स्वाध्याय स्वमन्थन स्वावलम्बन

UTTAR PRADESH RAJARSHI TANDON OPEN UNIVERSITY

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UGMM-13
Discrete Mathematics

FIRST BLOCK
Elementary Logic

Indira Gandhi National Open University

UP Rajarshi Tandon Open University

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UGMM-13
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Block

1

ELEMENTARY LOGIC

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DISCRETE MATHEMATICS

'Discrete' means 'finite sets', and in this course we shall discuss discrete objects. You must have come across many such objects, for instance, the set of students in IGNOU or the set of stars in the sky. But you would not have studied their nature systematically. The need to do so is dictated by the way the world is progressing technologically. For instance, to improve the efficiency of a computer programme, we need to study its speed and logical structure. This can be done by using the theory of combinatorics and graph theory, two major areas of discrete mathematics.

Discrete mathematics has vast applicability, which is why we have created this 4-credit course for you. In this course we have chosen to introduce you to only a few topics involving discrete objects, to give you a flavour of this recently evolving area of mathematics. The areas are symbolic logic, Boolean algebra, combinatorics and graph theory, which we cover in 4 blocks.

In Block 1, we show you how to differentiate between a sentence and a statement (or proposition). Then we look at various ways of combining propositions, and of finding whether these statements are true or not. After this we talk about a theory first studied by Aristotle (384-322 B.C.), and later evolved mathematically by the 19th century mathematicians Boole, De Morgan, Schroder and Frege. This is the theory of mathematical logic and the nature of mathematical proof. In this connection, it is necessary to mention the monumental work of A.N. Whitehead and Bertrand Russell, which they presented in their book 'Principia Mathematica' in 1913.

In the final unit of Block 1 we look at an important application of logic, namely, Boolean algebras and circuits.

In Blocks 2 and 3, we discuss combinatorics, or different ways of enumerating without actually counting. This theory was first developed by Pascal (1623-1662) and Jakob Bernoulli(1645-1705). We shall introduce you to various aspects of combinatorial reasoning, which underlies all analysis of computer systems, discrete operations research problems and finite probability. More specifically, you will study permutations and combinations, partitions of numbers, the pigeonhole principle, recurrence relations and generating functions. Of course, all these would be presented from an application-oriented point of view.

In Block 4, we take up elementary graph theory. The word "graph" is used to describe road maps, circuit diagrams, flow charts, etc., i.e., any structure that involves inter-connections between various parts of it. In the first unit of this block we introduce you to the basics of graph theory. Then, in the rest of the units, we discuss certain special graphs and colouring of graphs. The starting point of this theory is the solution of the Konigsberg seven-bridge problem, by Euler (1707-1783). Other mathematicians who have done a great deal towards developing and applying this theory are Hamilton (1805-1865), Arthur Cayley (1821-1895), Kirchoff (1824-1895), and, more recently, Appel and Haken (who proved the four-colour theorem)

Now a word about our notation. Each unit is divided into sections, which may be further divided into sub-sections. These sections/sub-sections are numbered sequentially, as are the exercises and important equations in a unit. Since the material in the different units is heavily interlinked, we will be doing a lot of cross-referencing. For this we will be using the notation Sec. x.y to mean Section y of Unit x. In each unit you will find several exercises (numbered E1, E2, ...) and examples (also numbered sequentially). We show the end of an example by *** after it.

Another compulsory component of this course are its **assignments** – Assignment 1 is based on Blocks 1 and 2, Assignment 2 is based on Blocks 3 and 4. Your academic counsellor will evaluate them and return them to you with detailed comments. Thus, the assignments are meant to be a teaching as well as an assessment aid.

As you can see from the introduction, the course is very elementary. Its only pre-requisite is the first-level course "Elementary Algebra" (MTE-04). Therefore, we are offering it at the second-level.

We hope you enjoy studying this course. If you have a problem in understanding any portion of it, please ask your academic counsellor for help. Also, if you feel like studying any topic in greater detail, you may consult:

1. *Elements of Discrete Mathematics, by C.L. Liu, McGraw-Hill, 1985*

2. *Graph Theory, by F. Harary, Narosa, 1995.*

These books are available at your study centre.

BLOCK 1 ELEMENTARY LOGIC

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

From 'Alice in Wonderland'
by Lewis Carroll

Logic is the study and analysis of the nature of the valid argument, the reasoning tool by which valid inferences can be drawn from a given set of facts and premises. It is the basis on which all the sciences are built. Logic was extensively studied and developed in ancient Greece. But the mathematical theory of logic, called symbolic logic, only came into its own in the 19^{th} century. This algebraic way of studying arguments was developed by the English mathematician George Boole (1815-1864).

In symbolic logic we study arguments. The basic building blocks of arguments are declarative sentences, called propositions or statements. In Unit 1 we introduce you to propositions and ways of combining them to form more complex propositions. We also introduce you to propositions that contain the quantifiers 'for every' and 'there exists'. In symbolic logic, the goal is to determine which propositions are true and which are false. A tool for finding this out is the truth table, which we shall also discuss in Unit 1.

In Unit 2 we look at paths of reasoning by which we can show that certain statements are true. Such arguments are called 'proofs'. In this unit we try to give you an understanding of why a proof is written the way it is. We expose you to several patterns of reasoning that make up different proofs. In this unit we also discuss mathematical induction, a fundamental tool for proving many propositions involving natural numbers.

The last unit of the block, Unit 3, is closely linked with Unit 1. In this unit you will see that the set of propositions along with certain operations, forms an algebraic structure called a Boolean algebra. You will also see the application of this theory for studying switches, gates and circuits.

Now, a few words about how we have presented the material. As you go through the units, you will find several examples, numbered sequentially through the unit. We end each example with * * * for your convenience. In the units you will also find several exercises (E1, E2,.....). The best way to absorb the material in the units is to try these exercises as and when you get to them.

After going through the unit, you must come back to the introduction, and check if you have achieved the objectives. Doing this will help you confirm that you are ready to go further.

NOTATIONS AND SYMBOLS

N	the set of natural numbers
R	the set of real numbers
$p \vee q$	p or q (p, q being statements)
$p \ominus q$	either p or q, but not both.
$p \wedge q$	p and q
$\sim p$	not p
$p \rightarrow q$	p implies q p is sufficient for q p only if q
$p \longleftrightarrow q$	p if and only if q p is necessary and sufficient for q p implies and is implied by q
$p \Rightarrow q$	if p is true, then q is true
$p \Longleftrightarrow q$	p is true if and only if q is true.
$p \equiv q$	p is equivalent to q
$\therefore$	therefore
iff	if and only if
$\forall$	for all
$\exists$	there exists
$\exists!$	there exists one and only one
$\mathcal{P}(X)$	set of all subsets of a set X
$\mathcal{B}$	two-element Boolean algebra
$\mathcal{B}^n$	$\mathcal{B} \times \mathcal{B} \times \ldots \times \mathcal{B}$ (n times)
$X(x_1, \ldots, x_k)$	Boolean expression in k-variables.
s.v.	state value
a par b	parallel connections of switches a and b
a ser b	series connections of switches a and b
CNF	conjunctive normal form
DNF	disjunctive normal form

UNIT 1 PROPOSITIONAL CALCULUS

Structure

1.1 INTRODUCTION

According to the theory of evolution, human beings have evolved from the lower species over many millenia. The chief asset that made humans "superior" to their ancestors was the ability to reason. How well this ability has been used for scientific and technological development is common knowledge. But no systematic study of logical reasoning seems to have been done for a long time. At least, the first such study that has been found is by the Greek philosopher Aristotle (384-322 BC). In a modified form, this type of logic seems to have been taught through the Middle Ages.

Then came a major development in the study of logic, its formalisation in terms of mathematics. It was mainly Leibniz (1646-1716) and George Boole (1815-1864) who seriously studied and developed this theory, called **symbolic logic**. It is the basics of this theory that we aim to introduce you to in this unit and the next one.

In the introduction to the block you have read about what symbolic logic is. Using it we can formalise our arguments and logical reasoning in a manner that can easily show if the reasoning is valid, or is a fallacy. How we symbolise the reasoning is what is presented in this unit.

More precisely, in Section 1.2 (i.e., Sec.1.2, in brief) we talk about what kind of sentences are acceptable in mathematical logic. We call such sentences statements or propositions. You will also see that a statement can either be true or false. Accordingly, as you will see, we will give the statement a truth value T or F.

In Sec.1.3 we begin our study of the logical relationship between propositions. This is called propositional calculus. In this we look at some ways of connecting simple propositions to obtain more complex ones. To do so, we use logical connectives like "and" and "or". We also introduce you to other connectives like "not", "implies" and "implies and is implied by". At the same time we construct tables that allow us to find the truth values of the compound statements that we get.

In Sec.1.4 we consider the conditions under which two statements are "the same". In such a situation we can safely replace one by the other.

And finally, in Sec.1.5, we talk about some common terminology and notation which is useful for quantifying the objects we are dealing with in a statement.

It is important for you to study this unit carefully, because the other units in this block are based on it. Please be sure to do the exercises as you come to them. Only then will you be able to achieve the following objectives.

Objectives

After reading this unit you should be able to

- distinguish between propositions and non-propositions;
- identify and use logical connectives;
- construct the truth table of any compound proposition;
- identify and use logically equivalent statements;
- identify and use logical quantifiers.

Let us now begin our discussion on mathematical logic.

1.2 PROPOSITIONS

Consider the sentence 'The President of India is a woman.' When you read this declarative sentence, you can immediately decide whether it is true or false. And so can anyone else. Also, it wouldn't happen that some people say that the statement is true and some others say that it is false. Everybody would have the same answer. So, this sentence is either **universally true** or **universally false**.

Similarly, 'An elephant weighs more than a human being.' is a declarative sentence which is either true or false, but not both. In mathematical logic we call such sentences **statements** or **propositions**.

On the other hand, consider the declarative sentence 'Women are more intelligent than men.'. Some people may think it is true while others may disagree. So, it is neither universally true nor universally false. Such a sentence is not acceptable as a statement or proposition in mathematical logic.

Note that a **proposition should be either uniformly true or uniformly false**. For example, 'An egg has protein in it.', and 'The Prime Minister of India has to be a man.' are both propositions, the first one true and the second one false.

Would you say that the following are propositions?

'Watch the film.'
'How wonderful!'
'What did you say?'

Actually, none of them are declarative sentences. (The first one is an order, the second an exclamation and the third is a question.) And therefore, none of them are propositions.

Now for some mathematical propositions! You must have studied and created many of them while doing mathematics. Some examples are

Two plus two equals four.
Two plus two equals five.
$x + y > 0$ for $x > 0$ and $y > 0$.
A set with n elements has 2^n subsets.

Of these statements, three are true and one false (which one?).

Now consider the algebraic sentence '$x + y > 0$'. Is this a proposition? Are we in a position to determine whether it is true or false? Not unless we know the values that x and y can take. For example, it is false for $x = 1, y = -2$ and true if $x = 1, y = 0$. Therefore,
'$x + y > 0$' is not a proposition, while
'$x + y > 0$ for $x > 0, y > 0$' is a proposition.

Why don't you try this short exercise now?

E1) Which of the following sentences are statements? What are the reasons for your answer?

i) The sun rises in the West.
ii) How far is Delhi from here?
iii) Smoking is injurious to health.
iv) There is no rain without clouds.
v) What a beautiful day!
vi) She is an engineering graduate.
vii) $2^n + n$ is an even number for infinitely many n.
viii) $x + y = y + x$ for all $x, y \in R$.
ix) Mathematics is fun.
x) $2^n = n^2$.

Usually, when dealing with propositions, we shall denote them by lower case letters like p, q, etc. So, for example, we may denote

'Ice is always cold.' by p, or
'$\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$' by q.
We shall sometimes show this by saying
p: Ice is always cold., or
q: $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$.

Now, given a proposition, we know that it is either true or false, but not both. If it is **true**, we will allot it the **truth value T**. If it is false, its **truth value will be F**. So, for example, the truth value of

'Ice melts at 30°C.' is F, while that of '$x^2 \geq 0$ for $x \in R$' is T.

Sometimes, as in the context of logic circuits (see Unit 3), we will use 1 instead of T and 0 instead of F.

Here are some exercises for you now.

E2) Give the truth values of the propositions in E1.

E3) Give two propositions each, the truth values of which are T and F, respectively. Also give two examples of sentences that are not propositions.

Let us now look at ways of connecting simple propositions to obtain compound statements.

1.3 LOGICAL CONNECTIVES

When you're talking to someone, do you use very simple sentences only? Don't you use more complicated ones which are joined by words like 'and', 'or', etc.? In the same way, most statements in mathematical logic are combinations of simpler statements joined by words and phrases like 'and', 'or', 'if ... then', 'if and only if', etc. These words and phrases are called **logical connectives**. There are 6 such connectives, which we shall discuss one by one.

1.3.1 Disjunction

Consider the sentence 'Alice or the mouse went to the market.'. This can be written as 'Alice went to the market or the mouse went to the market.' So, this statement is actually made up of two simple statements connected by 'or'. We have a term for such a compound statement.

Definition: The **disjunction** of two propositions p and q is the compound statement p or q, denoted by **p ∨ q**.
For example, 'Zarina has written a book or Singh has written a book.' is the disjunction of p and q, where
p : Zarina has written a book, and
q : Singh has written a book.

Similarly, if p denotes '$2 > 0$' and q denotes '$2 < 5$', then p ∨ q denotes the statement '2 is greater than 0 or 2 is less than 5.'.

Let us now look at how the truth value of p ∨ q depends upon the truth values of p and q. For doing so, let us look at the example of Zarina and Singh, given above. If even one of them has written a book, then the compound statement p ∨ q is true. Also, if both have written books, the compound statement p ∨ q is again true. Thus, if the truth value of even one out of p and q is T, then that of 'p ∨ q' is T. Otherwise, the truth value of p ∨ q is F. This holds for any pair of propositions p and q. To see the relation between the truth values of p, q and p ∨ q easily, we put this in the form of a table (Table 1), which we call a **truth table**.

Table 1 : Truth table for disjunction

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

How do we form this table? We consider the truth values that p can take — T or F. Now, when p is true, q can be true or false. Similarly, when p is false, q can be true or false. In this way there are 4 possibilities for the compound proposition p ∨ q. Given any of these possibilities, we can find the truth value of p ∨ q. For instance, consider the third possibility, i.e., p is false and q is true. Then, by definition, p ∨ q is true. In the same way, you can check that the other rows are consistent.

Let us consider an example.

Example 1: Obtain the truth value of the disjunction of 'The earth is flat.' and '$3 + 5 = 2$'.

Solution: Let p denote 'The earth is flat.' and q denote '$3 + 5 = 2$'. Then we know that the truth values of both p and q are F. Therefore, the truth value of $p \vee q$ is F.

* * *

Try an exercise now.

E4) Write down the disjunction of the following propositions, and give its truth value.
  i) $2 + 3 = 7$,
  ii) Radha is an engineer.

We also use the term '**inclusive or**' for the connective we have just discussed. This is because $p \vee q$ is true even when both p and q are true. But, what happens when we want to ensure that only one of them should be true? Then we have the following connective.

Definition: The **exclusive disjunction** of two propositions p and q is the statement **'Either p is true or q is true, but both are not true.'**. We denote this by $p \oplus q$.

So, for example, if p is '$2 + 3 = 5$' and q the statement given in E4(ii), then $p \oplus q$ is the statement 'Either $2 + 3 = 5$ or Radha is an engineer.'. This will be true only if Radha is not an engineer.

In general, how is the truth value of $p \oplus q$ related to the truth values of p and q? This is what the following exercise is about.

E5) Write down the truth table for $\oplus$. Remember that $p \oplus q$ is not true if both p and q are true.

Now let us look at the logical analogue of the coordinating conjunction 'and'.

.3.2 Conjunction

As in ordinary language, we use 'and' to combine simple propositions to make compound ones. For instance, '$1 + 4 \neq 5$ and Prof. Rao teaches Chemistry.' is formed by joining '$1 + 4 \neq 5$' and 'Prof. Rao teaches Chemistry' by 'and'. Let us define the formal terminology for such a compound statement.

Definition: We call the compound statement '**p and q**' the **conjunction** of the statements p and q. We denote this by $p \wedge q$.

For instance, '$3 + 1 \neq 7 \wedge 2 > 0$' is the conjunction of '$3 + 1 \neq 7$' and '$2 > 0$'. Similarly, '$2 + 1 = 3 \wedge 3 = 5$' is the conjunction of '$2 + 1 = 3$' and '$3 = 5$'.

Now, when would $p \wedge q$ be true? Do you agree that this could happen only when both p and q are true, and not otherwise? For instance, '$2 + 1 = 3 \wedge 3 = 5$' is not true because '$3 = 5$' is false.

So, the truth table for conjunction would be as in Table 2 alongside.

Table 2 : Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see how we can use the truth table above, consider an example.

Example 2: Obtain the truth value of the conjunction of '$2 \div 5 = 1$' and 'Padma is in Bangalore.'.

Solution: Let $p : 2 \div 5 = 1$, and q: Padma is in Bangalore.

Then the truth value of p is F. Therefore, from Table 2 you will find that the truth value of $p \wedge q$ is F.

* * *

Why don't you try an exercise now?

E6) Give the set of those real numbers x for which the truth value of $p \wedge q$ is T, where
$p : x > -2$, and $q : x + 3 \neq 7$

If you look at Tables 1 and 2, do you see a relationship between the truth values in their last columns? You would be able to formalise this relationship after studying the next connective.

1.3.3 Negation

You must have come across young children who, when asked to do something, go ahead and do exactly the opposite. Or, when asked if they would like to eat, say rice and curry, will say 'No', the 'negation' of yes! Now, if p denotes the statement 'I will eat rice.', how can we denote 'I will not eat rice.'? Let us define the connective that will help us do so.

Definition: The **negation** of a proposition p is '**not p**', denoted by $\sim p$.

For example, if p is 'Dolly is at the study centre.', then $\sim p$ is 'Dolly is not at the study centre.'. Similarly, if p is 'No person can live without oxygen.', $\sim p$ is 'At least one person can live without oxygen.'.

Now, regarding the truth value of $\sim p$, you would agree that it would be T if that of p is F, and vice versa. Keeping this in mind you can try the following exercises.

E7) Write down $\sim p$, where p is
i) $0 - 5 \neq 5$
ii) $n > 2$ for every $n \in N$.
iii) Most Indian children study till Class 5.

E8) Write down the truth table of negation.

Let us now discuss the conditional connectives, representing 'If $\cdots$, then $\cdots$' and 'if and only if'.

1.3.4 Conditional Connectives

Consider the proposition 'If Ayesha gets 75% or more in the examination, then she will get an A grade for the course.'. We can write this statement as 'If p, then q', where
p: Ayesha gets 75% or more in the examination, and
q: Ayesha will get an A grade for the course.
This compound statement is an example of the implication of q by p.

Definition: Given any two propositions p and q, we denote the statement 'If p, then q' by $p \rightarrow q$. We also read this as '**p implies q**', or '**p is sufficient for q**', or '**p only if q**'. We also call p the **hypothesis** and q the **conclusion**. Further, a statement of the form $p \rightarrow q$ is called a **conditional statement** or a **conditional proposition**.

So, for example, in the conditional proposition 'If m is in Z, then m belongs to Q.' the hypothesis is '$m \in Z$' and the conclusion is '$m \in Q$'.

Mathematically, we can write this statement as $m \in \mathbf{Z} \rightarrow m \in \mathbf{Q}$.

Let us analyse the statement $p \rightarrow q$ for its truth value. Do you agree with the truth table we've given below (Table 3)? You may like to check it out while keeping an example from your surroundings in mind.

Table 3 : Truth table for implication.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

You may wonder about the third row in Table 3. But, consider the example '$3 < 0 \rightarrow 5 > 0$'. Here the conclusion is true regardless of what the hypothesis is. And therefore, the conditional statement remains true. In such a situation we say that the **conclusion is vacuously true**.

Why don't you try this exercise now?

E9) Write down the proposition corresponding to $p \rightarrow q$, and determine the values of x for which it is false, where

$p : x + y = xy$ where $x, y \in \mathbf{R}$

$q : x \not< 0$ for every $x \in \mathbf{Z}$.

Now, consider the implication 'If Jahanara goes to Baroda, then she doesn't participate in the conference at Delhi.'. What would its converse be? To find it, the following definition may be useful.

Definition: The **converse** of $p \rightarrow q$ is $q \rightarrow p$. In this case we also say '**p is necessary for q**', or '**p if q**'.

So, in the example above, the converse of the statement would be 'If Jahanara doesn't participate in the conference at Delhi, then she goes to Baroda.'. This means that Jahanara's non-participation in the conference at Delhi is necessary for her going to Baroda.

Now, what happens when we combine an implication and its converse? To show '$p \rightarrow q$ and $q \rightarrow p$', we introduce a shorter notation.

Definition: Let p and q be two propositions. The compound statement

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

is the **biconditional** of p and q. We denote it by $p \longleftrightarrow q$, and read it as '**p if and only if q**'. We usually shorten 'if and only if' to iff.

We also say that '**p implies and is implied by q**', or '**p is necessary and sufficient for q**'.

The two connectives $\rightarrow$ and $\longleftrightarrow$ are called the **conditional connectives.**

For example, 'Sudha will gain weight if and only if she eats regularly.' means that 'Sudha will gain weight if she eats regularly and Sudha will eat regularly if she gains weight.'.

One point that may come to your mind here is whether there's any difference in the two statements $p \longleftrightarrow q$ and $q \longleftrightarrow p$. When you study Sec. 1.4 you will realise why they are inter-changeable.

Let us now consider the truth table of the biconditional, i.e., of the two-way implication. To obtain its truth values, we need to use Tables 2 and 3, as you will see in Table 4. This is because, to find the value of $(p \to q) \wedge (q \to p)$ we need to know the values of each of the simpler statements involved.

Table 4 : Truth table for two-way implication

p	q	$p \to q$	$q \to p$	$p \longleftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

As you can see from the last column of the table (and from your own experience), $p \longleftrightarrow q$ is true only when both p and q are true or both p and q are false. In other words, $p \longleftrightarrow q$ is true only when p and q have the same truth values. Thus, for example, 'Parimala is in America iff $2 + 3 = 5$' is true only if 'Parimala is in America.' is true.

Here are some related exercises.

E10) For each of the following compound statements, first identify the simple propositions p, q, r, etc., that are combined to make it. Then write it in symbols, using the connectives, and give its truth value.

i) If triangle ABC is equilateral, then it is isosceles.
ii) a and b are integers if and only if ab is a rational number.
iii) If Raza has five glasses of water and Sudha has four cups of tea, then Shyam will not pass the math examination.
iv) Mariam is in Class 1 or in Class 2.

E11) Write down two propositions p and q for which $q \to p$ is true but $p \longleftrightarrow q$ is false.

Now, how would you determine the truth value of a proposition which has more than one connective in it? For instance, does $\sim p \vee q$ mean $(\sim p) \vee q$ or $\sim (p \vee q)$? We discuss some rules for this below.

1.3.5 Precedence Rule

While dealing with operations on numbers, you would have realised the need for applying the BODMAS rule. According to this rule, when calculating the value of an arithmetic expression, we first calculate the value of the Bracketed portion, then apply Of, Division, Multiplication, Addition and Subtraction, in this order. While calculating the truth value of compound propositions involving more than one connective, we have a similar rule which tells us which connective to apply first.

Why do we need such a rule? Suppose we didn't have an order of preference, and want to find the truth value of, say, $\sim p \vee q$. Some of us may consider the value of $(\sim p) \vee q$, and some may consider $\sim (p \vee q)$. The truth values can be different in these cases. For instance, if p and q are both true, then $(\sim p) \vee q$ is true, but $\sim (p \vee q)$ is false. So, for the purpose of unambiguity, we agree to such an order or rule. Let us see what it is.

The rule of precedence: The order of preference in which the connectives are applied in a formula of propositions that has no brackets is

i) $\sim$
ii) $\wedge$
iii) $\vee$ and $\oplus$
iv) $\longrightarrow$ and $\longleftrightarrow$

Note that the 'inclusive or' and 'exclusive or' are both third in the order of preference. This means that you can apply either of them first, with the same end result. So, for instance, the truth values of $(p \oplus q) \vee r$ are the same as those of $p \oplus (q \vee r)$.

Similarly, the 'implication' and the 'biconditional' are both fourth in the order of preference.

To clearly understand how this rule works, let us consider an example.

Example 3: : Write down the truth table of $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$

Solution: We want to find the required truth value when we are given the truth values of p, q and r. According to the rule of precedence given above, we need to first find the truth value of $\sim r$, then that of $(q \wedge \sim r)$, then that of $(r \oplus q)$, and then that of either $p \rightarrow (q \wedge \sim r)$, or of $(q \wedge \sim r) \longleftrightarrow r \oplus q$, and finally the truth value of the remaining one. (We choose to apply '$\rightarrow$' before '$\longleftrightarrow$'.)

So, for instance, suppose p and q are true, and r is false. Then $\sim r$ will have value T, $q \wedge \sim r$ will be T, $r \oplus q$ will be T, $p \rightarrow (q \wedge \sim r)$ will be T, and hence, $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$ will be T.

You can check that the rest of the values are as given in Table 5. Note that we have 8 possibilities ($= 2^3$) because there are 3 simple propositions involved here.

Table 5 : Truth table for $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$

p	q	r	$\sim r$	$q \wedge \sim r$	$r \oplus q$	$p \rightarrow q \wedge \sim r$	$p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$
T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F

* * *

You may now like to try some exercises on the same lines.

E12) In Example 3, how will the truth values of the compound statement change if you first apply $\longleftrightarrow$ and then $\rightarrow$?

E13) In Example 3, if we replace $\oplus$ by $\wedge$, what is the new truth table?

E14) Form the truth tables of $p \wedge q \vee \sim r$ and $(p \wedge q) \vee (\sim r)$ and see where they differ.

E15) How would you bracket the following formulae to correctly interpret them? [For instance, $p \vee \sim q \wedge r$ would be bracketed as $p \vee ((\sim q) \wedge r)$.]

i) $\sim p \vee q$,
ii) $\sim q \rightarrow \sim p$,
iii) $p \rightarrow q \longleftrightarrow \sim p \vee q$,
iv) $p \ominus q \wedge r \rightarrow \sim p \vee q \longleftrightarrow p \wedge r$.

So far we have considered different ways of making new statements from old ones. But, are all these new ones distinct? Or are some of them the same? And "same" in what way? This is what we shall now consider.

1.4 LOGICAL EQUIVALENCE

'Then you should say what you mean', the March Hare went on.
'I do,' Alice hastily replied, 'at least ... at least I mean what I say — that's the same thing you know.'
'Not the same thing a bit!' said the Hatter. 'Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'

–from 'Alice in Wonderland'
by Lewis Carroll

In mathematics, as in ordinary language, there can be several ways of saying the same thing. In this section we shall discuss what this means in the context of logical statements.

$\sim q \rightarrow \sim p$ is the **contrapositive** of the proposition $p \rightarrow q$.

Consider the statements 'If Lala is rich, then he must own a car.', and 'If Lala doesn't own a car, then he is not rich.'. Do these statements mean the same thing? If we write the first one as $p \rightarrow q$, then the second one will be $(\sim q) \rightarrow (\sim p)$. How do the truth values of both these statements compare? We find out in the following table.

Table 6

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Consider the last two columns of Table 6. You will find that '$p \rightarrow q$' and '$\sim q \rightarrow \sim p$' have the same truth value for every choice of truth values of p and q. When this happens, we call them equivalent statements.

Definition: We call two propositions r and s **logically equivalent** provided they have the same truth value for every choice of truth values of the simple propositions involved in them. We denote this fact by $r \equiv s$.

So, from Table 6 we find that $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.

You can also check that $(p \longleftrightarrow q) \equiv (q \longleftrightarrow p)$ for any pair of propositions p and q.

As another example, consider the following equivalence that is often used in mathematics. You could also apply it to obtain statements equivalent to 'Neither a borrower, nor a lender be.'!

Example 4: For any two propositions p and q, show that
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

Solution: Consider the following truth table.

Table 7

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

You can see that the last two columns of Table 7 are identical. Thus, the truth values of $\sim (p \vee q)$ and $\sim p \wedge \sim q$ agree for every choice of truth values of p and q.

Therefore, $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

***

Fig. 1: Augustus De Morgan (1806-1871) was born in Madurai.

The equivalence you have just seen is one of **De Morgan's laws**. You have already come across these laws in the context of set operations in MTE-04.

The other law due to De Morgan is similar : $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

In fact, there are several such laws about equivalent propositions. Some of them are the following, where, as usual, p, q and r denote propositions.

a) **Double negation law** : $\sim (\sim p) \equiv p$

b) **Idempotent laws**: $p \wedge p \equiv p$,
$p \vee p \equiv p$

c) **Commutativity**: $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

d) **Associativity**: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

e) **Distributivity**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

We ask you to prove these laws now.

E16) Show that the laws given in (a)-(e) above hold true.

E17) Prove that the relation of 'logical equivalence' is an equivalence relation.

E18) Check whether $(\sim p \vee q)$ and $(p \rightarrow q)$ are logically equivalent.

The laws given above and the equivalence you have checked in E18 are commonly used, and therefore, useful to remember. You will also be applying them in Unit 3 in the context of switching circuits.

Let us now consider some propositional formulae which are always true or always false. Take, for instance, the statement 'If Bano is sleeping and Pappu likes ice-cream, then Bano is sleeping.'. You can draw up the truth table of this compound proposition and see that it is always true. This leads us to the following definition.

Definition: A compound proposition that is true for all possible truth values of the simple propositions involved in it is called a **tautology**. Similarly, a proposition that is false for all possible truth values of the simple propositions that constitute it is called a **contradiction**.

Let us look at some examples of such propositions.

Example 5: Verify that $p \wedge q \wedge \sim p$ is a contradiction and $p \rightarrow q \longleftrightarrow \sim p \vee q$ is a tautology.

Solution: Let us simultaneously draw up the truth tables of these two propositions below.

Table 8

p	q	$\sim p$	$p \wedge q$	$p \wedge q \wedge \sim p$	$p \rightarrow q$	$\sim p \vee q$	$p \rightarrow q \longleftrightarrow \sim p \vee q$
T	T	F	T	F	T	T	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	T	T
F	F	T	F	F	T	T	T

Looking at the fifth column of the table, you can see that $p \wedge q \wedge \sim p$ is a contradiction. This should not be surprising since $p \wedge q \wedge \sim p \equiv (p \wedge \sim p) \wedge q$ (check this by using the various laws given above).

And what does the last column of the table show? Precisely that $p \rightarrow q \longleftrightarrow \sim p \vee q$ is a tautology.

* * *

Why don't you try an exercise now?

E19) Let $\mathcal{T}$ denote a tautology (i.e., a statement whose truth value is always T) and $\mathcal{F}$ a contradiction. Then, for any statement p, show that

i) $p \vee \mathcal{T} \equiv \mathcal{T}$
ii) $p \wedge \mathcal{T} \equiv p$
iii) $p \vee \mathcal{F} \equiv p$
iv) $p \wedge \mathcal{F} \equiv \mathcal{F}$

Another way of proving that a proposition is a tautology is to use the properties of logical equivalence. Let us look at the following example.

Complementation law: $q \wedge \sim q$ is a contradiction.

Example 6: Show that $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Solution: $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv [(\sim p \vee q) \wedge \sim q] \rightarrow \sim p$, using E18, and symmetricity of $\equiv$.

$\equiv [(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \rightarrow \sim p$, by De Morgan's laws.

$\equiv [(\sim p \wedge \sim q) \vee \mathcal{F}] \rightarrow \sim p$, since $q \wedge \sim q$ is always false.

$\equiv (\sim p \wedge \sim q) \rightarrow \sim p$, using E18.

which is a tautology.
And therefore the proposition we started with is a tautology.

* * *

The laws of logical equivalence can also be used to prove some other logical equivalences, without using truth tables. Let us consider an example.

Example 7: Show that $(p \rightarrow \sim q) \wedge (p \rightarrow \sim r) \equiv \sim [p \wedge (q \vee r)]$.

Solution: We shall start with the statement on the left hand side of the equivalence that we have to prove. Then, we shall apply the laws we have listed above, or the equivalence in E18, to obtain logically equivalent statements. We shall continue this process till we obtain the statement on the right hand side of the equivalence given above. Now

$(p \rightarrow \sim q) \wedge (p \rightarrow \sim r)$

$\equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$, by E18

$\equiv \; \sim p \lor (\sim q \land \sim r)$, by distributivity
$\equiv \; \sim p \lor [\sim (q \lor r)]$, by De Morgan's laws
$\equiv \; \sim [p \land (q \lor r)]$, by De Morgan's laws

So we have proved the equivalence that we wanted to.

* * *

You may now like to try the following exercises on the same lines.

E20) Use the laws given in this section to show that
$\sim (\sim p \land q) \land (p \lor q) \equiv p$.

E21) Write down the statement 'If it is raining and if rain implies that no one can go to see a film, then no one can go to see a film.' as a compound proposition. Show that this proposition is a tautology, by using the properties of logical equivalence.

E22) Give an example, with justification, of a compound proposition that is neither a tautology nor a contradiction.

Let us now consider proposition-valued functions.

1.5 LOGICAL QUANTIFIERS

In Sec.1.2, you read that a sentence like 'She has gone to Patna.' is not a proposition, unless who 'she' is is clearly specified.

Similarly, '$x > 5$' is not a proposition unless we know the values of x that we are considering. Such sentences are examples of 'propositional functions'.

Definition: A **propositional function**, or a **predicate**, in a variable x is a sentence p(x) involving x that becomes a proposition when we give x a definite value from the set of values it can take. We usually denote such functions by p(x), q(x), etc. The set of values x can take is called the **universe of discourse**.

So, if p(x) is '$x > 5$', then p(x) is not a proposition. But when we give x particular values, say $x = 6$ or $x = 0$, then we get propositions. Here, p(6) is a true proposition and p(0) is a false proposition.

Similarly, if q(x) is 'x has gone to Patna.', then replacing x by 'Taj Mahal' gives us a false proposition.

Note that a predicate is usually not a proposition. But, of course, every proposition is a propositional function in the same way that every real number is a real-valued function, namely, the constant function.

Now, can all sentences be written in symbolic form by using only the logical connectives? What about sentences like 'x is prime and $x + 1$ is prime for some x.'? How would you symbolise the phrase 'for some x', which we can rephrase as 'there exists an x'? You must have come across this term often while studying mathematics. We use the symbol '$\exists$' to denote this quantifier, 'there exists'. The way we use it is, for instance, to rewrite 'There is at least one child in the class.' as
'$(\exists\, x$ in U$)$p(x)',
where p(x) is the sentence 'x is in the class.' and U is the set of all children.

$\exists$ is called the **existential quantifier**.

Now suppose we take the negative of the proposition we have just stated. Wouldn't it be 'There is no child in the class.'? We could symbolise this as 'for all x in U, q(x)' where x ranges over all children and q(x) denotes the sentence 'x is not in the class.', i.e., $q(x) \equiv \sim p(x)$.

We have **a mathematical symbol for the quantifier 'for all', which i** '∀'. So the proposition above can be written as '$(\forall x \in U)q(x)$', or '$q(x), \forall x \in U$'.

∀ is called the **universal quantifier**.

An example of the use of the existential quantifier is the true statement $(\exists x \in R)(x + 1 > 0)$, which is read as 'There exists an x in R for which $x + 1 > 0$.'.

Another example is the false statement
$(\exists x \in N)(x - \frac{1}{2} = 0)$, which is read as 'There exists an x in N for which $x - \frac{1}{2} = 0$.'.

An example of the use of the universal quantifier is
$(\forall x \notin N)(x^2 > x)$, which is read as 'for every x not in N, $x^2 > x$.'.
Of course, this is a false statement, because there is at least one $x \notin N, x \in R$, for which it is false.

We often use both quantifiers together, as in the statement called **Bertrand's postulate:**
$(\forall n \in N\backslash\{1\})(\exists x \in N)$ (x is a prime number and $n < x < 2n$).
In words, this is 'for every integer $n > 1$ there is a prime number lying strictly between n and 2n.'

As you have already read in the example of a child in the class,
$(\forall x \in U)p(x)$ is logically equivalent to $\sim (\exists x \in U)(\sim p(x))$. Therefore,
$\sim (\forall x \in U)p(x) \equiv \sim\sim (\exists x \in U)(\sim p(x)) \equiv (\exists x \in U)(\sim p(x))$.

This is one of the rules for negation that relate ∀ and ∃. The two rules are
$\sim (\forall x \in U)p(x) \equiv (\exists x \in U)(\sim p(x))$, and

$\sim (\exists x \in U)p(x) \equiv (\forall x \in U)(\sim p(x))$
where U is the set of values that x can take.

Now, consider the proposition
'There is a criminal who has committed every crime.'
We could write this in symbols as
$(\exists c \in A)(\forall x \in B)$(c has committed x)
where, of course, A is the set of criminals and B is the set of crimes (determined by law).
What would its negation be? It would be
$\sim (\exists c \in A)(\forall x \in B)$(c has committed x)
$\equiv (\forall c \in A)[\sim (\forall x \in B)$(c has committed x)
$\equiv (\forall c \in A)(\exists x \in B)$(c has not committed x).
We can interpret this as 'For every criminal, there is a crime that this person has not committed.'.

A predicate can be a function in two or more variables.

These are only some examples in which the quantifiers occur singly, or together. Sometimes you may come across situations (as in E22) where you would use ∃ or ∀ twice or more in a statement. It is in situations like this or worse [say, $(\forall x_1 \in U_1)(\exists x_2 \in U_2)(\exists x_3 \in U_3)(\forall x_4 \in U_4)\ldots(\exists x_n \in U_n)p$] where our rule for negation comes in useful. In fact, applying it, in a trice we can say that the negation of this seemingly complicated example is
$(\exists x_1 \in U_1)(\forall x_2 \in U_2)(\forall x_3 \in U_3)(\exists x_4 \in U_4)\ldots(\forall x_n \in U_n)(\sim p)$.

Why don't you try some exercises now?

E23) How would you present the following propositions and their negations using logical quantifiers? Also interpret the negations in words.

i) The politician can fool all the people all the time.
ii) Every real number is the square of some real number.
iii) There is a lawyer who never tells lies.

E24) Write down suitable mathematical statements that can be represented by the following symbolic propositions. Also write down their negations. What is the truth value of your propositions?

i) $(\forall x)(\exists y)p$
ii) $(\exists x)(\exists y)(\forall z)p$.

And finally, let us look at a very useful quantifier, which is very closely linked to $\exists$. You would need it for writing, for example, 'There is one and only one key that fits the desk's lock.' in symbols. The symbol is $\exists!\, x$ which stands for 'there is one and only one x' (which is the same as 'there is a unique x' or 'there is exactly one x').

So, the statement above would be $(\exists!\, x \in A)$(x fits the desk's lock), where A is the set of keys.

For other examples, try and recall the statements of uniqueness in the mathematics that you've studied so far. What about 'There is a unique circle that passes through three non-collinear points in a plane.'? How would you represent this in symbols? If x denotes a circle, and y denotes a set of 3 non-collinear points in a plane, then the proposition is
$(\forall y \in P)(\exists!\, x \in C)$(x passes through y).

Here C denotes the set of circles, and P the set of sets of 3 non-collinear points.

And now, some short exercises for you!

E25) Which of the following propositions are true (where x, y are in R)?

i) $(x \geq 0) \rightarrow (\exists\, y)(y^2 = x)$
ii) $(\forall x)(\exists!\, y)(y^2 = x^3)$
iii) $(\exists x)(\exists!\, y)(xy = 0)$
iv) $\sim (\exists x)(\exists!\, y)(x + y = 0)$.

Before ending the unit, let us take a quick look at what we have covered in it.

1.6 SUMMARY

In this unit we have considered the following points.

1. What a mathematically acceptable statement (or proposition) is.
2. The definition and use of logical connectives:
   Given propositions p and q,
   i) their disjunction is 'p or q', denoted by $p \vee q$;
   ii) their exclusive disjunction is 'either p or q', denoted by $p \oplus q$;
   iii) their conjunction is 'p and q', denoted by $p \wedge q$;
   iv) the negation of p is 'not p', denoted by $\sim p$;
   v) 'if p, then q' is denoted by $p \rightarrow q$;
   vi) 'p if and only if q' is denoted by $p \longleftrightarrow q$;
3. The truth tables corresponding to the 6 logical connectives.

4. Rule of precedence : In any compound statement involving more than one connective, we first apply '$\sim$', then '$\wedge$', then '$\vee$' and '$\oplus$', and last of all '$\longrightarrow$' and '$\longleftrightarrow$'.
5. The meaning and use of logical equivalence, denoted by '$\equiv$'.
6. The following laws about equivalent propositions:
   i) **De Morgan's laws** : $\sim (p \wedge q) \equiv \sim p \vee \sim q$
   $\sim (p \vee q) \equiv \sim p \wedge \sim q$
   ii) **Double negation law** : $\sim (\sim p) \equiv p$
   iii) **Idempotent laws**: $p \wedge p \equiv p$.
   $p \vee p \equiv p$
   iv) **Commutativity**: $p \vee q \equiv q \vee p$
   $p \wedge q \equiv q \wedge p$
   v) **Associativity**: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   vi) **Distributivity**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   vii) $(\sim p \vee q) \equiv p \longrightarrow q$ (ref. E18).
7. Logical quantifiers : 'For every' denoted by '$\forall$', 'there exists' denoted by '$\exists$', and 'there is one and only one' denoted by '$\exists!$'.
8. The rule of negation related to the quantifiers:
   $\sim (\forall x \in U)p(x) \equiv (\exists x \in U)(\sim p(x))$
   $\sim (\exists x \in U)p(x) \equiv (\forall x \in U)(\sim p(x))$

Now we have come to the end of this unit. You should have tried all the exercises as you came to them. You may like to check your solutions with the ones we have given below.

1.7 SOLUTIONS/ANSWERS

E1) (i), (iii), (iv), (vii), (viii) are statements because each of them is universally true or universally false.
(ii) is a question.
(v) is an exclamation.
The truth or falsity of (vi) depends upon who 'she' is.
(ix) is a subjective sentence.
(x) will only be a statement if the value(s) n takes is/are given.
Therefore, (ii), (v), (vi), (ix) and (x) are not statements.

E2) The truth value of (i) is F, and of all the others is T.

E4) The disjunction is
'2+3 = 7 or Radha is an engineer.'.
Since '2+3 = 7' is always false, the truth value of this disjunction depends on the truth value of 'Radha is an engineer.'. If this is T, then we use the third row of Table 1 to get the required truth value as T. If Radha is not an engineer, then we get the required truth value as F.

E5) Table 9: Truth table for 'exclusive or'

p	q	$p \ominus q$
T	T	F
T	F	T
F	T	T
F	F	F

E6) p will be a true proposition for $x \in]-2, \infty[$.
q will be a true proposition for $x \neq 4$.
Therefore $p \wedge q$ will be true for every x such that $x \in]-2, \infty[$ and $x \neq 4$, i.e., for $x \in]-2, 4[\cup]4, \infty[$.

E7) i) $0 - 5 = 5$
ii) 'n is not greater than 2 for every $n \in N$.', or 'There is at least one $n \in N$ for which $n \leq 2$.'
iii) There are some Indian children who do not study till Class 5.

E8) Table 10: Truth table for negation

p	~ p
T	F
F	T

E9) $p \rightarrow q$ is the statement 'If $x + y = xy$ for $x, y \in R$, then $x \not< 0$ for every $x \in Z$'.
In this case, q is false. Therefore, the conditional statement will be true if p is false also, and it will be false for those values of x and y that make p true.

So, $p \rightarrow q$ is false for all those real numbers x of the form $\frac{y}{y-1}$, where $y \in R\backslash\{1\}$. This is because if $x = \frac{y}{y-1}$ for some $y \in R\backslash\{1\}$, then $x + y = xy$, i.e., p will be true.

E10) i) $p \rightarrow q$, where p : ΔABC is equilateral, and q : ΔABC is isosceles.
If q is true, then $p \rightarrow q$ is true. If q is false, then $p \rightarrow q$ is true only when p is false. So, if ΔABC is an isosceles triangle, the given statement is always true. Also, if ΔABC is not isosceles, then it can't be equilateral either. So the given statement is again true.

ii) p : a is an integer.
q : b is an integer.
r : ab is a rational number.
The given statement is $(p \wedge q) \longleftrightarrow r$.
Now, if p is true and q is true, then r will be true.
If $p \wedge q$ is false, it can happen that r is still true.
So, $(p \wedge q) \longleftrightarrow r$ will be true if $p \wedge q$ is true, or when $p \wedge q$ is false and r is false.
In all the other cases $(p \wedge q) \longleftrightarrow r$ will be false.

iii) p : Raza has 5 glasses of water.
q : Sudha has 4 cups of tea.
r : Shyam will pass the math exam.
The given statement is $(p \wedge q) \rightarrow \sim r$.
This is true when $\sim r$ is true, or when r is true and $p \wedge q$ is false.
In all the other cases it is false.

iv) p : Mariam is in Class 1.
q : Mariam is in Class 2.
The given statement is $p \oplus q$.
This is true only when p is true or when q is true.

E11) There are infinitely many such examples. You need to give one in which p is true but q is false.

E12) Obtain the truth table. The last columns of your table and Table 5 will coincide.

E13) According to the rule of precedence, given the truth values of p, q, r you should first find those of $\sim r$, then of $q \wedge \sim r$, and $r \wedge q$, and $p \rightarrow q \wedge \sim r$, and finally of $(p \rightarrow q \wedge \sim r) \longleftrightarrow r \wedge q$.

Referring to Table 5, the values in the sixth and eighth columns will be replaced by

$r \wedge q$
T
F
F
F
T
F
F
F

and

$p \rightarrow q \wedge \sim r \longleftrightarrow r \wedge q$
F
F
T
T
T
F
F
F

E14) They should both be the same, viz.,

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee (\sim r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

E15) i) $(\sim p) \vee q$

ii) $(\sim q) \rightarrow (\sim p)$

iii) $p \rightarrow q \longleftrightarrow [(\sim p) \vee q]$

iv $[p \oplus (q \wedge r)] \rightarrow [(\sim p) \vee q] \longleftrightarrow (p \wedge r)$

E16) a)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

The first and third columns prove the double negation law.

c)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The third and fourth columns prove the commutativity of $\vee$.

The other laws can be similarly proved.

E17) For any three propositions p, q, r:

i) $p \equiv p$ is trivially true.

ii) if $p \equiv q$, then $q \equiv p$ ($\because$ if p has the same truth value as q for all choices of truth values of p and q, then clearly q has the same truth values as p in all the cases.)

iii) if $p \equiv q$ and $q \equiv r$, then $p \equiv r$ (reason as in (ii) above).

Thus, $\equiv$ is reflexive, symmetric and transitive.

$\because$ denotes 'because'.

E18)

p	q	$\sim p$	$\sim p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last two columns show that $[(\sim p) \vee q] \equiv (p \to q)$.

E19) i)

p	$\mathcal{T}$	$p \vee \mathcal{T}$
T	T	T
F	T	T

The second and third columns of this table show that $p \vee \mathcal{T} \equiv \mathcal{T}$.

iv)

p	$\mathcal{F}$	$p \wedge \mathcal{F}$
T	F	F
F	F	F

The second and third columns of the adjoining table show that $p \wedge \mathcal{F} \equiv$ F.

You can similarly check (ii) and (iii).

E20) $\sim (\sim p \wedge q) \wedge (p \vee q)$
$\equiv (\sim (\sim p) \vee \sim q) \wedge (p \vee q)$, by De Morgan's laws .
$\equiv (p \vee \sim q) \wedge (p \vee q)$, by the double negation law.
$\equiv p \vee (\sim q \wedge q)$, by distributivity
$\equiv p \vee \mathcal{F}$, where $\mathcal{F}$ denotes a contradiction
$\equiv p$, using E 19 .

E21) p: It is raining.
q: Nobody can go to see a film.
Then the given proposition is
$[p \wedge (p \to q)] \to q$
$\equiv p \wedge (\sim p \vee q) \to q$, since $(p \to q) \equiv (\sim p \vee q)$
$\equiv (p \wedge \sim p) \vee (p \wedge q) \to q$, by De Morgan's law
$\equiv \mathcal{F} \vee (p \wedge q) \to q$, since $p \wedge \sim p$ is a contradiction
$\equiv (\mathcal{F} \vee p) \wedge (\mathcal{F} \vee q) \to q$, by De Morgan's law
$\equiv p \wedge q \to q$, since $\mathcal{F} \vee p \equiv p$.
which is a tautology.

E22) There are infinitely many examples. One such is:
'If Venkat is on leave, then Shabnam will work on the computer.'.
This is of the form $p \to q$. Its truth values will be T or F, depending on those of p and q.

E23) i) $(\forall\, t \in [0, \infty[)(\forall\, x \in H)p(x, t)$ is the given statement where $p(x, t)$ is the predicate 'The politician can fool x at time t seconds.', and H is the set of human beings.
Its negation is $(\exists\, t \in [0, \infty[)(\exists\, x \in H)(\sim p(x, t))$, i.e., there is somebody who is not fooled by the politician at least for one moment.

ii) The given statement is
$(\forall\, x \in R)(\exists\, y \in R)(x = y^2)$.
Its negation is
$(\exists\, x \in R)(\forall\, y \in R)(x \neq y^2)$, i.e.,
there is a real number which is not the square of any real number.

iii) The given statement is
$(\exists\, x \in L)(\forall\, t \in [0, \infty[)p(x, t)$, where L is the set of lawyers and

p(x, t) : x does not lie at time t.
The negation is
$(\forall x \in L)(\exists t \in [0, \infty[)(\sim p)$
i.e., every lawyer tells a lie at some time.

E24) i) For example,
$(\forall x \in N)(\exists y \in Z)\left(\frac{x}{y} \in Q\right)$ is a true statement.
Its negation is
$(\exists x \in N)(\forall y \in Z)\left(\frac{x}{y} \notin Q\right)$
You can try (ii) similarly.

E25) (i), (iii) are true.
(ii) is false (e.g., for $x = -1$ there is no y such that $y^2 = x^3$).
(iv) is equivalent to $(\forall x \in R)[\sim (\exists! y \in R)(x + y = 0)]$, i.e., for every x there is no unique y such that $x + y = 0$. This is clearly false, because for every x there is a unique $y (= -x)$ such that $x + y = 0$.

UNIT 2 METHODS OF PROOF

Structure

2.1 INTRODUCTION

In the previous unit you studied about statements and their truth values. In this unit, we shall discuss ways in which statements can be linked to form a logically valid argument. Throughout your mathematical studies you would have come across the terms 'theorem' and 'proof'. In Sec. 2.2, we shall talk about what a theorem is and what constitutes a mathematically acceptable proof.

In Sec.2.3, we shall expose you to the different methods used for proving or disproving a statement. When you go through the different types of valid arguments, you will see how mathematicians think and build more mathematics on the basis of certain assumptions. The ideas in this section were formalised by the English mathematician Boole and the German logician Frege (1848 – 1925).

Fig.1: George Boole (1815 – 1864)

The principle of mathematical induction has a very special place in mathematics because of its simplicity and vast applicability. You will study this tool for proving statements in Sec.2.4.

Please go through this unit carefully. Not only is it important for studying this course, but its contents are part of the foundation on which all mathematical knowledge is built.

Objectives

After reading this unit you should be able to

- explain the terms 'theorem', 'proof' and 'disproof';
- describe the direct method and some indirect methods of proof;
- state and apply both forms of the principle of induction.

2.2 WHAT IS A PROOF?

Suppose I tell somebody, "I am stronger than you." The person is quite likely to turn around, look menacingly at me, and say, "Prove it!" What she or he really wants is to be convinced of my statement by some evidence. (In this case it would probably be a big physical push!)

Convincing evidence is also what the world asks for before accepting a scientist's predictions, or a historian's claims.

In the same way, if you want a mathematical statement to be accepted as true, you would need to provide **mathematically acceptable** evidence to support it. This means that you would need to show that the statement is **universally true**. And this would be done in the form of a logically valid argument.

Definition: An **argument** (in mathematics or logic) is a finite sequence of statements $p_1, \cdots, p_n, p$ such that $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow p$.
Each statement in the sequence, except the last one, (i.e., p_i for $i = 1, \cdots, n$) is called a **premise** (or an **assumption**, or a **hypothesis**). The final statement p is called the **conclusion**.

Let's consider an example of an argument that shows that a given statement is true.

Example 1: Give an argument to show that the mathematical statement 'For any two sets A and B, $A \cap B \subseteq A$' is true.

Solution: One argument could be the following.

Let x be an arbitrary element of $A \cap B$.
Then $x \in A$ and $x \in B$, by definition of '$\cap$'.
Therefore, $x \in A$.
This is true for every x in $A \cap B$.
Therefore, $A \cap B \subseteq A$, by definition of '$\subseteq$'.

* * *

The argument in Example 1 has a peculiar nature. The truth of each of the 4 premises and of its conclusion follows from the truth of the earlier premises in it. Of course, we start by assuming that the first statement is true. Then, assuming the definition of 'intersection', the second statement is true. The third one is true, whenever the second one is true because of the properties of logical implication. The fourth statement is true whenever the first three are true, because of the definition and properties of the term 'for all'. And finally, the last statement is true whenever all the earlier ones are. In this way we have shown that the given statement is true. In other words, we have proved the given statement, as the following definitions show.

Definitions: We say that a proposition p **follows logically from** propositions $p_1, p_2, \cdots$ and p_n if p must be true whenever $p_1, p_2, \cdots, p_n$ are true, i.e., $(p_1, \wedge p_2 \wedge \cdots \wedge p_n) \Longrightarrow p$.
[Here, note the use of the implication arrow '$\Rightarrow$'. For any two propositions r and s, '$r \Rightarrow s$' denotes 's is true whenever r is true.' Note that, using the contrapositive, this also denotes 'r is false whenever s is false'. Thus '$r \rightarrow s$' and '$r \Longrightarrow s$' are different except when both r and s are true or both are false.]

A **proof** of a proposition p is a mathematical argument consisting of a sequence of statements $p_1, p_2, \cdots, p_n$ from which p logically follows. So, p is the conclusion of this argument.

The statement that is proved to be true is called a **theorem**.

Sometimes, as you will see in Sec.2.3.3, instead of showing that a statement p is true, we try to prove that it is false, i.e., that $\sim p$ is true. Such a proof is called a **disproof** of p. In the next section you will read about some ways of disproving a statement.

Sometimes it happens that we feel a certain statement is true, but we don't succeed in proving it. It may also happen that we can't disprove it. Such statements are called **conjectures**. If and when a conjecture is proved, it would be called a theorem. If it is disproved, then its negative will be a theorem!

In this context, there's a very famous conjecture which was made by a mathematician Goldbach in 1742. He stated that :
For every $n \in N$, if n is even and $n > 2$, then n is the sum of two primes.

To this day, no one has been able to prove it or disprove it. To disprove it several people have been hunting for an example for which the statement is not true, i.e., an even number $n > 2$ such that n cannot be written as the sum of two prime numbers.

Now, as you have seen, a mathematical proof of a statement consists of one or more premises. These premises could be of four types:

i) a proposition that has been proved earlier (e.g., to prove that the complex roots of a polynomial in R[x] occur in pairs, we use the division algorithm); or

ii) a proposition that follows logically from the earlier propositions given in the proof (as you have seen in Example 1); or

iii) a mathematical fact that has never been proved, but is universally accepted as true (e.g., two points determine a line). Such a fact is called an **axiom (or a postulate)**;

iv) the definition of a mathematical term (e.g., assuming the definition of '$\subseteq$' in the proof of $A \cap B \subseteq A$).

You will come across more examples of each type while doing the following exercises, and while going through proofs in this course and other courses.

E1) Write down an example of a theorem, and its proof (of at least 4 steps), taken from school-level algebra. At each step, indicate which of the four types of premise it is.

E2) Is every statement a theorem? Why?

So far we have spoken about valid, or acceptable, arguments. Now let us see an example of a sequence of statements that will **not** form a valid argument. Consider the following sequence.

If Maya sees the movie, she won't finish her homework.
Maya won't finish her homework.
Therefore, Maya sees the movie.

Looking at the argument, can you say whether it is valid or not? Intuitively you may feel that the argument isn't valid. But, is there a formal logical tool that you can apply to check if your intuition is correct? What about truth tables? Let's see.

The given argument is of the form
$[(p \to q) \wedge q] \Rightarrow p$
where
p: Maya sees the movie, and
q: Maya won't finish her homework.

Let us look at the truth table related to this argument (see Table 1).

Table 1

p	q	$p \to q$	$(p \to q) \wedge p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	F

The last column gives the truth values of the premises. The first column gives the corresponding truth values of the conclusion. Now, the **argument will only be valid if whenever both the premises are true, the conclusion is true.** This happens in the first row, but **not** in the third row. Therefore, the argument is **not** valid.

Why don't you check an argument for validity now?

E3) Check whether the following argument is valid.
$(p \to q \vee \sim r) \wedge (q \to p) \Rightarrow (p \to r)$

You have seen that a proof is a logical argument that verifies the truth of a theorem. There are several ways of proving a theorem, as you will see in the next section. All of them are based on one or more **rules of inference**, which are different forms of arguments. We shall now present four of the most commonly used rules.

i) **Law of detachment** (or **modus ponens**)

'**Modus ponens**' is a Latin term which means 'method of affirmation'.

Consider the following argument:

If Kali can draw, she will get a job.
Kali can draw.
Therefore, she will get a job.

To study the form of the argument, let us take p to be the proposition 'Kali can draw.' and q to be the proposition 'Kali will get a job.'. Then the premises are $(p \to q)$ and p. The conclusion is q.
So, the form of the argument is

$\therefore$ denotes 'therefore'.

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array} \quad \text{i.e., } [(p \to q) \wedge p] \Rightarrow q.$$

Is this argument valid? To find out, let's construct its truth table (see Table 2).

Table 2: Truth table for $[(p \to q) \wedge p] \Rightarrow q$

p	q	$p \to q$	$(p \to q) \wedge p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

In the table, look at the second column (the conclusion) and the fourth column (the premises). Whenever the premises are true, i.e., in Row 1, the conclusion is true. Therefore, the argument is valid.

This form of valid argument is called the law of detachment because the conclusion q is detached from a premise (namely, $p \to q$). It is also called the **law of direct inference**.

ii) **Law of contraposition (or modus tollens)**
To understand this law, consider the following argument:

'Modus tollens' means 'method of denial'.

If Kali can draw, then she will get a job.
Kali will not get a job.
Therefore, Kali can't draw.

Taking p and q as in (i) above, you can see that the premises are $p \to q$ and $\sim q$. The conclusion is $\sim p$.
So the argument is

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}, \quad \text{i.e., } [(p \to q) \wedge \sim q] \Rightarrow \sim p.$$

If you check, you'll find that this is a valid form of argument.

There are two more rules of inference that most commonly form the basis of several proofs. The following exercise is about them.

E4) You will find three arguments below. Convert each of them into the language of symbols, and check if they are valid.

i) Either the eraser is white or oxygen is a metal.
The eraser is black.
Therefore, oxygen is a metal.

ii) If Madhu is a 'sarpanch', she will head the 'panchayat'.
If Madhu heads the 'panchayat', she will decide on property disputes.
Therefore, if Madhu is a 'sarpanch', she will decide on property disputes.

iii) Either Munna will cook or Munni will practise Karate.
If Munni practises Karate, then Munna studies.
Munna does not study.
Therefore, Munni will practise Karate.

