 Work Done by a Constant Force

Let a particle undergo a succession of displacements $\Delta\mathbf{l}_1, \Delta\mathbf{l}_2, \ldots, \Delta\mathbf{l}_n$ under the action of a constant force. Then the net displacement is $\Delta\mathbf{l} = \Delta\mathbf{l}_1 + \Delta\mathbf{l}_2 + \ldots + \Delta\mathbf{l}_n$ (see Fig. 3.2), and the work done is

$$W = \mathbf{F} \cdot \Delta\mathbf{l} = \mathbf{F} \cdot (\Delta\mathbf{l}_1 + \Delta\mathbf{l}_2 + \ldots + \Delta\mathbf{l}_n)$$

We can use the distributive law of scalar products (Eq. 1.14) and write

$$W = \mathbf{F} \cdot \Delta\mathbf{l}_1 + \mathbf{F} \cdot \Delta\mathbf{l}_2 + \ldots + \mathbf{F} \cdot \Delta\mathbf{l}_n. \tag{3.2}$$

Thus, the work done by a constant force for a succession of displacements is the sum of the work done by that force for individual displacements.

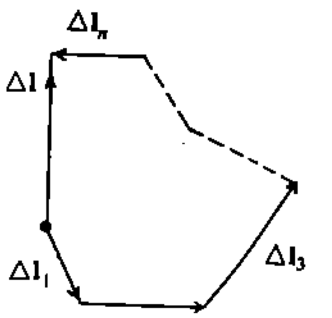


Fig. 3.2

However, in nature we come across many forces that vary with position. For example, let a unit positive charge be taken from point A to point B in the electrostatic field of a charge $+q$ (Fig. 3.3). q is located at the origin of a two-dimensional rectangular Cartesian coordinate system having x-and y-axes. The force experienced by a unit positive charge when placed at P is given by

$$\mathbf{F} = \frac{kq}{r^2}\hat{\mathbf{r}}, \tag{3.3}$$

where k is a constant dependent on the nature of the intervening medium. $\mathbf{OP} = \mathbf{r}$ and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. As the unit positive charge moves, the magnitude as well as the direction of $\mathbf{r}$ change. So $\mathbf{F}$ is a force which varies with position. How do we calculate the work done for such forces?

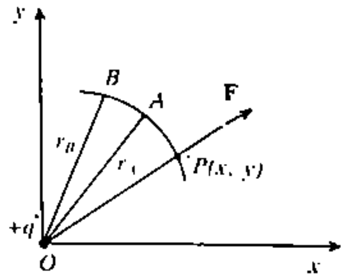


Fig. 3.3

3.2.2 Work Done by a Variable Force

A force of the type given by Eq. 3.3 can be expressed in general as

$$\mathbf{F} = \mathbf{F}(\mathbf{r}). \tag{3.4}$$

Let us now calculate the work done when a particle moves under the influence of this force from point A to B (Fig. 3.4).

The section of the path from A to B can be approximated by a zigzag polygon consisting of successive displacements $\Delta\mathbf{l}_1, \Delta\mathbf{l}_2, \ldots \Delta\mathbf{l}_n$, where the path AB is shown exaggerated. We can take each successive displacement $\Delta\mathbf{l}_i$ to be very small. Then $\mathbf{r}_i$ corresponding to each $\Delta\mathbf{l}_i$ will be effectively constant, so that $\mathbf{F}(\mathbf{r}_i)$ is also constant over that displacement. The work done for this displacement is given from Eq. 3.1 as $W_i = \mathbf{F}(\mathbf{r}_i).\ \Delta\mathbf{l}_i$.

Since work is a scalar quantity, the work done in going from A to B is the sum of the work done for each successive displacement, i.e.

$$W = \mathbf{F}(\mathbf{r}_1) \cdot \Delta\mathbf{l}_1 + \mathbf{F}(\mathbf{r}_2) \cdot \Delta\mathbf{l}_2 + \cdots + \mathbf{F}(\mathbf{r}_n) \cdot \Delta\mathbf{l}_n$$
$$= \sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{l}_i. \tag{3.5}$$

where the symbol Σ stands for the above summation.

We should expect to get a more significant value for zigzag paths if we fit the curve more closely. Therefore, it is natural to define work in the general case as the limit of such approximations as the lengths of the $\Delta\mathbf{l}_i$'s are made smaller and smaller, i.e.

$$W = \lim_{\Delta l_i \to 0}\left[\sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{l}_i\right] \tag{3.6}$$

From the concept of definite integration, the above limit is given by

$$W = \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{l}, \tag{3.7}$$

where A and B denote the initial and the final positions. This integral is called the **line integral** of $\mathbf{F}$ from A to B. So the integral of force with respect to the position variable over a certain path is the work done by the force over that path, provided the force depends on the position variable only.

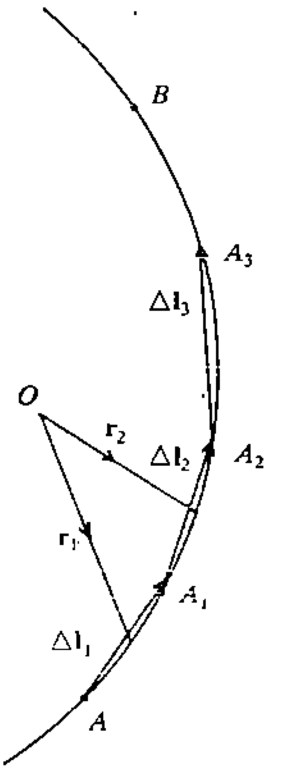


Fig. 3.4

For a two-dimensional system $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$ and $d\mathbf{l} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}$

$$\mathbf{F}.d\mathbf{l} = (F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}).(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}),$$

or

$$\mathbf{F}.d\mathbf{l} = F_x dx + F_y dy \tag{3.8}$$

Let us now evaluate the work done by a variable force.

Example 1

Let us determine W along the path AB for the electrostatic force **F** given by Eq. 3.3.

$$\mathbf{F} = \frac{kq}{r^2}\hat{\mathbf{r}} = \frac{kq}{r^2}\frac{\mathbf{r}}{r} \quad \left(\because \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}\right)$$

$$\text{or } \mathbf{F} = \frac{kq}{r^3}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}),$$

$$\text{or } F_x = \frac{kqx}{r^3},\ F_y = \frac{kqy}{r^3}.$$

Hence, from Eq. 3.8, we get

$$\mathbf{F}.d\mathbf{l} = \frac{kq}{r^3}(xdx + ydy)$$

$$= \frac{kq}{r^3}\left\{d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right)\right\}$$

$$= \frac{k}{2}\frac{q}{r^3}d(x^2 + y^2)$$

$$= \frac{kq}{2r^3}d(r^2) = \frac{kq}{2r^3}2r\,dr = \frac{kq}{r^2}dr.$$

From Eq. 3.7, we get

$$W = \int_{r_A}^{r_B} \frac{kq}{r^2}dr = kq\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \tag{3.9}$$

You can now try the following SAQ based on the ideas discussed so far.

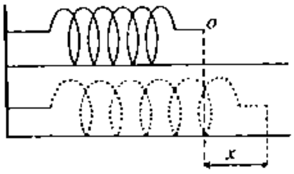


Fig. 3.5

SAQ 2

Suppose the equilibrium position of an end of a spring is O, (Fig. 3.5) and it is stretched through a length x. Due to elasticity a restoring force comes into play, which is proportional to the displacement x from the equilibrium position, i.e.

$$\mathbf{F} = -k_o\,\mathbf{x}.$$

k_o is a constant and the negative sign appears because the restoring force is directed opposite to the displacement. How much work is done in stretching the spring from the position $x = x_1$ to $x = x_2$?

We have thus discussed the meaning of the line integral of a force when it is a function of the position variable. We shall seek a more general meaning of the line integral of a force in the next section.

3.3 ENERGY

We have defined work. The capacity of a body to do work is called its **energy** and is always measured by the work the body is capable of doing. So the unit of energy is the same as that of work, i.e. joule. In nature, energy manifests itself in different forms — mechanical, heat, electrical, chemical, sound, light, etc. You have read about energy in general in FST-1, Block-4 (Sec. 17.3). In this unit we shall concentrate on mechanical energy. It can be of two kinds — kinetic energy and potential energy.

3.3.1 Kinetic Energy and Work-Energy Theorem

Kinetic energy (K.E.) is possessed by a body by virtue of its motion. For example, a moving car or a ball in motion has kinetic energy. We shall arrive at a quantitative measure of kinetic energy by applying Newton's second law of motion to Eq. 3.7. In the next few steps we shall be doing some algebra. It is necessary for obtaining the result which is physically very significant.

From Newton's second law, for a particle of mass m having a velocity v at a time t we have

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}),$$

or $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$, for a system of constant mass.

$$\text{Again } \mathbf{v} = \frac{d\mathbf{l}}{dt}$$

$$\text{and } \quad d\mathbf{l} = \frac{d\mathbf{l}}{dt}dt = \mathbf{v}\,dt$$

$$\mathbf{F}\cdot d\mathbf{l} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dt. \tag{3.10}$$

We shall now apply some algebra to simplify Eq. 3.10.

$$\text{Now, } \mathbf{v}\cdot\mathbf{v} = (v_x^2 + v_y^2 + v_z^2)$$

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d}{dt}(v_x^2 + v_y^2 + v_z^2)$$

$$= \frac{d}{dt}(v_x^2) + \frac{d}{dt}(v_y^2) + \frac{d}{dt}(v_z^2)$$

$$\frac{d}{dt}(v_x^2) = \frac{d}{dv_x}(v_x^2)\frac{dv_x}{dt} = 2v_x\frac{dv_x}{dt}$$

$$\text{Thus } \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt} + 2v_z\frac{dv_z}{dt}$$

$$= 2(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}})\cdot\left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right), \text{ from Eq. 1.15a.}$$

$$\therefore \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}.$$

Since dot product is commutative, we get

$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{d}{dt}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right). \tag{3.11}$$

Thus, from Eqs. 3.10 and 3.11, we get

$$\mathbf{F}\cdot d\mathbf{l} = m\frac{d}{dt}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)dt = m\,d\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right) = d\left(\frac{m}{2}\mathbf{v}\cdot\mathbf{v}\right).$$

$$\therefore \int_A^B \mathbf{F}\cdot d\mathbf{l} = \left|\tfrac{1}{2}m\,\mathbf{v}\cdot\mathbf{v}\right|_A^B \tag{3.12}$$

where A and B indicate the limits of position between which the definite integral has to be worked out. Since $\mathbf{v}.\mathbf{v} = v^2$, from Eq. 3.12, we get

$$\int_A^B \mathbf{F}\cdot d\mathbf{l} = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2. \tag{3.13}$$

We now want to interpret Eq.3.13. For the sake of simplicity let us consider $v_A = 0$. Then the right-hand of Eq. 3.13 becomes $^1/_2mv_B^2$. This in accordance with Eq. 3.7 represents the work done on the particle in its attaining a velocity v_B from rest. Thus, this work will be a measure of the energy the particle has acquired by virtue of its motion. So *$^1/_2mv^2$ is a measure of K.E. of a particle of mass m moving with a velocity* **v**. Now go back and look at the few steps worked out before Eq. 3.10. You will understand that the above analysis has been done for a system of constant mass. But $^1/_2mv^2$ is taken to be the measure of K.E. *also for systems having variable mass*. You have read about them in Sec. 2.3.3. Hence, the right-hand side of Eq. 3.13 represents the change in K.E. of the particle between the positions A and B and is expressed as

$$\int_A^B \mathbf{F}\cdot d\mathbf{l} = T_B - T_A. \tag{3.13a}$$

Thus, we have arrived at the general meaning of the line integral of a force which can be stated as follows:

The line integral of a force between two positions is equal to the change in K.E. of the particle in coming from the initial to the final position. Moreover, if **F** is a function of position, the line-integral of the force is equal to the work done by the force between these points. So we can now state the **work-energy theorem.**

The work done on a particle by the resultant force acting on it is always equal to the change in K.E. of the particle.

Let us supplement the above statement with a very simple example.

Example 2

A body of mass 1 kg and initial velocity 10 m s^{-1} is sliding on a horizontal surface. If the coefficient of kinetic friction between the body and the surface is 0.5, then find the

a) work done by friction when the body has traversed a distance of 5m along the surface,

b) the initial and the final kinetic energies of the body.

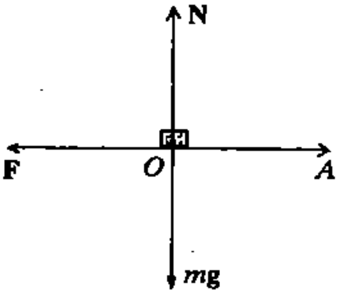


Fig. 3.6

(a) Refer to Fig. 3.6. Let the mass of the body be m. As we have seen in Sec. 2.2.2, the magnitude of the normal reaction $N = mg$ and that of the force of friction $= \mu_k N = \mu_k mg$. Let the displacement of the body be **d** in the direction OA. We shall assume that the force of friction is constant over that displacement. So Eq. 3.7 reduces to

$$W = \mathbf{F}.\mathbf{d}$$

Here **F**, being the force of kinetic friction, is opposite to **d**.

$$W = -Fd = -\mu_k mgd$$

$$W = -(0.5) \times (1\text{kg}) \times (9.8\text{m s}^{-2}) \times (5\text{m}) = -24.5\text{J}$$

b) Initial K.E. $= \frac{1}{2} mv^2 = \frac{1}{2} \times (1\text{kg}) \times (10 \text{ ms}^{-1})^2 = 50\text{J}$

We know from the work-energy theorem that

Work done = Final K.E. – Initial K.E.

$\therefore$ Final K.E. = Initial K.E. + W = 50J + (–24.5J) = 25.5J

SAQ 3

A truck and a car having equal K.E.s are travelling along a straight road. Equal braking forces are applied on them. Which one will travel farther before stopping?

We should realise that the work-energy the orem is not a new law. It is simply a relation between work and K.E. derived from Newton's second law of motion. In Example 2 and SAQ 3 we have used work-energy theorem. But for evaluating the left-hand side of Eq. 3.13 in these problems we have not effectively performed any integration. In general cases, we shall need to work out that integral for which the path of the particle must be known. If the path has a complicated geometry then it is not easy to determine W. However, there is a special kind of force for which W can be determined without the knowledge of the path of the particle. Only the initial and final positions need be known. Let us now discuss about this special kind of force, known as the conservative force. Through this discussion we shall also arrive at the concept of potential energy.

3.2.2 Conservative Force and Potential Energy

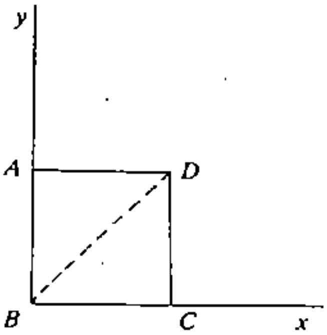


Fig. 3.7

Refer to Fig. 3.7. Let us consider four points A, B, C, D which are the vertices of a square of side L on a smooth vertical wall. A particle of mass m has to be taken from A to B in two ways

a) directly along the straight line AB,

b) along the path $ADCB$.

We shall calculate the work done by the force of gravity in these two cases. For this we shall take a two-dimensional rectangular Cartesian coordinate system with x and y-axes along BC and BA, respectively. As we shall use Eq. 3.7 let us write down the expression for **F** explicitly :

$$\mathbf{F} = -mg\,\hat{\mathbf{j}} \tag{3.14}$$

For case (a), the work done is given by

$$W_a = W_{AB} = \int_A^B \mathbf{F}\cdot d\mathbf{l} = \int_A^B -mg\hat{\mathbf{j}}\cdot(dy\hat{\mathbf{j}}) = \int_A^B mg\,dy$$

$$\text{or } W_{AB} = mg\,(y_B - y_A) = -mgL. \tag{3.15a}$$

For case (b), the work done is given by

$$W_b = W_{ADCB} = W_{AD} + W_{DC} + W_{CB}$$

$$= \int_A^D \mathbf{F}\cdot d\mathbf{l} + \int_D^C \mathbf{F}\cdot d\mathbf{l} + \int_C^B \mathbf{F}\cdot d\mathbf{l}$$

$$= \int_A^D -mg\hat{\mathbf{j}}\cdot dx\hat{\mathbf{i}} + \int_D^C -mg\hat{\mathbf{j}}\cdot(-dy\hat{\mathbf{j}}) + \int_C^B -mg\hat{\mathbf{j}}\cdot(-dx\hat{\mathbf{i}})$$

$$= 0 + \int_D^C mg\,dy + 0 = mg(y_C - y_D) = -mgL. \qquad (3.15b)$$

So $W_a = W_b$. In other words, the work done in taking the particle from A to B along two different paths is the same, i.e. the work done is independent of the path followed. Such a force is called a **conservative force.** It is defined as that force for which *the work done is independent of the path followed and depends only on the initial and final positions of the particle*. Like the force of gravity, electrostatic force is also conservative. You can nc work out an SAQ.

SAQ 4

a) Verify for Fig. 3.7 that $W_{AB} = W_{ADB}$

b) Prove that for a conservative force the work done around a closed path is zero.

There are forces for which the work done depends on the path followed, called **non-conservative forces**. Thus, for non-conservative forces, the work done around a closed path is non-zero. For example, friction is a non-conservative force.

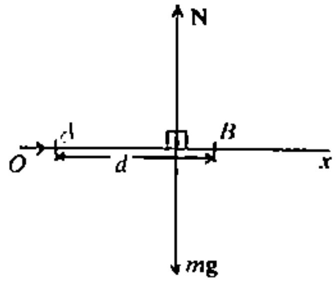


Fig. 3.8: When the motion is from A to B, the force of friction is along BA, i.e. opposite to $\hat{i}$ and when the motion is from B to A, the force of friction is along $\hat{i}$.

Refer to Fig. 3.8. Let us consider the motion of a particle over a fixed horizontal distance d, from A to B and back. This is a closed path. How much work is done by the frictional force acting over this path? The force of friction has magnitude $\mu_k N (=\mu_k mg)$ and is opposed to the direction of motion. Let us choose x-axis to be along the direction of motion. Therefore,

$$W_{AB} = \int_A^B \mathbf{F} \cdot d\mathbf{l} = \int_A^B (-\mu_k m_g)\hat{\mathbf{i}} \cdot dx\hat{\mathbf{i}} = -\mu_k mgd.$$

and $$W_{BA} = \int_B^A (\mu_k mg)\hat{\mathbf{i}} \cdot (-dx\hat{\mathbf{i}}) = -\mu_k mgd.$$

So, the total work done around the closed path $ABA = -2\mu_k mgd$. It is not zero.

Let us now go back to the discussion of conservative forces. In fact, before coming to this section, we have dealt with three conservative forces. These were the force of gravity, the electrostatic force between two charges (Example 1) and that of the spring-mass system (SAQ 2). In the first case, the work done in taking a particle of mass m from A to B can be given from Eq. 3.15a by

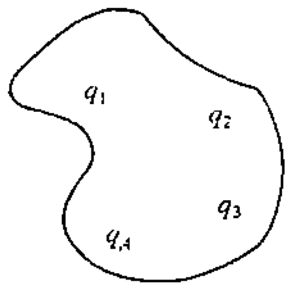
$$W = -(U_B - U_A), \text{ where } U = mgy. \qquad (3.16a)$$

Again from the results of Example 1 and SAQ 2, we get

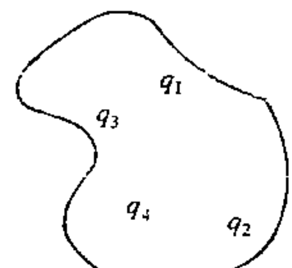
$$W = -(U_B - U_A), \text{ where } U = \frac{kq}{r}, \qquad (3.16b)$$

and $$W = -(U_2 - U_1), \text{ where } U = \tfrac{1}{2}k_o x^2 \qquad (3.16c)$$

We can see the similarities between the Eqs. 3.16a, 3.16b and 3.16c. In each case we are able to associate a quantity U, the negative of whose change gives the work done. In the first case U is dependent on the position y of the particle and in the second, on the position r of the unit positive charge. In the third case U depends on x. This is a variable which gives the displacement of the free end of the spring from its unstretched position. Thus, the value of x is indicative of the extent of stretching of the spring. So instead of saying that x is the displacement of the free end of the spring from its normal position we say that x measures the configuration of the system. Change in the value of x is a change in configuration of the spring. Similarly, if we have a system of charges (q_1, q_2, q_3, q_4) in an enclosure (Fig. 3.9a) then a change in their relative positions (Fig. 3.9b) also amounts to changing the configuration of the system. Thus, the work done by a force $\mathbf{F}$ in taking a system from A to B can be expressed as



(a)



(b)

Fig. 3.9

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{l} = -(U_B - U_A). \qquad (3.16d)$$

where U is a quantity depending on the configuration of the system. U is called the **potential energy** (or P.E.) of the system. It is defined as the energy a body possesses by virtue of its configuration. In order to measure P.E. we need to know the conservative force which gives rise to this P.E. We shall now see how P.E. is measured. From Eq. 3.16d, we get that if $U_A = 0$, then

$$U_B = -\int_A^B \mathbf{F} \cdot d\mathbf{l} = \int_B^A \mathbf{F} \cdot d\mathbf{l}. \qquad (3.16e)$$

Thus, the P.E. of a system in a certain configuration is measured by the work done by the concerned force in taking the system from that configuration to some standard configuration. Eq. 3.16e gives the P.E. at B with respect to the standard A. We have chosen A as the standard by putting $U_A = 0$. Let us now quickly go back to Eq. 3.15a. There we have $W_{AB} = -mgL$. Or the work done in taking the particle from B to A is mgL. So considering A as standard, the P.E. of the particle at B is mgL. You may now work out an SAQ to clarify the concepts about P.E.

SAQ 5

A peculiar spring is governed by a force law: $\mathbf{F} = -Cx^3\hat{\mathbf{i}}$ where C is a constant. What is the P.E. at x with the standard $U = 0$ at $x = 0$?

So, measurement of P.E. is never absolute. It is always determined with respect to some standard. You have also learnt that the knowledge of the corresponding force is essential for determining the P.E. Now let us try and see whether we can determine the concerned force or not if the P.E. is known.

An infinitesimal change in the value of P.E. is given in accordance with Eq. 3.16d as

$$dU = -\mathbf{F}.d\mathbf{l}. \tag{3.17}$$

Hence, for a simple case of one-dimensional force like in the case of spring, we have $dU = -Fdx$. Thus,

$$F = -\frac{dU}{dx}. \tag{3.18}$$

Eq. 3.18 indicates that *the conservative force is the negative of the rate of change of P.E. with respect to the position variable*. Let us take up an interesting application of Eq. 3.18.

Eq.3.18 indicates the connection between P.E. and equilibrium. If the total force acting on a body is zero, then it is in equilibrium. For a conservative force, this equilibrium means

$$\frac{dU}{dx} = 0. \tag{3.19}$$

This can occur in three-ways:

i) U is a minimum

ii) U is a maximum

iii) U is a constant independent of x.

At this stage let us recall the very simple fact that a ball falls if it is released. It does so due to gravity. But have you observed that in the process of falling, the P.E. of the ball decreases? Again if you stretch a spring and release it subsequently, it soon returns to its normal length. Thus, its P.E. also decreases. In fact, in nature all processes proceed to that configuration for which the P.E. of the concerned system gets minimised.

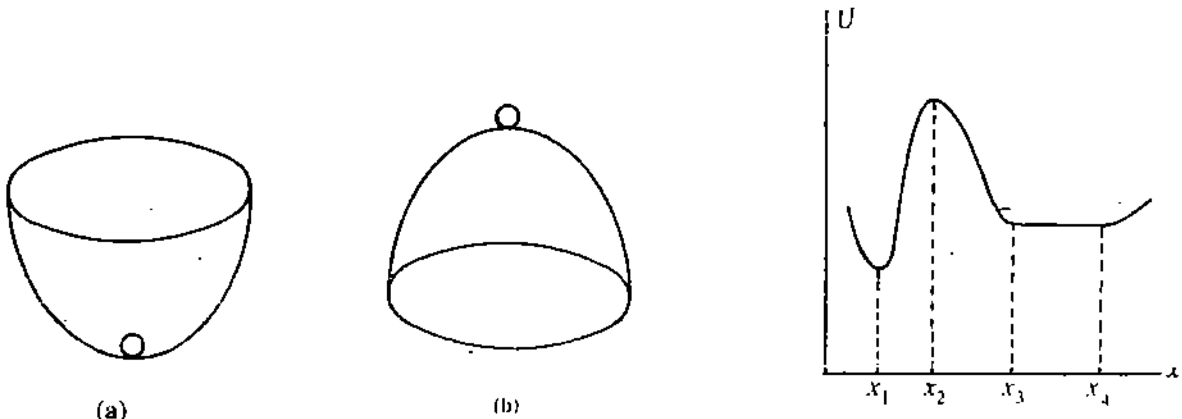


Fig. 3.10

Now, let us consider the situation in case (i) above. If this kind of an equilibrium is disturbed, the system tends to regain its equilibrium configuration. So it is called **stable equilibrium**. A ball resting at the bottom of a bowl (Fig. 3.10 a) provides the example of stable equilibrium. If an equilibrium of type (ii) is disturbed, the system does not return to its equilibrium configuration as a process cannot take a system to a configuration for which its

P.E. increases. So it is called **unstable equilibrium**. A ball resting at the top of a bowl (Fig. 3.10b) provides an example of unstable equilibrium. Case (iii) refers to a situation where the system will continue to remain at equilibrium even if its equilibrium is disturbed. This is called **neutral equilibrium.** A book resting on your table provides an example of neutral equilibrium. If you move it by giving a slight push it occupies another position on your table and remains in equilibrium there without bothering to come back to the previous position. Thus, if the variation of P.E. of a system is plotted with position variable x (as in Fig. 3.10c), then the system is in stable and unstable equilibrium at $x = x_1$ and $x = x_2$, respectively. It is in neutral equilibrium over the range $x_3 < x < x_4$.

SAQ 6

What kind of equilibria are the following?

a) A simple pendulum bob at its mean position.

b) A stick held vertically on the fingertip by a juggler.

So, we have seen that if the P.E. of a conservative system is known as a function of position then the corresponding force can be derived from it. We have also seen the role of P.E. in determining the nature of equilibrium of a body.

We shall now combine Eq. 3.16c with the work-energy theorem and arrive at the very important principle of conservation of energy.

3.3.3 Principle of Conservation of Energy

From Eqs. 3.13a and 3.16d, we get that

$$W = -(U_B - U_A) = T_B - T_A,$$

$$\text{or } T_A + U_A = T_B + U_B. \tag{3.20}$$

Since the points A and B are arbitrary, we conclude that for a system being acted upon by a conservative force, the sum of the kinetic energy and the potential energy is always a constant. We denote this constant as the total mechanical energy E of the system, i.e.

$$T + U = E = \text{a constant}. \tag{3.21}$$

Let us take up an example to illustrate Eq. 3.21.

Example 3

We had discussed projectile motion in Unit 2. Prove that in the absence of air-resistance, the sum total of K.E. and P.E. remains constant.