E5) Write down one example each of modus ponens and modus tollens.

As you must have discovered, the arguments in E4(i) and (ii) are valid. The first one is an example of a **disjunctive syllogism**. The second one is an example of a **hypothetical syllogism**.

Thus, a disjunctive syllogism is of the form

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array}, \quad \text{i.e., } [(p \vee q) \wedge \sim p] \Rightarrow q.$$

And, a hypothetical syllogism is of the form

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}, \quad \text{i.e., } [(p \to q) \wedge (q \to r)] \Rightarrow (p \to r).$$

Let us now see how different forms of arguments can be put together to prove or disprove a statement.

2.3 DIFFERENT METHODS OF PROOF

In this section we shall consider three different strategies for proving a statement. We will also discuss a method that is used only for disproving a statement.

Let us start with a proof strategy based on the first rule of inference that we discussed in the previous section.

2.3.1 Direct Proof

This form of proof is based entirely on modus ponens. Let us formally spell out the strategy.

Definition : A **direct proof** of $p \Rightarrow q$ is a logically valid argument that begins with the assumption that p is true and, in one or more applications of the law of detachment, concludes that q must be true.

So, to construct a direct proof of $p \Rightarrow q$, we start by assuming that p is true. Then, in one or more steps of the form $p \Rightarrow q_1, q_1 \Rightarrow q_2, \ldots, q_n \Rightarrow q$, we conclude that q is true. Consider the following examples.

Example 2: Give a direct proof of the statement 'The product of two odd integers is odd.'.

Solution: Let us clearly analyse what our hypotheses are, and what we have to prove.
We start by considering any two odd integers x and y. So our hypothesis is
p: x and y are odd.
The conclusion we want to reach is
q : xy is odd.
Let us first prove that $p \Rightarrow q$.
Since x is odd, $x = 2m + 1$ for some integer m.
Similarly, $y = 2n + 1$ for some integer n.
Then $xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1$.
Therefore, xy is odd.
So we have shown that $p \Rightarrow q$.

Now we can apply modus ponens to $p \wedge (p \Rightarrow q)$ to get the required conclusion.

Note that the essence of this direct proof lies in showing $p \Rightarrow q$.

* * *

Example 3: Give a direct proof of the theorem 'The square of an even integer is an even integer.'.

Solution: First of all, let us write the given statement symbolically, as
$(\forall x \in Z)(p(x) \Rightarrow q(x))$
where $p(x)$: x is even, and
$q(x)$: x^2 is even, i.e., $q(x)$ is the same as $p(x^2)$.

The direct proof, then goes as follows:

Let x be an even number (i.e., we assume $p(x)$ is true).
Then $x = 2n$, for some integer n (we apply the definition of an even number).
Then $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$.
$\therefore x^2$ is even (i.e., $q(x)$ is true).

Note that we have proved the statement for every x since we have treated x as an arbitrary even number and not a particular value.

* * *

Why don't you try an exercise now?

E6) Give a direct proof of the statement 'If x is a real number such that $x^2 = 9$, then either $x = 3$ or $x = -3$.'.

Let us now consider another proof strategy.

2.3.2 Indirect Proofs

In this sub-section we shall consider two roundabout methods for proving $p \Rightarrow q$.

Proof by contrapositive: In the first method, we use the fact that the proposition $p \Rightarrow q$ is logically equivalent to its **contrapositive** $(\sim q \Rightarrow \sim p)$, i.e.,
$(p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p)$.

For instance, 'If Ammu does not agree with communalists, then she is not orthodox.' is the same as 'If Ammu is orthodox, then she agrees with communalists.'.

Because of this equivalence, to prove $p \Rightarrow q$, we can, instead, prove $\sim q \Rightarrow \sim p$. This means that we can assume that $\sim q$ is true, and then try to prove that $\sim p$ is true. In other words, what we do to prove $p \Rightarrow q$ in this method is to assume that q is false and then show that p is false. Let us consider an example.

Example 4: Prove that 'If $x, y \in Z$ such that xy is odd, then both x and y are odd.', by proving its contrapositive.

Solution: Let us name the statements involved as below.
p : xy is odd
q : both x and y are odd.

So,
$\sim p$: xy is even, and
$\sim q$: x is even or y is even, or both are even.

We want to prove $p \Rightarrow q$, by proving that $\sim q \Rightarrow \sim p$. So we start by assuming that $\sim q$ is true, i.e., we suppose that x is even.
Then $x = 2n$ for some $n \in N$.
Therefore, $xy = 2ny$.
Therefore, xy is even, by definition.
That is, $\sim p$ is true.

So, we have shown that $\sim q \Rightarrow \sim p$. Therefore, $p \Rightarrow q$.

* * *

Why don't you try some related exercises now?

E7) Write down the contrapositive of the statement 'If f is a 1-1 function from a finite set X into itself, then f must be surjective.'.

E8) Prove the statement 'If x is an integer and x^2 is even, then x is also even.' by proving its contrapositive.

And now let us consider the other way of proving a statement indirectly.

Proof by contradiction: In this method, to prove q is true, we start by assuming that q is false (i.e., $\sim q$ is true). Then, by a logical argument we arrive at a situation where a statement is true as well as false, i.e., we reach a contradiction $r \wedge \sim r$ for some statement r. This means that the truth of $\sim q$ implies a contradiction, a statement that is always false. This can only happen when $\sim q$ is false also. Therefore, q must be true.

This method is called **proof by contradiction**. It is also called **reductio ad absurdum** (a Latin phrase) because it relies on reducing a given assumption to an absurdity.

Let us consider an example of the use of this method.

Example 5: Show that $\sqrt{5}$ is irrational.

Solution: Let us try and prove the given statement by contradiction. For this, we begin by assuming that $\sqrt{5}$ is rational. This means that there exist positive integers a and b such that $\sqrt{5} = \frac{a}{b}$, where a and b have no common factors.
This implies $a = \sqrt{5}b \Rightarrow a^2 = 5b^2 \Rightarrow 5|a^2 \Rightarrow 5|a$.
Therefore, by definition, $a = 5c$ for some $c \in Z$.
Therefore, $a^2 = 25c^2$.
But $a^2 = 5b^2$ also.
So $25c^2 = 5b^2 \Rightarrow 5c^2 = b^2 \Rightarrow 5|b^2 \Rightarrow 5|b$.

But now we find that 5 divides both a and b, which contradicts our earlier assumption that a and b have no common factor.
Therefore, we conclude that our assumption that $\sqrt{5}$ is rational is false, i.e., $\sqrt{5}$ is irrational.

* * *

We can also use the method of contradiction to prove an implication $r \Rightarrow s$. Here we can use the equivalence $\sim (r \rightarrow s) \equiv r \wedge \sim s$. So, to prove $r \Rightarrow s$, we can begin by assuming that $r \Rightarrow s$ is false, i.e., r is true and s is false. Then we can present a valid argument to arrive at a contradiction.

Consider the following example from plane geometry.

Example 6: Prove the following:
If two distinct lines L_1 and L_2 intersect, then their intersection consists of exactly one point.

Solution: To prove the given implication by contradiction, let us begin by assuming that the two distinct lines L_1 and L_2 intersect in more than one point. Let us call two of these distinct points A and B. Then, both L_1 and L_2 contain A and B. This contradicts the axiom from geometry that says 'Given two distinct points, there is exactly one line containing them.'.

Therefore, if L_1 and L_2 intersect, then they must intersect in only one point.

* * *

The contradiction rule is also used for solving many logical puzzles by discarding all solutions that reduce to contradictions. Consider the following example.

Example 7: There is a village that consists of two types of people -- those who always tell the truth, and those who always lie. Suppose that you visit the village and two villagers A and B come up to you. Further, suppose
A says, "B always tells the truth," and
B says, "A and I are of opposite types."
What types are A and B?

Solution: Let us start by assuming A is a truth-teller.
∴ What A says is true.
∴ B is a truth-teller.
∴ What B says is true.
∴ A and B are of opposite types.
This is a contradiction, because our premises say that A and B are both truth-tellers.
∴ The assumption we started with is false.
∴ A always tells lies.
∴ What A has told you is a lie.
∴ B always tells lies.
∴ A and B are of the same type, i.e., both of them always lie.

* * *

Here are a few exercises for you now. While doing them you would realise that there are situations in which all the three methods of proof we have discussed so far can be used.

E9) Use the method of proof by contradiction to show that

i) $\sqrt{3}$ is irrational,

ii) For $x \in R$, if $x^3 + 4x = 0$, then $x = 0$.

E10) Prove E 9(ii) directly as well as by the method of contrapositive.

E11) Suppose you are visiting the village described in Example 7 above. Another two villagers C and D approach you. C tells you, "Both of us always tell the truth," and D says, "C always lies." What types are C and D?

There can be several ways of proving a statement.

Let us now consider the problem of showing that a statement is false.

2.3.3 Counterexamples

Suppose I make the statement 'All human beings are 5 feet tall.'. You are quite likely to show me an example of a human being standing nearby for whom the statement is not true. And, as you know, the moment we have even one example for which the statement $(\forall x)p(x)$ is false [i.e., $(\exists x)(\sim p(x))$ is true], then the statement is false.

An example that shows that a statement is false is a **counterexample** to such a statement. The name itself suggests that it is an example to counter a given statement.

A common situation in which we look for counterexamples is to disprove statements of the form $p \rightarrow q$. From Unit 1, you know that $\sim (p \rightarrow q) \equiv p \wedge \sim q$. Therefore, a counterexample to $p \rightarrow q$ needs to be an example where $p \wedge \sim q$ is true, i.e., p is true and $\sim q$ is true, i.e., the hypothesis p holds but the conclusion q does not hold.

For instance, to disprove the statement 'If n is an odd integer, then n is prime.', we need to look for an odd integer which is not a prime number. 15 is one such integer. So, $n = 15$ is a counterexample to the given statement.

Notice that **a counterexample to a statement p proves that** p is false, i.e., **$\sim$ p is true.**

Let us consider another example.

Example 8: Disprove the following statement:
$(\forall a \in R)(\forall b \in R)[(a^2 = b^2) \Rightarrow (a = b)]$.

Solution: A good way of disproving it is to look for a counterexample, that is, a pair of real numbers a and b for which $a^2 = b^2$ but $a \neq b$. Can you think of such a pair? What about $a = 1$ and $b = -1$? They serve the purpose.

In fact, there are infinitely many counterexamples. (Why?)

* * *

Now, an exercise!

E12) Disprove the following statements by providing a suitable counterexample.

i) $\forall\, x \in Z, x \in N$.

ii) $(x + y)^n = x^n + y^n \;\forall\, n \in N, x, y \in Z$.

iii) $f : N \to N$ is 1-1 iff f is onto.

(**Hint:** To disprove $p \Leftrightarrow q$ it is enough to prove that $p \Rightarrow q$ is false or $q \Rightarrow p$ is false.)

There are some other strategies of proof, like a **constructive proof**, which you will come across in the appendix to Unit 11 and in other mathematics courses. We shall not discuss this method here.

Other proof-related adjectives that you will come across are vacuous and trivial.

A **vacuous proof** makes use of the fact that if p is false, then $p \to q$ is true, regardless of the truth value of q. So, to vacuously prove $p \to q$, all we need to do is to show that p is false. For instance, suppose we want to prove that 'If $n > n + 1$ for $n \in Z$, then $n^2 = 0$'. Since '$n > n + 1$' is false for every $n \in Z$, the given statement is **vacuously true**, or **true by default**.

Similarly, a **trivial proof** of $p \to q$ is one based on the fact that if q is true, then $p \to q$ is true, regardless of the truth value of p. So, for example, 'If $n > n + 1$ for $n \in Z$, then $n + 1 > n$' is trivially true since $n + 1 > n \;\forall\, n \in Z$. The truth value of the hypothesis (which is false in this example) does not come into the picture at all.

Here's a chance for you to think up such proofs now!

E13) Give one example each of a vacuous proof and a trivial proof.

And now let us study a very important technique of proof for statements that are of the form $p(n), n \in N$.

2.4 PRINCIPLE OF INDUCTION

In a discussion with some students the other day, one of them told me very cynically that all Indian politicians are corrupt. I asked him how he had reached such a conclusion. As an argument he gave me instances of several politicians, all of whom were known to be corrupt. What he had done was to formulate his general opinion of politicians on the basis of several particular instances. This is an example of **inductive logic**, a process of reasoning by which general rules are discovered by the observation of several individual cases. Inductive reasoning is used in all the sciences, including mathematics. But in mathematics we use a more precise form.

Precision is required in mathematical induction because, as you know, a statement of the form $(\forall n \in N)p(n)$ is true only if it can be shown to be true for each n in N. (In the example above, even if the student is given an example of one clean politician, he is not likely to change his general opinion.)

How can we make sure that our statement p(n) is true for each n that we are interested in? To answer this, let us consider an example.

Suppose we want to prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for each $n \in N$. Let us call p(n) the predicate '$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$'. Now, we can verify that it is true for a few values, say, $n = 1, n = 5, n = 10, n = 100$, and so on. But we still can't be sure that it will be true for some value of n that we haven't tried.

But now, suppose we can show that if p(n) is true for some $n, n = k$ say, then it will be true for $n = k + 1$. Then we are in a very good position because we already know that p(1) is true. And, since p(1) is true, so is $p(1 + 1)$, i.e., p(2), and so on. In this way we can show that p(n) is true for every $n \in N$. So, our proof boils down to two steps, namely,

i) Checking that p(1) is true;

ii) Proving that whenever p(k) is true, then $p(k + 1)$ is true, where $k \in N$.

This is the principle that we will now state formally, in a more general form.

Principle of Mathematical Induction (PMI): Let p(n) be a predicate involving a natural number n. Suppose the following two conditions hold:

i) p(m) is true for some $m \in N$;

ii) If p(k) is true, then $p(k + 1)$ is true, where $k(\geq m)$ is any natural number.

Then p(n) is true for every $n \geq m$.

Looking at the two conditions in the principle, can you make out why it works? (As a hint, put $m = 1$ in our example above.)

Well, (i) tells us that p(m) is true. Then putting $k = m$ in (ii), we find that $p(m + 1)$ is true. Again, since $p(m+1)$ is true, $p(m + 2)$ is true, and so on.

Going back to the example above, let us complete the second step. We know that p(k) is true, i.e., $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. We want to check if $p(k + 1)$ is true. So let us find

$$\begin{aligned} 1 + 2 + \cdots + (k + 1) &= (1 + 2 + \cdots + k) + (k + 1) \\ &= \frac{k(k+1)}{2} + (k + 1), \text{ since } p(k) \text{ is true} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

So, $p(k + 1)$ is true.

And so, by the principle of mathematical induction, we know that p(n) is true for every $n \in N$.

What does this principle really say? It says that if you can walk a few steps, say m steps, and if at each stage you can walk one more step, then you can walk any distance. It sounds very simple, but you may be surprised to know that the technique in this principle was first used by Europeans only as late as the 16th century by the Venetian F. Maurocylus (1494 - 1575). He used it to show that $1 + 3 + \cdots + (2n - 1) = n^2$. Pierre de Fermat (1601 - 1665) improved on the technique and proved that this principle is equivalent to the following often-used principle of mathematics.

The Well-ordering Principle: Any non-empty subset of **N** contains a smallest element.

You may be able to see the relationship between the two principles if we reword the PMI in the following form.

The term 'mathematical induction' was first used by De Morgan.

Principle of Mathematical Induction (Equivalent form): Let $S \subseteq N$ be such that

i) $m \in S$

ii) For each $k \in N, k \geq m$, the following implication is true: $k \in S \Rightarrow k+1 \in S$.

Then $S = \{m, m+1, m+2, \cdots\}$.

Can you see the equivalence of the two forms of the PMI? If you take $S = \{n \in N \mid p(n) \text{ is true }\}$, then you can see that the way we have written the principle above is a mere rewrite of the earlier form.

Now, let us consider an example of proof using PMI.

Example 9: Use mathematical induction to prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1) \ \forall \ n \in N.$$

Note that p(n) is a predicate, not a statement, unless we know the value of n.

Solution: We call p(n) the predicate

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1).$$

Since we want to prove it for every $n \in N$, we take $m = 1$.

Step 1: p(1) is $1^2 = \frac{1}{6}(1+1)(2+1)$, which is true.

Step 2: Suppose, for an arbitrary $k \in N, p(k)$ is true, i.e.,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k}{6}(k+1)(2k+1) \text{ is true.}$$

Step 3: To check if the assumption in Step 2 implies that $p(k+1)$ is true. Let's see.

$p(k+1)$ is $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k+1}{6}(k+2)(2k+3)$

$$\iff (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 = \frac{k+1}{6}(k+2)(2k+3)$$

$$\iff \frac{k}{6}(k+1)(2k+1) + (k+1)^2 = \frac{k+1}{6}(k+2)(2k+3),$$

since p(k) is true.

$$\iff \frac{k+1}{6}[k(2k+1) + 6(k+1)] = \frac{k+1}{6}(k+2)(2k+3)$$

$\iff 2k^2 + 7k + 6 = (k+2)(2k+3)$, dividing throughout by $\frac{k+1}{6}$, which is true.

So, p(k) is true implies that p(k + 1) is true.

So, both the conditions of the principle of mathematical induction hold. Therefore, its conclusion must hold, i.e., p(n) is true for every $n \in N$.

* * *

Have you gone through Example 9 carefully? If so, you would have noticed that the proof consists of three steps:

Step 1 (called the basis of induction): Checking if p(m) is true for some $m \in N$.

Step 2 (called the **induction hypothesis**): Assuming that p(k) is true for an arbitrary $k \in N, k \geq m$.

Step 3 (called the **induction step**): Showing that p(k + 1) is true, by a direct or an indirect proof.

Now let us consider an example in which $m \neq 1$.

Example 10: Show that $2^n > n^3$ for $n \geq 10$.

Solution: We write p(n) for the predicate '$2^n > n^3$'.

Step 1: For $n = 10, 2^{10} = 1024$, which is greater that 10^3. Therefore, p(10) is true.

Step 2: We assume that p(k) is true for an arbitrary $k \geq 10$. Thus, $2^k > k^3$.

Step 3: Now, we want to prove that $2^{k+1} > (k+1)^3$. Note that

$$\begin{aligned} 2^{k+1} &= 2.2^k > 2.k^3, \text{by our assumption} \\ &> \left(1+\frac{1}{10}\right)^3.k^3, \text{ since } 2 > \left(1+\frac{1}{10}\right)^3 \\ &\geq \left(1+\frac{1}{k}\right)^3.k^3, \text{ since } k \geq 10 \\ &= (k+1)^3. \end{aligned}$$

Thus, p(k + 1) is true if p(k) is true for $k \geq 10$.

Therefore, by the principle of mathematical induction, p(n) is true $\forall\, n \geq 10$.

* * *

Why don't you try to apply the principle now?

E14) Use mathematical induction to prove that
$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}\ \forall\, n \in N$.

E15) Show that for any integer $n > 1$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.
(Hint : The basis of induction is p(2).)

Before going further a note of warning! To prove that p(n) is true $\forall\, n \geq m$, both the basis of induction as well as the induction step must hold. If even one of these conditions does not hold, we cannot arrive at the conclusion that p(n) is true $\forall\, n \geq m$.

For example, suppose p(n) is $(x+y)^n \leq x^n + y^n\ \forall\, x, y \in R$. Then p(1) is true. But Steps 2 and 3 do not hold. Therefore, p(n) is not true for every $n \in N$. (Can you find a value of n for which p(n) is false?)

As another example, take p(n) to be the statement '$1 + 2 + \cdots + n < n$'. Then, if p(k) is true, so is p(k + 1) (prove it!). But the basis step does not hold for any $m \in N$. And, as you can see, p(n) is false.

Now let us look at a situation in which we may expect the principle of induction to work, but it doesn't. Consider the sequence of numbers 1, 1, 2, 3, 5, 8, $\cdots$. These are the **Fibonacci numbers**, named after the Italian mathematician Fibonacci. Each term in the sequence, from the third term on, is obtained by adding the previous 2 terms. So, if a_n is the nth term, then $a_1 = 1, a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}\ \forall\, n \geq 3$. Suppose we want to show that $a_n < 2^n\ \forall\, n \in N$ using the PMI. Then, if p(n)

is the predicate $a_n < 2^n$, we know that p(1) is true.
Now suppose we know that p(k) is true for an arbitrary $k \in N$, i.e., $a_k < 2^k$. We want to show that $a_{k+1} < 2^{k+1}$, i.e., $a_k + a_{k-1} < 2^{k+1}$. But we don't know anything about a_{k-1}. So, how can we apply the principle of induction in the form that we have stated it? In such a situation, a stronger, more powerful, version of the principle of induction comes in handy. Let's see what this is.

Principle of Strong Mathematical Induction: Let p(n) be a predicate that involves a natural number n. Suppose we can show that

i) p(m) is true for some $m \in N$, and

ii) whenever $p(m), p(m+1), \cdots, p(k)$ are true, then $p(k+1)$ is true, where $k \geq m$.

Then we can conclude that p(n) is true for all natural numbers $n \geq m$.

Why do we call this principle stronger than the earlier one? This is because, in the induction step we are making more assumptions, i.e., that **p(n) is true for every n lying between m and k**, not just that p(k) is true.

In using the strong form we often need to check Step 1 for more than one value of n.

Let us now go back to the Fibonacci sequence. To use the strong form of the PMI, we take m = 1. We have seen that p(1) is true. We also need to see if p(2) is true. This is because we have to use the relation $a_n = a_{n-1} + a_{n-2}$, which is valid for $n \geq 3$.

Now that we know that both p(1) and p(2) are true, let us go to the next step. In Step 2, for an arbitrary $k \geq 2$, we assume that p(n) is true for every n such that $1 \leq n \leq k$, i.e., $a_n < 2^n$ for $1 \leq n \leq k$.

Finally, in Step 3, we must show that $p(k+1)$ is true, i.e., $a_{k+1} < 2^{k+1}$. Now

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \\ &< 2^k + 2^{k-1}, \text{ by our assumption in Step 2.} \\ &= 2^{k-1}(2+1) \\ &< 2^{k-1}.2^2 \\ &= 2^{k+1} \end{aligned}$$

$\therefore p(k+1)$ is true.
$\therefore p(n)$ is true $\forall n \in N$.

Though the "strong" form of the PMI appears to be different from the "weak" form, **the two are actually equivalent**. This is because each can be obtained from the other. So, we can use either form of mathematical induction. In a given problem we use the form that is more suitable. For instance, in the following example, as in the case of the one above, you would agree that it is better to use the strong form of the PMI.

Example 11: Use induction to prove that any integer $n \geq 2$ is either a prime or a product of primes.

Solution: Here p(n) is the predicate 'n is a prime or n is a product of primes.'.

Step 1 (**basis of induction**) : Since 2 is a prime, p(2) is true.

Step 2 (**induction hypothesis**): Assume that p(n) is true for any integer n such that $2 \leq n \leq k$, i.e., $p(3), p(4), \cdots, p(k)$ are true.

Step 3 (**induction step**): Now consider $p(k+1)$. If $k+1$ is a prime, then $p(k+1)$ is true. If $k+1$ is not a prime, then $k+1 = rs$, where $2 \leq r \leq k$ and $2 \leq s \leq k$. But, by our induction hypothesis, p(r) is true and p(s) is true. Therefore, r and s are either primes or products of primes. And therefore, $k+1$ is a product of primes. So, $p(k+1)$ is true.

Therefore, p(n) is true $\forall n \geq 2$.

* * *

Why don't you try some exercises now?

E16) If $a_1, a_2, \cdots$ are the terms in the Fibonacci sequence, use the weak as well as the strong forms of the principle of mathematical induction to show that $a_n > \frac{3}{2} \forall n \geq 3$. Which form did you find more convenient?

E17) Consider the following "proof" by induction of the statement 'Any n marbles are of the same size.', and say why it is wrong.

Basis of induction : For n = 1, the statement is clearly true.

Induction hypothesis : Assume that the statement is true for n = k.

Induction step : Now consider any k + 1 marbles $1, 2, \cdots, k+1$. By the induction hypothesis the k marbles $2, 3, \cdots, k+1$ are of the same size. Therefore, all the k + 1 marbles are of the same size.

Therefore, the given statement is true for every n.

E18) Prove that the following result is equivalent to the principle of mathematical induction (strong form):

Let $S \subseteq N$ such that

i) $m \in S$

ii) If $m, m+1, m+2, \cdots, k$ are in S, then $k+1 \in S$.

Then $S = \{n \in N | n \geq m\}$.

E19) To prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1 \; \forall n \in N$, which form of the principle of mathematical induction would you use, and why? Also, prove the inequality.

With this we come to the end of our discussion on various techniques of proving or disproving mathematical statements. Let us take a brief look at what you have read in this unit.

2.5 SUMMARY

In this unit you have studied the following points.

1. What constitutes a proof of a mathematical statement, including 4 commonly used rules of inference, namely,
   i) law of detachment (or modus ponens) : $[(p \rightarrow q) \wedge p] \Rightarrow q$
   ii) law of contraposition (or modus tollens) : $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$
   iii) disjunctive syllogism : $[(p \vee q) \wedge \sim p] \Rightarrow q$
   iv) hypothetical syllogism : $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$
2. The description and examples of a direct proof, which is based on modus ponens.
3. Two types of indirect proofs : proof by contrapositive and proof by contradiction.
4. The use of counterexamples for disproving a statement.
5. The "strong" and "weak" forms of the principle of mathematical induction, and their equivalence with the well-ordering principle.

2.6 SOLUTIONS/ANSWERS

E1) For example,

Theorem: $(x+y)^2 = x^2 + 2xy + y^2$ for $x, y \in \mathbb{R}$.

Proof: For $x, y \in \mathbb{R}$, $(x+y)^2 = (x+y)(x+y)$ (by definition of 'square') $(x+y)(x+y) = x(x+y) + y(x+y)$ (by distributivity, which has been proved earlier)

$x(x+y) + y(x+y) = x^2 + 2xy + y^2$ (again by distributivity, and by definition of addition and multiplication of algebraic terms).

Therefore, $(x+y)^2 = x^2 + 2xy + y^2$ (using an earlier proved statement that $a = b$ and $b = c$ implies that $a = c$).

E2) No, not unless it has been proved to be true.

E3)

p	q	r	$\sim r$	$q \vee \sim r$	$p \to q \vee \sim r$ (premises)	$q \to p$ (premises)	$p \to r$ (conclusion)
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	F
T	F	T	F	F	F	T	T
T	F	F	T	T	T	T	F
F	T	T	F	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	F	F	T	T	T
F	F	F	T	T	T	T	T

The premises are true in Rows 1, 2, 4, 7, 8. So, the argument will be valid if the conclusion is also true in these rows. But this does not happen in Row 2, for instance. Therefore, the argument is invalid.

E4) i) Let p : The eraser is white,
q : Oxygen is a metal.
Then the argument is

$p \vee q$

$\sim p$

$\therefore q$

Its truth table is given alongside.

p	q (conclusion)	$\sim p$ (premises)	$p \vee q$ (premises)
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

All the premises are true only in the third row. Since the conclusion in this row is also true, the argument is valid.

ii) The argument is $(p \to q) \wedge (q \to r) \Longrightarrow (p \to r)$
where p: Madhu is a 'sarpanch',
q : Madhu heads the 'Panchayat',
r : Madhu decides on property disputes.
This is valid because, whenever both the premises are true, so is the conclusion (see the following table.)

p	q	r	$p \to q$ (premise)	$q \to r$ (premise)	$p \to r$ (conclusion)
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

iii) The argument is
$[(p \vee q) \wedge (q \to r) \wedge \sim r] \Longrightarrow q$
where p: Munna will cook.
q: Munni will practise Karate.
r: Munna studies.
This is not valid, as you can see from Row 4 of the following truth table.

p	q (conclusion)	r	$\sim r$ (premise)	$p \vee q$ (premise)	$q \to r$ (premise)
T	T	T	F	T	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	T	T	F
F	F	T	F	F	T
F	F	F	T	F	T

E6) We need to prove $p \Longrightarrow q$, where
p: $x \in R$ such that $x^2 = 9$, and
q: $x = 3$ or $x = -3$.
Now, $x^2 = 9 \Longrightarrow \sqrt{x^2} = \pm\sqrt{9} \Longrightarrow x = \pm 3$.
Therefore, p is true and $(p \Longrightarrow q)$ is true, allows us to conclude that q is true.

E7) If f is not surjective, then f is not a 1-1 function from X into itself.

E8) We want to prove $\sim q \Longrightarrow \sim p$, where
p: $x \in Z$ such that x^2 is even,
q: x is even.
Now, we start by assuming that q is false, i.e., x is odd.
Then $x = 2m + 1$ for some $m \in Z$.
Therefore, $x^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$
Therefore, x^2 is odd, i.e. p is false.

Thus, $\sim q \Longrightarrow \sim p$, and hence, $p \Longrightarrow q$.

E9) i) This is on the lines of Example 5.

ii) Let us assume that $x^3 + 4x = 0$ and $x \neq 0$. Then $x(x^2+4) = 0$ and $x \neq 0$. Therefore, $x^2 + 4 = 0$, i.e., $x^2 = -4$. But $x \in R$ and $x^2 = -4$ is a contradiction. Therefore, our assumption is false. Therefore, the given statement is true.

E10) **Direct proof**: $x^3 + 4x = 0 \Longrightarrow x(x^2+4) = 0$
$\Longrightarrow x = 0$ or $x^2 + 4 = 0$
$\Longrightarrow x = 0$, since $x^2 \neq -4 \; \forall \, x \in R$.

Proof by contrapositive: Suppose $x \neq 0$. Then $x(x^2+4) \neq 0$ for any $x \in R$.
$\therefore x^3 + 4x \neq 0$ for every $x \in R$.
So we have proved that 'For $x \in R, x \neq 0 \Longrightarrow x^3 + 4x \neq 0$.'.
That is, 'For $x \in R, x^3 + 4x = 0 \Longrightarrow x = 0$.'.

E11) Suppose C tells the truth. Therefore, D always tells the truth. Therefore, C always lies, which is a contradiction. Therefore, C can't be a truth-teller, i.e., C is a liar. Therefore, D is a truth-teller.

E12) i) What about $x = 0$, or $x = -1$, or $\cdots$?

ii) Take $n = 2, x = 1$ and $y = -1$, for instance.

iii) Here we can find an example f such that f is 1-1 but not onto, or such that f is onto but not 1-1.
Consider $f : N \to N : f(x) = x + 10$. Show that this is 1-1, but not surjective.

E13) i) **Theorem**: The area of every equilateral triangle of side a and perimeter 2a is divisible by 3.
Proof: Since there is no equilateral triangle that satisfies the hypothesis, the proposition is vacuously true.

ii) **Theorem**: If a natural number c is divisible by 5, then the perimeter of the equilateral triangle of side c is 3c.
Proof: Since the conclusion is always true, the proposition is trivially true.

E14) Let p(n) be the given predicate.

Step 1: $p(1) : 1 \leq 2 - 1$, which is true.

Step 2: Assume that p(k) is true for some $k \geq 1$, i.e., assume that
$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.$$

Step 3: To show that p(k + 1) is true, consider
$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} = \left(1 + \frac{1}{4} + \cdots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2}$$
$$\leq \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2}, \text{ by Step 2.}$$

Now, $\left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

iff $\frac{1}{(k+1)^2} \leq \frac{1}{k} - \frac{1}{k+1}$

iff $k \leq k + 1$, which is true.

Therefore, $\left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$

Therefore, p(k + 1) is true.

Thus, by the PMI, p(n) is true $\forall \, n \in N$.

E15) p(2) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$, which is true.

Now, assume that p(k) is true for some $k \geq 2$. Then

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}, \text{ since p(k) is true.}$$
$$= \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$
$$> \sqrt{k+1}, \text{ since } \sqrt{k+1} > \sqrt{k}.$$

Hence p(k + 1) is true.

$\therefore$ p(n) is true $\forall\, n \geq 2$.

E16) We shall apply the strong form of the PMI here.

Let p(n) : $a_n > \frac{3}{2}$.

Step 1: p(3) and p(4) are true.

Step 2: Assume now that for $k \in \mathbb{N}, k \geq 3$, p(n) is true for every n such that $3 \leq n \leq k$.

Step 3: We want to show that p(k + 1) is true. Now

$$a_{k+1} = a_k + a_{k-1} > \frac{3}{2} + \frac{3}{2}, \text{ by Step 2}$$
$$> \frac{3}{2}.$$

$\therefore$ p(k + 1) is true.

Thus, p(n) is true $\forall\, n \geq 3$.

In this case, you will be able to use the weak form conveniently too since $a_k > \frac{3}{2}$ is enough for showing that p(k + 1) is true.

Thus, **in this case the weak form is more appropriate** since fewer assumptions give you the same result.

E17) The problem is at the induction step. The first marble may be a different size from the other k marbles. So, we have not shown that p(k + 1) is true whenever p(k) is true.

E18) With reference to the statement of the strong form of the PMI, let $S = \{n \in \mathbb{N} | p(n) \text{ is true}\}$.

Then you can show how the form in this problem is the same as the statement of the strong form of the PMI.

E19) Let p(n) : $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1$.

The weak form suffices here, since the assumption that p(k) is true is enough to prove that p(k + 1) is true. We don't need to assume that p(1), p(2), $\cdots$, p(k − 1) are also true to show that p(k + 1) is true. Let's prove that p(n) is true $\forall\, n \in \mathbb{N}$.

Now, p(1) : $1 \leq 2 - 1$, which is true.

Next, assume that p(k) is true for some $k \in \mathbb{N}$.

Then $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq (2\sqrt{k} - 1) + \frac{1}{\sqrt{k+1}}$, since p(k) is true.

Now $2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$

$\iff 2(\sqrt{k+1} - \sqrt{k}) \geq \frac{1}{\sqrt{k+1}}$

$\iff 2(k + 1 - \sqrt{k(k+1)}) \geq 1$

$\iff 1 \geq 0$, which is true.
$\therefore$ p(k + 1) is true.
$\therefore$ p(n) is true $\forall$ n $\in$ N.

UNIT 3 BOOLEAN ALGEBRA AND CIRCUITS

Structure

Page No.

3.1 INTRODUCTION

In the previous two units you have read about the elementary aspects of symbolic logic. C.E.Shannon, the founder of information theory, observed an analogy between the functioning of switching circuits and certain operations of logical connectives. In 1938 he gave a technique based on this analogy to **express and manipulate** simple switching circuits algebraically. Later, the discovery of some new solid state devices (called **electronic switches** or **logic gates**) helped to modify these algebraic techniques and, thereby, paved a way to solve numerous problems related to digital systems algebraically.

Fig. 1: Claude Shannon, who made the first major contribution in applied Boolean algebra in 1938.

In this unit, we shall discuss the symbolic logic techniques which are required for the algebraic understanding of circuits and computer logic. In Sec.3.2, we shall introduce you to **Boolean algebras** with the help of certain examples based on objects you are already familiar with. You will see that such algebras are apt for describing operations of logical circuits used in computers.

In Sec.3.3, we have discussed **Boolean expressions**. In Sec.3.4, we look at the linkages that they have with logic circuits.

In Sec.3.5, you will read about how to express the overall functioning of a circuit mathematically in terms of certain suitably defined functions called **Boolean functions**. In this section we shall also consider a simple circuit **design problem** to illustrate the applications of the relationship between Boolean function and circuits.

Let us now consider the objectives of this unit.

Objectives

After reading this unit you should be able to

- define and give examples of Boolean algebras, expressions and functions;
- obtain the disjunctive normal form (DNF) and the conjunctive normal form (CNF) of a Boolean expression;

- give mathematical interpretations of the functioning of logic gates;
- obtain and simplify the Boolean expression representing a circuit;
- construct a circuit for a Boolean expression;
- design and simplify some simple circuits using Boolean algebra techniques.

3.2 BOOLEAN ALGEBRAS

How do you react to the questions: Is it possible to design an electric/electronic circuit without actually using switches (or logic gates) and wires? Can a circuit be redesigned, without defeating its purpose, to get a simpler circuit with the help of pen and paper only?

Relax! The answer to both these questions is 'Yes'. What allows us to give this reply is the concept of **Boolean algebras**. Before we start a formal discussion on these type of algebras, let us take another look at the objects treated in Unit 1.

As before, let the letters p, q, r,... denote statements (or propositions). We write $\mathcal{S}$ for the set of all propositions. As you may recall, a tautology $\mathcal{T}$ (or a contradiction $\mathcal{F}$) is any proposition which is always true (or always false, respectively). By abuse of notation, we shall let $\mathcal{T}$ **denote the set of all tautologies and** $\mathcal{F}$ **denote the set of all contradictions.** Thus, $\mathcal{T} \subseteq \mathcal{S}, \mathcal{F} \subseteq \mathcal{S}$.

A binary operation on a non-empty set X is a function $f : X \times X \to X$.

You already know from Unit 1 that, given two propositions p and q, both $p \wedge q$ and $p \vee q$ are again propositions. And so, by the definition of a binary operation, you can see that both $\wedge$ (**conjunction**) and $\vee$ (**disjunction**) are binary operations on the set $\mathcal{S}$, where we are writing $\wedge(p, q)$ as $p \wedge q$ and $\vee(p, q)$ as $p \vee q$, $\forall\, p, q \in \mathcal{S}$.

Again, since $\sim p$ is also a proposition, the operation $\sim$ (**negation**) defines a unary function $\sim: \mathcal{S} \to \mathcal{S}$. Thus, the set of propositions $\mathcal{S}$, with these operations, acquires an algebraic structure.

As is clear from Sec.1.3 of Unit 1, under these three operations, the elements of $\mathcal{S}$ satisfy **associative** laws, **commutative** laws, **distributive** laws and **complementation** laws.
Also, by E19 of Unit 1, you know that $p \vee \mathcal{F} = p$ and $p \wedge \mathcal{T} = p$, for any proposition p. These are called the **identity** laws.

The set $\mathcal{S}$ with the three operations and properties listed above is a particular case of an algebraic structure which we shall now define.

Definition: A **Boolean algebra** B is an algebraic structure which consists of a set X ($\neq \phi$) having two binary operations (denoted by $\vee$ and $\wedge$), one unary operation (denoted by $'$) and two specially defined elements O and I (say), which satisfy the following five laws for all x, y, z $\in$ X.

B1. **Associative Laws:** $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

B2. **Commutative Laws:** $x \vee y = y \vee x$, $x \wedge y = y \wedge x$

B3. **Distributive Laws:** $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

B4. Identity Laws: $x \vee O = x$, $x \wedge I = x$

B5. Complementation Laws: $x \wedge x' = O$, $x \vee x' = I$.

We write this algebraic structure as $B = (X, \vee, \wedge, ', O, I)$, or simply B, if the context makes the meaning of the other terms clear. The two operations $\vee$ and $\wedge$ are called the **join operation** and **meet operation**, respectively. The unary operation $'$ is called the **complementation**.

From our discussion preceding the definition above, you would agree that the set $\mathcal{S}$ of propositions is a Boolean algebra, where $\mathcal{T}$ and $\mathcal{F}$ will do the job of I and O, respectively. Thus, $(\mathcal{S}, \wedge, \vee, \sim, \mathcal{F}, \mathcal{T})$ is an example of a Boolean algebra.

We give another example of a Boolean algebra below.

Example 1: Let X be a non-empty set, and $\mathcal{P}(X)$ denote its power set, i.e., $\mathcal{P}(X)$ is the set consisting of all the subsets of the set X. Show that $\mathcal{P}(X)$ is a Boolean algebra.

Solution: We take the usual set-theoretic operations of intersection ($\cap$), union ($\cup$), and complementation (^{c}) in $\mathcal{P}(X)$ as the three required operations. Let ϕ and X play the roles of O and I, respectively. Then, from MTE-04 you can verify that all the conditions for $(\mathcal{P}(X), \cup, \cap, {}^c, \phi, X)$ to be a Boolean algebra hold good.

For instance, the identity laws (B4) follow from two set-theoretic facts, namely, 'the intersection of any subset with the whole set is the set itself' and 'the union of any set with the empty set is the set itself'. On the other hand, the complementation laws (B5) follow from another set of facts from set theory, namely, 'the intersection of any subset with its complement is the empty set' and 'the union of any set with its complement is the whole set'.

* * *

Yet another example of a Boolean algebra is based on **switching circuits**. For this, we first need to elaborate on the functioning of ordinary switches in a mathematical way. In fact, we will present the basic idea which helped the American, C.E.Shannon, to detect the connection between the functioning of switches and Boole's symbolic logic.

You may be aware of the functioning of a simple on-off switch which is commonly used as an essential component in the electric (or electronic) networking systems. A switch is a device which allows the current to flow only when it is placed in the ON position, i.e., when the gap is closed by a conducting rod. Thus, the ON position of a switch is one state of a switch, called a **closed state**. The other state of a switch is the **open state**, when it is placed in the OFF position. So, a switch has two stable states.

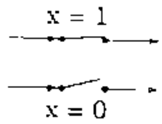


Fig.2: OFF-ON position

There is another way to talk about the functioning of a switch. We can denote a switch by x, and use the values 0 and 1 to depict its two states, i.e., to convey that x is **open** we write $x = 0$, and to convey that x is closed we write $x = 1$ (see Fig.2).

These values which denote the state of a switch x are called the **state-values** (s.v., in short) of that switch.

We shall also write x' for a switch which is always in a state opposite to x. So that

$$x \text{ is open} \Rightarrow x' \text{ is closed} \quad \text{and} \quad x \text{ is closed} \Rightarrow x' \text{ is open.}$$

The switch x′ is called the **invert** of the switch x. For example, the switch a′ shown in Fig.3 is an invert of the switch a.

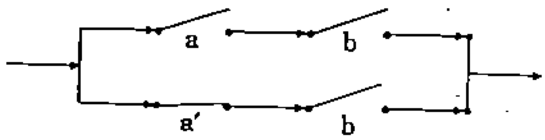


Fig. 3: a′ is the invert of a.

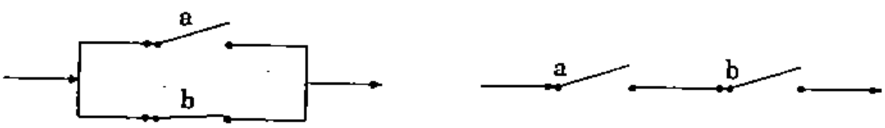
Table 1: s.v. of x′.

x	x′
0	1
1	0

Table 1 alongside gives the state value of x′ for a given state value of the switch x. These values are derived from the definition of x′ and our preceding discussion.

Note that the variable x that denotes a switch can only take on 2 values, 0 and 1. Such a variable (which can only take on two values) is called a **Boolean variable**. Thus, if x is a Boolean variable, so is x′.

Now, in order to design a circuit consisting of several switches, there are two ways in which two switches can be connected: **parallel connections** and **series connections** (see Fig.4).



(i) Parallel connection

(ii) Series connection

Fig. 4: Two ways of connecting switches.

From Fig.4(i) above, you can see that in case of a parallel connection of switches a and b (say), current will flow from the left to the right extreme if **at least one** of the two switches is closed. Note that 'parallel' **does not** mean that both the switches are in the same state.

On the other hand, current can flow in a series connection of switches only when **both** the switches a and b are closed (see Fig.4(ii)).

Given two switches a and b, we write **a par b** and **a ser b** for these two types of connections, respectively.

In view of these definitions and the preceding discussion, you can see that the state values of the connections a par b and a ser b, for different pairs of state values of switches a and b, are as given in the tables below.

Table 2: State values of a par b and a ser b.

s.v. of a	s.v. of b	s.v. of a par b
0	0	0
0	1	1
1	0	1
1	1	1

s.v. of a	s.v. of b	s.v. of a ser b
0	0	0
0	1	0
1	0	0
1	1	1

We have now developed a sufficient background to give you the example of a Boolean algebra which is based on switching circuits.

Example 2: The set $S = \{0, 1\}$ is a Boolean algebra.

Solution: Take ser and par in place of $\wedge$ and $\vee$, respectively, and inversion(′) instead of $\sim$ as the three required operations in the definition of a Boolean algebra,. Also take 0 for the element O and 1 for the element I in

this definition.
Now, using Tables 1 and 2, you can check that the five laws B1-B5 hold good. Thus, (S, par, ser, ′, 0, 1) is a Boolean algebra.

* * *

A Boolean algebra whose underlying set has only two elements is very important in the study of circuits. We call such an algebra a **two-element Boolean algebra**. Throughout the unit, we denote this algebra by $\mathcal{B}$. From this Boolean algebra we can build many more, as in the following example.

Example 3: Let $\mathcal{B}^n = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} = \{(e_1, e_2, \ldots, e_n) \mid \text{each } e_i = 0 \text{ or } 1\}$, for $n \geq 1$, be the Cartesian product of n copies of $\mathcal{B}$. For $i_k, j_k \in \{0, 1\}$ $(1 \leq k \leq n)$, define

$$(i_1, i_2, \ldots, i_n) \bigwedge (j_1, j_2, \ldots, j_n) = (i_1 \wedge j_1, i_2 \wedge j_2, \ldots, i_n \wedge j_n),$$
$$(i_1, i_2, \ldots, i_n) \bigvee (j_1, j_2, \ldots, j_n) = (i_1 \vee j_1, i_2 \vee j_2, \ldots, i_n \vee j_n), \quad \text{and}$$
$$(i_1, i_2, \ldots, i_n)' = (i'_1, i'_2, \ldots, i'_n).$$

Then $\mathcal{B}^n$ is a Boolean algebra, for all $n \geq 1$.

Solution: Firstly observe that the case $n = 1$ is the Boolean algebra $\mathcal{B}$.

Now, let us write $0 = (0, 0, \ldots, 0)$ and $I = (1, 1, \ldots, 1)$, for the two elements of $\mathcal{B}^n$ consisting of n-tuples of 0's and 1's, respectively. Using the fact that $\mathcal{B}$ is a Boolean algebra, you can check that $\mathcal{B}^n$, with operations as defined above, is a Boolean algebra for every $n \geq 1$.

* * *

The Boolean algebras $\mathcal{B}^n$, $n \geq 1$, (called **switching algebras**) are very useful for the study of the hardware and software of digital computers.

We shall now state, without proof, some other properties of Boolean algebras, which can be deduced from the five laws (B1-B5).

Theorem 1: Let $B = (S, \vee, \wedge, ', O, I)$ be a Boolean algebra. Then the following laws hold $\forall\, x, y \in S$.

a) **Idempotent laws :** $x \vee x = x$, $x \wedge x = x$.
b) **Absorption laws :** $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$.
c) **Involution law :** $(x')' = x$.
d) **De Morgan's laws :** $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$.

In fact, you have already come across some of these properties for the Boolean algebra S of propositions in Unit 1. In the following exercise we ask you to verify them.

E1) a) Verify the identity laws and absorption laws for the Boolean algebra $(\mathcal{S}, \wedge, \vee, \sim, \mathcal{T}, \mathcal{F})$ of propositions.

b) Verify the absorption laws for the Boolean algebra $(\mathcal{P}(X), \cup, \cap, {}^c, \phi, X)$.

In Theorem 1, you may have noticed that for each statement involving $\vee$ and $\wedge$, there is an analogous statement with $\wedge$ (instead of $\vee$) and $\vee$ (instead of $\wedge$). This is not a coincidence, as the following definition and result shows.

Definition : If p is a proposition involving $\sim$, $\wedge$ and $\vee$, the **dual of** p, denoted by p^d, is the proposition obtained by replacing each occurrence of $\wedge$ (and/or $\vee$) in p by $\vee$ (and/or $\wedge$, respectively) in p^d.
For example, $x \vee (x \wedge y) = x$ is the dual of $x \wedge (x \vee y) = x$.

Now, the following principle tells us that if a statement is proved **true**, then we have simultaneously proved that its dual is true.

Theorem 2 (The principle of duality): If s is a theorem about a Boolean algebra, then so is its dual s^d.

It is because of this principle that the statements in Theorem 1 look so similar.

Let us now see how to apply Boolean algebra methods to circuit design. For this purpose we shall introduce the necessary mathematical terminology and ideas in the following section.

3.3 BOOLEAN EXPRESSIONS

In Unit 2, you learnt how a compound statement can be formed by combining some propositions $p_1, p_2, \dots, p_n$ (say) with the help of logical connectives $\wedge, \vee$ and $\sim$.

Analogously, while expressing circuits mathematically, we identify each circuit in terms of some Boolean variables. Each of these variables represents either a simple switch or an input to some electronic switch.

'Recursive' means defining elements of a set in terms of previousely defined elements of the set.

Definition : Let $B = (X, \vee, \wedge, ', O, I)$ be a Boolean algebra. A **Boolean expression** in variables $x_1, x_2, \dots, x_k$ (say), each taking their values in the set X, is defined recursively as follows:

i) Each of the variables $x_1, x_2, \dots, x_k$, as well as the elements **O** and **I** of the Boolean algebra B are Boolean expressions.

ii) If X_1 and X_2 are previously defined Boolean expressions, then $X_1 \wedge X_2, X_1 \vee X_2$, and X_1' are also Boolean expressions.

For instance, $x_1 \wedge x_3'$ is a Boolean expression because so are x_1 and x_3'. Similarly, because $x_1 \vee x_2$ is a Boolean expression, so is $(x_1 \vee x_2) \wedge (x_1 \wedge x_3')$.

If X is a Boolean expression in n variables $x_1, x_2, \dots, x_n$ (say), we write this as $X = X(x_1, \dots, x_n)$.

Each variable x_i and its complement $x_i', 1 \le i \le k$, is called a **literal**. For example, in the Boolean expression

$$X(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (x_1 \wedge x_3'),$$

there are three literals, namely, x_1, x_2, and x_3'.

In the context of designing a circuit or redesigning a circuit with fewer electronic switches, we need to consider techniques for **minimising** Boolean expressions. In the process, we shall be using the concepts defined below.

Definition : A Boolean expression in k variables $x_1, x_2, \dots, x_k$ is called

i) a **minterm** if it is of the form $y_1 \wedge y_2 \wedge \dots \wedge y_k$;

ii) a **maxterm** if it is of the form $y_1 \vee y_2 \vee \dots \vee y_k$;

where each y_j is a literal (i.e. it is either an x_i or an x_i'), for $1 \le i \le k$, and $y_i \ne y_j$ for $i \ne j$.

Thus, a minterm (or a maxterm) in k variables is a meet (or a join, respectively) of exactly k distinct variables. For example, $x_1 \wedge x_2'$ (and $x_1' \vee x_2$) is a minterm (a maxterm, respectively) in the two variables x_1 and x_2.

Definition : A Boolean expression involving k variables is in **disjunctive normal form** (**DNF**, in short) if it is a join of distinct minterms, each one involving exactly k variables.

For instance, the Boolean expression in 2 variables

$$X(x_1, x_2) = (x_1' \wedge x_2') \vee (x_1 \wedge x_2') \vee (x_1' \wedge x_2)$$

is in DNF because it is a join of three minterms, namely, $x_1' \wedge x_2'$, $x_1 \wedge x_2'$, and $x_1' \wedge x_2$, where each one of these involves exactly two variables.

Observe that each minterm in a DNF should involve all the k variables in the expression $X(x_1, x_2, \ldots, x_k), k \geq 2$. For instance, the Boolean expression

$$X(x_1, x_2, x_3) = (x_1' \wedge x_2) \vee (x_1 \wedge x_2' \wedge x_3)$$

is not in DNF because $x_1' \wedge x_2$ is not a minterm of all the three variables. However, since we can write

$$\begin{aligned}(x_1' \wedge x_2) &= (x_1' \wedge x_2) \wedge I && \text{(by Identity law)}\\ &= (x_1' \wedge x_2) \wedge (x_3 \vee x_3') && \text{(by Complementation law)}\\ &= (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3'), && \text{(by Distributive law)}\end{aligned}$$

and so, the expression $X(x_1, x_2, x_3)$ with this change for $x_1' \wedge x_2$ is in the disjunctive normal form.

Indeed, using similar techniques, any Boolean expression ($\neq O$) can be written in disjunctive normal form. Let us work out an example to illustrate this technique.

Example 4: Obtain a disjunctive normal form for the expression

$$X(x_1, x_2, x_3) = (x_1' \wedge x_2) \vee (x_1 \wedge x_3).$$

Solution: We can write

$$\begin{aligned}x_1' \wedge x_2 &= (x_1' \wedge x_2) \wedge I && \text{(Identity law)}\\ &= (x_1' \wedge x_2) \wedge (x_3 \vee x_3') && \text{(Complementation law)}\\ &= (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3') && \text{(Distributive law)}\end{aligned}$$

Also,

$$\begin{aligned}x_1 \wedge x_3 &= (x_1 \wedge x_3) \wedge I && \text{(by Identity law)}\\ &= (x_1 \wedge x_3) \wedge (x_2 \vee x_2') && \text{(by Complementation law)}\\ &= (x_1 \wedge x_3 \wedge x_2) \vee (x_1 \wedge x_3 \wedge x_2') && \text{(by Distributive law)}\\ &= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3). && \text{(by commutativity of } \wedge \text{)}\end{aligned}$$

Hence the required disjunctive normal form of the given expression $X(x_1, x_2, x_3)$ in three variables is given by

$$(x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3).$$

* * *

Why don't you try an exercise now?

E2) Obtain the disjunctive normal form of the Boolean expression
$$X(x_1, x_2, x_3) = (x_1 \vee x_2)' \vee (x_1' \wedge x_3).$$

The conjunctive normal form is another important type of expression which is analogous to the concept of DNF.

Definition : A Boolean expression in k variables is said to be in **conjunctive normal form** (CNF, in short) if it is a meet of maxterms, each of which involves all the k variables.

For instance, the Boolean expression

$$X(x_1, x_2, x_3) = (x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2 \vee x_3'),$$

is in CNF because it is the meet of maxterms $(x_1' \vee x_2 \vee x_3)$, $(x_1' \vee x_2' \vee x_3)$ and $(x_1' \vee x_2 \vee x_3')$. Note that all 3 variables are involved in each maxterm.

Let us consider an example of how to obtain the CNF of a Boolean expression.

Example 5: Obtain the CNF of the Boolean expression

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2)' \wedge (x_1' \wedge x_3)'$$

Solution: We have

$$\begin{aligned}
(x_1 \wedge x_2)' &= x_1' \vee x_2' && \text{(De Morgan's Law)}\\
&= (x_1' \vee x_2') \vee O && \text{(Identity law)}\\
&= (x_1' \vee x_2') \vee (x_3 \wedge x_3') && \text{(Complementation law)}\\
&= (x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3') && \text{(Distributive law)}
\end{aligned}$$

Similarly, you can check that

$$(x_1' \wedge x_3)' = (x_1 \vee x_2' \vee x_3') \wedge (x_1 \vee x_2 \vee x_3').$$

Thus, the required CNF of the expression $X(x_1, x_2, x_3)$ given here is

$$(x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3') \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_1 \vee x_2 \vee x_3').$$

* * *

Try the following exercise now.

E3) Obtain the CNF of the Boolean expression

$$X(x_1, x_2, x_3) = ((x_1 \wedge x_2') \vee (x_1' \wedge x_3'))'.$$

As we have said earlier, in the context of simplifying circuits, we need to reduce Boolean expressions to simpler ones. 'Simple' means that the expression has fewer connectives, and all the literals involved are distinct. We illustrate this technique now.

Example 6: Reduce the following Boolean expressions to a simpler form.

(a) $X(x_1, x_2) = (x_1 \wedge x_2) \wedge (x_1 \wedge x_2')$;

(b) $X(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_3)$.

Solution: (a) Here we can write

$$\begin{aligned}
(x_1 \wedge x_2) \wedge (x_1 \wedge x_2') &= ((x_1 \wedge x_2) \wedge x_1) \wedge x_2' && \text{(Associative law)}\\
&= (x_1 \wedge x_2) \wedge x_2' && \text{(Absorption law)}\\
&= x_1 \wedge (x_2 \wedge x_2') && \text{(Associative law)}\\
&= x_1 \wedge O && \text{(Complementation law)}\\
&= O. && \text{(Identity law)}
\end{aligned}$$

Thus, in its simplified form, the expression given in (a) above is O, i.e., a null expression.

(b) We can write

$$\begin{aligned}
&(x_1 \wedge x_2) \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_3)\\
&= [x_1 \wedge \{x_2 \vee (x_2' \wedge x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Distributive law)}\\
&= [x_1 \wedge \{(x_2 \vee x_2') \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Distributive law)}\\
&= [x_1 \wedge \{I \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Complementation law)}\\
&= [x_1 \wedge (x_2 \vee x_3)] \wedge (x_1 \wedge x_3) && \text{(Identity law)}\\
&= [(x_1 \wedge x_2) \vee (x_1 \wedge x_3)] \wedge (x_1 \wedge x_3) && \text{(Distributive law)}\\
&= [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee [(x_1 \wedge x_3) \wedge (x_1 \wedge x_3)] && \text{(Distributive law)}\\
&= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3) && \text{(Idemp.,\& assoc. laws)}\\
&= x_1 \wedge [(x_2 \wedge x_3) \vee x_3] && \text{(Distributive law)}\\
&= x_1 \wedge x_3 && \text{(Absorption law)}
\end{aligned}$$

Thus, the simplified form of the expression given in (b) is $(x_1 \wedge x_3)$.

* * *

Now you should find it easy to solve the following exercise.

E4) Simplify the Boolean expression

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3) \vee (x_2 \wedge x_3).$$

With this we conclude this section. In the next section we shall give an important application of the concepts discussed here.

3.4 LOGIC CIRCUITS

If you look around, you would notice many electric or electronic appliances of daily use. Some of them need a simple switching circuit to control the auto-stop (such as in a stereo system). Some would use an auto-power off system used in transformers to control voltage fluctuations. Each circuit is usually a combination of on-off switches, wired together in some specific configuration. Nowadays certain types of **electronic blocks** (i.e., solid state devices such as transistors, resistors and capacitors) are more in use. We call these electronic blocks **logic gates**, or simply, **gates**. In Fig. 5 we have shown a box which consists of some electronic switches (or logic gates), wired together in a specific manner. Each line which is entering the box from the left represents an independent power source (called **input**), where all of them need not supply voltage to the box at a given moment. A single line coming out of the box gives the final **output** of the circuit box. The output depends on the type of input.

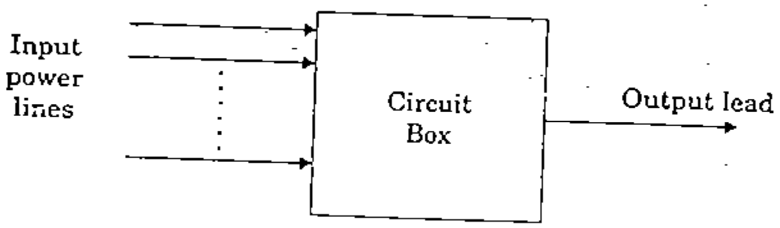


Fig. 5 : A logic circuit.

This sort of arrangement of **input power lines**, a **circuit box** and **output lead** is basic to all electronic circuits. Throughout the unit, any such interconnected assemblage of logic gates is referred to as a **logic circuit**.

As you may know, computer hardwares are designed to handle only two levels of voltage, both as inputs as well as outputs. These two levels, denoted by 0 and 1, are called **bits** (an acronym for binary digits). When the bits are applied to the logic gates by means of one or two wires (input leads), the output is again in the form of voltages 0 and 1. Roughly speaking, you may think of a gate to be on or off according to whether the output voltage is at level 1 or 0, respectively.