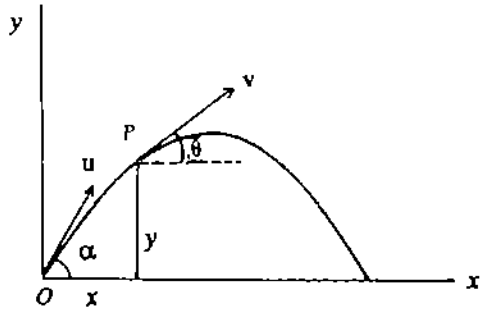


Fig. 3.11: Projectile Motion

Refer to Fig. 3.11. Since the magnitude of the velocity of projection is u, the K.E. of the projectile at O is $\frac{1}{2}mu^2$. Let us take the horizontal level through O as the reference for determining P.E. So, P.E. at O is zero.

$$\text{Hence at } O,\ \text{K.E.} + \text{P.E.} = \frac{1}{2} mu^2 \tag{3.22}$$

At the point $P(x, y)$, the velocity vector makes an angle θ with the horizontal direction. There is no acceleration in the horizontal direction.

Hence, the horizontal component of velocity remains unchanged and the vertical component experiences an acceleration **g** downwards. So

$$v \cos \theta = u \cos \alpha, \tag{3.23a}$$

$$v^2 \sin^2 \theta = u^2 \sin^2 \alpha - 2gy. \tag{3.23b}$$

From Eq. 3.23a, we get

$$v^2 \cos^2 \theta = u^2 \cos^2 \alpha. \tag{3.23c}$$

Adding Eqs. 3.23b and 3.23c, we have

$$v^2 = u^2 - 2gy. \qquad (3.23d)$$

Now, K.E. at $P = \frac{1}{2}mv^2 = \frac{1}{2}\ m\,(u^2 - 2gy)\ =\ \frac{1}{2}mu^2 - mgy.$ (3.24a)

And P.E. at $P = mgy.$ (3.24b)

Hence, K.E. + P.E. $= \frac{1}{2}\ mu^2$, at P. (3.25)

You must have noted that Eq. 3.23d appears to be the same as the equation we get in studying linear motion under gravity. But there is a difference. In linear motion, v and u are along the same direction, while in this case they are not. So you should not feel that proving this equation is a futile exercise.

Thus, from Eqs. 3.22 and 3.25, we get that the sum-total of K.E. and P.E. remains constant.

At this stage let us go back to Example 2. There we had seen that the final K.E. is less than the initial K.E., whereas the initial and final P.E.s are the same, as the body was moving on a horizontal surface. So we cannot say that (K.E. + P.E.) is a constant. This means that Eq. 3.21 does not hold. Now let us try to find out what has happened to the loss in K.E. You must have noted that there is a non-conservative force in the form of friction and the work done by that force is −24.5 J. The negative sign indicates that the work done against the force of friction is 24.5 J which is exactly equal to the loss in K.E. Now, is the work done against the force of friction lost? Well, the answer is that it is not lost. It has been dissipated in the form of heat energy. This energy which amounts to heating the body and the surface is not useful at all. But still the energy is *not lost* in the true sense of the term. Through Example 2 we may conclude that

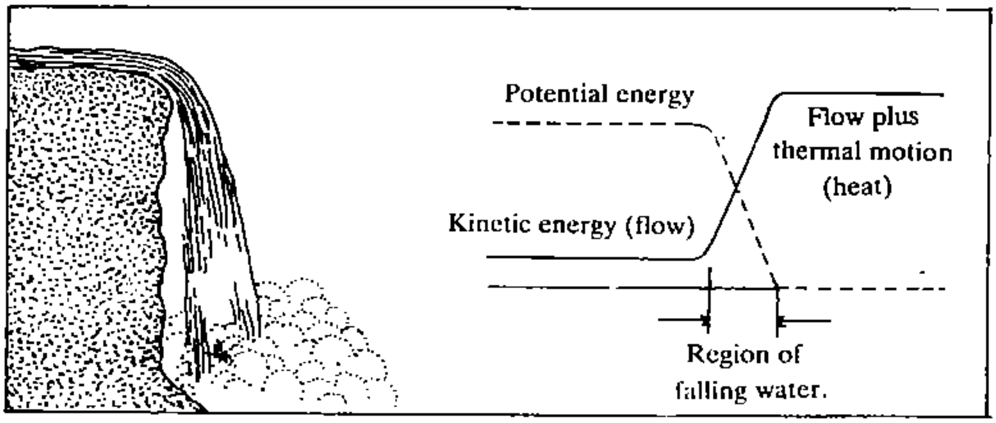
K.E. + P.E. + (work done against Friction) = a constant

or (Total mechanical energy) + (Energy dissipated) = a constant.

Thus, we arrive at the statement of the *principle of conservation of energy:*

> *Energy cannot be created nor can it be destroyed but it can be transformed from one form to another, the total amount of energy in the universe remaining constant.*

Let us take up an application of this principle.



(a) (b)

Fig. 3.12

The water at the top of the waterfall (Fig. 3.12 a) has gravitational P.E. which in falling is converted into K.E. So, P.E. decreases and at its cost the K.E. increases, maintaining the sum of K.E. and P.E. constant. Here we assume that the water particles do not experience any friction with the bed of the fall. However, on coming to the foot of the fall the P.E. becomes zero. The energy is solely kinetic. But what happens to this K.E. later? A part of K.E. remains with the water as it continues to flow and the rest can be used to drive the motor of a turbine and hydroelectricity can be generated. In any case some energy is always dissipated. The variation of different forms of energy is depicted in the graph in Fig. 3.12 b.

Whenever there is a conversion of energy from one form to another this principle is applicable. For example, in a lead acid cell chemical energy is converted to electrical energy whereas in a solar cell light energy is converted to electrical energy. The quantity of energy in one form is always equivalent to the quantity in the converted form.

In nature we always come across non-conservative forces. For example, in the case of a body falling freely under gravity there is some air resistance. When a body is sliding along an inclined plane there is friction. But if these forces are negligible then the principle of conservation of energy can be used in the form of Eq. 3.21 as a good approximation. We shall now see such an application.

3.3.4 Energy Diagrams

Fig. 3.13 shows the track of a car having negligible friction. How fast must the car be moving at point *P* if it is to reach point *R*? What happens if it is going slower than this?

As the friction is negligible the principle of conservation of energy will provide answers to our questions. Let us consider the lowest point on the track as our reference for determining P.E. The energy at *P* is ($\frac{1}{2} mv_P^2 + mgh_P$), and this is the total mechanical energy everywhere, as mechanical energy is conserved. To reach the point *S*, the car must clear the highest peak at *R*, where P.E. is mgh_R. If it is just able to do so, its K.E. is extremely close to zero on the peak. From the principle of conservation of energy we have

$$mgh_R = \frac{1}{2} mv_P^2 + mgh_P,$$

$$\text{or} \quad \frac{1}{2} mv_P^2 = mg(h_R - h_P). \tag{3.26}$$

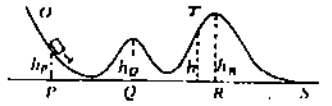


Fig. 3.13

In other words, the initial K.E. must be at least equal to the difference in P.E. between the highest and the initial point. Eq. 3.26 may be solved for v_P to get the minimum speed required at *P* to get to *R*. What happens if the car is moving a little slower? Then it would not be able to reach the top of the second peak but will reverse direction before that from a point *T* where its K.E. becomes zero. If it occurs at a height *h*, then

$$mgh = \frac{1}{2} mv_P^2 + mgh_P. \tag{3.27}$$

It will head back clearing peak *Q*, then down and up past point *P* to a point *O* where its K.E is again zero. You can now quickly work out an SAQ.

SAQ 7

What is the height of the point *O* above the reference level in Fig. 3.13?

O and *T* are called the **turning points** set by the value of total energy. With still lower speed, the car won't clear peak *Q* and its motion will be confined to the first valley alone.

Fig. 3.14 is a drawing of the actual track followed by the car. But, because gravitational potential energy near earth's surface is directly proportional to height, we can also regard it as a plot of potential energy versus position: a **potential energy curve**. We can understand the car's motion graphically by plotting the car's total energy on the same graph as the potential energy curve. Since total energy *E* is constant, the total energy curve is a straight, horizontal line. Fig. 3.14 shows the potential energy curve and the total energy curve for several values of the total energy. These graphs tell us immediately about the motion of the car. In Fig 3.14a, the total energy E_a exceeds the potential energy at peak *R*, therefore, the car will reach *R* with kinetic energy to spare, and will make it all the way to *S*. In Fig 3.14b the total energy E_b is less than the potential energy at *R*. The car must stop when its total energy is entirely potential. This happens when the total energy curve intersects the potential energy curve; the points of intersection are the turning points that bound the car's motion. In Fig. 3.14c, the total energy E_c is still lower, and the turning points are closer together. We say that the car is trapped in the **potential well** between its turning points.

We know that for a conservative system K.E. + P.E. = *E*, the total energy. i.e.

$$\frac{1}{2} mv^2 + U = E,$$

$$\text{or } v = \sqrt{\frac{2}{m}(E - U)}. \tag{3.28}$$

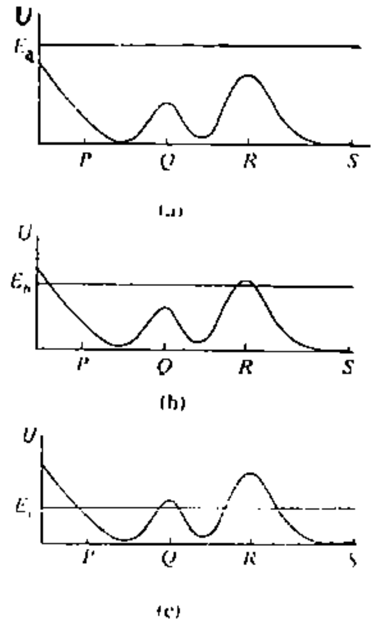


Fig. 3.14

If *U* > *E*, then from Eq. 3.28, *v* is imaginary and motion is not possible. But if *U* < *E* then motion is possible.

In both Figs. 3.14b and 3.14c the car's total energy *E* exceeds the potential energy in the rightmost region, so that motion in this region is possible. But starting at point *P*, the car is blocked from this region, because near peak *R*, *U* > *E*. In the case of Fig. 3.14c, peak *Q* also poses a situation (i.e. *U* > *E*) that keeps the car out of the valley between *Q* and *R* where motion is possible. Such a peak which does not allow motion by having *U*>*E* is called a **potential barrier**. We shall use the terminologies 'potential wells' and 'barriers' widely in our courses on Quantum, Atomic and Molecular Physics, Nuclear Physics and Solid State/ Materials Science.

We have thus studied a few applications of the principle of conservation of energy. We have studied another conservation principle, that of linear momentum, in Unit 2. We shall now take up the study of collisions which involve both these conservation principles.

3.4 ELASTIC AND INELASTIC COLLISIONS

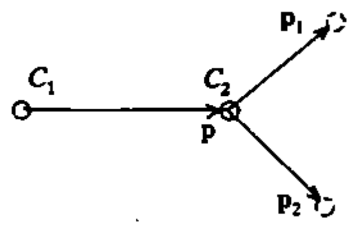


Fig. 3.15

We know that certain metallic surfaces emit electrons when ultra-violet light is made to fall upon it. This process can be explained through the study of collisions. It can also throw light upon several aspects of other processes like generation of X-rays and artificial radioactivity. However, let us begin our study on collisions by thinking of a coin hit by another on the surface of a table. You may try this yourself with the help of two coins. What do you observe? A situation of the type is shown in Fig. 3.15. The striking coin C_1 and the struck coin C_2 move on two sides of the original line of motion of C_1. Let the linear momentum of C_1 be $\mathbf{p}$ immediately before it hits C_2. Let the momenta of C_1 and C_2 be $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, after the collision. We shall assume that the surface of the table is reasonably smooth so that the frictional forces can be neglected. Then during the collision no external force acts on the system of coins. So its linear momentum is conserved, i.e.

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2. \quad (3.29)$$

The principle of conservation of linear momentum holds for all sorts of collisions provided no external force acts. The principle of conservation of energy also holds. Since the coins are always on a table whose surface is horizontal, their P.E. is always zero with respect to the table top as standard. This means that the K.E. of C_1 before collision is equal to the sum total of the K.E.s of C_1 and C_2 after collision and any amount of energy in the form of heat or sound, which might have dissipated due to collision. We must remember that we have assumed that the surface of the table is frictionless. Now if, the kinetic energy of the system remains constant in a collision process, then it is said to be an **elastic collision**. In other words, the condition of an elastic collision of the coins is given by

$$T = T_1 + T_2, \quad (3.30)$$

where T is the K.E. of C_1 before collision and T_1, T_2 are the K.E.s of C_1, C_2, respectively after collision. If the kinetic energy of the system does not remain constant then the collision is said to be **inelastic**. But the total energy is conserved in both the cases. Let us first discuss elastic collisions.

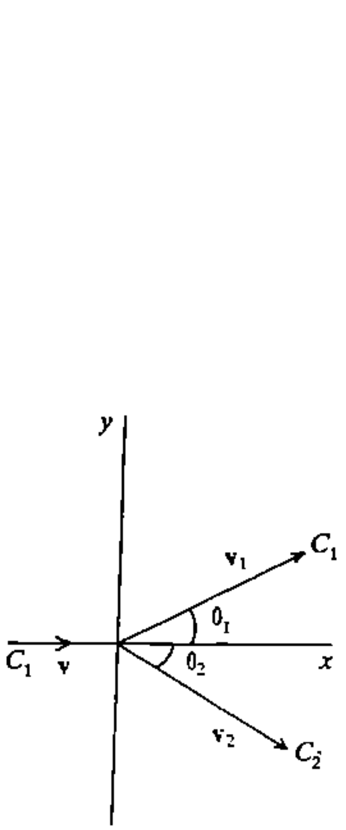


Fig. 3.16

Study of elastic collisions

We understand that in the case of an elastic collision both the equations 3.29 and 3.30 will hold. We shall now apply them to obtain the angular separation between the directions of motion of the coins after the collision. We shall primarily be applying some results of trigonometry. This kind of an analysis will be required for solving any problem on collision in two dimensions. For this we redraw Fig. 3.15 as shown in Fig. 3.16. We take the x-axis along the original direction of motion of C_1 and y-axis perpendicular to it. C_1 is referred to as the *projectile* and C_2 as the *target*. Let the masses of C_1 and C_2 be m_1 and m_2, respectively. Before collision, C_1 is moving with a velocity $\mathbf{v}$. Let the velocities of C_1 and C_2 after collision be $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. $\mathbf{v}_2$ is called the *recoil velocity and* θ_2, the *angle of recoil*. Eq. 3.29 can be written as,

$$m_1 \mathbf{v} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2.$$

Using Eqs. 1.3a and 1.3d, we have

$$m_1 v\hat{\mathbf{i}} = m_1 (v_1 \cos\theta_1 \hat{\mathbf{i}} + v_1 \sin\theta_1 \hat{\mathbf{j}}) + m_2 (v_2 \cos\theta_2 \hat{\mathbf{i}} - v_2 \sin\theta_2 \hat{\mathbf{j}}).$$

or $$m_1 v\hat{\mathbf{i}} = (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)\hat{\mathbf{i}} + (m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2)\hat{\mathbf{j}}.$$

From Eqs. 1.5 and 1.6, we get

$$m_1 v = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \quad (3.31a)$$

and $$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2. \quad (3.31b)$$

Again from Eq. 3.30, we get

$$m_1 v^2 = m_1 v_1^2 + m_2 v_2^2 \quad (3.32)$$

First we shall eliminate θ_1 between Eqs. 3.31a and 3.31b, i.e. we have

$$(m_1 v_1)^2 = (m_2 v_2 \sin\theta_2)^2 + (m_1 v - m_2 v_2 \cos\theta_2)^2$$

or $$m_1^2 v_1^2 = m_2^2 v_2^2 + m_1^2 v^2 - 2 m_1 m_2 v v_2 \cos\theta_2.$$

We shall now use Eq. 3.32 to get an expression of v_2 in terms of v and θ_2. Using Eq. 3.32 we get,

$$0 = m_2^2 v_2^2 + m_1 m_2 v_2^2 - 2m_1 m_2 v v_2 \cos\theta_2 .$$

This is effectively a quadratic in v_2 of which $v_2 = 0$ is a trivial solution. We disregard that and write only the acceptable solution

$$v_2 = \frac{2m_1 m_2 v \cos\theta_2}{m_2^2 + m_1 m_2} = \frac{2\alpha v \cos\theta_2}{1+\alpha}. \qquad (3.33)$$

where $\alpha = \frac{m_1}{m_2}$.

Again, from Eqs. 3.31a and 3.31b, we get

$$\frac{m_1 v_1 \sin\theta_1}{m_1 v_1 \cos\theta_1} = \frac{m_2 v_2 \sin\theta_2}{m_1 v - m_2 v_2 \cos\theta_2}.$$

Multiplying the numerator and denominator of the right-hand side by $2\cos\theta_2$, we get,

$$\tan\theta_1 = \frac{m_2 v_2 \sin 2\theta_2}{2m_1 v \cos\theta_2 - 2m_2 v_2 \cos^2\theta_2}.$$

Using Eq. 3.33, we get

$$\tan\theta_1 = \frac{m_2 v_2 \sin 2\theta_2}{(m_1 + m_2) v_2 - 2m_2 v_2 \cos^2\theta_2} = \frac{\sin 2\theta_2}{\alpha - \cos 2\theta_2}. \qquad (3.34)$$

Eq. 3.34 indicates that the relation between θ_1 and θ_2 is strongly dependent on α. We shall discuss the extreme case when $\alpha \gg 1$, (i.e. $m_1 \gg m_2$).

Since $\sin 2\theta_2$ and $\cos 2\theta_2$ lie between -1 and $+1$, in this case we have $\tan\theta_1 \to 0$ or $\theta_1 \to 0$. And as $\theta_1 \to 0$ we get from Eq. 3.31b that $\theta_2 \to 0$ also. So when the projectile is much heavier than the target then both move along the same straight line as that of the initial direction of the projectile.

SAQ 8

Prove that a) $\theta_1 + 2\theta_2 = 180°$ when $\alpha \ll 1$

and b) $\theta_1 + \theta_2 = 90°$ when $\alpha = 1$.

We shall now study an application of inelastic collision.

Ballistic pendulum

The ballistic pendulum is a device for measuring the velocity of a bullet. The pendulum is a large wooden block of mass M hanging vertically by two cords. A bullet of mass m strikes the pendulum with an initial velocity v_i and gets embedded in it. The final velocity v_f of the system after collision is much less than that of the bullet before collision. This final velocity can easily be determined, so that the initial velocity of the bullet can be computed by applying the law of conservation of linear momentum. Initial linear momentum of the bullet $= m\,v_i$. Linear momentum of the system after collision $= (m + M)\,v_f$

So from conservation of linear momentum,

$$m v_i = (m + M)\, v_f. \qquad (3.35)$$

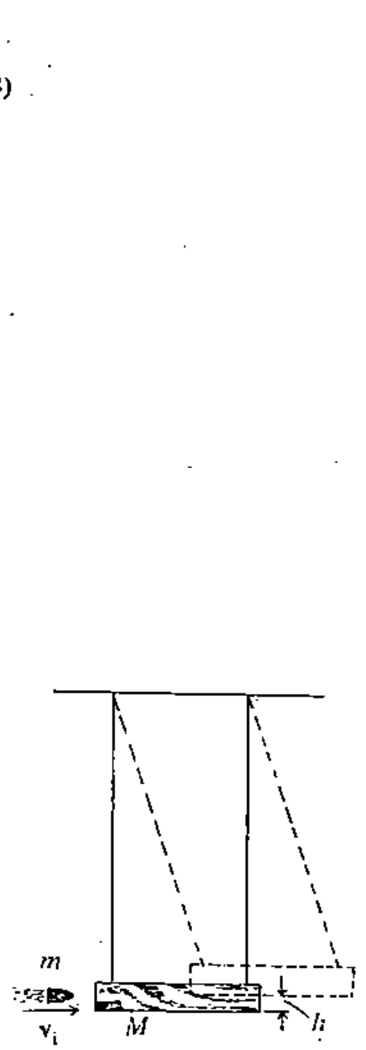


Fig. 3.17

The K.E. before collision is $\frac{1}{2} m v_i^2$ and the total K.E. after collision is

$$\frac{1}{2}(m + M) v_f^2 = \frac{1}{2}\frac{m^2 v_i^2}{m + M}$$ from Eq. 3.35. Now, we have

$$(\text{K.E. before collision}) - (\text{K.E. after collision}) = \frac{1}{2} m v_i^2 \left(1 - \frac{m}{m + M}\right) = \frac{1}{2}\frac{mM}{m + M} v_i^2,$$

which is a positive quantity. This means that there is a loss of K.E. Hence, the collision is inelastic. However, the K.E. after collision makes the wooden block swing up to a maximum height, h as shown in Fig. 3.17. Our task is to determine v_i in terms of the known parameters m, M and h. The K.E. of the bullet and the block is used up in raising the block through a height h. So the block and bullet acquire a P.E. equal to $(M + m)gh$. The K.E. of the block and bullet $= \frac{1}{2}(M + m) v_f^2 = (M + m)gh$, from the principle of conservation of energy. So $v_f^2 = 2gh$ or $v_f = \sqrt{2gh}$. Hence from Eq. 3.35, we get

$$v_i = \frac{M + m}{m}\sqrt{2gh}. \qquad (3.36)$$

You may like to make an estimate of v_i in the form of a numerical example in the following SAQ.

SAQ 9

In a ballistic pendulum, the masses of the bullet and the block are 5 g and 2 kg, respectively. After being struck by the bullet the block along with the bullet is raised through 0.5 cm. Find the velocity of the bullet. ($g = 9.8\ m\ s^{-2}$)

How long has it taken you so far to go through Unit 3? May be something like 3 hours. Some of your friends might have taken 4 hours to complete the same matter and some may again have taken only 2 hours. So all of you have covered the study material to the same extent, but your rates of working are different. This throws some light on the following question! Why do you feel more exhausted when you run up the staircase at constant speed than when you walk up the same at a constant speed? In each case you exert an average force exactly equal to your weight and do so over a fixed distance. And as the product of force and distance is the work done, the same amount of work is done in each case. But what matters is the rate at which work is done. We shall discuss this aspect now.

3.5 POWER

Power is defined as the rate of doing work. If an amount of work ΔW is done in a time Δt, then the **average power** $\bar{P}$ is

$$\bar{P} = \frac{\Delta W}{\Delta t}. \tag{3.37}$$

If the rate varies with time, we define **instantaneous power** as the average power taken in the limit of arbitrarily small time interval Δt:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}. \tag{3.38}$$

Eqs. 3.37 and 3.38 show that the unit of power is joule s^{-1}, whose other name is watt. You may now try a very simple SAQ.

SAQ 10

A man ascends to Badrinath Temple from Joshimath, a vertical rise of 1,500 m. His mass is 60 kg. He takes 5 h. A 1,500 kg car drives up the motorable road for the same vertical rise in 1h. What is the average power exerted in each case if we neglect friction for the sake of simplicity? Assume that the man and the car maintain constant speed.

We have not seen how P depends on the applied force **F**. For this we go back to Eq. 3.1. Using Eqs. 3.1 and 3.38 we get

$$P = \lim_{\Delta t \to 0} \frac{\mathbf{F} \cdot \Delta \mathbf{l}}{\Delta t} = \mathbf{F} \cdot \lim_{\Delta t \to 0} \frac{\Delta \mathbf{l}}{\Delta t},$$ (as the process of taking dot product with **F** is independent of Δt).

or $$P = \mathbf{F} \cdot \frac{d\mathbf{l}}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{3.39}$$

So we have obtained the expression of P in the form of a scalar product. You can apply your knowledge of scalar product to work out the following SAQ.

SAQ 11

A man weighing 60 kg is riding his 15 kg bicycle. What power must he supply to maintain a steady speed of 20 $km h^{-1}$ (a) on a level ground and (b) while going up a 5° incline if the frictional force is 30N in each case?

Let us now summarise what we have learnt in this unit.

3.6 SUMMARY

- The work done by a force over a path from A to B is given by

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{l}.$$

- Work-Energy theorem:
  The work done on a particle by the resultant force acting on it is always equal to the change in kinetic energy of the particle.
- If the work done by a force on a particle in taking it from one point to another is independent of the path followed and dependent only on the initial and final positions of the path then the force is said to be conservative.

- The work done by a conservative force **F** in taking a particle from A to B is given by

$$\int_A^B \mathbf{F}\cdot d\mathbf{l} = -(U_B - U_A)$$

where U_A and U_B are the potential energies of the particle at the positions A and B, respectively.
- In the absence of non-conservative forces the total mechanical energy of a system is conserved.
- Principle of conservation of energy: Energy cannot be created, nor can it be destroyed but it can be transformed from one form to another, the total amount of energy in the universe remaining constant.
- In any collision process, the linear momentum and the total energy are conserved. In an elastic collision kinetic energy is conserved, whereas in an inelastic collision it is not.
- Power P is defined as $P = \dfrac{dW}{dt}$

For a constant force **F**, power is $P = \mathbf{F}\cdot\mathbf{v}$.

3.7 TERMINAL QUESTIONS

1. Cite two examples in which you might think you are doing work but from the point of view of physics you are not doing so.
2. An electron is projected with an initial speed of 3.24×10^5 m s^{-1} directly towards a proton which is at rest. The electron is initially at a very large distance from the proton. At what distance from the proton does the electron's speed become instantaneously equal to twice its initial value? [Hint: Use Eq. 3.9 along with the work-energy theorem. Take the value of k in Eq. 3.9 equal to 9×10^9 Nm^{2}C^{-2}.]
3. In a nuclear collision, an alpha-particle A of mass 4 units is incident with speed v on a stationary helium nucleus B of mass 4 units (Fig. 3.18). After collision A moves in the direction BC with a speed $v/2$ at an angle of 60° with the initial direction AB, and the helium nucleus moves along BD. Calculate the speed of the He nucleus along BD and the angle θ.
4. Refer to Fig. 3.19. The variations of potential and total energy with position are shown for a particle executing simple harmonic oscillation. Indicate the turning points. Also indicate the point where the velocity of the oscillator is maximum.