Three basic types of logic gates are an AND-gate, an OR-gate and a NOT-gate. We shall now define them one by one.

Table 3: Output of AND-gate.

x_1	x_2	$x_1 \wedge x_2$
0	0	0
1	0	0
0	1	0
1	1	1

Definition : Let the Boolean variables x_1 and x_2 represent any two bits. An **AND-gate** receives inputs x_1 and x_2 and produces the output, denoted by $x_1 \wedge x_2$, as given in Table 3 alongside. The standard pictorial representation of an AND-gate is shown in Fig.6 . From the first three rows of Table 3, you can see that whenever the voltage in any one of the input wires of the AND-gate is at level 0, then the output voltage of the gate is also at level 0.

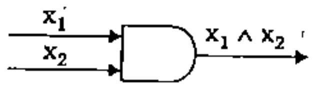


Fig. 6: Diagrammatic representation of an AND-gate

You have already encountered such a situation in Unit 1. In the following exercise we ask you to draw an analogy between the two situations.

E5) Compare Table 3 with Table 2 of Unit 1. How would you relate $x_1 \wedge x_2$ with $p \wedge q$, where p and q denote propositions?
(**Hint:** Take T for 1 and F for 0 in Table 3 above.)

Table 4: Output of OR-gate.

x_1	x_2	$x_1 \vee x_2$
0	0	0
0	1	1
1	0	1
1	1	1

Let us now consider another elementary logic gate.

Definition : An **OR-gate** receives inputs x_1 and x_2 and produces the output, denoted by $x_1 \vee x_2$, as given in Table 4. The standard pictorial representation used for the OR-gate is as shown in Fig.7.

Fig. 7: Diagrammatic representation of an OR-gate

From Table 4 you can see that the situation is the other way around from that in Table 3, i.e., the output voltage of an OR-gate is at level 1 whenever the level of voltage in even one of the input wires is 1. What is the analogous situation in the context of propositions? The following exercise is about this.

E6) Compare Table 4 with Table 1 of Unit 1. How would you relate $x_1 \vee x_2$ with $p \vee q$, where p and q are propositions?

Table 5: Output of NOT-gate

x	x
0	1
1	0

And now we will discuss an electronic realisation of the invert of a simple switch about which you read in Sec.3.2.

Definition : A NOT-gate receives bit x as input, and produces an output denoted by x', as given in Table 5. The standard pictorial representation of a NOT-gate is shown in Fig.8 below.

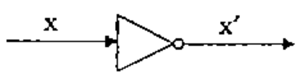


Fig. 8: Diagrammatic representation of a NOT-gate

If you have solved E5 and E6, you would have noticed that Tables 3 and 4 are the same as the truth tables for the logic connectives $\wedge$ (conjunction) and $\vee$ (disjunction). Also Table 3 of Unit 1, after replacing T by 1 and F by 0, gives Table 5. This is why the output tables for the three elementary gates are called logic tables. You may find it useful to remember these logic tables because they are needed very often for computing the logic tables of logic circuits.

Another important fact that these logic tables will help you prove is given in the following exercise.

E7) Let $B = \{0, 1\}$ consist of the bits 0 and 1. Show that B is a Boolean algebra, i.e., that the bits 0 and 1 form a two-element Boolean algebra.

As said before, a logic circuit can be designed using elementary gates, where the output from an AND-gate, or an OR-gate, or a NOT-gate is used as an input to other such gates in the circuitry. The different levels of voltage in these circuits, starting from the input lines, move only in the direction of the arrows as shown in all the figures given below. For instance, one combination of the three elementary gates is shown in Fig.9.

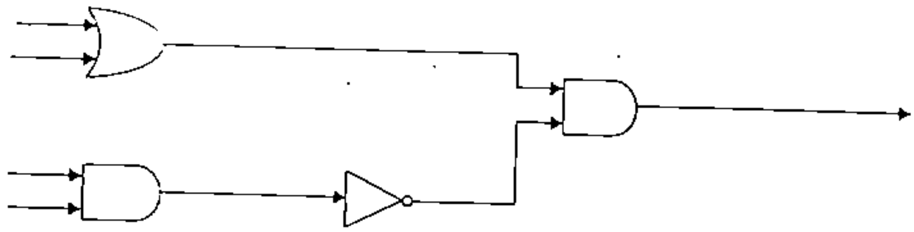


Fig. 9: A logic circuit of elementary gates.

Now let us try to see the connection between logic circuits and Boolean expressions. We first consider the elementary gates. For a given pair of inputs x_1 and x_2, the output in the case of each of these gates is an expression of the form $x_1 \wedge x_2$ or $x_1 \vee x_2$ or x'.

Next, let us look at larger circuits. Is it possible to find an expression associated with a logic circuit, using the symbols $\wedge$, $\vee$ and $'$? Yes, it is. We will illustrate the technique of finding a Boolean expression for a given logic circuit with the help of some examples. But first, note that the output of a gate in a circuit may serve as an input to some other gate in the circuit, as in Fig. 9. So, to get an expression for a logic circuit the process always moves in the direction of the arrows in the circuitry. With this in mind, let us consider some circuits.

Example 7: Find the Boolean expression for the logic circuit given in Fig.9 above.

Solution: In Fig.9, there are four input terminals. Let us call them x_1, x_2, x_3 and x_4. So, x_1 and x_2 are inputs to an OR-gate, which gives $x_1 \vee x_2$ as an output expression (see Fig. 9(a)).

Similarly, the other two inputs x_3 and x_4, are inputs to an AND-gate. They will give $x_3 \wedge x_4$ as an output expression. This is, in turn, an input for a NOT-gate in the circuit. So, this yields $(x_3 \wedge x_4)'$ as the output expression. Now, both the expressions $x_1 \vee x_2$ and $(x_3 \wedge x_4)'$ are inputs to the extreme right AND-gate in the circuit. So, they give $(x_1 \vee x_2) \wedge (x_3 \wedge x_4)'$ as the final output expression, which represents the logic circuit.

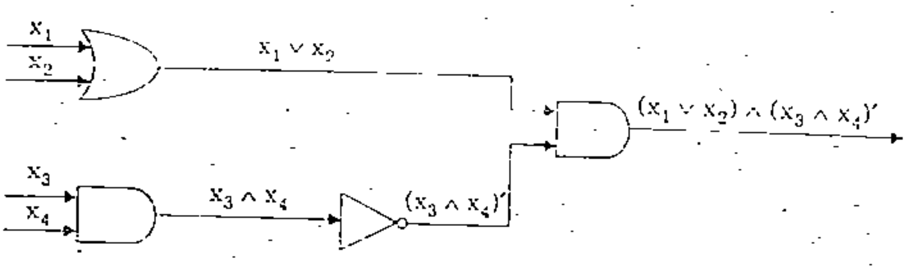


Fig. 9(a)

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You have just seen how to find a Boolean expression for a logic circuit. For more practice, let us find it for another logic circuit.

Example 8: Find the Boolean expression C for the logic circuit given in Fig. 10.

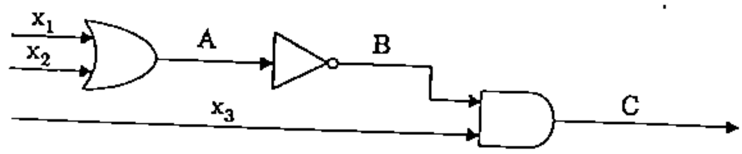


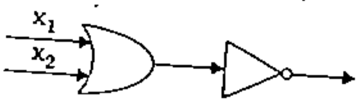
Fig. 10

Solution: Here the first output is from an OR-gate, i.e., A is $x_1 \vee x_2$. This, in turn, serves as the input to a NOT-gate attached to it from the right. The resulting bit B is $(x_1 \vee x_2)'$. This, and x_3, serve as inputs to the extreme right AND-gate in the circuit given above. This yields an output expression $(x_1 \vee x_2)' \wedge x_3$, which is C, the required expression for the circuit given in Fig.10.

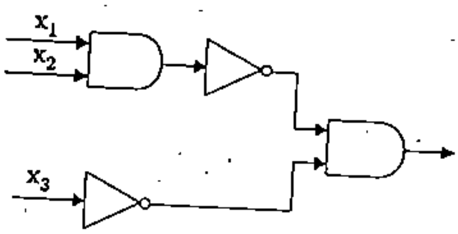
* * *

Why don't you try to find the Boolean expressions for some more logic circuits now?

E8) Find the Boolean expression for the output of the logic circuits given below.



(a)



(b)

So far, you have seen how to obtain a Boolean expression that represents a given circuit. Can you do the converse? That is, can you construct a logic circuit corresponding to a given Boolean expression? In fact, this is done when a circuit designing problem has to be solved. The procedure is quite simple. We illustrate it with the help of some examples.

Example 9: Construct the logic circuit represented by the Boolean expression $(x_1' \wedge x_2) \vee (x_1 \vee x_3)$, where x_i $(1 \leq i \leq 3)$ are assumed to be inputs to that circuitry.

Solution: Let us first see what the portion $(x_1' \wedge x_2)$ of the given expression contributes to the complete circuit. In this expression the literals x_1' and x_2 are connected by the connective $\wedge$ (AND). Thus the circuit corresponding to it is as shown in Fig.11(a), by the definitions of NOT-gate and AND-gate.

Similarly, the gate corresponding to the expression $x_1 \vee x_3$ is as shown in Fig.11(b) above. Finally, note that the given expression has two parts, namely, $x_1' \wedge x_2$ and $x_1 \vee x_3$, which are connected by the connective $\vee$ (OR). So, the two logic circuits given in Fig.11, when connected by an OR-gate, will give us the circuit shown in Fig.12.

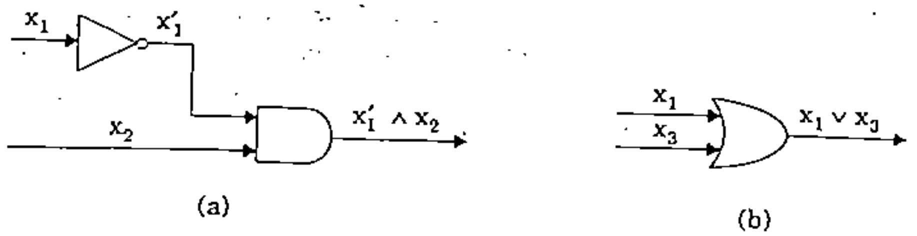


Fig. 11: Logic circuits for the expressions $x_1' \wedge x_2$ and $x_1 \vee x_3$.

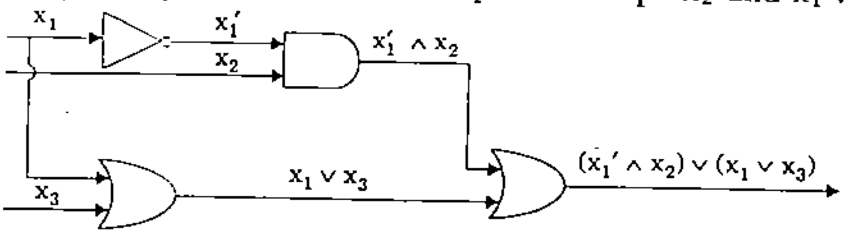


Fig. 12: Circuitry for the expression $(x_1' \wedge x_2) \vee (x_1 \vee x_3)$

This is the required logic circuit which is represented by the given expression.

* * *

Example 10: Given the expression $(x_1' \vee (x_2 \wedge x_3')) \wedge (x_2 \vee x_4')$, find the corresponding circuit, where x_i $(1 \leq i \leq 4)$ are assumed to be inputs to thé circuitry.

Solution: We first consider the circuits representing the expressions $x_2 \wedge x_3'$ and $x_2 \vee x_4'$. They are as shown in Fig.13(a).

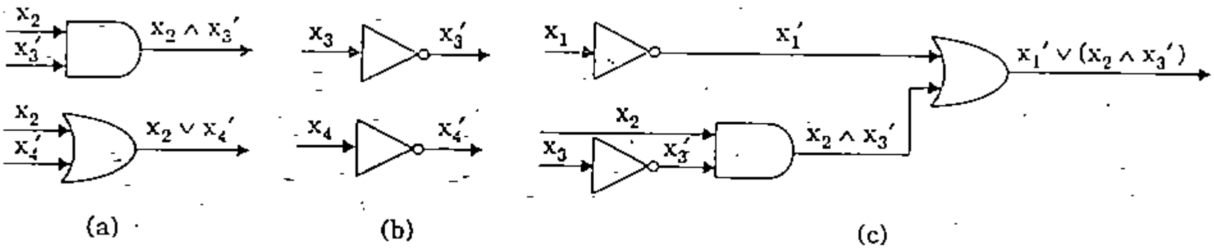


Fig.13: Construction of a logic circuitry.

Also you know that the literals x_3' and x_4' are outputs of the NOT-gate. So, these can be represented by logic gates as shown in Fig.13(b). Then the circuit for the part $x_1' \vee (x_2 \wedge x_3')$ of the given expression is as shown in Fig.13(c). You already know how to construct a logic circuit for the expression $x_2 \vee x_4'$.

Finally, the two expressions $(x_1' \vee (x_2 \wedge x_3'))$ and $(x_2 \vee x_4')$ being connected by the connective $\wedge$ (AND), give the required circuit for the given expression as shown in Fig.14.

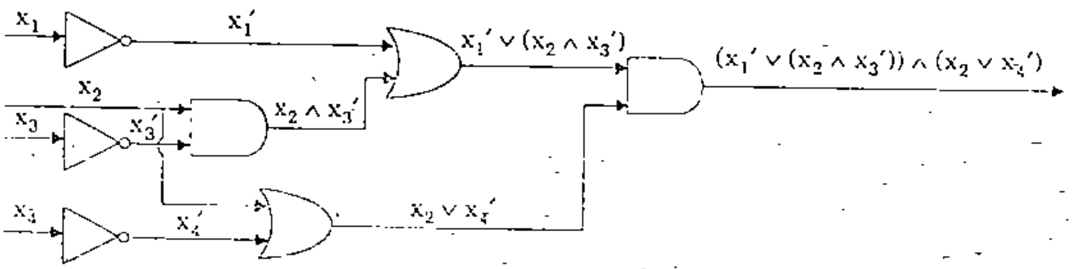


Fig. 14: Circuitry for the expression $(x_1' \vee (x_2 \wedge x_3)) \wedge (x_2 \vee x_4')$.

* * *

Why don't you try to solve some exercises now?

…ading to the expression $x_1' \wedge (x_2 \vee x_3')$.
… obtain the logic table for the

… correspondence between logic circuits … der about the utility of this. The … us understand the overall … nd how, consider the circuit given in … puts bits x_1, x_2, and x_3 as three … have two values only, namely, 0 or 1, … se inputs have at any moment of … xpression $(x_1 \vee x_2)' \wedge x_3$, which … values of the 3-tuple (x_1, x_2, x_3).

… lerstand the functioning of the … in which the settings of x_1, x_2 and … $x_1 = x_3 = 0$ and $x_2 = 1$. Then we know that $x_1 \vee x_2 = 0 \vee 1 = 1$ (see the second row of Table 3 given earlier). Further, using the …gic table of a NOT-gate, we get $(x_1 \vee x_2)' = 1' = 0$. Fi…lly, from Table 3, we get $(x_1 \vee x_2)' \wedge x_3 = 0 \wedge 1 = 0$. Thus, the expression $(x_1 \vee x_2)' \wedge x_3$ has value 0 for the set of values $(0, 1, 0)$ of input bits (x_1, x_2, x_3). **Thus, if x_1 and x_3 are closed, while x_2 is open, the circuit remains closed.**

Using similar arguments, you can very easily calculate the other values of the expression $(x_1 \vee x_2)' \wedge x_3$ in the set

$$\{0,1\}^3 = \{(x_1, x_2, x_3) \mid x_i = 0 \text{ or } 1,\ 1 \le i \le 3\}$$

of values of input bits. We have recorded them in Table 6.

Observe that the row entries in the first three columns of Table 6 represent the different values which the input bits (x_1, x_2, x_3) may take. Each entry in the last column of the table gives the output of the circuit represented by the expression $(x_1 \vee x_2)' \wedge x_3$ for the corresponding set of values of (x_1, x_2, x_3). For example, if (x_1, x_2, x_3) … then the level of voltage in the output lead is at a level 0 (see the third row of Table 6).

You should verify that the values in the other rows are correct.

Table 6: Logic table for the expression $(x_1 \vee x_2)' \wedge x_3$.

x_1	x_2	x_3	$x_1 \vee x_2$	$(x_1 \vee x_2)'$	$(x_1 \vee x_2)' \wedge x_3$
[illegible]	[illegible]	[illegible]	[illegible]	1	0
[illegible]	[illegible]	[illegible]	[illegible]	1	1
[illegible]	[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0

…cuit given in E8(b) above.

… expression representing a circuit … ween the state (or level) of voltage in … ut lead of that logic circuitry. This leads us the concept of Boolean functions, which we will now discuss.

3.5 BOOLEAN FUNCTIONS

In the last section you studied that an output expression is not merely a device for representing an interconnection of gates. It also defines output values as a function of input bits. This provides information about the overall functioning of the corresponding logic circuit. So, this function gives us a relation between the inputs to the circuit and its final output .

This is what helps us to understand control over the functioning of logic circuits from a mathematical point of view. To explain what this means, let us reformulate the logic tables in terms of functions of the input bits.

Let us first consider the Boolean expression

$$X(x_1, x_2) = x_1 \wedge x_2',$$

where x_1 and x_2 take values in $B = \{0, 1\}$. You know that all the values of this expression, for different pairs of values of the variables x_1 and x_2, can be calculated by using properties of the Boolean algebra B. For example,

$$0 \wedge 1' = 0 \wedge 0 = 0 \quad \Rightarrow \quad X(0, 1) = 0.$$

Similarly, you can calculate the other values of $X(x_1, x_2) = x_1 \wedge x_2'$ over B.

In this way we have obtained a function $f : B^2 \to B$, defined as follows:

$$f(e_1, e_2) = X(e_1, e_2) = e_1 \wedge e_2', \quad \text{where} \quad e_1,\ e_2 \in \{0, 1\}.$$

So f is obtained by replacing x_i with e_i in the expression $X(x_1, x_2)$. For example, when $e_1 = 1$, $e_2 = 0$, we get $f(1, 0) = 1 \wedge 0' = 1$.

More generally, each Boolean expression $X(x_1, x_2, \ldots, x_k)$ in k variables, where each variable can take values from the two-element Boolean algebra B, defines a function $f : B^k \to B : f(e_1, \ldots, e_k) = X(e_1, \ldots, e_k)$.
Any such function is called a Boolean function.

Thus, each Boolean expression over $B = \{0, 1\}$ gives rise to a Boolean function. In particular, corresponding to each circuit, we get a Boolean function. Therefore, the logic table of a circuit is just another way of representing the Boolean function corresponding to it.

For example, the logic table of an AND-gate can be obtained using the function $\wedge : B^2 \to B : \wedge(e_1, e_2) = e_1 \wedge e_2$.

To make matters more clear, let us work out an example.

Example 11: Let $f : B^2 \to B$ denote the function which is defined by the Boolean expression $X(x_1, x_2) = x_1' \wedge x_2'$. Write the values of f in tabular form.

Solution: f is defined by $f(e_1, e_2) = e_1' \wedge e_2'$ for $e_1, e_2 \in \{0, 1\}$. Using Tables 3, 4 and 5, we have

$f(0, 0) = 0' \wedge 0' = 1 \wedge 1 = 1$, $\quad f(0, 1) = 0' \wedge 1' = 1 \wedge 0 = 0$,
$f(1, 0) = 1' \wedge 0' = 0 \wedge 1 = 0$. $\quad f(1, 1) = 1' \wedge 1' = 0 \wedge 0 = 0$.

We write this information in Table 7.

Table 7: Boolean function for the expression $x_1' \wedge x_2'$.

e_1	e_2	e_1'	e_2'	$f(e_1, e_2) = e_1' \wedge e_2'$
0	0	1	1	$1 \wedge 1 = 1$
0	1	1	0	$1 \wedge 0 = 0$
1	0	0	1	$0 \wedge 1 = 0$
1	1	0	0	$0 \wedge 0 = 0$

* * *

Why don't you try an exercise now?

E12) Find all the values of the Boolean function $f : \mathcal{B}^2 \to \mathcal{B}$ defined by the Boolean expression $(x_1 \wedge x_2) \vee (x_1 \wedge x_3')$.

Let us now consider the Boolean function $g : \mathcal{B}^2 \to \mathcal{B}$, defined by the expression $X(x_1, x_2) = (x_1 \vee x_2)'$.
Then $g(e_1, e_2) = (e_1 \vee e_2)'$, $e_1, e_2 \in \mathcal{B}$.
So, the different values that g will take are

$$g(0,0) = (0 \vee 0)' = 0' = 1, \qquad g(0,1) = (0 \vee 1)' = 1' = 0,$$
$$g(1,0) = (1 \vee 0)' = 1' = 0, \qquad g(1,1) = (1 \vee 1)' = 1' = 0.$$

In tabular form, the values of g can be presented as in Table 8.

Table 8: **Boolean function of the expression $(x_1 \vee x_2)'$.**

e_1	e_2	$e_1 \vee e_2$	$g(e_1, e_2) = (e_1 \vee e_2)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

By comparing Tables 7 and 8, you can see that $f(e_1, e_2) = g(e_1, e_2)$, for all $(e_1, e_2) \in \mathcal{B}^2$. So f and g are the same function.

What you have just seen is that **two (seemingly) different Boolean expressions can have the same Boolean function specifying them.** Note that if we replace the input bits by propositions in the two expressions involved, then we get logically equivalent statements. This may give you some idea of how the two Boolean expressions are related. We give a formal definition below.

Definition : Let $X = X(x_1, x_2, \ldots, x_k)$ and $Y = Y(x_1, x_2, \ldots, x_k)$ be two Boolean expressions in the k variables $x_1, \ldots, x_k$. We say X is **equivalent** to Y over the Boolean algebra $\mathcal{B}$, and write $X \equiv Y$, if both the expressions X and Y define the same Boolean function over $\mathcal{B}$, i.e.,

$$X(e_1, e_2, \ldots, e_k) = Y(e_1, e_2, \ldots, e_k), \text{ for all } e_i \in \{0, 1\}.$$

So, the expressions to which f and g (given by Tables 7 and 8) correspond are equivalent.

Why don't you try an exercise now?

E13) Show that Boolean expressions

$$X = (x_1 \wedge x_2) \vee (x_1 \wedge x_3') \quad \text{and} \quad Y = x_1 \wedge (x_2 \vee x_3')$$

are equivalent over the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$.

So far you have seen that given a circuit, we can define a Boolean function corresponding to it. You also know that given a Boolean expression over $\mathcal{B}$, there is a circuit corresponding to it. Now, you may ask:

- Given a Boolean function $f : \mathcal{B}^n \to \mathcal{B}$, is it always possible to get a Boolean expression which will specify f over B? The answer is 'yes', i.e., for every function $f : \mathcal{B}^n \to \mathcal{B}$ $(n \geq 2)$ there is a Boolean expression (in n variables) whose Boolean function is f itself.

In fact, the disjunctive (and conjunctive) normal forms described in Sec.3.3 are precisely the expressions which will come in handy here.

To help you understand the underlying procedure, consider the following examples.

Example 12: Let $f : \mathcal{B}^2 \to \mathcal{B}$ be a function which is defined by

$$f(0,0) = 1, \quad f(1,0) = 0, \quad f(0,1) = 1, \quad f(1,1) = 1.$$

Find the Boolean expression (in DNF) specifying the function f.

Solution: The procedure involved for the construction of a Boolean expression (in DNF) which will specify the given function f is given in the following three steps.

Step-I: Collect all the pairs of values $v_i = (e_{i1}, e_{i2})$ for which $f(e_{i1}, e_{i2}) = 1$, where $(e_{i1}, e_{i2}) \in \mathcal{B}^2 \; \forall \; i$. In this case these are

$$v_1 = (0,0), \quad v_2 = (0,1) \text{ and } v_3 = (1,1).$$

Step-II: Write a minterm $m_i = y_{i1} \wedge y_{i2}$ for each pair v_i of these values, namely, (0,0), (0,1) and (1,1), where, for $1 \le i \le 3$, $1 \le j \le 2$,

$$y_{ij} = \begin{cases} x_j, & \text{if } e_{ij} = 1, \\ x_j', & \text{if } e_{ij} = 0. \end{cases}$$

Now, because $v_1 = (0,0)$ i.e. $e_{11} = 0$ and $e_{12} = 0$, so, we have

$$m_1 = y_{11} \wedge y_{12} = x_1' \wedge x_2',$$

by the definition of y_{11} and y_{12} given above.
Similarly, you can see that

$$m_2 = x_1' \wedge x_2 \quad \text{and} \quad m_3 = x_1 \wedge x_2.$$

Step-III: The join of the three minterms m_1, m_2 and m_3 gives the expression of the type

$$X(x_1, x_2) = m_1 \vee m_2 \vee m_3 = (x_1' \wedge x_2') \vee (x_1' \wedge x_2) \vee (x_1 \wedge x_2),$$

which is the required Boolean expression (in DNF) whose Boolean function is the same as the given function f (see the exercise given below).

* * *

You can complete Example 12, by doing the following exercise.

E14) In the previous example, show that $X(e_1, e_2) = f(e_1, e_2)$, $\forall e_1, e_2 \in \mathcal{B}$.

In Example 12, you saw how to obtain an expression (in DNF) for a given function $f : \mathcal{B}^2 \to \mathcal{B}$. In the next example, you will see how to obtain the expression in CNF.

Example 13: Let $g : \mathcal{B}^2 \to \mathcal{B}$ be a function which is defined by

$$g(0,0) = 0, \quad g(1,0) = 1, \quad g(0,1) = 0, \quad g(1,1) = 1.$$

Find the Boolean expression in CNF which specifies the function g.

Solution: The procedure to obtain a Boolean expression in CNF specifying the function g is given in the following three steps.

Step-I: Collect all pair of values $v_i = (e_{i1}, e_{i2})$ such that $g(v_i) = 0$, where $(e_{i1}, e_{i2}) \in \mathcal{B}^2 \; \forall \; i$. Here two such pairs are given by $v_1 = (0,0)$ and $v_2 = (0,1)$.

Step-II: Write a maxterm $M_i = y_{i1} \vee y_{i2}$ for each pair $v_i = (e_{i1}, e_{i2})$ of these two, where, for $1 \le i, j \le 2$,

$$y_{ij} = \begin{cases} x_j, & \text{if } e_{ij} = 1, \\ x_j', & \text{if } e_{ij} = 0. \end{cases}$$

Now, because $v_1 = (0,0)$ i.e. $e_{11} = 0$ and $e_{12} = 0$, we have

$$M_1 = y_{11} \vee y_{12} = x_1' \vee x_2',$$

using the definition of y_{11} and y_{12} given above.

Similarly, $M_2 = x_1' \vee x_2$.

Step-III: Finally, the meet of these two maxterms M_1 and M_2 give the expression

$$X(x_1, x_2) = M_1 \wedge M_2 = (x_1' \vee x_2') \wedge (x_1' \vee x_2),$$

which is the required Boolean expression in CNF specifying the function g .(Verify this!)

* * *

The following theorems are simple generalisations of the procedures illustrated in the previous two examples. (We shall not prove them here.)

Theorem 3: Let $f : \mathcal{B}^n \to \mathcal{B}$ $(n \geq 1)$ be a function and $v_i = (e_{i1}, e_{i2}, \dots, e_{in})$ $(1 \leq i \leq k)$ be those elements of the Boolean algebra $\mathcal{B}^n$ for which $f(v_i) = 1$. For each such v_i, set $m_i = y_{i1} \wedge \cdots \wedge y_{in}$, where

$$y_{ij} = \begin{cases} x_j, & \text{if } e_{ij} = 1, \\ x_j', & \text{if } e_{ij} = 0. \end{cases} \quad \text{for } j = 1, \dots, n.$$

Then $X(x_1, x_2, \dots, x_n) = m_1 \vee m_2 \vee \cdots \vee m_k$ is a Boolean expression (in DNF) whose Boolean function is the same as the function f.

Theorem 4: Let $g : \mathcal{B}^n \to \mathcal{B}$ be a function and $v_i = (e_{i1}, e_{i2}, \dots, e_{in})$, $(1 \leq i \leq k)$ be those elements of the Boolean algebra $\mathcal{B}^n$ for which $f(v_i) = 0$. For each such v_i, set $M_i = y_{i1} \vee \cdots \vee y_{in}$, where

$$y_{ij} = \begin{cases} x_j, & \text{if } e_{ij} = 1, \\ x_j', & \text{if } e_{ij} = 0. \end{cases} \quad \text{for } j = 1, \dots, n,$$

Then $X(x_1, x_2, \dots, x_n) = M_1 \wedge M_2 \wedge \cdots \wedge M_k$ is a Boolean expression (in CNF) whose Boolean function is the same as the function g.

Remark: To get a Boolean expression for a Boolean function h (say), we should first see how many v_i's there are at which $h(v_i) = 0$, and for how many v_i's $h(v_i) = 1$. **If the number of values for which the function h is 0 is less than the number of values at which h is 1, then we shall choose to obtain the expression in CNF, and not in DNF.** This will give us a shorter Boolean expression, and hence, a simpler circuit. For similar reasons, we will prefer DNF if the number of values at which h is 0 is more.

Why don't you apply Theorems 3 and 4 now?

E15) Find the Boolean expressions, in DNF or in CNF (keeping in mind the remark made above), for the functions defined in tabular form below.

(a)

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

(b)

x_1	x_2	x_3	$g(x_1, x_2, x_3)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

Boolean functions tell us about the functioning of the corresponding circuit. Therefore, circuits represented by two equivalent expressions should essentially do the same job. We use this fact while redesigning a circuit to create a simpler one. In fact, in such a simplification process of a circuit, we write an expression for the circuit and then evaluate the same (over two-element Boolean algebra B) to get the Boolean function. Next, we proceed to get an equivalent, simpler expression. Finally, the process terminates with the construction of the circuit for this simpler expression. Note that, as the two expressions are equivalent, the circuit represented by the simpler expression will do exactly the same job as the circuit represented by the original expression.

Let us illustrate this process by an example in some detail.

Example 14: Design a logic circuit capable of operating a central light bulb in a hall by three switches x_1, x_2, x_3 (say) placed at the three entrances to that hall.

Solution: Let us consider the procedure stepwise.

Step 1: To obtain the function corresponding to the unspecified circuit.

To start with, we may assume that the bulb is off when all the switches are off. Mathematically, this demands a situation where $x_1 = x_2 = x_3 = 0$ implies $f(0, 0, 0) = 0$, where f is the function which depicts the functional utility of the circuit to be designed.

Let us now see how to obtain the other values of f. Note that every change in the state of a switch should alternately put the light bulb on or off. Using this fact repeatedly, we obtain the other values of the function f.

Now, if we assign the value (1,0,0) to (x_1, x_2, x_3), it brings a single change in the state of the switch x_1 only. So, the light bulb must be on. This can be written mathematically in the form $f(1, 0, 0) = 1$. Here the value 1 of f stands for the on state of the light bulb.

Then, we must have $f(1, 1, 0) = 0$, because there is yet another change, now in the state of switch x_2.

You can verify that the other values of $f(x_1, x_2, x_3)$ are given as in Table 9.

Table 9: Function of a circuitry for a three-point functional bulb.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
1	0	0	1
1	1	0	0
1	1	1	1
0	1	0	1
0	1	1	0
0	0	1	1
1	0	1	0

Step 2: To obtain a Boolean expression which will specify the function f. Firstly, note that the number of 1's in the last column of Table 9 are fewer than the number of 0's. So we shall obtain the expression in DNF (instead of CNF).

By following the stepwise procedure of Example 12, and using Theorem 3, you can see that the required Boolean expression is given by

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3') \vee (x_1' \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3).$$

At this stage we can directly jump into the construction of the circuit for this expression (using methods discussed in Sec.3.3). But why not try to get a simpler circuit?

Step 3 : To simplify the expression $X(x_1, x_2, x_3)$ **given above.** Firstly, observe that

$$\begin{aligned}(x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3) &= x_1 \wedge [(x_2' \wedge x_3) \vee (x_2 \wedge x_3)] \\ &= x_1 \wedge [(x_2' \vee x_2) \wedge x_3] \\ &= x_1 \wedge (1 \wedge x_3) \\ &= x_1 \wedge x_3,\end{aligned}$$

by using distributive, complementation and identity laws (in that order). Similarly, you can see that

$$(x_1' \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_3) = (x_2' \vee x_1) \wedge x_3.$$

We thus have obtained a simpler (and equivalent) expression, namely,

$$X(x_1, x_2, x_3) = (x_1' \wedge x_2 \wedge x_3') \vee [(x_2' \vee x_1) \wedge x_3],$$

whose Boolean function is same as the function f. (Verify this!)

Step 4: To design a circuit for the expression obtained in Step 3. Now, the logic circuit corresponding to the simpler (and equivalent) expression obtained in Step 3 is as shown in Fig.15.

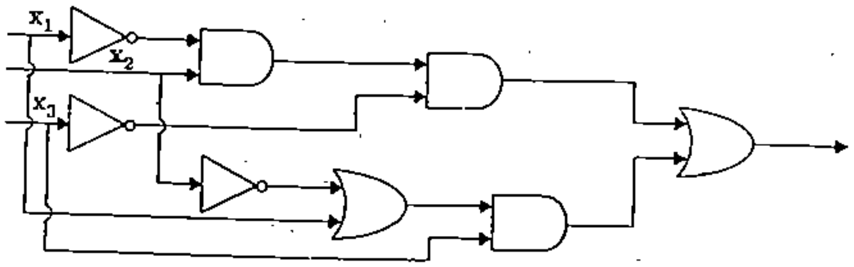


Fig. 15: A circuit for the expression $(x_1' \wedge x_2 \wedge x_3') \vee ((x_2' \vee x_1) \wedge x_3)$.

So, in 4 steps we have designed a 3-switch circuit for the hall.

* * *

We can't claim that the circuit designed in the example above is the simplest circuit. How to get that is a different story and is beyond the scope of the present course.

Why don't you try an exercise now?

E16) Design a logic circuit to operate a light bulb by two switches, x_1 and x_2 (say).

We have now come to the end of our discussion on applications of logic. Let us briefly recapitulate what we have discussed here.

3.6 SUMMARY

In this unit we have considered the following points.

1. The definition and examples of a Boolean algebra. In particular, we have discussed the two-element Boolean algebra $B = \{0, 1\}$, and the switching algebras $B^n, n \geq 2$.
2. The definition and examples of a Boolean expression.
3. How to write a Boolean expression in disjunctive normal form (DNF) or

in conjunctive normal form (CNF).

4. The three elementary logic gates, namely, AND-gate, OR-gate and NOT-gate; and the analogy between their functioning and operations of logical connectives.

5. The method of construction of a logic circuit corresponding to a given Boolean expression, and vice-versa.

6. How to obtain the logic table of a Boolean expression, and its utility in the understanding of the overall functioning of a circuit.

7. The method of simplifying a Boolean expression.

8. The method of construction of a Boolean function $f : \mathcal{B}^n \to \mathcal{B}$, corresponding to a Boolean expression, and the concept of equivalent Boolean expressions.

9. The method of obtaining a Boolean expression (in CNF or DNF) for a given function $f : \mathcal{B}^n \to \mathcal{B}, n \geq 2$.

10. Examples of the use of Boolean algebra techniques for constructing a logic circuit which can function in a specified manner.

3.7 SOLUTIONS/ANSWERS

E1) a) In E19 of Unit1, you have already verified the Identity laws. Let us proceed to show that the propositions $p \vee (p \wedge q)$ and p are logically equivalent. It suffices to show that the truth tables of both these propositions are the same. This follows from the first and last columns of the following table.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
F	F	F	F
F	T	F	F
T	F	F	T
T	T	T	T

Similarly, you can see that the propositions $p \wedge (p \vee q)$ and p are equivalent propositions. This establishes the absorption laws for the Boolean algebra $(\mathcal{S}, \wedge, \vee, ', \mathcal{T}, \mathcal{F})$.

b) Let A and B be two subsets of the set X. Since $A \cap B \subseteq A$, $(A \cap B) \cup A = A$. Similarly, as $A \subseteq A \cup B$, we have $(A \cup B) \cap A = A$. Thus, both the forms of the absorption laws hold good for the Boolean algebra $(\mathcal{P}(X), \cup, \cap, {}^c, X,)$.

E2) Observe that

$$\begin{aligned}(x_1 \vee x_2)' &= x_1' \wedge x_2' && \text{(De Morgan's Laws)}\\ &= (x_1' \wedge x_2') \wedge I && \text{(Identity Law)}\\ &= (x_1' \wedge x_2') \wedge (x_3 \vee x_3') && \text{(Complementation Law)}\\ &= (x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3') && \text{(Distributive Law)}\end{aligned}$$

Similarly, you can see that

$$\begin{aligned}x_1' \wedge x_3 &= (x_1' \wedge x_3) \wedge (x_2 \vee x_2') = (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2')\\ &= (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3).\end{aligned}$$

Thus, the DNF of the Boolean expression $X(x_1, x_2, x_3)$ is given by

$$(x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3).$$

E3) We have

$$\begin{aligned}((x_1 \wedge x_2') \vee (x_1' \wedge x_3'))' &= (x_1 \wedge x_2')' \wedge (x_1' \vee x_3')' \\ &= (x_1' \vee (x_2')') \wedge ((x_1')' \wedge (x_3')') \\ &= (x_1' \vee x_2) \wedge (x_1 \wedge x_3)\end{aligned}$$

Now

$$\begin{aligned}(x_1' \vee x_2) &= (x_1' \vee x_2) \vee O \\ &= (x_1' \vee x_2) \vee (x_3 \wedge x_3') \\ &= (x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3').\end{aligned}$$

Similarly, it can be seen that

$(x_1 \vee x_3) = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2' \vee x_3)$. Thus, the CNF of the given Boolean expression is

$$(x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3') \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2' \vee x_3).$$

E4) We can write

$$\begin{aligned}X(x_1, x_2, x_3) &= ((x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3)) \vee (x_2 \wedge x_3) \\ &= (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \quad \text{(by Absorption law)} \\ &= x_2 \wedge (x_1 \vee x_3) \quad \text{(by Distributive law)}\end{aligned}$$

This is the simplest form of the given expression.

E5) Take the propositions p and q in place of the bits x_1 and x_2, respectively. Then, when 1 and 0 are replaced by T and F in Table 3 here, we get the truth table for the proposition $p \wedge q$ (see Table 2 of Unit 1).

This establishes the analogy between the functioning of the AND-gate and the conjunction operation on the set of propositions.

E6) Take the propositions p and q in place of the bits x_1 and x_2, respectively. Then, when 1 and 0 are replaced by T and F in Table 4 here, we get the truth table for the proposition $p \vee q$ (see Table 1 of Unit 1).

This establishes the analogy between the functioning of the OR-gate and the disjunction operation on the set of propositions.

E7) Firstly, observe that the information about the outputs of the three elementary gates, for different values of inputs, can also be written as follows:

$0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1;$ (see Table 3)
$0 \vee 0 = 0,\ 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1;$ and (see Table 4)
$0' = 1,\ 1' = 0.$ (see Table 5)

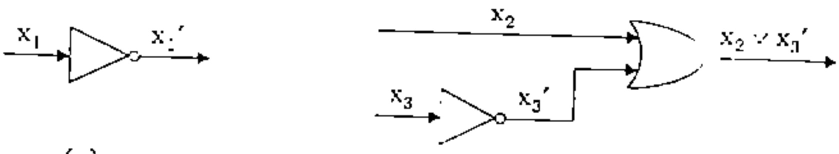
Clearly, then both the operations $\wedge$ and $\vee$ are the binary operations on B and $' : B \to B$ is a unary operation. Also, we may take 0 for O and 1 for I in the definition of a Boolean algebra.

Now, by looking at the logic tables of the three elementary gates, you can see that all the five laws B1-B5 are satisfied. Thus, B is a Boolean algebra.

E8) a) Here x_1 and x_2 are inputs to an OR-gate, and so, we take $x_1 \vee x_2$ as input to the NOT-gate next in the chain which, in turn, yields $(x_1 \vee x_2)'$ as the required output expression for the circuit given in (a).

b) Here x_1 and x_2 are the inputs to an AND-gate. So, the expression $x_1 \wedge x_2$ serves as an input to the NOT-gate, being next in the chain. This gives the expression $(x_1 \wedge x_2)'$ which serves as one input to the extreme right AND-gate. Also, since x_3' is another input to this AND-gate (coming out of a NOT-gate), we get the expression $(x_1 \wedge x_2)' \wedge x_3'$ as the final output expression which represents the circuit given in (b).

9) You know that the circuit representing expressions x_1 and $x_2 \vee x_3'$ are as shown in Fig. 16 (a) and (b) below.



(a) (b)

Fig. 16

Thus, the expression $x_1' \wedge (x_2 \vee x_3')$, being connected by the symbol $\wedge$, gives the circuit corresponding to it as given in Fig.17 below.

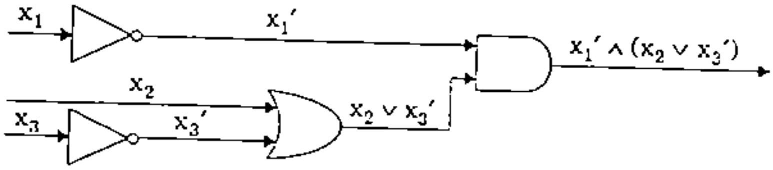


Fig. 17: A logic circuit for the expression $x_1' \wedge (x_2 \vee x_3')$

) You can easily see, by following the arguments given in E9, that the circuit represented by the expression $x_1 \vee (x_2' \wedge x_3)$ is as given in Fig. 18.

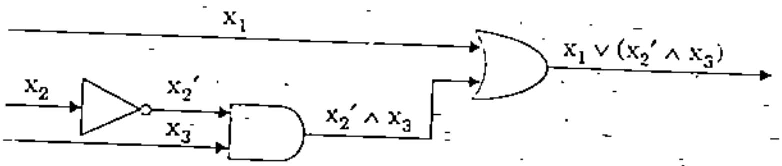


Fig. 18

The logic table of this expression is as given below.

x_1	x_2	x_3	x_2'	$x_2' \wedge x_3$	$x_1 \vee (x_2' \wedge x_3)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	0
1	0	0	1	0	1
0	1	1	0	0	0
1	1	0	0	0	1
1	0	1	1	1	1
1	1	1	0	0	1

Since the output expression representing the circuit given in E8(b) is found to be $(x_1 \wedge x_2)' \wedge x_3'$, the logic table for this circuit is as given below.

x_1	x_2	x_3	$x_1 \wedge x_2$	$(x_1 \wedge x_2)'$	x_3'	$(x_1 \wedge x_2)' \wedge x_3'$
0	0	0	0	1	1	1
0	0	1	0	1	0	0
0	1	0	0	1	1	1
1	0	0	0	1	1	1
0	1	1	0	1	0	0
1	1	0	1	0	1	0
1	0	1	0	1	0	0
1	1	1	1	0	0	0

E12) Because the expression $(x_1 \wedge x_2) \vee (x_1 \wedge x_3')$ involves three variables, the corresponding Boolean function, f (say) is a three variable function, i.e. $f : \mathcal{B}^3 \to \mathcal{B}$. It is defined by

$$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e_3'), \; e_1, e_2 \text{ and } e_3 \in \mathcal{B}.$$

Now, you can verify that the values of f in tabular form are as given in the following table.

e_1	e_2	e_3	$e_1 \wedge e_2$	e_3'	$e_1 \wedge e_3'$	$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e_3')$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	0	1	0	0
1	0	0	0	1	1	1
0	1	1	0	0	0	0
1	1	0	1	1	1	1
1	0	1	0	0	0	0
1	1	1	1	0	0	1

E13) To show that the Boolean expressions X and Y are equivalent over the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, it suffices to show that the Boolean functions f and g (say) corresponding to the expressions X and Y, respectively, are the same. As you can see, the function f for the expression X is calculated in E12 above.
Similarly, you can see that the Boolean function g for the expression Y in tabular form is as given below.

x_1	x_2	x_3	x_3'	$x_2 \vee x_3'$	$g(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3')$
0	0	0	1	1	0
0	0	1	0	0	0
0	1	0	1	1	0
1	0	0	1	1	1
0	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	0	0
1	1	1	0	1	1

Comparing the last columns of this table and the one given in E12 above, you can see that
$f(e_1, e_2, e_3) = g(e_1, e_2, e_3) \; \forall \; e_1, e_2, e_3 \in \mathcal{B} = \{0, 1\}$. Thus, X and Y are equivalent.

E14) Firstly, let us evaluate the given expression $X(x_1, x_2, x_3)$ over the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$ as follows:

$$\begin{aligned}
X(0,0) &= (0' \wedge 0') \vee (0' \wedge 0) \vee (0 \wedge 0) \\
&= (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) \\
&= 1 \vee 0 \vee 0 = 1 = f(0,0); \\
X(1,0) &= (1' \wedge 0') \vee (1' \wedge 0) \vee (1 \wedge 0) \\
&= (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \\
&= 0 \vee 0 \vee 0 = 0 = f(1,0); \\
X(0,1) &= (0' \wedge 1') \vee (0' \wedge 1) \vee (0 \wedge 1) \\
&= (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1)
\end{aligned}$$

$$= 0 \vee 1 \vee 0 = 1 = f(0,1);$$

$$\begin{aligned} X(1,1) &= (1' \wedge 1') \vee (1' \wedge 1) \vee (1 \wedge 1) \\ &= (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ &= 0 \vee 0 \vee 1 = 1 = f(1,1). \end{aligned}$$

It thus follows that $X(e_1, e_2) = f(e_1, e_2) \; \forall \; e_1, e_2 \in B = \{0, 1\}$.

E15) a) Observe from the given table that, among the two values 0 and 1 of the function $f(x_1, x_2, x_3)$, the value 1 occurs the least number of times. Therefore, by the remark made after Example 13, we would prefer to obtain the Boolean expression in DNF.

To get this we will use Theorem 3 and the stepwise procedure adopted in Example 12. firstly observe that

$v_1 = (e_{11}, e_{12}, e_{13}) = (1, 1, 1), v_2 = (e_{21}, e_{22}, e_{23}) = (1, 0, 0)$ and $v_3 = (e_{31}, e_{32}, e_{33}) = (0, 0, 0)$,

are the three triplets of values v_i for which $f(v_i) = 1, \; 1 \leq i \leq 3$. Then, the three minterms m_1, m_2 and m_3 (say) corresponding to these three values v_1, v_2 and v_3, respectively, are given by

$$\begin{aligned} m_1 &= y_{11} \wedge y_{12} \wedge y_{13} \\ &= x_1 \wedge x_2 \wedge x_3; \qquad (\text{because } e_{11} = e_{12} = e_{13} = 1) \\ m_2 &= y_{21} \wedge y_{22} \wedge y_{23} \\ &= x_1 \wedge x_2' \wedge x_3'; \qquad (\text{because } e_{21} = 1 \text{ and } e_{22} = e_{23} = 0) \\ m_3 &= y_{31} \wedge y_{32} \wedge y_{33} \\ &= x_1' \wedge x_2' \wedge x_3'. \qquad (\text{because } e_{31} = e_{32} = e_{33} = 0) \end{aligned}$$

Finally, the required Boolean expression in DNF is given by

$$\begin{aligned} X(x_1, x_2, x_3) &= m_1 \vee m_2 \vee m_3 \\ &= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2' \wedge x_3'). \end{aligned}$$

b) By the given table, among the two values 0 and 1 of the function $g(x_1, x_2, x_3)$, the value 0 has the least number of appearances. So we would prefer to obtain the corresponding Boolean expression in CNF.

To get that we will use Theorem 4 and the stepwise procedure adopted in Example 13. firstly, observe that

$v_1 = (e_{11}, e_{12}, e_{13}) = (1, 0, 1), v_2 = (e_{21}, e_{22}, e_{23}) = (0, 1, 1)$ and $v_3 = (e_{31}, e_{32}, e_{33}) = (0, 1, 0)$,

are the three triplets of values v_i for which $g(v_i) = 0, \; 1 \leq i \leq 3$. Then, the three maxterms M_1, M_2 and M_3 (say) corresponding to these three values v_1, v_2 and v_3, respectively, are given by

$$\begin{aligned} M_1 &= y_{11} \vee y_{12} \vee y_{13} \\ &= x_1 \vee x_2' \vee x_3 \qquad (\text{because } e_{11} = e_{13} = 1 \text{ and } e_{12} = 0) \\ M_2 &= y_{21} \vee y_{22} \vee y_{23} \\ &= x_1' \vee x_2 \vee x_3 \qquad (\text{because } e_{21} = 0 \text{ and } e_{22} = e_{23} = 1) \\ M_3 &= y_{31} \vee y_{32} \vee y_{33} \\ &= x_1' \vee x_2 \vee x_3'. \qquad (\text{because } e_{31} = e_{33} = 0 \text{ and } e_{32} = 1) \end{aligned}$$

Finally, the required Boolean expression (in CNF) is given by

$$X(x_1, x_2, x_3) = M_1 \wedge M_2 \wedge M_3$$
$$= (x_1 \vee x_2' \vee x_3) \wedge (x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x_3'$$

E16) Let g denote the function which depicts the functional utility of the circuit to be designed. We may assume that the light bulb is off when both the switches x_1 and x_2 are off, i.e., we write $g(0,0) = 0$. Now, by arguments used while calculating the entries of Table 9, you can easily see that all the values of the function g are as given below:

$$g(0,0) = 0,\ g(0,1) = 1,\ g(1,0) = 1,\ g(1,1) = 0.$$

Thus, proceeding as in the previous exercise, it can be seen that the Boolean expression (in DNF), which yields g as its Boolean function, is given by the expression

$$X(x_1, x_2) = (x_1' \wedge x_2) \vee (x_1 \wedge x_2'),$$

because $g(0,1) = 1$ and $g(1,0) = 1$.

Finally, the logic circuit corresponding to this Boolean expression is as shown in Fig. 19.

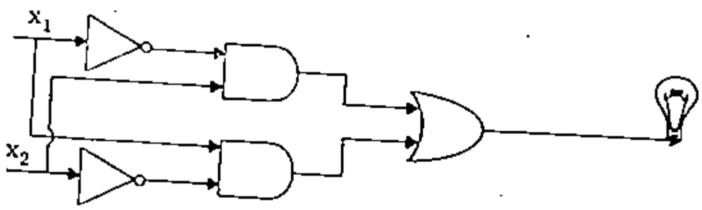


Fig. 19

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2

BASIC COMBINATORICS

Have you ever thought about how a communication engineer can find the total number of distinct ways in which a fixed number of dots and dashes can be used for telegraphic communication? Or, how we can count the number of primes less than or equal to a given number? Enumeration problems such as these are what we discuss in combinatorics. The techniques discussed in this block are termed combinatorial. We use them to study the problem of determining the size, and in some cases also the structure, of various sets that arise in such diverse applications as games, probability, computer programme analysis and mathematics itself.

This block consists of three units. Unit 4 deals with permutations and combinations, the binomial and multinomial theorems, and combinatorial probability.

In this context, you might find it interesting to note that the notion of permutation can be found in the Hebrew work "Sefer Yetzirah" (i.e., The Book of Creation), a manuscript written by a mystic some time between 200 and 600. Also, the 'binomial theorem', which everybody is so familiar with, first appeared in the work of Euclid (300 BC). What is of further historical interest is that Blaise Pascal (1623 – 1662), published in the 1650s a treatise dealing with the relationships among binomial coefficients, combinations, and polynomials. These results were used by Jakob Bernoulli (1645-1705) to prove the general form of the binomial theorem.

In the next unit of this block, Unit 5, we discuss partitions of natural numbers and counting the number of ways of distributing a finite number of objects into a finite number of containers, usually called boxes. It was Leonard Euler (1707-1783) who advanced the study of partitions of integers in his 1740 two-volume opus, "Introduction in Analysin Infinitorum".

The last unit of this block, Unit 6, deals with the pigeonhole principle and the principle of inclusion and exclusion. The latter principle has an interesting history, being found in different manuscripts under such names as the "Sieve Method" or the "Principle of Cross Classification". A set theoretic version of this principle, which concerned itself with set unions and intersection, is found in "Doctrine of Chances" (1718), a text on probability theory by Abraham De Moivre (1667 – 1754). Somewhat earlier, in 1713, Pierre Remond de Montmort (1678 – 1719) used the idea behind the principle in his solution of the problem generally known as "le probleme des rencontres" (derangements).

On the other hand, the pigeonhole principle has no clear-cut mathematical origin. This is also known as the Dirichlet-drawer principle, after the name of the famous German mathematician Dirichlet (1805 – 1859). The more sophisticated generalisation of this principle culminated in the 1930 paper of F. Ramsey.

The combinatorial reasoning that you will study in this block underlies all analysis of computer systems, discrete operations research problems and finite probability. Our discussion of combinatorics doesn't end with the three units here. We continue this study in the next block, also consisting of three units.

Before we end, a note of advice! If you want to really get to grips with the content of this block, you must attempt Miscellaneous Exercises given at the end of each unit. Doing this, will help you understand the underlying reasoning better, and hence enjoy the theory of combinatorics.

NOTATION AND SYMBOLS

$n!$	$n(n-1)\ldots 2.1$
$P(n,r)$	$\frac{n!}{(n-r)!}$
$C(n,r)$	$\frac{n!}{(n-r)!r!}$
$P(n,r_1,r_2,\ldots,r_n)$	$\frac{n!}{r_1!r_2!\ldots r_k!}$
$n(A), \lvert A\rvert$	The cardinality of the set A
$\mathcal{P}(X)$	The powerset of the set X
$P(A)$	The probability of the event A
P_n	The number of partitions of the natural number n
P_n^k	The number of partitions of n with exactly k parts
Q_n^k	The number of partitions of n with k or fewer parts
$P_n(k)$	The number of partitions of n with no part larger than k
$P_n^{(d)}$	The number of distinct partitions of n
$P_n^{(o)}$	The number of odd partitions of n
$s(n,k), 0 \le k \le n$	The Stirling number of the first kind
$S_n^m (n \ge m)$	The Stirling number of the second kind
$[x]_n$	$x(x-1)(x-2)\ldots(x-n+1)$, i.e., falling factorial
B_n	Bell number
$W(A)$	The sum of the weights of objects possessing all the properties in the set A
$N(p_1)$	The number of objects having the property p_1
$N(p_1,p_2)$	The number of objects that have the property p_1 and p_2
$N(p_1',p_2')$	The number of objects that do not have the properties p_1 and p_2
$W(\phi)$	The sum of the weights of all the N objects
$E(0)$	The weight of all objects not possessing any of the properties p or equivalently possessing exactly o properties
d_n	The number of derangements of the numbers 1 to n

UNIT 4 COMBINATORICS — AN INTRODUCTION

Structure

4.1 INTRODUCTION

Combinatorics deals with arrangements of objects according to some pattern (listing) and counting the number of ways it can be done. Mostly it deals with finite number of objects and finite number of ways of arranging them according to some pattern. Sometimes even infinite number of objects and infinite number of ways in which they can be arranged are considered.

We present a few basic formulae involving permutations and combinations. We have discussed binomial and multinomial expansions. At the end we present some applications to probability theory. To get an idea of the type of counting problems we present a few simple examples below:

Example a: Consider the 26 letters of English alphabet. Find the number of words (not necessarily meaningful) of length 3.

The words can be enumerated as aaa, aab, aac, ..., zzz. Clearly the number of words is $26 \times 26 \times 26$.

This is an example with finite number of objects arranged in a finite number of ways.

Example b: In the previous example consider the number of all possible words of finite length.

As the length of words is not bounded, clearly, the number of words is infinite. This is an example with finite number of objects arranged in infinite number of ways.

Example c: Consider the set of all positive integers. How many of them are less than 100?

The answer is clearly 99. In this case we have infinite number of objects arranged in a finite number of ways.

Example d: Consider the set of all positive integers. How many of them are prime?

The answer is 'infinity' as there are infinity of prime numbers.

Example e: Suppose a mail-order company sells six styles of slack. Each style is available in 8 lengths, six waist sizes, and four colours. How many different kinds of slacks does the company have to stock?

The answer is $6 \times 8 \times 6 \times 4 = 1152$ kinds of slacks.

We will be mainly interested in arranging a finite number of objects in a finite number of ways.

Objectives

After reading this unit you should be able to:

- know the contents of the subject of combinatorics;
- use factorials;
- know what permutations and combinations are;
- perform calculations in permutations;
- perform calculations in combinations;
- expand in binomial series;
- expand in multinomial series;
- use the ideas to calculate combinatorial probabilities.

4.2 THE MULTIPLICATION PRINCIPLE AND THE ADDITION PRINCIPLE

We now discuss two fundamental principles of counting called **Multiplication Principle and Addition Principle**. There is one principle called multiplication principle which is more-general than permutations. There are various ways of explaining this principle. Suppose that a task/procedure consists of a sequence subtasks or steps say subtask 1 ...., subtask 2...., subtask k. Furthermore suppose that subtask 1 can be performed in n_1 ways, subtask 2 can be performed in n_2 ways after the subtask 1 has been performed, subtask 3 can be performed in n_3 ways after the subtask 1 and subtask 2 have been performed and so on. Then the number of ways the whole task can be performed is $n_1 \cdot n_2 \ldots n_k$. Let us take the model of boxes and objects filling them. Suppose there are m boxes. Suppose the first box can be filled up in k (1) ways. For every way of filling the first box, suppose there are k (2) ways of filling the second box. Then the two boxes can be filled up in k (1) · k (2) ways. In general, if for every way of filling the first (r − 1) boxes the rth box can be filled up in k (r) ways, for r = 2, 3 ..., m, then the total number of ways of filling all them boxes is k (1) · k (2) ... k (m) ways.

This principle can handle many situations which the simple permutation cannot. It is easily seen that the formula for P (n, r) has been derived using this principle.

Just as we have multiplication principle, there is another fundamental principle called the addition principle. Suppose that a task consists of performing exactly one subtask from among a collection of disjoint (mutually-exclusive) subtasks, says subtask 1, subtask 2, ..., subtask k. (i.e. the task is performed if either the subtask 1 is performed or subtask 2, ... or subtask k is performed). Further suppose that subtask i can be performed in n_i ways, i = 1, 2 ..., k then the number of ways the task can be performed is the sum $n_1 + n_2 + \ldots n_k$. Suppose we want to enumerate some combinatorial arrangements. If there are k classes $C_1, C_2, \ldots, C_k$ of grouping these arrangements, such that, these classes are mutually exclusive and exhaustive in the sense that every arrangement falls precisely in one of the classes, then the total number of arrangements is equal to the sum of the number of arrangements belonging to these k classes.

Example 1: There are three political parties P_1, P_2, and P_3. The party P_1 has 4 members, P_2 has 5 members and P_3 has 6 members in an assembly. Suppose we want to select two persons, both from the same party to become president and vice president of a government organisation. In how many ways can this be done?

Solution: From P_1 we can do it in 4.3 = 12 ways by the multiplication principle. From P_2 it can be done in 5.4 = 20 ways. From P_3 it can be done in 6.5 = 30 ways. By the addition principle the number ways of doing it is 12 + 20 + 30 = 62 ways.

Though the addition principle seems to be simple, together with the multiplication principle, quite a number of combinatorial enumeration can be done with them.

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4.3 PERMUTATIONS

Permutations are ordered arrangement of objects. More specifically, if we are given a number of objects, permutation of them taken k at a time (where k is not more than the number of objects) consists of arranging k of them on a line, the order in which they are arranged being important. (linear arrangement)

Example 2: The permutations of a, b, c, d taken 2 at a time are ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc. They are 12 in number. Note that ab and ba are considered different even though they consist of same two objects.

4.3.1 Notations

We need a notation for writing product of consecutive integers from 1. The products 1, 1 × 2, 1 × 2 × 3, 1 × 2 × 3 × 4, etc. can be written compactly by 1!, 2!, 3!, 4!, etc. respectively. These are read as 'one factorial', 'two factorial', 'three factorial', 'four factorial' etc.

In general we write 1 × 2 × 3 × .... × n as n! and read it as 'n factorial' for every positive (integer) n.

E1) Evaluate 15!/12!.

E2) Compute (3 + 4) ! and 3! + 4!. Are they equal?

E3) If m and n are positive integers show that $(m + n)! \geq m! + n!$.

E4) Compute $\frac{n!}{(n-r)!}$ for n = 20 and r = 17.

E5) If n couples are at a dance, in how many ways can the men and women be paired for a single dance?

Suppose n and r are two positive integers with $r \leq n$. Then the number of permutations of n distinguishable objects taken k at a time is denoted interchangeably as P (n, r), ^nP_r, P^n_r, $_nP_r$. We will use the notation P (n, r).

What is the value of P (n, r)? To answer this question consider r boxes arranged in a line. Choose one object out of n and place it in the first box. This can be done in n ways. Then from the remaining (n – 1) objects choose one and place it in the second box. The first two boxes can be filled in n (n – 1) ways. We continue this operation till the rth box is filled. The number of ways of doing this is n (n – 1) (n – 2) .... (n – r + 1). Thus we have provide that

$$P(n, r) = n(n-1) \ldots (n-r+1).$$

If we see the expression for P (n, r) it is clear that is obtained by omitting the last (n – r) terms (n – r) (n – r – 1) .... 3.2.1 from n (n – 1) (n – 2) .... 3.2.1. Thus we have proved that

$$P(n, r) = n!/(n-r)!.$$

We state this as theorem.

Theorem 1 The number of r-permutation from an n-set, where $0 \leq r \leq n$, is given by

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$
$$= \frac{n!}{(n-r)!}$$

In particular, the number of permutation of an n-set, $n \geq 0$, is given by

$$P(n, n) = n!$$

Example 3: $P(6, 4) = 6.5.4.3 = 6!/(6-4)!$

* * *

We defined factorials only for positive integers. Thus at this stage 0! or zero-factorial has no meaning. But consider P (n, n). Clearly its value is n!. On the other hand $P(n, n) = n!/(n-n)!$ by the earlier formula derived. Hence, if at all 0! has to be defined, it can only be 1. At this stage we will take, by **definition** that $0! = 1$. This will be consistently used everywhere and hence no logical difficulty will arise. In particular we will have $P(n, 0) = 1$ and $P(0, 0) = 1$ though apparently no serious interpretations except mathematical necessity consistency can be given in support of these identities.

Distinguishable and indistinguishable objects: In defining the concept of permutation we assumed that the objects were distinguishable. What does it mean and what is the need for it? Going back to the example of permutations of a, b, c, d taken two at a time, namely, ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc, supposed $d = c = b$, which means merely that we don't distinguish between the three objects b, c, d. Then the 12 permutations change to ab, ba, ab, ba, ab, ba, bb, bb, bb, bb, bb, bb. These comprise repeated permutations ab and ba together with bb. Are we going to consider them as permutations? Later on we will consider permutations in which repetitions are allowed. But for the present we have to assume that all the objects are distinguishable and no permutation contains any repeated object.

4.3.2 Circular Permutations

Permutation of objects is usually thought of as linear arrangement of objects; this means that we visualize arrangement of objects in a line. But there is a variant in which the objects are arranged along the circumference of a circle. We observe the objects in the clockwise direction. On the circumference there is no preferred origin and hence the permutations abcd, bcda, cdab, dabc will look exactly-alike (see Fig. 1). If we consider all the n! permutations of n things each permutation will be indistinguishable from the $(n-1)$ more obtained by the process of transferring the object at the first position to the last position repeatedly in linear permutation, that is, if arrangements are considered the same when one can be obtained from the other by rotation. Thus as circular permutations we will have exactly $n!/n = (n-1)!$. Thus we have shown that the number of circular permutations of n things taken all at a time is $(n-1)!$

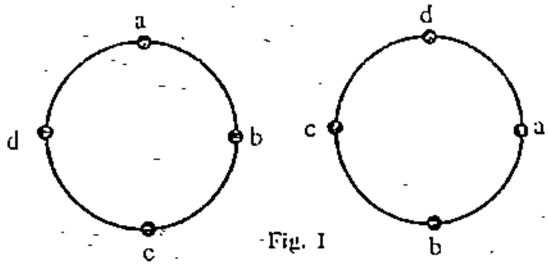

Fig. 1

Example 4: In how many ways is it possible to seat eight person at a round-table?

Solution: Clearly we need the circular permutations of 8 things. Hence the answer is $7! = 5040$.

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Example 5: In the proceeding question, what would be the answer if a certain pair among the eight persons (i) must not sit in adjacent seats? (ii) must sit in adjacent seats?

Solution: From 5040 we have to subtract the number of cases in which the pair of person sit together. If we consider the pair as forming one unit, we have 6! circular permutations, i.e. $(7-1)!$ But even as unit they can be arranged in two ways. Hence the required answer is $7! - 6! - 6! = 3600$ for part (i).

* * *

Example 6: Suppose there are five married couples and they (10 people) are made to seat about a round table so that neither two men nor two women sit together. (the sexes alternate) Find the number of circular arrangements.

Solution: Five females can be made to sit about a round table in $(5-1)! = 4!$ ways. One male can be seated in between two females. There are five positions and hence they can be made to sit in 5! ways. By multiplication principle, the total number of ways of seating arrangement is $4! \times 5! = 2800$.

* * *

Example 7: If seven people are seated about a round table, how many circular arrangements are possible if any will not have the same neighbours in any two arrangements.

Solution: The following two distinct arrangements show that each has same neighbours. Hence the total number of circular arrangements

$$= (7-1)! \times \frac{1}{2} = 360$$

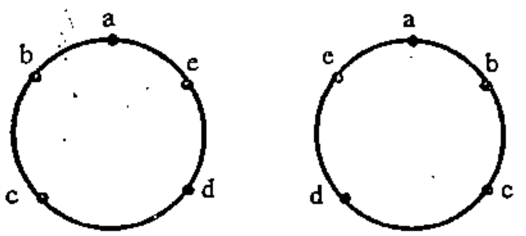

Fig. 2

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Example 8: If there are 7 gents and 5 ladies, how many circular arrangements are possible if the ladies do not sit adjacent to each other.

Solution: The seven gents can be seated first. This can be done in 6! ways. The ladies can sit in between two gents. There are seven places to sit. The ladies can sit in P (7, 5) ways. Hence the answer is $6! \times P(7, 5)$.

* * *

Example 9: How many numbers between 10 and 99 have distinct digits?

Solution: One may be tempted to give the answer as P (10, 2). But these will also include the cases with 0 in the first position. The correct answer is 9.9 = 81. For, the first position can be filled up in 9 ways (by any digit other than 0). After filling the first position the second position can be filled up in 9 ways (by any of the 9 digits other than that occupying the first position. Note that this will include 0 also)

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You may now try the following exercises.

E6) How many licence plates can be made if each should have 3 letters with no letter repeated? What will be the answer if the letters can be repeated?

E7) How many integers between 100 and 999 consist of distinct even digits?

E8) Consider all the numbers between 100 and 999 that have distinct digits. How many of them are odd?

E9) Verify that P (15, 2) = P (7, 3) and P (5,5) = P (6, 3).

E10) How many integers of five digits greater that 65000 have the following two properties: (i) the digits of the number are distinct: (ii) the digits 0 and 1 do not occur in the number?

4.3.3 Permutation of Objects not Necessarily Distinct

We have shown that there are P (n, r) ways to choose r objects from a set of n distinct objects and arrange them in linear order. In this section we consider the same problem with the relaxed condition that some of the objects in the collection may be indistinguishable, that is, we discuss arrangements of a collection of objects with repeated objects such as the collections a, b, π, b, b, π, a. Suppose there are n things of which m_1 belong to category 1, m_2 belong to category 2, etc. m_k belong to category k, with the categories mutually exclusive and exhaustive so that $m_1 + m_2 ... + m_k = n$. Then the number of distinct permutations of these n things is $\frac{n!}{m_1! m_2! ... m_k!}$. This follows from the fact that any permutation is unaffected if the objects belonging to category 1 are permuted among themselves in $m_1!$ ways, objects belonging to category 2 are permuted among themselves in $m_2!$ ways, ..., objects belonging to category k are permuted themselves in $m_k!$ ways. More precisely we have the following theorem:

Theorem 2 If there are n objects classified into k distinct types, with m_1 identical objects of the first type, m_2 identical objects of the second type ... and m_k identical objects of the kth type where $m_1 + m_2 + ... m_k = n$, then the number of arrangements of these n objects, denoted by $P(n; m_1, m_2, ..., m_k) = \frac{n!}{m_1! m_2! ... m_k!}$.

Proof: Let x be the number of such permutations. If the objects of category i are considered distinct, then they can be arranged amongst themselves in $m_i!$ ways where i = 1, 2, ... k. By multiplication principle, the total number of permutation of n distinct things taken all at a time is $x m_1! m_2! ... m_k!$ But this is precisely n! when there are n distinct objects. Hence $x m_1! m_2! ... m_k! = n!$ That is $x = n!/m_1! m_2! ... m_k!$

Example 9: How many 9-lettered words (not necessarily meaningful) can be formed using all the letters of the word CHARIVARI?

Solution: In the word CHARIVARI C, H, V occur once. A, R, I each occurs twice. Hence we can form 9!/1!1!1!2!2!2! = 45360 words.

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E11) How many permutations are there of the letters taken all at a time of the words (a) ASSESSES, (b) PATTIVEERANPATTI?

4.4 COMBINATIONS

Permutation refers to the ordered arrangement of objects. But combination is a selection of a specified number of objects from a store of distiguishable objects. Suppose there are n distinct objects and we want a selection of r objects, where $r \le n$, and the order of the objects in the selection does not matter. This is called a combination of n things taken r at a time. The number of ways of doing this is represented interchangeably by $_nC_r$, ^nC_r, C^n_r, $\binom{n}{r}$ and C (n, r). We will use the notation C (n, r) for typographical convenience and also in conformity with the notation P (n, r) for permutation. We can conveniently read C (n, r) as 'n choose r' to emphasize the fact that only 'choice' is involved but not ordering. One of the main difficulties in permutations and combinations is in determining which one is to be used in a particular situation. This calls for logical thinking. Only practice brings out the difference between them. In terms of ideas of set theory C (n, r) is the number of subsets of size r from a set containing n elements. From this point of view, clearly, C (n, n) is 1, for every positive integer n.

4.4.1 Formula for C (n, r)

Let us get a relationship between P (n, r) and C (n, r). With n distinct objects, C (n, r) counts the number of ways of choosing r of them without regard to the order. Any one of these choices is simply a set of r objects. Such a set can be ordered in r! ways. Thus, to each combination there corresponds r! permutations and hence there are r! times as many permutations as there are combinations. Hence, by the multiplication principle we get $P(n, r) = r!\, C(n, r)$ or $C(n, r) = P(n, r)/r! = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!\, r!}$. Thus we have proved.

Theorem 3 The number of r-combination from an n-set, where $0 \le k \le n$, is given by

$$C(n, r) = \frac{n!}{(n-r)!\, r!}$$

Theorem 4 $C(n, r) = C(n, n-r)$.

Proof: For every choice of r things from n there corresponds uniquely a choice of $n-r$ things from n, consisting of the leftovers. This one-to-one correspondence shows there numbers must be the same. This proves the theorem. Another proof of the theorem is to observe that the formula for C (n, r) is unaltered if r is changed to $n-r$.

Even though we have a formula for C (n, r) in terms of factorials, in practice, we use the following obvious identity.

$$C(n, r) = \frac{n(n-1)\ldots r \text{ factors}}{r(r-1)\ldots r \text{ factors}}$$

In the above expression there are r factors in the numerator as well as in the denominator. Hence if r is less than $(n-r)$ we use the formula as it is. On the other hand, if r is greater than $n-r$, we use the expression with $n-r$ factors in the numerator as well as the denominator.

Example 10: $C(10, 3) = \frac{10.9.8}{3.2.1}$. But $C(10, 8) = \frac{10.9}{2.1}$.

* * *

The numbers C (n, r)'s are also called the binomial coefficients as they occur as coefficients of x^r in the expansion of $(1 + x)^n$ in ascending powers of x. But we will consider the expansions later. At this stage some numerical examples are in order.

Example 11: Evaluate C (6, 2), C (7, 4) and C (9,3).

Solution: $C(6, 2) = \frac{6.5}{2.1} = 15$. $C(7, 4) = \frac{7.6.5}{3.2.1} = 35$. Here we have made use of the fact that $C(7, 4) = C(7, 3)$. $C(9, 3) = \frac{9.8.7}{3.2.1} = 84$.

* * *

At this stage certain values of C (n, r) that are easy to get are given.

$C(n, n) = C(n, 0) = P(n, 0) = 1$. $C(n, 1) = C(n, n-1) = P(n, 1) = n$.

4.4.2 Combinations with Repetition

Let us consider the following example: Suppose five friends stop at a sweet shop where each of them has one of the following: a samosa, a tikki, and a vada. How many different purchases are possible? Let s, t, and v represent samosa, tikki, vada, respectively. In the following table we have listed some possible purchases in first column and in second column we have shown another representation of each purchase.

1. s	s	t	t	t	x	x	|	x	x	x	|
2. s	s	s	s	s	x	x	x	x	x	|	|
3. v	v	v	t	t	|	x	x	|	x	x	x
4. v	v	t	t	s	x	|	x	x	|	x	x

Here each x to the left of the first bar (|) represents a s. each x between the first and second bars represents a t; the x's to the right of the second bar stands for v's. Any order will consists of five x's and two |'s. Conversely any sequence of five x's and two |'s represents an order. By this a correspondence has been established between two collections of objects, where we know how to count the numbers in one collection. But the number of sequences of five x's and two |'s is just the number of two positions in the sequence for the |'s. Hence the answer is C (7.2) or C (7, 5) = $\frac{7!}{5!2!}$.

When repetitions are allowed, for n distinct objects an arrangement of size r of these objects can be obtained in n^r ways, for any integer $r \geq 0$. Let us now discuss a comparable problem for combination. When we wish to select, with repetition, r of n distinct objects, we are considering all arrangements of r of one kind (say x's) and n – 1 of the other kind (say |'s) as (n – 1) |'s are needed to separate n types and their number is $\frac{(n+r-1)!}{r!\,(n-1)!}$ = C (n + r – 1, r) as shown below.

Theorem 5 Let n and r be natural numbers. Then the number of solutions in natural numbers to the equation $x_1 + x_2 + ... + x_n = r$ or equivalently the number of ways to choose r objects from a collection of n objects with repetition allowed is C (n + r – 1, r).

Proof: Consider the set of all strings of length n + r – 1 containing exactly r stars and n – 1 bars. The cardinality of this set is C (n + r – 1, r). Now we demonstrate how such strings corresponds to solution of the equation $x_1 + ... + x_n = r$. Then n – 1 bars in the string divide the string into n substrings of stars. The number of stars in these n substrings are the values of x_1 through of x_n. Since there are r stars altogether, the sum is r. There is one-to-one correspondence between strings and solutions and the theorem is proved.

Example 12: A boy wants to buy some pet birds. The pet store sells parrots, bulbuls and mainas. How many different selections are possible if the boy wants to take home six birds.

Solution: Here r = 6, n = 3. Hence number of possible selection of pet birds

= C (6 + 3 – 1, 6) = C (8, 6) = $\frac{8 \times 7}{2}$ = 28.

In fact we are enumerating all arrangements of 8 symbols consisting of six x's and two bars.

* * *

Example 13: Determine all integers solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7 \text{ where } x_i \geq 0 \text{ for all } 1 \leq i \leq 4$$

Solution: The solution of the equation corresponds to a selection, with repetition, of size 7 from a collection of size 4. Hence there are C (4 + 7 – 1,7) = 120 solutions. (n = 4, r = 7)

* * *

We conclude with the following remark. Let us recognize the equivalence of the following:

(a) The number of integer solutions of the equation $x_1 + x_2 + ... + x_n = r$, $x_i \geq 0$, $1 \leq i \leq n$.

(b) The number of selections, with repetition, of size r from a collection of size n.

(c) The number of ways r identical objects can be distributed among n distinct containers. (Please refer to unit-5).

4.5 THE BINOMIAL EXPANSION

Sum of two distinct symbols like $a + b$, $p + q$, $x + y$, etc. is called a binomial, the binomial expansion refers to the expansion of a positive integral power of such a binomial assuming that the symbols stand for real numbers or complex numbers. Elementary multiplication gives the following expansions readily.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example 14: Let us take the last identity. On the right hand side we have the six terms a^5, $5a^4b$, $10a^3b^2$, $10a^2b^3$, $5ab^4$, and b^5. Our aim is to explain the significance of the coefficients 1, 5, 10 10, 5, 1.

Let us consider

$$(a + b)^5 = (a + b)(a + b)(a + b)(a + b)(a + b)$$

Suppose we want the coefficient of a^3b^2 in this expansion. Clearly every term can be got by selecting one term from the binomial in each of the 5 parentheses. To get a^3b^2 we have to select a from 3 of the parentheses and b from the remaining 2 parentheses. Clearly the parentheses for a can be chosen in C (5, 3) ways, which is precisely 10.

* * *

The above argument can be extended to get the coefficient of a^rb^{n-r} in the expansion of $(a + b)^n$. From the n parentheses representing $(a + b)^n$ we have to select r for a and the remaining $(n - r)$ for b. This can be done precisely in C (n, r) ways. Thus the coefficient of a^rb^{n-r} in the expansion of $(a + b)^n$ is C (n, r). In view of the fact that C (n, r) = C (n, n – r) the coefficients of a^rb^{n-r} and $a^{n-r}b^r$ will be equal. Clearly r can take only the values 0, 1, 2, ... n. We also have C (n, 0) = C (n, n) = 1 as the coefficients of a^n and b^n. Hence we have established the binomial expansion

$$(a + b)^n = a^n + C(n, 1)\, a^{n-1}b + C(n, 2)\, a^{n-2}b^2 + \ldots$$
$$\ldots + C(n, r)\, a^{n-r}b^r + \ldots + b^n.$$

4.5.1 Pascal's Formula for C (n, r)

An interesting property of the binomial coefficients makes it easy to tabulate their values. The formula is as follows.

Theorem 6 For all positive integers n and all r such that $1 \le r \le n$,

$$C(n + 1, r) = C(n, r) + C(n, r - 1).$$

Proof: The left hand side of the identity represents the number of ways of choosing r things out of (n + 1) distinct things. Suppose we select an object from (n + 1) and mark it. Then the number of combinations in which the marked thing is absent is clearly C (n, r) as we have then to choose r things out of the unmarked things. The number of combinations in which the marked thing is present is C (n, r – 1) as we have to choose (r – 1) objects from the n unmarked things and attach the marked thing to it to make up r things. Pascal's formula now follows from the fact that the sum of the last two numbers mentioned must be equal to C (n + 1, r). Alternative algebraic proof:

$$C(n, r) + C(n, r - 1) = \frac{n!}{(n - r)!\,r!} + \frac{n!}{(n - r + 1)!\,(r - 1)!}$$

$$= \frac{n!}{r!\,(n + 1 - r)!}\,(n - r + 1 + r) = C(n + 1, r).$$

Pascal's Triangle: Pascal's identity formula gives us a recursive way to calculate the binomial coefficients, since it tells us the value of C (n, r) in terms of binomial coefficients with a smaller value of n. The base cases are C (n, 0) = C (n, n) = 1 for all $n \geq 0$, since Theorem 3 only applies for $1 \leq r \leq n$. This recursive approach allows us to form Pascal's triangle, the display of the binomial coefficients shown in the figure given below.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

Pascal's triangle.

Fig. 3

The nth row of Pascal's triangle gives the binomial coefficients C (n, r) as r goes from 0 (at the left) to n (at the right); the top row, consisting of just the number 1, is for n = 0. The left and right borders are all 1's, reflecting the fact that C (n, 0) = C (n, n) = 1 for all n. Each entry in the interior of the Pascal's triangle is the sum of the two entries immediately above it to the left and right. We call this property the Pascal property. For example, each 15 in row 6 (remember that we are starting the count of rows at 0) is the sum of the 10 and the 5 immediately above it.

The diagonals of Pascal's triangle are also interesting; they correspond to constant values of r. The left edge, consisting of all 1's, corresponds to r = 0, reflecting the fact that C (n, 0) = 1. The diagonal parallel to the left edge but moved one unit to the right reads (from the top down) 1, 2, 3, 4, 5, ..., reflecting the fact that C (n, 1) = n for $n \geq 1$. The next diagonal to the right, reading 1, 3, 6, 10, 15, ..., reflects the fact that C (n, 2) = n (n – 1)/2 for $n \geq 2$; these numbers are called "triangular" numbers, and their differences increase by 1 as we move down the diagonal.

4.5.2 Some Identities Involving Binomial Coefficients

Identity 1: By setting a = b = 1 in the binomial expansion of $(a + b)^n$ we get

$$C(n, 0) + C(n, 1) + C(n, 2) + \ldots\ C(n, n-1) + C(n, n) = 2^n$$

This identity has an interpretation worth knowing. Suppose there is a set with n elements. How many distinct subsets can be formed from this set? The number of subsets containing precisely r elements is C (n, r). Hence the total number of subsets is $\sum_{r=0}^{n} C(n, r) = 2^n$ by the identity. Thus we have proved the following:

The number of distinct subsets of a set with n elements is 2^n

Identity 2: By setting a = 1, b = –1 in the expansion of $(a + b)^n$ we get

$$C(n, 0) + C(n, 1) + C(n, 2) + \ldots + (-1)^n + C(n, n) = 0$$

By taking all the negative terms to the right hand side we get

$$\sum_{r\ \text{even}} C(n, r) = \sum_{r\ \text{odd}} C(n, r) = 2^{n-1}$$

and this has the interpretation that the number of subsets of a set with n elements with even number of terms is equal to the number of subsets with odd number of terms.

E12) Show that C (n, m) C (m, k) = C (n, k) C (n – k, m – k)

E13) Prove that C (k, k) + C (k + 1, k) + C (k + 2, k) + ... + C (n, k) = C (n + 1, k + 1) for all natural numbers $k \le n$.

4.6 THE MULTINOMIAL EXPANSION

In analogy with binomial which is a sum of two symbols, we have multinomial which is a sum of several distinct symbols (atleast three, but finite). Multinomial expansion refers to the expansion of a positive integral power of a multinomial. Specifically we will consider the expansion of $(a_1 + a_2 + \ldots a_m)^n$. For the expansion we can use the same technique as we use for the binomial expansion. We consider the nth power of the multinomial as the product of n factors each of which is the multinomial. Every term in the expansion can be obtained by picking one symbol from each factor and multiplying them. Clearly any term will be of the form $a_1^{r_1}a_2^{r_2} \ldots a_m^{r_m}$ where $r_1, r_2, \ldots$, are non-negative integers adding to n. Such a term is obtained by selecting a_1 from r_1 factors, a_2 from r_2 factors from among the remaining $(n - r_1)$ parentheses, and so on. This can be done in $C(n, r_1) \cdot C(n - r_1, r_2) \cdot C(n - r_1 - r_2, r_3) \ldots C(n - r_1 - r_2 - \ldots - r_{m-1}, r_m)$ ways. This is easily seen to simplify to $\frac{n!}{r_1!r_2!\ldots r_m!}$. Thus we have shown that the multinomial expansion is

$$(a_1 + a_2 + \ldots + a_m)^n = \sum \frac{n!}{r_1!r_2!\ldots r_m!} a_1^{r_1}a_2^{r_2} \ldots a_m^{r_m}$$

Where the summations is over all non-negative integers $r_1, r_2, \ldots r_m$ adding to n.

4.6.1 A Notation for Multinomial Coefficients

We saw that the coefficient of $a_1^{r_1}a_2^{r_2} \ldots a_m^{r_m}$ in the expansion of $(a_1 + a_2 + \ldots + a_m)^n$ is $\frac{n!}{r_1!r_2!\ldots r_m!}$. This is called a multinomial coefficient in analogy with the binomial coefficient. We have a convenient notation to represent the multinomial coefficient as $C(n; r_1, r_2, \ldots, r_m)$. This is also represented by many authors as $\binom{n}{r_1, r_2, \ldots r_m}$.

Example 15: What is the coefficient of $x^2y^2z^2t^2u^2$ in the expansion of $(x + y + z + t + u)^{10}$?

Solution: Clearly the coefficient is C (10: 2, 2, 2, 2, 2) = $10!/(2!)^5$.

* * *

Example 16: What is the sum of the coefficients of all terms in the expansion of $(a + b + c)^7$?

Solution: The required answer is

$$\sum \frac{7!}{r!s!t!}$$

Where the summation is over all non-negative integers r, s, t adding to n. But it is also the value of

$$\sum \frac{7!}{r!s!t!} a^r b^s c^t$$

evaluated at a = b = c = 1. Thus answer is $(1 + 1 + 1)^7 = 3^7$.

* * *

E14) Write expansion of $(a + b + c)^4$ in full.

E15) What is the value of $\sum \frac{8!}{r!s!t!} 2^r 3^s 4^t$ where the summation is over all r, s, t, non-negative integers adding to 8?

E16) Show that

$C(n, r_1)\, C(n - r_1, r_2)\, C(n - r_1 - r_2, r_3) \ldots C(n - r_1 - r_2 - \ldots - r_{m-1}, r_m) =$

$\frac{n!}{r_1! r_2! \ldots r_m!}$ if $r_1 + r_2 + \ldots + r_m = n$.

4.7 APPLICATIONS TO COMBINATORIAL PROBABILITY

Historically, counting problems have been closely associated with probability. The probability of getting at least 6 heads on 10 flips of a fair coin, the probability of finding a defective bulb in a sample of 25 bulbs if 5 percents of the bulbs from which the sample was drawn are defective — all these probabilities are essentially counting problems. The famous Pascal's triangle for binomial coefficients discussed in section 4.5.1 was developed by Pascal around 1650 while analysing some gambling probabilities.

4.7.1 Elements of Classical Probability Theory

Suppose we have a finite set X with N elements. The collection of all subsets of X is represented by $\mathcal{P}(X)$ or simply by $\mathcal{P}$. The elements of $\mathcal{P}$ are called events. The null set Ø is called the impossible event, and the set X itself is called the sure event. Let us represent the number of elements of a finite set A, also referred to as the cardinality of A, by n (A).

Definition: If by some random mechanism we can ensure that all the n (X) cases are equally likely, which merely means that there is nothing to prefer one cases for the other, then, the probability of the event A in $\mathcal{P}$, represented by P (A) is the ratio $\frac{n(A)}{n(X)} = \frac{n(A)}{N}$, defined by the French mathematician Laplace.

Note that the assumption of 'equally likely' is fundamental in the Classical Probability Theory. Usually this is ensured by considering experiments in which all the n (X) cases are given equal possibility for selection. An experiment is a clearly defined procedure that produces one of a given set of outcomes. These outcomes (n (X) cases) are called elementary events and the set of all elementary events is called the sample space of the experiment. When we consider coin tossing we assume that the coin is unbiased, which merely means that head and tail are equally likely in a toss. The toss itself is considered a random mechanism ensuring 'equally likely' outcomes. There are coins that are 'loaded', which means that one side of the coin may be heavier than the other. Such coins are excluded from our purview. Probability theory has been developed to consider even cases where X is a infinite set. But we will not consider those cases. In the absence of any statement about the 'equally likely' case we always assume it to be the case.

Some Consequences: As n (Ø) = 0 it follows that P (Ø) = 0. By definition n(X) = N and hence P (X) = 1. If A and B are two events, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ implies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. In particular, if A and B are mutually exclusive (which means that A and B have no common elements), then $P(A \cup B) = P(A) + P(B)$. Note that $A \cup B$ can be interpreted as atleast one of the events from A and B. Thus we have proved.

4.7.2 Addition Theorem in Probability

If A and B are two mutually exclusive events then the probability of their union is the sum of the probabilities of A and B.

Corollary: Suppose A is an event. Then the probability of A^c (also denoted by A'), the event complementary to A, or the event 'not A' is 1 – P (A).

The reason is that the events A and A^c are mutually exclusive and exhaustive and hence $A \cup A^c = X$ and $P(A) + P(A^c) = 1$. It is easily seen by a similar argument that if the events $A_1, A_2, \ldots, A_m$ are pairwise disjoint (mutually exclusive), then the probability of the union of A_i's is the sum of the probabilities of A_i's. This is the generalised addition theorem in probability. The subject matter of Combinatorial Probability Theory is the computation of probabilities of events in finite sets, where all elements are equally likely. The probabilities of events are solely determined by the cardinalities of the events and the cardinality of the master set X. The difficulty in the calculation of probability is then merely the difficulty in calculating the cardinality of the events. Events are usually described by certain properties of some points in X and determining the event and its cardinality can be quite difficult at times.

Example 17: A die is rolled once. What are the probabilities of the following events (i) even number (ii) atleast 2 (iii) atmost 2 (iv) atleast 10?

Solution: If we call the events as A, B, C, and D, then we have X = {1, 2, 3, 4, 5, 6}, A = {2, 4, 6}, B = {2, 3, 4, 5, 6}, C = {1, 2}, and D = { }. Hence n (X) = 6, n (A) = 3, n (B) = 5, n (C) = 2, n (D) = 0 leading to the answers P (A) = 3/6, P (B) = 5/6, P (C) = 2/6, P (D) = 0.

* * *

Example 18: A coin is tossed twice. What is the probability of getting atleast one head?

Solution: In this case X can be described by

{{H, H}, {H,T}, {T, H} {T, T}}

For example the pair {H, T} represents the case when head occurs in the first toss and tail in the second. In our problem the event A consists of the cases

{H, T}, {T, H}, {H, H}.

Thus n (A) = 3, n (X) = 4. Hence P (A) = 3/4.

* * *

Example 19: A coin is tossed n times. What is the probability of getting exactly r heads?

Solution: If H and T represent head and tail respectively, then X consists of sequences of length n that can be formed using only the letters H and T. Clearly n (X) = 2^n. The event A consists of those cases in which there are precisely r H's. Obviously n (A) = C (n, r). Hence the required probability is C (n, r)/2^n.

* * *

Example 20: What is the probability of getting a total of 7 when two dice are thrown?

Solution: If x and y denote the numbers that come up on the two dice, then clearly X consists of 36 pairs (x, y) where x and y can take any value from 1 to 6. The required event A of getting total of 7 consists of the 6 cases

(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)

Thus n (A) = 6, n (X) = 36 and hence P (A) = n (A)/n (X) = 6/36 = 1/6.

* * *

Example 21: Two dice, one red and one white, are rolled. What is the probability that the white die turns up a smaller number than the red die?

Solution: As in the previous example, if the number on the red die is x and that on the white is y, then X consists of the 36 pairs (x, y), where x and y can be any integer from {1, 2, 3, 4, 5, 6}. For the event A we need $x < y$. Clearly for x = 1, 2, 3, 4, 5, y can be $x + 1, x + 2, \ldots 6$, i.e., $6 - x$ in number. Thus

$$n(A) = \sum_{x=1}^{5} (6 - x) = 5 + 4 + 3 + 2 + 1 = 15$$

by the addition principle. Hence P (A) = 15/36 = 5/12.

* * *

Example 22: If a five digit number is chosen at random, what is the probability that the product of the digits is 20?

Solution: X is the collection of all 5 digit numbers. If any one of them is abcde, than a can be from 1 to 9. But b, c, d, e can be from 0 to 9. Thus by the multiplication principle $n(X) = 9.10^4 = 90000$. For the elements of A we need to have a. b. c. d. e = 20. Clearly 20 can be factored in only two ways, viz., (i) 1.1.1.4.5 and (2) 5.2.2.1.1 as the product of five factors. Of course the numbers can be permuted to give all possible cases for A. The numbers 5, 4, 1, 1, 1 can be permuted in 5!/1!1!3! = 20 ways and the numbers 5, 2, 2, 1, 1 can be permuted in 5!/1!2!2! = 30 ways. So, n (A) = 20 + 30 = 50, giving P (A) = 50/90000 = 1/1800.

* * *

There are several other methods for solving combinatorial problems which will be taken up in the next two units. Let us now summarise what we have covered in this unit.

4.8 SUMMARY

In this unit we have discussed the nature of combinatorics. Specifically we covered the following:

1. Did some problems involving addition principle and multiplication principle without explicitly naming these principles.
2. Introduced multiplication principle and did some problems using it.
3. Introduced addition principle and did some problems using it.
4. Defined permutations and derived formula for calculating them.
5. Did some numerical problems on permutations.
6. Introduced circular permutations.
7. Introduced the idea of permutations of objects not necessarily distinct.
8. Introduced the concept of combinations and derived a formula for calculating the number of combinations.
9. Did some problems in combinations.
10. Derived a formula for combination with repetition.
11. Derived the formula for binomial expansion and did some problems using it.
12. Introduced Pascal's formulae and Pascal's triangle for binomial coefficients.
13. Extended the idea of binomial expansion to multinomial expansion.
14. Introduced the classical combinatorial probability.
15. Derived addition theorem of probability.
16. Did a number of numerical problems in probability.

4.9 SOLUTIONS/ANSWERS

E1) 15!/12! = 15.14.13.12!/12! = 15.14.13 = 2730.

E2) $(3 + 5)! = 7! = 5040$. But $3! + 4! = 6 + 24 = 6 + 24 = 30$. Obviously the two numbers are not equal.

E3) $(m + n)! = (m + n)(m + n - 1) \ldots (m + 1)\, m!$.

$(m + n)! - m! = m![(m + n)(m + n - 1) \ldots (m + 1) - 1] \geq m!\,[n! + m^n - 1]$

$(m + n)! - m! - n! \geq m![n! + m^n - 1] - n! = n!\,(m! - 1) + m!\,(m^n - 1) \geq 0$.

E4) $\dfrac{n!}{(n-r)!}$ for $n = 20$ and $r = 3$ is $20!/17! = 20.19.18 = 6840$.

E5) Suppose we number the men as 1, 2, 3, ..., n. Then the first man can be paired with any from n, the second can be paired with any from the remaining $(n - 1)$ women, and so on. Hence the number of ways of pairing is $n(n - 1) \ldots 1$.

E6) By multiplication principle the answer is 26.25.24 if the letters cannot be repeated and 26.26.26 if the letters cannot be repeated.

E7) By multiplication principle the number of integers between 100 and 999 with all digits even is $4.5.5 = 100$ (note that the first one cannot be zero, but the second and third digits can be 0 too)

E8) For a number to be odd the last digit should be odd. The last position can be filled up in 5 ways. If the second position is filled up by 0, then the first position can be filled up in 8 ways. Thus the number of odd numbers with 0 in the middle position and all digits distinct is 40, by the multiplication principle. If the second position is filled up by a digit other than 0 then it can be done in 8 ways. Then the first position can be filled up in 7 ways, so, the number of odd numbers with all digits distinct with the middle digit not zero is $5.8.7 = 280$. Thus by addition principle the answer is $40 + 280 = 320$.

E9) $P(15, 2) = 15.14 = 210$ and $P(7, 3) = 7.6.5 = 210$.

$P(5, 5) = 5.4.3.2.1 = 120$ and $P(6, 3) = 6.5.4 = 120$.

E10) We will divide the required numbers into two classes. Class I consists of those numbers with first digit 6. Class II consists of those numbers that have first digit greater than 6. In class I the number of elements is 1.4.6.5.4 (the first digit is chosen in 1 way the second can be only from 5, 7, 8, 9, the third in 6 ways, etc). The number of elements in class I is thus 480. In class II we have $3.7.6.5.4 = 2520$. Thus by addition principle the required answer is $480 + 2520 = 3000$.

E11) a) In the word 'ASSESSES' we have A once, E twice, and S five times. Thus the number of permutations is

$8!/1!2!5! = 8.7.6/2 = 168$.

b) In the word 'PATTIVEERANPATTI' R, N and V occur once, P, E and I occur twice, A thrice and T four times. Thus the required number of permutations is $16!/1!1!1!2!2!2!3!4! = 455.11!$.

E12) The left side counts the ways to select a group of m people chosen from a set of n people and then select a subset of k leaders, say, of this group. Equivalently, the right side counts the ways to select the subset of k leaders from the set of n people first and then select the remaining $m - k$ member of the group from the remaining $n - k$ people.

E13) One can prove this by induction (by inducting on the variable n). The base case is trivial, since if $n = 0$, then $k = 0$ as well, and the equation reduces to $C(0, 0) = C(1, 1)$, which is true. The induction step is proved by pascal's formula/identity and the induction hypothesis

E14) $(a + b + c)^4 = (a^4 + b^4 + c^4) + 4(a^3b + ab^3 + a^3c + ac^3 + b^3c + bc^3) + 6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + ab^2c + abc^2)$. The coefficients 1, 4, 6, 12 are precisely $4!/4!0!0!$, $4!/3!1!0!$, $4!/2!2!0!$, $4!/2!1!1!$, the multinomial coefficients

E15) Clearly $\sum \dfrac{8!}{r!s!t!} 2^r 3^s 4^t$ is the expansion of $(2 + 3 + 4)^8$. Hence the required answer is 9^8.

E16) We can prove this by induction on m. Assume that the result is true for m. Consider the left hand side with (m + 1) factors. By induction the product of the last m factors is $\frac{(n-r_1)!}{r_2!r_3!\ldots r_{m+1}!}$. Thus the left hand side reduces

to $\frac{n!}{r_2!(n-r_1)!}\ \frac{(n-r_1)}{r_2!r_3!\ldots r_{m+1}!}$ and this is equal to $\frac{n!}{r_1!r_2!\ldots r_{m+1}!}$.

This shows that the result is true for (m + 1) also. But the result reduces to expansion for $C(n, r_1)$ when m = 2 and hence true for m = 2. By induction the result is true for all positive integral m > 1.

4.10 MISCELLANEOUS EXERCISES

E1) How many "words" can be formed using the letters of IGNOU (each at most once):

a) if all the five letters must be used;

b) if some (or all) of the letters may be omitted?

E2) In how many ways 52 cards can be arranged as a deck?

E3) How many "words" can be formed using four X's and two Y's?

E4) How many committees of four people can be chosen from a club with ten members?

E5) If Bill wants to take two maths courses and two history courses, and there are five suitable maths courses and four suitable history courses available, in how many ways can he choose the four courses?

E6) How many words can be formed using all the letters of MISSISSIPPI?

E7) Find C (20, 3), C (10, 2) and C (10, 8).

E8) Find C (15; 5, 3, 2, 5) and C (15; 5, 5, 3, 2).

E9) If a bakery has five kinds of cookies, in how many ways can a dozen be chosen?

E10) Jack has six toys and wants to trade two toys with Jim, who has eight toys. In how many ways can they trade?

E11) In how many ways can five A's and 7 B's be lined up so that no two A's are adjacent?

E12) The morse code is made up of marks called dots and dashes. "Q", for example, is (--.-). Is it possible to make up such a code so that every letter of the alphabet is represented by at most three marks? At most four?

E13) In how many orders can six people be seated at a round table if one of the people hates one of the other five people and refuses to sit beside him or her?

E14) If a committee of ten people contains four women, in how many ways can a subcommittee of five be chosen if that subcommittee must by law contain atleast one woman?

E15) If a single card is drawn from a standard deck, what is the probability that it is red or a face card?

E16) What is the probability that in eight tosses of a fair coin there will be exactly four heads? At lest four heads?

E17) Among thirty people chosen at random, what is the probability that at least two of them have the same birthday?

E18) In the expansion of $(x + y)^n$ there is a term of the form Ax^5y^m, A a constant. What is A and what is m?

E19) What is the coefficient of x^7 in $(2 + 3x)^{10}$?

E20) Prove the following identities by writing out both sides in factorials and simplifying.

(a) $\frac{n+1}{r+1} C(n, r) = C(n+1, r+1)$

(b) $C(n, m) \cdot C(m, r) = C(n, r) \cdot C(n-r, m-r)$.

E21) Show that $\sum_{r=0}^{n} C(n, r) \cdot C(m, k+r) = C(n+m, n+k)$.

E22) What is the coefficient of x^4 in the expansion of $(1 + x + 2x^2)^5$?

E23) Use binomial theorem to find $\sum C(n, r) k^r$ an arbitrary number.

E24) In a ten-question true-false exam, a student must achieve six correct answers to pass. If she selects his answers randomly, what is the probability that she will pass?

4.11 SOLUTIONS TO MISCELLANEOUS EXERCISES

E1) (a) $P(5, 5) = 5! = 120$. (b) As the number of r-lettered words is $P(5, r)$ the answer is

$\sum_{r=0}^{5} P(5, r) = 1 + 5 + 5.4 + 5.4.3 + 5.4.3.2 + 5.4.3.2.1 = 326$. Note that this includes a null word.

E2) Answer is $P(52, 52) = 52!$. This is a huge number.

E3) Answer is $\frac{6!}{2!4!} = 15$.

E4) The answer is $C(10, 4) = \frac{10.9.8.8}{4.3.2.1} = 210$.

E5) The answer is $C(5, 2). C(4, 2) = 60$.

E6) The answer is $\frac{11!}{1!2!4!4!} = 35650$.

E7) The answer is $C(20, 3) = \frac{20.19.18}{3.2.1} = 1140$. $C(10, 2) = \frac{10.9}{2.1} = 45$.
$C(10, 8) = C(10, 2) = 45$.

E8) The answer is $C(15; 5, 3, 2, 5) = \frac{15!}{5!3!2!5!}$ and $C(15; 5, 5, 3, 2)$ is same as $C(15; 5, 3, 2, 5)$, the previous answer.

E9) If the cookies are called A, B, C, D, E, the problem is to choose 12 letters from them taking any of them any number of times. Clearly it is the coefficient of x^{12} in $(1 + x + x^2 + \ldots)^5$. It is the coefficient of x^{12} in $(1-x)^{-5}$ and it is $C(9, 5) = 126$.

E10) The answer is $C(6, 2). C(8, 2) = 15.28 = 420$

E11) When we line up 5 A's there are 6 gaps within which each A lies. Tp keep them separate we should put respectively a, b, c, d, e, f, g B's with the restriction $a \geq 0$, $b, c, d, e \geq 1$, $f \geq 0$, and $a + b + c + d + e + f = 7$. The answer is the coefficient of x^7 in
$(1 + x + x^2 + \ldots)(x + x^2 + \ldots)^4 (1 + x + x^2 + \ldots)$. It is the coefficient of x^3 in $(1 + x + x^2 + \ldots)^3 = (1-x)^{-3}$ The coefficient is $C(5, 3) = 10$.

E12) With 3 marks we can form $2 + 4 + 8 = 14$ letters only. With 4 marks we can form $2 + 4 + 8 + 16 = 30$. As we have only 26 letters 4 marks should suffice.

E13) The answer is $5! - 2.4! = 120 - 48 = 72$.

E14) The answer is $C(10, 5) - C(6, 5)$.

E15) There are 26 red cards. Of the remaining the face cards are 8 (2 A's, 2 K's, 2 Q's and 2 J's). Thus there are 34 cards favourable to our event. Hence the probability is 34/52 = 17/26.

E16) Probability of exactly 4 heads is $C(8, 4)/2^8 = 35/128$. Probability of atleast 4 heads is $[C(8, 4) + C(8, 5) + C(8, 6) + C(8, 7) + C(8, 8)] / 2^8$. It is $(70 + 56 + 28 + 8 + 1)/2^8 = 163/256$.

E17) There are possible 365 birthdays (assuming a non-leap year). The complementary event is that all the 30 will have different birthdays. Probability of this is $C(365, 30)/365^{30}$. This comes to more than 0.5.

E18) As every term has degree n in x, y, m should be (n – 5). Coefficient of x^5y^{n-5} in $(x + y)^n$ is $C(n, 5)$. Thus $A = C(n, 5)$.

E19) The term containing x^7 in $(2 + 3x)^{10}$ is $C(10, 3) \cdot 2^3 \cdot (3x)^7$. Hence the answer is $C(10, 3).8.3^7$.

E20) (a) $$\frac{n+1}{r+1} \cdot \frac{n!}{r!\,(n-r)!} = \frac{(n+1)!}{(r+1)!\,(n+1-r-1)!} = C(n+1, r+1)$$

(b) LHS $$\frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!} = \frac{n!\,(n-r)!}{r!\,(n-r)!\,(m-r)!\,(n-m)!} = \text{RHS}$$

E21) LHS is the coefficient of x^{-k} in the expansion of $(1 + x)^n \cdot \left(1 + \frac{1}{x}\right)^m$ equal to coefficient of x^{m-k} in the expansion of $(1 + x)^{m+n}$ which is $C(n + m, m - k) = C(n + m, n + k)$.

E22) $(1 + x + 2x^2)^5 = \sum \frac{1^r x^s (2x^2)^t 5!}{r!s!t!}$ by multinomial expansion. x^4 occurs for the following cases with coefficients. (a) r = 3, s = 0, t = 2. This gives 5!4/(3!0!2!) = 40. (b) r = 2, s = 2, t = 1. This gives 5!2/(2!2!1!0) = 60. (c) r = 1, s = 4, t = 0. This gives 5!1/(4!1!0!) = 5. The required answer is 40 + 60 + 5 = 105.

E23) The answer is $(k + 1)^n$.

E24) The answer is same as the probability of getting atleast 6 heads in 10 tosses of a true coin. Hence the answer is $C(10, 6)/2^{10} + C(10, 7)/2^{10} + C(10, 8)/2^{10} + C(10, 9)/2^{10} + C(10, 10)/2^{10}$. It simplifies to (210 + 120 + 45 + 10 + 1)/1024 = 193/512.

UNIT 5 PARTITIONS AND DISTRIBUTIONS

Structure Page No.

5.1 INTRODUCTION

In this unit we will be mainly discussing partition of a natural number, partition of a n-set and counting the number of ways of distributing a finite number of objects into a finite number of containers, usually called boxes. The objects themselves may be described as balls. The counting depends upon two things.

(1) The balls are distinguishable or indistinguishable.
(2) The containers may be distinguishable or indistinguishable.

We will see them in detail in this unit. In the process we have introduced Stirling numbers of first and second kind and Bell numbers. We have also introduced some recurrence relations and generating functions concerning partitions and these numbers.

Objectives

After reading this unit you should be able to:

- know what an integer partition is and how to count the number of partitions of an integer;
- understand the problems involved in distribution of objects in containers;
- count the number of ways of distributing distinguishable objects into distinguishable containers;
- count the number of ways of distributing distinguishable objects into indistinguishable containers;
- count the number of ways of distributing indistinguishable objects into distinguishable containers;
- count the number of ways of distributing indistinguishable objects into indistinguishable containers;
- calculate Stirling numbers of second kind;
- calculate Bell numbers;
- calculate Stirling number of first kind.

5.2 INTEGER PARTITIONS

Suppose S is a set with n objects. Any collection of non-empty, disjoint subsets of S with union S is called the partition of S. For example, if S = {a, b, c, d}, then {{a, b}, {c}, {d}} is a partition of S, {{c, d}, {a}, {b}} is another partition of S. Whenever we consider a set, the elements of the set are considered distinct. If some elements repeat, then the collection is no longer a set but a multiset. Consider a multiset with n elements: single element repeated n times. How can we define its partition? If S = {a, a, a, a} then the partitions {{a, a}, {a},{a}} and {{a}, {a}, {a, a}} are identical as order is not relevant in a collection. It is clear that a partition of a multiset is completely determined if the number of elements in each subcollection forming the partition is known. Moreover the order in which these numbers are given is totally irrelevant. This brings us to the definition of the partition of a positive integer. Any representation of n as a sum of positive integers in non-increasing order is called a partition of n i.e. We consider the partitioning of a positive integer n into positive summands and seek the number of such partitions, without regard to order. Since order is to be ignored, we have followed the convention of writing the summands in non increasing order. For example the partitions of 5 are (a) 5, (b) 4 + 1, (c) 3 + 2, (d) 3 + 1 + 1, (e) 2 + 2 + 1, (f) 2 + 1 + 1 + 1, and (g) 1 + 1 + 1 + 1 + 1. If P_n represents the number of partitions of the integer n, then we have shown that $P_5 = 7$. In any partition of n the numbers constituting the sum are called the parts. For example, in 2 + 2 + 1, the parts are 2, 2, and 1. It has three parts. Among the partitions of 5, we have 1 with 1 part, 2 with 2 parts, 2 with 3 parts, 1 with 4 parts, and 1 with 5 parts. The number of partitions of n with exactly k parts is represented by P_n^k. Thus we have $P_5^1 = 1$, $P_5^2 = 2$, $P_5^3 = 2$, $P_5^4 = 1$, $P_5^5 = 1$.

5.2.1 Recurrence Relation for P_n^k

First of all let us define what we mean by a recurrence relation.

Definition: Let $\{a_n: n \geq 0\}$ be a sequence of real or complex numbers. A recurrence relation is simply an expression of the form $a_n = F(a_{n-1}, a_{n-2}, \ldots n)$, where F is any function of the variables $a_{n-1}, a_{n-2}, \ldots$, and n. In other words, it permits/allows us to compute the nth term of a sequence from one or more of the preceding terms. We shall mainly deal with such function F which are polynomials and depend on only finitly many variables $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$, and n. Further discussion on recurrence relation has been taken up in Block 3.

Theorem 1 We have

$$P_n^1 + P_n^2 + \ldots + P_n^k = P_{n+k}^k,$$

$$P_n^1 = P_n^n = 1.$$

Proof: The second formula is obvious from the definition. We will prove the first formula. Let M be the set of partitions of n having k or less parts; each partition belonging to M may be considered as a k–tuples. Define on M the mapping

$$(p_1, p_2, \ldots, p_m, 0, 0, \ldots, 0) \mapsto (p_1 + 1, p_2 + 1, \ldots, p_m + 1, 1, 1, \ldots, 1).$$

M is mapped into the set M′ of partitions of n + k into exactly k parts. This mapping is bijective, since (1) two distinct k–tuples of M are mapped onto two distinct k–tuples of M′, (2) every k–tuple of M′ is the image of a k–tuple of M. Therefore,

$$|M| = P_n^1 + \ldots + P_n^k = |M'| = P_{n+k}^k.$$

From these formulas the P_n^k's may be calculated recursively, e.g.,

P_n^k	k = 1	2	3	4	5	6
n = 1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	2	1	1	0	0
5	1	2	2	1	1	0
6	1	3	3	2	1	1

Let Q_n^k denote the number of partitions of n with k or fewer parts. Clearly $Q_n^1 = 1$ for every n, $Q_n^n = P_n$. For the case n = 5 we have $Q_5^1 = 1$, $Q_5^2 = 3$, $Q_5^3 = 5$, $Q_5^4 = 6$, and $Q_5^5 = 7$. Let $P_n(k)$ denote the number of partitions of n with no part larger than k. We can think of $P_n(k)$ as a function of two variables, n and k. Then we have $P_5(1) = 1$, $P_5(2) = 3$, $P_5(3) = 5$, $P_5(4) = 6$, and $P_5(5) = 7$. Clearly for any n, we must have $P_n(n) = P_n$. In the example given, peculiarly, we have $P_5(k) = Q_5^k$ for every k. Is it true in general that $P_n(k) = Q_n^k$ for every k and n? To show that this result is true we represent a partition by its Ferrer's graph.

5.2.2 Ferrer's Graph

Suppose the parts of a partition are $s_1, s_2, \ldots, s_m$. Then the Ferrer's graph of the partition consists of m rows of dots, the first row with s_1 dots, the second row with s_2 dots, and so on. This graph uses rows of dots to represent a partition of an integer where the number of dots per row does not increase as we go from any row to one below it. In each row the dots are left-justified. For example, the partition 5 + 4 + 3 + 2 of 14 is represented by its Ferrer's graph as follows. (Fig. (a)):

14 = 5 + 4 + 3 + 2   14 = 4 + 4 + 3 + 2 + 1

(a)   (b)

Fig. 1

Clearly, if we change the rows into columns in a Ferrer's graph we get the Ferrer's graph of another partition of the same number. The new partition so obtained is called the conjugate partition. Clearly, for every partition there is a unique conjugate. Moreover, the conjugate of the conjugate partition is the original partition. But the size of the largest part in a partition is the number of parts in the conjugate partition. Thus we have a one-one correspondence between the partitions of an integer with no part larger than k and the partitions of n with almost k parts. Thus $P_n(k) = Q_n^k$. Thus we have proved.

Theorem 2 For any two integers n, k, $k \le n$, the number of partitions of n with at most k parts is equal to the number of partitions with no part larger than k.

E1) Evaluate P_1, P_3, and P_5

E2) Evaluate P_n^1 and Q_n^1.

E3) Evaluate Q_5^2, Q_6^2 ..., in general, Q_n^2

E4) Show that $P_n^n = P_n^{n-1} = 1$.

5.2.3 A Recurrence Relation for the Number of Partitions

We are going to see, how $P_n(k)$ depends on values of P will smaller arguments where both n and k are viewed as arguments.

Theorem 3 For any positive integers n and k, $1 < k < n$ we have

$$P_n(k) = P_n(k-1) + P_{n-k}(k)$$

Proof: $P_n(k)$ counts the number of partitions of n with parts not larger than k. We can group these partitions into two classes. (i) those having k as a part; (ii) those not having k as a part. The partitions of type (ii) are clearly $P_n(k-1)$ in number. In partition of type (i) clearly one part is k and hence the other parts constitute a partition of n − k with no part larger than k, and hence must be $P_{n-k}(k)$. Adding the two numbers we obtain a proof of the theorem.

Note that the above recurrence relation can be used to get the value of $P_n(k)$ for any combination of n and k, if we note that $P_n(1) = 1$, since $n = 1 + 1 + \ldots + 1$ (n times) and $P_n(k) = P_n(n)$, if $k > n$. For example, to calculate $P_6(4)$ we use the recurrence relation repeatedly to get, $P_5(4) = P_6(3) + P_2(4)$. But

$$P_2(4) = P_2(2) = 2$$
$$P_6(3) = P_6(2) + P_3(3) = P_6(1) + P_4(2) + P_3(3) = 1 + 3 + 3 = 7, \text{ giving}$$
$$P_6(4) = 9.$$

Note that $P_n(k) - P_n(k-1) = Q_n^k - Q_n^{k-1}$. This means that the number of partitions in which the largest part is k is equal to the number of partitions of n with exactly k parts. Thus we have proved.

Theorem 4 For every n, k, $k \leq n$, the number of partitions of n with exactly k parts is equal to the number of partitions of n with k as the largest part.

Example 1: If the conjugate partition of a partition is itself then it is said to be self-conjugate. Exhibit a self-conjugate partition of 6.

Solution: $6 = 3 + 2 + 1$. It is easily seen that it is self-conjugate.

* * *

Example 2: Show that if a partition of n is $p_1 + p_2 + \ldots p_k$, then its conjugate partition is $q_1 + q_2 + \ldots q_r$, where r is p_1 and q_i is the number of p_j's that are at least i.

Solution: If we construct the Ferrer's graph for the partition then the statements made in the problem are transparently seen.

* * *

Example 3: Show that a number of the form $n(n+1)/2$ always has a self-conjugate partition.

Solution: We can write $n(n+1)/2$ as $1 + 2 + 3 + \ldots + n$. When the order of this is reversed it is a partition of the number $n(n+1)/2$. Evidently the partition is self conjugate.

* * *

Example 4: Exhibit the partitions of the numbers from 2 to 8 which have only the numbers 1 and/or 2 as parts.

Solution: For 2 the partitions are 1 + 1 and 2. For 3 the partitions are 1 + 1 + 1 and 2 + 1. For 4 the partitions are 1 + 1 + 1 + 1, 2 + 1 + 1, and 2 + 2. For 5 the partitions are 1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1, and 2 + 2 + 1. For 6 the partitions are 1 + 1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1, 2 + 2 + 1 + 1, and 2 + 2+ 2. For 7 the partitions are 1 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 1 + 1+ 1 + 1 + 1, 2 + 2 + 1 + 1 + 1, and 2 + 2 + 2 + 1. For 8 the partitions are 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1, and 2 + 2 + 2 + 2.

* * *

Example 5: How many partitions of 2n are there which have only the numbers 1 and/or 2 as parts?

Solution: The maximum number of parts that are 2 is clearly n for the number 2n. Hence the number of required partitions is n + 1.

* * *

E5) How many partitions of 2n + 1 are there which have only the numbers 1 and/or 2 as parts.

E6) How many partitions of 2n are there which have only one or two not necessarily distinct parts?

5.2.4 Generating Function for P_n's

Let us define Ordinary Generating functions :

Definition. The generating function for the sequence of real (or complex) numbers, $\{a_n\}_{n=0}^{\infty}$ is given by the expression $A(x) = \sum_{k=1}^{\infty} = a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \ldots$ – a formal power series in x.

More about this have been discussed in unit 8 (Block 3). The series $P(x) = \sum_{n=1}^{\infty} P_n x^n$ is called the generating function for P_n's. The reason for this is that the coefficient of x^n in this series is the number P_n.

Theorem 5 The generating function for P_n's is $(1-x)^{-1}.(1-x^2)^{-1}(1-x^3)^{-1}\ldots$

Proof: Consider the product

$$(1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + x^9 + \ldots)\ldots$$

In this product we can find the coefficient of x^n as follows. Take one term each from each parenthesis and multiply to get x^n. The term from the first parenthesis gives the number of 1's in its exponent, the term from the second gives the number of 2's in its exponent and so on, in a partition of n. We can thus get all the partitions of n by considering the coefficient of x^n in it. This proves our result.

Calculation of P_n's for small values of n:

Suppose we want to calculate P_n for $n \le 6$. It is enough to retain relevant parts of the generating function as we need only coefficient of x^n for $n \le 6$. The relevant part is as follows:

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)(1 + x^2 + x^4 + x^6)(1 + x^3 + x^6)(1 + x^4)(1 + x^5)(1 + x^6)$$

We multiply this out and retain only powers of x at most 6. From the first two parentheses the required terms are $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6$. From the last four parentheses the required terms are $1 + x^3 + x^4 + x^5 + 2x^6$. Multiplying now the two series obtained and retaining only terms upto x^6 we get $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6$. Thus we get the following:

$$P_0 = 1,\ P_1 = 1,\ P_2 = 2,\ P_3 = 3,\ P_4 = 5,\ P_5 = 7,\ P_6 = 11.$$

Example 6: Show that the generating function for the number of partitions of n in which every part is at least 2 is $(1-x)\,P(x)$. Hence show that the number of partitions of n in which every part is at least 2 is $P_n - P_{n-1}$.