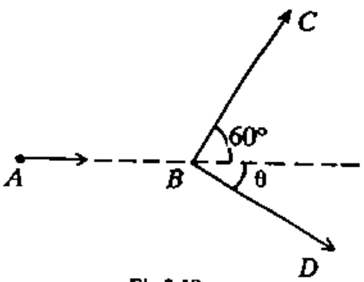


Fig 3.18

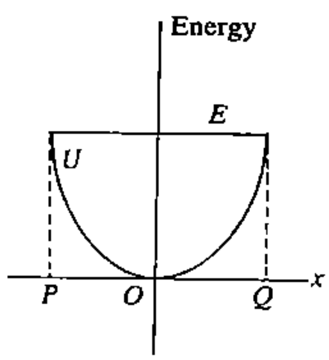


Fig. 3.19

3.8 ANSWERS

SAQs

1. Tension in the string of a simple pendulum. See Fig. 3.20. It acts perpendicular to the displacement of the bob, which is always along the tangent to the circular path.
2. $\mathbf{F} = -k_0 x\hat{\mathbf{i}};\ d\mathbf{l} = dx\hat{\mathbf{i}},\quad \mathbf{F}\cdot d\mathbf{l} = -k_0 x dx.$

$$W = -\int_{x_1}^{x_2} k_0 x dx = -\frac{k_0}{2}(x_2^2 - x_1^2).$$

3. From work-energy theorem we know that work done = change in K.E. Let the distances travelled by the truck and car be x_1 and x_2, respectively. Their initial K.E.s are the same and final K.E.s are zero. So change in K.E. is same for both. Let the magnitude of the equal braking force be F. Thus, we have $Fx_1 = Fx_2$

or $x_1 = x_2$. So they travel equal distances before stopping.
4. a) Refer to Fig. 3.21.

$$W_{ADB} = W_{AD} + W_{DB} = \int_A^D \mathbf{F}\cdot d\mathbf{l} + \int_D^B \mathbf{F}\cdot d\mathbf{l}$$

Now,

$$\int_A^D \mathbf{F}\cdot d\mathbf{l} = \int_A^D -mg\hat{\mathbf{j}}\cdot dx\hat{\mathbf{i}} = 0$$

And

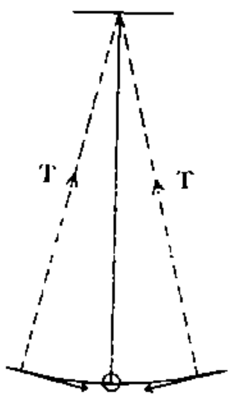


Fig. 3.20

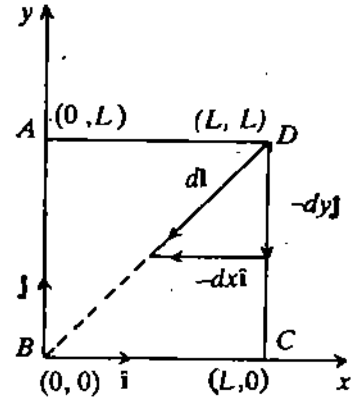


Fig 3.21

$$\int_D^B \mathbf{F}\cdot d\mathbf{l} = \int_D^B -mg\hat{\mathbf{j}}\cdot(-dx\hat{\mathbf{i}} - dy\hat{\mathbf{j}}) = \int_D^B mg\,dy = mg\int_L^0 dy = -mgL$$ and we know from Eq. 3.15a that $W_{AB} = -mgL$

So $W_{AB} = W_{ADB}$.

b) Refer to Fig. 3.22. Let us consider two points A and B in space. We now join A and B by two paths ADB and ACB, so that we get a closed path $ADBCA$. We know that for a conservative force, $W_{ADB} = W_{ACB}$

or $W_{ADB} = -W_{BCA}$ $\quad \therefore\ W_{ADB} + W_{BCA} = 0$

or $W_{ADBCA} = 0$. Hence, the work round a closed path is zero.

5. We shall use Eq. 3.16 to determine the P.E. Here B corresponds to the point having x-coordinate equal to x and A to $x = 0$. $\mathbf{F} = -Cx^3\hat{\mathbf{i}}$, $d\mathbf{l} = dx\hat{\mathbf{i}}$

So the required P.E. is $U = -\int_0^x (-Cx^3\hat{\mathbf{i}})\cdot(dx\hat{\mathbf{i}}) = \int_0^x Cx^3 dx = \frac{1}{4}Cx^4$

6. a) Stable b) Unstable

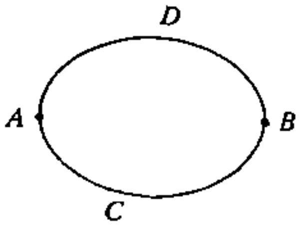


Fig 3.22

7. It is again h as K.E. becomes zero when P.E. $= mgh$.

8. a) If $\alpha \ll 1$, then from Eq. 3.34, we get $\tan\theta_1 = -\tan 2\theta_2 = \tan(180° - 2\theta_2)$

   or $\theta_1 = 180° - 2\theta_2 \quad \therefore \theta_1 + 2\theta_2 = 180°$

   b) If $\alpha = 1$, then from Eq. 3.34, we get $\tan\theta_1 = \dfrac{\sin 2\theta}{1 - \cos 2\theta_2}$

   or $\tan\theta_1 = \dfrac{2\sin\theta_2\cos\theta_2}{2\sin^2\theta_2} = \cot\theta_2 = \tan(90° - \theta_2)$

   $\therefore\ \theta_1 = 90° - \theta_2$, or $\theta_1 + \theta_2 = 90°$.

9. We shall use Eq. 3.36. There we put $M = 2$ kg, $m = 0.005$ kg, $g = 9.8$ m s^{-2} and $h = 5\times10^{-3}$ m and get $v_i = 125.5$ m s^{-1}.

10. $P_{man} = \left(\dfrac{\Delta W}{\Delta t}\right)_{man} = \dfrac{(60\ \text{kg})\ (9.8\ \text{m s}^{-2})\ (1500\ \text{m})}{(5\times3600)\text{s}} = 49\ \text{W}$.

    $P_{car} = \left(\dfrac{\Delta W}{\Delta t}\right)_{car} = \dfrac{(1500\ \text{kg})\ (9.8\ \text{m s}^{-2})\ (1500\ \text{m})}{(1\times3600)\text{s}} = 6125\text{W}$.

11. $20\ \text{km h}^{-1} = \dfrac{20\times1000}{3600}\ \text{m s}^{-1} = \dfrac{50}{9}\ \text{m s}^{-1}$

    Now $P = \mathbf{F}\cdot\mathbf{v}$, where $\mathbf{F}$ = force applied.

    a) On level ground $\mathbf{F}$ is equal and opposite to force of friction $\mathbf{F}_o$.

    So, $P = \mathbf{F}\cdot\mathbf{v} = F_o v = (30\ \text{N})\ (50/9\ \text{m s}^{-1}) = 166.7$ W.

    b) On the slope $\mathbf{F}$ has to oppose $\mathbf{F}_o$ and the component of mg down the plane, (as shown in Fig. 3.23). So $\mathbf{F}$ must act up the plane and is equal in magnitude to

    $(F_o + mg\sin\theta)$

    $P = (F_o + mg\sin\theta)v$

    $= [30\ \text{N} + (75\ \text{kg})\ (9.8\ \text{m s}^{-2})\ (\sin 5°)]\ (50/9\ \text{m s}^{-1})$

    $= [30\ \text{N} + 64\ \text{N}]\ (50/9\ \text{m s}^{-1}) = 522.2$ W

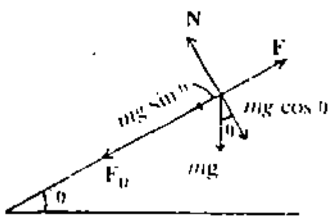


Fig. 3.23

Terminal Questions

1. Example 1. A boy reads a book continuously for a long time sitting on a chair.

   Example 2. A cashier maintains the account of total transaction during the banking hours of a day.

2. Let the initial speed of the electron be v. Its change in K.E. $= \frac{1}{2}m_e\{(2v)^2 - v^2\} = \frac{3}{2}m_e v_i^2$ where m_e = mass of the electron. According to the work-energy theorem this change in K.E. is equal to the work done in bringing it from a large distance (which we shall assume to be infinite) to a point whose distance is x metres from the proton. Our task is to determine x. The above work done is equal to the product of the magnitude of charge

on an electron and the work done in bringing a unit positive charge from infinity to the said point. Let the magnitude of charges on a proton and an electron be e. Using Eq. 3.9 and putting $r_A = \infty$, $r_B = x$ and $q = e$ we get that the work done is equal to

$$(-e) \times ke\left(0 - \frac{1}{x}\right) = \frac{ke^2}{x}$$

From work-energy theorem,

$$\frac{3}{2} m_e v^2 = \frac{ke^2}{x} \quad or\ x = \frac{2ke^2}{3m_e v^2}. \tag{3.40}$$

For our problem, $k = 9 \times 10^9$ N m^2 C^{-2} $v = 3.24 \times 10^5$ m s^{-1}. Now, putting the values of k, v, m_e and e in Eq. 3.40, we get $x = 1.6 \times 10^{-9}$ m.

3. Refer to Fig. 3.24. This is similar to Fig. 3.16. We shall write Eqs. 3.31a and 3.31b by putting $\theta_1 = 60°$, $\theta_2 = \theta$, $v = v$, $v_1 = v/2$, $v_2 = u$. We have

$$m_1 v = m_1 \frac{v}{2} \cos 60° + m_2 u \cos\theta. \tag{3:41a}$$

$$0 = m_1 \frac{v}{2} \sin 60° - m_2 u \sin\theta. \tag{3.41b}$$

Our task is to determine u in terms of v, and the value of θ. For our problem $m_1 = m_2 = 4$ units. So we have, $4v = v + 4u \cos\theta$ or $4u \cos\theta = 3v$. (3.41c)

and $0 = \sqrt{3}v - 4u \sin\theta$ or $4u \sin\theta = \sqrt{3}v$.

Squaring and adding Eqs. 3.41c and 3.41d, we get

$$16u^2 = 12v^2 \quad \text{or} \quad u = \frac{\sqrt{3}}{2} v. \tag{3.41d}$$

Dividing Eq. 3.41d by Eq. 3.41c, we get $\tan\theta = 1/\sqrt{3}$ or $\theta = 30°$.

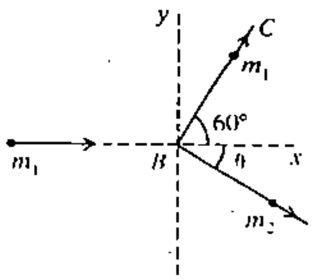


Fig. 3.24

4. Since $U = E$ at P and Q, the K.E.s of the particle at these points are zero. So P and Q are the turning points. For answering the second part we shall use Eq. 3.28. Since E is a constant, v is maximum when U is minimum, i.e. zero. So the velocity of the particle is maximum at O.

UNIT 4 ANGULAR MOTION

Structure

4.1 INTRODUCTION

In Units 1 to 3 you have studied some important concepts in mechanics, such as displacement, velocity, acceleration, force, linear momentum, work and energy. You have also studied two important conservation principles: Conservation of linear momentum and Conservation of energy. However, our development of the concepts of mechanics so far has been restricted in one important respect. We have not developed techniques to describe and analyse the angular motion of particles, in particular their rotational motion.

You may say that we have studied the problems of uniform circular motion and projectile motion using these concepts. But the world is full of objects that undergo rotational motion: From rotating galaxies to orbiting planets, from merry-go-rounds, bicycle wheels and flywheels to rotating ballerinas (dancers) and acrobats. In principle, we can analyse all such motions using Newton's laws by applying them to each particle of the object undergoing angular motion. But in practice it is a difficult task, especially for extended bodies, because the particles number in thousands. What we need is a simple method for treating the angular motion of an object as a whole.

In most cases, we can study the angular motion of an object in terms of the angular motion of a point on it. Therefore, in this unit we shall study the angular motion of a particle and develop related concepts, such as angular displacement, angular velocity, angular acceleration, torque and angular momentum. Using these concepts, we shall study angular motion of rigid bodies in Unit 9. In the next unit, we will turn our attention to gravitation and other forces in nature.

Objectives

After studying this unit you should be able to:

- compute angular displacement, angular velocity and angular acceleration of a particle undergoing angular motion
- express displacement, radial and transverse velocities, and radial and transverse acceleration using plane polar coordinates
- relate the kinematical variables of angular motion and linear motion in their vector forms
- solve problems related to the concepts of torque, rotational kinetic energy and angular momentum of a particle
- apply the law of conservation of angular momentum.

4.2 KINEMATICS OF ANGULAR MOTION

Let us begin our study of angular motion by considering a particle moving in a circle about a fixed axis passing through the centre and perpendicular to the plane of the circle. (Fig. 4.1a).

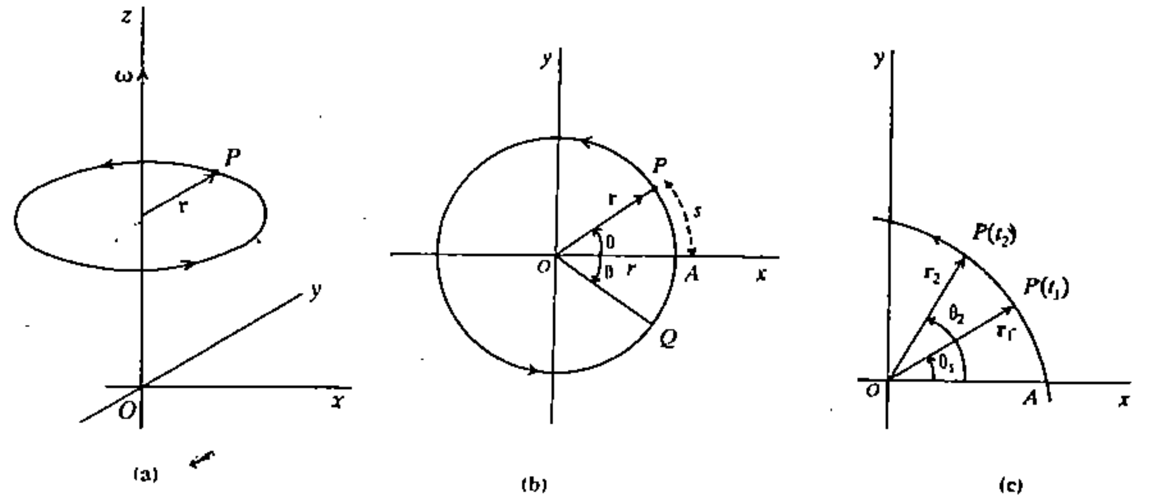


Fig. 4.1: (a) A particle P rotating anticlockwise in a circle about a fixed axis, known as the axis of rotation; (b) the angular position θ of the particle at an instant t; (c) the particle P undergoes an angular displacement $\Delta\theta$ $(= \theta_2 - \theta_1)$ in time $\Delta t (= t_2 - t_1)$.

As you know from Sec. 1.4, we need only a two-dimensional frame of reference to describe this motion (Fig. 4.1b). The angle θ is the angular position of the particle at P with respect to the reference axis, namely the x-axis. By convention, we take θ to be positive for anticlockwise rotation and negative for clockwise rotation. It is given, in radians, by the relation

$$\theta = \frac{s}{r} \tag{4.1}$$

You are perhaps more familiar with the unit of degrees for measuring angles. The unit of radians is related to degrees by the following formula:

$360° = 2\pi$ rad;
$\pi = 3.1415927 ...$

where s is the arc length shown in Fig. 4.1 b and r the magnitude of the position vector **r** of the particle. If the particle rotates more than once, then θ will take the increased value accordingly. For example, let the particle be at P at the instant t after completing two rotations around the circle starting from A. Then its angular position at the instant t will be given by the angle $(2 \times 2\pi + \theta) = (4\pi + \theta)$. Now, let the particle rotate anticlockwise. Let its angular positions at time t_1 and at a later time t_2 be θ_1 and θ_2, respectively (see Fig. 4.1 c). The angular displacement of the particle will be $\theta_2 - \theta_1 = \Delta\theta$ during the time interval $t_2 - t_1 = \Delta t$. Notice that we have used the term 'angular displacement'. Is this a vector quantity like linear displacement? Let us find out and discuss angular displacement in somewhat greater detail.

4.2.1 Angular Displacement

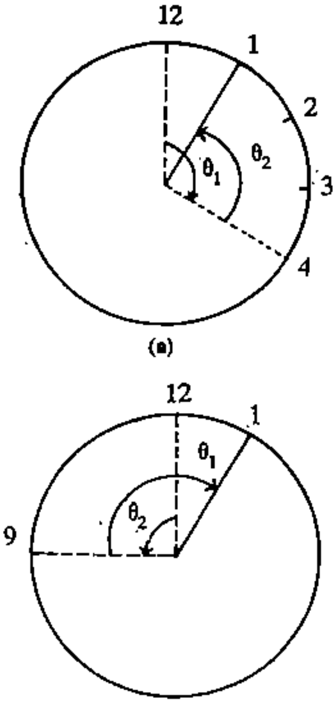
If we say that angular displacement is a vector, then, firstly, alongwith a magnitude it should have a direction. Secondly, angular displacements should add like vectors. As you can see, the magnitude of the angular displacement is the angle through which the particle turns. What is the direction of angular displacement?

In a sense the idea of a direction is associated with angular motion. We have both clockwise and anticlockwise rotations. Let us represent an anticlockwise rotation of say, θ rad by an arrow of a certain length pointing in a certain direction. Then a rotation of $-\theta$ rad will be an arrow of the same length, but pointing in the opposite direction. But in what direction should the first arrow point?

It obviously cannot be the direction of the particle's position vector at its final angular position. Why? See Fig. 4.1b again. For an anticlockwise rotation through an angle θ, the direction of angular displacement would be OP. But for a clockwise rotation through the same angle, its direction will be OQ. So, two equal and opposite rotations (clockwise and anticlockwise) of any magnitude will not in general be antiparallel. Thus, with this choice of directions, angular displacements will not be vectors.

Then how can we define the direction of angular displacement? You must have handled a screw-gauge at school. There the rotational motion of the screw is translated into the forward motion of the screw-head which takes place along a straight line. This straight line can define the direction of the rotational motion of the screw. This straight line is essentially the axis of rotation of the screw.

So we can define the direction of angular displacement to be along the axis of rotation. But how do we represent a clockwise or an anticlockwise rotation along the axis of rotation?



(a) (b)

Fig. 4.2

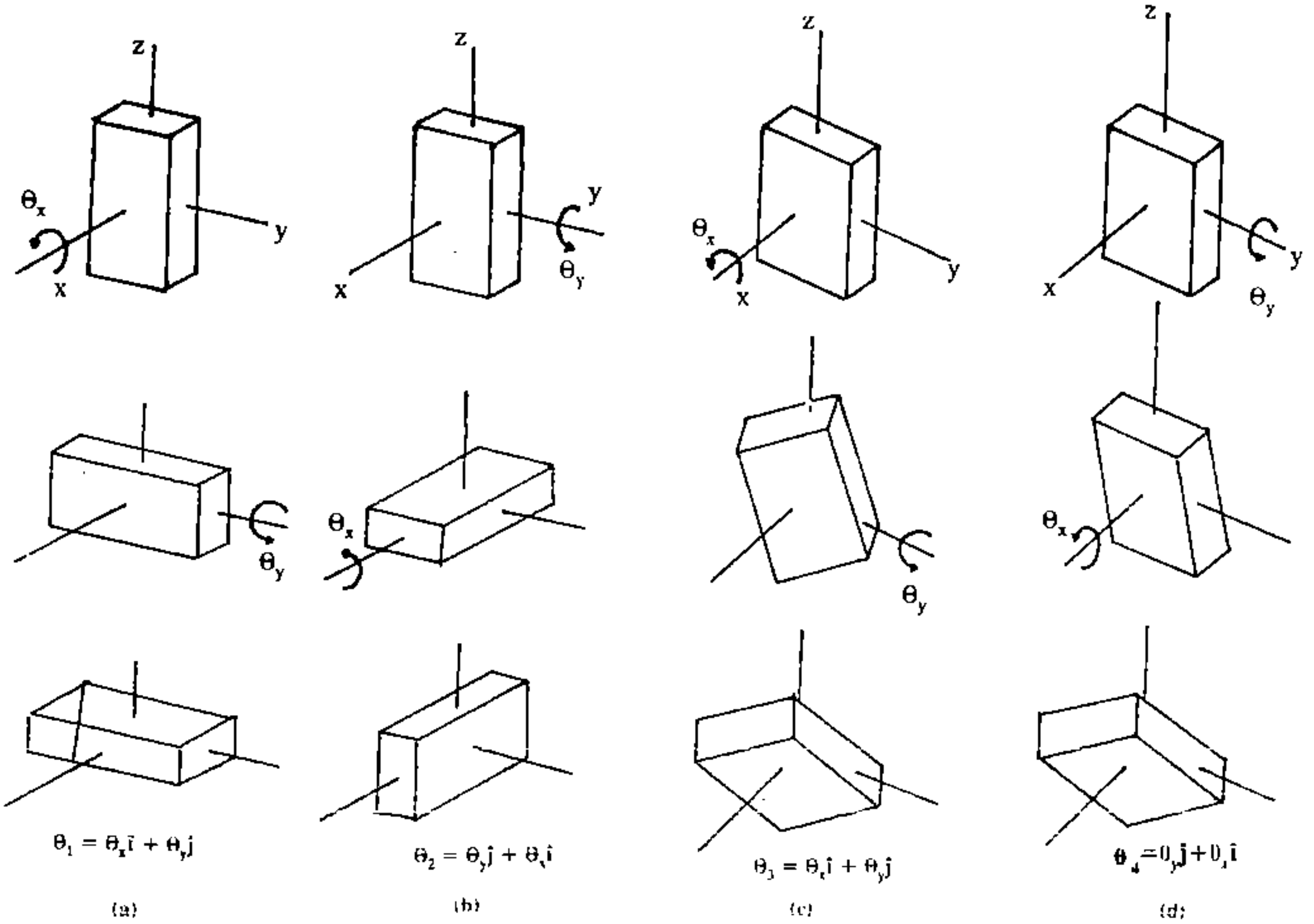
We follow the right-hand rule to make the choice. We curl the fingers of our right-hand around the axis, in the direction of rotation of the particle. The extended thumb points along the direction of the angular displacement (see Fig. 1.9b). Thus, for the particle of Fig. 4.1 (a), the direction of θ will be along the positive z-axis. In Fig. 4.1 (b), the direction of θ will be perpendicular to the page and the point up out of the page.

SAQ 1

What would be the magnitude and direction of the angular displacement in a clockwise rotation of a hand of a clock from 5 to 9?

Having specified the direction of the angle turned by a rotating particle, let us see whether it satisfies the laws of vector addition. Let us consider the commutative law of vector addition: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. What happens in the two-dimensional case when the particle remains in the same plane while rotating about a fixed axis? You can find the answer with the help of a clock as shown in Fig. 4.2. In Fig. 4.2a starting from 12, the clockhand is given a clockwise rotation $\boldsymbol{\theta}_1 = 2\pi/3$ rad and then an anticlockwise rotation $\boldsymbol{\theta}_2 = \pi/2$ rad to get the resultant $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$. In Fig. 4.2b the order of rotation is reversed : starting from 12, the clockhand is first rotated anticlockwise by $\pi/2$ rad and then clockwise by $2\pi/3$ rad, giving $\boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$. The resultant is the same. Now perform a similar exercise with different magnitudes of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$.

What do you conclude? Clearly, if the particle remains in the same plane and rotates about a fixed axis, the angular displacement is a vector quantity. Does this law hold for rotations in three dimensions? Study Fig. 4.3 and perform the rotations with the help of a book for an answer.



$\theta_1 = \theta_x \hat{i} + \theta_y \hat{j}$ $\theta_2 = \theta_y \hat{j} + \theta_x \hat{i}$ $\theta_3 = \theta_x \hat{i} + \theta_y \hat{j}$ $\boldsymbol{\theta}_4 = \theta_y \hat{\mathbf{j}} + \theta_x \hat{\mathbf{i}}$

(a) (b) (c) (d)

Fig. 4.3: Rotation through finite angles: (a) The book is rotated by an angle of $\pi/2$ rad anticlockwise around the x-axis ($\theta_x \hat{i}$) and then by $\pi/2$ anticlockwise around the y-axis ($\theta_y \hat{j}$). The resultant is $\theta_1 = \theta_x \hat{i} + \theta_y \hat{j}$; (b) the rotations are the same but in reverse order, i.e. $\theta_2 = \theta_y \hat{j} + \theta_x \hat{i}$. Clearly, $\theta_1 \neq \theta_2$. Rotation through infinitesimal angles: (c) the book is rotated by a small angle, say $\pi/36$ rad anticlockwise around x and y=axes; (d) the rotations are the same but in reverse order. In this case $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$. In all these figures, the origin of the coordinate axes remains at the centre of the book, and the axes remain parallel to themselves.

What is the answer? Finite angular displacements in three dimensions are *not* vector quantities, but *three-dimensional infinitesimal angular displacements are vectors.*

Having defined the angular displacement and studied its vector nature, you are ready to learn about angular velocity and angular acceleration.

4.2.2 Angular Velocity and Angular Acceleration

The average angular speed of a particle undergoing angular displacement $\Delta\theta$ in time Δt is

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \tag{4.2a}$$

If $\Delta\boldsymbol{\theta}$ is infinitesimal, then $\bar{\boldsymbol{\omega}}$ will be a vector. It will be in the same direction as $\Delta\boldsymbol{\theta}$ and we will call it **average angular velocity**. When the angular speed changes with time, we define **instantaneous angular velocity** as

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\theta}}{\Delta t} = \frac{d\boldsymbol{\theta}}{dt} \tag{4.2 b}$$

$d\boldsymbol{\theta}$ is a vector as it is an infinitesimal angular displacement. We can write $d\boldsymbol{\theta} = \frac{d\boldsymbol{\theta}}{dt} dt$. Since dt is a scalar, $\frac{d\boldsymbol{\theta}}{dt}$ will be a vector, i.e. the instantaneous angular velocity $\boldsymbol{\omega}$ is a vector quantity. Its direction lies along the axis of rotation and its sense is given by the right-hand rule. Study Fig. 4.4 to understand the vector nature of $\boldsymbol{\omega}$ better.

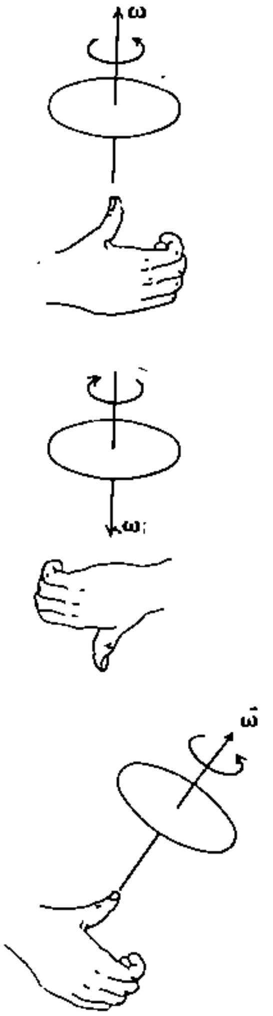


Fig. 4.4: The direction of the angular velocity is given by the right-hand rule.