Solution: Clearly $(1-x)\,P(x) = (1-x^2)^{-1}(1-x^3)^{-1}\ldots$. The RHS clearly gives in the coefficient of x^n those partitions of n in which each part is at least is 2. This proves the result. Coefficient of x^n on RHS is coefficient of x^n on $(1-x)\,P(x)$ and it is clearly $P_n - P_{n-1}$

* * *

The generating function for $P_n^{(d)}$'s, the number of ways to express n as a sum of district integers, is given by

$$P^{(d)}(x) = (1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^k)\ldots$$

$$= \prod_{i=1}^{\infty}(1 + x^i) = \frac{(1-x^2)}{(1-x)}\cdot\frac{(1-x^4)}{(1-x^2)}\cdot\frac{(1-x^6)}{(1-x^3)}\cdot\frac{(1-x^8)}{(1-x^4)}$$

$$= \frac{1}{(1-x)}\cdot\frac{1}{(1-x^3)}\cdot\frac{1}{(1-x^5)}\ldots$$

The generating function for $P_n^{(0)}$s, the number of ways to express n as a sum of odd integers, is given by

$$P^{(0)}(x) = (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)\times$$

$$(1 + x^5 + x^{10} + \ldots)(1 + x^7 + x^{14} + \ldots)\ldots$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7} \cdots\cdots$$

Since $P^{(d)}(x) = P^{(o)}(x)$ we have $P_n^{(d)} = P_n^{(o)}$.

You will come across further discussion and problems concerning partition of integer in unit 8 (Generating functions) of Block 3.

5.3 DISTRIBUTIONS

By distribution we mean the distribution of several objects into several containers. In a colourful representation we talk about distributing balls among boxes. Here there are possible cases.

1 The balls could all be distinguishable and the boxes all distinguishable.

2. The balls could be distinguishable and the boxes indistinguishable.

3. The balls could be indistinguishable and the boxes distinguishable.

4. The balls could be indistinguishable and the boxes too indistinguishable.

In each of these four cases we will be counting the number of such distributions. It is quite possible that there are mixed cases apart from these four 'pure' cases. Wherever possible we will mention them and treat them with the methods developed for these for cases.

A general guideline for modelling distribution problem is, distribution of distinct objects corresponds to arrangement and distribution of identical objects corresponds to selection. We will give illustrative examples to cover the four cases.

(A) There are twenty students and four colleges. In how many ways the students could be accommodated in the four colleges?

In this example the students are clearly distinguishable and the colleges are also distinguishable. This comes under case (1).

(B) An employer wants to distribute 100 one-rupee notes among 6 employees. Find the number of ways of doing this.

Though the one-rupee notes can be distinguished by their distinct numbers, we don't consider them distinguishable as far as their use is concerned. Hence this is a case of distributing indistinguishable objects among distinguishable boxes. Here the employees, considered as distinguishable, are the 'boxes'. This falls under case (3).

(C) Suppose we want to group 100 student into 10 groups of 10 each for the purpose of medical examination. Then the groups as such are indistinguishable though the students in them are distinguishable. Hence this falls under case (2).

(D) There are 1000 one-rupee notes. In how many ways they can be bundled into 20 bundles?

As before the rupee notes are considered indistinguishable. Clearly the bundles are by themselves not distinguishable, only the contents may vary. Hence this falls under case (4).

5.3.1 Distinguishable Objects into Distinguishable Containers

There is a special interpretation for this case. As the objects are distinguishable they can be considered as elements of a set, say O. The containers are also distinguishable forming a set, say C. Any distribution f can be now considered as a mapping from O into C. When there is no restriction on the way the objects are distributed into containers (that is, with any number of objects per container), it is clear that the number of ways of doing this is m^n, where n is the number of objects, the cardinality $|O|$ of the set O, and m is the number of containers, the cardinality of the set C. This follows from the multiplication principle if we note that each object can be distributed into the containers in m ways.

The set of all mappings from a set A into a set B is represented by B^A. Thus we have shown that the cardinality of the set B^A is $|B|^{|A|}$

Example 7: Show that the number of words of length n on an alphabet of m letters is m^n.

Solution: Note that the letters of the alphabet can be used any number of times in a word. A word of n letters can be considered as n ordered boxes each holding a letter from the alphabet. The boxes become distinguishable because they 'ordered'. The letters of the alphabet are clearly distinguishable. Thus any word of length n is equivalent to setting up a map from the boxes to the letters. Clearly the number of ways of doing this is m_n. There could be a confusion here. The boxes are taken as objects and the letter of the alphabet are taken as containers. (boxes are not the containers!)

* * *

Example 8: Suppose we have a set S with m objects. An n-sample from this set S is ordered arrangement of n letters taken from S with replacement at every draw in n draws. Show that the number of n-samples from an m-set is m^n.

Solution: Clearly every n-sample is a word of length n from the 'alphabet' S containing m letters. The result now follows from the previous Example.

* * *

5.3.2 Generating Function Approach

Suppose we have two letters {a, b}. If we formally expand $(a + b)^3$ by multiplying one letter taken from each of the three parentheses without changing the order of the factors, then we get the following terms. aaa, aab, aba, abb, baa, bab, bba, bbb. Clearly these are the 3-lettered words from the alphabet {a, b}. Thus the expansion $(a + b)^3$ can be considered as generating all these words. Clearly the number of such words can be calculated by replacing a, and b by 1. This gives $2^3 = 8$. In general, if there are m-letters $\{a_1, a_2, \ldots a_m\}$, then the generating function for all the word of length n from the alphabet of m-letters is $(a_1 + a_2 + \ldots + a_m)^n$ and the number of words can be got by replacing all the letters by 1 in this generating function and this clearly is m^n.

The generating function approach is very useful in enumerating combinatorial objects and counting the number of ways of doing it. As an example consider the following.

Example 9: Find the number of five-lettered words over the alphabet {a, b} in which the second letter is b and the fourth letter is a.

Solution: The generating function for all these words is clearly

$(a + b)\ b\ (a + b)\ a\ (a + b)$. These words can got by formally expanding it preserving the order of factors. The number of such words is got by replacing a, b by 1. Clearly the answer is $2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 = 8$.

* * *

Now here are some exercises for you to solve.

E7) Find the number of three-lettered words that can be formed with the English alphabet with 26 letters. How many of them end in x? How many of them have a vowel in the middle position?

E8) How many five-digit numbers are even? How many five-digit numbers are composed of only odd digits?

E9) There are 4 ladies and 5 gents. A committee of three, a president, a vice-president, and a secretary has to be formed from them. In how many ways this can be done in the following cases.

(a) The vice-president should be a lady?

(b) Exactly one of vice-president or secretary should be a lady?

(c) There is at least one lady in the committee?

5.3.3 Containers with at Most One Object

Suppose we want the distribution of m distinguishable objects into n distinguishable containers with the extra condition that no container should contain more than one object. It is clear that this is impossible if $n < m$. On the other hand if $n \geq m$ then we can get all these arrangements by first choosing m containers to contain exactly one object and then permuting the m objects among the chosen containers. Clearly this can be done in $C(n, m) \cdot m! = n(n-1) \ldots (n-m+1) = P(n, m)$.

Thus we get a new interpretation for $P(n, m)$. Note that $n(n-1) \ldots (n-m+1)$ is also called a falling factorial and is represented by $[n]_m$. If $m > n$, then $[n]_m$ is interpreted as zero. Thus we have proved.

Theorem 6 The number of ways of distributing m distinguishable objects into n distinguishable containers such that no container contains more than one object is $[n]_m$.

E10) Show that the number of m-lettered words with distinct letters over an alphabet with n letters is $[n]_m$.

E11) Show that the number injective mappings from an m-set into an n-set is $[n]_m$.

5.3.4 Distinguishable Objects into Indistinguishable Containers

To get the number of ways distributing n distinguishable objects into m indistinguishable containers we need the number when exactly k of the containers are occupied. This takes us to Stirling numbers of second kind.

5.3.5 Stirling Numbers of the Second Kind

Suppose $n \geq m$. The number of distributions of n distinguishable objects into m indistinguishable containers such that no container is empty is represented by S_n^m. This number is called the Stirling number of the second kind. This is also the number of partitions of a set of n objects into m classes clearly, that is, we define the Stirling numbers of the second kind as follows: For natural numbers n and m, S_n^m is the number of partitions of an n-set into exactly m parts (recall that the parts have to be non-empty).

Clearly $S_n^m = 0$ if $n < m$, for, if the number of containers exceeds the number of objects, then it is impossible to have all the containers non-empty.

It can be shown that $S_n^m = \dfrac{1}{m!} \sum_{k=0}^{m} (-1)^k C(m, m-k)(m-k)^n$

5.3.6 Recurrence Relation for S_n^m

Theorem 7 If $1 < m \leq n$, then $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

Proof: Let us mark one of the $n+1$ objects and consider the partition of the $n+1$ objects into m classes.

Case (1) The marked object forms a class of one element. Then the remaining n objects will form $(m-1)$ classes in S_n^{m-1} ways.

Case (2) The marked object occurs with at least one more element in a class. The number of such partitions is mS_n^m, for we can first form a partition of m classes with the n unmarked objects and then attach the marked object to one of these m classes.

By addition principle we now get $S_{n+1}^m = S_n^{m-1} + mS_n^m$. Note that, by definition, we have $S_n^n = 1$, and $S_n^m = 0$ if $m > n$. Also, trivially we may define $S_n^m = 0$ if $m < 0$ or $n < 0$. With this interpretation of S_n^m it is now easy to see that we have, for $1 \leq m \leq n$, $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

5.3.7 A Generalisation of the Recurrence Relation for Stirling Numbers of Second Kind

Theorem 8 $S^m_{n+1} = \sum_{k=0}^{n} C(n, k).S^{m-1}_k$

Proof: Let us mark one object in a set of (n + 1) objects. Suppose the marked object is present in a class with (n – k + 1) elements. This is possible in $C(n, n-k)\ S^{m-1}_k$ ways. For we can choose n – k more objects to go with the marked object in C (n, n – k) ways. The remaining k objects can be distributed into (m – 1) classes in S^{m-1}_k ways. The result now follows from the addition principle by allowing k to vary from 0 to n.

Note: Though the statement of the previous theorem is formally correct, it is to be remembered that on RHS k can meaningfully take only values $m - 1 \leq k \leq n$, other terms being zero. But the statement of the theorem is simpler than this, in a way.

Stirling Numbers and onto Functions

Suppose N = { 1, 2, ... n} and M = {1, 2, ..., m}. Then the number of onto functions from N to M is precisely $S^m_n m!$. This follows from the fact that if f is an onto function from N to M, then the inverse images, $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)$ constitute partition of N into m classes. The factor m! comes because S^m_n represents only partition, where the order of partition is immaterial, but in functions it cannot be ignored. Concerning this, you may refer to section 6.3.2 of unit 6.

5.3.8 Generating Function for Stirling Numbers of Second Kind

If x is a variable, then an ordinary power of x is x^n for positive integral n. We also have factorial power of x, written $[x]_n$ for any positive integral n, defined by

$$[x]_n = x(x-1)(x-2)\ldots(x-n+1).$$

It is also called a falling factorial. It was Stirling who discovered the relation between ordinary powers of x and the factorial power of x through Stirling numbers.

The Stirling numbers of the first kind are defined as follows: For n a positive integer and $0 \leq k \leq n$, s (n, k) is the coefficient of x^k in the expansion of the "falling factorial" with n factors. That is $[x]_n = x(x-1)(x-2)\ldots(x-n+1) = \sum_{i=0}^{n} s(n, i)\, x^i$.

In fact his result is as follows concerning Stirling number of second kind.

Theorem 9 $x^n = \sum_{j=0}^{n} S^j_n [x]_j$

Proof: If n > 0, let F (N, J) denote the number of onto functions from an n-element set N to the set J. If we add the numbers F (N, J) for all subsets J of a set M, we get the total number of functions from N to M. We will write this as $\sum_{J \subset M} F(N, J)$. But the number of functions from N to M is clearly m^n. Thus we have,

$$m^n = \sum_{J \subset M} F(N, J) = \sum_{j=0}^{m} C(m, j)\, F(N, \{1, 2, \ldots, j\})$$

$$= \sum_{j=0}^{m} C(m, j) j! S^j_n = \sum_{j=0}^{m} S^j_n [m]_j$$

But $[m]_j = 0$, for j > m and $S^j_n = 0$ if j > n. Hence we can write

$$= \sum_{j=0}^{m} S^j_n [m]_j = \sum_{j=0}^{n} S^j_n [m]_j$$

Thus we have proved that $m^n = \sum_{j=0}^{n} S^j_n [m]_j$. Now consider the equation $x^n = \sum_{j=0}^{n} S^j_n [x]_j$. This is polynomial equation in x of degree n. But by our previous proof, this equation is satisfied by x = 0, 1, 2, ... n. But a polynomial equation of degree n in x cannot have more than n roots unless it is an identity. Thus our equation is in fact an identity. Thus we have proved that for all real x,

$$x^n = \sum_{j=0}^{n} S_n^j [x]_j .$$

Example 10: Express x^4 in terms of falling factorials and hence get S_4^m for $m = 0, 1, 2, 3, 4$.

Solution: $x^4 - [x]_4 = x^4 - (x^4 - 6x^3 + 11x^2 - 6x) = 6x^3 - 11x^2 + 6x$

Thus $x^4 - [x]_4 - 6[x]_3 = 7x^2 - 6x$ and hence $x^4 - [x]_4 - 6[x]_3 - 7[x]_2 = x$ or $x^4 = [x]_4 + 6[x]_3 + 7[x]_2 + [x]_1$.

The coefficients are $S_4^0 = 0$, $S_4^1 = 1$, $S_4^2 = 7$, $S_4^3 = 6$, $S_4^4 = 1$. Note that we can also do this by writing $x^4 = a[x]_4 + b[x]_3 + c[x]_2 + d[x]_1$, and determining the constants a, b, c, d by substituting x = 1, 2, 3, 4 successively on both sides.

* * *

E12) Write down the partitions of [1, 2, 3, 4] into two parts. Hence calculate S_4^2.

Example 11: Calculate S_3^2 and S_3^3.

Solution: $S_3^2 = S_2^1 + 2.\ S_2^2 = 1 + 2 * 1 = 3$. We have used here the obvious fact that $S_n^1 = 1$.

$S_3^3 = 1$ as we have $S_n^n = 1$ for every positive n.

* * *

Now we have everything required to prove

Theorem 10 The number of ways of distributing n distinguishable objects into m indistinguishable containers is $S_n^1 + S_n^2 + \ldots + S_n^m$.

Proof: When we distribute n distinguishable objects into m indistinguishable containers there are m cases. Case (k) is exactly k containers are occupied (the rest being empty). Here k can vary from 1 to m. The number of distributions in case (k) is clearly S_n^k. The theorem now follows from addition principle.

5.3.9 Bell Numbers

The number of distribution of n distinguishable objects into n indistinguishable containers is called the nth Bell number (after the American mathematician E.T. Bell) and is represented by B_n. Also B_n is the number of partitions of a set with n elements say of {1, 2, ..., n} (that is, B_n also is the number of different equivalence relations on a set with n elements).

Corollary:

$$B_n = S_n^1 + S_n^2 + \ldots + S_n^n.$$

Example 12: Calculate B_4

Solution: By definition $B_4 = S_4^1 + S_4^2 + S_4^3 + S_4^4$. But $S_4^1 = 1$, $S_4^4 = 1$.

$S_4^2 = 7$, for $S_4^2 = S_3^1 + 2.\ S_3^2 = 1 + 2.\ 3 = 7$. $S_4^3 = 6$, for

$S_4^3 = S_3^2 + 3.\ S_3^3 = 3 + 3.\ 1 = 6$. Thus $B_4 = 1 + 7 + 6 + 1 = 15$.

* * *

Theorem 11 $B_{n+1} \sum_{k=0}^{n} C(n, k).\ B_k$

Proof: Using recurrence relation for S_n^m we get

$$B_{n+1} = \sum_{m=1}^{n+1} S_{n+1}^m = \sum_{m=1}^{n+1} \sum_{k=0}^{n} C(n, k)\, S_k^{m-1} =$$

$$\sum_{k=0}^{n} C(n, k) \sum_{m=1}^{n+1} S_k^{m-1} = \sum_{k=0}^{n} C(n, k) \cdot B_k$$

In the above proof we have used the obvious facts that $S_k^0 = 0$ and $S_k^{m-1} = 0$, if $m - 1 > k$.

Note that we take $B_0 = 1$ by definition in this formula.

Example 13: Calculate successively $B_1, B_2, \ldots, B_6$.

Solution: $B_1 = S_1^1 = 1$.

$B_2 = C(1, 0) \cdot B_0 + C(1, 1) \cdot B_1 = 1 + 1 = 2$.

$B_3 = C(2, 0) \cdot B_0 + C(2, 1) \cdot B_1 + C(2, 2) \cdot B_2 = 1 + 2.1 + 1.2 = 5$

$B_4 = C(3, 0) \cdot B_0 + C(3, 1) \cdot B_1 + C(3, 2) \cdot B_2 + C(3, 3) \cdot B_3 = 1 + 3.1 + 3.2 + 5 = 15$.

$B_5 = C(4, 0) \cdot B_0 + C(4, 1) \cdot B_1 + C(4, 2) \cdot B_2 + C(4, 3) \cdot B_3 + C(4, 4) \cdot B_4 = 1 + 4.1 + 6.2 + 4.5 + 15 = 52$.

$B_6 = C(5, 0) \cdot B_0 + C(5, 1) \cdot B_1 + C(5, 2) \cdot B_2 + C(5, 3) \cdot B_3 + C(5, 4) \cdot B_4 + C(5, 5) \cdot B_5 = 1 + 5.1 + 10.2 + 10.5 + 5.15 + 52 = 203$.

* * *

5.3.10 Indistinguishable Objects into Distinguishable Containers

Suppose there are m indistinguishable objects and n distinguishable containers. As the objects are indistinguishable the distributions depend only on the *number* of objects in the n containers. As the containers are distinguishable they can be assumed to be arranged in a line. Hence the number of distributions is the number of ways of writing the number m as the sum $x_1 + x_2 + \ldots x_n$, where the x_i's are non-negative integers. But the number of distributions can be got more easily as follows.

As the objects are indistinguishable we can label all of them as X. Arrange m X's along a line. Introduce n – 1 breaks among the m + 1 spaces separating the X's (including the space before the first X and the space after the last X). The number of breaks put in any space is not restrained. Now we have n + m – 1 elements, m of them being X's and n – 1 of them break symbols. The n – 1 break symbols separate the X's into $x_1, x_2, \ldots, x_n$ X's. Thus the number of such distributions is $C(n + m - 1, m)$. Thus we have proved that

Theorem 12 The number of distributions of m indistinguishable objects into n distinguishable containers is $C(n + m - 1, m)$. (with any number of objects for container).

Corollary: The number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_n = m$ is $C(n + m - 1, m)$.

Incidentally, we note that the number of distributions of m indistinguishable objects into n distinguishable containers with at most one object per container is $C(n, m)$. Concerning this, refer to section 4.4.2 of unit 1.

Example 14: Show that the number of combinations of n distinguishable objects taken m times repetitions of the objects being allowed is $C(n + m - 1, m)$.

Solution: If the objects are $\{X_1, X_2, \ldots, X_n\}$ then we can choose X_1, x_1 times X_2, x_2 times, ... X_n, x_n times, such that $x_1 + x_2 + \ldots + x_n = m$. Clearly this can be done in $C(n + m - 1, m)$ ways.

* * *

Another approach: Suppose we get the number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_n = m$. Then we can get the number of distributions of m indistinguishable objects into n distinguishable containers as the two numbers are equal. Consider $(1 + t + t^2 + \ldots)^n$. This expression can be treated as the product of n expression $(1 + t + t^2 + \ldots)$. In this the coefficient of t^m is got by taking t with exponent x_1 from the first bracket, t with exponent x_2 from the second bracket, etc., t with exponent x_n from the nth bracket and multiplying all the terms, provided the x_i's add up to m. Thus the number of solutions of equation $x_1 + x_2 + \ldots + x_n = m$ is equal to the coefficient of t^m in the expansion of $(1 + t + t^2 (1 - t)^{-n} + \ldots)n$ (which is $(1 - t)^{-n}$) and it is $C(n + m - 1, m)$. Note that we used binomial expansion with negative integral exponent. The negative binomial expansion is as follows. For n, a positive integer, we have

$$(1 + x)^{-n} = \sum_{r=0}^{\infty} C(n + r - 1, r)(-1)^r x^r.$$

Some authors use the notation $C(-n, m)(-1)^m$ for $C(n + m - 1, m)$ for this reason.

Thus the number of distributions of m indistinguishable objects into n distinguishable containers is $C(n + m - 1, m)$.

Example 15: How many distinct solutions are there of $x + y + z + w = 10$ (a) in non negative integers? and (b) in positive integers?

Solution: (a) Clearly the answer is $C(4 + 10 - 1, 10)$ and this reduces to $C(13, 3) = 13.12.11/6 = 286$. (b) We want here x, y, z, w to be positive. Hence we can write them respectively as $X + 1$, $Y + 1$, $Z + 1$, $W + 1$, where X, Y, Z W are non-negative. Hence we want the number of non-negative solutions of the equation $X + 1 + Y + 1 + Z + 1 + W + 1 = 10$ or $X + Y + Z + W = 6$. The answer now is $C(4 + 6 - 1, 6) = C(9, 6) = C(9, 3) = 84$.

* * *

Example 16: Show that the number of positive solutions of the equation $x_1 + x_2 + \ldots + x_n = m$ is $C(m - 1, m - n)$.

Solution: If a positive solution is $x_1, x_2, \ldots x_n$, then it can be written as $X_1 + 1$, $X_2 + 1, \ldots X_n + 1$, where X_i's are non-negative. Thus the required number is the number of non-negative solutions of $X_1 + X_2 + \ldots + X_n + n = m$, or the number of non-negative integral solutions of $X_1 + X_2 + \ldots + X_n = m - n$ and this is clearly equal to $C(n + m - n - 1, m - n) = C(m - 1, m - n)$.

* * *

5.3.11 Indistinguishable Objects into Indistinguishable Containers

Suppose there are n indistinguishable objects and m indistinguishable containers. Any distribution is determined purely by the unordered m-tuple of non-negative integers with sum n. This is equivalent to the number of non-increasing sequences of length m of non-negative integers with sum n. But this is precisely the number of partitions of the integer n with at most m parts, viz., $P_n^1 + P_n^2 + \ldots P_n^m = Q_n^m = P_n(m)$. We have already seen this in partitions of integers.

With this we have come to the end of this unit. Let us take a quick look at what we have studied in this unit.

5.4 SUMMARY

In this unit we have covered the following:

1. We have started our discussion with partition of a natural number (unordered).
2. While discussing partitions you have been introduced to the concepts of recurrence relations and generating functions.
3. Concerning distributions of several objects into several containers, four cases have been discussed.
4. In the process we have introduced Bell numbers and Stirling numbers of second kind (also of the first kind).

5.5 SOLUTIONS/ANSWERS

E1) P_1 is obviously 1 as 1 can be partitioned in only one way. For calculating P_3 we can write all the partitions as follows 1 + 1 + 1, 2 + 1, 3. Thus $P_3 = 3$. The partitions of 5 are

1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1, 2 + 2 + 1, 3 + 1 + 1, 3 + 2, 4 + 1, 5. Thus $P_5 = 7$.

E2) P_n^1 is the number of partitions of n with just one part. This is the partition {n} only. Thus $P_n^1 = 1$. Q_n^1 is the number of partitions of n into at most one part. Clearly this is 1.

E3) Q_5^2 is the number of partitions of n into at most 2 parts. That is, $Q_5^2 = P_5^1 + P_5^2$. But $P_5^1 = 1$. The partitions of P_5^2 are 4 + 1, 3 + 2 and hence 2 in number. Thus $Q_5^2 = 3$. In a similar manner P_6^2 consist of 5 + 1, 4 + 2, 3 + 3. Thus $Q_6^2 = 4$. In general, to calculate P_n^2 we have to write n as x + y, where $x \geq y$. If n is odd, say (2r + 1), the partitions are (2r) + 1, (2r − 1) + (2r − 2) + 3, ..., (r + 1) + r, clearly r = (n − 1)/2) in number. But if n is even, say, 2r, then the partitions are

(2r − 1) + 1, (2r − 2) + 2, ..., (r) + r, clearly r (= n/2) in number. Thus, if n is odd, $Q_n^2 = (n-1)/2 + 1$ and if n is even, $Q_n^2 = n/2 + 1$.

E4) The only partition of P_n^n consists of sum of n 1's. Thus $P_n^n = 1$. If we want P_n^{n-1}, clearly the only way is the sum of one 2 and (n − 2) 1's. Thus $P_n^{n-1} = 1$.

E5) The partitions of 2n + 1 with each part 2 or 1 can be enumerated by considering the number of 2's appears. Clearly, we can have at most n 2's. Corresponding to r 2's, there is a unique partition, for r = 0, 1, 2, ..., n. Thus there are n + 1 such partitions.

E6) We need precisely Q_{2n}^2. According to E3, $Q_n^2 = n/2 + 1$ if n is even. Thus $Q_{2n}^2 = n + 1$.

E7) The 26 letters are distinguishable objects. We have to fill in three distinguishable containers, viz. the first, second, and third positions of a three-lettered words. The solution clearly is 26^3. If the last letter is to be x, the number is only $26^2 \times 1 = 676$. If the middle letter is a vowel, then by multiplication principle, the answer is $26 \times 5 \times 26 = 3380$.

E8) In a five digit number we do not want the first digit to be 0. Hence the number of 5-digit numbers is, by multiplication principle 9.10.10.10.10 = 90000. The number of 5-digit numbers composed of only odd digits (i.e, 1, 3, 5, 7, 9) is clearly $5^5 = 3125$.

E9) (a) We can choose a lady for vice-president in 4 ways. To fill the remaining 2 positions we can select 2 from the remaining 8 persons in $8 \times 7 = 56$ ways. Hence the required number is $4 \times 56 = 224$.

(b) If vice-president is a lady (chosen in 4 ways) others can be selected in $5 \times 4 = 20$. Similarly case applies to secretary being a lady. Hence by addition and multiplication principle the answer is $20 \times 4 + 20 \times 4 = 160$.

(c) Without any restriction three can be selected in $9 \times 8 \times 7 = 504$ ways. If no lady is to be selected, then it can be done in $5 \times 4 \times 3 = 60$ ways. What we need is the complement of this. Thus the required answer is $504 - 60 = 444$.

E10) If the alphabet has n letters, m-lettered words with distinct letters can be formed, by multiplication principle, in $n(n-1)(n-2)\ldots(n-m+1) = [n]_m$ ways.

E11) In an injective mapping images of distinct elements should be distinct. There are n possible images for the first element of the m-set, n − 1 possible images for the second and so on. Hence the number of such mappings is $n(n-1)\ldots(n-m+1) = [n]^m$.

E12) The set {1, 2, 3, 4} can be partitioned into two parts in the following ways.

{1}, {2, 3, 4}; {2}, {1, 3, 4}; {3} {1, 2, 4}; {4}, {1, 2, 3}; {1, 2}, {3, 4}; {1, 3}; {2, 4}; {1, 4}, {2, 3}. Thus there are 7 cases. Thus we have $S_5^2 = 7$.

5.6 MISCELLANEOUS EXERCISES

1. In how many ways may n identical chemistry books, r identical mathematics book, s identical physics books, t identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf).

2. A store sells eight different kinds of candy. In how many ways may you choose a bag of 15 pieces?

3. In how many ways may 40 students be grouped into four groups of five each for a discussion section led by a graduate student and two groups of 10 each for a section led by a professor?

4. In how many ways may 100 distinct beads be used to make three necklaces with 20 beads and four necklaces with 10 beads?

5. If m identical dice and n identical coins are thrown, how many results may be distinguished?

6. Express the polynomial $3x^4 + 2x^2 + 1$ in terms of the factorial polynomials $[x]_4$, $[x]_3$ etc.

7. A computer has eight jobs to divide among five different slave computers. Assuming each slave gets at least one job, in how many ways may this be done? Answer the same question assuming the five slave computers are indistinguishable to the master computer.

8. How many functions are there from an eight-element set onto a four element set?

9. Show that $S_n^{n-1} = C(n, 2)$.

10. Show that $S_n^2 = 2^{n-1} - 1$.

11. Show that $P_m^k = \sum_{i=1}^{k} P_{m-k}^i$.

12. Find the conjugate partitions for the following? (6, 5, 5, 3), (5, 4, 3, 2, 1), (8, 6, 6, 4, 2, 2).

13. Show that the number of partitions of n into 2 parts is n/2 if n is even and (n – 1)/2 if n is odd.

14. Show that $P_n - P_{n-1}$ is the number of partitions of n into parts greater than 1.

15. Show that $P_{n+2} + P_n \geq 2P_{n+1}$.

5.7 SOLUTIONS TO MISCELLANEOUS EXERCISES

1. The bookshelves are distinguishable. Assuming all the books to be distinguishable, the distribution can be done in N^3 ways, where $N = n + r + s + t$. Once the distribution is made we know that as books belonging to the same subject are not distinguishable the number of distinct distribution is only $\frac{N^3}{n!r!s!t!}$.

2. We have only to find how many of each kind is to be bought to make up a total of 15. This is the number of non-negative integral solutions of $x_1 + x_2 + \ldots + x_8 = 15$. Clearly the answer is $C(15 + 8 - 1, 15) = C(22, 15)$.

3. Let us first of all divide 40 students into two groups of 20 each. This can be done in $C(40, 20)$ ways. The first 20 may now be grouped into 4 groups of 5 each in $20!/(5!)^4$ ways. The second group 20 can be grouped into two groups of 10 each in $C(20, 10)$. Thus the required number of ways is $C(40, 20)[20!/(5!)^4]$ $C(20, 10)$. This simplifies to $\frac{40}{(10!)^2(5!)^4}$.

4. The selection of beads for the 7 necklaces can be made in $\frac{100}{(20!)^3(10!)^4}$ ways.

   Having made the selection a necklace with k distinct beads can be made in (k – 1)!

ways as circular permutation. Thus the required answer is $\frac{100!}{(20!)^3(10!)^4}$ $(19!)^3$ $(9!)^4$. This reduces to $\frac{100!}{20^3 10^4}$.

5. In each die there are 6 possible results and in each coin 2 results. The results from m dice will be an unordered m tuple of numbers from 1 to 6. This is just distinguished by the number of 1's number of 2's etc. This is just the number of solutions of $x_1 + x_2 + \ldots + x_6 = m$. This is $C(m + 6 - 1, m) = C(m + 5, 5)$. Similarly the n coins give rise to $C(n + 1, 1)$. Hence the required answer is $(n + 1) C(m + 5, 5)$.

6. We have to write $3x^4 + 2x^2 + 1$ as $a[x]_4 + b[x]_3 + c[x]_2 + d[x]_1 + e$. One easy method of doing this is to give successively the values $x = 0, 1, 2, 3, 4$ and equate both sides. When $x = 0$ we have $1 = e$. When $x = 1$ we get $6 = d + e$ giving $d = 5$. When $x = 2$ we get $57 = 2d + 2c + e$ giving $c = 23$. When $x = 3$ we get $262 = e + 3d + 6c + 6b$ giving $b = 18$. When $x = 4$ we get $801 = e + 4d + 12c + 24b + 24a$ giving $a = 3$. Thus we finally have $3x^4 + 2x^2 + 1 = 3[x]_4 + 18[x]_3 + 23[x]_2 + 5[x]_1 + 1$.

7. If the slave computers are distinguishable, then the required number is the number of positive solutions of $x_1 + x_2 + \ldots + x_5 = 8$ and this is same as the number of non-negative solutions of $x_1 + \ldots + x_5 = 3$ and this is as $C(5 + 3 - 1, 3) = C(7, 3)$. If the slave computers are indistinguishable, then the required answer is the number of partitions of 8 with exactly 5 parts and it is P_8^5.

8. The number of onto functions from 8-element set to a 4-element is $S_8^4 4!$.

9. S_n^{n-1} is the number of partitions of n things into $(n - 1)$ non-empty classes. Clearly $(n - 2)$ classes will be singletons and one class doubleton. For this we have just to choose two elements to form a doubleton and this can be done in $C(n, 2)$ ways.

10. S_n^2 is the number of partitions of an n-set into two non-empty classes. For this we have to select members of one class. This can be any non-empty subset of the n-set other than the entire set. But the total number of subsets is 2^n. Hence the required answer is $(2^n - 2)/2 = 2^{n-1} - 1$. We have to divide by 2 as the two classes are unordered.

11. We have to prove that $P_m^k = \sum_{i=1}^{k} P_{m-k}^i$. But this is merely a restatement of the recurrence relation for P_n^k's.

12. By constructing Ferrer's graph we can easily see that the conjugate partitions of

(6, 5, 5, 3), (5, 4, 3, 2, 1), (8, 6, 6, 4, 2, 2)

are respectively

(4, 4, 4, 3, 3, 1), (5, 4, 3, 2, 1), (6, 6, 4, 4, 3, 3, 1, 1).

13. The number of partitions of n into 2 parts is the number of positive integral solutions of $x + y = n$ with the condition $x \geq y$. If n is even, then x can be $n/2$, $n/2 + 1$, …, $n - 1$, precisely $n/2$ in number. If n is odd, then $x = y$ is impossible and we will have $x > y$. Hence x can be $(n + 1)/2$, …, $n - 1$, precisely $(n - 1)/2$ in number.

14. $P_n - P_{n-1}$ is the difference between the number of partitions of n and the number of partitions of $(n - 1)$. Consider a partition of $n - 1$. Adding 1 as an extra part we get a partition of n. Hence there is one-one correspondence between the partitions of $n - 1$ and the partitions of n in which 1 is a part. The result now follows.

15. The inequality can be written as $P_{n+2} - P_{n+1} \geq P_{n+1} - P_n$. By the previous problem LHS is the number of partitions of $n + 2$ with parts greater than 1. As the number of such partitions increase with n the result follows.

UNIT 6 MORE ABOUT COUNTING

Structure

6.1 INTRODUCTION

In this unit we are going to discuss / to learn about the pigeon-hole principle, the principle of inclusion-exclusion and application of these two principles to combinatorial problems.

Objectives

After reading this unit you should be able to:

- apply the pigeon-hole principle to problems;
- count the number of combinatorial objects with the help of the principle of inclusion-exclusion.

6.2 PIGEON-HOLE PRINCIPLE

A patently obvious fact about finite sets, known as Pigeon-Hole principle is a transparently simple principle with surprisingly large number of applications in combinatorics.

Suppose there are 10 boxes and 11 objects. If the objects are each placed in some box or the other in an arbitrary manner, then, at least one box will have more than one object in it. On the face of it this assertion is clearly true, as we cannot ensure at most one object in every box if the number of objects exceeds the number of boxes. No formal proof seems necessary. This principle is called the pigeon-hole principle. Formally we state the pigeon hole principle.

Pigeon-Hole Principle: If there are n boxes and (n + 1) objects, then for any assignment of objects to the boxes, there will always be a box with more than one object in it, or if m pigeons occupy n pigeonholes and $m > n$, then there is at least one pigeon hole with two or more pigeons roosting in it.

A variant of this used in most of the examples is as follows: If $nm + 1$ objects are distributed among m boxes, then at least one box will have more than n objects — known as generalized pigeon-hole principle.

The Generalized Pigeon-hole Principle — Some Variants:

Theorem 1. Let k and n be positive integers. If k balls are put into n boxes, then some box contains at least $\lceil k/n \rceil$ balls ($x \le \lceil x \rceil < x + 1$).

Proof: If each box contained fewer than [k/n] balls, then there would be at most n ([k/n] – 1) balls altogether. But
n([k/n] – 1) < n ((k/n) + 1 – 1) = k, a contradiction.

For example if 479 students are enrolled in Discrete Mathematics, and if there are 9 sections of the course being offered, then some section has at least $\left[\frac{479}{9}\right]$ = [53.2] = 54 students in it.

Theorem 2. If a finite set S is partitioned into k sets, then at least one of the sets has $\frac{|S|}{k}$ or more elements.

Proof: Let $A_1, \ldots A_k$ be the sets in the portion of set S. Then average value of $|A_i|$ is $\frac{1}{k}[|A_1 + \ldots + |A_k|] = \frac{|S|}{k}$. So the largest A_i has at least this many elements.

Theorem 3. Consider a function f: S → T where S and T are finite sets satisfying $|S| > r.|T|$. Then at least one of the sets $f^{\leftarrow}(t)$ has more than r elements. ($f^{\leftarrow}(t)$ denotes the inverse image of the set $\{t\} = \{x \in S : f(x) = t\}$)

Proof: The family $\{f^{\leftarrow}(t) : t \in T\}$ partitions S into k sets with $k \leq |T|$. By the principle just shown above, some set of $f^{\leftarrow}(t)$ has at least $\frac{|S|}{k}$ members. Since $\frac{|S|}{k} \geq \frac{|S|}{|T|} > r$ by hypothesis; such a set $f^{\leftarrow}(t)$ has more than r elements.

When r = 1, this principle states that if f : S → T and $|S| \geq |T|$, then at least one of the sets $f^{\leftarrow}(t)$ has more than one element, that is f is not injective.

Example 1: Assuming that friendship is mutual show that in any group of people we can always find two persons with the same number of friends in the group.

This appears rather surprising. If there are n persons in the group, then let the number of friends in the group of the ith person be f(i). Clearly f(i) can take values only between 0 and (n – 1). If some f(i) is 0, it means that the ith person does not have any friends in the group. In this case no f(i) can be (n – 1). Thus only one of the values 0 or (n – 1) may be present among the f(i)'s. Thus the n f(i)'s can take only (n–1) distinct values. By pigeon hole principle two f(i)'s must be equal.

* * *

Example 2: If 5 points are chosen at random within or on the boundary of an equilateral triangle of side 1 inch., show that we can find two points at a distance of at most 1/2 inch.

Solution: Divide the triangle into four equilateral triangles of side 1/2 inch by joining the midpoints of the sides by three line segments. These four triangles may now be considered as boxes and the five points as objects. By pigeon hole principle we can find a smaller triangle with two points in it. Clearly distance between these two points is at most 1/2 inch.

* * *

Example 3: Given any ten different positive integers less than 107 show that there will be two disjoint subsets with the same sum.

Solution: The highest numbers we could be given would be 97, 98, ..., 106 which add up to 1015. So, consider pigeon-holes marked 0, 1, 2, ..., 1015. The set of 10 positive integers have 2^{10} = 1024 subsets. Put a subset in the pigeon-hole marked with the sum of the numbers in the set. The 1024 subsets have to be put in 1016 pigeon-holes. So, some pigeon-hole will have more than one subset with the same sum. Two of them, though having the same sum, may not be disjoint. But by dropping the common elements in them, we are left with disjoint subsets with the same sum.

* * *

Here are some exercises for you to do.

E1) If 10 points are chosen in an equilateral triangle of side 3 cms., show that we can find two points at a distance of at most 1 cm.

E2) On 11 occasions a pair of persons from a group of 5 was called for a function. Show that some pair of persons must have attended the functional at least twice.

E3) Four persons were found in a queue independently on 25 occasions. Show that at least on two occasions they must have been in the queue in the same order.

Example 4: Show that every sequence on $n^2 + 1$ distinct integers contains either an increasing subsequence of $n + 1$ numbers or a decreasing subsequence of $n + 1$ numbers.

Solution: Let the sequence be $a_1, a_2, \ldots, a_{n^2+1}$. Suppose there is no increasing subsequence of $n + 1$ numbers. For each a_k let $s(k)$ be the length of the longest increasing subsequence beginning at a_k. Since all $n^2 + 1$ of the $s(k)$'s are between 1 and n, some label, say, m must be used at least $n + 1$ times. Since by the generalised pigeon-hole principle, at least $\left[\frac{n^2+1}{n}\right] = n + 1$ of these numbers are the same (the $s(k)$'s are the pigeons, and the numbers from 1 to n are the pigeon holes). Now if $i < j$ and $s(i) = s(j)$, then $a_i > a_j$. Otherwise a_i followed by the longest increasing subsequence starting at a_j would be increasing, subsequence of length $s(j) + 1$ starting at a_i, a contradiction since $S(i) = S(j)$. Then $n + 1$ integers a_k for which $s(k) = m$ must form a decreasing subsequence of length at least $n + 1$.

* * *

Example 5: If we take n integers, not necessarily distinct, then show that sum of some of these numbers is a multiple of n.

Solution: Let S(m) be the sum of the first m of these numbers. If for some l, m, $l < m$, $S(m) - S(l)$ is divisible by n, then, $a_{l+1} + a_{l+2} + \ldots + a_m$ is a multiple of n. This also will mean that S(l) and S(m) leave the same remainder when divided by n. If we cannot find such pairs, then it means that the numbers S(1), S(2), …, S(n) leave different remainders when divided by n. But there being only n possible remainders, viz 0, 1, 2, …, (n – 1), one of these numbers must leave a remainder of 0. This means one of the sums S(i) is divisible by n. This completes the proof. In fact we have proved tha one of the sums of consecutive terms is divisible by n.

* * *

You may try some more exercises now.

E4) If any set of 11 integers is chosen from 1, …, 20, show that we can find among them one of them dividing another.

E5) If 100 balls are placed in 15 boxes, show that two of the boxes must have the same number of balls.

E6) If $a_1, a_2, \ldots, a_n$ is a permutation of 1, 2, …, n and n is odd, show that the produc $(a_1 - 1)(a_2 - 2) \ldots (a_n - n)$ must be even.

We conclude by stating some extensions of pigeon hole principle and, some more exercises. Some extensions of Pigeon-Hole Principle:

1. Suppose we put infinity of objects in a finite number of boxes. Then atleast one box must have infinity of objects.

   This follows from the fact that if every box contains only a finite number of objects, then the total number of objects must be finite.

2. Let $A_1, A_2, \ldots, A_k$ be subsets of the finite set S such that each element of S is ir at least t of the sets A_i. Then the average number of elements in the A_i's is at lea $t \cdot \frac{|S|}{k}$.

This generalized version allows the sets A_i to overlap.

The Pigeon-Hole Principle, Theorem 2, is the special case $t = 1$.

E7) Every positive integer is given one of the seven colours in VIBGYOR. Show that atleast one of the colours must have been used infinite number of times.

E8) Let A be some fixed 10-element subset of $\{1, 2, ..., 50\}$. Show that A possesses two different 5-element subsets, the sum of whose elements are equal.

E9) The positive integers are grouped into 100 sets. Show that at least one of the sets has an infinity of even numbers. Is it necessary that at least one set should have infinity of even numbers and infinity of odd numbers?

6.3 INCLUSION-EXCLUSION PRINCIPLE

Let us illustrate this principle with an example first.

In a club with 54 members, 34 play tennis, 22 play golf, and 10 play both. There are 11 playing handball, of whom 6 play tennis also, 4 playing golf also, and 2 play both tennis and golf. How many play none of the three sports?

Let S represent the set of all members of the club. Let T represent the set of tennis playing members, G represent the set of golf playing members, and H represent the set of handball playing members. Let us represent the number of elements in A by $|A|$. Consider the number $|S|-|T|-|G|-|H|$. Is this the answer to the problem? No, for those who are in T as well as G have been subtracted twice. To compensate for this double subtraction we may now consider the number $|S|-|T|-|G|-|H|+|T\cap G|+|G\cap H|+|H\cap T|$. Is this the answer? No, for those playing all the three games have been subtracted thrice and then added thrice. But those members must have totally excluded. Hence we now consider the number. $|S|-|T|-|G|-|H|+|T\cap G|+|G\cap H|+|H\cap T|-|T\cap G\cap H|$. This is the correct answer. This reduces to $54-32-22-11+10+6+4-2=5$.

In this formula we make alternately inclusions and exclusions to arrive at the correct answer. This is a simple case of the principle of inclusion and exclusion. It is also known as the sieve principle. The reason for this is that we subject the objects to sieves of progressively finer mesh to arrive at a certain grading.

The Inclusion-Exclusion Principle will tell us the size of a union in terms of the sizes of various intersections'.

To calculate the size of $A_1 \cup A_2 \cup \ldots \cup A_n$. Calculate the sizes of all possible intersections of sets from $A_1, A_2, \ldots, A_n$. Add the results obtained by intersecting an odd number of the sets, and then subtract the results obtained by intersecting an even number of the sets.

The Inclusion-Exclusion Principle is ideally suited to situations in which (i) We just want the size of $A_1 \cup A_2 \ldots \cup A_n$, not a listing of its elements and (ii) Multiple interesections are fairly easy to count.

We will consider the inclusion-exclusion formula in its generality in the following theorem:

Theorem 4. Suppose we have a set of N objects and a set of n properties $p_1, p_2, \ldots p_n$ which can be applied to these objects. By a property we mean any distinguishing criterion by which we can say whether an object satisfies the criterion or not. Let us suppose that every object is assigned a weight. Let P represent the set of properties. If A is a subset of P let W(A) represent the sum of the weights of objects possessing all the properties in A (possessing possibly some other properties not in A as well). Then we have the formula,

$$E(0) = W(\emptyset) - \sum_{A \subset P, |A|=1} W(A) + \sum_{A \subset P, |A|=2} W(A) - \ldots + (-1)^n W(P). \qquad (1)$$

where $W(\emptyset)$ is the sum of the weights of all the N objects and E(0) is the sum of the weights of all objects not possessing any of the properties in P, or equivalently possessing exactly 0 properties.

The above formula gives the sum of the weights of objects possessing none of the properties in P.

Proof: Consider an object which possesses exactly r properties in P. Let us see how many times its weight is considered on the RHS of the formula. Clearly its weights is considered only in those terms where A is contained in the set of the r properties. In $W(\emptyset)$ it is considered once. In $\sum_{A \subset P, |A| = 1} W(A)$ it is considered (actually subtracted) r times. In the next term it is added C (r, 2) times and so on. Thus in the sum it appears

$$C(r, 0) - C(r, 1) + C(r, 2) - C(r, 3) + \ldots + (-1)^r C(r, r)$$

times. But this sum is 0 if r > 0 and 1 if r = 0. Thus the weight of any object which possesses none of the properties is added precisely once in the sum. This proves the correctness of the formula.

Note that $0 = (1-1)^r = 1 - C(r, 1) + C(r, 2) - \ldots + (-1)^r C(r, r)$ if r > 0.

But if r = 0, then $(1-1)^0 = 1$.

Corollary 1: If we take the weight of every object as 1, we get the number of objects possessing none of the properties in P from the formula by

$$N(p_1' p_2' \ldots p_n') = N - \sum_{i=1}^{n} N(p_i) + \sum N(p_i p_j) - \ldots + (-1)^n N(p_1 p_2 \ldots p_n) \qquad (2)$$

Where $N(p_i)$ denotes the number of objects that have the properties p_i, $N(p_i p_j)$ denotes the number of objects that have the properties p_i and p_j, $N(P_i')$ denotes the number of objects that do not have the property p_i etc.

Unless specified otherwise, assume that the weight of each object is 1. Then sum of the weights of a collection will be exactly equal to the cardinality of the collection.

Let n(A) denote the number of elements in the set A (which we have also denoted by |A|). Also we denote $A_1 \cap A_2 \cap A_3'$ by $A_1 A_2 A_3'$ where A_3' is the complement of the set A_3. An element is in $A_1' A_2' \ldots A_n'$ if it is in none of the sets A_i i = 1, 2, ... n.

Corollary 2: Let $A_1, A_2, \ldots, A_n$ be n sets in a universe $\mathcal{U}$ of N elements. Let S_k denote the sum of the sizes of all k-tuple intersection of the A_i's. Then

$$n(A_1' A_2' \ldots A_n') = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n. \qquad (3)$$

Corollary 3: Let $A_1, A_2, \ldots A_n$ be sets in the universe $\mathcal{U}$. Then

$$n(A_1 \cup A_2 \ldots \cup A_n) = S_1 - S_2 + S_3 - \ldots - (-1)^{k-1} S_k \ldots (-1)^{n-1} S_n. \qquad (4)$$

We will illustrate the use of the formula.

Example 6: Find the sum of the numbers from 1 to 25 which are not divisible by 2 or 3

Solution: Let us give a weight of r for the integer r from {1, 2, ..., 25}. Then we have find the sum of the weights of all objects not possessing the two properties (1) divisibility by 2, and (2) divisibility by 3. In this case we have

W(0) = 1 + 2 + ... + 25 = 325.

W(1) = 2 + 4 + ... + 24 = 2(1 + 2 + ... + 12) = 156. (sum of the weights of all objects possessing property (1)).

W(2) = 3 + 6 + ... + 24 = 3(1 + 2 + ... + 8) = 109. Also

W(1, 2) = 6 + 12 + 18 + 24 = 60. The required answer is, by the sieve formula, 325 – 156 – 108 + 60 = 121.

We illustrate the application of corollary 2 and 3 by the following example.

Example 7: How many ways are there to distribute r distinct objects into five (distinct) boxes with (i) at least one empty box? (ii) no empty box ($r \geq 5$)?

Solution: Let $\mathcal{U}$ be all distributions of r distinct objects into five boxes. Let A_i denote the set of distributions with ith box being empty. Then the required number of distributions with at least one empty box is $n(A_1 \cup A_2 \ldots \cup A_5)$. We have $N = 5^r$, $n(A_i) = 4^r = (5-1)^r$), the number of distributions with each object giving into one of the remaining four boxes, $n(A_i A_j) = 3^r = ((5-2)^r)$, and so forth. Thus by Corollary 3 above, we have

$n(A_1 \cup \ldots \cup A_5) = S_1 - S_2 + S_3 - S_4 + S_5$

$= C(5, 1)\, 4^r - C(5, 2)\, 3^r + C(5, 3) 2^r (- C(5, 4) 1^r + 0$

Also $n(A'_1 A'_2 \ldots A'_5) = 5^r - C(5, 1)\, 4^r + C(5, 2)\, 3^r - C(5, 3) 2^r + C(5, 4) 1^r$ by Corollary 2.

* * *

Example 8: How many solutions are there to the equation $x + y + z + w = 20$ in positive integers $x \le 6$, $y \le 7$, $z \le 8$, $w \le 9$?

Solution: To use inclusion-exclusion, we let the objects be solutions (in positive integers) of the equation. A solution has property p_1 if $x > 6$, property p_2 if $y > 7$, property p_3 if $z > 8$, and p_4 if $w > 9$. Then what we need is precisely E_0. The total number of positive solutions to the equation is $C(20 - 1, 4 - 1) = C(19, 3)$. Thus $W(\emptyset) = C(19, 3)$. Similarly

$W(p_1) = C(20 - 6 - 1, 4 - 1) = C(13, 3)$, $W(p_2) = C(12, 3)$, $W(p_3) = C(11, 3)$,

$W(p_4) = C(10, 3)$, $W(p_1p_2) = C(20 - 6 - 7 - 1, 4 - 3) = C(6, 3)$, $W(p_1p_3) = C(5, 3)$, and so on. By inclusion-exclusion we obtain

$$E(0) = C(19, 3) - C(13, 3) - C(12, 3) - C(11, 3) - C(10, 3)$$
$$+ C(6, 3) + C(5, 3) + C(4, 3) + C(4, 3) + C(3, 3)$$
$$= 969 - 286 - 220 - 165 - 120$$
$$+ 20 + 10 + 4 + 4 + 1$$
$$= 217.$$

* * *

Now you may try the following exercises:

E10) How many numbers from 0 to 999 are not divisible by either 5 or 7?

E11) Eight people enter an elevator. At each of four floor stops at least one person leaves the elevator. After four floor stops the elevator is empty. In how many ways can this be done?

E12) How many six-digit numbers contain exactly three different digits?

6.3.1 Application to Number Theory — Euler's Totient Function

Let m be a positive integer whose distinct prime factors are $p_1, p_2, \ldots, p_n$. Then the number of integers between 1 and m which are relatively prime to m (have no common factor other than 1) equals

$$m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_n}\right)$$

(This expression is usually denoted by $\phi(m)$ and it defines Euler's totient function in number theory.)

Solution: Let the objects be $\{1, 2, 3, \ldots, m\}$ and for $1 \le i \le n$ let 'property i' be that a number is divisible by p_i. Then the integers in that set which are relatively prime to m are precisely those which have none of the properties $1, 2, \ldots, n$. Hence the answer is (by formula (1))

$$m$$
$$- W(1) - W(2) - \ldots - W(n)$$
$$+ W(1, 2) + W(1, 3) + \ldots + W(n - 1, n)$$
$$- W(1, 2, 3) - W(1, 2, 4) - \ldots - W(n - 2, n - 1, n)$$
$$+ \ldots$$
$$\vdots$$
$$(-1)^n W(1, 2, \ldots, n)$$

But $W(i) = \frac{m}{p_i}$, $W(i, j) = \frac{m}{p_i p_j}$, etc. Hence the required number is

$$m - \sum_{i=1}^{n} \frac{m}{p_i} + \sum_{i,j} \frac{m}{p_i p_j} - + \ldots + (-1)^n \frac{m}{p_1 p_2 \cdots p_n}$$

But the above expression is clearly equal to

$$m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_n}\right).$$

6.3.2 Application to onto Maps

We will show that the number of functions from an m-element set onto a k-element set is $\sum_{i=0}^{k} (-1)^i C(k, i)(k-i)^m$. $(m \geq k \geq 1)$

To prove this we will define the objects to be all the mappings from M, an m-element set to K, a k-element set. For these objects we will define k properties. The ith property is that a mapping does not have the ith element in K as an image. Clearly the number of objects is k^m. The number of mappings excluding a specific set of i elements in K is $(k-i)^m$ and there are C (k, i) such sets. An application of inclusion-exclusion principle now gives the required answer.

Stated more precisely, these are

$$k^m - C(k, 1)(k-1)^m + C(k, 2)(k-2)^m - \ldots + (-1)^{k-1} C(k, k-1)\, 1^m$$

different subjective functions from M onto K.

Example 9: How many functions are there from a five-element set onto a three-element set?

Solution: The answer is $\sum_{i=0}^{k} (-1)^i C(k, i)(k-i)^m$, for m = 5 and k = 3. Thus the required answer is $3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 243 - 96 + 3 = 150$.

* * *

Theorem 1: The number of partitions of an m-element set into k classes is

$$\frac{1}{k!} \sum_{i=0}^{k} (-1)^i C(k, i)(k-i)^m.$$

Proof: If the k classes are distinguishable the number of partitions would have been the same as the number of functions from an m-element set onto a k-element set. As the classes are indistinguishable we have to divide this number by k!. From the previous application to onto maps the result follows. We thus have an explicit formula for S_m^k.

Example 10: What is the Stirling number S_5^3?

Solution: We have already seen that the number of functions from a 5-element set onto a three-element set is 150. By the previous theorem the answer is 150/3! = 25.

* * *

Example 11: Suppose A, B, C are three finite subsets of a set X. Show that
$|A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |A \cap C| - |A \cap B \cap C| = 0.$

Solution: Consider the set of objects as the set $A \cup B \cup C$. Let the property p_1 be, not a member of A, the property p_2, not a member of B, and p_3, not a member of C. Then, clearly the number of elements which are in none of the three sets is counted by the expression in the problem. This number is clearly 0, as we are considering *only* the elements in $A \cup B \cup C$.

* * *

6.3.3 Application of the Principle of Inclusion-Exclusion to Probability

An important application of the principle of inclusion-exclusion is used in probability. Suppose in a probability space $A_1, A_2, \dots, A_n$ are n events. Then we have,

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \dots < i_r \le n} P(A_{i_1} A_{i_2} \dots A_{i_r})$$

Proof: It must be noted that in the above formula AB means $A \cap B$, and $A_1 \cup A_2 \cup \dots \cup A_n$ means atleast one of the events $A_1, A_2, \dots, A_n$ happens.

To every elementary event let us give the weight equal to its probability. The ith property is that the elementary event belongs to the event A_i. We will then have $W(\emptyset) = 1$.

By DeMorgan's law, we have $A'_1A'_2 \dots A'_n$ is the complement of $A_1 \cup A_2 \cup \dots \cup A_n$. But the principle of inclusion-exclusion gives

$$P(A'_1A'_2 \dots A'_n) = 1 - \sum_{r=1}^{n} (-1)^r \sum_{1 \le i_1 \le i_2 < \dots < i_r \le n} P(A_{i_1}A_{i_2} \dots A_{i_r})$$

The result now follows from the fact that

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(A'_1A'_2 \dots A'_n)$$

6.3.4 Application to Derangements

The expression $a_1a_2 \dots a_n$ is called a permutation of 1, 2, ..., n if all the a_i's are distinct and are from {1, 2, ..., n}. A permutation $a_1a_2 \dots a_n$ is called a derangement if $a_i \ne i$ for i = 1, 2, ..., n. Thus 231 is a derangement, while 321 is not, because 2 is in its natural position.

The problem now is to find d_n, the number of derangements of the numbers 1 to n. Let the set of all permutations of 1 to n be our objects, and let us give a weight of 1 to each of these objects. The property p_i is that the number i occurs in the ith position of the permutation. Then d_n is precisely E(0) or $N(p'_1 \dots p'_n)$. It follows that $W(p_i) = (n-1)!$, i = 1, 2, ..., n, $W(p_i\, p_j) = (n-2)!$, i, j = 1, 2, ..., n, $i \ne j$

Clearly $W(p_{i_1}\, p_{i_2} \dots p_{i_r}) = (n-r)!$, for after fixing the i_jth position with i_j, for j = 1, 2, ..., r, we can fill the remaining (n – r) positions with the remaining (n – r) numbers in (n – r)! ways. By the principle of inclusion-exclusion we have

$$d_n = E(0) = n! - C(n, 1)(n-1)! + C(n, 2)(n-2)! - \dots + (-1)^n C(n, n)0!$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right)$$

Note: The expression $\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right)$ is the beginning of the expansion for e^{-1}. Even for moderately large value of n, d_n is very close to $n!e^{-1} = 0.36788\, n!$.

Further we have the following formula:

For a set of n objects, the number of permutations in which (i) a subset of r objects are deranged can be computed by the formula

$$n! - C(r, 1)(n-1)! + C(r, 2)(n-2)! - \dots + (-1)^r C(r, r)(n-r)! \qquad (5)$$

(ii) exactly r elements are in their natural position is $C(n, r)\, d_{n-r}$ (6)

Example 12: Let n books be distributed to n children. The books are returned and distributed to the children again later on. In how many ways the books be distributed so that no child will get the same book twice?

Solution: $(n!)^2 e^{-1}$ since corresponding to each first distribution, there are $(n!)e^{-1}$ ways of distribution.

* * *

Example 13: If ten people check their hats, and the drunken hat-check girl returns the hats randomly to the people, what is the probability that no one gets the right hat?

Solution: The number of cases favourable to the event is clearly d_{10}. The total number of cases is 10!. Thus the probability that none will get the right hat is $d_{10}/10! \approx 0.36788$.

* * *

You may now try the following exercises.

E13) In how many ways can the integers 1, 2, 3, ..., 7, 8 and 9 be permuted such that no odd integer will be in its natural position.

E14) Find the number of permutations in which exactly four of the nine integers (1, 2, ..., 9) are in their natural positions (exactly five integers are dearranged).

With this we have come to the end of this unit which is the last unit of this block. Let us now summarise what we have covered in this unit.

6.4 SUMMARY

After going through this unit, you have studied the following:

1. Pigeon hole principle stated in several equivalent forms.
2. Several types of generalized pigeon-hole principle.
3. Various applications of the pigeon-hole principle.
4. Principle of inclusion and exclusion-various formulae.
5. Various application of the principle of inclusion and exclusion.

6.5 SOLUTIONS/ANSWERS

E1) By drawing lines parallel to the sides and through the points trisecting each side, we can divide the equilateral triangle into 9 equilateral triangles of side 1 cm. Thus if 10 points are chosen, atleast two of them must lie in one of the 9 triangles.

E2) 5 persons can be paired in C(5, 2) = 10 ways. Hence if pairs are invited 11 times, at least one pair must have been invited twice or more times by pigeon-hole principle.