If the angular speed of the particle in Fig. 4.1 c is not constant, then it has an angular acceleration. If $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are the instantaneous angular velocities of the particle at times t_1 and t_2, respectively, then the **average angular acceleration** $\bar{\boldsymbol{\alpha}}$ of the particle P is defined as

$$\bar{\boldsymbol{\alpha}} = \frac{\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1}{t_2 - t_1} = \frac{\Delta\boldsymbol{\omega}}{\Delta t} \tag{4.3 a}$$

The **instantaneous angular acceleration** is

$$\boldsymbol{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\omega}}{\Delta t} = \frac{d\boldsymbol{\omega}}{dt} \tag{4.3 b}$$

What is the direction of the angular acceleration? Study Fig. 4.5. If the angular velocity changes only in magnitude but not in direction, then $\boldsymbol{\omega}$ simply increases or decreases.

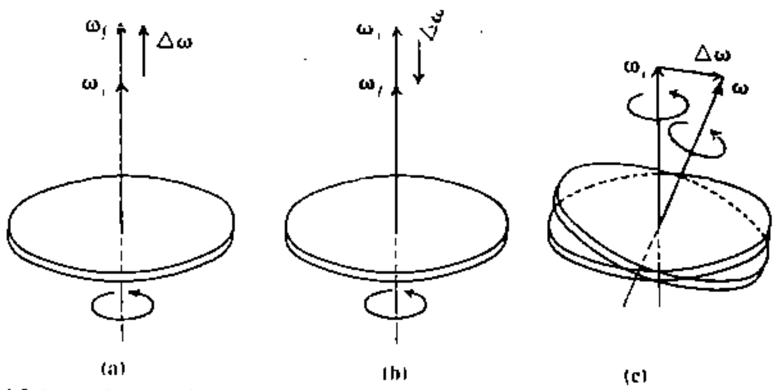


Fig. 4.5: (a) An increase in angular speed alone leads to a change $\Delta\boldsymbol{\omega} (= \boldsymbol{\omega}_f - \boldsymbol{\omega}_i)$ in the angular velocity that is parallel to $\boldsymbol{\omega}_i$. So α is also parallel to $\boldsymbol{\omega}$. Here $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_f$ are the initial and final angular velocities, respectively; (b) a decrease in angular speed means that $\Delta\boldsymbol{\omega}$ and hence $\boldsymbol{\alpha}$ are antiparallel to $\boldsymbol{\omega}_i$; (c) when angular velocity changes only in direction, the change $\Delta\boldsymbol{\omega}$ and hence $\boldsymbol{\alpha}$ is perpendicular to angular velocity.

Therefore $\boldsymbol{\alpha}$ which has a direction along $\Delta\boldsymbol{\omega}$ lies parallel or antiparallel to the axis of rotation (see Figs. 4.5 a and b). When $\boldsymbol{\omega}$ changes only in direction, the angular acceleration vector is perpendicular to $\boldsymbol{\omega}$ (see Fig. 4.5 c). Work out the following SAQ to prove this yourself.

SAQ 2

Show that $\boldsymbol{\alpha}$ is perpendicular to $\boldsymbol{\omega}$, if ω is a constant. [Hint: For $\boldsymbol{\alpha}$ to be perpendicular to $\boldsymbol{\omega}$, $\boldsymbol{\alpha} \cdot \boldsymbol{\omega} = 0$. Since ω is a constant, $\frac{d}{dt}(\omega^2) = \frac{d}{dt}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) = 0$.]

In most general cases, both the direction and magnitude of the angular velocity may change, in which case $\boldsymbol{\alpha}$ is neither parallel nor perpendicular to $\boldsymbol{\omega}$.

You must have observed by now that the rotation of a particle about a fixed axis has a correspondence with the translation of a particle along a fixed direction. The kinematical variables θ, ω and α for angular motion are analogous to x, v and a for linear motion: θ corresponds to x, ω to v and α to a. You are already familiar with the relations between kinematical variables x, v, a and t for linear motion with constant acceleration. In the same manner we can derive the four equations linking θ, ω, α and t for constant angular acceleration. We are stating these relations in Table 4.1 without giving their proof.

Table 4.1: Angular and linear position, speed and acceleration

Linear Quantity or Equation	Angular Quantity or Equation
Position x	Angular position θ
Speed $v = \frac{dx}{dt}$	Angular speed $\omega = \frac{d\theta}{dt}$
Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Equations for Constant Acceleration	
$\bar{v} = \frac{1}{2}(v_0 + v)$	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Notice that you get the second set of equations merely by substituting θ for x, ω for v, α for a and the initial angular velocity ω_0 for v_0, the initial linear velocity. We have seen that a correspondence exists between linear and angular kinematical variables. Can we establish a relationship between the two sets of variables for angular motion? The answer is yes. We will find that these relations are easier to derive if we use plane polar coordinates.

4.2.3 Relating Linear and Angular Kinematical Variables

In your school mathematics courses, you may have studied plane polar coordinates r and θ of the point $P(x, y)$, shown in Fig. 4.6a. These are related to x and y by the equations:

$$x = r\cos\theta,\ y = r\sin\theta, \tag{4.4a}$$

giving $$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x}. \tag{4.4b}$$

You also know that

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}. \tag{4.5}$$

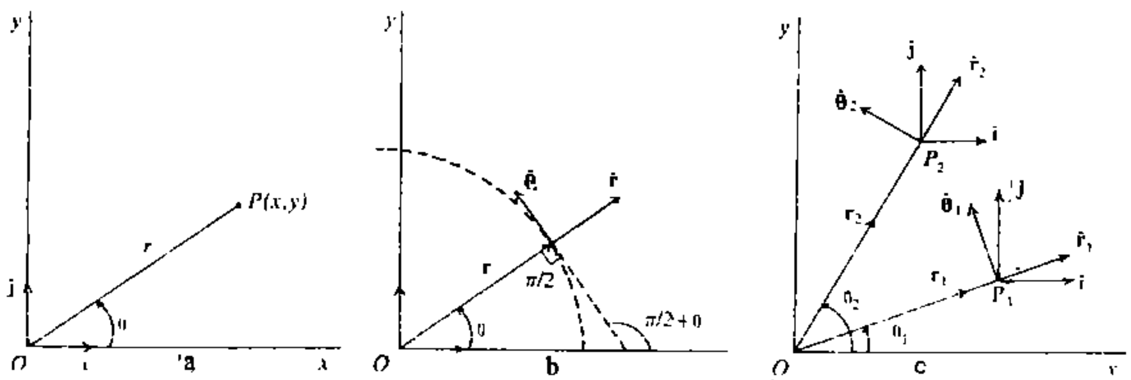


Fig. 4.6: (a) Plane-polar coordinates r and θ; (b) unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the plane-polar coordinate system; (c) unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ have different directions at points P_1 and P_2, i.e. they vary with the position of the particle.

We now introduce two new unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, perpendicular to each other which point in the direction of increasing r and in the sense of increasing angle θ, respectively (see Fig. 4.6b). There is an important difference between the two sets of unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ and $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$: $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ have fixed directions but the directions of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with the position of the particle as you can see in Fig. 4.6c. Since $\hat{\mathbf{r}}$ is a unit vector along $\mathbf{r}$, we can write

$$\mathbf{r} = r\,\hat{\mathbf{r}}. \tag{4.6}$$

We can use Eqs. 4.4, 4.5 and 4.6 to find the relationship between $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$. From Eqs. 4.4, 4.5 and 4.6, we get

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{1}{r}(r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}),$$

$$\text{or } \hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}. \qquad (4.7a)$$

So a unit vector in the direction making an angle θ with the positive x-axis is $\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. $\hat{\boldsymbol{\theta}}$ is a unit vector making an angle $(\pi/2 + \theta)$ with positive x-axis (see Fig. 4.6b). So in order to obtain $\hat{\boldsymbol{\theta}}$ we replace θ in the expression of $\hat{\mathbf{r}}$ by $(\pi/2 + \theta)$.

$$\text{So, } \hat{\boldsymbol{\theta}} = \cos(\theta + \pi/2)\,\hat{\mathbf{i}} + \sin(\theta + \pi/2)\,\hat{\mathbf{j}},$$

$$\text{or } \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \qquad (4.7b)$$

Notice that although $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with position, they depend only on θ, and not on r. Before proceeding further, we suggest that you try the following SAQ to become used to the polar coordinates:

In the text, whenever we use the terms 'velocity' and 'acceleration', we mean 'linear velocity' and 'linear acceleration'.

SAQ 3

a) Show that the results $|\hat{\mathbf{r}}| = 1$, $|\hat{\boldsymbol{\theta}}| = 1$, and $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$ are consistent with Eqs. 4.7.

b) If $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}}$ and $\mathbf{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}}$, then prove that $\mathbf{A}\cdot\mathbf{B} = A_r B_r + A_\theta B_\theta$; where r' and θ's of **A** and **B** refer to the same point in the space.

c) Show that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$.

Now that you are familiar with the plane polar coordinates let us first derive the expressions of velocity and acceleration for circular motion in terms of these coordinates. In Sec. 1.4, you have studied these relations for *uniform* circular motion. You know that for constant ω, $v = \omega r$ and $a_R = \frac{v^2}{r} = \omega^2 r$. Let us now consider circular motion with variable angular speed.

Velocity and acceleration for circular motion in polar coordinates

Recall that $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. Now, we have from Eq. 4.6,

$$\mathbf{v} = \frac{d(\mathbf{r})}{dt} = \frac{d(r\hat{\mathbf{r}})}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = 0 + r\frac{d\hat{\mathbf{r}}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} \qquad \left(\text{Since } r \text{ is a constant, } \frac{dr}{dt} = 0\right)$$

Notice that $\frac{d\hat{\mathbf{r}}}{dt}$ is non-zero. Let us now evaluate it.

Differentiating Eq. 4.7a with respect to time we get

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos\theta)\hat{\mathbf{i}} + \frac{d}{dt}(\sin\theta)\hat{\mathbf{j}}, \text{ since } \hat{\mathbf{i}} \text{ and } \hat{\mathbf{j}} \text{ are constant unit vectors,}$$

$$= -\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}$$

$$= \dot{\theta}\,(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$$

where we have written $\frac{d\theta}{dt}$ as $\dot{\theta}$. Using Eq. 4.7b, we get

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad (4.8)$$

Thus, for circular motion

$$\mathbf{v} = r\,\dot{\theta}\,\hat{\boldsymbol{\theta}},$$

$$\text{or } \mathbf{v} = r\,\omega\,\hat{\boldsymbol{\theta}}, \qquad (4.9)$$

since $\omega = \frac{d\theta}{dt}$. Thus, the velocity of a particle moving in a circle has the magnitude ωr. It is directed along $\hat{\boldsymbol{\theta}}$, which is along the tangent to the circle. You can see that Eq. 4.9 holds for uniform circular motion also.

Again differentiating Eq. 4.9 with respect to time, we get the acceleration for circular motion in plane polar coordinates:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dr}{dt}\,\dot{\theta}\hat{\boldsymbol{\theta}} + r\frac{d\dot{\theta}}{dt}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}.$$

$$= r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}, \text{ where } \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d^2\theta}{dt^2}$$

$$\text{or } \mathbf{a} = r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

To evaluate $\frac{d\hat{\theta}}{dt}$, we differentiate Eq. 4.7b with respect to time:

$$\frac{d\hat{\theta}}{dt} = -\frac{d}{dt}(\sin\theta)\,\hat{\mathbf{i}} + \frac{d}{dt}(\cos\theta)\,\hat{\mathbf{j}}$$

$$= -\cos\theta\frac{d\theta}{dt}\hat{\mathbf{i}} - \sin\theta\frac{d\theta}{dt}\hat{\mathbf{j}}$$

$$= -\dot{\theta}(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}).$$

Using Eq. 4.7a, we get

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}}. \qquad (4.10)$$

So, the acceleration of a particle moving in a circle is

$$\mathbf{a} = r\ddot{\theta}\hat{\theta} - r(\dot{\theta})^2\hat{\mathbf{r}}$$

Since $\omega = \dot{\theta}$ and $\alpha = \frac{d\omega}{dt} = \ddot{\theta}$, we get

$$\mathbf{a} = -\omega^2 r\,\hat{\mathbf{r}} + \alpha r\,\hat{\theta}, \qquad (4.11\text{ a})$$

$$= -a_R\hat{\mathbf{r}} + a_T\hat{\theta}. \qquad (4.11\text{ b})$$

$$\text{or } \mathbf{a} = \mathbf{a}_R + \mathbf{a}_T. \qquad (4.11\text{ c})$$

Thus, for circular motion $\mathbf{a}$ has a radial component $\mathbf{a}_R$ opposite to $\hat{\mathbf{r}}$ in direction, which gives the negative sign. It also has a transverse component $\mathbf{a}_T$ along $\hat{\theta}$. You can see that the transverse component $\mathbf{a}_T$ vanishes for uniform circular motion. You may now like to work out an SAQ to concretise these ideas.

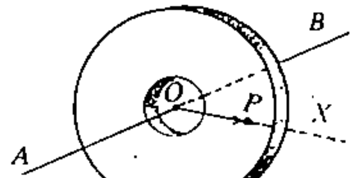


Fig. 4.7: A grindstone rotating about a fixed axis *AOB*. The particle *P* on its rim executes circular motion.

SAQ 4

A grindstone of radius 0.5 m is rotating anticlockwise at a constant angular acceleration α of 3.0 rad s^{-2} (Fig. 4.7). Start from a reference horizontal line *OX* at time $t = 0$, when the grindstone is at rest and find the following for a particle *P* situated at the rim of the grindstone.

a) Its angular displacement and angular velocity 2.0 s later.

b) Its linear velocity, radial and transverse acceleration at the end of 2.0 s.

In Eqs. 4.6 to 4.11 we have expressed vectors **r**, **v** and **a** in terms of scalars θ, ω and α. What is the relation between the vectors **r**, **v**, **a** and θ, ω, α? Let a particle rotate in a circle about the z-axis. The vectors **r**, **v**, **a** and ω, α will be as shown in Fig. 4.8. Let the angle between ω and r be ϕ. Then, since $\angle PCO = 90°$, the radius CP of the circle will be $r\sin\phi$, and

$$v = \omega r\sin\phi$$

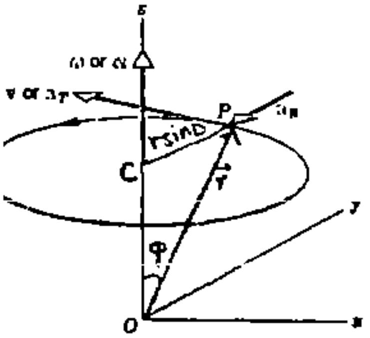


Fig. 4.8: Vectors r, v, a_T, a_R, ω and α for a particle rotating in a circle about the z-axis.

If we now sweep ω into **r** through the smaller angle between them and use the right-hand rule, we find that the extended thumb points towards **v**. This gives the relation

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \qquad (4.12\text{a})$$

$$\text{Now, } \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})$$

Since $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \left(\frac{d\mathbf{A}}{dt}\right) \times \mathbf{B} + \mathbf{A} \times \left(\frac{d\mathbf{B}}{dt}\right)$, we get

$$\mathbf{a} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}.$$

We can once again prove that

$$\mathbf{a}_T = \boldsymbol{\alpha} \times \mathbf{r}, \qquad (4.12\text{b})$$

$$\mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v}, \text{ giving} \qquad (4.12\text{c})$$

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_R. \qquad (4.12\text{d})$$

Eq. 4.12b follows from the same reasoning as we used for **v**. $a_T = \alpha r \sin\phi$, and its direction is obtained from the right-hand rule applied to $\boldsymbol{\alpha}$ and **r**. Now

$$a_R = \omega^2 r \sin\phi = \omega\,(\omega r \sin\phi) = \omega v.$$

The direction of $\mathbf{a}_R$ is along *PC*. It is the same direction in which the right-hand thumb points if $\boldsymbol{\omega}$ is swept into **v** through the smaller angle.

Let us now express **r**, **v** and **a** in terms of plane polar coordinates for any *general angular motion* of a particle about a fixed axis of rotation.

Eq. 4.6 for **r** holds good for any kind of angular motion. For velocity we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}.$$

Using Eq. 4.8, we get

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = \mathbf{v}_R + \mathbf{v}_T. \qquad (4.13a)$$

$$\text{where } \mathbf{v}_R = \dot{r}\hat{\mathbf{r}},\ \mathbf{v}_T = r\dot{\theta}\hat{\boldsymbol{\theta}} \qquad (4.13b)$$

Similarly, acceleration **a** is given as :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$

where we have used Eqs. 4.8 and 4.10. Thus,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \qquad (4.14)$$

Eq. 4.14 means that the acceleration for general angular motion has two components. One is along $\hat{\mathbf{r}}$ and is called the **radial component**. The other is perpendicular to $\hat{\mathbf{r}}$ and is called the **transverse component.**

Eqs. 4.6 to 4.14 enable us to describe the motion of a particle undergoing angular motion either in angular variables or in linear variables. You may wonder why we need angular variables for describing angular motion, when they appear more complicated. The answer is that the angular description is more useful than the linear description when we discuss angular motion. For example, it is much more convenient to use these equations to find out the orbits of planets. You will see this in Unit 6. Similarly, for describing the motion of a rotating body we will have to consider the motion of various points on it. It is clear from Eqs. 4.6 to 4.14 that different points on the body will not have the same linear displacement, velocity or acceleration. But *all* points on a body rotating about a fixed axis (which does not pass through the body) have the same angular displacement, velocity or acceleration at any instant. Therefore, we can describe the motion of the whole body in a simple way if we use angular variables $\boldsymbol{\theta}$, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$. You will appreciate this point better when you study Unit 9. We end this section on the kinematics of angular motion with an example and an SAQ.

Example 1: Acceleration of a bead on a spoke of a wheel

A bead moves outward with a constant speed u along the spoke of a rotating wheel. It starts from the centre at time $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is constant. Find the velocity and acceleration of the bead.

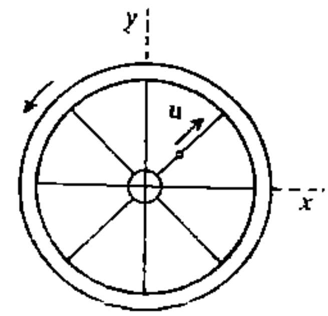


Fig. 4.9: Acceleration of a bead on a wheel's spoke

Let us choose the reference frame as shown in Fig. 4.9. Here $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position r can be obtained by integrating with respect to t the relation $\dot{r} = u$.

$$\int dr = \int u\,dt$$

or $r = ut + c$, where c = constant of integration.

At $t = 0$, $r = 0$. Thus, $c = 0$.

From Eq. 4.13

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$= u\hat{\mathbf{r}} + ut\,\omega\hat{\boldsymbol{\theta}}$$

$$= \mathbf{v}_R + \mathbf{v}_T$$

We find that the magnitude of radial velocity is constant, whereas that of the transverse velocity increases linearly with time.

The acceleration is given by Eq. 4.14:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$
$$= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.$$

The magnitude of transverse acceleration is also constant.

SAQ 5

A particle moves outward along a spiral. Its trajectory is given by $r = C\theta$, where C is a constant equal to $(1/\pi)$ m rad^{-1}. θ increases in time according to $\theta = \frac{\alpha t^2}{2}$, where α is a constant.

a) Find the velocity and acceleration of the particle.

b) Show that the radial acceleration of the particle is zero when $\theta = \frac{1}{\sqrt{2}}$ rad.

[Hint: Use Eqs. 4.7, 4.13 and 4.14.].

So far we have described angular motion. We will now study the causes of angular motion.

4.3 DYNAMICS OF ANGULAR MOTION

As we have seen earlier circular motion is the simplest kind of angular motion. There are numerous examples of circular motion in nature. Many satellites are in circular orbits, the orbits of planets are nearly circular. The earth's daily rotation carries you around in circular motion, Pieces of rotating machinery, cars rounding curves etc., describe circular motion. Let us see what forces cause a particle to execute circular motion.

4.3.1 Circular Motion

We will first consider the case of uniform circular motion about which you have read in Sec. 1.4 of Unit 1. Recall that in this case, the particle moves in a circle with a constant angular speed. Thus, both r and ω are constant. The force $\mathbf{F}$ is given by Newton's second law as $\mathbf{F} = m\mathbf{a}$.

We use the expression of $\mathbf{a}$ from Eq. 4.11. In this case α is zero as ω is constant. So we get

$$\mathbf{F} = -ma_R\,\hat{\mathbf{r}} = -mr\omega^2\hat{\mathbf{r}} = -\frac{mv^2}{r}\hat{\mathbf{r}}. \quad (4.15)$$

You can recognise the term $\frac{v^2}{r}$ as the centripetal acceleration of Eq. 1.30c. The force defined by Eq. 4.15 has a magnitude mv^2/r and is directed toward the centre of the circle. The negative sign in Eq. 4.15 appears because $\mathbf{F}$ is opposite to $\mathbf{r}$ in direction. This is called the **centripetal force**. What does Eq. 4.15 mean? It means that for an object of mass m to be in uniform circular motion, a net force $-\frac{mv^2}{r}\hat{\mathbf{r}}$ must act on the object. Whenever we see an object in uniform circular motion, we know that a net force of this magnitude must be acting. Some physical mechanism like gravity, tension in a string, an electric or magnetic force, friction etc. must provide this force. For example, the giant planet Jupiter circles the Sun at a speed of 13 kms^{-1}. The gravitational force keeps it in its approximately circular path. Similarly, when a tiny sports car rounds a tight curve, the centripetal force needed to keep it in a circular path is provided by the frictional force between its tyres and the roadbed, and also by the banking of the road. Protons circle around an accelerator ring because a magnetic force provides the centripetal force.

Example 2

A geostationary satellite is held in its orbit by the force of gravitation. What is its height above the surface of the earth?

You may have studied about geostationary satellites in Unit 29 of the Foundation Course FST 1. You may know that its time period of rotation is 24 h which is the same as the period of rotation of the Earth about its axis. Now, the centripetal force needed to keep the satellite in its path is provided by the force of gravitation between the Earth and the satellite. So, if

m_s and m_E are the masses of the satellite and the Earth, respectively, and r the radius of the satellite's orbit, then

$$\frac{m_s v^2}{r} = \frac{G m_s m_E}{r^2}$$

where v is orbital velocity of the satellite given as $v = \frac{2\pi}{T} r$,

and T = Time period of rotation = 24 h = $24 \times 60 \times 60$ s

So, we get. $\frac{4\pi^2}{T^2} r = \frac{Gm_E}{r^2}$ or $r^3 = \frac{Gm_E T^2}{4\pi^2}$.

Putting $r = R_E + h$, where R_E = the radius of earth and h = height of the satellite above the surface of earth, we get

$$h = \left(\frac{Gm_E T^2}{4\pi^2}\right)^{1/3} - R_E. \quad (4.16)$$

Substituting the values of G, m_E and R_E and putting $T = 24 \times 60 \times 60$ s, we get

$h = 3.59 \times 10^6$ m = 35900 km.

SAQ 6

Suppose the moon were held in orbit not by gravitation of the Earth but by the tension in a massless cable. Estimate the magnitude of the tension in the cable.

What is the force for circular motion in which the angular speed of the particle changes? For example, the rotary motion of a particle on a record turntable spinning up from rest to full speed, or a ball swung in a vertical circle. In this case, we again use Eq. 4.11 for a and obtain

$$\mathbf{F} = m\mathbf{a} = \mathbf{F}_R + \mathbf{F}_T \quad (4.17a)$$

where $\mathbf{F}_R = -mr\omega^2\hat{\mathbf{r}} = -\frac{mv^2}{r}\hat{\mathbf{r}}$, and (4.17b)

$$\mathbf{F}_t = mr\alpha\,\hat{\boldsymbol{\theta}}. \quad (4.17c)$$

Thus, for non-uniform circular motion the force has a finite transverse component in addition to the radial or centripetal component. You have studied in Sec. 1.4 of Unit 1 that the centripetal acceleration and, therefore, the centripetal force changes only the *direction* of velocity, and not its magnitude. What effect does the transverse force have on the particle?

Role of transverse force

The transverse force gives the particle a finite angular acceleration: the greater the force, the greater is α, and greater the rate at which angular speed increases. In other words, this force makes the particle turn faster and faster, if it continues to act. What do you think will happen to the rotating object if this force stopped acting?

If $\mathbf{F}_T$ is zero and $\mathbf{F}_R$ continues to act, the particle will continue to rotate in a circle but with zero angular acceleration, i.e. at constant angular speed. Thus, to keep a particle moving in a circle at a constant angular speed, only a centripetal force is needed. Only if you want to increase or decrease the rate at which the particle is rotating, you have to apply a transverse force in a direction perpendicular to the radius. Suppose you want to start rotating a wheel, (potter's wheel or bicycle wheel), or a grindstone or a merry-go-round, which is initially at rest (see Fig 4.10). You will have to apply a transverse force because you want to change its angular speed from zero to some positive value. You also need a centripetal force to make it move in a circle. Hence, you apply a force which is not exactly perpendicular to the radius but along the direction of the resultant F of the radial and transverse forces, i.e. tilted a little towards the centre of the object.

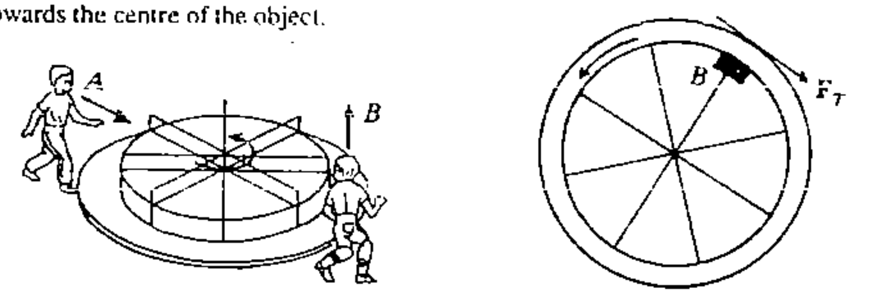


Fig. 4.10: (a) A transverse force along B is needed alongwith the radial force along A to set the merry-go-round moving; (b) you apply a retarding transverse force F_T while braking a bicycle wheel.

Activity

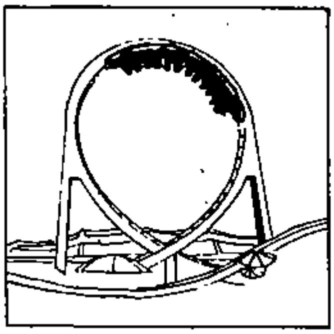
Try to rotate a merry-go-round, a grindstone or a bicycle wheel yourself. What is the direction in which you apply the force? Draw the direction on Fig. 4.10a.

We have seen that a transverse force is needed to increase the angular speed of a rotating object. The same force but in opposite direction would be required to reduce the angular speed of the object. This is what happens when you apply brakes while riding a bicycle. The surface of the brake B comes in contact with the rim of the wheel which rotates in an anticlockwise direction (see Fig. 4.10 b). It produces a transverse frictional force $\mathbf{F}_T$ in the opposite direction, decreasing the angular speed of the wheel..

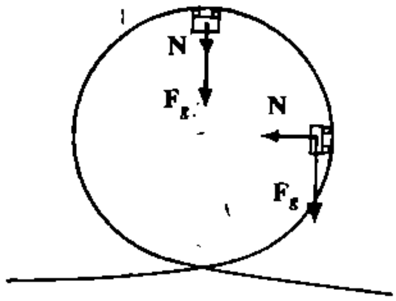
Actually, friction is always present between a rotating wheel and the shaft or axle about which it rotates. Therefore, left to itself it will stop rotating, sooner or later due to friction. This is the same as in straight line motion where a force of friction slows down a moving object till it stops.

Example 3

A roller coaster has a Loop-the-Loop section of radius r (Fig.4.11(a)). What should the speed of a train be if it is not to leave the track even at the top of the loop?



(a)



(b)

Fig. 4.11: (a) Loop-the-Loop roller coaster is a winding train track in amusement parks. Forces on the train include gravity and the normal force of reaction between the train and the track. The resultant of these forces provides the centripetal force to keep the train moving on a circular path;(b) at the top of the loop the net force on the passengers points downwards.

What are the forces acting on the train and the track? These are gravity and the normal force of reaction, between the train and track. The train will stay on the track only as long as the normal force of reaction between the train and the track remains non-zero. The forces are shown in the Fig. 4.11(b) at two points on the loop. The net force at any point is related to acceleration by Newton's second law:

$$\mathbf{F}_g + \mathbf{N} = m\mathbf{a}.$$

Let us, for convenience, choose a coordinate system with the positive direction downward. At the top of the loop, the vertical component of the force equation becomes

$$mg + N = ma = \frac{mv^2}{r},$$

so that

$$v^2 = gr + \frac{Nr}{m}.$$

Now, if N is to remain non-zero at the top of the loop, then

$$(v^2 - gr) > 0,$$

i.e. $v^2 > gr$,

or $v > \sqrt{gr}$.

Therefore, for the train to be in contact with the track even at the top of the loop, its speed should always be greater than $\sqrt{gr}$. So for a typical roller coaster for which $r = 6$m, say,

$\sqrt{gr} = \sqrt{(9.8\ \mathrm{ms^{-2}})(6\mathrm{m})} = 7.7\ \mathrm{ms^{-1}}$. The train's speed, therefore, should always be greater than 7.7 $\mathrm{ms^{-1}}$ in this case.

SAQ7

A level road has a turn of 95 m radius of curvature. What is the maximum speed with which a car can negotiate this turn (a) when the road is dry and the coefficient of static friction is 0.88 and (b) when the road is snow-covered and the coefficient of static friction is 0.21?

[Hint: The frictional force between tyres and road provides the car's acceleration.]

4.3.2 Angular Motion in General

Let us now determine the force acting on a particle executing accelerated angular motion.

From Newton's second law, using Eq. 4.14 we have

$$\mathbf{F} = m\mathbf{a} = m[\ddot{r} - r\dot{\theta}^2]\hat{\mathbf{r}} + m[r\ddot{\theta} + 2\dot{r}\dot{\theta}]\hat{\boldsymbol{\theta}} = \mathbf{F}_R + \mathbf{F}_T, \quad (4.18a)$$

where $\mathbf{F}_R$ is the radial force which acts along $\hat{\mathbf{r}}$ and has a magnitude

$$F_R = m(\ddot{r} - r\dot{\theta}^2). \quad (4.18b)$$

and $\mathbf{F}_T$ is a transverse force which acts perpendicular to $\hat{\mathbf{r}}$ and has a magnitude

$$F_T = m[r\ddot{\theta} + 2\dot{r}\dot{\theta}]. \quad (4.18c)$$

Equations 4.18 are very general. They can be used to solve any problem of motion in two dimensions, such as planetary motion. These expressions may look a little complicated to you. Don't let this put you off. All that we need to understand is this: We can use plane polar coordinates to describe any two-dimensional motion.Then, such a motion may be seen as a combination of straight line motion along the radius vector and a rotation about the origin of the frame of reference. The straight line motion is accelerated due to a radial force. The rotation, which is also an accelerated motion is the result of transverse force. For most situations Eqs. 4.18 are reduced to a simple form.

So far we have applied Newton's second law to study the angular motion of a particle. However, if the rotating object were a rigid body, then applying Newton's laws to determine the motion of every particle in it would be too cumbersome. Can we, instead, formulate an analogous law that deals directly with rotational quantities? For doing this, we need the analogues of force, linear momentum and acceleration for angular motion. We have seen that the angular acceleration is the rotational analogue of linear acceleration. What is the rotational analogue of force? The answer is torque, which we will now study.

4.3.3 Torque

Perform the following activity to understand what torque is.

Activity

Open a door by applying a force near its edge along the door's plane. Then try to open it by pushing it at the same point in a direction perpendicular to or at an angle with the door's plane. Next push it at a point near the hinges, with roughily the same force. In which case does the door open more quickly?

You can repeat this activity to open a book or to open a rusty nut with a spanner.

You would have noticed that in all cases, the job was easier if you applied the force at the point farthest away from the axis of rotation and also in a direction perpendicular to the plane of the door, or the book, or the arm of the spanner. So, how easily an object rotates depends not only on the force but also on the point and on the angle at which the force is applied, i.e. it depends on the torque. We define the torque for a single particle observed from an inertial frame of reference as follows:

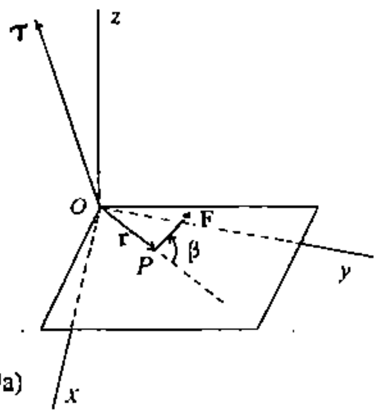


Fig.4.12: A force F is applied to a particle P, displaced r relative to the origin. F makes an angle β with r. The direction of torque is perpendicular to the plane containing r and F with the sense given by right-hand rule.

If a force **F** acts on a particle at a point P which has a position vector **r**, the torque $\boldsymbol{\tau}$ acting on the particle with reference to the origin O is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \quad (4.19a)$$

Torque is a vector quantity (Fig. 4.12). Its magnitude is given by

$$\tau = rF \sin \beta, \quad (4.19b)$$

where β is the angle between **r** and **F**. Its direction is normal to the plane formed by **r** and **F**. Thus $\boldsymbol{\tau}$ and **F** are always perpendicular to each other. The unit of torque is newton-metres. Now, if we substitute **F** from Eq. 4.18 (a), we get

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_R + \mathbf{r} \times \mathbf{F}_T. \quad (4.19c)$$

Since $\mathbf{F}_R$ is parallel to **r**, their cross product will be zero.

$$\therefore \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}_T. \quad (4.19d)$$

It is important to realise that torque and force are entirely different quantities. The concept of torque provides a relation between the applied force and the tendency of a body to rotate.

For one thing, torque depends on the origin but force does not. You produce greater torque for the same force, if you apply the force at greater distances from the pivot point or the origin. Again, for a given force and distance r, the torque is greatest when $\mathbf{r}$ and $\mathbf{F}$ are at right angles (see Fig. 4.13a).

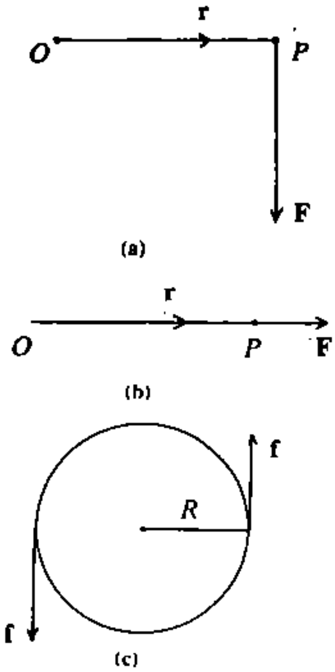


Fig 4.13:(a) The torque is greatest with F and r at right angles; (b) zero when they are collinear; (c) there can be a torque on a system with zero net force.

The torque becomes zero when $\mathbf{r}$ and $\mathbf{F}$ are along the same line (Fig. 4.13b). Thus, even if torque is zero, the external force need not be zero. The torque is also zero if the force acts on the point or along the axis, about which the particle is rotating. This is because in such a case, the vector $\mathbf{r}$ will be zero. Torque is obviously zero if the external force itself is zero. There can also be a torque on a system with zero net force (Fig. 4.13c). In general there will be both torque and force.

Let us now find out the torque acting on a particle in circular motion in the xy-plane. Using Eq. 4.17 c, and Eq. 4.19 d we get

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{r} \times mr\alpha\hat{\boldsymbol{\theta}} \\ &= mr^2\alpha(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}), \text{ since } \mathbf{r} = r\hat{\mathbf{r}} \\ &= mr^2\alpha\hat{\mathbf{k}}, \text{since } \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}, \text{ from SAQ 3(c)}. \\ &= mr^2\boldsymbol{\alpha},\end{aligned}$$

since $\alpha\hat{\mathbf{k}}$ is simply the angular acceleration vector $\boldsymbol{\alpha}$. (4.20)

Let us compare Eq. 4.20 with Newton's second law $\mathbf{F} = m\mathbf{a}$. The torque is the product of the angular acceleration α and a quantity mr^2. On comparison we can say that this quantity mr^2 is the rotational analogue of the mass. We call the quantity mr^2 the **rotational inertia** or **moment of inertia** and represent it by the symbol I. Rotational inertia has the units kg m^2 and accounts both for mass of the particle and for the location of the particle relative to the axis of rotation. You know that the inertial mass is a measure of the body's resistance to change in its state of motion. In the same way, rotational inertia is a measure of the body's resistance to change in its rotational motion. Note that I would change if we change the axis of rotation. In contrast m is a constant. Substituting I for mr^2 in Eq. 4.20, we can write for circular motion of a particle of mass m about a fixed axis of rotation

$$\tau = I\,\alpha, \qquad (4.21a)$$

$$\text{where } I = mr^2. \qquad (4.21b)$$

This equation is similar to Newton's second law. We can deduce the same kinds of things from it as we did from equation $\mathbf{F} = m\mathbf{a}$. For instance, for constant I, the angular acceleration is directly proportional to the applied torque. In the absence of torque, an object continues to move at a constant angular speed. And, the same torque will produce greater angular acceleration for an object of smaller moment of inertia. You can now apply the Eqs. 4.21a and 4.21b to solve a problem in which the torque acts to change the particle's angular velocity.

SAQ 8

You may have studied in Block 3 of Foundation Course FST 1 that a neutron star is an extremely dense, rapidly spinning object that results from the collapse of a star at the end of its life. A neutron star of mass 15×10^{30} kg has a rotational inertia of 45×10^{36} kg m^2 about an axis of rotation passing through its centre. The neutron star's rotation rate slowly decreases as a result of torque associated with magnetic forces. If the rate of change in its angular speed is 5×10^{-5} rad s^{-2}, what is the magnitude of magnetic torque?

Another question concerning angular motion is whether we can express the kinetic energy of a rotating particle in terms of the angular variables? Yes, we can. Let us see how to do it.

4.3.4 Kinetic Energy of Rotation

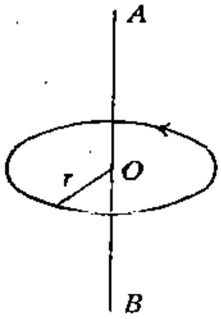


Fig. 4.14

Let us consider a particle of mass m moving in a circle of radius r about a fixed axis of rotation AOB (see Fig. 4.14). Let its angular speed about the axis be ω. Its kinetic energy is

$$\begin{aligned}\text{K.E.} &= \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2, \\ &= \frac{1}{2}mr^2\omega^2\end{aligned}$$

Thus, using Eq. 4.21b, we get

$$K_{Rot} = \frac{1}{2}I\omega^2. \qquad (4.22)$$

This is also termed as the **kinetic energy of rotation** of the body.

So far, we have studied some concepts of angular motion. We have seen that an analogy exists between the kinematics and dynamics of linear and angular motion. This analogy would be complete if we could define a physical quantity corresponding to linear momentum. Indeed, there is such a quantity called angular momentum. We will discuss angular momentum especially so as to arrive at another very important conservation law.

4.4 ANGULAR MOMENTUM

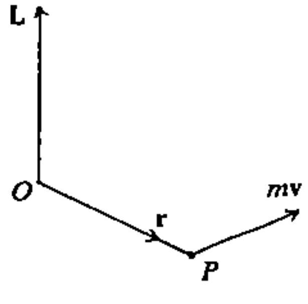


Fig. 4.15

We know that the torque on a particle due to a force **F** is given as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Since $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ from Newton's second law, we get

$$\boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}.$$

$$\text{Now, } \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times (m\mathbf{v}) + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \quad (\because \mathbf{p} = m\mathbf{v})$$

$$= \mathbf{0} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \quad (\because \mathbf{v} \times m\mathbf{v} = \mathbf{0}).$$

So, we can write,

$$\boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}).$$

We define the angular momentum **L** of the particle with respect to the origin O to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (4.23a)$$

Thus, angular momentum is a vector with magnitude

$$L = rp \sin \gamma, \qquad (4.23b)$$

where γ is the angle between **r** and **p**. The direction of **L** is perpendicular to the plane formed by **r** and **p**. It is determined by the right-hand rule (see Fig. 4.15). Although **L** has been drawn through the origin, this location has no special significance. Only the direction and magnitude of **L** are important. The unit of angular momentum is kg m^2 s^{-1}. Thus, the expression for torque becomes

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \qquad (4.24)$$

You can see that this relation is analogous to Newton's second law. We can also relate angular momentum to angular velocity. Let a particle of mass m move anticlockwise in the xy-plane about a fixed axis of rotation perpendicular to the plane with a linear momentum **p**. Then you know that its angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times \mathbf{v}.$$

Using Eq. 4.6 for **r** and Eq. 4.13 for **v** we get

$$\mathbf{L} = m\mathbf{r} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}),$$

$$= 0 + mr^2\dot{\theta}(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}), \text{ since } \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0. \qquad (4.25a)$$

$$\therefore \mathbf{L} = mr^2\,\dot{\theta}\hat{\mathbf{k}}$$

Now mr^2 is the moment of inertia I of the particle and $\dot{\theta}\hat{\mathbf{k}}$ is the angular velocity vector $\boldsymbol{\omega}$.

Therefore, we can write

$$\mathbf{L} = I\,\boldsymbol{\omega}. \qquad (4.25b)$$

Notice that this equation is analogous to $\mathbf{p} = m\mathbf{v}$.

Let us now work out an example on angular momentum.

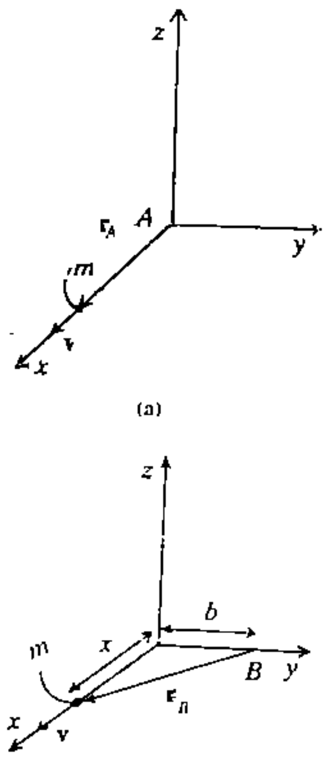


Fig. 4.16

Example 4: Angular momentum of a particle in uniform motion

A block of mass m and negligible dimensions moves at a constant speed v in a straight line (see Fig. 4.16). What is its angular momentum $\mathbf{L}_A$ about the origin A and its angular momentum $\mathbf{L}_B$ about the origin B?

Let the particle move along the x-axis, i.e., $\mathbf{v} = v\,\hat{\mathbf{i}}$. As shown in Fig. 4.16(a), the position vector of the particle with respect to A is

$$\mathbf{r}_A = x\hat{\mathbf{i}}$$

Since $\mathbf{r}_A$ is parallel to $\mathbf{v}$, their cross product is zero and

$$\mathbf{L}_A = \mathbf{r}_A \times m\mathbf{v} = m\mathbf{r}_A \times \mathbf{v} = \mathbf{0}.$$

The particle's angular momentum with respect to B is

$$\mathbf{L}_B = m\mathbf{r}_B \times \mathbf{v}.$$

We can write

$$\mathbf{r}_B = x\hat{\mathbf{i}} - b\hat{\mathbf{j}}$$

where x is the component of $\mathbf{r}_B$ parallel to $\mathbf{v}$ and b its component perpendicular to $\mathbf{v}$.

Since $\hat{\mathbf{i}} \times \mathbf{v} = \mathbf{0}$, only $b\hat{\mathbf{j}}$ contributes to $\mathbf{L}_B$. Thus,

$$\mathbf{L}_B = m(x\hat{\mathbf{i}} - b\hat{\mathbf{j}}) \times v\hat{\mathbf{i}} = \mathbf{0} - mbv\hat{\mathbf{j}} \times \hat{\mathbf{i}}$$

$$\mathbf{L}_B = mbv\hat{\mathbf{k}}.$$

Thus, $\mathbf{L}_B$ lies in the positive z-direction and has a magnitude mbv. This example shows how $\mathbf{L}$ depends on the choice of the origin. Further, for the particle moving in a straight line, b is constant. Therefore, the angular momentum of a particle moving at a constant speed in a straight line remains constant.

So, the torque acting on such a particle is zero.

Another idea brought out by the above example is this: Do not think that the quantities $\boldsymbol{\omega}$, $\mathbf{L}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ can be defined, or have meaning only for angular motion. Any moving object can possess an angular velocity, angular acceleration, angular momentum and torque about an origin. What is more, the same object can have different values for these quantities about different origins.

SAQ 9

A particle of mass m falls from rest in the earth's gravitational field according to Galileo's law $z = z_o - \frac{1}{2}gt^2$. Its horizontal coordinates are $x = x_o$, $y = 0$.

a) Determine the position vector $\mathbf{r}$ and velocity $\mathbf{v}$ of the particle at time t.

b) Find the angular momentum $\mathbf{L}$ as a function of time about the origin.

c) Determine the torque acting on the particle about the origin. (Hint: $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$)

4.4.1 Conservation of Angular Momentum and its Applications

What happens when the net external torque on the particle is zero? Eq. 4.24 becomes

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \mathbf{0}$$

i.e. $\mathbf{L}$ = constant.

Thus, we get the principle of conservation of angular momentum. *The angular momentum of a particle remains constant both in magnitude and direction if no net external torque acts on it.*

Constant angular momentum implies that the particle's motion is confined to a fixed plane normal to $\mathbf{L}$. This is because by definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{L}$ is normal to the plane containing $\mathbf{r}$ and $\mathbf{p}$. Since $\mathbf{L}$ is constant in direction, $\mathbf{r}$ and $\mathbf{v}$ will lie in a fixed plane normal to the constant vector $\mathbf{L}$. So we need to use only a two-dimensional coordinate system to study the particle's motion. The principle of conservation of angular momentum applies to systems ranging from subatomic particles to huge rotating galaxies. Let us study some applications of the law of conservation of angular momentum to understand it better.

Pointing a Satellite

Angular momentum conservation is used to steer a satellite, i.e. to point it in any desired direction. For this purpose wheels are fixed inside the satellite. Each wheel has a motor and brakes to start and stop its rotation. When a wheel starts rotating, the satellite rotates in the opposite direction to conserve the angular momentum. After the satellite has rotated through the desired angle, the wheel is stopped and the satellite also stops rotating. Three wheels are normally used so that the satellite can be pointed in any direction. The motors and brakes run on electricity generated through solar energy, so there is no fuel to run out.

It is also because of the conservation of angular momentum that a satellite's axis of rotation remains fixed in space. Satellites are usually rotationally isolated bodies. So the net torque acting on them is zero. Thus, the direction of **L** and hence the direction of the axis of rotation remains fixed. Therefore, spinning the satellite gives it a stability in orbit (Fig. 4.17).

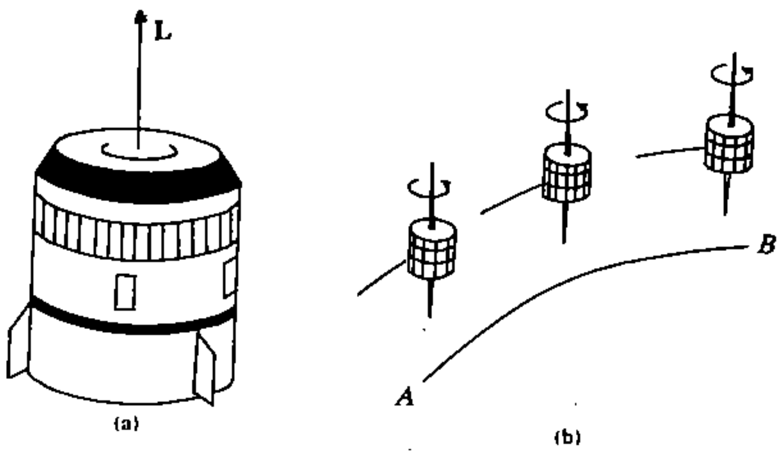


Fig. 4.17: (a) For a rotationally isolated satellite, L remains constant in magnitude and its direction remains fixed in space. Thus, the axis of rotation remains fixed as it is along L; (b) the fact that the axis of rotation remains fixed in space for constant L is used for stabilisation of a satellite by spinning it. *AB* shows a section of the earth's surface.

Angular acceleration accompanying contraction of a string

An object of mass m is attached to a string and is rotated in a horizontal plane (the plane of the dashed line in Fig. 4.18).

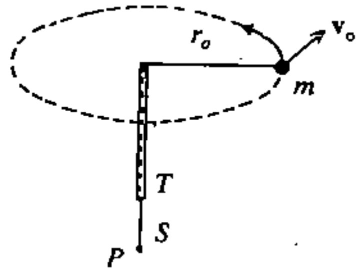


Fig. 4.18: Mass m describes circular motion of radius r_0 and velocity $\mathbf{v}_0$. It is connected to a string *S* which passes through a tube *T*. The radius of the circle can be shortened by pulling on the string at *P*.

The object rotates with velocity $\mathbf{v}_0$ when the radius of the circle is r_0. It is seen that as the string is shortened by pulling it in, the object speeds up even as it rotates. Why does the object speed up?

The force on the object due to the string is radial. Here we are neglecting the force of gravity. Thus, the net external torque on the object is zero and its angular momentum is conserved. Therefore, as the string is shortened, the angular momentum should remain constant. The magnitude of the initial angular momentum of the object when the radius of the circle is r_0 is $|m\mathbf{r}_0 \times \mathbf{v}_0| = mr_0v_0 \sin 90° = mr_0v_0$.
The magnitude of the object's angular momentum when the radius of the circle is shortened to r is $|m\mathbf{r} \times \mathbf{v}| = mrv \sin 90° = mrv$.

Since angular momentum is constant, $mr_0 v_0 = mrv$.

This gives

$$v = \frac{v_0 r_0}{r}.$$

As r is smaller than r_0, v will be greater than v_0, that is the object will speed up

Let us now summarise the unit.

4.5 SUMMARY

- Infinitesimal angular displacements are vectors. The angular velocity and angular acceleration vectors are defined as

  $$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}, \quad \boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$$

  The directions of angular displacement and angular velocity vectors are taken along the axis of rotation, and their sense is determined by the right-hand rule.

- Plane polar coordinates can be used to describe angular motion in two dimensions and to express the relationship between kinematical variables of linear and angular motion.

- For *uniform circular motion, r and ω are constant.*

$$\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = r\omega\hat{\boldsymbol{\theta}},\ \mathbf{a}_R = -\frac{v^2}{r}\hat{\mathbf{r}}.$$

- For *circular motion r is constant, ω varies giving a finite α and*

$$\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = r\omega\hat{\boldsymbol{\theta}},\ \mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}} + \alpha r\hat{\boldsymbol{\theta}}.$$

- For *general angular motion, r is a variable*

$$\mathbf{r} = r\hat{\mathbf{r}},\ \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = \mathbf{a}_R + \mathbf{a}_t.$$

- The vector forms of these relationships are

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r},\ \mathbf{a}_R = \boldsymbol{\omega} \times \mathbf{v},\ \mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}.$$

- Torque and moment of inertia are the analogues of force and inertial mass for angular motion. The torque acting on a particle displaced by **r** under the influence of force **F** around the origin is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

- There exists a relationship analogous to Newton's second law between torque and angular momentum:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt},\ \text{where } \mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

- For a particle of mass m moving in a circle of radius r around a fixed axis of rotation $\tau = I\alpha$ where $I = mr^2$ is its moment of inertia.

- The kinetic energy of a particle of mass m rotating with an angular speed ω is

$$K_{Rot} = \frac{1}{2}I\omega^2$$

- If the net external torque acting on a system is zero, the angular momentum of the system is constant both in magnitude and direction. This is the principle of conservation of angular momentum and it has many applications.

4.6 TERMINAL QUESTIONS

1. Take a rectangular coordinate system. A particle moves parallel to x-axis with a constant speed v. Show that the magnitude of its angular velocity varies inversely as the square of its distance from the origin. Also obtain an expression for the magnitude of its angular acceleration.

2. A particle of mass 5g moves in a plane with constant radial speed $\dot{r} = 4$ m s^{-1}. The angular velocity is constant and has magnitude $\dot{\theta} = 2$ rad s^{-1}. When the particle is 3 m from the origin, find the (a) velocity, (b) acceleration and (c) kinetic energy of the particle.

3. A particle of mass m moves along a space curve defined by $\mathbf{r} = 6t^4\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + (4t^3 - 5)\hat{\mathbf{k}}$. Find its (a) angular momentum, (b) torque and (c) kinetic energy of rotation about the origin.

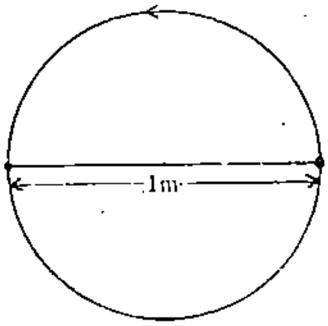


Fig. 4.19

4. Two objects of mass 20g and 30g are connected by a light rod of length 1m and move in a horizontal circle as shown in Fig. 4.19. The speed of each is 2 m s^{-1}. (a) What is the total angular momentum of the objects about the centre? (b) If the rod contracts uniformly to half of its original length, will the speed of the objects change? If so, by how much?

4.7 ANSWERS

SAQs

1. The magnitude of the rotation will be $2\pi/3$ rad. Its direction will be perpendicular to the face of the clock pointing away from you if you are holding it face up.

2. Since ω is constant, $\frac{d\omega}{dt} = 0$, or $d\frac{(\omega^2)}{dt} = \frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = 0.$

Since $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$.

$$\frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = \frac{d\boldsymbol{\omega}}{dt}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\frac{d\boldsymbol{\omega}}{dt} = 2\frac{d\boldsymbol{\omega}}{dt}\cdot\boldsymbol{\omega}, \ (\because \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A})$$

$$\therefore \frac{d}{dt}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}) = 2\boldsymbol{\alpha}\cdot\boldsymbol{\omega}, \text{ or } 2\boldsymbol{\alpha}\cdot\boldsymbol{\omega} = 0$$

This implies that $\boldsymbol{\alpha}$ is perpendicular to $\boldsymbol{\omega}$.