E3) Four persons can be arranged in a line in 4! = 24 ways. Hence if we consider 25 occasions, atleast on two occasions the same ordering in the queue must have been found by pigeon-hole principle.

E4) Consider the following grouping of numbers.

{1, 2, 4, 8, 16}, {3, 9, 18}, {5, 15},

{6, 12}, {7, 14}, {10, 20}, {11}, {13}, {17}, {19}

There are 10 groupings exhausting all the 20 integers from 1 to 20. If 11 numbers are chosen it is impossible to select at most one from each group. So two numbers have to be selected from some group. Obviously one of them will divide the other.

E5) Suppose $x_1, x_2, \ldots, x_{15}$ are the number of balls in the 15 boxes in the increasing order, assuming that all these numbers are different. Then, clearly, $x_i \geq i-1$ for $i = 1, 2, \ldots, 15$. But then,
$\sum_{i=1}^{15} x_i \geq 14 \cdot 15/2 = 105$. But the total number of balls is only 100, a contradiction. Thus the x_i's cannot all be different.

E6) In the sequence $a_1, a_2, ..., a_n$, there are $(n + 1)/2$ odd numbers and $(n - 2)/2$ even numbers as n is odd. Hence it is impossible to pair all a_i's with numbers from 1, 2, ..., n with opposite parity (evenness and oddness). Hence, in at least one pair (i, a_i) both the numbers will be of the same parity. This means that the factor $(a_i - i)$ is even and hence the product is even.

E7) Consider the seven colours as containers and the numbers getting the respective colour their contents. Then we have a distribution of infinite number of objects in 7 container. Hence, by the extension of pigeon-hole principle, at least one container must have infinity of objects. The colour of that container must have been used infinite number of times.

E8) Let $\mathcal{H}$ be the family of 5-element subsets B of A. For each B in $\mathcal{H}$, let f (B) be the sum of the numbers in B. Obviously we must have $f(B) \geq 1 + 2 + 3 + 4 + 5 = 15$ and $f(B) \leq 46 + 47 + 48 + 49 + 50 = 240$. Hence $f: \mathcal{H} \to T$ where $T = \{15, 16, ..., 240\}$. Since $|T| = 226$ and $|\mathcal{H}| = C(10, 5) = 252$, by the generalised pigeon hole principle (3) $\mathcal{H}$ contains different sets with the same image under f, that is, different sets the sums of whose elements are equal.

E9) The 100 collections can be considered as containers. There are infinity of even numbers. When these even numbers are distributed into 100 containers, at least one container must have infinity of them.

A container need not contain infinity of even and infinity of odd numbers. For, if we put all odd numbers in the first container and all even numbers in the second, leaving other 98 containers empty, then, no container has infinity of odd *and* infinity of even numbers.

E10) Let the objects be the integers 0, 1, ..., 999. Let p_1 be the property that a number is divisible by 5. Let p_2 be the property that a number is divisible by 7. Let the weight of each of these numbers be 1. Then we need precisely the sum of the weights of the objects possessing none of the properties p_1, p_2. $W(\emptyset) = 1000$. $W(p_1) = 200$. For, the numbers divisible by 5 are 0, 5, 10, ..., 995, exactly 200 numbers. $W(p_2) = 143$. For, the numbers divisible by 7 are 0, 7, 14, ..., 994, exactly 143 numbers. $W(p_1\ p_2) = 29$, for the numbers divisible by 5 as well as 7 are principle the answer is $1000 - 200 - 143 + 29 = 686$.

E11) The answer to this problem is clearly the number of functions from an 8-set onto a 4-set. 8-set is the set of people and the 4-set is the set of floors. This number is

$$\sum_{i=0}^{4} C(4, i)(4 - i)^8 = 4^8 - 4 \cdot 3^8 + 6 \cdot -2^8 - 4 \cdot 1^8.$$

E12) We can choose three digits in C (10, 3) = 120 ways. The number of 6-digit numbers using all the three numbers is same as the number of functions from a 6-set onto a 3-set and this number is $36 - 3 \cdot 26 + 3 \cdot 1^6 = 540$. Hence the answer is $120 \cdot 540 = 64800$. But this will include numbers starting with 0 also.

E13) 1, 3, 5, 7, 9 are the odd integers.
By formula (5) the required number of ways is
$9! - C(5, 1)8! + C(5, 2)7! - C(5, 3)6! + C(5, 4)5! - C(5, 5)4!$

E14) By formula (6), the required number of permutation is $C(9, 4)d_{9-4} = C(9, 4)d_5$.

6.6 MISCELLANEOUS EXERCISES

E1) A group of couples sits around a circular table for a group-discussion on marital problems. In how many ways may the group be seated so that no husband and wife sit together?

E2) A used car dealer has 18 cars on-the-lot. Nine of them have an automatic transmission, 12 have power steering, and 8 have power brakes. Seven have both automatic transmission and power steering, four have automatic transmission and power brakes, and five have power steering and power brakes. Three cars have power steering and brakes and automatic transmission. How many cars have automatic transmission only? How many cars are "stripped?".

E3) A bookcase has five shelves each with 10 books on it. Each shelf contains books on one of five different subjects. In how many ways may the books be removed for dusting and returned to the shelves so that each subject still has a shelf of its own, even though no shelf has a book previously on it?

E4) In a club there are 10 people who play tennis and 15 who play squash; 6 of them play both. How many play at least one of the sports?

E5) In a club there are 10 people who play tennis, 15 who play squash and 12 who play badminton. Of these, 5 play tennis and squash, 4 play tennis and badminton and 3 play squash and badminton; and of these just 2 people play all three sports. How many people play at least one of the three sports?

E6) How many numbers from 2 to 1000 are perfect squares, perfect cubes or any higher power?

E7) Suppose that you are given n + 1 different positive integers less than or equal to 2n. Show that

(i) there exists a pair of them which add upto 2n + 1,

(ii) there must exist two which are relatively prime

E8) If n + 1 positive integers are less than or equal to 2n, show that we can find two of them such that one is a multiple of the other.

E9) Prove that in any n + 1 integers there will be a pair which differs by a multiple of n.

E10) Every day I put 1 rupee or 2 rupees into a piggy-bank and the total is m rupees after n days. Show that for any integer k with $1 \le k \le 2n - m$ there will have been a period of consecutive days during which the total amount put into the piggy-bank was exactly k rupees.

E11) Prove that, given any positive integer n, some multiple of it must be of the form 99 ... 900 ... 0.

E12) How many integers between 1 and 10000 are divisible by at least one of 2, 3 and 5.

E13) How many integers between 1 and 10000 are divisible by at least one of 2, 3, 5 and 7? Deduce that there are at most 2288 prime numbers less than 10000.

6.7 SOLUTION TO MISCELLANEOUS EXERCISES

E1) We assume the couples are numbered as couple 1 through couple n. Let $N = \{1, 2, \ldots, n\}$. Let the property P_i be "husband and wife i sit together", for $i = 1, 2, \ldots, n$. Let P be the set of n properties. Let us assign a weight of 1 to every object. What we need is clearly E_0. To get W (A), $A \subset P$, $|A| = r$, we seat the r couples together and allow the remaining 2n – 2r people to sit down at the remaining places; we are in effect arranging 2n – r units around a circular table. This can be done in (2n – r – 1)! ways. Now each couple can occupy its two chairs in two ways. Therefore, after the 2n – r unit are assigned to places, there are 2^r ways for the units to be placed in their assigned places. Thus, there are $2^r(2n - r - 1)!$ ways to seat the people. Thus

$$E(0) = \sum_{A \subset P} (-1)^{|A|} W(A) = \sum_{i=0}^{n} (-1)^i C(n, i) 2^i (2n - i - 1)!$$

E2) Let the properties T, S, B represent 'with automatic transmission' 'with power steering', 'with power brakes', respectively. Here the cars are our objects. Then, we have, W (Ø) = 18.

W (T) = 9, W (S) = 12, W (B) = 8, W (T, S) = 7, W (T, B) = 4, W (S, B) = 5, W (T, S, B) = 7. 'Stripped' will be E (0) in number. Thus, 'Stripped' = 18 – 9 – 12 – 8 + 7 + 4 + 5 – 3 = 2. For getting number of cars with automatic transmission only, consider those with automatic transmission and with no other property. Clearly the number required is W (T) – W (T, S) – W(T, B) + W (T, S, B) – 9 – 7 – 4 + 3 = 1.

E3) The arrangements of interest here are arrangements of books on shelves. If we take the property j, "shelf j" gets the same subject it had last time," then for a given set I of i shelves, there are 10! ways to return the books of a given subject to each shelf in I, giving $(10!)^i$ ways to fill these shelves. Next there are (5 – i) other shelves and there are (5 – i)! ways to assign subjects to shelves so that the shelves in I-and perhaps some others - get their original subjects back. Then there are $(10!)^{5-i}$ ways to assign the books to these shelves. Thus we have $(10!)^5$ (5 – i)! arrangements having at least the properties in I, so W (I) = $(10!)^5$ (5 – i)!. Since we want E_0 we apply the inclusion-exclusion principle to get, with K = {1, 2, 3, 4, 5},

$$E(0) = \sum_{I \subset K} (-1)^{|I|} (10!) (5 - |I|)!$$

$$= \sum_{i=0}^{5} (-1)^5 C(5, i) (5 - i)!$$

$$= 5! (10!)^5 \sum_{i=0}^{5} \frac{(-1)^i}{i!}.$$

E4) The required number is 10 + 15 – 6 = 19.

E5) The required number is 10 + 15 + 12 – 5 – 4 – 3 + 2 = 27.

E6) Consider the objects {2, 3, ..., 1000} and let a member of this have 'property i' if it equals the ith power of some integer. Since $2^{10} > 1000$ there are no tenth powers in the set and the only properties which really concern are properties 2, 3, ..., 9.

W (2) = $[(1000)^{1/2}] - 1$ = 30, W (3) = $[(1000)^{1/3}] - 1$ = 9.

W (2, 3) = W (6) = $[(1000)^{1/6}] - 1$ = 2, W (2, 4) = W(4) = 4

W (2, 3, 4) = W (12) = 0, W (2, 3, 6) = W (6) = 2, ... where [x] denotes

the 'integer part' of x. Continuing in this way gives the number of object with at least one of the properties as

30 + 9 + 4 + 2 + 2 + 1 + 1 + 1 – 2 – 4 – 2 – 1 – 2 – 1 – 1 + 2 + 1 = 40.

E7) (i) Let the numbers be $a_1, a_2, \ldots a_{n+1}$. These numbers are distinct and lie between 1 and 2n. Let us suppose that we cannot find a pair of them with sum 2n + 1. If we define $b_i = 2n + 1 - a_i$ for i = 1, 2, ... n + 1, then each b_i is a positive integer less than equal to 2n. No b_i can be an a_j. Thus the collection $a_1, b_1, a_2, b_2, \ldots a_{n+1}, b_{n+1}$ has (2n + 2) distinct integers all between 1 and 2n. This is clearly impossible by pigeon-hole principle. This contradiction shows that some pair must have a sum of 2n + 1.

(ii) We claim that two of the numbers must be consecutive integers. Let the (n + 1) numbers arranged in increasing order be $a_1, \ldots\ldots a_{n+1}$. If not two numbers are consecutive integers, then $a_{i+1} - a_i \geq 2$ for i = 1, 2, ... n. Adding these we get $a_{n+1} - a_1 \geq 2n$, and this is impossible. Thus two of the numbers must be consecutive integers. These are obviously prime to each other.

E8) If all the n + 1 numbers are not distinct, then two of them would be equal, and one is trivially a multiple of the other. Thus we can assume that the numbers are distinct. Let us consider pigeon-holes marked 1, 3, 5, ..., 2n – 1. We put a number in the collection of n + 1 given numbers in a pigeon-hole marked r if r is the largest odd number dividing the number. There being only n pigeon-holes two of the numbers should fall in the same pigeon-hole. Those two numbers have the same odd number as the maximum odd divisor, r say. Then the two numbers should be of the form $r \cdot 2^a$, $r \cdot 2^b$, where $a \le b$. Clearly $r \cdot 2^a$ divides $r \cdot 2^b$.

E9) Let the integers be $a_1, a_2, \ldots a_{n+1}$. Let us suppose that difference of no two of them is divisible by n. Consider the n differences $a_1 - a_i$ for i = 2, 3, ... n. When these differences are divided by n the remainders can only be from 0, 1, 2, ..., n − 1. But 0 is excluded by our assumption. Hence two of the remainders must be equal by pigeon-hole principle. Suppose $a_1 - a_i$ and $a_1 - a_j$ leave the same remainder. Then their difference $a_i - a_j$ will be divisible by n.

E10) Let the total upto ith day be t_i, i = 1, 2, ...n. Let $1 \le k \le 2n - m$. Consider the 2n numbers, $t_1, t_2, \ldots t_n, t_1 + k, t_2 + k, \ldots, t_n + k$. Clearly all these 2n numbers lie in the interval [1, 2n − 1]. By pigeon-hole principle two of these must be equal, say t_i and $t_j + k$, then $t_i - t_j = k$.

E11) Consider the n + 1 numbers 1, 10, 10^2, ... 10^n. Let the remainders when these numbers are divided by n be $r_0, r_1, \ldots, r_n$ respectively. These r's can take only the values 0, 2, ..., n − 1. Thus, by pigeon-hole principle, two of them must be the same, say r_a, r_b. This means, n divides $10^b - 10^a$, assuming that b > a. But $10^b - 10^a$ is exactly of the form 99 ... 900 ... 0.

E12) Let the objects be the numbers 1 to 10000. Let A, B, C be the properties, (i) divisible by 2, (ii) divisible by 3 and (iii) divisible by 5. Then the number of numbers not divisible by any of the three numbers is given by

$$E(0) = 10000 - W(A) - W(B) - W(C)$$
$$+ W(AB) + W(BC) + W(AC) - W(ABC).$$

But W(A) = 5000, W(B) = 3333, W(C) = 2000.

W(AB) = 1666, W(BC) = 666, W(AC) = 1000, W(ABC) = 333. Thus

$$E(0) = 10000 - 5000 - 3333 - 2000$$
$$+ 1666 + 666 + 1000 - 333 = 2666.$$ The required answer is

$$10000 - 2666 = 7334.$$

E13) As in the previous problem we define A, B, C, D. Then

$$E(0) = 10000 + 1666$$
$$+ 1000 + 714 + 666 + 476 + 285 + 47$$

− (5000 + 3333 + 2000 + 1428 + 333 + 238 + 142 + 95) = 2285. Thus the required number is 10000 − 2285 = 7715. Also we have 2285 numbers not divisible by 2, 3, 5 and 7. The prime numbers could be found only among these 2285 numbers and of course 2, 3, 5 and 7. But we have to omit 1. Thus there could be at most 2285 + 4 − 1 = 2288 prime numbers.

NOTES

NOTES

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UGMM-13
Discrete Mathematics

Block

3

RECURRENCES

BLOCK 3 RECURRENCES

Suppose you were a communication technologist and needed to create a single error detecting code of a particular type. You would then need to find the number of binary sequences with an even number of 0's (or 1's). How would you do this? One of the simplest ways to solve this problem and other counting problems is by using recurrence relations. What are these relations?

A recurrence relation, or recurrence (in short), is an equation that expresses a given problem for n objects in terms of the same problem posed for less than n objects. For example, let us consider the most famous example of a recurrence relation/equation, which is also the first recurrence relation found in mathematical texts. The problem is:

How many pairs of rabbits are produced after n months if we start with one pair of rabbits exactly one month old, and if every month, to each pair of rabbits more than one month old a new pair is born?

Let f(n) denote the number of pairs of rabbits present at the beginning of a month $n, n \geq 1$. In Unit 7 you will see that the recurrence relation $f(n) = f(n-1)+f(n-2)$, for $n \geq 3$, with $f(0) = 1$ and $f(1) = 1$ describes the situation. In this unit we discuss, in great detail, several problems that lead to recurrence relations, and give you an indication of how they can be solved.

Recurrence relations have been used in various ways from the time of Fibonacci (1170-1250). Jakob Bernoulli (1654-1705), his nephew Daniel Bernoulli (1700-1782), James Stirling (1692-1770), Euler, and other mathematicians of the late seventeenth and early eighteenth centuries also used recurrence relations extensively to solve problems in analysis and combinatorics. More recently, recurrence relations have been used in diverse fields like economics, psychology and sociology.

When we talk of "using recurrences", what do we mean? Is it enough to merely state the problem as a recurrence relation? After all, the problem has to be solved. So, what is important is to be able to solve recurrence relations, that is, to find explicit formulas for recursively defined functions. This is what we aim to do in Units 8 and 9.

In Unit 8, we introduce you to the theory of combinatorial generating functions, developed in the late eighteenth century, and first discussed by Laplace in his 1812 classic 'Theorie Analytiques des Probabilité. A generating function is merely a simple and sophisticated mathematical model for a counting problem. By using this, one can solve complicated counting problems, some of which cannot be solved by the combinatorial arguments of Block 2. In this unit we have shown how generating functions can be used to model selection and arrangement problems, as well as partition problems. In the context of recurrences, we have discussed how to solve them by using generating functions. This method of solution was introduced by De Moivre and James Stirling (1692-1770).

In Unit 9 we take up four other techniques for solving recurrence relations. As you will see, we have largely confined ourselves to solutions of only one type of recurrence relation. Sometimes, though, other types of recurrences can be reduced to this type and solved by the methods given in the unit.

By the time you reach the end of the block, we hope that you will be familiar with various aspects of recurrence relations. We also hope that you would appreciate the simplicity and elegance of the methods using recurrences for solving counting problems.

UNIT 7 RECURRENCE RELATIONS

Structure

.1 INTRODUCTION

the previous block, you have learnt to solve various types of combinatorial oblems using a varied set of tools. However, there are many kinds oblems that have to do with counting which cannot be tackled only with e techniques we have presented so far. To give you one such example, look the problem of counting the number of ways of filling up n boxes labelled 2, ... , n, with 0's and 1's, in such a way that no two adjacent boxes have a To solve this, and several such problems, we need to use the idea of currence relations'.

e basic method to solve counting problems that seem to resist a solution ng the basic counting tools is the method of recurrences. As a first step, ı will need to set up the required recurrence relation satisfied by the blem. This is quite similar to the problem of learning how to get to the ı rung of a ladder from the $(n-1)$ th rung. It is for this reason that you y expect to be able to verify solutions to recurrence relations by use of thematical Induction. The second and final step is to actually solve the urrence. There is a large number of techniques available to us to do this, l we will study them in the following Units.

e first two sections deal with problems that may be solved by means of h relations. We aim to give you some insight into how to set up irrences in these sections. The next section deals with various notations definitions that you may come across in the course of this block. Finally, discuss 'divide and conquer relations', used in Computer Science.

jectives

er reading this unit, you should be able to

efine a recurrence relation;

ive examples of recurrence relations;

et up recurrence relations;

pply divide and conquer algorithm.

7.2 THREE RECURRENT PROBLEMS

Let us begin by exploring three sample problems that will give you an idea of what is to follow. They have two characteristics in common: each has been investigated repeatedly for centuries, and each has a solution based on the idea of recurrences. This means that the solution to each problem depends on the solution to smaller instances of the same problem.

Fig.1: Fibonacci (1170-1250)

Problem 1(Rabbits and the Fibonacci numbers): Have you heard of the problem of breeding rabbits, originally posed by **Leonardo di Pisa**, also known as **Fibonacci**, in 1202 in his book **Liber abaci**? The problem is: one pair of rabbits, one male and one female, are left on an island. These rabbits begin breeding at the end of two months and produce a pair of rabbits of opposite sex at the end of each month thereafter. Can you determine the number of pairs of rabbits after n months assuming no rabbit dies on this island?

Let $\mathcal{F}_n$ denote the number of pairs of rabbits after n months. Then $\mathcal{F}_1 = 1$. Since the pair does not breed in the second month, $\mathcal{F}_2 = 1$ as well. To find the number of pairs after n months, we must add the number of pairs after n − 1 months to the number of pairs born in the nth month. But the newborns come from pairs at least two months old, i.e. from the pairs that already existed after n − 2 months; there are $\mathcal{F}_{n-2}$ of these. Therefore, the sequence $\{\mathcal{F}_n | n \geq 1\}$ meets the condition $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ if $n \geq 3$, together with $\mathcal{F}_1 = 1 = \mathcal{F}_2$. This sequence is called the **Fibonacci**, sequence, and the $\mathcal{F}_n$ are called **Fibonacci numbers.**

So, have we solved the problem? Not quite; but it uniquely defines the sequence we seek, describing its members in terms of some previous members. We can also define $\mathcal{F}_n$ as a function of n, as in the following exercise.

E1) Using induction, verify that $\sqrt{5}\,\mathcal{F}_n = (\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n$, $n \geq 1$.

Now let us consider another important recurrent problem.

Problem 2 (The Tower of Hanoi): This problem is invented by the French mathematician **Edouard Lucas** in 1883. We are given a tower of eight discs, initially stacked in decreasing size on one of three pegs.

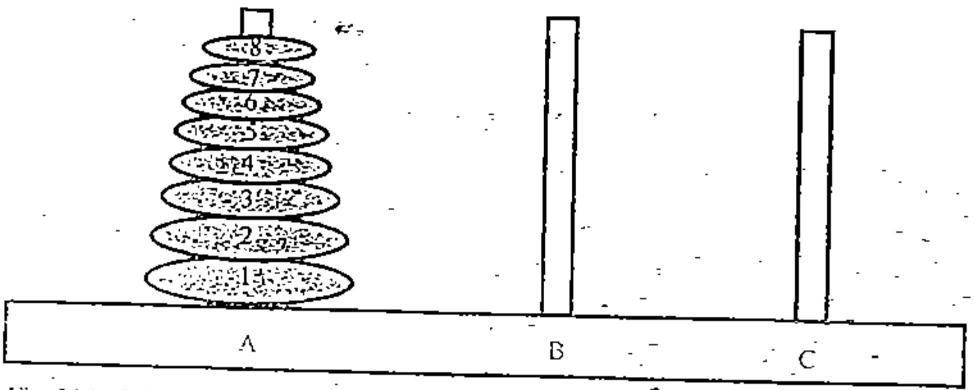

Fig. 2(a): Initial position for the towers of Hanoi-problem.

The objective is to transfer the entire tower to one of three pegs, moving only one disc at a time without ever moving a larger disc onto a smaller one. Lucas furnished this toy with a legend about a much larger Tower of Brahma, which supposedly had 64 discs of pure gold resting on three diamond needles. At the beginning of time, he said, God placed these golden discs on the first needle and said that a group of priests should transfer them to a third, according to the rules above. The Tower will crumble and the world come to an end once the task is finished.

Let us generalise this problem and see what happens if we have n discs instead. Let us say that T_n is the minimum number of moves that will transfer n disks from one peg to another under the rules. Clearly, $T_1 = 1$, and $T_2 = 3$ (why?). A little bit of experimentation on three disks leads us to the general strategy: we first transfer the $n-1$ smallest to a different peg (requiring T_{n-1} moves), then move the largest (requiring one move; remember, it must move!), and finally transfer the $n-1$ smallest back onto the largest (requiring another T_{n-1} moves). Thus, we can transfer n discs

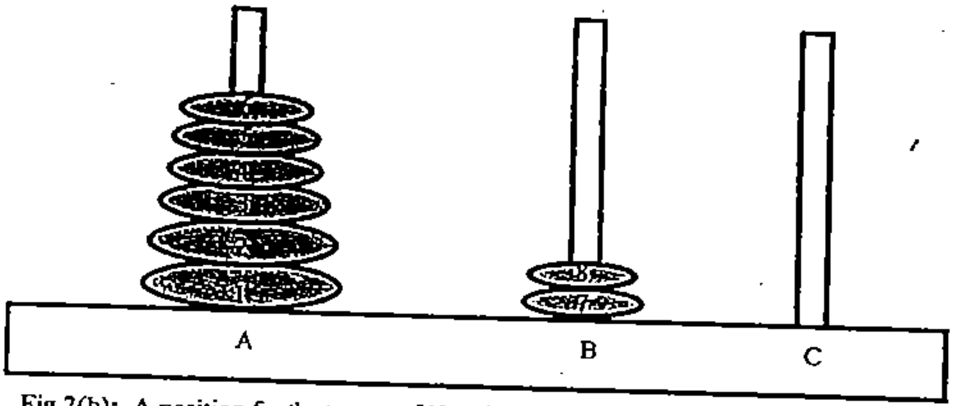

Fig.2(b): A position for the towers of Hanoi Problem after three moves.

or $n \geq 2$) in at most $2T_{n-1} + 1$ moves. So, $T_n \leq 2T_{n-1} + 1$, if $n \geq 2$. Why ave we used " $\leq$ " instead of " $=$ " here? Our construction proves only that $T_{n-1} + 1$ moves are enough but can we do better? The answer is "No". At me point, we must move the largest disc. When we do, the $n-1$ smallest ust be on a single peg (why?), and it has taken at least T_{n-1} moves to put em there. And, before moving the largest disc for the last time, we must ansfer the $n-1$ smallest discs (which must again be on a single peg) back ito the largest; this too requires T_{n-1} moves. Hence, $T_n \geq 2T_{n-1} + 1$ if ≥ 2.

with the first example, we shall postpone solving the recurrence relation st obtained to Unit 9. Incidentally, once you have done the following ercise, you will note that the priests will require a minimum of $^{4} - 1 = 18\,446\,744\,073\,709\,551\,615$ moves to transfer the golden disks. Even the rate of one move per second, it will take them more than 5×10^{11} ırs to solve the puzzle, so the world should survive a while longer!

) Using induction, show that $T_n = 2^n - 1$, $n \geq 1$.

---

w let us consider the third problem we had in mind. This recurrent blem has a geometric flavour.

oblem 3 (Lines in the plane): We wish to determine the maximum number of regions, L_n, into which the plane is cut by n straight lines. As with our previous examples, we will content ourselves with merely being able to express L_n in a recurrence equation, leaving the task of solving it for Unit 9. Looking at the first few cases will convince you that a picture would help. We have pictured the situation for n = 1 and 2 . We suggest you make a

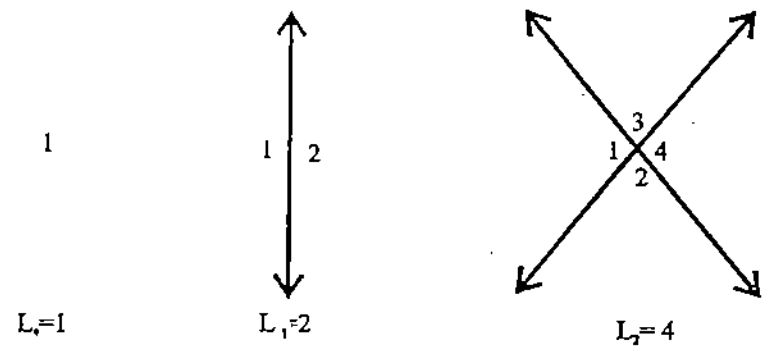

Fig.3

drawing in the case n = 3. The answer for three lines will tell you that the initial guess (which you may be tempted to make looking at the one line and the two line case) that $L_n = 2^n$ needs to be revised. Suppose we have already broken up the plane into L_{n-1} regions by means of $n-1$ lines. We should insert the nth line in a manner so as to increase the number of regions by as much as possible. A little bit of playing around will convince you that the number of regions increase by k precisely when the nth line splits k of the previous regions. This will take place only if it hits the previous lines in $k-1$ different places. However, two lines can intersect in at most one point, so that the new line can intersect the $n-1$ old lines in at most $n-1$ different points. So $k-1 \leq n-1$, i.e., This establishes the upper bound $L_n \leq L_{n-1} + n$ for $n \geq 2$.

But, can we achieve this upper bound? To do this, we place the n th line in such a way that it is not parallel to any of the others (hence intersects them all), and such that it does not pass through any of the existing points of intersection (hence it intersects them at different points). This establishes the recurrence for L_n, namely, $L_n = L_{n-1} + n$ for $n \geq 2$, with $L_1 = 2$.

Here's an exercise for defining L_n in terms of n.

E3) Using induction, show that $L_n = \frac{1}{2}n(n+1) + 1$, $n \geq 1$.

You will have noticed that in each of the three problems, we have been able to express the nth term of a sequence in terms of one or more previous terms and a function of n. This gives you a method to compute the terms of the sequence accurately, given enough time. At times, if the relation between the terms is in a reasonably nice form, we can even "solve" the recurrence, that is, express the n th term as a function of n. You will learn how to solve these three recurrences by the methods discussed in Unit 9.

Let us now consider some more recurrent problems.

7.3 MORE RECURRENCES

You have been exposed to some famous recurrent problems in the previous section. In this section, we shall take another look at setting up recurrence relations for combinatorial problems of the kind you would have encountered in the previous block or elsewhere. You will find that in trying to determine the recurrence, we are really attempting to describe the counting inductively. In most cases, you will see that the recurrence relation leads to an alternate method of solution, although the methods themselves will be dealt with in Unit 9.

Problem 4: We begin by considering the problem of sorting a list of n numbers into increasing order. Let us denote by c_n the number of comparisons made in sorting n items. To find the smallest element of the list, we will need to make $n-1$ comparisons (pick the first two items in the list, choose the smaller one, compare this to the third item, and continue the procedure). If we now exchange the first element with the smallest, we are left to carry out the process on $n-1$ items. Since the number of comparisons necessary on the remaining $n-1$ items is c_{n-1}, the total number of comparisons is given by $c_n = c_{n-1} + n - 1$, $n \geq 2$, with $c_1 = 0$.

E4) Using the recurrence relation for c_n, show that $c_n = \frac{1}{2}n(n-1)$, $n \geq 1$.

Problem 5: You may recall that the set of all subsets of any non-empty set, S, is called its **power set**, and denoted by $\mathcal{P}(S)$. Let us determine a recurrence relation satisfied by $s_n = |\mathcal{P}(S)|$, where $|S| = n$. Let us take $S = \{1, 2, \ldots, n\}$. Now, any subset, A, of S either contains the number n or does not. Let us consider these two mutually exclusive case separately and count the number of such subsets, A. If $n \in A$, then $A = A' \cup \{n\}$, where A' is a subset of $\{1, 2, \ldots, n-1\}$. So, there are as many subsets A as there are subsets A'. Since $A' \subset \{1, 2, \ldots, n-1\}$, there are s_{n-1} such subsets A. On the other hand, if $n \notin A$, then, in fact, A is a subset of $\{1, 2, \ldots, n-1\}$, and there are s_{n-1} of these too. Combining these, we see that $s_n = s_{n-1} + s_{n-1} = 2s_{n-1}$, $n \geq 1$, with $s_0 = 1$.

E5) Using the recurrence relation for s_n, show that $s_n = 2^n$, $n \geq 0$.

Problem 6: Recall that a **bijection** is a **one-one, onto** mapping of a set onto itself. It is quite easy to determine directly the number of bijections of an n-set (a set with n elements). We will, however, be looking at a recurrence relation satisfied by the number of bijections, b_n, of any n-set, say $\{1, 2, \ldots, n\}$. To begin with, if f is any such bijection, f(n) could be any one of the n elements of the set $\{1, 2, \ldots, n\}$. But now we must map the elements of $\{1, 2, \ldots, n-1\}$ bijectively to $\{1, 2, \ldots, n\} \setminus \{f(n)\}$; there are b_{n-1} ways of doing this, and hence that many choices for the function f. Notice that each choice of f(n) leads to a bijection of an $(n-1)$-set. In all then, $b_n = nb_{n-1}$, $n \geq 2$, with $b_1 = 1$.

E6) Using the recurrence relation for b_n, show that $b_n = n!$, $n \geq 1$.

We end this section by looking again at the problem of the missing hats, discussed towards the end of the previous block.

Problem 7: You may recall that the problem is to determine the number of **derangements** of n objects, d_n, and that we had employed the method of **Inclusion-Exclusion** to solve it.

Recall that d_n counts the number of permutations of n objects that leave no object fixed. Any such permutation is called a **derangement**. Let us begin by labelling the objects serially: $1, 2, \ldots, n$. In any such derangement of n objects, 1 gets sent to some i, where $i \neq 1$. Two cases arise: for the same

derangement, either i gets sent back to 1 or it does not. In the first case, we can leave out 1 and i from the original set and obtain a derangement of $n-2$ objects; there are d_{n-2} such possibilities. In the second case, we may omit 1 from the original set to obtain a derangement of $n-1$ objects; there are d_{n-1} such possibilities. Therefore, assuming 1 gets sent to i, there is a total of $d_{n-1}+d_{n-2}$ possibilities. Observing that i could have been any number between 2 and n, we conclude that $d_n = (n-1)(d_{n-1}+d_{n-2})$ for $n \geq 3$. To complete the recurrence relation, we note that $d_1 = 0$ and $d_2 = 1$.
You will have noticed that to compute d_n one needs to know the values of the two preceding terms. Can we get to compute d_n on the basis of the value of only one preceding term, d_{n-1}? To explore this, let us write the recurrence in the form $d_n - nd_{n-1} = -[d_{n-1} - (n-1)d_{n-2}]$. We now observe that the expression on the right hand side within the brackets is got from the expression on the left hand side by merely replacing n by $n-1$. If we write $D_n = d_n - nd_{n-1}$, we have the simplified expression $D_n = -D_{n-1}$. But then $D_{n-1} = -D_{n-2}$, and so $D_n = D_{n-2}$. Continuing this procedure, we arrive at $D_n = (-1)^{n-2}D_2 = (-1)^n[d_2 - 2d_1] = (-1)^n$. Therefore, we have $d_n = nd_{n-1} + (-1)^n$ if $n \geq 2$, with $d_1 = 0$.

E7) Using either recurrence relation for d_n in the discussion above, show that

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!},\ n \geq 1.$$

We end this section with a few problems in which you are required to set up the recurrence equation.

E8) For each $n \geq 1$, define $a_n = \sum_{k=0}^{n} C(n+k, 2k), b_n = \sum_{k=0}^{n} C(n+k, 2k+1)$, with $a_0 = 1$, $b_0 = 0$. Show that for each $n \geq 0$, $a_{n+1} = a_n + b_{n+1}$, $b_{n+1} = a_n + b_n$.

E9) Derive the recurrence relation for the number of ways to parenthesise the expression $x_1 + x_2 + \cdots + x_n$ so that only two terms will be added at a time. For example, the expression $((x_1 + x_2) + x_3)$ is fully parenthesized, but $(x_1 + x_2) + x_3$ is not.

E10) Set up a recurrence relation for the determinant of the $n \times n$ matrix with 1 along the main diagonal and with 1 on either side of the main diagonal in each row and zero elsewhere.

E11) Set up a recurrence relation for the n-digit sequences of numbers using only the integers $\{0, 1, 2, 3\}$ having an even number of 0's.

E12) Show that the number of r-permutations of n distinct objects, $P(n, r)$, satisfies the recurrence relation
$P(n, r) = P(n-1, r) + rP(n-1, r-1)$, $n \geq 1$, $r \geq 1$.

E13) Let S_r^n denote the Stirling numbers of the second kind, that is, the number of ways to distribute r distinct objects into n nondistinct boxes with no box left empty. Show that S_r^n satisfies the recurrence relation $S_{r+1}^n = S_r^{n-1} + nS_r^n$; $1 < n < r$.

E14) Let $f(n,k)$ denote the number of ways of selecting k numbers from the n numbers $1, 2, \ldots, n$ so that no two consecutive numbers are selected. Find a recurrence relation for $f(n,k)$, and using the conditions $f(n,1) = n$ and $f(n,n) = 0$, verify that $f(n,k) = C(n-k+1, k)$.

E15) Let t_n be the number of incongruent triangles with integral sides and perimeter n. Show that

$$t_n = \begin{cases} t_{n-3} & \text{if n is even;} \\ t_{n-3} + \dfrac{n + (-1)^{(n+1)/2}}{4} & \text{if n is odd.} \end{cases}$$

E16) Suppose n unit circles are drawn on the plane such that each circle intersects any of the other circles at exactly two points and no three circles meet at the same point. Derive a recurrence relation for the number, r_n, of regions into which the plane is divided by the n circles.

In the next section, we give all relevant definitions and introduce the notations.

7.4 DEFINITIONS

We hope you have got a fairly good idea of what a "recurrence relation" is, as well as how to set it up by now. It is time to formalise the procedure and set up a more rigorous mathematical pedestal for it. A **recurrence relation** is a formula that counts the number of ways to do a procedure involving n objects in terms of the number of ways to do it with fewer objects. The formal definition is as follows:

Definition: Let $\{a_n : n \geq 0\}$ be a sequence of real or complex numbers. A **recurrence relation** (or a **recurrence equation**) is an expression of the form

$$a_n = F(a_{n-1}, a_{n-2}, \ldots, n)$$

where F is a function of some of the variables $a_{n-1}, a_{n-2}, \ldots, a_0, n$. Note that all the a_is need not occur in the expression.

In other words, it allows us to compute the n th term of a sequence from one or more of the preceding terms. The symbol "F" merely denotes a (any) function, and the variables are (some or all of) the preceding terms in the sequence as also n. For our purposes, we shall only deal with such functions which are **polynomials** and depend on only **finitely** many variables $a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n$.

Definition: The **order** of the recurrence relation defined by

$$a_n = F(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n)$$

is k, where a_n depends on one or more of the previous k terms and k is the smallest such integer. We do not define an order for recurrence relations of the form $a_n = F(a_{n-1}, a_{n-2}, \ldots, n)$ that depend on each of its previous terms.

Therefore, if we can compute the n th term of a sequence from the preceding k terms, but not from the preceding k − 1 terms, we define the order to be k.

Definition: The **degree** of the recurrence relation is the degree of F, treated as a polynomial in its variables excluding n. If F is not a polynomial in its variables, no degree is assigned to the recurrence relation.

A recurrence relation of degree one is also called **linear**, one of degree two **quadratic**, and so on, just like we have in the case of polynomials. After all, the notion of "degree" is tied up with the degree of the defining polynomial F.

Definition: A recurrence relation is called to be **homogeneous** if it contains no term that depends only on the variable n. A recurrence relation that is not homogeneous is said to be **non-homogeneous** or **inhomogeneous**.

Thus, for a recurrence to be called homogeneous, every term defining the recurrence must contain at least one of the preceding terms of the sequence. Usually, the term **homogeneous** is used for linear recurrences regardless of its order.

Examples:

1. $a_n = 3a_{n-1} + n^2$ is **nonhomogeneous** of order 1 and degree 1.
2. $a_n = na_{n-2} + 2^n$ is **nonhomogeneous** of order 2 and degree 1.
3. $a_n = \sqrt{a_{n-1}} + a_{n-2}^2$ is **homogeneous** of order 2, but has no degree.
4. $a_n = a_{n-1} + a_{n-2} + \cdots + a_0$ is **homogeneous**, has no order, but has degree 1.
5. $a_n = a_{n-1}^2 + a_{n-2}a_{n-3}a_{n-4}$ is **homogeneous** of order 4 and degree 3.
6. $a_n = \sin a_{n-1} + \cos a_{n-2} + \sin a_{n-3} + \cdots + e^n$ is **nonhomogeneous**, has no order and no degree.
7. $a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_{n-k}(n)a_{n-k} + g(n)$ represents the general form of a linear k th order recurrence relation ($f_{n-k}(n) \not\equiv 0$). It is homogeneous if $g(n) = 0$ for each n, and **nonhomogeneous** otherwise.
8. $a_n = a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-1}a_0 (n \geq 2)$ with $a_0 = 0$, and $a_1 = 1$ is a nonlinear recurrence relation.
9. $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$ is a recurrence relation in two variables n and k. Taking $a_{n,k} = C(n,k)$, the given relation is nothing but Pascal's identity with initial conditions $a_{n,0} = C(n,0) = a_{n,n} = C(n,n) = 1$ for all $n \geq 0$ and $a_{n,k} = 0, k \geq n$.
10. $a_{n,k} = a_{n-2,k-1} + a_{n-3,k-1} + a_{n-4,k-1}$, with initial conditions $a_{2,1} = a_{3,1} = a_{4,1} = 1$ and $a_{k,1} = 0$ otherwise, is a recurrence relation in two variables (This is the recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box).
11. $a_n = a_{n/2} + 1$ with $a_1 = 0$ (n a power of 2) is a nonlinear recurrence relation.

You must have observed while looking at the various examples above that the recurrence relation alone will not define for you the terms of the sequence. To be able to do this, one needs to know where to begin the sequence. If a_n is defined in terms of a_{n-1} alone, deciding the value for a_0 (or, a_1, or where ever you wish to begin the sequence) uniquely describes the sequence for you. More generally, in case of a k th order recurrence, one needs to know the first k terms, typically $a_0, \ldots, a_{k-1}$ of the sequence in order to uniquely define the sequence. A well-defined linear recurrence relation of degree k consists of a recurrence part and initial conditions for k consecutive values.

Definition: A k th order recurrence relation has **initial conditions** provided the values of one or more of the terms $a_0, a_1, \ldots, a_{k-1}$ are known.

Definition: A function f(n) is said to be a **general solution** to the recurrence relation if it satisfies the recurrence equation.
A function g(n) is said to be the **particular solution** to a recurrence relation if it satisfies the recurrence equation, together with the initial conditions.

Please note that there are infinitely many "general solutions" to any recurrence relation without initial condition(s), one for each set of values for the initial terms, but only one "solution" once the first k initial terms are fixed for recurrence relations of order k. You have been verifying that given functions are indeed solutions to the recurrences of the previous two sections. We give a few more examples of a simpler nature. The solution of recurrence relations will be discussed in unit 9.

Examples:

1. The general solution to $a_n = a_{n-1}$ is $a_n = c$, where c is any constant, but if in addition $a_0 = 1$, then the solution is $a_n = 1, n \geq 0$.
2. The general solution to $a_n = a_{n-1} + 1$ is $a_n = c + n$, where c is any constant; if $a_0 = 0$, then the solution is $a_n = n, n \geq 0$.
3. The general solution to $a_n = ka_{n-1}$ is $a_n = ck^n$, where c is any constant; if $a_0 = 1$, then the solution is $a_n = k^n, n \geq 0$.
4. The general solution to $a_n = a_{n-1} + a_{n-2}$ is
$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$, where c_1, c_2 are any constants.
If $a_1 = 1, a_2 = 3$, then the particular solution is
$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 1$.
5. The general solution to $a_n - \frac{n}{n-1}a_{n-1} = n^3$ with $a_1 = 1$, is
$a_n = \frac{n^2(n+1)(2n+1)}{6}$

In the concluding section, we discuss some common types of recurrence relations that result from divide and conquer algorithms.

1.5 DIVIDE AND CONQUER RELATIONS

This is a decomposition algorithm that solves a problem of size $n \in Z^+$ by

1. breaking it up into a number of instances of the same kind of problem with smaller input parameter (several smaller nonoverlapping subproblems of approximately equal size),
2. solving these subproblems, and
3. use their solutions to construct a solution for the original problem of size n.

We shall be specially interested in cases where n is a power of 2.
We discuss a few classic examples of such algorithms to motivate you i.e. see recurrence relations that result from divide and conquer algorithm.

Problem 8: In a tennis tournament, each entrant plays a match in the first round. Next all winners from the first round play a second-round match. Winners continue to move on to the next round, until finally only one player is left as the tournament winner. Assuming that tournaments always involve

$n = 2^k$ players, for some k, find the recurrence relation for the number of rounds in a tournaments of n players.

The recurrence relation for a_n, the number of rounds, is $a_n = a_{n/2} + 1$. since after $a_{n/2}$ rounds there remains only two players, the winners of the subtournament of the first n/2 players and the subtournament of the second n/2 players. One more rounds picks the tournament winner from the two remaining players. Here $a_1 = 0$ (with one player, zero tournament. You will see in unit 9 that its solution is given by $a_n = \log_2 n$

Problem 9: Suppose that A is a sorted array of n elements, and we wish to determine whether some number x is in the list. The straightforward sequential search in which x in successively compared to $A[1], A[2], \cdots, A[n]$ requires n comparisons in the worst case.

Consider the following divide and conquer binary search algorithm for determining whether or not a number x is in a sorted list kept in some array.

If the array has one element, then compare x to that element the required number of comparisons (a_n) for n = 1, is one, that is $a_1 = 1$.

If the array has more than one element, get the element M in the 'middle' of the array. If x is greater than or equal to M then call the algorithm recursively on the "Second half'" of the array. Otherwise call the algorithm on the "first half" of the array.

This is called binary search because it successively throws away "half" the remaining possible elements until it has found the one entry that could be x. Suppose $n = 2^k$, and the required number of comparisons be a_n. Then $a_1 = 1$ and $a_n = a_{n/2} + 1 (n \geq 2)$.

If n = 1, then the list has one element and it requires one comparison. Otherwise, we have one comparison with M (It represents the overhead required to break the problem into half and $a_{n/2}$ comparisons in the recursive call to solve the subproblem

Very often a list of n names must be alphabetized, or a list of n numbers must be rearranged into ascending order. Such an alphabetization or rearrangement is called a simple sort

Suppose that we have a pair of lists each with m (m and n) elements. which are already sorted. The two sorted lists may be merged into one sorted list of 2m (m + n) elements. This is called **merge sort**, That is, given sorted lists $A = (a_1, a_2 \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_m)$, with $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_m$, we wish to produce a list $C = (c_1, c_2 \cdots, c_{n+m})$ which contains all the elements of the two lists A and B, totally sorted so that $c_1 \leq c_2 \leq \cdots \leq c_{n+m}$. We briefly indicate the way the list C is obtained. We begin with our left index finger pointing to a_1 and our right index finger pointing to b_1. We compare the two numbers and find the smaller. The smaller number is put in the list C and we advance the index finger that was pointing to that number. We repeat the process-compare numbers being pointed at, put the smaller one as the next element of C, and advance that finger-until the list C has been filled completely. Every time the comparison of a_j to b_k is made another element is correctly placed in the merged list. Hence the number of comparisons required is $2m - 1$ $(m + n - 1)$

Sorting algorithm for an array A of n elements say $x_1, x_2, \cdots, x_n$: In simple sorting algorithm, to determine the smallest number in the list, we first compare x_1 to x_2, the smaller of the two in compared to x_3, the result is compared to x_4 and so on. Add this number to what is to be the final sorted list and create a new list of $n-1$ elements by removing this number from the original list. To find the second smallest number in the original list, apply the process described above to the list of $n-1$ elements left after the first step and so on. (refer Problem 4 in section 7.3)

Problem 10: We now derive the recurrence relation of the merge sort algorithm for sorting an array of n numbers. In merge sort algorithm we have the following steps: If A has one element then it is already sorted else we divide the array in "half": i.e, we split the original list in half. recursively sort the first half and recursively sort the second half.

Merge the two halves together to form a single array. Suppose $n = 2^k$. $a_1 = 0$. The number of comparisons used by merge sort satisfies the recurrence relation

$a_n = 2a_{n/2} + n - 1 \quad (n \geq 2)$.

where $2a_{n/2}$ term represents the total comparisons required for the subproblems; and the $n-1$ term in the overhead required to combine the results i.e. merging two sorted lists each containing n/2 elements $(2 \times \frac{n}{2} - 1)$. The solution is given by $a_n = n \log_2(n) - n + 1 = k2^k - 2^k + 1$.

We call a recurrence a divide and conquer recurrence if it has the form $a_n = ba_{n/a} + d(n)$ for integer $a \geq 1$, where b is a constant and d is a function of n. We have considered the cases where $a = 2$.

E17) Finding the nth power of an interger i, by successive multiplications by i requires $n-1$ multiplications. Assuming $n = 2^k$, describe a divide and conquer algorithm such that if a_n is the number of multiplications to find the nth power, then $a_n = a_{n/2} + 1$. Is the algorithm desirable given that the solution is given by $a_n = \log_2(n)$?

E18) Finding the product of a list of n integers by successive multiplication requires $n-1$ multiplications. Assuming $n = 2^k$, describe a divide and conquer algorithm for which the number of multiplications a_n satisfies the recurrence relation
$a_n = 2a_{n/2} + 1$

E19) To multiply two n-digit numbers, one must do normally n^2 digit-times-digit multiplications. Use a divide and conquer algorithm to do better when n is a power of 2.

With this we have come to the end of this unit. Next two units will deal with the methods of solving recurrence relations. Now let us take a quick look at what we have discussed in this unit.

7.6 SUMMARY

In this Unit recurrence relations you have studied the following points:

1) You have come across several examples of recurrence relations, drawn from well-known problems and from routine exercises in combinatorics.

2) You should have a fairly good idea of how to set up recurrence relations after having read this unit.

3) You should also be familiar with the different definitions that have been introduced to you.

4) Finally you have learnt of setting up of recurrence relations with the help of divide and conquer algorithm.

7.7 SOLUTIONS/ANSWERS

E1) It is easy to check that $\mathcal{F}_1 = 1 = \mathcal{F}_2$. With $\alpha \doteq \frac{1+\sqrt{5}}{2}$, $\beta \doteq \frac{1-\sqrt{5}}{2}$, we observe that α, β are solutions to the equation $x^2 - x - 1 = 0$. If $n \geq 3$, $\sqrt{5}\,(\mathcal{F}_{n-1} + \mathcal{F}_{n-2}) = (\alpha^{n-1} - \beta^{n-1}) + (\alpha^{n-2} - \beta^{n-2}) = \alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1) = \alpha^{n-2}.\alpha^2 - \beta^{n-2}.\beta^2 = \alpha^n - \beta^n = \sqrt{5}\,\mathcal{F}_n$, as desired.

E2) Observe that $T_1 = 1$. If $n \geq 2$, $2T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1 = T_n$, verifying the formula.

E3) Note that $L_1 = 2$. If $n \geq 2$, $L_{n-1} + n = \frac{1}{2}(n-1)n + 1 + n = \frac{1}{2}n(n+1) + 1 = L_n$, as required.

E4) It is easy to see that $c_1 = 0$. If $n \geq 2$, $c_{n-1} + n - 1 = \frac{1}{2}(n-1)(n-2) + n - 1 = \frac{1}{2}n(n-1) = c_n$, as desired.

E5) Observe that $s_0 = 1$. If $n \geq 1$, $2s_{n-1} = 2.2^{n-1} = 2^n = s_n$, verifying the formula.

E6) We note that $b_1 = 1$. If $n \geq 2$, $nb_{n-1} = n.(n-1)! = n! = b_n$, as required.

E7) We check that $d_1 = 0, d_2 = 1$. To verify the first order recurrence relation, note that if $n \geq 2$,

$$\begin{aligned} nd_{n-1} + (-1)^n &= n.(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} + (-1)^{n-1} \\ &= n!\left(\sum_{i=0}^{n}\frac{(-1)^i}{i!} - \frac{(-1)^n}{n!}\right) + (-1)^n \\ &= d_n \end{aligned}$$

as desired.

In case of the second order recurrence relation, if $n \geq 1$,

$$\begin{aligned} (n-1)(d_{n-1} + d_{n-2}) &= (n-1)\left[(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} + (n-2)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}\right] \\ &= n.(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} \\ &\quad -(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} \\ &\quad +(n-1)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} \\ &= n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} - (n-1)!\frac{(-1)^{n-1}}{(n-1)!} \\ &= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} = d_n, \end{aligned}$$

as desired.

E8) Writing $a_n = \sum_{k=1}^{n} C(n+k, 2k) + 1$, we get

$$\begin{aligned} a_{n+1} - a_n &= \sum_{k=1}^{n} C(n+k+1, 2k) - \sum_{k=1}^{n} C(n+k, 2k) + 1 \\ &= \sum_{k=1}^{n} C(n+k, 2k-1) + 1 \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{n-1} C(n+k+1, 2k+1) + 1 \\ &= \sum_{k=0}^{n} C(n+k+1, 2k+1) \\ &= b_{n+1}. \end{aligned}$$

Likewise, with $b_n = \sum_{k=0}^{n} C(n+k, 2k+1)$,

$$\begin{aligned} b_{n+1} - b_n &= \sum_{k=0}^{n} C(n+k+1, 2k+1) - \sum_{k=0}^{n-1} C(n+k, 2k+1) \\ &= \sum_{k=0}^{n-1} C(n+k+1, 2k+1) - \sum_{k=0}^{n-1} C(n+k, 2k+1) + 1 \\ &= \sum_{k=0}^{n-1} C(n+k, 2k) + 1 \\ &= \sum_{k=0}^{n} C(n+k, 2k) \\ &= a_n \end{aligned}$$

E9) If the number of ways to parenthesize the expression $x_1 + x_2 + \cdots + x_n$ is a_n, the required number for the two subexpressions $x_1 + \cdots + x_k$ and $x_{k+1} + \cdots + x_n$ are a_k and a_{n-k}, respectively. It follows that there are $a_k a_{n-k}$ ways to parenthesize the total expression, with $k \geq 1$.
Therefore, the recurrence relation satisfied by a_n is
$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1},\ n \geq 2$, with $a_1 = 1$.
Using the fact $a_0 = 0$, we can extend this to
$a_0 = 0, a_1 = 1, a_n = a_n a_0 + a_{n-1}a_1 + \ldots + a_1 a_{n-1} + a_{n-1} + a_0 a_n (n \geq 2)$

E10) Let Δ_n denote the required $n \times n$ determinant. Expanding about the first row, we get Δ_{n-1} minus the determinant which when expanded about its first row yields Δ_{n-2}. The corresponding recurrence relation is $\Delta_n = \Delta_{n-1} - \Delta_{n-2},\ n \geq 3$, with $\Delta_1 = 1, \Delta_2 = 0$.

E11) Let a_n denote the number of n-digit sequences containing an even number of 0's. Then there are a_{n-1} $(n-1)$-digit sequences that have an even number of 0's and $4^{n-1} - a_{n-1}$ $(n-1)$-digit sequences that have an odd number of 0's. To each of the a_{n-1} sequences that have an even number of 0's, the digit 1, 2 or 3 can be appended to yield sequences of length n that contain an even number of 0's. To each of the $4^{n-1} - a_{n-1}$ sequences that have an odd number of 0's, the digit 0 must be appended to yield sequences of length n that contain an even number of 0's. Therefore, for $n \geq 2$, $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$, with $a_1 = 3$.

E12) Of the n distinct objects, pick any one object and call it "special". Then, the number of r-permutations in which this "special" object does not appear is $P(n-1, r)$ because this is the number of r- permutations of the remaining $n-1$ objects. On the other hand, if the "special" object does appear, the number of r-permutations is $rP(n-1, r-1)$ because the "special" object could be in any one of r positions between objects or at either end, and we have then to determine the number of $(r-1)$-permutations of $n-1$ objects. Combining the two, we get the required recurrence.

E13) This is somewhat similar to the previous one; choose a "special" object first. The box containing this object either contains no other object or contains at least one more. In the first case, we need to distribute r distinct objects into $n-1$ nondistinct boxes, with no empty box; the number of ways in which to do this is S_r^{n-1}. Otherwise, the special "object" may be placed into any one of the n (nondistinct) boxes (there are n choices), and we still need to distribute r objects into n nondistinct boxes, with no empty box; there are S_r^n such choices for each choice of the box the "special" object is placed in. Combining the two cases gives the recurrence relation.

E14) We first show that $f(n,k) = f(n-1,k) + f(n-2,k-1)$ for $1 \le k \le n-1$. If the number 1 is among the k numbers chosen, we have to choose $k-1$ more numbers from the $n-2$ numbers $3, 4, \ldots, n$. If the number 1 is not among the k numbers selected, we have to choose k numbers from the $n-1$ numbers $2, 3, \ldots, n$.
To verify the formula, we induct on $n+k$. The base case is trivial. From the induction hypothesis, we have
$f(n,k) = f(n-1,k) + f(n-2,k-1) = C(n-k,k) + C(n-k,k-1) = C(n-k+1,k)$ as desired.

E15) A triangle may be formed with integral sides a, b, c, with $a \ge b \ge c$ if and only if $b + c \ge a + 1$. Therefore, given a triangle with sides a, b, c, there exists a triangle with sides $a-1, b-1, c-1$ if and only if $b + c \ge a + 2$. Conversely, given a triangle with sides $a-1, b-1, c-1$, there always exists a triangle with sides a, b, c. Hence, the difference between the number of triangles with integral sides a, b, c and those with integral sides $a-1, b-1, c-1$ is precisely the number of ordered triples (a, b, c), $a \ge b \ge c$, such that $b + c = a + 1$.
Let us count the number of such triples. If $b + c = a + 1$, the perimeter, $n = a + b + c = 2a + 1$ must be an odd integer, so that there are as many triangles with perimeter n as there are with perimeter $n-3$. This proves the recurrence for the even case. If n is odd, then $a = \frac{n-1}{2}$ and c must satisfy the inequalities $1 \le c \le b = a - c + 1$, or $1 \le c \le \lfloor \frac{a+1}{2} \rfloor = \lfloor \frac{n+1}{4} \rfloor$. Thus, in this case,

$$t_n - t_{n-3} = \left\lfloor \frac{n+1}{4} \right\rfloor = \frac{n + (-1)^{(n+1)/2}}{4}.$$

E16) A bit of experimentation will yield the formulae $r_1 = 2, r_2 = 4, r_3 = 8$ and $r_4 = 14$. Suppose that we have drawn $n-1$ unit circles that divide the plane into r_{n-1} regions. The n th circle intersects these $n-1$ circles at $2(n-1)$ points, that is, the n th circle will be divided into $2(n-1)$ arcs. Since each of these arcs will divide one of the r_{n-1} regions into two, we have the recurrence $r_n = r_{n-1} + 2(n-1)$, $n \ge 2$.

E17) Divide n by 2, find $i^{\frac{n}{2}}$, and square it. Thus $a_n = a_{\frac{n}{2}} + 1$. The algorithm is desirable since $\log_2(n)$ grows more slowly than n.

E18) Divide n by 2. Find the product of first $\frac{n}{2}$ integers and the product of last $\frac{n}{2}$ integers. Multiply two products obtained. Thus $a_n = 2a_{\frac{n}{2}} + 1$

E19) Let n be a power of 2. Let the two n-digit numbers be A and B. We split each of these numbers into two $\frac{n}{2}$-digit parts:

$A = A_1 10^{\frac{n}{2}} + A_2$ and
$B = B_1 10^{\frac{n}{2}} + B_2$ (like $1235 = 12 \times 100 + 35$)

Then $A.B = A_1B_1 10^n + A_1B_2 10^{\frac{n}{2}} + A_2B_1 10^{\frac{n}{2}} + A_2B_2$

We need only to make three $\frac{n}{2}$-digit multiplications, $A_1.B_1, A_2.B_2$ and $(A_1 + A_2) \cdot (B_1 + B_2)$ to determine $A \cdot B$ since
$A_1 \cdot B_2 + B_2 \cdot A_1 = (A_1 + A_2) \cdot (B_1 + B_2) - A_1 \cdot B_1 - A_2 \cdot B_2$

Actualy $(A_1 + A_2)$ or $(B_1 + B_2)$ may be $(\frac{n}{2} + 1)$-digit numbers but this slight variation does not effect the general magnitude of our solution (like $1295 = 12 \times 10^2 + 95$)

If a_n represents the number of digit-times-digit multiplications needed to multiply two n-digit numbers by the above procedure, this gives the recurrence relation

$$a_n = 3a_{\frac{n}{2}}$$

a_n is proportional to $n^{\log_2 3} = n^{1.6}$ – a substantantial improvement over n^2

UNIT 8 GENERATING FUNCTIONS

Structure

Page No.