3. a) i) $|\hat{\mathbf{r}}| = \sqrt{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}} = \sqrt{(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\cdot(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}$

$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

ii) $|\hat{\boldsymbol{\theta}}| = \sqrt{\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}} = \sqrt{(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})\cdot(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

iii) $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\cdot(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

b) $\mathbf{A}\cdot\mathbf{B} = (A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}})\cdot(B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}})$

$$= A_rB_r(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) + A_rB_\theta(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}}) + A_\theta B_r(\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{r}}) + A_\theta B_\theta(\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}})$$

$$= A_rB_r(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) + A_\theta B_\theta(\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}) \qquad (\because \hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{r}} = 0)$$

From a.i) and ii) $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = 1$

Therefore, $\mathbf{A}\cdot\mathbf{B} = A_rB_r + A_\theta B_\theta$

c) $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\times(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$

$$= \cos^2\theta(\hat{\mathbf{i}}\times\hat{\mathbf{j}}) - \sin^2\theta(\hat{\mathbf{j}}\times\hat{\mathbf{i}}) \qquad (\because \hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \mathbf{0})$$

$$= (\cos^2\theta + \sin^2\theta)\hat{\mathbf{k}} = \hat{\mathbf{k}} \qquad (\because \hat{\mathbf{i}}\times\hat{\mathbf{j}} = -\hat{\mathbf{j}}\times\hat{\mathbf{i}} = \hat{\mathbf{k}})$$

4. We shall use the equations for constant angular acceleration given in Table 4.1 along with Eqs. 4.6 to 4.11. Hence $r = 0.5\text{m}$, $\alpha = 3.0 \text{ rad s}^{-2}$.

a) The magnitude of the angular displacement is given by

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\alpha t^2, \ \because \omega_o = 0 \text{ in this case}$$

or $\theta = \frac{1}{2}(3 \text{ rad s}^{-2})\times(2^2\text{s}^2) = 6 \text{ rad}$

The direction of θ will point along the axis of rotation from O to A (Fig. 4.7). The angular speed

$$\omega = \omega_o + \alpha t = \alpha t = 3 \text{ rad s}^{-2}\times 2\text{s}$$

or $\omega = 6 \text{ rad s}^{-1}$

and the direction of angular velocity is along OA.

b) The linear velocity $\mathbf{v}$ is given by

$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r\omega\hat{\boldsymbol{\theta}}$ since r is constant,

or $\mathbf{v} = 0.5\text{m}\times 6 \text{ rad s}^{-1}\hat{\boldsymbol{\theta}}$ since $\omega = 6 \text{ rad s}^{-1}$ at $t = 2\text{s}$.

So the linear velocity of the particle has a magnitude 3m s^{-1} and is directed along the tangent at that point.

Radial acceleration $\mathbf{a}_R = -\omega^2 r\hat{\mathbf{r}} = -(6 \text{ rad s}^{-1})^2\times 0.5 \text{ m}\hat{\mathbf{r}}$

$$= -18 \text{ m s}^{-2}\hat{\mathbf{r}}.$$

Transverse acceleration $\mathbf{a}_T = \alpha r\hat{\boldsymbol{\theta}}$

$$= (3.0 \text{ rad s}^{-2})\times(0.5\text{m})\hat{\boldsymbol{\theta}} = 1.5 \text{ m s}^{-2}\hat{\boldsymbol{\theta}}$$

You must note that radian is the unit of angle which is dimensionless and hence its multiplication with any other unit leaves it unchanged.

5 The trajectory of the particle is given by

$$r = C\theta = \left(\frac{1}{\pi}\text{ m rad}^{-1}\right)\left(\frac{\alpha}{2}t^2 \text{ rad}\right) = \frac{\alpha t^2}{2\pi}\text{ m}$$

a) The velocity $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ from Eq. 4.13 a.

Since $\dot{r} = \frac{dr}{dt} = \frac{\alpha t}{\pi}\text{ m s}^{-1}$ and $\dot{\theta} = \frac{d\theta}{dt} = \alpha t \text{ rad s}^{-1}$

$$\mathbf{v} = \left(\frac{\alpha t}{\pi}\hat{\mathbf{r}} + \frac{\alpha t^2}{2\pi}\cdot \alpha t\,\hat{\boldsymbol{\theta}}\right)\text{m s}^{-1}$$

$$= \frac{\alpha t}{\pi}\left(\hat{\mathbf{r}} + \frac{\alpha t^2}{2}\hat{\boldsymbol{\theta}}\right)\text{m s}^{-1}$$

Acceleration $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$

Since $\ddot{r} = \frac{d\dot{r}}{dt} = \frac{\alpha}{\pi}\text{ms}^{-2}$, $\ddot{\theta} = \alpha\,\text{rad s}^{-2}$

$$\mathbf{a} = \left(\frac{\alpha}{\pi} - \frac{\alpha t^2}{2\pi}\cdot\alpha^2 t^2\right)\hat{\mathbf{r}} + \left(\frac{\alpha t^2}{2\pi}\cdot\alpha + \frac{2\alpha t}{\pi}\cdot\alpha t\right)\hat{\boldsymbol{\theta}}$$

$$= \left(\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4\right)\hat{\mathbf{r}} + \frac{5}{2\pi}\alpha^2 t^2\hat{\boldsymbol{\theta}}.$$

b) $a_R = 0$ means that

$$\frac{\alpha}{\pi} - \frac{\alpha^3}{2\pi}t^4 = 0, \quad \text{or } \alpha^3 t^4 = 2\alpha.$$

Since $\alpha \neq 0$, we get $\alpha^2 t^4 = 2$ or $\left(\frac{\alpha t^2}{2}\right)^2 \cdot 2 = 1$.

i.e. $\theta^2 = \frac{1}{2}\text{rad}^2$ giving $\theta = \frac{1}{\sqrt{2}}$ rad.

6. The tension in the massless cable holding the moon will provide the centripetal force $\frac{mv^2}{r}$. Now, if the moon were held by the force of gravitation between the earth and the moon then,

$$\frac{mv^2}{r} = \frac{Gmm_E}{r^2}$$

where m and m_E are the masses of the moon and the earth, respectively, and r is the mean distance between the moon and the earth.

So, the tension T in the cable $= \frac{mv^2}{r} = \frac{Gmm_E}{r^2}$

$$\text{or } T = \frac{(6.673\times10^{-11}\,\text{Nm}^2\text{kg}^{-2})\times(7.35\times10^{22}\,\text{kg})\times(5.97\times10^{24}\,\text{kg})}{(3.85\times10^{8}\,\text{m})^2}$$

$$= 1.98\times10^{20}\,\text{N}.$$

Compare this with the tensions in cables required to lift cars or trucks which are of the order of 20,000 N.

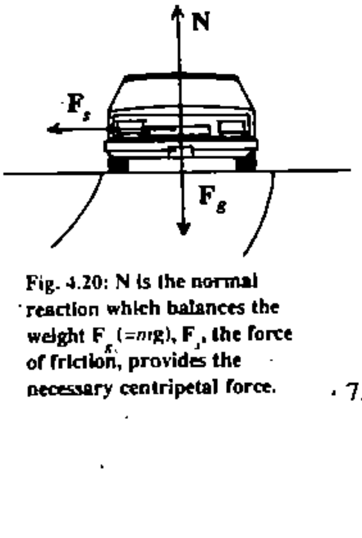


Fig. 4.20: N is the normal reaction which balances the weight F_g (=mg), F_s, the force of friction, provides the necessary centripetal force.

7. The centripetal force is provided by the force of friction between the car's tyres and the road (Fig. 4.20).

The magnitude of the force of friction $F_s = \mu_s N = \mu_s mg$, where m = the mass of the car.

So $\mu_s mg = \frac{mv^2}{r}$, where v = the maximum possible speed of the car.

$$\therefore v = \sqrt{\mu_s r g}\,.$$

a) For the dry road, $v = \sqrt{(0.88)(95\,\text{m})(9.8\,\text{ms}^{-2})} = 29\,ms^{-1}$,

b) For the snow covered road $v = \sqrt{(0.21)(95\,\text{m})(9.8\,\text{ms}^{-2})} = 14\,\text{ms}^{-1}$,

If this speed is exceeded, the car must move in a path of greater radius which means it will go off the road.

8. The torque of an object is related to its angular acceleration by $\tau = I\alpha$. Here

$I = 45\times10^{36}$ kg m^2 and $\alpha = 5\times10^{-3}$ rad s^{-2} .

Therefore, the magnitude of the magnetic torque $= (45\times10^{36}\,\text{kg m}^2)\times(5\times10^{-3}\,\text{rad s}^{-2})$ $= 2.2\times10^{33}$ newton-metres.

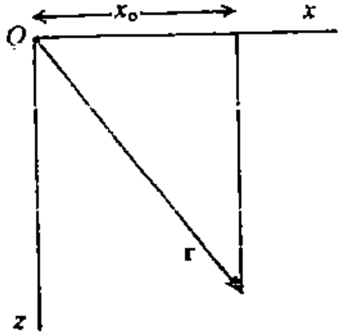


Fig. 4.21

9. a) Refer to Fig. 4.21. The position vector **r** of the particle with respect to origin at time t is

$$\mathbf{r} = x_0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}.$$

$$= x_0\hat{\mathbf{i}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}.$$

Its velocity at time t is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -gt\hat{\mathbf{k}}.$$

b) The angular momentum

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \mathbf{r}\times m\mathbf{v} = m(\mathbf{r}\times\mathbf{v})$$

or $\mathbf{L} = m\left[x_0\hat{\mathbf{i}} + \left(z_0 - \frac{1}{2}gt^2\right)\hat{\mathbf{k}}\right]\times\left[-gt\hat{\mathbf{k}}\right]$

$$= m\left[x_0gt\hat{\mathbf{j}} + 0\right] \quad \left[\because \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}, \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0\right]$$

$$= mx_0gt\hat{\mathbf{j}}.$$

c) The torque acting on the particle about the origin is

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = mx_0g\hat{\mathbf{j}}$$

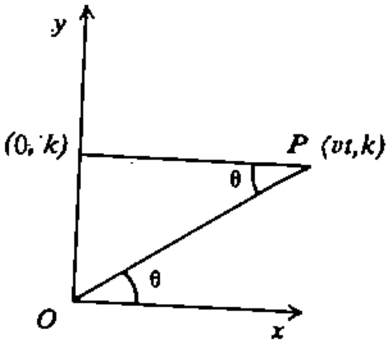


Fig. 4.22

Terminal Questions

1. Refer to Fig. 4.22. Let the particle be at P at the time t. Its distance from the y-axis is then equal to vt. So $\theta = \tan^{-1}\ (k/vt)$. The magnitude of its angular velocity is given by

$$\omega = \frac{d\theta}{dt} = \left[\frac{1}{1 + k^2/v^2t^2}\right]\left(-\frac{k}{vt^2}\right) = \frac{-kv}{k^2 + v^2t^2}.$$

or, $\omega = \frac{\text{a constant}}{OP^2}$. So, ω is inversely proportional to OP^2.

The magnitude of angular acceleration is given by

$$\alpha = \frac{d\omega}{dt} = \frac{2kv^3t}{(k^2 + v^2t^2)^2}.$$

2. a) The linear velocity of the particle is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Here $\dot{r} = 4\ \text{m s}^{-1}$, $\dot{\theta} = 2\ \text{rad s}^{-1}$ and $r = 3\text{m}$. So,

$$\mathbf{v} = (4\ \text{m s}^{-1})\,\hat{\mathbf{r}} + (3\ \text{m}\times 2\ \text{rad s}^{-1})\hat{\boldsymbol{\theta}}$$

$$= (4\hat{\mathbf{r}} + 6\hat{\boldsymbol{\theta}})\ \text{m s}^{-1} = \mathbf{v}_R + \mathbf{v}_T$$

The magnitude of the velocity $v = \sqrt{v_R^2 + v_T^2}$

$$= \sqrt{4^2 + 6^2}\ \text{m s}^{-1} = 2\sqrt{13}\ \text{m s}^{-1}$$

Its direction is given from Fig. 4.23 by

$\tan\phi_1 = \frac{v_T}{v_R} = \frac{6}{4} = 1.5$ where

ϕ_1 is the angle which $\mathbf{v}$ makes with $\mathbf{r}$.

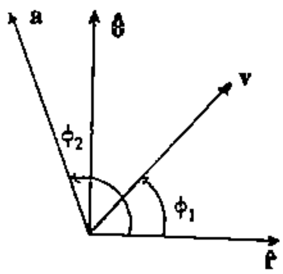


Fig. 4.23

b) From Eq. 4.14 acceleration $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

Since $\dot{r}$ and $\dot{\theta}$ are constant, $\ddot{r} = 0$, $\ddot{\theta} = 0$.

So $\mathbf{a} = (-r\dot{\theta}^2)\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\theta}$

$$= (-3\text{m}\times 4\ \text{rad}^2\text{s}^{-2})\hat{\mathbf{r}} + (2\times 4\text{m s}^{-1}\times 2\ \text{rad}^2\text{s}^{-1})\hat{\theta}$$

$$= -12\text{m s}^{-2}\hat{\mathbf{r}} + 16\text{m s}^{-2}\hat{\boldsymbol{\theta}}.$$

Its magnitude $a = \sqrt{(12\times 12) + (16\times 16)}\ \text{m s}^{-2} = 20\ \text{m s}^{-2}$ and its direction is given by $\tan\phi_2 = \frac{a_T}{a_R} = \left(-\frac{16}{12}\right) = -1.3$, where ϕ_2 is the angle which $\mathbf{a}$ makes with $\hat{\mathbf{r}}$ (see Fig. 4.23).

c) The kinetic energy of the particle is $K_{Rot} = \frac{1}{2}I\omega^2$

$$= \frac{1}{2}mr^2\dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{5}{1000}\text{ kg}\right)\cdot(3\text{m})^2(2\text{ rad s}^{-1})^2$$
$$= 0.1 \text{ joule}$$

3. Since $\mathbf{r} = 6t^4\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + (4t^3 - 5)\hat{\mathbf{k}}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 24t^3\hat{\mathbf{i}} - 6t\hat{\mathbf{j}} + 12t^2\hat{\mathbf{k}}$$

Angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times\mathbf{v}$

or $\mathbf{L} = m[6t^4\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + (4t^3 - 5)\hat{\mathbf{k}}]\times[24t^3\hat{\mathbf{i}} - 6t\hat{\mathbf{j}} + 12t^2\hat{\mathbf{k}}]$

$= m[-36t^5\hat{\mathbf{k}} - 72t^6\hat{\mathbf{j}} + 72t^5\hat{\mathbf{k}} - 36t^4\hat{\mathbf{i}} + (96t^6 - 120t^3)\hat{\mathbf{j}} + (24t^4 - 30t)\hat{\mathbf{i}}]$

$= m[-(12t^4 + 30t)\hat{\mathbf{i}} + (24t^6 - 120t^3)\hat{\mathbf{j}} + 36t^5\hat{\mathbf{k}}]$

b) Torque $\tau = \frac{d\mathbf{L}}{dt} = m[-(48t^3 + 30)\hat{\mathbf{i}} + (144t^5 - 360t^2)\hat{\mathbf{j}} + 180t^4\hat{\mathbf{k}}]$

c) Kinetic energy of rotation $= \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$

$$= \frac{1}{2}m[576t^6 + 36t^2 + 144t^4]$$
$$= 18mt^2[16t^4 + 4t^2 + 1].$$

4. a) Refer to Fig. 4.24. The total angular momentum of the system

$= (0.02\text{ kg})(2\text{m s}^{-1})(0.5\text{ m}) + (0.03\text{ kg})(2\text{m s}^{-1})(0.5\text{m})$

$= 0.05\text{ kg m}^2\text{s}^{-1}$.

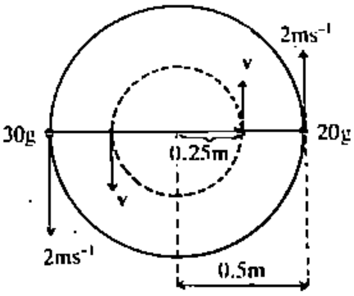


Fig. 4.24

b) Since no external torque acts on the system, the angular momentum remains conserved. As the rod is light, we shall assume it to be massless. As the particles remain connected by the rod the magnitudes of their velocities must be same (=v,say). When the rod gets contracted to half its original length the radius of the circular path (shown dotted) becomes 0.25 m (Fig. 4.24). So, the total angular momentum

$= (0.02\text{ kg})(v)(0.25\text{m}) + (0.03\text{ kg})(v)(0.25\text{m})$

$= 0.05\times 0.25\ v\text{ kg m}$

From the principle of conservation of angular momentum, we get

$0.05\text{ kg m}^2\text{ s}^{-1} = 0.05\times 0.25\ v\text{ kg m}$.

or $v = 4\text{m s}^{-1}$. So, speed of each particle becomes 4m s^{-1} i.e. double the original value.

UNIT 5 GRAVITATION

5.1 INTRODUCTION

In the previous four units you have studied linear as well as angular motion of a variety of objects. However, by and large we restricted our study to motion of objects on the earth. We did discuss some examples of motion of heavenly bodies but they lacked in details for want of the knowledge of gravitation. Therefore, we shall study gravitation in this unit.

We shall start from the familiar Kepler's laws of planetary motion to arrive at the law of universal gravitation. We shall then develop the concept of gravitational field and potential and use them to revisit the ideas of earth's gravity, and escape velocity. Finally, we shall visualise the gravitational force as a fundamental force in nature. Alongwith that we shall discuss, in brief, the electroweak and strong forces which are the other basic forces in nature.

In Block 2, we shall apply the concepts of mechanics developed in this block to motion under central conservative forces, systems of many particles and rigid bodies. We shall also study motion in accelerating frames of reference.

Objectives

After studying this unit you should be able to:

- apply the law of gravitation
- infer that the law of gravitation is universally true
- compute gravitational intensity and potential
- solve problems related to the variation of acceleration due to gravity with the height, depth and latitude of a place
- derive expression for velocity of escape
- distinguish between the fundamental forces in nature.

5.2 LAW OF GRAVITATION

You must be aware that the 'Law of Gravitation' was formulated by Sir Isaac Newton. The popular story goes like this:

Newton was sitting under a tree from which an apple fell and struck him on his head. This gave him the necessary impetus to discover the law. There could have been another part in the story: Newton was staring at the moon when the apple hit him (Fig. 5.1)! Newton's stroke of genius was that he realised that *the force which causes apples to fall to the ground is of the same kind as the force which causes the moon to orbit the earth.* In fact, the law of gravitation did not strike Newton in his first effort. He was looking for the answers to many questions related to wide-ranging topics from the 'Law of Falling Bodies' due to Galileo to Kepler's 'Laws of Planetary Motion'. Let us first arrive at the law of gravitation using Kepler's laws (Fig. 5.2). We shall then examine its universality

Fig. 5.1: Newton realised that all objects in the universe whether on earth or in heavens move under the influence of the same force of gravity.

through the discussion of the motion of the falling apple and that of the moon around the earth.

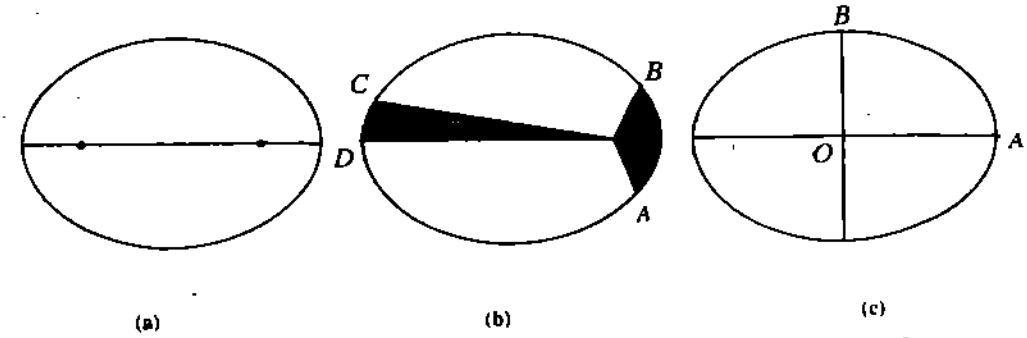


Fig. 5.2: Kepler's laws: (a) All planets move around the sun in elliptic orbits with the sun at one focus; (b) the equal area law: the line joining a planet and the sun sweeps out equal areas in equal intervals of time; (c) the square of the time period of revolution of a planet around the sun is directly proportional to the cube of the semi-major axis of the elliptic orbit. Here OA = semi-major axis, OB = semi-minor axis.

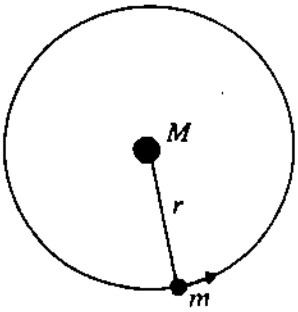


Fig. 5.3: Planet of mass m moving around the sun of mass M.

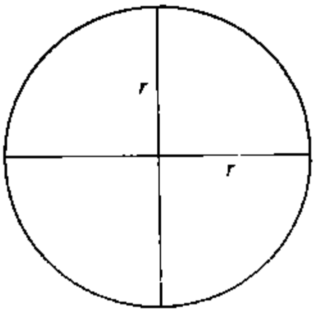


Fig. 5.4: A circle is a special case of an ellipse.

5.2.1 Arriving at the Law

Let us make an approximation and consider the orbit of a planet to be circular rather than elliptic. Let a planet of mass m revolve round the sun of mass M in a circular orbit of radius r with a linear speed v (Fig. 5.3). Let us further assume that m and M are point masses as their sizes are much smaller compared to the distance between their centres. The planet for being in the orbit demands a centripetal force (as discussed in Unit 4) whose magnitude is given by $F = \frac{mv^2}{r}$. Since, the time-period of revolution of the planet is $T = \frac{2\pi r}{v}$, we get

$$F = \frac{4\pi^2 mr}{T^2}. \tag{5.1}$$

Now, a circle is a special case of an ellipse (Fig. 5.4) whose semi-major axis is equal to its radius. According to Kepler's third law $T^2 = Cr^3$, where C is a constant. So from Eq. 5.1 we get $F = \frac{4\pi^2 m}{Cr^2} = \frac{km}{r^2}$, where $k = \frac{4\pi^2}{C}$ = a constant. (5.2)

Eq. 5.2 gives an expression of F, the centripetal force necessary for the planet for being in the circular orbit. F is a force experienced by the planet towards the centre of circular orbit, i.e. towards the sun. With these ideas in mind you may like to try an SAQ.

SAQ 1

Define the radius vector **r** as originating from the sun and ending at the planet and write down the expression for the vector **F**.

We have not yet arrived at the law of gravitation. We may rewrite Eq. 5.2 as

$$F_{planet} = \frac{k_{sun} m}{r^2} \tag{5.2a}$$

This indicates that the force is on the planet due to sun. At this stage let us recall the good old third law of motion. We know that the sun experiences the same force due to the planet as the planet due to the sun. So following Eq. 5 2a we have

$$F_{planet} = F_{sun} = \frac{k_{sun} m}{r^2} = \frac{k_{planet} M}{r^2}, \tag{5.2b}$$

$$\text{or } k_{sun} m = k_{planet} M, \text{ i. e. } \frac{k_{sun}}{M} = \frac{k_{planet}}{m}. \tag{5.3}$$

So we get the nature of dependence of k on the mass of the respective celestial object. k is directly proportional to its mass and the constant of proportionality is called the '**Universal Gravitational Constant**' and is denoted by G. We have not yet explained why we call the constant '*universal*'. We shall take that up soon. Going back to Eq. 5.3 we get,

$k_{sun} = MG$ and hence from Eq. 5.2b we have

$$F_{planet} = F_{sun} = \frac{GMm}{r^2}. \tag{5.4}$$

So the force between a planet and the sun is one of mutual attraction and is proportional to the product of their masses and inversely proportional to the square of the distance between them.

If you go back and read the paragraph before Eq. 5.1 you will remember that m and M were considered as point masses. Keeping this in view we consider earth and apple A to be point masses, having masses M_e and M_a, respectively. Let the distance between them be r, i.e. $r = R_e + h$, (see Fig. 5.5), where

R_e = Radius of earth

h = Height of the point above the earth's surface from which the apple falls.

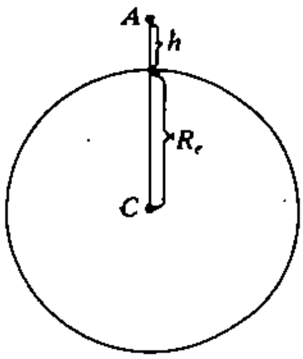


Fig. 5.5: An apple (A) at a distance ($R_e + h$) from the centre of the earth (C).

Here, the apple experiences a force of attraction due to the earth. Its magnitude (according to Eq. 5.4) is given by

$$F = \frac{GM_eM_a}{r^2}. \qquad (5.5)$$

Now $r^2 = (R_e + h)^2 \approx R_e^2$, as h is much smaller in comparison to R_e. So $F = \frac{GM_eM_a}{R_e^2}$.

Thus, the acceleration of the apple towards the earth is

$$a = \frac{F}{M_a} = \frac{GM_e}{R_e^2}. \qquad (5.6)$$

Study Eq. 5.6 carefully. Everything on its right-hand side is a constant on the earth and the left-hand side stands for the acceleration of an object falling near the surface of the earth. Now, as you are very well aware of the *'Law of Falling Bodies'* due to Galileo try the following SAQ.

SAQ 2

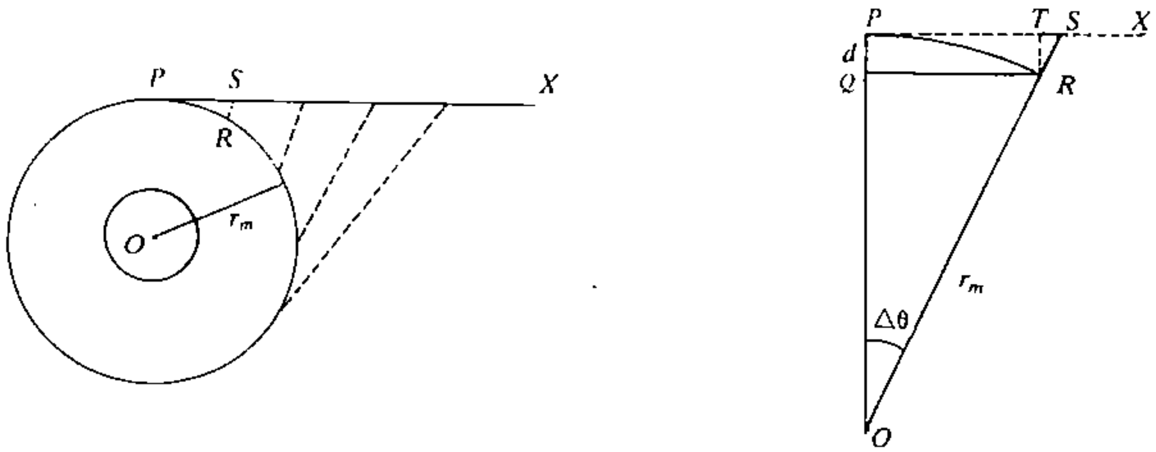
Show that Eq. 5.6 agrees with the 'Law of Falling Bodies'. [*Hint*: The acceleration of a falling body near the surface of earth is a constant irrespective of the body.]

Let us now apply Eq. 5.6 to analyse the fact that the moon falls towards the earth very much as the apple does.

5.2.2 Moon's Rotation about the Earth

Let the moon be at any position P in its orbit (Fig. 5.6a).

If no force were acting on the moon, it would have travelled along a straight line PX tangent to the orbit at P. Instead it follows the circular path of radius r_m about the centre of the earth.



(a) (b)

Fig. 5.6 (a) In absence of gravity the moon would have followed the straight path PX. However, in its circular orbit it can be regarded as falling away from the straight path (shown by dashed lines). Inner circle represents earth and the outer circle represents the orbit of the moon around the earth; (b) $\Delta\theta = \omega\ \Delta t$, $\omega = \frac{2\pi}{T}$ = the angular velocity of moon's rotation about the earth where T is the time-period of rotation.

Now let the moon travel from P to R in a very short time Δt. Let its linear velocity be v. In the absence of any force it would have travelled a distance $v\Delta t$ (= PS, say) along PX. Its motion along the circular arc PR can, therefore, be considered as a fall towards the earth through a distance SR. ST and RQ are drawn perpendicular to PX and OP, respectively. Since PR is infinitesimal, $SR = TR = PQ = d$, say. So, effectively the fall in the time interval Δt is d.

Now, $d = r_m(1-\cos\Delta\theta) = 2r_m \sin^2\left(\frac{\Delta\theta}{2}\right) \approx \frac{r_m}{2}(\Delta\theta)^2$, as $\sin\frac{\Delta\theta}{2} = \frac{\Delta\theta}{2}$ for small $\Delta\theta$.

Since $\Delta\theta = \omega(\Delta t) = \frac{2\pi}{T}(\Delta t)$,

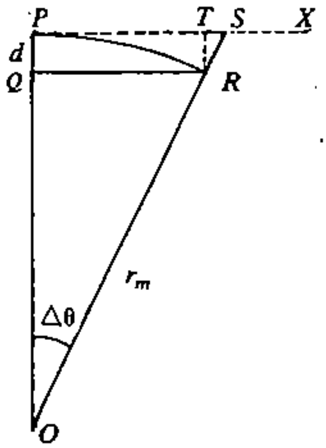


Fig. 5.6b repeated for ready reference

$$d = \frac{r_m}{2}\left(\frac{2\pi}{T}\Delta t\right)^2 = \frac{2\pi^2 r_m}{T^2}(\Delta t)^2. \quad (5.7)$$

Now, r_m and T are known to have the values 3.85×10^8 m and 2.4×10^6 s, respectively. If we put $\Delta t = 1$s, d is found to be 1.3×10^{-3} m. This means that while moon turns for 1s around the earth, the distance by which it falls towards the earth is a little over 1 mm. If the acceleration of the moon towards the earth is a_m in ms^{-2}, then the distance through which the moon falls in 1 s is given by

$$d = \frac{1}{2}a_m(1\text{s})^2 = 1.3\times10^{-3}\text{m}, \quad (5.8)$$

or $a_m = 2.6\times10^{-3}$m s^{-2}.

Now, the acceleration a of a freely falling object near the surface of the earth has an experimentally determined approximate value of 9.8 m s^{-2}. This gives a value for the ratio of the two accelerations:

$$a_m/a = 2.6\times10^{-4}. \quad (5.9)$$

Note that in the estimation of this ratio, the law of gravitation has not been used at all. Now using Eq. 5.6, we get

$$\frac{a_m}{a} = \frac{GM_e / r_m^2}{GM_e / R_e^2} = \left(\frac{R_e}{r_m}\right)^2: \quad (5.10)$$

Now, putting the values of R_e and r_m we have $a_m/a = 2.7\times10^{-4}$ which agrees reasonably well with Eq. 5.9.

Newton argued that this could not be a coincidence. There is a force of attraction between two objects that is proportional to the product of their masses and inversely proportional to the square of the distance between the two. The law is indeed universally applicable to all objects in the universe, be these dust particles or stars and galaxies. Hence the constant G is universal. Its value is 6.673×10^{-11} Nm^2 kg^{-2} and is same for all pairs of particles. So we are now in a position to state the *'Law of Universal Gravitation'*:

Newton's theory of gravitation was the culmination of two centuries of scientific revolution that began in 1543 through Copernicus. After that the works of Tycho Brahe, Kepler and in particular Galileo provided the necessary launching pad for Newton's law.

The force between any two particles having masses m_1 and m_2 separated by a distance r is attractive, acting along the line joining the particles and has the magnitude,

$$F = G\frac{m_1 m_2}{r^2} \quad (5.11)$$

But force as we know is a vector quantity. Hence, we must take care of the direction of F. So we shall rewrite Eq. 5.11 vectorially. Refer to Fig. 5.7. Let $\mathbf{r}_{12}$ be the position vector of m_2 with respect to m_1, i.e it points from m_1 to m_2. The gravitational force $\mathbf{F}_{12}$, exerted by m_1 on m_2 is given by

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}. \quad (5.12)$$

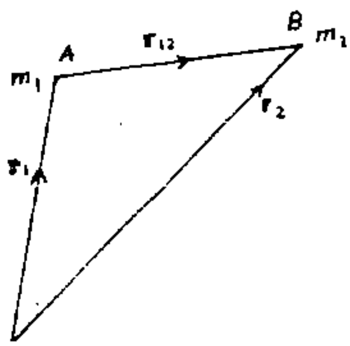


Fig. 5.7 : $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$

$\hat{\mathbf{r}}_{12}$ is the unit vector from m_1 to m_2. The minus sign indicates that the force on m_2 due to m_1, is directed opposite to $\mathbf{r}_{12}$, it being a force of attraction. Likewise $\mathbf{F}_{21}$, the force experienced by m_1 due to m_2 will be directed along $\mathbf{r}_{12}$. We know from Newton's third law of motion that, $\mathbf{F}_{21} = -\mathbf{F}_{12}$. So we have from Eq. 5.12

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}. \quad (5.12a)$$

However, did you note one point? Eq. 5.12 was stated for two point masses. For dealing with situations like attraction between earth and moon we considered them as point masses as the distance between them was much greater compared to their sizes. But otherwise we may have to calculate the force between a sphere and a point mass, say for example the force between earth and a particle. To tackle such a problem we need to know the 'Principle of Superposition'.

5.3 PRINCIPLE OF SUPERPOSITION

Eq. 5.12 gives us the force between two point masses. If there are several masses, like m_1, m_2 and m_3 as shown in Fig. 5.8, how would we calculate the gravitational force on one of them, say m_1? If only m_1 and m_2 were present, the force on m_1 due to m_2 would be

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\hat{\mathbf{r}}_{21}.$$

Similarly if only m_1 and m_3 were present, the force on m_1 due to m_3 would be

$$\mathbf{F}_{31} = -G\frac{m_1 m_3}{r_{31}^2}\hat{\mathbf{r}}_{31}.$$

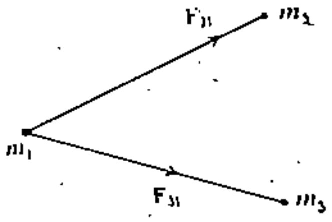


Fig. 5.8 : $\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31}$

Now, if both m_2 and m_3 are attracting m_1, the total force on m_1 is the vector sum of $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$, i.e.

$$\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31},$$

$$\text{or } \mathbf{F}_1 = -G\frac{m_1 m_2}{r_{21}^2}\hat{\mathbf{r}}_{21} - G\frac{m_1 m_3}{r_{31}^2}\hat{\mathbf{r}}_{31}. \quad (5.13)$$

This is the **superposition principle** according to which *the resultant force on a mass is the vector sum of the individual forces acting on it.* We shall apply this principle to determine the gravitational force due to an extended body. But before we go into that discussion, you may like to try an SAQ.

SAQ 3

Show that the location of the point between two fixed masses m_1 and m_2 at which a mass m does not feel any resultant gravitational force due to them is independent of m.

If the force between a point mass O and an extended body (Fig. 5.9) is required, we can apply the principle of superposition. But the problem will be complicated as the number of particles is very large. Effectively we shall have to perform integration. In order to get rid of this difficulty we take resort to the concept of Gravitational Potential, which we shall discuss next.

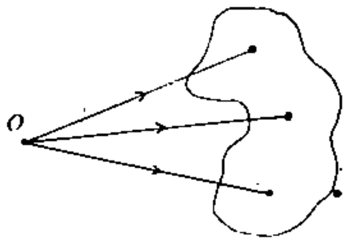


Fig. 5.9 : Force between a point mass and an extended body.

5.4 GRAVITATIONAL FIELD AND POTENTIAL

Let us consider a particle of mass m_1 placed at some point. We place another particle of mass m_2 at a distance r from it. So each particle experiences a force of attraction due to the other. If the distance between the particles is changed then also there will be a force. In other words, however large the value of r might be, there will be some force of attraction. We say that m_1 modifies the space around it in some way and sets up a field of influence, called the **gravitational field.**

The strength of a gravitational field is given by its **intensity.** The intensity of the gravitational field due to a mass M at a distance r from it is given by the force experienced by a unit mass placed at that point. Hence, the intensity at point P due to mass M, say at O, (see Fig. 5.10) will be given by

$$\mathbf{E} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \quad (5.14)$$

where $\hat{\mathbf{r}}$ is the unit vector along OP, and $OP = r$. The force $\mathbf{F}$ experienced by a mass m at P due to M at O is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}. \quad (5.14a)$$

Thus, from Eq. 5.14 we get $\mathbf{F} = m\mathbf{E}$. (5.14 b)

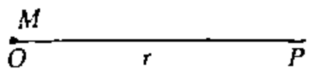


Fig. 5.10: Gravitational intensity at P due to M at O.

If mass m is now removed from this point and placed at a larger distance then work has to be done against this attractive force. In Unit 3, we have dealt with a similar case. Look up Sec. 3.2.2 and try the following SAQ.

SAQ 4

Show that the work done in bringing a mass m from a point Q to a point P in the gravitational field of a mass M placed at O (Fig. 5.11) is given by

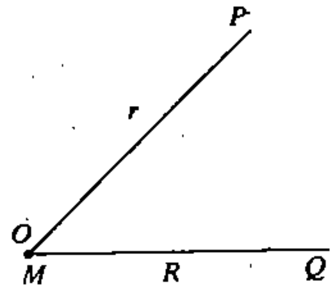


Fig. 5.11

Since the above work has to be performed, the mass m acquires a P.E. called the **gravitational potential energy**. This P.E. is mutual to the masses m and M. By convention, the gravitational potential energy U of a mass m in the field of mass M at a distance r is measured as the negative of the work done in bringing m from infinity to the said point. So we can get U by putting $R = \infty$ in the expression of $-W$ of Eq. 5.15, i.e.

$$U = -\frac{GMm}{r}. \qquad (5.16)$$

We shall introduce another term called *gravitational potential* of a mass M at a distance r. It is defined as the negative of the work done in bringing a unit mass from infinity to that point.

SAQ 5

Using Eqs. 5.14a and 5.16, verify that

$$\mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}}.$$

The meaning of Eq. 5.17 is that the *gravitational force of attraction between two masses can be obtained as the negative space-rate of change of their gravitational P.E.* If you look back at Eq. 3.18 you will find that we have discussed this aspect about a conservative force field. And the gravitational force is indeed a conservative one.

At this stage we shall recall the problem of determination of force between a point mass and an extended body. We were worried over the complication involved in performing the vector sum of the individual forces. Eq. 5.17 provides us with a way out. We can determine the overall gravitational P.E. by taking the sum of individual P.E.s. This sum can be obtained quite conveniently as potential energy is a scalar quantity. We can then use Eq. 5.17 to determine the force.

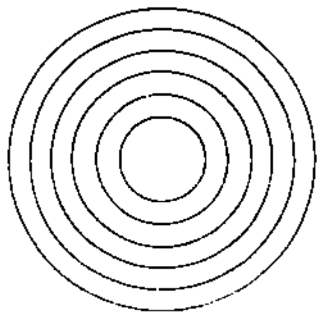


Fig. 5.12

We shall apply whatever we have learnt so far to the study of the force of gravity. For that we shall consider earth to be a sphere, neglecting its equatorial bulge. So let us take up the problem of determining the force of attraction between a sphere and a point mass. In the course of this discussion we shall have to do quite a bit of mathematical calculations. But do not let that put you off.

You might have noticed that when an onion is peeled off, thin layers come out one after another till we reach its central part when almost nothing remains. Similarly, we may consider a solid sphere as an aggregate of several concentric thin spherical shells as shown in Fig. 5.12. We shall first find out the gravitational potential energy of a point mass due to a spherical shell and then go over to the study of spheres.

5.4.1 Gravitational P.E. due to a Spherical Shell

Let a point mass m be placed at a distance r from the centre of a spherical shell of mass M_s and radius R_s. Let us calculate the gravitational potential energy of the mass due to the shell when i) $r > R_s$ ii) $r < R_s$.

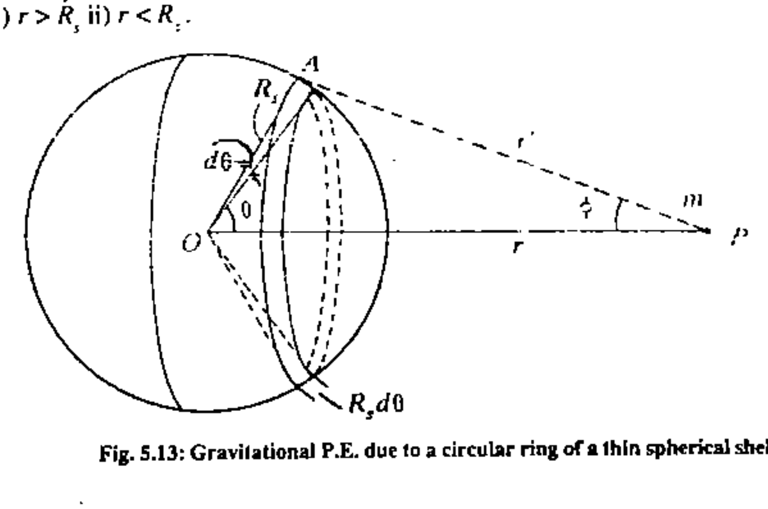


Fig. 5.13: Gravitational P.E. due to a circular ring of a thin spherical shell

Refer to Fig. 5.13. We first consider a ring-like portion of the shell contained between the directions θ and $\theta + d\theta$ with respect to the axis OP. Let it be of infinitesimal width so that every point on it is at the same distance, say r', from P. The angular width of the ring is $d\theta$, its width is $R_s d\theta$, and its radius is $R_s \sin\theta$. What is its mass? The mass per unit area of the shell is $\sigma = M_s/4\pi R_s^2$. The ring's mass is then $M_{ring} = \sigma dA$, dA being the surface area of the ring. The circumference of the ring is $2\pi R_s \sin\theta$. Its area is, therefore, given by

$$dA = (2\pi R_s \sin\theta)\ R_s\, d\theta, \tag{5.18}$$

$$\text{or} \quad M_{ring} = \frac{M_s}{4\pi R_s^2}(2\pi R_s \sin\theta)\ R_s d\theta = \frac{M_s}{2}\sin\theta d\theta. \tag{5.19}$$

We shall now determine the gravitational P.E. at P due to this ring. The ring is made up of a large number of point masses each having mass equal to, say δM.

The gravitational P.E. at P due to one such point mass is $-\dfrac{Gm\delta M}{r'}$. So the gravitational P.E. at P due to the ring will be $dU_{ring} = \Sigma\left[-\dfrac{Gm\delta M}{r'}\right]$, where the summation (Σ) extends over all the points on the ring. Here G is a constant. Again as every point on the ring is at the same distance r' from P, r' is also a constant for the points on the ring. So

$$dU_{ring} = \Sigma\left[-\frac{Gm\delta M}{r'}\right] = -\frac{Gm}{r'}\Sigma\delta M = -\frac{Gm}{r'}M_{ring}.$$

On using Eq. 5.19 we get

$$dU_{ring} = -\frac{GmM_s \sin\theta\, d\theta}{2r'}. \tag{5.20}$$

The shell can be imagined to be made up of such rings having a common axis OP. Since P.E. is a scalar quantity we shall integrate Eq. 5.20 to get the gravitational P.E. U of the shell. On the right hand-side of Eq. 5.20 we now have two variables θ and r'. It would be convenient if we can express it in terms of a single variable. For this we shall consider the relation between r', r and R_s. From triangle OAP we have,

$$r'^2 = r^2 + R_s^2 - 2rR_s \cos\theta. \tag{5.21}$$

On differentiating with respect to θ we get

$$2r'\frac{dr'}{d\theta} = 2rR_s \sin\theta,$$

$$\text{or} \quad \frac{dr'}{rR_s} = \frac{\sin\theta\ d\theta}{r'} \tag{5.22}$$

Hence, from Eq. 5.20, we get

$$dU_{ring} = -\frac{GmM_s}{2rR_s}dr' \tag{5.23}$$

And on integrating Eq. 5.23, we get for the entire spherical shell

$$U = -\frac{GM_s m}{2rR_s}\int_{r_1'}^{r_2'} dr', \tag{5.24}$$

where r_1' and r_2' are, respectively, the minimum and maximum values of r'. Now study Fig. 5.14. For a point P outside the shell, i.e. for $r > R_s$,

$$r_1' = r - R_s,\ r_2' = r + R_s \tag{5.25a}$$

and for a point P inside the shell, i.e. for $r < R_s$

$$r_1' = R_s - r,\ r_2' = R_s + r. \tag{5.25b}$$

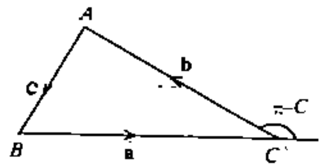
So the gravitational potential energy

$$U = -\frac{GmM_s}{2rR_s}(r_2' - r_1') = -\frac{GmM_s}{2rR_s}2R_s, \quad r > R_s \ \text{(from 5.25a)}$$

$$\text{or} \quad U = -\frac{GmM_s}{r},\ r > R_s. \tag{5.26}$$

The force on mass m is given by Eq. 5.17 as

$$\mathbf{F} = -\left(\frac{dU}{dr}\right)\hat{\mathbf{r}} = -\frac{GmM_s}{r^2}\hat{\mathbf{r}}. \tag{5.27}$$



In the above figure
$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$
i.e. $\mathbf{a} + \mathbf{b} = -\mathbf{c}$
or $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b}) = c^2$
or $a^2 + b^2 + 2\mathbf{a}.\mathbf{b} = c^2$
or $a^2 + b^2 + 2ab\cos(\pi - C) = c^2$
or $c^2 = a^2 + b^2 - 2ab\cos C$

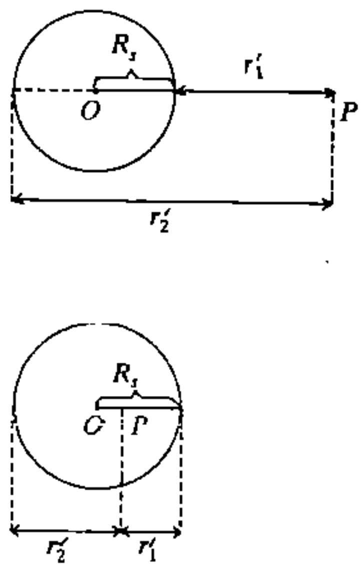


Fig. 5.14

The negative sign indicates that it is a force of attraction. The negative sign in the expression of U in Eq. 5.26 also indicates that it is attractive. Comparing Eqs. 5.12 and 5.27, it can be said that the shell behaves as a point mass having the same mass as that of the shell and located at its centre.

For $r < R_s$, we use Eq. 5.25b to get

$$U = -\frac{GmM_s}{2rR_s}(r_2 - r_1) = -\frac{GmM_s}{2rR_s}2r$$

$$\text{or } U = -\frac{GmM_s}{R_s} = \text{a constant} = U_0, \text{ say}, r < R_s. \qquad (5.28)$$

From Eq. 5.17, we get $\mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}} = 0.$ (5.29)

So P.E. of a mass placed at any point within the shell remains constant and the gravitational force on it is zero.

You can now apply the concepts you have learnt in working out the following SAQ.

SAQ 6

Draw a graph of U vs. r for the spherical shell. Take the range of r as $r = 0$ to $r = 2R_s$. Explain using physical argument whether U should be continuous at $r = R_s$ or not. Does your graph agree with your argument?

So far we have determined the gravitational P.E. of a point mass due to a spherical shell. Let us now extend these ideas to the case of a solid sphere.

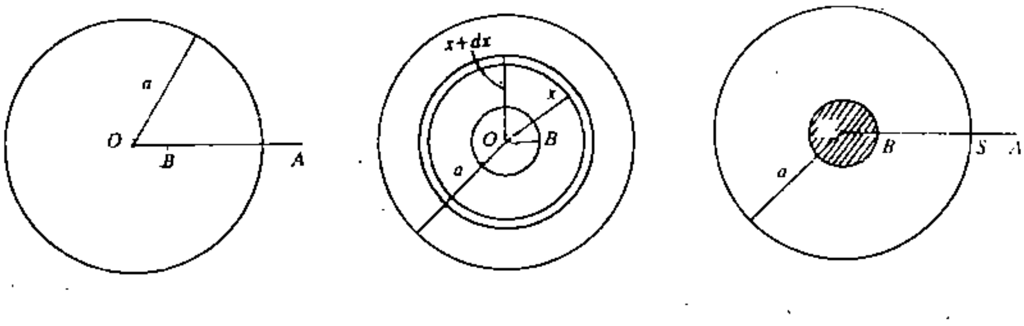
5.4.2 Gravitational P.E. due to a Solid Sphere

We have seen earlier that a solid sphere is an aggregate of concentric spherical shells. The determination of a gravitational P.E. due to a solid sphere at an external point is a straightforward application of the ideas of Sec. 5.4.1 So you may like to work it out yourself.

SAQ 7

Refer to Fig. 5.15a. A solid sphere of mass M and radius a has been shown. Show that the gravitational potential energy of a mass m at A ($OA = r$) due to the sphere is given by

$$U_A = -\frac{GMm}{r}. \qquad (5.30)$$



(a) (b) (c)

Fig. 5.15

We took up the problem of the gravitational force on a mass due to a sphere with a view to studying the variation of earth's gravity. For this we not only need to know about the force experienced by a mass placed external to a sphere but also by a mass placed inside it. So let us work out the following example.

Example 1

Refer to Fig. 5.15a. Show that the gravitational P.E. of a mass m at B ($OB = r$) due to the solid sphere of mass M and radius a is given by

$$U_B = -\frac{GMm}{2a^3}(3a^2 - r^2). \qquad (5.31)$$

See Fig. 5.15b. The point B is on the surface of a solid sphere of radius r and on the inner surface of a thick spherical shell included between radii r and a. These two will contribute to the P.E. of m at B. Let us name the contributions as U_{B1} and U_{B2}. From the result of SAQ 7, we have $U_{B1} = -\frac{GM_1 m}{r}$, where M_1 is the mass of the inner sphere of radius r:

$$M_1 = \frac{M}{\frac{4}{3}\pi a^3}\left(\frac{4}{3}\pi r^3\right) = \frac{Mr^3}{a^3}. \tag{5.32}$$

Hence, from Eq. 5.32, $U_{B1} = -\frac{GMmr^2}{a^3}$. (5.33)

For determining U_{B2} we consider the concentric shell included between radii x and $x + dx$. The volume of this shell is

$4\pi x^2 dx$ and its mass $= \frac{M}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3M}{a^3} x^2 dx$.

Since dx is infinitesimal, we can consider this thin shell equivalent to a spherical shell of radius x. Hence, from Eq. 5.28 we have the P.E. of m due to this shell at B as

$$dU_{B2} = -\frac{Gm\left(\frac{3M}{a^3}x^2 dx\right)}{x} = -\frac{3GMm}{a^3} x\,dx. \tag{5.34}$$

But the thick shell is made up of a number of such thin shells with radii ranging from r to a. So in order to get U_{B2}, we shall have to integrate Eq. 5.34. Thus,

$$U_{B2} = -\int_r^a \frac{3GMm}{a^3} x\,dx = -\frac{3GMm}{2a^3}(a^2 - r^2). \tag{5.35}$$

Now, $U_B = U_{B1} + U_{B2}$.

From Eqs. 5.33 and 5.35, we get

$$U_B = -\frac{GMm}{a^3}\left[r^2 + \frac{3}{2}(a^2 - r^2)\right] = -\frac{GMm}{2a^3}(3a^2 - r^2)$$ which is Eq. 5.31.

Now, we can calculate the force of attraction on the mass m due to the solid sphere of mass M and radius a. Refer to Fig. 5.15c. When m is placed external to the sphere at A (i.e. $r = OA$) we have from Eqs. 5.17 and 5.30 that the force is

$$\mathbf{F}_A = -\frac{d}{dr}\left(-\frac{GMm}{r}\right)\hat{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}}. \tag{5.36}$$

This is the same as Eq. 5.14a. Eq. 5.36 signifies that the force of attraction due to a solid sphere on a mass m placed external to it is *the same as that due to a point mass having the same mass as that of the sphere and located at its centre.*

When m is placed inside the sphere at B (i.e. $r = OB$) we have from Eqs. 5.17 and 5.31 that the force is

$$\mathbf{F}_B = -\frac{d}{dr}\left\{-\frac{GMm}{2a^3}(3a^2 - r^2)\right\}\hat{\mathbf{r}} = -\frac{GMm}{a^3} r\hat{\mathbf{r}}. \tag{5.37}$$

Now, using Eq. 5.32 we may write Eq. 5.37 as

$$\mathbf{F}_B = -\frac{GM_1 m}{r^2}\hat{\mathbf{r}}. \tag{5.38}$$

Now, refer to Fig. 5.15c. Eq. 5.38 signifies that the force experienced by m at B is the same as that due to a point mass at O having mass M_1 which is, incidentally, the mass of the sphere with radius OB. So we can infer that when the point mass m is placed *inside a solid sphere it experiences a force of attraction only due to the spherical mass shown shaded in Fig. 5.15c. The thick spherical shell (shown unshaded in Fig. 5.15c) does not contribute to the force of attraction.* Now, putting $r = a$ in each of Eqs. 5.36 and 5.37, we get the force of attraction when m is placed at S on the surface of the sphere as

$$\mathbf{F}_S = -\frac{GMm}{a^2}\hat{\mathbf{r}}. \tag{5.39}$$

We shall now use the results of Eqs. 5.36, 5.37 and 5.39 to study the variation of earth's gravity.

5.4.3 Gravity and its Variation

The phenomenon of attraction between the earth and any other body is called **gravity**. Due to such an attraction a body experiences an acceleration towards the centre of the earth. This is known as the *acceleration due to gravity*, and is denoted by g. We shall study how **g** at a place varies with altitude and depth.