8.1 INTRODUCTION

The theory of generating functions is an outstanding example of the beauty of mathematics revealed through its theories whose simplicity and power provides an intuitive understanding of a wide range of problems. Though it is based on simple polynomial arithmetic, it provides a unified approach to questions in many fields. In this unit we will discuss their use in combinatorics, including recurrences.

A **generating function** is simply a formal power series of the type $\sum_{n=0}^{\infty} a_n z^n$, where the coefficients a_n are the terms of a sequence of numbers representing solution of a combinatorial problem and the exponents of the symbol z depicts certain enumerative objectives of that problem.

In Sec.8.2. we shall explain the concept and some elementary uses of generating functions. In Sec.8.3, we shall introduce you to a particular type of generating functions which are used to solve arrangement problems in combinatorics.

In Sec.8.4, we shall explore the power of the generating functions as a tool when, for example, it is used to evolve some combinatorial identities, solve some combinatorial problems involving general integer equations, find the number of partitions and solve certain recurrence relations.

Objectives

After reading this unit, you should be able to

- define and construct generating functions for various types of combinatorial problems;
- identify the generating function associated with a sequence;
- identify the exponential generating function associated with a sequence;
- use generating functions to evolve identities involving combinatorial coefficients;

- use generating functions to solve general integer equation problems, certain problems in the theory of partitions and linear recurrence relations.

8.2 GENERATING FUNCTIONS

As you have seen in previous units, the solution of a combinatorial problem (in most cases) is a number sequence. In some cases these numbers can be obtained explicitly by simple combinatorial arguments. But in many other situations these numbers are linked by certain relations (see Unit 7), and so, we need to do more to get them explicitly.

The beauty of generating functions is that it helps us to solve many such problems using simple algebraic operations on polynomials (possibly infinite). The basic idea here is that we identify sequence of numbers with a power series (a type of infinte polynomial) which on applying certain simple (algebraic) operations assumes a form from where we can easily read out the desired numbers as coefficients.

There are many occasions when terms of a sequence (which otherwise may represent the solution to a combinatorial problem) appear as coefficients in some power series. We illustrate this point with the help of following example.

Example 1: Determine the number of integer solutions to linear equation

$$X_1 + X_2 = 3, \text{ with } 0 \le X_1 \le 1 \text{ and } 0 \le X_2 \le 2.$$

Solution: By explicit enumeration, the possible values are given below.

X_1	X_2	Sum
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	3

Thus, there are two ways to obtain a sum of 1 (also 2) and one way to obtain the sum 3.

Now consider the following product of polynomials:

$$(z^0 + z^1)(z^0 + z^1 + z^2),$$

where the exponents of symbol z in the first factor correspond to the possible values of X_1 and in the second factor to the possible values of X_2. On expanding this product, we get

$$\begin{aligned}(z^0 + z^1)(z^0 + z^1 + z^2) &= (z^0z^0 + z^0z^1 + z^0z^2 + z^1z^0 + z^1z^1 + z^1z^2)\\ &= 1 + 2z + 2z^2 + z^3.\end{aligned}$$

Adding the exponents of the symbol z after multiplication corresonds to considering the sum of the values of X_1 and X_2.

We note that the coefficient of z^r, $1 \le r \le 3$, in this expression gives the number of integer solution to $X_1 + X_2 = r$, with $0 \le X_1 \le 1$ and $0 \le X_2 \le 2$. In particular, because the coefficient of z^3 in the above expression is 1, and so, there is only one pair of values viz. (1,2), which satisfy the given linear equation.

* * *

Suppose we intend to find non-negative integer solution to the linear equation

$$X_1 + X_2 + X_3 = 10, \text{ with } 0 \le X_1 \le 4,\ X_2 > 0, \text{ and } X_3 \ge 0.$$

Then, by arguments given in the example above, we take the product of the following three polynomials:

$$(1 + z + z^2 + z^3 + z^4)(z + z^2 + \dots)(1 + z + z^2 + \dots).$$

In the above product, both second and third factors are infinite because there is no upper bound on X_1 and X_2. Also, second factor does not contain the constant term owing to the fact that $X_2 > 0$. Then, as before, coefficient of z^{10} in the above expression will give us a solution to the linear equation given above.

For finding the coefficients of a power series, we often use the following results.

Result 1: (Binomial Theorem)

a) $$(1+z)^n = \begin{cases} \sum_{r=0}^{n} C(n,r)z^r, & \text{if } n \ge 0 \\ \sum_{r=0}^{\infty} C(n,r)z^r, & \text{if } n < 0. \end{cases}$$

b) $$(1-z)^{-n} = (1 + z + z^2 + \dots)^n = 1 + \sum_{r=1}^{\infty} C(n-1+r, r)z^r.$$

Result 2: $$\frac{1-z^n}{1-z} = 1 + z + z^2 + \dots + z^{n-1},\ z \ne 1.$$

Next, we illustrate the technique of identifying the power series associated with a combinatorial problem with the help of following example.

Example 2: Find a power series associated with the problem where we have to find the number of ways to select a dozen pieces of fruit from 5 apples, 10 bananas and 15 coconuts.

Solution: To begin with, let us use the letters A, B and C for apples, bananas and coconuts, respectively. So, if we select k apples, ℓ bananas and m coconuts, then we must have $k + \ell + m = 12$, with the restriction that $0 \le k \le 5, 0 \le \ell \le 10$ and $0 \le m \le 15$. Let us see what we could do to set up the problem using the symbols A, B and C.

Here you may think of A^k to denote k apples, B^ℓ denoting ℓ bananas and C^m denoting m coconuts, then we have picked the correct number of pieces provided the degree (i.e. the sum $k + \ell + m$) of the term $A^k B^\ell C^m$ equals 12. Thus, to find the required number of ways of selecting a dozen pieces of fruit, you simply have to find the number of terms in the expansion

$$(A^0 + A^1 + \dots + A^5)(B^0 + B^1 + \dots + B^{10})(C^0 + C^1 + \dots + C^{15}) \qquad (1)$$

whose degree equals 12. This will be the sum of the coefficients of all the terms $A^k B^\ell C^m$ in (1) such that $k + \ell + m = 12$ i.e. of $A^0B^0C^{12}, A^0B^1C^{11}$, etc.

At this point it is important to observe that any selection of fruits with the given restriction on the numbers k, ℓ and m corresponds to precisely one term in this product. For instance, if you pick 3 apples, 4 bananas and 5 coconuts, the corresponding term in the product (1) is $A^3B^4C^5$. And conversely, the term AB^2C^9 represents the choice of 1 apple, 2 bananas and 9 coconuts. Thus product (1) when expanded as $\sum_{i,j,k} a_{ijk} A^i B^j C^k$, gives the required (finite) power series for the given problem.

* * *

Now, since our real interest is in the degree of $A^kB^\ell C^m$ (i.e. in the sum $k+\ell+m$), we may as well replace each of these symbols in (1) by a common symbol, say z. Then, as before, we are led to determine the coefficient of z^{12} in the following product of polynomials:

$$(1+z+\cdots+z^5)(1+z+\cdots+z^{10})(1+z+\cdots+z^{15}).$$

Here, now we don't need to look into the possible ways in which A^k, B^ℓ, and C^m add up to 12 fruits.

Next, let us ask a similar question for the problem given in the following example.

Example 3: How can a power series be associated with the problem in which we have to find the number of selections of fruits if we have Rs.50 with us and it is given that an apple costs Rs.5, a banana Rs.2 and a coconut Rs.3.

Solution: Since here we don't have any restriction on the number of pieces of fruit, the required power series (in terms of money) is of the form

$$(A^0+A^5+A^{10}+\cdots)(B^0+B^2+B^4+\cdots)(C^0+C^3+C^6+\cdots),$$

which is the product of three polynomials (infinite because there is no restriction on the number of pieces of fruit). Because an apple costs Rs.5, so, purchase of k apples would mean that we have to spend Rs.5k. Similarly, purchase of ℓ bananas and m coconuts will amount to spending Rs.$(2\ell+3m)$. Thus purchase of $(k+\ell+m)$ fruits correspond to the term $A^{5k}B^{2\ell}C^{3m}$ in the above product of three polynomials. Also because we have Rs.50 only, we must have $5k+2\ell+3m=50$. On the other hand, each term $A^{5k}B^{2\ell}C^{3m}$ (with $5k+2\ell+3m=50$) in the above series gives a choice for purchasing k apples, ℓ bananas and m coconuts.

Thus, in view of given cost of the apple, banana and coconut, powers of symbols A, B and C in the first, second and third polynomials are multiples of 5, 2 and 3, respectively. As before, in this expression we seek the number of terms with degree 50. However, by our discussion following Example 2, we may replace each of these symbol by a common symbol z (say) then the required number is given by the coefficient of z^{50} in the expression

$$(1+z^5+z^{10}+\cdots)(1+z^2+z^4+\cdots)(1+z^3+z^6+\cdots). \qquad (*)$$

Hence this product on expansion gives the power series associated with the above problem.

* * *

In above example, if we impose some restrictions on our selection of the fruits, then there will be a relative change in the associated power series (*). This is what we want you to see in the following exercise.

E1) Find the power series associated with the problem given in Example 3,

a) when all our selections are required to have 1 apple at least;

b) when each selection has to have at least one fruit of each type.

Above you have seen how to associate a power series with a combinatorial problem whose solution is given by certain coefficients of that series. Certain series can be written in a functional form which we call as closed form. For example, it follows from binomial theorem (see R1 given above) that

$(1+z)^n$ $(n<0)$ is the closed form (or a functional form) of the power series $\sum_{r=0}^{\infty} C(n,r)z^r$.

A functional form (or closed form) of a series associated with a sequence is called its **generating function**. A formal definition of the generating function is given below.

Definition: The generating function $A(z)$ (say) for the sequence of real (or complex) numbers, $\{a_0, a_1, \ldots, a_n, \ldots\}$ is given by the power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \cdots + a_n z^n + \cdots.$$

Thus, the $(n+1)$ th term a_n of the sequence $\{a_n\}_{n\geq 0}$ is simply the coefficient of z^n in $A(z)$. As said before, the generating function thus serves the purpose of identifying the different terms of a sequence by different powers of the symbol z.

To distinguish them from exponential generating functions (which we will define in the next section), they are sometimes called **ordinary generating functions**.

For example, the associated power series of the constant sequence $\{a, a, \cdots\}$ is

$$\begin{aligned} a + az + az^2 + \ldots &= a(1 + z + z^2 + \ldots) \\ &= a(1-z)^{-1}. \quad \text{(by binomial theorem)} \end{aligned}$$

Thus $a(1-z)^{-1}$ is the generating function (i.e. a closed form) of the constant sequence $\{a, a, \cdots\}$.

More generally, let $G(z)$ be the generating function of the geometric progression $\{ar^n\}_{n\geq 0}$, i.e.,

$$G(z) = a + (ar)z + (ar^2)z^2 + \ldots.$$

Then

$$\begin{aligned} G(z) - a &= rz[a + (ar)z + (ar^2)z^2 + \ldots] \\ &= rzG(z), \end{aligned}$$

which gives, on simplification, $G(z) = a/(1-rz)$.

Why don't you try an exercise now?

E2) In the following verify that
   (a) the generating function for the finite geometric progression $\{a, ar, ar^2, \ldots, ar^{k-1}\}$ is $a(1-r^kz^k)/(1-rz)$.
   (b) the generating function for the sequence of binomial coefficients $\{C(k,0), C(k,1)a, C(k,2)a^2, \ldots\}$ is $(1+az)^k$.
   (c) the generating function for the sequence of binomial coefficients $\{C(k-1,0), C(k,1)a, C(k+1,2)a^2, \ldots\}$ is $(1-az)^{-k}$.

Note that the generating function for a finite sequence is the generating function for a corresponding infinite sequence which can be obtained by setting to zero every term not previously defined. Thus for a finite polynomial $a_0 + a_1z + a_2z^2$ we write

$$a_0 + a_1z + a_2z^2 + 0.z^3 + 0.z^4 + \cdots.$$

Now let us see how the technique of associating a series with a sequence is helpful in solving a combinatorial problem. We try to understand this with the help of following example.

Example 4: Determine the number of subsets of a set of n elements, $n \geq 0$.

Solution: Let s_n denote the number of subsets that a set of n elements can have. In the previous unit, you have seen that the recurrence relation satisfied by the sequence $\{s_n\}$ is given by

$s_n = 2s_{n-1}$ if $n \geq 1$ and $s_0 = 1$. (see Problem 5 of Unit 7)

Let S(z) stands for the generating function of the sequence $\{s_n\}_{n\geq 0}$. So we can write

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} s_n z^n = 1 + \sum_{n=1}^{\infty} s_n z^n \\ &= 1 + 2\sum_{n=1}^{\infty} s_{n-1} z^n \qquad \text{(by definition of } s_n, n \geq 1) \\ &= 1 + 2z \sum_{n=0}^{\infty} s_n z^n = 1 + 2z.S(z), \end{aligned}$$

$$\text{i.e.}\quad S(z) = 1 + 2z\,S(z).$$

Solving last equation for S(z), we get

$$S(z) = \frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n. \qquad \text{(by binomial theorem)}$$

Two symbolic series $\sum a_n z^n$ and $\sum b_n z^n$ are said to be equal iff $a_n = b_n$, $\forall n$.

Finally, comparing the coefficients of z^n on both sides of above equation, we get $s_n = 2^n$, $n \geq 0$. Thus the number of subsets of a set of n element is 2^n, $\forall n$.

* * *

As you have seen in above example, some (algebraic) operations are needed at the middle stage of the process while writing the general term of a sequence explicitly. These operations on generating functions, which we are defining below, have a crucial role to play in solving combinatorial problems.

Aside from the usual operations of addition, subtraction, multiplication and division of series, we will also find the need for differentiating or integrating a power series. It is important to observe that, while performing last two operations, our aim is to associate with the object $\frac{d}{dz}\left(\sum a_n z^n\right)$ (and $\int \left(\sum a_n z^n\right) dz$) a new power series as given in the right hand side of O_3 (and O_4, respectively).

O_1.(Sum and Difference) $\sum a_n z^n \pm \sum b_n z^n = \sum (a_n \pm b_n) z^n$;

O_2.(Multiplication) $\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$;

O_3.(Differentiation) $\frac{d}{dz}\left(\sum a_n z^n\right) = \sum (n+1) a_{n+1} z^n$;

O_4.(Integration) $\int \left(\sum a_n z^n\right) dz = \sum \frac{a_n}{n+1} z^{n+1}$.

O_5.(Division) $(\sum a_n z^n) / (\sum b_n z^n) = \sum c_n z^n$

$$\iff (\sum b_n z^n)(\sum c_n z^n) = \sum a_n z^n, \text{ i.e., } a_n = \sum_{k=0}^{n} b_k c_{n-k}.$$

The quotient of two power series defined in O_5 above is via the product in the usual manner. Infact, there is no really convenient expression for the quotient.

Next, let us now look at some general results which provide connection between the generating functions of various sequences, terms of which are related in some manner to each other. These results are particularly useful when we know the generating functions of some of these, and wants to find the same for others.

For instance, following lemma can help you to obtain generating function of the product of two sequences if the generating functions of individual sequences are known.

Lemma 1: If A(z) is the generating function for the sequence $\{a_n\}_{n\geq 0}$ and B(z) is the generating function for the sequence $\{b_n\}_{n\geq 0}$, then $A(z) \times B(z)$ is the generating function for the sequence $\{c_n\}_{n\geq 0}$, where

$c_n = \sum_{k=0}^{n} a_k b_{n-k},\ n \geq 0.$

Proof: The proof readily follows from the definition of multiplication of power series (see O_2 above). We have, by definition,

$$\begin{aligned}
A(z) \times B(z) &= \left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) \\
&= a_0\left(\sum_{k=0}^{\infty} b_k z^k\right) + a_1\left(\sum_{k=0}^{\infty} b_k z^{k+1}\right) + \ldots \\
&= \sum_{j=0}^{\infty} a_j \sum_{k=0}^{\infty} b_k z^{j+k} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n,
\end{aligned}$$

where at each step of the process we collected the coefficients of same powers of z. The proof of the lemma is then complete by using the definition of c_n as given in the statement.

Why don't you use it to solve the following exercise?

E3) Prove the binomial identity $\sum_{j=0}^{k} C(m,j)C(n,k-j) = C(m+n,k)$, using Lemma 1. Hence deduce the binomial identity $\sum_{j=0}^{k} C(k,j)^2 = C(2k,k)$.

We next prove another useful lemma of similar nature.

Lemma 2: Suppose that the sequence $\{a_n\}_{n\geq 0}$ has the generating function A(z). Then generating function B(z) (say) for the sequence $\{b_n\}_{n\geq 0}$, where $b_n = a_n - a_{n-1}$, for $n \geq 1$, and $b_0 = a_0$, is given by

$B(z) = (1-z)A(z).$

Proof: By definition, the generating function for the sequence $\{b_n\}$ is

$$\begin{aligned}
B(z) &= \sum_{n=0}^{\infty} b_n z^n \\
&= b_0 + \sum_{n=1}^{\infty} b_n z^n \\
&= a_0 + \sum_{n=1}^{\infty} a_n z^n - z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} \quad \text{(using definition of } b_n\text{)} \\
&= a_0 + [A(z) - a_0] - z.A(z) \quad \text{(by definition of } A(z)\text{)} \\
&= (1-z)A(z).
\end{aligned}$$

This completes the proof of the lemma.

E4) (a) Use Lemma 2 to find the generating function A(z) (say) for the sequence in arithmetic progression $\{a, a+d, a+2d, \dots\}$.

(b) Suppose that A(z) is the generating function for the sequence $\{a_n\}_{n\geq 0}$. Show that the generating function S(z) (say) for the sequence $\{s_n\}$ of its partial sums viz. $s_n = \sum_{k=0}^{n} a_k$, $(n \geq 0)$ is given by $S(z) = \frac{A(z)}{1-z}$.

(c) Use (b) to find the generating function for the sequence $\{1, 3, 6, \dots\}$.

We next look at a problems which you might have solved earlier by different methods. Using generating functions, we shall give you alternative methods of solving them. This is an example involving the sum of k-th power of the first n natural numbers which we denote by σ_n^k,

$$\text{i.e.} \qquad \sigma_n^k = 1^k + 2^k + \dots + n^k = \sum_{i=1}^{n} i^k, \quad k \geq 1.$$

You already know how a formula for σ_n^k $(1 \leq k \leq 3)$ can be verified by induction (see Unit 2). Let us see how generating function technique makes this task easier. You will see this in operation for the evaluation of $\sigma_n^2 = \sum_{j=1}^{n} j^2$ in the following example.

Example 5: Find the sum σ_n^2 of squares of the first n natural numbers.

Solution: Differentiating the binomial function $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=0}^{\infty} jz^{j-1} = (1-z)^{-2} \qquad \text{(see } O_3\text{)}.$$

Multiplying this by z on both sides, we get

$$\sum_{j=1}^{\infty} jz^j = z(1-z)^{-2}.$$

Repeating this process of first differentiating and then multiplying by z, we get

$$A(z) = \sum_{j=1}^{\infty} j^2 z^j = z(1+z)(1-z)^{-3},$$

where we write A(z) for the generating function of the sequence $\{j^2\}_{j\geq 1}$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \sigma_k^2 z^k &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j^2 \right) z^k \\ &= \frac{A(z)}{(1-z)} \qquad \text{(by E 4(b))} \\ &= z(1+z)(1-z)^{-4}.\end{aligned}$$

Therefore, σ_n^2 is the coefficient of z^n in the series which can be obtained by expanding the function $z(1+z)(1-z)^{-4}$. However, because

$$z(1+z)(1-z)^{-4} = z(1-z)^{-4} + z^2(1-z)^4,$$

this is the same as looking for the sum of coefficients of z^{n-1} and z^{n-2} in the expanded form o inomial function $(1 - z)^{-4}$. Thus, in view of binomial identity C(n, k) = n − k), we have

$$\sigma_n^2 = C(n \div \quad \cdot C(n+1, 3) = n(n+1)(2n+1)/6.$$

* * *

Try the followin ise now.

E5) Find the s of the first n natural numbers, using generating functions.

So far, you lear to identify generating functions and use them to solve some simple co rial problems. However, there are several combinatorial p which are hard to crack by using these functions. This is particul e of problems that involve arrangements (in which order plays a c ole) and distributions of distinct objects (see Block 2 for more details e next section we introduce you to a slightly different kind of generati ction which will prove useful for sovling these type of problems.

8.3 Expo al Generating Functions

In this section, ll study a modified form of the series we discussed in the last sectio derstand the difference, let us consider the problem of finding the nu three-letter words i.e., a string of three letters which can be formed two-alphabet set {a, b} (say), with the restriction that not all letters words are identical.

Thus, we may er two a's and one b or two b's and one a to form all the three-lett s out of the two-element set {a, b}. Each of these two possibilities (i iscussion of permutations of objects, not necessarily distinct, in B give 3!/2!1! = 3 distinct words viz. aab, aba, baa in the first case, and b, abb in the second, for a total of six words.

An ordered pair (x,y) positive integers is a sol tion to the linear equat $m + n = 3$, iff $x + y = 3$

Now could w t the number of distinct possibilities in above problem is merely the of positive integer solutions to the linear equation $m + n = 3$, if of this as using m a's and n b's, where $m, n \geq 1$? This would have l we had not been interested in the position of a and b, in which cas aba would mean the same to us. But this is not the case. We are ring the number of three-letter words i.e., different strings of th s. So, the position of the letters is important. Consequentl uld like each integer solution to contribute not 1 but 3 (so total is 3 total number of words.

Now, as we count the number of three letter words, we should look for the coeffi z^3 in a series that counts $(m+n)!/m!n!$ each time $z^m z^n = z^3$ a that. So, we try the product

$$\left(\frac{z}{1!} \quad \frac{z}{1!} + \frac{z^2}{2!}\right) = \frac{z^2}{1!1!} + \frac{z^3}{1!2!} + \frac{z^3}{2!1!} + \frac{z^4}{2!2!}.$$

For r = 1, 2. coefficient of z^r in this is term of the form $1/m!n!$, where $m + n = r$, n We need to multiply this by $(m+n)!$ in order to get the answer we a g for. Since the coefficient of z^3 in the above expansion is 1, we end tiplying this by 3! to get a right answer to above problem.

An expone erating function is precisely the power series of this type. A formal d is given below.

Definition: The exponential generating function $A_{exp}(z)$ (say) for the sequence of real or complex numbers $\{a_0, a_1, \dots, a_n, \dots,\}$ is given by the power series

$$A_{exp}(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k = a_0 + \frac{a_1}{1!} z + \dots + \frac{a_n}{n!} z^n + \dots .$$

As you can see, the nth term a_n of the given sequence is no longer the coefficient of z^n in $A_{exp}(z)$, rather it is n! times that coefficient.

For example, the exponential generating function for the constant sequence $\{1, 1, 1, \dots\}$ is given by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \dots .$$

Does it remind you of some function? Of course, it resembles exponential function with which you are familiar but here z is just a symbol and not a variable. It is this resemblance from where these type of generating functions have derived their name.

Try the following exercise now.

E6) Find the exponential generating function of the sequence $\{P(n,k)\}_{k=1}^{n}$, for a fixed $n \in \mathbb{N}$ where P(n,k) denotes the number of k-permutations of n objects.

As before, let us try to identify the exponential generating functions associated with the combinatorial problem given in the following example:

Example 6: Show that the exponential generating function associated with the problem of finding the number of ways to choose some subset of m objects and distribute them into n boxes in such a way that the order in the same box is counted, is given by $e^z(1-z)^{-n}$.

Solution: First of all, recall from Unit 5 that there are C(m, k) ways of choosing k out of the m objects, and then $n(n+1)\cdots(n+k-1)$ ways to arrange them into n boxes. Thus, the total number of ways to choose some subset of m objects and distribute them into n boxes in such a way that the order in the same box is counted, are

$$C(m,0) + \sum_{k=1}^{m} n(n+1)\cdots(n+k-1)\, C(m,k)$$

$$= m! \left[\frac{1}{m!} + \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \times n(n+1)\cdots(n+k-1)\right].$$

Here we may take n to be fixed, and consider this a sequence in m alone. Therefore, the coresponding exponential generating function for this sequence is

$$\sum_{m=0}^{\infty} \left[\frac{1}{m!} + \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \times n(n+1)\cdots(n+k-1)\right] z^m,$$

which, in turn, is a product of the series

$$\left(\sum_{m=0}^{\infty} \frac{1}{m!} z^m\right) \text{ and } \left(1 + \sum_{m=1}^{\infty} \frac{n(n+1)\cdots(n+m-1)}{m!} z^m\right). \quad \text{(see } O_2\text{)}$$

Now the first series equals e^z (by definition), while the second equals

$(1-z)^{-n}$, by binomial theorem. Hence we have obtained the associated exponential generating function, as claimed.

* * *

Why don't you try an exercise now?

E7) Show that the exponential generating function of the sequence $\{B_n\}_{n=0}^{\infty}$ of Bell numbers, satisfying the recurrence

$$B_n = \sum_{k=0}^{n} C(n,k)B_k,\ n \geq 2 \text{ with } B_0 = 1,$$

is $z/(e^z - 1)$.

Let us work out few examples to get a feeling about some elementary uses of the exponential generating functions in solving combinatorial problems.

Example 7: Find the number of bijections on a set of n elements, $n \geq 1$.

Solution: Let b_n denote the number of bijections on a set of n elements, $n \geq 1$. Recall from the previous unit (Problem 6) that the recurrence relation satisfied by the sequence $\{b_n\}$ is given by

$$b_n = nb_{n-1} \text{ if } n \geq 2 \text{ and } b_1 = 1.$$

Since we do not know b_0, we will ignore this term. The exponential generating function B(z) (say) of the sequence $\{b_n\}$) is given by

$$B(z) = \frac{b_1}{1!}z + \frac{b_2}{2!}z^2 + \frac{b_3}{3!}z^3 + \ldots + \frac{b_r}{r!}z^r + \ldots .$$

Then

$$\begin{aligned} B(z) &= \sum_{n=1}^{\infty} \frac{b_n}{n!} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{nb_{n-1}}{n!} z^n \quad \text{(by definition of } b_n, n \geq 2) \\ &= z + z \sum_{n=1}^{\infty} \frac{b_n}{n!} z^n = z + z.B(z). \end{aligned}$$

Solving for B(z), we get

$$B(z) = z/(1-z) = \sum_{n=1}^{\infty} z^n. \qquad \text{(by binomial theorem)}$$

So, by comparing coefficients of z^n, we get from the last equality $b_n = n!, n \geq 1$.

* * *

At times, the exponential generating functions are also useful in calculating the sum of an infinite series. Let us see an example of this.

Example 8: Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{(k+1)^2}{k!} = \frac{1^2}{0!} + \frac{2^2}{1!} + \cdots + \frac{(n+1)^2}{n!} + \cdots ,$$

using exponential generating functions.

Solution: Multiply by z on the both side of $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, we get

$$ze^z = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}.$$

This equation when differentiated once, gives

$$(1+z)e^z = \sum_{n=0}^{\infty} \frac{(n+1)z^n}{n!}. \qquad \text{(see } O_3)$$

Having already got one $n+1$ term in the numerator suggests that we are on the right track. We repeat the first two steps viz. multiply each side of the last equation by z and then differentiate, we get

$$(1+3z+z^2)e^z = \sum_{n=0}^{\infty} \frac{(n+1)^2 z^n}{n!}.$$

The rest of the job is easy. Put $z = 1$ in the last equation to get $5e = \sum_{n=0}^{\infty} (n+1)^2/n!$. Therefore, the required sum of the given series is 5e.

* * *

Why don't you try an exercise now?

E8) Using exponential generating functions, find the number d_n of derangements of n objects. (see Unit 6 & 7 for more detailes on derangements.)

In the previous two sections, you have seen some elementay use of two type of generating functions. In the next section, we shall give some more applications of generating functions.

8.4 Applications

In Sec.8.2, for certain problems, we only talked about the generating function and didn't attempted to solve them. For instance, this is the case with Example 2 and Example 3, because to solve them, you need to be familiar with methods of solving linear equations. We shall discuss methods involving generating functions for solving a linear equation here. Also, we shall discuss the use of generating functions in solving certain problems about partitions, which you have studied in Unit 5. Finally, in this section, you will see how different type of recurrences are solved by using generating functions.

So let us start by applying generating functions to solve some simple combinatorial identities, particularly those that involve binomial coefficients.

8.4.1 Combinatorial Identities

By binomial theorem:

$$(1+z)^n = \sum_{k=0}^{n} C(n,k)z^k, \tag{2}$$

we know that $(1+z)^n$ is the generating function of the finite sequence $\{C(n,k)\}_{k=0}^{n}$. We shall use this to evolve some combinatorial identities given in the following two examples.

Example 9: Prove the binomial identity

$C(n,1)+3C(n,3)+5C(n,5)+\cdots = n2^{n-2} = 2C(n,2)+4C(n,4)+6C(n,6)+\cdots$

Solution: Differentiating both sides of (2) with respect to z, we get

$$n(1+z)^{n-1} = \sum_{k=0}^{\infty} kC(n,k)z^{k-1}.$$

Now setting $z = 1$ and $z = -1$ in the resulting expression, we get

$$\sum_{k=1}^{\infty} kC(n,k) = n2^{n-1}, \text{ and} \qquad (3)$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} kC(n,k) = 0, \text{ respectively.} \qquad (4)$$

Shifting negative terms to the r.h.s. in (4), we have
$C(n,1) + 3C(n,3) + 5C(n,5) + \cdots = 2C(n,2) + 4C(n,4) + 6C(n,6) + \cdots$.
Now on adding terms $2C(n,2), 4C(n,4), 6C(n,6) \ldots$ so on, to both sides of above identity, we get

$$\sum_{n=1}^{\infty} kC(n,k) = 2[2C(n,2) + 4C(n,4) + 6C(n,6) + \ldots]. \qquad (5)$$

From this, using (3), it follows that r.h.s. of (5) equals $\frac{n2^{n-1}}{2} = n2^{n-2}$. With this we have established the binomial identity stated above.

* * *

Our next application concerns k-permutations of a set of n elements. By E12 of Unit 7, you know that the number of k-permutations of n distinct objects, $P(n,k)$, satisfies the recurrence relation

$$P(n,k) = P(n-1,k) + kP(n-1,k-1), \; n,k \geq 1. \qquad (6)$$

Example 10: For fixed n, find an explicit formula for $P(n,k)$ by making use of its exponential generating function, $\mathcal{P}_{\exp}(z;n)$ (say) as defined below.

$$\mathcal{P}_{\exp}(z;n) = \sum_{k=0}^{\infty} (P(n,k)/k!)\, z^k.$$

Solution: Using (6) and the definition of $\mathcal{P}_{\exp}(z;k)$, we have

$$\sum_{k=1}^{\infty} \frac{P(n,k)}{k!} z^k = \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + \sum_{k=1}^{\infty} \frac{kP(n-1,k-1)}{k!} z^k$$

i.e.

$$\sum_{k=1}^{\infty} \frac{P(n,k)}{k!} z^k = \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + z\sum_{k=1}^{\infty} \frac{kP(n-1,k-1)}{(k-1)!} z^{k-1}$$

$$\Rightarrow \quad \mathcal{P}_{\exp}(z;n) - P(n,0) = [\mathcal{P}_{\exp}(z;n-1) - P(n-1,0)] + z\mathcal{P}_{\exp}(z;n-1)$$

$$\Rightarrow \quad \mathcal{P}_{\exp}(z;n) = (1+z)\,\mathcal{P}_{\exp}(z;n-1) \quad (\text{as } P(n,0) = P(n-1,0))$$

$$\Rightarrow \quad \mathcal{P}_{\exp}(z;n) = (1+z)^n\,\mathcal{P}_{\exp}(z;0) = (1+z)^n. \quad (\text{by iteration})$$

Since the coefficient of z^k in $(1+z)^n$ is $C(n,k)$ (by binomial theorem), it follows by comparing coefficients, that

$$\frac{P(n,k)}{k!} = C(n,k) \Rightarrow P(n,k) = k!C(n,k) = \frac{n!}{(n-k)!}.$$

Of course, if $k > n$, $C(n,k) = 0$, and hence $P(n,k) = 0$ then. So, we have obtained P(n,k), explicitly.

* * *

Try the following exercise now.

E9) Evaluate, using generating function technique, the sum $\sum_{k=1}^{n} k3^kC(n,k)$.

We next consider the application of generating functions to general integer equations.

8.4.2 Linear Equations

Generating functions are also particularly handy when one is looking for non-negative integer solutions to linear equations of the type $a_1 + a_2 + \ldots + a_k = n$. You may recall that we showed earlier (see Theorem 5 of Unit 4) that this equals $C(n+k-1, k-1)$ by elementary counting techniques. If, on the other hand, each a_i is a positive integer, then the number of such solutions equals $C(n-1, k-1)$ (see Example 16 of Unit 5).

Generating functions often provide a simpler way to solve such equations. This is illustarted in the following example.

Example 11: Find the number of non-negative integer solution of the linear equation

$$a_1 + a_2 + \cdots + a_k = n,$$

using generating function techniques.

Solution: In the first case, where each $a_j \geq 0$, the required number is the coefficient of z^n in the following product of polynomials (see discussion following Example 1)

$$(1 + z + z^2 + \cdots)\cdots(1 + z + z^2 + \cdots). \quad \text{(k times)}$$

Each term of this product equals $(1-z)^{-1}$ (by binomial theorem) and the coefficient of z^n in $(1-z)^{-k}$ is

$$C(n+k-1, n) = C(n+k-1, k-1);$$

If each $a_j \geq 1$ instead, we seek the coefficient of z^n in the expansion

$$(z + z^2 + z^3 + \cdots)\cdots(z + z^2 + z^3 + \cdots). \quad \text{(k times)}$$

Each term of this product equals $z(1-z)^{-1}$ (by binomial theorem) and the coefficient of z^n in $z^k(1-z)^{-k}$ is the coefficient of z^{n-k} in $(1-z)^{-k}$. This equals $C((n-k)+k-1, n-k) = C(n-1, k-1)$.

Of course, this means that there is no solution if $n < k$, as should be the case.

* * *

If, in above example, we require that one or more of the solutions a_j are bounded at both ends, and if we allow a_j to be negative, then the number of solutions, even for $k = 2$ or 3 becomes a tedious computation. The method of generating functions is just what you could use for such problems. We illustrate this in the following example.

Example 12: Find the number of integer solutions to $a_1 + a_2 + a_3 = n$, where $-1 \leq a_1 \leq 1, 1 \leq a_2 \leq 3$ and $a_3 \geq 3$.

Solution: Let us bring this into the situation of Example 11. For this, we put $b_1 = a_1 + 1$ and $b_3 = a_3 - 3$. Then our problem is same as looking for the number of integer solutions to

$$b_1 + b_2 + b_3 = n - 2, \text{ where } 0 \leq b_1 \leq 2, 1 \leq b_2 \leq 3 \text{ and } b_3 \geq 0.$$

Now, in view of these bounds on b_i's, it follows that associated generating function is given by

$$(1+z+z^2)(z+z^2+z^3)(1+z+z^2+\cdots) = \frac{1-z^3}{1-z} \times \frac{z(1-z^3)}{1-z} \times \frac{1}{1-z},$$

by using binomial theorem and **R2**. As before, we want the coefficient of z^{n-2} in this expansion, which is same as the coefficient of z^{n-3} in

$$(1-z^3)^2(1-z)^{-3} = (1-z)^{-3} - 2z^3(1-z)^{-3} + z^6(1-z)^{-3}.$$

We leave it for you to check that the answer is:
$C(n-1,2) - 2C(n+2,2) + C(n+5,2)$.

This simplifies to 9 if $n \geq 7$. What would it be if $n < 7$? Consider the two cases: $4 \leq n < 7$ and $n = 3$, separately, you can get the answer easily. Of course, it is 0 for $n < 3$.

* * *

The technique adopted in the example given above is no different if we have more than three summands or if the bounds we had are more general. In principle, therefore, we are in a position to find the number of integer solutions to

$$a_1 + a_2 + \cdots + a_k = n, \text{ with } m_j \leq a_j \leq M_j, m_j, M_j \in \mathbb{Z}. \quad (1 \leq j \leq k)$$

Why don't you check your understanding of it by attempting the following exercise?

E10) How many integer solutions are there to $a_1 + a_2 + a_3 + a_4 + a_5 = 25$ with $a_k > k$ for each k, $1 \leq k \leq 5$?

Another illustration of the use of generating functions is in the mathematical theory of partitions – historically one of the first problems studied with generating functions. We shall talk about this next.

8.4.3 Partitions

We shall only see one aspect of partitions, namely, their connection with generating functions. You already had some exposure of them in Unit 5. Here we will go a little deeper. For this, we should first define the sequence of **partitions**, P_n.

Definition: The nth term of the sequence $\{P_n\}$, $n \geq 1$, counts the number of ways in which n can be expressed as a sum of positive integers such that the order of the summands (parts) is not important. We define $P_0 = 1$.

For example, $P_4 = 5$ since $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 = 4$. So, partitioning n is the same as distributing n non-distinct objects into n non-distinct boxes, with the empty box allowed (e.g. 4=3+1+0+0). In terms of linear equations discussed above, P_n is the number of non-negative integer solutions to the integer equation

$$X_1 + X_2 + \cdots + X_k + \cdots = n, \quad X_i = ia_i \quad \text{(vi)},$$

where a_k denotes the number of k's in the partition.

Let us look at the form that the generating function, $\mathcal{P}(z)$ of the sequence $\{P_n\}_{n\geq 0}$ must take.

Note that, in the above linear equation, for each integer $k \geq 1$, we may use none, one or more k's according to the value of $a_k \geq 0$. There is no other restriction on a_k's. Therefore, for each term $X_i = i.a_i$ $(a_i \geq 0)$ the corresponding term in the associated generating function is simply $(1 + z^k + z^{2k} + \cdots)$. Therefore, on taking the product for all $i \geq 1$, we obtain

$$\mathcal{P}(z) = \prod_{k=1}^{\infty} (1 + z^k + z^{2k} + \cdots) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k}.$$

Generating functions of related sequences are not any harder to determine. They play a significant role in proving identities involving partitions. We illustrate this with the help of the following example.

Example 13: Show that every nnonnegative integer can be written as a unique sum of distict powers of 2.

Solution. The generating function for the sequence $\{a_n\}$, where a_n denote the number of ways n can be written as sum of distinct powers of 2, is

$$(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots$$

Now, we have

$$\begin{aligned}(1 - z)(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots \\ &= (1 - z^2)(1 + z^2)(1 + z^4)(1 + z^8) \cdots \\ &= (1 - z^4)(1 + z^4)(1 + z^8) \cdots \\ &= \ldots \\ &= (1 - z^{2^n})(1 + z^{2^n}) \cdots \\ &= 1. \qquad \text{(assuming } |z| < 1)\end{aligned}$$

Thus, in view of operation O_5 and binomial theorem, it follows that

$$(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots = \frac{1}{1 - z} = 1 + z + z^2 + \ldots$$

From this, by comparing coefficients, we conclude that the coefficient of z^n in the l.h.s. of the equation is 1. Hence, the number a_n of partitions of n into distinct parts of of size $1, 2, 4, 8, 16, \ldots$, so on, is 1. In other words, every non-negative integer can be uniquely expressed as the sum of distinct powers of 2.

***

Why don't you try the following exercise now?

E11) Show that the generating function for the sequence of the number of partitions of n with:

a) parts each of which is at most m is $\prod_{k=1}^{m} (1 - z^k)^{-1}$;

b) unequal parts is $\prod_{k=1}^{\infty} (1 + z^k)$;

c) parts each of which is odd is $\prod_{k=1}^{\infty} (1 - z^{2k-1})^{-1}$

E12) Find the generating function for the sequence of the number of partitions of n

i) into primes;

ii) into distinct primes.

Next, we shall discuss one of the most important uses of generating functions, viz., its utility as a tool to solve the recurrence relations.

8.4.4 Recurrence Relations

In Unit 7, you have learnt how to set up recurrences for a combinatorial problem. Though we had not talked about how to solve them, we gave you some solutions, which you verified.

For solving a recurrence, we need to know the terms of a sequence explicitly. In other words, for a sequence $\{a_n\}$ that satisfies a given recurrence, we shall use its generating function $A(z)$ (say) to find an explicit formula for a_n in terms of n.

As is clear from the solution of Example 4, Example 7 and Example 10, we can write the procedure involved as an **algorithm**, in steps as follows:

1. Express a_n in terms of the previous terms of the sequence, as an equation valid for all integers $n \geq n_0$, for some n_0. (Usually, the recurrence relation is already in that form.)
2. Multiply both sides of the equation by z^n and sum all the resulting equations over all $n \geq n_0$. The left-hand side yields the generating function for a_n minus at most a finite number of terms, while the right-side is to be algebraically simplified so that it becomes an expression involving $A(z)$. Here $A(z)$ is the generating function associated with the sequence $\{a_n\}$.
3. Solve the resulting equation for $A(z)$.
4. Expand the closed form of $A(z)$ into a power series (by using the binomial theorem) and read off the coefficient of z^n. This gives an explicit expression for a_n, for all n.

Note that the Step 2, where we are required to express the r.h.s. in terms of $A(z)$, is very important. A certain degree of algebraic simplification is needed there.

Let us try to understand the steps of above algorithm with the help of following example.

Example 14: Find the maximum number of regions, L_n, into which the plane is cut by n straight lines. (Problem 3 of Unit 7)

Solution: The recurrence relation satisfied by the sequence $\{L_n\}$ is $L_n = L_{n-1} + n$ for $n \geq 2$, and $L_1 = 2$. If the same recurrence were to hold for $n \geq 1$ instead, then L_0 must equal 1. (This actually makes sense for if there is no line, there is only one region.) So we need not apply Step 1 of the algorithm.

Starting the sequence at L_0 instead, the generating function $L(z)$ (say) of the sequence $\{L_n\}_{n\geq 0}$ is given by

$$L(z) = \sum_{n=0}^{\infty} L_n z^n.$$

Step 2: Now, by using the above recurrence relation, we get

$$\begin{aligned} L(z) &= 1 + \sum_{n=1}^{\infty} (L_{n-1} + n)z^n \\ &= 1 + z\sum_{n=1}^{\infty} L_{n-1}z^{n-1} + z\sum_{n=1}^{\infty} nz^n \\ &= 1 + z\sum_{n=0}^{\infty} L_n z^n + z\sum_{n=1}^{\infty} n z^{n-1} \\ &= 1 + z.L(z) + \frac{z}{(1-z)^2}. \end{aligned}$$

Step 3: Solving for L(z) the last equation, we get

$$L(z) = \frac{1}{1-z} + \frac{z}{(1-z)^3}.$$

Step 4: So, using binomial theorem, we get

$$L(z) = \sum_{n=0}^{\infty} \{1 + \frac{1}{2}n(n+1)\}z^n.$$

Finally, equating coefficients of z^n on both sides of the last equation, we get

$$L_n = \tfrac{1}{2}n(n+1) + 1,\ n \geq 1.$$

With this the algorithm terminates and we have obtained an explicit formula for L_n, $\forall n$.

* * *

Why don't you try an exercise now?

E13) Using Theorem 1, find the n th term, $\mathcal{L}_n$ of the Lucas sequence given by $\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2}$, $n \geq 3$, with $\mathcal{L}_1 = 1, \mathcal{L}_2 = 3$.

We next consider the sequence of Fibonacci numbers $\{\mathcal{F}_n\}$ which satisfy the recurrence relation

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} \text{ if } n \geq 3, \text{ and } \mathcal{F}_1 = 1 = \mathcal{F}_2. \quad \text{(Problem 1 of Unit 7)}$$

Example 15: Find the generating function associated with the sequence of Fibonacci sequence $\{\mathcal{F}_n\}_{n\geq 1}$. Then deduce a formula for $\mathcal{F}_n$, $n \geq 1$.

Solution: We write $\mathcal{F}(z)$ for the associated generating function. Then, by definition, we have

$$\begin{aligned}
\mathcal{F}(z) &= \sum_{n=1}^{\infty} \mathcal{F}_n z^n \\
&= z + z^2 + \sum_{n=3}^{\infty} (\mathcal{F}_{n-1} + \mathcal{F}_{n-2})z^n \qquad \text{(Step 2)} \\
&= z + z^2 + z\sum_{n=2}^{\infty} \mathcal{F}_n z^n + z^2 \sum_{n=1}^{\infty} \mathcal{F}_n z^n \\
&= z + z^2 + z[\mathcal{F}(z) - z] + z^2.\mathcal{F}(z).
\end{aligned}$$

Then $(1 - z - z^2)\mathcal{F}(z) = z$. Therefore, with $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$,

$$\begin{aligned}
\mathcal{F}(z) &= \frac{z}{(1-\alpha z)(1-\beta z)} \qquad \text{(by solving equation } z^2 + z - 1 = 0) \\
&= \frac{1}{\alpha - \beta}\left(\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right) \\
&= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n - \beta^n)z^n. \quad \text{(by binomial theorem)}
\end{aligned}$$

Comparing coefficients of z^n now gives $\mathcal{F}_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, for all $n \geq 1$.

* * *

Try the folowing exercise now.

E14) Solve the recurrence relation $T_n = 2T_{n-1} + 1$ if $n \geq 2$ and $T_1 = 1$, using generating functions technique. (see Tower of Hanoi problem in Unit 7)

If you have understood the steps that we followed in solving the recurrence relation involving Fibonacci sequence in previous example, then it should not be difficult for you to understand the proof of the following general result.

Theorem 1: The generating function, denoted by $U(z)$, for a general linear, homogeneous recurrence relation with constant coefficients, of order k,

$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}, n \geq k$, with $u_0 = c_0, \ldots, u_{k-1} = c_{k-1}$,

satisfies the equation

$$(1 - a_1 z - a_2 z^2 - \cdots - a_k z^k) U(z) = c_0 + \sum_{n=1}^{k-1} (c_n - a_1 c_{n-1} - \cdots - a_n c_0) z^n.$$

Proof: We have, by definition,

$$\begin{aligned} U(z) &= \sum_{n=0}^{\infty} u_n z^n \\ &= (u_0 + u_1 z + \cdots + u_{k-1} z^{k-1}) + \sum_{n=k}^{\infty} u_n z^n \\ &= (c_0 + \cdots + c_{k-1} z^{k-1}) + \sum_{n=k}^{\infty} (a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}) z^n \\ &= (c_0 + \cdots + c_{k-1} z^{k-1}) + a_1 z \sum_{n=k}^{\infty} u_{n-1} z^{n-1} + \cdots + a_k z^k \sum_{n=k}^{\infty} u_{n-k} z^{n-k} \\ &= (c_0 + \cdots + c_{k-1} z^{k-1}) + a_1 z[U(z) - c_0 - c_1 z - \cdots c_{k-2} z^{k-2}] + \cdots + a_k z^k U(z) \\ &= p_{k-1}(z) + [a_1 z + a_2 z^2 + \cdots + a_k z^k] U(z), \end{aligned}$$

where $p_{k-1}(z) = c_0 + \sum_{n=1}^{k-1} (c_n - a_1 c_{n-1} - \cdots - a_n c_0) z^n$ is a polynomial of degree at most $k-1$ (How?).

Further simplification gives

$$[1 - a_1 z - a_2 z^2 - \cdots - a_k z^k] U(z) = p_{k-1}(z).$$

This completes the proof of the theorem.

A first conclusion that you can easily deduce from the theorem above, is given in the following result.

Corollary 1: The generating function of linear, homogeneous recurrence relations with constant coefficients given in Theorem 1 is a rational function, $p(z)/q(z)$, with the numerator, $p(z)$, a polynomial of degree at most one less than the order of the recurrence.

Also observe that $1 + q(z)$ is equal to the polynomial obtained from the r.h.s. of the given recurrence relation given in Theorem 1 by replacing u_{n-i} with z^i $(1 \leq i \leq k)$. While applying this corollary, you need to pay careful attention to the form of $q(z)$. You should not try to memorize $p(z)$ at all. After all, once you know $q(z)$, $p(z)$ can be obtained by multiplying $q(z)$ by the generating series $\sum_{n=0}^{\infty} u_n z^n$.

Let us employ Theorem 1 and Corollary 1 to solve the following recurrence relation.

Example 16: Solve the third-order recurrence

$$u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3} = 0, \ n \geq 3,$$

with the initial conditions $u_0 = 6, u_1 = 17$ and $u_2 = 53$.

Solution: We denote by $U(z)$ the generating function for the sequence $\{u_n\}$. Then, by Theorem 1, we know that

$$(1-9z+26z^2-24z^3)U(z)=p(z)$$

is a polynomial of degree 4 in z. Now, a little more calculations will lead you to conclude that

$$(1-9z+26z^2-24z^3)U(z)=(1-2z)(1-3z)(1-4z)U(z)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} u_n z^n - 9\sum_{n=0}^{\infty} u_n z^{n+1} + 26\sum_{n=0}^{\infty} u_n z^{n+2} - 24\sum_{n=0}^{\infty} u_n z^{n+3} \\
&= u_0 + (u_1 - 9u_0)z + (u_2 - 9u_1 + 26u_0)z^2 \\
&\qquad + \sum_{n=3}^{\infty} (u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3})z^n \\
&= 6 - 37z + 56z^2, \text{ by using the given recurrence relation.}
\end{aligned}$$

Therefore,

$$U(z) = (6-37z+56z^2)/(1-2z)(1-3z)(1-4z).$$

Decomposing the r.h.s. into partial fractions, we then get

$$U(z) = 3(1-2z)^{-1} + (1-3z)^{-1} + 2(1-4z)^{-1}.$$

Now using binomial theorem and comparing coefficients of z^n in the resulting series with the series on the l.h.s. viz. U(z), we get

$$u_n = 3\cdot 2^n + 3^n + 2\cdot 4^n,\ n \geq 0.$$

* * *

Try the following exercise now.

E15) Determine the generating function for the sequence $\{t_n\}_{n=0}^{\infty}$ given by the recurrence relation $(n \geq 3)$

$$t_n = \begin{cases} t_{n-3}, & \text{if n is even;} \\ t_{n-3} + \dfrac{n+(-1)^{(n+1)/2}}{4}, & \text{if n is odd,} \end{cases}$$

where, t_n denotes the number of incongruent triangles with integral sides and perimeter n. You may also take $t_0 = t_1 = t_2 = 0$.

In yet another situation, let us next consider the case of nonhomogenous recurrences viz. when the nonhomogenous term(s) are either of the types r^n $(r \in \mathbb{C})$ or n^k $(k \in \mathbb{N} \cup \{0\})$. Below we consider the case when it is of the form r^n. The method of generating functions, and in particular Theorem 1, can still be of use to good effect as the following example shows.

Example 17: Solve the third-order nonhomogeneous linear recurrence with constant coefficients viz. $u_n - 3u_{n-2} - 2u_{n-3} = an + b\cdot 2^n$ in terms of the initial conditions u_0, u_1 and u_2.

Solution: Write U(z) for the generating function of the sequence $\{u_n\}_{n\geq 0}$, then

$$(1-3z^2-2z^3)U(z) = (1+z)^2(1-2z)U(z)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} u_n z^n - 3\sum_{n=0}^{\infty} u_n z^{n+2} - 2\sum_{n=0}^{\infty} u_n z^{n+3} \\
&= u_0 + u_1 z + (u_2 - 3u_0)z^2 + \sum_{n=3}^{\infty} (u_n - 3u_{n-2} - 2u_{n-3})z^n \\
&= u_0 + u_1 z + (u_2 - 3u_0)z^2 + az\sum_{n=3}^{\infty} nz^{n-1} + b\sum_{n=3}^{\infty} (2z)^n
\end{aligned}$$

$$= (u_0 - b) + (u_1 - a - 2b)z + (u_2 - 3u_0 - 2a - 4b)z^2 - \frac{a}{(1-z)^2} - \frac{a}{1-z} + \frac{b}{1-2z}.$$

The rest of the calculation is tedious, but routine. We employ partial fractions, to get U(z) in the form

$A(1-z)^{-1} + B(1-z)^{-2} + C(1+z)^{-1} + D(1+z)^{-2} + E(1-2z)^{-1} + F(1-2z)^{-2}$,

for some choice of A, ... , F. In terms of these constants,

$u_n = A + B(n+1) + C(-1)^n + D(-1)^n(n+1) + E.2^n + F.2^n(n+1),\ n \geq 0.$

* * *

Try the following exercise now.

E16) Use Theorem 1 to solve the recurrence $a_n - 3a_{n-1} - 10a_{n-2} = 28 \times 5^n$ for $n \geq 2$, with $a_0 = 25$ and $a_1 = 120$.

It is sometimes possible to solve even non-linear recurrences with the help of generating functions. We illustrate this by solving a recurrence about which you have read before in Unit 7.

Example 18: Solve the recurrence relation

$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2a_{n-2} + a_1a_{n-1},\ n \geq 2$, with $a_n \geq 0 (\forall n)$ and $a_1 = 1$.

Solution: In order to extend the validity of the given recurrence to $n \geq 1$, we define $a_0 = 0$. If we denote its generating function by A(z), we get

$$\begin{aligned} \sum_{n=2}^{\infty} a_n z^n &= \sum_{n=2}^{\infty} (a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2a_{n-2} + a_1a_{n-1})z^n \\ \Rightarrow \quad A(z) - a_1 z - a_0 &= \{A(z)\}^2 - (a_1a_0 + a_0a_1)z - {a_0}^2 \quad \text{(by } O_2\text{)} \\ \Rightarrow \quad \{A(z)\}^2 - A(z) + z &= 0 \\ \Rightarrow \quad A(z) &= \frac{1 \pm \sqrt{1-4z}}{2}. \end{aligned}$$

Now, using Binomial Theorem, the coefficient of z^n in $(1-4z)^{1/2}$ is equal to

$$\frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n,$$

which you can easily simplify to $\frac{-2}{n}C(2n-2, n-1)$.

We choose the solution $A(z) = (1 - \sqrt{1-4z})/2$, so that the terms a_n are non negative. For $n \geq 1$, we thus have

$$a_n = \frac{1}{n}C(2n-2, n-1) = \frac{(2n-2)!}{(n-1)!n!}.$$

* * *

So far we have discussed the use of generating functions in various areas. Regarding linear recurrence relations, we have seen how useful they are for finding solutions of such equations. There are several other methods for solving equations of this kind. We shall discuss them in the next unit. For now, let us summarise what we have covered in this unit.

8.5 SUMMARY

In this unit we have covered the following points.

1. Generating functions, both **ordinary and exponential**, are defined by analysing certain combinatorial problems.
2. Some elementary uses of generating functions are illustrated through examples.
3. Applications of generating functions in solving combinatorial identities are illustrated.
4. Generating functions are employed in determining the number of integer solutions to linear equations in general, and to some results on partitions of integers.
5. Some linear, homogeneous (as well as non-homogenous) recurrence equations with constant coefficients are solved.
6. How to use generating function to solve certain non-linear recurrence relations.

8.6 SOLUTIONS/ANSWERS

Note: In all the following solutions, we will skip some steps and you are encouraged to work out the individual steps to ensure understanding of the computational procedure. In most cases, previous blocks will also be helpful.

E1) a) The associated power series is

$$(z^5 + z^{10} + \dots)(1 + z^2 + z^4 + \dots)(1 + z^3 + z^6 + \dots).$$

Here the first polynomial does not contain the constant term because of the given condition.

b) Since each k, ℓ amd m are positive by given condition, and so, for a choice of $(k + \ell + m)$ fruits (with $5k + 2\ell + 3m = 50$), the associated power series is

$$(z^5 + z^10 + \dots)(z^2 + z^4 + \dots)(z^3 + z^6 + \dots).$$

E2) a) The generating function for the finite geometric progression is

$$\sum_{n=0}^{k-1} ar^n z^n = a \sum_{n=0}^{k-1} (rz)^n = a(1 - r^k z^k)/(1 - rz), \text{ by result R2.}$$

b) Replacing z by az in the binomial theorem, it follows that $(1 + az)^k$ is the generating function for the sequence $\{C(k, n)a^n\}_{n=0}^{\infty}$, if k is negative. This gives solution of (b).

c) Replacing a and k by their negatives in the expansion for $(1 + az)^k$ given in (b), we get $(1 - az)^{-k} = \sum_{n=0}^{\infty} C(-k, n)(-1)^n a^n z^n$, where $C(-k, n)$ denotes the term $(-k)(-k-1)\cdots(-k-(n-1))/n! = (-1)^n k(k+1)\cdots(k+n-1)/n! = (-1)^n C(n+k-1, n)$. Therefore, we get $(1 - az)^{-k} = \sum_{n=0}^{\infty} C(k + (n-1), n)a^n z^n$. Thun (c) follows.

E3) For negative m and n, since $(1 + z)^m$ is the generating function for the sequence $\{C(m, k)\}_{k=0}^{\infty}$ and $(1 + z)^n$ is the generating function for $\{C(n, k)\}_{k=0}^{\infty}$, the function $(1 + z)^m(1 + z)^n$ is the generating function for the sequence with k th term $\sum_{j=0}^{k} C(m, j)C(n, k-j)$ by Lemma 1. However, $(1 + z)^{m+n}$ is the generating function for $\{C(m + n, k)\}_{k=0}^{\infty}$. Hence the first identity.

The second identity follows from the first by taking $m = n = k$ and using the identity $C(n, k) = C(n, n-k)$.

E4) a) Write $a_n = a + nd, n \geq 0$. Then $a_n - a_{n-1} = d, \forall\, n \geq 1$, and $a_0 = a$. Let $\{b_n\}$ denote the sequence, where $b_0 = a$ and $b_n = d, \forall n \geq 1$.
By definition,
$B(z) = a + dz + dz^2 + \ldots = a + zd[1 + z + z^2 + \ldots] = a + dz(1-z)^{-1}$,
which is the generating function for the sequence $\{b\}_{n\geq 1}$. Thus, by Lemma 2, $B(z) = (1-z)A(z)$
$\Rightarrow \quad A(z) = a(1-z)^{-1} + z\,d(1-z)^{-2} = \{a + (d-a)z\}(1-z)^{-2}$.

b) Since $a_n = s_n - s_{n-1}$, for $n \geq 1$, and $a_0 = s_0$, so. we have

$$(1-z)S(z) = A(z). \qquad \text{(by Lemma 2)}$$

Finally, proof is complete by using the definition O_5 of quotients of series.

c) The n th term of the given sequence is the n th partial sum of the sequence $\{1, 2, 3, \ldots,\}$ whose generating function A(z) (say) is $(1-z)^{-1}$, by (a). Hence, by (b), the generating function for the sequence $\{1, 3, 6, \ldots\}$ equals $(1-z)^{-3}$.