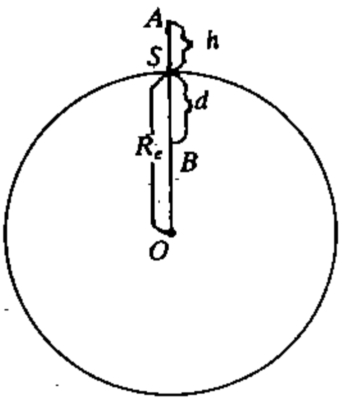


Fig. 5.16

Refer to Fig. 5.16. We consider the positions of a particle of mass m at A and B, respectively, where $SA = h$ = the altitude of A and $SB = d$ = the depth of B. S is a point on the surface of the earth. $OS = R_e$ = the radius of earth. Let the mass of earth be M_e. Let the forces of attraction experienced by m at A, B and S be denoted by $\mathbf{F}_A$, $\mathbf{F}_B$ and $\mathbf{F}_S$, respectively. In each of Eqs. 5.36, 5.37 and 5.39, we put $M = M_e$, $a = R_e$. Then we put $r = R_e + h, R_e - d$ in the Eqs. 5.36 and 5.37, respectively, to get the magnitudes of the force as

$$F_S = \frac{GM_e m}{R_e^2},\quad F_A = \frac{GM_e m}{(R_e + h)^2},\quad F_B = \frac{GM_e m}{R_e^3}(R_e - d). \tag{5.40}$$

Let the magnitudes of acceleration due to gravity on the surface of earth, and at points A and B be denoted by g_S, g_A and g_B. Then

$$g_S = \frac{F_S}{m} = \frac{GM_e}{R_e^2} = g_0, \text{ say}, \tag{5.41}$$

$$g_A = \frac{F_A}{m} = \frac{GM_e}{(R_e + h)^2}, \tag{5.42}$$

$$g_B = \frac{F_B}{m} = \frac{GM_e(R_e - d)}{R_e^3}. \tag{5.43}$$

From Eqs. 5.41 and 5.42, we get

$$g_A = \frac{g_0 R_e^2}{(R_e + h)^2}, \tag{5.42a}$$

$$\text{or } g_A = g_0\left(1 + \frac{h}{R_e}\right)^{-2} = g_0\left(1 - \frac{2h}{R_e}\right), \text{ for } (h \ll R_e). \tag{5.42b}$$

From Eqs. 5.41 and 5.43, we get

$$g_B = \frac{g_0}{R_e}(R_e - d). \tag{5.43a}$$

From Eqs. 5.42a and 5.43a we get that the acceleration due to gravity varies inversely as the square of the distance from the centre of earth for points above the surface of earth, and directly as the distance from the centre of earth for points below the surface of earth. Now you can work out an SAQ on Eqs. 5.42a and 5.43a.

SAQ 8

a) Plot a graph of g vs. r with r ranging from 0 to $2R_e$.

b) By what percentage of its value at sea-level does g increase or decrease when one goes to i) an altitude of 2500 km and ii) Kolar Gold Field at a depth of 3000 m.

We have discussed the variation of g with altitude and depth. It also varies with latitude due to the rotation of the earth about its axis. We are stating the formula for this variation without proof, which will be given in Unit 10.

$$g(\lambda) = g_e + \omega^2 R \sin^2 \lambda, \tag{5.44}$$

where $g(\lambda)$ = Value of g on the surface of earth at a place having latitude λ

g_e = Value of g on equator = 9.7805 m s^{-2}

ω = Angular speed of rotation of earth.

We have discussed how g is affected due to several factors. We understand from Eqs. 5.42 and 5.42a that at any finite distance from the surface of earth, g is non-zero. So the effect of gravity can be felt at any point irrespective of its distance from the centre of the earth. But we shall see that at any position a particle may be made to escape from the bounds of earth's attraction if it is provided with a certain minimum velocity. This is called **the velocity of escape**. This concept applies to any spherical celestial object. We shall now derive an expression for it.

5.4.4 Velocity of Escape

Let us consider a particle of mass m at a distance r from the centre of a huge spherical body of mass M (see Fig. 5.17). Its gravitational P.E. at this position is $U = -GMm/r$. Read the paragraph after Eq. 5.27 and you will realise the significance of the negative sign of U. It indicates that the mass m is bound by the attraction of the body of mass M.

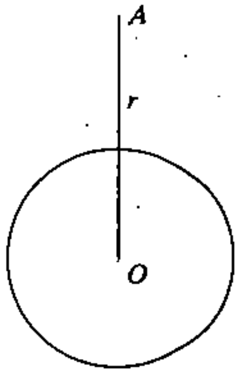


Fig. 5.17

Now, if the particle is to become free from the bounds of the gravitational attraction of the body then it must be provided with an external energy E ($\geq U$). Thereby, its total energy $(E + U)$ becomes non-negative. Thus, the particle ceases to remain bound and escapes from the attraction of M. If E is provided in the form of K.E. then $E = \frac{1}{2}mv^2$, where v is the velocity given to the particle. Accordingly the condition becomes

$$\frac{1}{2}mv^2 + U \text{ is not negative}$$

$$\text{or } \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right) \geq 0, \text{ i.e. } v^2 \geq \frac{2GM}{r} \text{ or } v \nless \sqrt{\frac{2GM}{r}}.$$

Hence, $\sqrt{\frac{2GM}{r}}$ is the required minimum velocity and it is the expression for the *velocity of escape* (v_e) which we can see is independent of m, the mass of the particle. Thus.

$$v_e = \sqrt{\frac{2GM}{r}}. \qquad (5.45)$$

If the particle were originally on the surface of earth, then $r = R_e$, $M = M_e$ and from Eqs. 5.45 and 5.41, we get

$$v_e = \sqrt{\frac{2GM_e}{R_e}} = \sqrt{2g_0R_e}. \qquad (5.46)$$

Now, taking $g_0 = 9.8 \text{ m s}^{-2}$, we get $v_e = 1.1 \times 10^4 \text{ m s}^{-1} = 11 \text{ km s}^{-1}$ — a velocity that will take you from Srinagar to Kanyakumari in about five minutes! So, now you can work out a simple SAQ.

SAQ 9

Find the velocity of escape on the surface of moon.

So far we have dealt with the phenomenon of gravitation and some of its applications. Newton's law of gravitation was the fountainhead of all the discussion. But now we raise the question — 'why at all there is a force of attraction between any two material bodies?' Does Newton's law provide an answer? It cannot because the gravitational force between two bodies exists naturally. Such a force is called a '*Fundamental Force in Nature*'. There are three different kinds of fundamental forces in nature. We shall now discuss briefly about them.

5.5 FUNDAMENTAL FORCES IN NATURE

The three kinds of fundamental forces are (i) *gravitational* (ii) *electroweak* and (iii) *strong*. You have read in detail about (i) which acts on all matter as you have seen so far. It varies inversely as the square of the distance but its range is infinite. This force is responsible for holding together the planets and stars and in overall organisation of solar system and galaxies.

The second kind is the **electroweak force**. It includes the forces of electromagnetism and the so-called weak nuclear force. We shall discuss about the latter towards the end of this section. But let us first identify the electromagnetic forces. The force between two charged particles at rest (electrostatics) or in motion(electrodynamics) comes under the purview of electromagnetic forces. The electrostatic force between two charges obeys the inverse square law like the gravitational force between two masses. However, there is an important dissimilarity. Charges can be of two kinds— positive and negative. If the charges are of opposite kind the force between them is *attractive* and if they are of the same kind, then the force between them is *repulsive*. It can be shown that the gravitational force between an electron and a proton in a hydrogen atom is 10^{39} times weaker than the electrostatic force between them. Thus, we get a comparative estimate of the strengths of gravitational and electrostatic force.

Now, let us come to the case of moving charges. We know that charges in motion give rise to electric current. You have also studied Oersted's experiment in your school science courses. From this experiment we understand that a current carrying conductor is equivalent to a magnet. This is the meeting point of electricity and magnetism and hence the word 'electromagnetic' got associated with this field of force. The forces that one comes across in daily life, like friction, tension, etc. can be explained from the standpoint of the electromagnetic force field.

Now, if we make an estimate of the relative strengths of the repulsive electrostatic and the attractive gravitational force between two protons in a nucleus we shall find that the former is 10^{36} imes larger than the latter. So how is it that the protons in an atomic nucleus, stay together instead of flying away? The answer lies in the third kind of fundamental force known as the **strong (nuclear) force** that exists between the protons inside the nucleus which is strongly attractive, much stronger than the electrostatic force between them. As the nucleus also contains neutrons, which are as tightly bound as the protons, this force must also exist between two neutrons as well as between neutrons and protons. Unlike the gravitational and the electromagnetic forces, the nuclear force acts only when the nucleons (protons and neutrons) are very close to each other (10^{-15}m or less). The nuclear forces decrease very rapidly with distance, so rapidly that a nucleon only interacts with its closest neighbours. You will study in detail about the nuclear forces in the Nuclear Physics course.

The strong nuclear force as we have seen just now accounts for the binding of atomic nuclei. But this cannot account for processes like radioactive beta decay about which once again you will read in the Nuclear Physics course. This can be explained from the point of view of the so-called *weak nuclear force*. It is much weaker than the electromagnetic force at nuclear distance but still greater by a factor of 10^{34} than the gravitational force. Just a few years ago, this weak force was listed separately from the electromagnetic force. However a theory was proposed which led to the unification of the weak forces and the electromagnetic forces and hence the name 'electroweak' forces.

We shall now give the different characteristics of the fundamental forces in nature in Table 5.1.

Table 5.1: Some Characteristics of the Three Fundamental Forces

Force		Relative strength	Range	Importance
Strong nuclear		1	10^{-13}cm	Holds nucleons together
Electroweak	Electromagnetic	10^{-2}	Infinite	Controls everyday phenomena—friction, tension etc.
	Weak nuclear	10^{-5}	10^{-15} cm	Nuclear transmutation
Gravitational		10^{-39}	Infinite	Organises large-scale phenomena and universe

Now let us sum up what you have learnt in this unit.

5.6 SUMMARY

- Newton's law of universal gravitation states that any two particles in the universe exert an attractive force on each other, given by

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} = -\mathbf{F}_{21},$$

where $\mathbf{F}_{12}$ is the force exerted by m_1 on m_2 and $\hat{\mathbf{r}}_{12}$ is the unit vector directed from m_1 to m_2, along the line joining the two masses.

- **Any mass creates about itself a field of influence called the gravitational field. The intensity E and the potential *U* of a gravitational field due to a point mass *M* at a point are given by**

$$\mathbf{E} = -\frac{GM}{r^2}\hat{\mathbf{r}} \text{ and } U = -\frac{GM}{r},$$

where $\hat{r}$ is the unit vector along the line joining the mass M to the point

- The gravitational force of attraction due to a solid sphere experienced by a point mass placed external to it is the same as that due to a point mass placed at the centre of the sphere and whose mass is equal to that of the sphere.

  When the point mass is placed inside the sphere, it experiences force of attraction only due to a concentric spherical mass on whose surface it lies. The matter contained in the shells external to this point mass does not contribute at all to the force of attraction.
- The value of acceleration due to gravity at the points above and below the surface of earth varies, respectively, as the inverse square of and directly as the distance of the point from the centre of earth.
- The minimum velocity that an object of mass m at a distance r from the centre of a spherical body of mass M must have so that it can escape from the bounds of the gravitational attraction of M is called its escape velocity. Its value is $\sqrt{2GM/r}$.
- Gravitational force is fundamental force in nature. There are two other kinds of fundamental forces — the electroweak force and the strong nuclear force.

5.7 TERMINAL QUESTIONS

1. The weight of a body on the surface of the earth is 900N. What will be its weight on the surface of Mars whose mass is 1/9 and radius 1/2 that of the earth?
2. The Gravitational P.E. of an object of mass m at a height h above the surface of earth is equal to $-GM_e m/(R_e + h)$ according to Eq. 5.30. Show that it is consistent with the expression '$mg_0 h$' for Gravitational P.E., where g_0 is the value of acceleration due to gravity on the surface of earth.
3. Three bodies A, B, C of masses 5×10^6 kg each are arranged in space at the vertices of an equilateral triangle of side 2 km (see Fig. 5.18). How much work should be done to separate them to infinite distance apart?

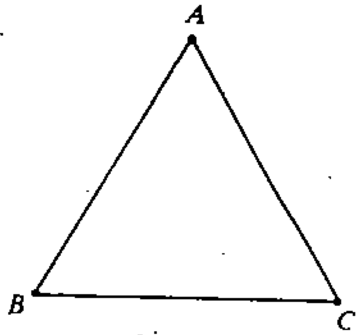


Fig. 5.18

5.8 ANSWERS

SAQs

1 Refer to Fig. 5.19. S and P refer to the positions of the sun and the planet, respectively.

$$\mathbf{F} = -\frac{km}{r^2}\hat{\mathbf{r}}.$$

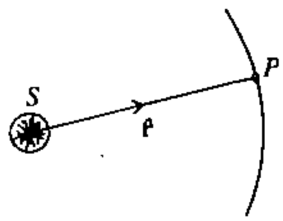


Fig. 5.19

2. Since a is constant near the surface of earth, we have for any object falling freely from rest for t s the following relations:

$$v = at,\ s = \tfrac{1}{2}\ at^2,$$

or v is proportional to t and s is proportional to t^2 which are consistent with the law of falling bodies.

3. Refer to Fig. 5.20. Let the distance between m_1 and m_2 be a. Let m be at a distance x from m_1 when the resultant gravitational force on m due to m_1 and m_2 is zero. Then in this situation the magnitudes of forces of attraction between m, m_1 and m, m_2 must be same. Hence

$$\frac{Gm_1 m}{x^2} = \frac{Gm_2 m}{(a-x)^2} \quad \text{or} \quad \frac{a-x}{x} = \sqrt{\frac{m_2}{m_1}} = b, \text{ say.}$$

Here b is the value of the positive square root of m_2/m_1 as $x < a$.

$$\therefore x = \frac{a}{b+1} = a \text{ constant independent of } m.$$

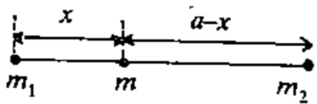


Fig 5.20

4. Required work done $W = \int_Q^P \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$

$$\mathbf{F}\cdot d\mathbf{r} = -\frac{GMm}{r^2}\hat{\mathbf{r}}\cdot d\mathbf{r} = -\frac{GMm}{r^2}dr \quad \text{(as explained in Sec. 3.2.2).}$$

$$\therefore W = -\int_R^r \frac{GMm}{r^2}dr = GMm\left[\frac{1}{r}\right]_R^r = -GMm\left(\frac{1}{R} - \frac{1}{r}\right).$$

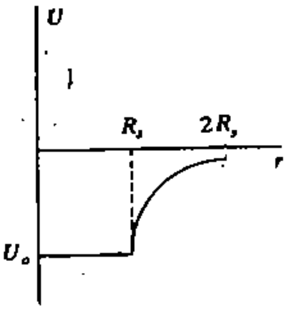


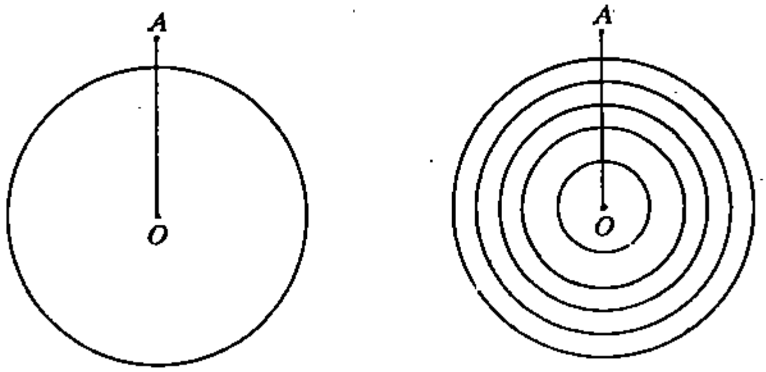
Fig 5.21 : U vs r graph; $U_o = -\frac{GmM_s}{R_s}$

5. From Eq. 5.16, $U = -\frac{GMm}{r}$. $\therefore \frac{dU}{dr} = \frac{GMm}{r^2}$.

   Again, from Eq. 5.14a, $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$ $\therefore \mathbf{F} = -\frac{dU}{dr}\hat{\mathbf{r}}$

6. U vs. r graph is shown in Fig. 5.21. If U is discontinuous at $r = R_s$, then $\frac{dU}{dr}$ becomes infinite at that point. This indicates that in accordance with Eq. 5.17 the gravitational force of attraction has to be infinite at the boundary of the spherical shell. But that is absurd. So U vs. r must be continuous everywhere as shown by the graph.

7. Refer to Fig. 5.22a. The P.E. has to be determined at A. See Fig. 5.22b.



(a) (b)

Fig. 5.22

The solid sphere as we have seen earlier can be considered as an aggregate of a number of concentric spherical shells of masses, say m_1, m_2, m_3, ... where

$$m_1 + m_2 + m_3 + ... = M \qquad (5.47)$$

The point A is at a distance r from the centre of each shell. From Eq. 5.26, the P.E. due to the shells will be given by

$$U_1 = -\frac{Gm_1m}{r},\ U_2 = -\frac{Gm_2m}{r},\ U_3 = -\frac{Gm_3m}{r} \text{ and so on}$$

Hence, the P.E. due to the sphere at A is

$$U_A = U_1 + U_2 + U_3 + \ldots$$
$$= -\frac{Gm}{r}(m_1 + m_2 + m_3 + \ldots)$$

$\therefore$ From Eq. 5.47, $U_A = -\frac{GMm}{r}$.

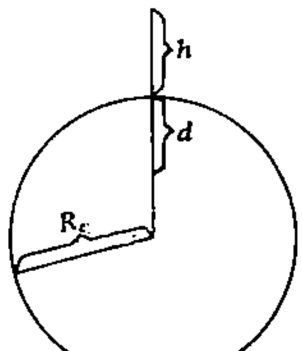


Fig. 5.23

8. a) Refer to Fig. 5.23. For $r > R_e$, $r = R_e + h$ and for $r < R_e$, $r = R_e - d$. From Eqs. 5.43 and 5.42,

$$g = g_0\frac{r}{R_e} \quad \text{for } r < R_e$$

and

$$g = g_0\frac{R_e^2}{r^2} \text{ for } r > R_e$$

See Fig. 5.24 for variation of g with r.

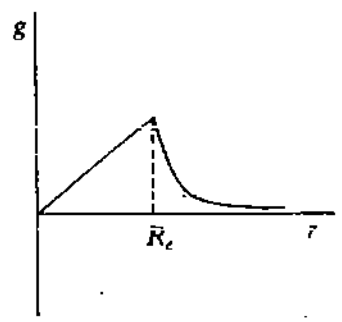


Fig. 5.24

b) (i) $g = g_0\left(\frac{R_e}{R_e + h}\right)^2$. R_e = 6,370 km, h = 2,500 km.

$$\therefore \left(\frac{R_e}{R_e + h}\right)^2 = \left(\frac{6370}{8870}\right)^2 = 0.5157$$

$g = g_0(0.5157)$

Percentage decrease $= \frac{g_0 - g}{g_0} \times 100 = 48.4$

(ii) $g = \frac{g_0}{R_e}(R_e - d), d = 3$ km. $R_e - d = 6367$ km.

$$\therefore g = g_0 \frac{6367}{6370} = g_0\,(0.9995)$$

$\therefore$ Percentage decrease = 0.05.

9. The velocity of escape on moon $v_{em} = \sqrt{\frac{2GM_m}{R_m}}$.

where M_m = mass of the moon, R_m = radius of the moon. Putting these values of G, M_m and R_m, we get $v_{em} = 2.37 \times 10^3$ m s^{-1}.

Terminal Questions

1. Mass of earth = M_e, Radius of earth = R_e, Mass of body = m. Newton's law of gravitation gives

$$F = G\frac{M_e m}{R_e^2} = 900 \text{ N}$$

Mass of Mars = $\frac{M_e}{9}$, Radius of Mars = $\frac{R_e}{2}$.

Suppose weight of the body on the surface of Mars = x

$$\therefore x = \frac{G\frac{M_e}{9}m}{\left(\frac{R_e}{2}\right)^2} = \frac{4}{9}\frac{GM_e m}{R_e^2} = \frac{4}{9} \times 900\,\text{N} = 400 \text{ N}$$

2. Refer to Fig. 5.25. A is a point on the surface of earth and $AB = h$. For the point A, $h = 0$. So using the result given in the question we have the values of P.E.s at A and B as

$$U_A = -\frac{GM_e m}{R_e},\; U_B = -\frac{GM_e m}{R_e + h}, \text{ respectively.}$$

So the P.E. of the object with respect to the surface of earth is

$$U_{BA} = U_B - U_A = -GM_e m\left(\frac{1}{R_e + h} - \frac{1}{R_e}\right) = \frac{GM_e mh}{(R_e + h)R_e}$$

But usually $h \ll R_e$ $\therefore (R_e + h)R_e \simeq R_e^2$. Hence, $U_{BA} = \frac{GM_e}{R_e^2} mh$.

From Eq. 5.41, $U_{BA} = mg_0 h$

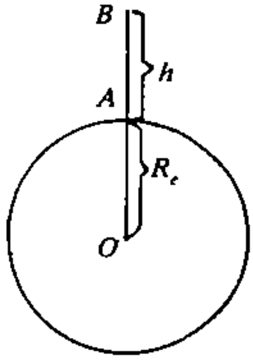


Fig. 5.25

3. The gravitational P.E. of a mass m_1 in the field of m_2 when they are separated by a distance r is $\left(-\frac{Gm_1 m_2}{r}\right)$. Refer to Fig. 5.26. Let the masses kept at the corners of the equilateral triangle be m and its side a. So the overall gravitational P.E. of the system is given by

$$U = \left(-\frac{Gmm}{a}\right) + \left(-\frac{Gmm}{a}\right) + \left(-\frac{Gmm}{a}\right)$$

or $U = -\frac{3Gm^2}{a}$, where $m = 5 \times 10^6$ kg, $a = 2 \times 10^3$ m

$$U = -\frac{3 \times (6.673 \times 10^{-11}\,\text{Nm}^2\text{kg}^{-2}) \times (25 \times 10^{12}\,\text{kg}^2)}{2 \times 10^3\,\text{m}} = -2.5 \text{ J}$$

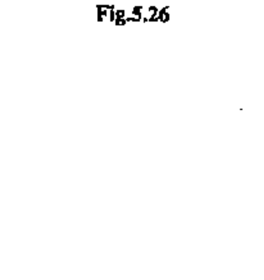


Fig. 5.26

Since, the energy is negative the system is bound with an energy of 2.5J. So in order to take them infinite distance apart an external energy 2.5 J is required.

FURTHER READING

1. *Mechanics, Berkeley Physics Course* — Volume I; C. Kittel, W.D. Knight, M.A. Ruderman, A.C. Helmholz, B.J. Moyer; Asian Student Edition, McGraw-Hill International Book Company, 1981.

2. *An Introduction to Mechanics*; D. Kleppner, R.J. Kolenkow; International Student Edition, McGraw-Hill International Book Company, 1984.

3. *Introduction of Classical Mechanics*; A.P. French, M.G. Ebison; Van Nostrand Reinhold (UK) Co Ltd, 1986.

4. *Physics Part I*; Robert Resnick and David Halliday; Wiley Eastern Ltd, 1988.
5. *The Mechanical Universe*, Mechanics and Heat, Advanced Edition; S.C. Frautschi, R.P. Olenick, T.M. Apostol, D.L. Goodstein; Cambridge University Press, 1986.
6. *Physics Volume I*; R. Wolfson, J.M. Pasachoff; Little, Brown and Company, 1987.

A list of commonly occurring quantities in the block along with their unit symbols, special names (if any) and dimensions is given below. Dimensions are given in terms of length [L], mass [M], time [T], temperature [K], and charge [Q].

Quantity	SI Unit		Dimensions
	Special name (if any)	Symbol	
Displacement		m	[L]
Velocity		$m\ s^{-1}$	$[LT^{-1}]$
Acceleration		$m\ s^{-2}$	$[LT^{-2}]$
Angular displacement	radian	rad	
Angular velocity		$rad\ s^{-1}$	$[T^{-1}]$
Angular acceleration		$rad\ s^{-2}$	$[T^{-2}]$
Angular momentum		$kg\ m^2 s^{-1}$	$[ML^2T^{-1}]$
Force	newton	N	$[MLT^{-2}]$
Work, Energy	joule	J	$[ML^2T^{-2}]$
Power	watt	W	$[ML^2T^{-3}]$
Gravitational potential		$J\ kg^{-1}$	$[L^2T^{-2}]$
Gravitational Intensity		$N\ kg^{-1}$	$[LT^{-2}]$
Momentum, Impulse		$kg\ m\ s^{-1}$	$[MLT^{-1}]$
Period		s	[T]
Moment of inertia		$kg\ m^2$	$[ML^2]$
Area		m^2	$[L^2]$
Volume		m^3	$[L^3]$
Density		$kg\ m^{-3}$	$[ML^{-3}]$
Torque		N m	$[ML^2T^{-2}]$
Temperature	kelvin	K	[K]
Electric charge	coulomb	C	[Q]
Electric current	ampere	A	$[T^{-1}Q]$

Table of Constants

Physical Constants

Symbol	Quantity	Value
c	speed of light in vacuum	2.998×10^{8} m s^{-1}
μ_0	permeability of free space	1.257×10^{-6} N A^{-2}
ε_0	permittivity of free space	8.854×10^{-12} C^{2}N^{-1}m^{-2}
$1/4\pi\varepsilon_0$		8.988×10^{9} N m^{2}C^{-2}
e	charge of the proton	1.602×10^{-19}C
$-e$	charge of the electron	-1.602×10^{-19}C
h	Planck's constant	6.626×10^{-34}J s
$\hbar$	$h/2\pi$	1.055×10^{-34}J s
m_e	electron rest mass	9.109×10^{-31}kg
$-e/m_e$	electron charge to mass ratio	-1.759×10^{11}C kg^{-1}
m_p	proton rest mass	1.673×10^{-27}kg
m_n	neutron rest mass	1.675×10^{-27}kg
R	Rydberg constant	1.097×10^{7}m^{-1}
a_0	Bohr radius	5.292×10^{-11}m
N_A	Avogadro constant	6.022×10^{23}mol^{-1}
R	Universal gas constant	8.314 J K^{-1}mol^{-1}
k_B	Boltzmann constant	1.381×10^{-23}J K^{-1}
G	Universal gravitational constant	6.673×10^{-11}N m^{2}kg^{-2}

Astrophysical Data

Celestial Body	Mass (kg)	Mean Radius(m)	Mean distance from the centre of Earth (m)
Sun	1.99×10^{30}	6.96×10^{8}	1.50×10^{11}
Moon	7.35×10^{22}	1.74×10^{6}	3.85×10^{8}
Earth	5.97×10^{24}	6.37×10^{6}	0

NOTES