E5) Differentiating the binomial function $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=0}^{\infty} jz^{j-1} = (1-z)^{-2} \qquad \text{(see } O_3\text{)}.$$

On multiplying this by z both sides, we get
$A(z) = \sum_{j=1}^{\infty} jz^j = z(1-z)^{-2}$,
where we write A(z) for the generating function of the sequence $\{j\}_{j\geq 1}$.
Then

$$\begin{aligned}\sum_{k=1}^{\infty} \sigma_k^1 z^k &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j\right) z^k \\ &= \frac{A(z)}{(1-z)} \qquad \text{(by E4(b))} \\ &= z(1-z)^{-3}.\end{aligned}$$

Therefore, σ_n^1 is the coefficient of z^n in the series which can be obtained by expanding the function $z(1-z)^{-3}$. However, this is the same as looking for the coefficients of z^{n-1} in the expanded form of the binomial function $(1-z)^{-3}$. Thus, in view of binomial identity $C(n, k) = C(n, n-k)$, we have

$$\sigma_n^1 = C(n+1, n-1) = C(n+1, 2) = n(n+1)/2.$$

E6) By definition, exponential generating function of the sequence $\{P(n.k)\}_{k=1}^{n}$ is

$$\sum_{k=0}^{\infty} \frac{P(n,k)}{k!} z^k = \sum_{k=0}^{\infty} C(n,k)z^k = (1+z)^n.$$

E7) The exponential generating function for the sequence of Bell numbers,

$\mathcal{B}_{exp}(z)$ equals $\sum_{n=0}^{\infty}(B_n/n!)\,z^n$. Now,

$$\begin{aligned}(e^z-1)\mathcal{B}_{exp}(z) &= \left(\sum_{n=0}^{\infty}\frac{1}{n!}z^n\right)\left(\sum_{n=0}^{\infty}\frac{B_n}{n!} \quad\quad \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \\ &= \left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\cdot\frac{B_{n-k}}{(n-k} \quad\quad -\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \\ &= \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\sum_{k=0}^{n}C(n,k)\right. \quad\quad z^n\right]-\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \\ &= B_0z+\left[\sum_{n=2}^{\infty}\frac{1}{n!}\left\{\sum_{k=0}^{n} \quad\quad B_{n-k}-B_n\right\}z^n\right] \\ &= B_0z = z.\end{aligned}$$

E8) A first-order recurrence equation that the sequenc satisfies is given by

$$d_n = nd_{n-1}+(-1)^n,\ n\geq 2,\ \text{with}\ d_1=0, d_2 \quad\quad \text{(see Problem 7 of U}$$

In order that the recurrence also hold for $n=1$, ... e $d_0=1$.

Then, with $D_{exp}(z)=\sum_{n=0}^{\infty}(d_n/n!)\,z^n$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{d_n}{n!}z^n &= \sum_{n=1}^{\infty}\frac{nd_{n-1}}{n!}z^n+\sum_{n=}^{\infty} \quad\quad z^n \\ \Rightarrow\quad D_{exp}(z)-d_0 &= z\,D_{exp}(z)+(e^{-z}- \\ \Rightarrow\quad D_{exp}(z) &= \frac{e^{-z}}{1-z}.\end{aligned}$$

Now the coefficient of z^n in the expansion of e^{-z} ... $(-1)^n/n!$, and so, the coefficient of z^n in the expansion of D_{exp} ... $\sum_{k=0}^{n}(-1)^k/k!$ (see E3(b)). It then follows that $d_n = n!\sum_{k=0}^{n}$ ... $!$, $\forall n$, by comparing coefficients of z^n.

E9) Differentiating and then multiplying by z on bo ... of the identity $\sum_{k=0}^{n}C(n,k)z^k=(1+z)^n$, we get $\sum_{k=1}^{n}kC(n,$ ... $z(1+z)^{n-1}$. Putting $z=3$ yields $\sum_{k=1}^{n}k3^kC(n,k)=3\times 4^n$

E10) Since the required generating function is $(z^2+z^3+z^4+\cdots)(z^3+z^4+z^5+\cdots)(z^4+z^5$ ... $)(z^5+z^6+z^7+\cdots)(z^6+z^7+z^8+\cdots)=z^{20}(1+z+z^2+\cdots)$ ... number of integer solutions is the coefficient of z^8 in $(1-z)^{-5}$, w ... ls $\binom{12}{4}=495$.

E11) a) The contribution to the generating functi ... part k is $(1+z^k+z^{2k}+\cdots)$. Since $1\leq k\leq m$, th ... generating function is $\prod_{k=1}^{m}(1+z^k+z^{2k}+\cdots)=\prod$ ... $z^k)^{-1}$.

b) If we use unequal parts, no part k may b ... The corresponding term in the generating fu ... $+z^k)$, so that k may be used at most once. Therefore, th ... ng function is $\prod_{k=1}^{\infty}(1+z^k)$.

c) The contribution from the odd part, 2k - $(1+z^{2k-1}+z^{2(2k-1)}+\cdots)$. Thus, the re ... enerating function is $\prod_{k=1}^{\infty}(1+z^{2k-1}+z^{2(2k-1)}+\cdots)=\prod$ ... $z^{2k-1})^{-1}$.

E12) i) By above discussion, the required genera ... ction is $(1+z^{p_1}+z^{2p_1}+\ldots)(1+z^{p_2}+z^{2p_2}+\ldots$

ii) Similarly, here generating function will be
$(1+z^{p_1})(1+z^{p_2})\cdots$

E13) We set $\mathcal{L}_0 = \mathcal{L}_2 - \mathcal{L}_1 = 2$, so that the recurrence is valid for $n \ge 2$. By Theorem 1, $(1-z-z^2)L(z) = \mathcal{L}_0 + (\mathcal{L}_1 - \mathcal{L}_0)z = 2 - z$. Therefore, $L(z) = (1-\alpha z)^{-1} + (1-\beta z)^{-1}$, where $\alpha + \beta = 1 = -\alpha\beta$. Comparing the coefficients of z^n, we get $\mathcal{L}_n = \alpha^n + \beta^n$, $n \ge 0$.

E14) Defining $T_0 = 0$, so that the recurrence is valid for $n \ge 1$, and writing $T(z)$ for the generating function of $\{T_n\}_{n=0}^{\infty}$, we have $T(z) = \sum_{n=0}^{\infty} T_n z^n = T_0 + 2\sum_{n=1}^{\infty} T_{n-1} z^n + \sum_{n=1}^{\infty} z^n = 2z.T(z) + z/(1-z)$. Therefore, $T(z) = z/(1-z)(1-2z) = (1-2z)^{-1} - (1-z)^{-1}$, and hence $T_n = 2^n - 1$, $n \ge 0$, by comparing coefficients after applying binomial theorem on r.h.s. of the last equality.

E15) Let $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Then,

$$\begin{aligned}
T(z) &= (t_0 + t_1 z + t_2 z^2) + \sum_{n=3}^{\infty} t_{n-3} z^n \\
&\qquad + \sum_{n=1}^{\infty} \frac{2n+1+(-1)^{n+1}}{4} z^{2n+1} \\
&= z^3.T(z) + \frac{z}{4}\sum_{n=1}^{\infty}(2n+1)z^{2n} + \frac{z^3}{4}\sum_{n=0}^{\infty}(-1)^n z^{2n} \\
\Rightarrow (1-z^3)\,T(z) &= \frac{z}{4}\frac{d}{dz}\sum_{n=1}^{\infty} z^{2n+1} + \frac{z^3}{4(1-z^2)} \\
&= \frac{3z^3}{4(1-z^3)^2} + \frac{z^3}{4(1-z^2)} \\
\Rightarrow \quad T(z) &= \frac{z^3(4-3z^2-2z^3+z^6)}{4(1-z^3)^3(1-z^2)}.
\end{aligned}$$

E16) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Then,

$$\begin{aligned}
(1-3z-10z^2)A(z) &= a_0 + (a_1 - 3a_0)z + \\
&\quad \sum_{n=2}^{\infty}(a_n - 3a_{n-1} - 10a_{n-2})z^n \\
&= 25 + 45z + 28\sum_{n=2}^{\infty}(5z)^n \\
&= (25 - 80z + 475z^2)/(1-5z).
\end{aligned}$$

Using partial fractions, we get
$A(z) = (25 - 80z + 475z^2)/(1+2z)(1-5z)^2 =$
$15(1+2z)^{-1} - 10(1-5z)^{-1} + 20(1-5z)^{-2}$. Equating coefficients of z^n, we get
$a_n = 15(-2)^n - 10\cdot 5^n + 20(n+1)5^n = 15(-2)^n + (10+20n)5^n$, $n \ge 0$.

UNIT 9 SOLVING RECURRENCES

Structure Page No.

9.1 INTRODUCTION

In the two previous units of this block, you have studied about setting up recurrences and how to solve them by the use of generating functions. In this unit we concentrate on other methods of finding solutions of recurrence equations.

To begin with, we shall develop the general theory for solving a linear homogeneous recurrence with constant coefficients. Following this, we shall discuss some general theory for solving a linear non-homogeneous recurrence whose non-homogeneous part is a polynomial or an exponential function. We shall conclude the unit by illustrating several techniques developed for solving recurrences which may otherwise be hard to solve by more standard methods. We shall also look at examples of real-life applications of the theory we discuss.

As you can see, this unit is closely linked with Unit 7. So, please glance over that unit again before going further.

Let us now clearly spell out the objectives of this unit.

Objectives

After reading this unit, you should be able to

- find the characteristic polynomial, equation and roots of a linear, homogeneous recurrence relation with constant coefficients;
- solve any linear, homogeneous recurrence relation with constant coefficients;
- solve linear, non-homogeneous recurrences with constant coefficients when the non-homogenous part is either a polynomial or an exponential function;
- solve recurrence relations by the method of inspection/telescopic sums/iteration/substitution, wherever applicable.

9.2 LINEAR HOMOGENEOUS RECURRENCES

You would recall from Unit 7 that the general form of a linear, non-homogeneous recurrence of order k is

$u_n = f_1(n)u_{n-1} + f_2(n)u_{n-2} + \cdots + f_k(n)u_{n-k} + g(n),\ n \geq k,$

where each f_j and g is a function of n. It is homogeneous if g is identically zero, and non-homogeneous otherwise.

Now, let us assume that g is non-zero. Then, associated with the non-homogeneous recurrence is the homogeneous recurrence

$u_n = f_1(n)u_{n-1} + f_2(n)u_{n-2} + \cdots + f_k(n)u_{n-k},\ n \geq k,$

which we get by simply setting to zero the non-homogeneous part.

Let us concentrate on recurrences whose homogeneous parts are linear. You know that the most general linear homogeneous equation with constant coefficients is

$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k},\ n \geq k,$ (1)

where the c_i are constants, i.e., $c_i \in \mathbf{C}\ \forall\ i$.

Related to this is an equation that we shall now define.

Definitions: The characteristic equation, or **auxiliary equation**, of the linear, homogeneous recurrence (1) is the equation

$z^k - c_1z^{k-1} - c_2z^{k-2} - \cdots - c_{k-1}z - c_k = 0.$ (2)

The roots of the characteristic equation (2) are called the **characteristic roots** of (1).

The **multiplicity of a characteristic root** α of (1) is the greatest integer m such that $(z-\alpha)^m$ is a factor of the **characteristic polynomial** of (1), i.e., of $z^k - c_1z^{k-1} - \cdots - c_k$.

Notice that the characteristic equation is simply obtained by setting the mth term of the sequence $\{u_n\}$ equal to z^m in the recurrence, and simplifying.

For instance, the characteristic equation of the recurrence

$u_{n+2} = 2u_n - u_{n-2},\ \ n \geq 2,$ is

$z^{n+2} = 2z^n - z^{n-2}$, i.e., $z^4 = 2z^2 - 1$.

Therefore, the characteristic roots of this recurrence are 1 and -1, both with multiplicity 2.

Now, given the characteristic roots of a recurrence, how do we solve it? As you know from Unit 8, solving a recurrence means finding a sequence $\{a_n\}$ that satisfies it, where a_n is a function of n. Often if we can find such a sequence, then we shall (somewhat carelessly!) say a_n is a solution.

Now, to try and understand how to solve recurrences like (1), let us consider the recurrence

$a_n = 16a_{n-2}$.

From Unit 8, you know that its solution is of the form

$a_n = A(4)^n + B(-4)^n$, where A and B are constants. Observe that 4 and -4 are the roots of the characteristic equation, $z^2 = 16$, of the given recurrence. Both these roots have multiplicity 1.

Now let us consider the recurrence

$a_{n+2} = 2a_{n+1} + 4a_n - 8a_{n-1}$.

You can check that its characteristic polynomial is $z^3 - 2z^2 - 4z + 8$, i.e., $(z-2)^2(z+2)$.
So, its characteristic roots are 2 (with multiplicity 2) and -2 (with multiplicity 1).
By applying the techniques of Unit 8, you can also check that the general solution of the given recurrence is
$a_n = (A_0 + A_1 n)(2)^n + B_0(-2)^n, A_0, A_1, B_0 \in \mathbf{C}$.
We can write this as
$a_n = A'_0 C(n,0)2^n + A'_1 C(1+n,1)2^n + B_0 C(n,0)(-2)^n, A'_0, A'_1, B_0 \in \mathbf{C}$.

Have these examples given you an inkling of the general form of the solution of (1) in terms of its characteristic roots? Match your conclusions with the following theorem.

Theorem 1: A sequence $\{a_n\}$ satisfies the linear, homogeneous recurrence relation with constant coefficients
$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k}, n \geq k,$

if and only if each a_n is a sum of expressions of the form
$b_0 C(n,0)\alpha_i^n + b_1 C(1+n,1)\alpha_i^n + \ldots + b_{m_i-1} C(m_i - 1 + n, m_i - 1)\alpha_i^n$
where $\alpha_1, \alpha_2, \cdots$ are the characteristic roots of multiplicity $m_1, m_2, \cdots$, respectively, and the b_js are constants.

Proof: We recall from Theorem 1 in Unit 8 that the generating function, $U(z)$, of the sequence $\{u_n\}$ is of the form $p(z)/q(z)$, where p and q are polynomials with $\deg p < \deg q$, and $q(z) = 1 - c_1 z - c_2 z^2 - \cdots - c_k z^k$.
Now, $z^k - c_1 z^{k-1} - c_2 z^{k-2} - \cdots - c_{k-1} z - c_k = \prod_i (z - \alpha_i)^{m_i}$

$$\iff z^k\left[1 - c_1\left(\frac{1}{z}\right) - c_2\left(\frac{1}{z}\right)^2 - \cdots - c_k\left(\frac{1}{z}\right)^k\right] = z^k \prod_i \left(1 - \frac{\alpha_i}{z}\right)^{m_i}$$

$$\iff 1 - c_1 t - c_2 t^2 - \cdots - c_k t^k = \prod_i (1 - \alpha_i t)^{m_i}, \text{ where we put } t = \frac{1}{z}.$$

$$\therefore U(z) = \frac{p(z)}{q(z)}, \text{ where } q(z) = \prod_i (1 - \alpha_i z)^{m_i} \text{ and } \deg p < \deg q.$$

So, using partial fractions, we can express $U(z)$ as a linear combination of terms of the form $(1 - \alpha_i z)^{-j-1}$, where $0 \leq j \leq m_i - 1$. Since the coefficient of z^n in the expansion of $(1-\alpha_i z)^{-j-1}$ equals $C(-j-1, n)\alpha_i^n$, i.e., $C(j+n, j)\alpha_i^n$, the theorem follows.

For $j \geq 0$, $C(-j, n) = (-1)^n C(j + n - 1, n)$.

Note that each a_n in the theorem is actually a finite linear combination of terms of the form $n^j\alpha^n$, where α is a characteristic root of multiplicity m, and $0 \leq j \leq m - 1$. This is because the binomial coefficients $C(j+n, j)$ are themselves polynomials of degree j in the variable n. It is often easier to express the solution in this form, as for instance, **when the characteristic roots are all distinct, i.e., of multiplicity one.** In this situation, the form of the solution sequence is

$$u_n = \sum_{j=1}^{k} A_j \alpha_j^n, \ n \geq 0,$$

where the α_js are the characteristic roots and A_js are constants that are to be determined by the initial conditions.

Let us look at some examples of how Theorem 1 can be applied. While doing so, let us see how the solution depends on the initial conditions.

Example 1: Solve the recurrence $a_n = 4a_{n-2}$, where
a) $a_0 = 4, a_1 = 6$

b) $a_0 = 6, a_2 = 20$
c) $a_1 = 6, a_2 = 20$.

Solution: The roots of the characteristic equation of the recurrence, $z^2 = 4$, are ± 2. Thus, by Theorem 1, the general solution is of the form $a_n = A(2)^n + B(-2)^n$, where A and B are arbitrary constants.

a) Now, if $a_0 = 4$ and $a_1 = 6$, then the general solution gives us
$A + B = 4$ and $2A - 2B = 6$.
$\therefore A = \frac{7}{2}, B = \frac{1}{2}$.
So the solution is $a_n = 7(2)^{n-1} - (-2)^{n-1}$.

b) If $a_0 = 6$ and $a_2 = 20$, the general solution yields
$A + B = 6$ and $4A + 4B = 20$.
Since these equations are inconsistent, there is **no solution.**

c) If $a_1 = 6, a_2 = 20$, we get
$2(A - B) = 6$ and $4(A + B) = 20$.
So, $A = 4$, $B = 1$, and the solution is
$a_n = 4(2)^n + (-2)^n$.

* * *

In the example above you have seen how important the initial conditions are. You have also seen that sometimes these conditions can be such that no solution is possible.

Now consider a second order linear homogeneous recurrence with constant coefficients that you solved in Unit 8 by making use of generating functions. This equation can also be solved by applying Theorem 1, as you will just see.

Example 2: Obtain the solution for the recurrence relation satisfied by the Fibonacci sequence (see Problem 1, Unit 7).

Solution: Recall that the Fibonacci sequence $\{\mathcal{F}_n\}$ satisfies

$$\mathcal{F}_n - \mathcal{F}_{n-1} - \mathcal{F}_{n-2} = 0 \text{ if } n \geq 3, \text{ and } \mathcal{F}_1 = 1 = \mathcal{F}_2. \quad (3)$$

The characteristic equation, $z^2 - z - 1 = 0$, has distinct roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Therefore, by Theorem 1, for some constants A and B,

$$\mathcal{F}_n = A\alpha^n + B\beta^n, \ n \geq 1. \quad (4)$$

This is the general solution for the recurrence (3).
As you have seen in the previous example, the values of A and B depend on the initial conditions, i.e., the first two terms of the sequence.

Since $\mathcal{F}_1 = 1, (4) \implies 1 = A\alpha + B\beta$.

Since $\mathcal{F}_2 = 1, (4) \implies 1 = A\alpha^2 + B\beta^2$.

Also, since α and β are roots of $z^2 - z - 1 = 0$,

$\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$.

So, we get

$1 = A\alpha^2 + B\beta^2 = A(\alpha + 1) + B(\beta + 1) = (A\alpha + B\beta) + (A + B) = 1 + (A + B)$.
Therefore, $A + B = 0$.
Therefore, $A(\alpha - \beta) = 1$, and

$$\mathcal{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}, \ n \geq 1.$$

* * *

Now consider an example in which no initial conditions are given.

Example 3: Solve the sixth order linear, homogeneous recurrence relation

$u_n + u_{n-1} - 11u_{n-2} - 13u_{n-3} + 26u_{n-4} + 20u_{n-5} - 24u_{n-6} = 0.$

Solution: The first step is to identify the characteristic roots together with their multiplicities. The characteristic equation is
$z^6 + z^5 - 11z^4 - 13z^3 + 26z^2 + 20z - 24 = 0$
i.e., $(z-1)^2(z-3)(z+2)^3 = 0$.
Since the root 1 is of multiplicity two, the root 3 of multiplicity one and the root (-2) of multiplicity three, by Theorem 1 we know that u_n is a linear combination of the six terms

$C(0+n,0)\cdot 1^n, C(1+n,1).1^n, C(0+n,0)3^n, C(0+n,0)(-2)^n, C(1+n,1)(-2)^n$ and $C(2+n,2)(-2)^n$,

i.e., $u_n = a + b(1+n) + c.3^n + d(-2)^n + e(1+n)(-2)^n + f.\dfrac{(1+n)(2+n)}{2}.(-2)^n$,

where a, …, f are constants which can be determined if any six consecutive terms (typically, the first six) of the sequence are known. Since no initial conditions are given, we can only simplify the expression to the form
$u_n = A + Bn + C.3^n + (D + En + Fn^2)(-2)^n$, where A, …, F are constants.

* * *

So far we have solved linear recurrence relations by using Theorem 1. Now let us solve a non-linear recurrence relation, by reducing it to a linear relation.

Example 4: Solve the recurrence $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ and $a_0 = 2$. Also find a_8.

Solution: The given recurrence is a quadratic relation. But, if we put $b_n = a_n^2$, the relation becomes
$b_{n+1} = 5b_n$, $b_0 = 4$.
From Theorem 1, you know that its solution is
$b_n = A(5)^n$, A a constant.
Now, $b_0 = 4 \Rightarrow A = 4$.
$\therefore b_n = 4(5)^n$.
Since a_n is the positive square root of b_n,
$a_n = 2(5)^{n/2}$ for $n \geq 0$.
$\therefore a_8 = 1250$.

* * *

Why don't you try some exercises now?

E1) Find the general solution of the recurrence relation
$a_n = 3a_{n-1}$.

E2) Determine constants c_1 and c_2 such that the recursion
$u_n + c_1u_{n-1} + c_2u_{n-2} = 0$ has the characteristic roots $1 \pm \sqrt{-1}$.

E3) Find the solution of the following recurrence equation satisfied by P_n^2, the number of partitions of n into two parts in non-increasing order:
$P_n^2 = P_{n-1}^2 + P_{n-2}^2 - P_{n-3}^2, n \geq 3, P_1^2 = 0, P_2^2 = 1, P_3^2 = 1$

In this section we have seen some ways of solving linear homogeneous recurrences with constant coefficients. You have also seen how some non-linear recurrences can be reduced to such linear recurrences, and hence solved. Let us now see how to use what we have discussed here for solving non-homogeneous recurrences with constant coefficients.

9.3 LINEAR NON-HOMOGENEOUS RECURRENCES

In this section we shall look at some general theory pertaining to finding solutions of equations like $u_n = 3u_{n-2} + 3n^5 - 2^n$. More generally, we shall study equations of the form

$$u_n = c_1u_{n-1} + c_2u_{n-2} + \ldots + c_ku_{n-k} + g(n), n \geq k. \quad (5)$$

Looking at (5), you may wonder if the solutions of (1) and (5) are linked. The following theorems tell us something about this.

Theorem 2: If $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences, each satisfying the non-homogeneous recurrence (5), then $\{d_n\}$, with $d_n = a_n - b_n$, $n \geq 0$, satisfies the associated homogeneous recurrence (1).

Proof: Since $\{a_n\}$ and $\{b_n\}$ satisfy (5), and $d_n = a_n - b_n$, for $n \geq 0$ we get

$$\begin{aligned} d_n &= a_n - b_n \\ &= [c_1a_{n-1} + \cdots + c_ka_{n-k} + g(n)] - [c_1b_{n-1} + \cdots + c_kb_{n-k} + g(n)] \\ &= c_1d_{n-1} + \cdots + c_kd_{n-k}. \end{aligned}$$

This shows that $\{d_n\}$ satisfies (1), i.e., we have proved the statement.

Now, can you see how we can use Theorem 2 along with Theorem 1 to find the general form of any solution of (5)? The following result explicitly answers this question.

A particular solution of (5) is any sequence $\{a_n\}$ that satisfies (5).

Theorem 3: Every solution of the recurrence (5) is of the form $a_n + b_n$, where a_n is any particular solution of (5) and b_n is any solution of its associated homogeneous recurrence (1).

Proof: Let a_n be any particular solution of (5). Now, Theorem 2 tells us that the difference of any two solutions of (5) is a solution of (1).
So, every solution u_n of (5) satisfies $u_n - a_n = b_n$, where b_n satisfies (1). Therefore, $u_n = a_n + b_n$, where a_n is a particular solution of (5) and b_n is a solution of (1).

We have proved the two theorems above only for linear recurrence relations with constant coefficient. But they hold true in the general case also. This is what the following exercise is about.

E4) State and prove the analogues of Theorems 2 and 3 for general recurrences of the form
$u_n = f_1(n)u_{n-1} + f_2(n)u_{n-2} + \cdots + f_k(n)u_{n-k} + g(n)$,
where the f_is and g are functions of n.

In view of the two theorems above, to solve (5) we must look for any one solution of (5) and the general form of the solution of (1). Let us consider an example.

Example 5: Find the complete solution of the recurrence
$a_n = 3a_{n-1} - 4n, n \geq 1$.

Solution: The required solution, as Theorem 3 says, is the sum of the general solution of $a_n = 3a_{n-1}$ and any solution of the given recurrence.

From E1 you know that the general solution of $a_n = 3a_{n-1}$ is
$a_n = b.3^n$,
where b is a constant.

Now, let us consider the non-homogeneous part too. We have
$a_n = 3a_{n-1} - 4n$.
Let us see if a_n can be of the form $An + B, A, B \in C$.
If it is, then
$An + B = 3[A(n-1) + B] - 4n = n(3A - 4) - 3A + 3B$.
Comparing the coefficients of n, we get
$A = 3A - 4$ and $B = 3B - 3A$,
i.e., $A = 2$ and $B = 3$.
So, $a_n = 2n + 3$ works, and hence is a **particular solution** of the given recurrence.
So, the total solution of the recurrence will be
$a_n = b.3^n + 2n + 3, b \in C$.

* * *

In the example above, we have obtained a particular solution by guess work. In many cases we need to use such an approach. Unlike the homogeneous case, there is no general method to obtain a particular solution for a non-homogeneous recurrence. But there are techniques available for certain recurrences, including the one given in Example 5. The following theorems tell us about two special cases.

Theorem 4: A particular solution of (5) with non-homogeneous part an^d, where a is a known constant and $d \in N$, is of the form

i) $A_0 + A_1 n + A_2 n^2 + \cdots + A_d n^d$, if 1 is not a characteristic root of (5);

ii) $A_0 n^m + A_1 n^{m+1} + \cdots + A_d n^{m+d}$, if 1 is a characteristic root of (5) with multiplicity m,

where $A_0, A_1, \cdots, A_d$ are constants.

Theorem 5: A particular solution of (5) with non-homogeneous part ar^n (where a is a known constant) is of the form

i) Ar^n, if r is not a characteristic root of (5);

ii) $An^m r^n$, if r is a characteristic root of (5) with multiplicity m,

where A is a constant.

We will not prove these results here, but shall look at a few examples of their use. You may recall having encountered some of these examples in previous units.

Example 6: Find the solution to the recurrence in Problem 3 of Sec. 7.2, namely, $L_n = L_{n-1} + n, n \geq 2$, with $L_1 = 2$.

Solution: Observe that 1 is the only characteristic root of this recurrence. So, the general solution to the homogeneous part of this recurrence is simply $a.1^n = a$, where a is a constant.

Now, the non-homogeneous part of the recurrence is n. So, applying Theorem 4(ii) with $m = 1$ and $d = 1$, we see that a particular solution of this recurrence is of the form
$A_0 n + A_1 n^2, A_0, A_1 \in C$.
To find the values of A_0 and A_1, we set $L_n = A_0 n + A_1 n^2$ in the recurrence relation to get

$$\begin{aligned} A_0 n + A_1 n^2 &= A_0(n-1) + A_1(n-1)^2 + n \\ &= (-A_0 + A_1) + (A_0 - 2A_1 + 1)n + A_1 n^2 \end{aligned}$$

Comparing the constant terms and the coefficients of n, we get
$0 = -A_0 + A_1,\ A_0 = A_0 - 2A_1 + 1$.

Therefore, $A_0 = A_1 = \frac{1}{2}$.

Now, taking the sum of both the solutions, we get

$$L_n = a + \frac{n(n+1)}{2}.$$

The initial condition $L_1 = 2$ tells us that $a = 1$, so that

$$L_n = 1 + \frac{n(n+1)}{2},\ n \geq 1.$$

* * *

Example 7: Rani takes a loan of R rupees which is to be paid back in T months. If I is the interest rate per month for the loan, what constant payment P must she make at the end of each period?

Solution : Let a_n denote the amount Rani owes at the end of the nth month, i.e., after the nth payment. Then the problem can be written as $a_{n+1} = a_n + Ia_n - P,\ \ 0 \leq n \leq T-1, a_0 = R, a_T = 0.$

So, the homogeneous part contributes $b(1+I)^n$ to the solution, b being a constant.

Using Theorem 5(i), with $r = 1$, we see that the non-homogeneous part contributes A, a constant.
But then, putting $a_n = A$ in our recurrence relation, we get
$A = A(1+I) - P \Rightarrow A = P/I.$

Thus, $a_n = b(1+I)^n + P/I.$
Then, $a_0 = R \Rightarrow b + P/I = R \Rightarrow b = R - P/I$
Also, $a_T = 0 \Rightarrow b(1+I)^T + P/I = 0$

$$\therefore P = \frac{IR(1+I)^T}{[1-(1+I)^T]}$$

* * *

Example 8: Solve the recurrence $u_n = au_{n-1} + c.a^n$, $n \geq 1$, where a and c are known constants.

Solution: Using Theorem 5, we get

$$\begin{aligned} u_n &= A.a^n + Bna^n, \text{ A and B being constants.} \\ &= a^n(A + Bn) \text{ for } n \geq 0. \end{aligned}$$

* * *

Now, here are some simple exercises for you.

E5) Solve the recurrence $T_n = 2T_{n-1} + 1$, $n \geq 2$, with $T_1 = 1$ (see Problem 2, Sec. 7.2).

E6) The population of a species of snails in a certain lake triples every year. Starting with 1000 such snails, and finding 1500 of them the following year, 200 are removed from the lake to increase them in other lakes. Similarly, at the end of every year 200 are removed. If a_n represents the snail population in the lake after n years, find and solve a recurrence relation for $a_n, n > 0$.

Now let us consider a result which tells us how to find a particular solution for recurrences with non-homogeneous parts which are linear combinations of n^d and r^n, r a constant.

Theorem 6 (Superposition Principle): If $\{a_n\}$ is a solution of
$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k} + g_1(n)$
and $\{b_n\}$ is a solution of

$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k} + g_2(n)$,
then, for constants A and B, $Aa_n + Bb_n$ is a solution of
$u_n = c_1u_{n-1} + \cdots + c_ku_{n-k} + Ag_1(n) + Bg_2(n)$.

Proof: For $n \geq k$, we have $Aa_n + Bb_n$
$= A[c_1a_{n-1} + \cdots + c_ka_{n-k} + g_1(n)] + B[b_{n-1} + \cdots + c_kb_{n-k} + g_2(n)]$
$= c_1(Aa_{n-1} + Bb_{n-1}) + \cdots + c_k(Aa_{n-k} + Bb_{n-k}) + \{Ag_1(n) + Bg_2(n)\}$.

This means that $Aa_n + Bb_n$ is a solution of (5) with $g(n) = Ag_1(n) + Bg_2(n)$.

In view of Theorem 6, we can combine the results of Theorems 4 and 5 to get solutions of non-homogeneous recurrences like the following one.

Example 9: Obtain the general solution of the recurrence
$v_n - 7v_{n-1} + 12v_{n-2} = 5.2^n - 4.3^n$, $n \geq 2$.

Solution: Since there are no initial conditions and the equation is of second order, we can only expect a general solution involving two constants.

To begin with, the homogeneous part $v_n - 7v_{n-1} + 12v_{n-2} = 0$ has the characteristic polynomial $z^2 - 7z + 12$, i.e., $(z-3)(z-4)$. Consequently, its general solution is of the form $a.3^n + b.4^n$, where $a, b \in C$.

Now let's consider the non-homogeneous part. It consists of two terms, one of which is a power of one of the characteristic roots. By Theorems 5 and 6, we must set $v_n = c.2^n + dn.3^n$ in order to find a particular solution.
When we do this, the recurrence relation gives us
$2^{n-2}c(4 - 14 + 12) + 3^{n-2}d\,[9n - 21(n-1) + 12(n-2)] = 5.2^n - 4.3^n$.
$\Rightarrow 2^{n-1}(c - 10) = 3^{n-1}(d - 12)$
Since this equality is true for every $n \geq 1$, we see that $2^{n-1} | (d - 12)$ for every $n \geq 1$. This can only be true if $d - 12 = 0$, i.e., $d = 12$. This forces $c - 10 = 0$ to be true, i.e., $c = 10$.
Putting all this information together, we get
$v_n = 10.2^n + (a + 12n)3^n + b.4^n$, where $a, b \in C$.

***

Let us go back to Theorem 6 for a moment. Will the superposition principle be true for linear homogeneous recurrences too? Actually, it will, and we have been using this fact quite a lot. Try and pinpoint where we have first used it for such recurrences.

Here are some exercises now.

E7) If the recurrence $u_n + c_1u_{n-1} + c_2u_{n-2} = an + b$ has a general solution $u_n = A.2^n + B.5^n + 3n - 5$, find a, b, c_1 and c_2.

E8) Solve the recurrence $v_n - 7v_{n-1} + 16v_{n-2} - 12v_{n-3} = 2^n + 3^n$, with the initial terms $v_0 = 1, v_1 = 0, v_2 = 1$.

So far we have seen how to solve (5) if $g(n)$ is of the form an^d, ar^n or a linear combination of terms of these types. There is one more type of non-homogeneous part that we shall discuss now.

Theorem 7: A particular solution of (5) with non-homogeneous part an^dr^n, where a and r are known constants and $d \in N$, is of the form

i) $Ar^n(A_0 + A_1n + \cdots + A_dn^d)$, if neither r nor 1 are characteristic roots of (5);

ii) $An^mr^n(A_0 + A_1n + \cdots + A_dn^d)$, if either r or 1 (but not both) is a characteristic root of (5) with multiplicity m;

iii) $An^{m_1+m_2}r^n(A_0 + A_1n + \cdots + A_dn^d)$, if r and 1 both are characteristic roots of (5) with multiplicities m_1 and m_2, respectively,

where $A, A_0, A_1, \cdots, A_d$ are constants.

As before, we shall not prove this result, but shall show how it can be applied.

Example 10: Find a linear homogeneous recurrence with constant coefficients for which the characteristic roots are 1 with multiplicity two, -1 with multiplicity three and 2 with multiplicity five. Further, assume that the non-homogeneous part is a linear combination of $n(-1)^n, n^2.2^n$ and 3^n plus a polynomial of degree three.

Solution: We wish to solve a recurrence which has 10 characteristic roots. So, it is of the form

$$u_n = c_1u_{n-1}+\cdots+c_{10}u_{n-10}+(a_0+a_1n+a_2n^2+a_3n^3)+bn(-1)^n+c.n^22^n+d.3^n,$$

where we know that the characteristic polynomial of the homogeneous part is $(z-1)^2(z+1)^3(z-2)^5$,

i.e., $z^{10} - c_1z^9 - \cdots - c_9z - c_{10} \equiv (z-1)^2(z+1)^3(z-2)^5$.

So, by Theorem 1, the form of the general solution to the homogeneous part will be

$$(A_0 + A_1n).1^n + (B_0 + B_1n + B_2n^2)(-1)^n + (C_0 + C_1n + \cdots + C_4n^4)2^n, \quad (6)$$

where the As, Bs and Cs are constants.

Now, by Theorem 4, you know that the form of the particular solution corresponding to the third degree polynomial is

$n^2(D_0 + D_1n + D_2n^2 + D_3n^3)$, where the Ds are constants.

From Theorem 7 you know that the form of the solution corresponding to $bn(-1)^n$ is $n^5(-1)^n(E_0 + E_1n)$, and to

$cn^2.2^n$ is $n^7.2^n(F_0 + F_1n + F_2n^2)$, where the Es and Fs are constants.

From Theorem 5, you know that the part of the solution corresponding to $d.3^n$ is $G.(3)^n$, G being a constant.

Thus, the particular solution is of the form

$$\left.\begin{array}{l} n^2(D_0 + D_1n + D_2n^2 + D_3n^3)x + n^5(-1)^n(E_0 + E_1n) \\ \quad +n^7(2^n)(F_0 + F_1n + F_2n^2) + G(3^n) \end{array}\right] \quad (7)$$

Therefore, the complete solution is the sum of the expressions in (6) and (7).

* * *

Here's an exercise of the same type for you.

E9) Find a recurrence relation with constant coefficients for which the characteristic roots are 3 with multiplicity 1 and -2 with multiplicity 2. The relation also has a non-homogeneous part which is a linear combination of $2^n, n(-1)^n$ and a polynomial of degree 2.

In this section we have considered some general methods for tackling special kinds of non-homogeneous recurrences. While studying them you would have noticed that the solution of the non-homogeneous part is dependent on whether a characteristic root of the recurrence occurs in this part.

Now that you have studied this section and Unit 8, can you solve all the problems given in Unit 7? What about the 'divide and conquer' problem? To solve this problem and other recurrences with non-homogeneous parts different from the ones looked at in this section, we need to look at some other solution techniques. Let us do so now.

9.4 SOME OTHER METHODS

In the previous section we have seen how to deal with two kinds of non-homogeneous parts of linear recurrences. There are many other kinds of recurrences that we can solve by some special methods. We shall look at four of these methods in this section.

9.4.1 Method of Inspection

One simple way of solving a recurrence is to write down enough terms in the sequence until one feels comfortable in guessing the solution. However, unless the pattern of the sequence is fairly straightforward, it is not easy to make a good guess. Usually, if one has made a correct guess here, the principle of mathematical induction (see Unit 2) can be used to prove the guess. Let us consider an example.

Example 11: Solve, by inspection, the recurrence relation $a_n = a_{n-1} + n!n$ if $n \geq 1$, and $a_0 = 0$.

Solution: If we compute the first five terms of this sequence, we get $0, 1, 5, 23$ and 119. Can you see what the nth term might be? Does adding one to each term in the sequence help? Doing so would give us a sequence that you would recognise, i.e., $(n+1)!$. So our initial guess is $a_n = (n+1)! - 1$.

Having done the initial work of making a guess, let us attempt to prove it by using induction on n.
The base case is easy to check: $a_0 = (0+1)! - 1 = 0$.
If we are to assume the result for $n = k$, for some $k \geq 0$, then

$$\begin{aligned} a_{k+1} &= a_k + (k+1)!(k+1) = [(k+1)! - 1] + (k+1)!(k+1) \\ &= (k+1)!(k+2) - 1 = (k+2)! - 1, \text{ as we hoped.} \end{aligned}$$

This completes the proof by induction, and proves our guess.

* * *

Here's an exercise for you now.

E10) Use the method of inspection to solve the recurrence $b_n = b_{n-1} + 4n^3 - 6n^2 + 4n - 1$ for $n \geq 1$ with $b_0 = 0$.

Let us now consider another method for solving recurrences.

9.4.2 Method of Telescoping Sums

This neat method is useful for solving recurrences of the form $u_n = u_{n-1} + g(n)$, particularly if $\sum_{n=1}^{k} g(n)$ is easy to find. More generally, it can be used to evaluate sums of series and products.

This method is based on the fact that the sum of the first N terms of a series whose nth term is of the form $a_n - a_{n-1}$ is simply $(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_{N-1} - a_{N-2}) + (a_N - a_{N-1}) = a_N - a_0$.

In much the same manner, the product of the first N terms of a series with nth term a_n/a_{n-1} is a_N/a_0 provided, of course, that none of the a_k's is zero.

A sum of the form $\sum_i [f(i+1) - f(i)]$ is called **telescoping** in analogy with the thickness of a collapsed telescope, which is the difference between the outer radius of the outermost tube and the inner radius of the innermost tube.

Though this method appears easy, it is not often that this method can be applied, and often not easy to see how to use it even when it can be! Let us see a few instances of where it can be applied.

Example 12: Solve the linear recurrence
$a_n - a_{n-1} = \mathcal{F}_{n+2}.\mathcal{F}_{n-1}$, $n \geq 1$,
where $a_0 = 2$ and $\mathcal{F}_i$ denotes the ith Fibonacci number.

Solution: From Example 2, you know that
$\mathcal{F}_{n+2}.\mathcal{F}_{n-1} = (\mathcal{F}_{n+1} + \mathcal{F}_n)(\mathcal{F}_{n+1} - \mathcal{F}_n) = \mathcal{F}_{n+1}^2 - \mathcal{F}_n^2$.
So, for $n = 1, 2, \cdots$, the recurrence gives us the following equations:

$$\begin{aligned} a_1 - a_0 &= \mathcal{F}_2^2 - \mathcal{F}_1^2 \\ a_2 - a_1 &= \mathcal{F}_3^2 - \mathcal{F}_2^2 \\ a_3 - a_2 &= \mathcal{F}_4^2 - \mathcal{F}_3^2 \\ \vdots & \qquad \vdots \\ a_n - a_{n-1} &= \mathcal{F}_{n+1}^2 - \mathcal{F}_n^2 \end{aligned}$$

On adding these equations, we find that
$a_n - a_0 = \mathcal{F}_{n+1}^2 - \mathcal{F}_1^2$
$\iff a_n = 2 + \mathcal{F}_{n+1}^2 - 1 = \mathcal{F}_{n+1}^2 + 1$

* * *

The next example should be familiar to you. Recall, from Sec.8.2, that σ_n^k denotes the sum of the kth powers of the first n positive integers.

Example 13: Compute σ_n^1, σ_n^2 and σ_n^3, using the method of telescoping sums.

Solution: To find σ_n^1, we sum both sides of the identity
$(k+1)^2 - k^2 = 2k + 1$ from $k = 1$ to $k = n$. On doing so, we get

$$(n+1)^2 - 1 = \sum_{k=1}^{n} \{(k+1)^2 - k^2\} = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 2\sigma_n^1 + n.$$

$\therefore \sigma_n^1 = n(n+1)/2$.

Let us now find σ_n^2 and σ_n^3.

Summing both sides of the identities $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ and $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ from $k = 1$ to $k = n$, we obtain

$$\begin{aligned} (n+1)^3 - 1 &= \sum_{k=1}^{n} \{(k+1)^3 - k^3\} = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 3\sigma_n^2 + 3\sigma_n^1 + n, \text{ and} \\ (n+1)^4 - 1 &= \sum_{k=1}^{n} \{(k+1)^4 - k^4\} = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 4\sigma_n^3 + 6\sigma_n^2 + 4\sigma_n^1 + n. \end{aligned}$$

From the first of these equations, and using the value of σ_n^1 from above, $\sigma_n^2 = n(n+1)(2n+1)/6$.

Plugging in the values of σ_n^1 and σ_n^2 into the second equation, we now obtain $\sigma_n^3 = \{n(n+1)/2\}^2$.

* * *

While going through the example above, you may have felt that there is a much simpler method to compute σ_n^1. But the advantage of using telescoping sums is that it also works for computing σ_n^k for larger values of k, where the simpler method does not.

Now you can try and obtain the general formula for $\sigma_n^k, k \geq 1$.

E11) Find a recurrence relation satisfied by the sequence $\{\sigma_n^k\}_k$, and **hence** compute σ_n^4.

Let us now look at Problem 7 of Unit 7, namely, the number of derangements on k symbols, d_k.

Example 14: Solve the recurrence

$d_k = kd_{k-1} + (-1)^k$ if $k \geq 2$, with $d_1 = 0$.

Solution: Looking at the recurrence, it doesn't seem to be in the form in which we can apply the method of telescoping sums to solve it. But we can alte. it slightly to bring it into a suitable form. We simply divide each term by k!, and get

$$\frac{d_k}{k!} - \frac{d_{k-1}}{(k-1)!} = \frac{(-1)^k}{k!}.$$

Now we can apply the method since the terms are such that if we write down the equations from $k = 2$ to $k = n$, and add them, most of the terms will get cancelled. We will only be left with

$$\frac{d_n}{n!} - \frac{d_1}{1!} = \sum_{k=2}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Therefore,

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},\ n \geq 1.$$

* * *

In the next example we see how 'telescoping products' can be used for solving recurrences.

Example 15: Solve the recurrence $a_n = n^3 a_{n-1}, n \geq 1, a_0 = 2$.

Solution : Let us put $k = 1, 2, \cdots, n$, in the equation $\frac{a_k}{a_{k-1}} = k^3$.

We get

$$\begin{aligned} \frac{a_1}{a_0} &= 1^3 \\ \frac{a_2}{a_1} &= 2^3 \\ \vdots & \quad \vdots \\ \frac{a_n}{a_{n-1}} &= n^3. \end{aligned}$$

Multiplying these equations, we get

$\frac{a_n}{a_0} = (n!)^3$

$\Longrightarrow a_n = 2(n!)^3$.

* * *

The technique in the example above can be used more generally for non-homogeneous recurrences of the type

$a_n = f(n)a_{n-1} + g(n)$, where $f(n) \neq 0$ for all n.

Let us consider an example of this.

Example 16: Solve the recurrence $u_n = \frac{1}{n}u_{n-1} + \frac{1}{n!}, n \geq 1, u_0 = 1$.

Solution : The homogeneous part of this recurrence is $\frac{a_n}{a_{n-1}} = \frac{1}{n}$. Using the method of telescoping products, we get

$a_n = \frac{1}{n}\frac{1}{n-1}\cdots\frac{1}{1} = \frac{1}{n!}$.

Now, suppose that the solution of the given recurrence is of the form $u_n = a_n b_n$, where $b_0 = 1$. Then

This method can be applied here since $\frac{1}{n} \neq 0 \; \forall \; n \geq 1$.

$$\begin{aligned} a_n b_n &= \frac{1}{n}a_{n-1}b_{n-1} + \frac{1}{n!} \\ &= a_n b_{n-1} + \frac{1}{n!}, \text{ since } a_n = \frac{1}{n}a_{n-1} \\ \Rightarrow b_n &= b_{n-1} + \frac{\frac{1}{n!}}{a_n} = b_{n-1} + 1, \text{ since } a_n = \frac{1}{n!}. \end{aligned}$$

Now, we can use the method of telescoping sums to solve the recurrence $b_n = b_{n-1} + 1, b_0 = 1$. We get
$b_n = n + 1$.
Therefore, $u_n = a_n b_n = \frac{n+1}{n!}$.

* * *

Can you clearly spell out the steps we have gone through in Example 16? To obtain a solution of the recurrence $u_n = f(n)u_{n-1} + g(n)$, the steps are:

Step 1 : See if $f(n) \neq 0 \forall n$. Only then can this method be applied.

Step 2 : Find the solution $\{a_n\}$ for the homogeneous part of the recurrence. So,
$a_n = f(n)a_{n-1} \forall n \geq 1$.

Step 3 : Assume that the solution of the given recurrence is of the form $u_n = a_n b_n$.
Then

$$\begin{aligned} a_n b_n &= f(n)a_{n-1}b_{n-1} + g(n) \\ &= a_n b_{n-1} + g(n) \end{aligned}$$

Therefore, $b_n = b_{n-1} + g(n)/a_n$.
Here is where we use the fact $f(n) \neq 0 \forall n$. (How?)

Step 4: Solve the recurrence
$b_n = b_{n-1} + \frac{g(n)}{a_n}$
by whichever method you find suitable.

Step 5 : Then the solution to the given recurrence is $u_n = a_n b_n$.

Here are some exercises now.

E12) Show that $C(2n, n)$ is a solution of the recurrence
$x_n = \frac{2(2n-1)}{n} x_{n-1}, n \geq 1$.

E13) Use the method of telescoping sums and products to solve the recurrence $a_n = n^3 a_{n-1} + (n!)^2$ if $n \geq 1$ and $a_0 = 1$.

E14) Solve the recurrence $a_n = (n!n)a_{n-1}, n \geq 1, a_0 = 5$.

Let us now see how telescoping sums can be used efficiently to sum an infinite series. Although this is not an example involving recurrence relations, you would get some idea of how this method can be applied to different problems.

Example 17: Use the method of telescoping sums to sum the infinite series
$$\frac{3}{1.2.3}+\frac{5}{2.3.4}+\frac{7}{3.4.5}+\cdots+\frac{2n+1}{n(n+1)(n+2)}+\cdots$$

Solution: The central idea behind telescoping sums is the expression of the nth term as a difference of successive terms of a sequence. We would have been able to apply this had the nth term in this summation been a product of only two terms in the denominator. But don't worry! Let us try and extend the idea.

With three terms in the denominator, we first express the nth term as a partial fraction:
$$\frac{2n+1}{n(n+1)(n+2)}=\frac{1/2}{n}+\frac{1}{n+1}-\frac{3/2}{n+2}.$$
Now, if a_i denotes the ith term of the series, then
$$\begin{aligned}S_n &= a_1+a_2+\cdots+a_n\\ &= \left(\frac{1/2}{1}+\frac{1}{2}-\frac{3/2}{3}\right)+\left(\frac{1/2}{2}+\frac{1}{3}-\frac{3/2}{4}\right)+\left(\frac{1/2}{3}+\frac{1}{4}-\frac{3/2}{5}\right)\\ &\quad+\cdots+\left(\frac{1/2}{n}-\frac{1}{n+1}-\frac{3/2}{n+2}\right)\end{aligned}$$
Since $\frac{-3/2}{n}+\frac{1}{n}+\frac{1/2}{n}=0$, cancelling groups of such terms, we get
$$\begin{aligned}S_n &= \left(\frac{1/2}{1}+\frac{1}{2}\right)+\left(\frac{1/2}{2}\right)+\left(\frac{-3/2}{n+1}\right)+\left(\frac{1}{n+1}-\frac{3/2}{n+2}\right)\\ &= \frac{5}{4}-\frac{1/2}{n+1}-\frac{3/2}{n+2}.\end{aligned}$$
Therefore, $S=\lim_{n\to\infty} S_n = 5/4$.

* * *

Why don't you try some exercises now?

E15) By methods of this sub-section, solve the recurrence $nx_n=(n-2)x_{n-1}+1,\ n\geq 1$, where $x_0=0$.

E16) Using the method of telescoping sums, prove the following Fibonacci identities:

a) $\sum_{k=1}^{n}\mathcal{F}_k=\mathcal{F}_{n+2}-1$;

b) $\sum_{k=1}^{n}\mathcal{F}_{2k-1}=\mathcal{F}_{2n}$;

c) $\sum_{k=1}^{n}\mathcal{F}_k^2=\mathcal{F}_n.\mathcal{F}_{n+1}$;

d) $\sum_{k=2}^{\infty}\mathcal{F}_k/(\mathcal{F}_{k-1}\mathcal{F}_{k+1})=2$;

e) $\sum_{k=2}^{\infty}(\mathcal{F}_{k-1}\mathcal{F}_{k+1})^{-1}=1$.

And now we shall consider another very commonly used method for solving recurrences.

9.4.3 Method of Iteration

Iteration means 'to repeat'. In a sense, this is what we do in this method. More precisely, we successively express the nth term u_n in terms of some or all of the previous (n − 1) terms $u_0, u_1, \cdots, u_{n-1}$, using the recurrence equation again and again. While doing so, we try and find an emerging pattern which can help us find u_n explicitly as a function of n.

To see how this method works, let us look at an example.

Example 18: Solve the recurrence relation given by $u_n = 2u_{n-1} + 2^n - 1$, where $n \geq 1$ and $u_0 = 0$.

Solution: Replacing n by n − 1, n − 1 by n − 2, ⋯ and so on in the recurrence equation, we get

$$\begin{aligned}
u_n &= 2u_{n-1} + 2^n - 1 \\
&= 2(2u_{n-2} + 2^{n-1} - 1) + 2^n - 1 \\
&= 2^2u_{n-2} + 2.2^n - (1+2) \\
&= 2^2(2u_{n-3} + 2^{n-2} - 1) + 2.2^n - (1+2) \\
&= 2^3u_{n-3} + 3.2^n - (1+2+2^2) \\
&\vdots \\
&= 2^n u_0 + n.2^n - (1+2+2^2+\cdots+2^{n-1}) \\
&= (n-1)2^n + 1, \text{since } 1+2+2^2+\cdots+2^{n-1} = \frac{2^n-1}{2-1}.
\end{aligned}$$

***

In the example above, we began with the recurrence relation and reached an expression for the nth term in terms of n. **In principle** this method always works. But, it is not always easy to apply because the computation can sometimes get out of hand.

Why don't you try the following exercise now? You shouldn't have any difficulty in the computation. In fact, you may find it easier to solve by this method than by the method you used earlier.

E17) Solve the recurrence $u_n = \frac{1}{n}u_{n-1} + \frac{1}{n!}$, $n \geq 1$, with $u_0 = 1$ by the method of iteration.

Let us now consider an example which can be solved by iteration as well as by first solving the recurrence for a_k and then summing the series. Let us solve it by using the former method.

Example 19: Sum the first n terms of the series whose kth term, a_k, satisfies the recurrence $a_k = 3a_{k-1} + 1$, and whose initial term, is $a_1 = 2$.

Solution: From the recurrence, we find that

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n-1} a_k + (3a_{n-1} + 1) \\
&= \sum_{k=1}^{n-2} a_k + (1+3)(3a_{n-2} + 1) + 1 \\
&= \sum_{k=1}^{n-3} a_k + (1+3+3^2)(3a_{n-3} + 1) + \{1 + (1+3)\} \\
&= \sum_{k=1}^{n-4} a_k + (1+3+3^2+3^3)(3a_{n-4} + 1) + \{1 + (1+3) \\
&\qquad + (1+3+3^2)\} \\
&\vdots
\end{aligned}$$

$$= a_1 + (1+3+\cdots+3^{n-2})(3a_1+1) + \{1+(1+3) + \cdots + (1+3+\cdots+3^{n-3})\}$$

$$= 2(1+3+\cdots+3^{n-1}) + \frac{1}{2}\sum_{k=1}^{n-1}(3^k-1), \text{since } a_1 = 2$$

and $1+3+\cdots+3^{k-1} = \dfrac{3^k-1}{3-1}$.

$$= \frac{5.3^n - 3 - 2n}{4}.$$

* * *

You may like to try and solve a similar problem now.

E18) Use the method of iteration to find the sum of the first $n+2$ terms of the series whose kth term, u_k, satisfies the recurrence $u_k = u_{k-1} + k$, and whose initial term is $u_1 = 1$.

Now let us discuss the fourth method of this section.

9.4.4 Method of Substitution

So far we have seen several techniques for solving a variety of recurrences, linear and non-linear. But there are some that defeat our entire arsenal. For instance, we are ill equipped to handle even some of the simplest non-linear recurrences and linear recurrences with non-constant coefficients. It is in some of these cases that we can call upon a substitution to extricate ourselves from this position.

The method of substitution is used to change the given recurrence to a form that can then be readily solved by one of the previously discussed techniques. As you might expect, the hard part is to figure out what the substitution should be. Let us see how this method works through examples related to the divide-and-conquer relations.

Example 20: Solve the recurrence relation of Problem 8 of Unit 7, namely, $a_n = a_{n/2} + 1$ for $n = 2^k, k \geq 1, a_1 = 0$.

Solution: Let us put $a_{2^k} = u_k$. Then the given recurrence becomes
$u_k = u_{k-1} + 1, u_0 = 0$.
Now, we can apply the method of telescoping sums, to get the solution
$u_n = u_0 + n = n$, i.e., $a_{2^n} = n$, i.e., $a_m = \log_2 m$ for $m \geq 1$.

* * *

Example 21: Solve the recurrence obtained by 'merge sort' in Sec. 7.4, namely, $a_n = 2a_{n/2} + n - 1, n = 2^k, k \geq 1, a_1 = 0$.

Solution : As in the previous example, we put $a_{2^k} = u_k$. Then the recurrence becomes
$u_k = 2u_{k-1} + 2^k - 1, u_0 = 0$.
Now, as in Example 18, we get
$u_k = (k-1)2^k + 1$,
i.e., $a_{2^k} = (k-1)2^k + 1$,
i.e., $a_n = (\log_2 n - 1)n + 1$.

* * *

Here are some recurrences for you to solve now.

E19) Using an appropriate substitution, solve the recurrence

$$y_n = \frac{n-1}{n} y_{n-1} + \frac{1}{n}, \; n \geq 1, \text{ where } y_0 = 5.$$

E20) Solve the recurrence $t_n = 3t_{n/2} + n^2$, $t_1 = 2$, by substitution, and specify for which values of n the conversion is valid.

Let us look at another example, one which makes it a natural candidate for the substitution technique.

Example 22: Solve the second order, non-linear recurrence $x_n = (2\sqrt{x_{n-1}} + 3\sqrt{x_{n-2}})^2$, $n \geq 2$, with the initial conditions $x_0 = 1, x_1 = 4$.

Solution: Looking at the recurrence, you probably feel that we have not developed the tools to solve an equation of this type. Let's see if we can transform this into a linear recurrence. Let us make the substitution $y_n = \sqrt{x_n}$, $n \geq 0$. (Note that this substitution is valid because each x_n is non-negative.)

The substitution does not quite make the recurrence linear, but at least it gets rid of the square root symbol—the problem now becomes
$y_n^2 = (2y_{n-1} + 3y_{n-2})^2$, $n \geq 2$, with $y_0 = 1, y_1 = 2$.
Extracting square roots of each side, we now get
$y_n = 2y_{n-1} + 3y_{n-2}$, $n \geq 2$.

This is a second order linear recurrence with constant coefficients, and can be solved by standard methods discussed earlier. We leave it to you to verify that the solution is
$y_n = A.3^n + B(-1)^n$, $n \geq 0$, for some constants A, B.

Using the initial conditions, we further get
$A + B = 1$ and $3A - B = 2$, so that $A = 3/4$ and $B = 1/4$.

$$\therefore x_n = y_n^2 = \frac{\{3^{n+1} + (-1)^n\}^2}{16}, \; n \geq 0.$$

* * *

As a final example, we look at another non-linear recurrence with an exponential-type relation between the terms.

Example 23: Solve the recurrence given by $x_n = x_{n-1}^7 / x_{n-2}^{12}$, together with the initial conditions $x_0 = 1$ and $x_1 = 2$.

Solution: Taking the logarithm to any convenient base (we are only going to be dealing with positive numbers in this sequence after all!) reduces the right side of this to a form we can quite easily handle:

$$\log_2 x_n = 7 \log_2 x_{n-1} - 12 \log_2 x_{n-2}.$$

Now, let $y_n = \log_2 x_n$. The sequence $\{y_n\}$ satisfies the recurrence

$$y_n - 7y_{n-1} + 12y_{n-2} = 0.$$

So its characteristic roots are 3 and 4.
Therefore, $y_n = a.3^n + b.4^n$, $n \geq 0, a, b \in C$.
Now, the initial conditions $x_0 = 1$ and $x_1 = 2$ yield $y_0 = 0$ and $y_1 = 1$.
Putting $n = 0$ and 1 in $y_n = a.3^n + b.4^n$ gives $a = -1, b = 1$.
Therefore, $y_n = 4^n - 3^n$.
Thus, $x_n = 2^{y_n} = 2^{4^n - 3^n}$, $n \geq 0$.

* * *

Observe that the choice of base in the example above does not alter the final answer, as indeed it must not! We chose 2 because x_0 and x_1 are both powers of two.

Now, for some exercises.

E21) Find the solution of $\sqrt{x_n} - 5\sqrt{x_{n-1}} + 6\sqrt{x_{n-2}} = 0$, $n \geq 2$, where $x_0 = 4$ and $x_1 = 25$.

E22) Solve the recurrence $x_n = 4n(n-1)x_{n-2} + \frac{5}{9}n!(3^n)$ for $n \geq 2$, if $x_0 = 1$ and $x_1 = -1$.

E23) Let $\{u_n\}$ satisfy the non-homogeneous recurrence

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + c_3 u_{n-3} + c_4 u_{n-4} + g(n),$$

such that the associated homogeneous recurrence has $(z-2)(z-3)(z-4)^2$ as its characteristic polynomial, and $\{g(n)\}$ satisfies a fifth order linear homogeneous recurrence with constant coefficients whose characteristic polynomial is $(z-2)^2(z-3)(z-5)^2$. Determine u_n.

E24) Let $\{v_n\}$ satisfy the second order recurrence

$$v_n + b_1 v_{n-1} + b_2 v_{n-2} = 5r^n,$$

with b_1, b_2 and r as constants. Prove that the sequence also satisfies the third order homogeneous linear recurrence with constant coefficients having $(z^2 + b_1 z + b_2)(z - r)$ as its characteristic polynomial.

E25) Let $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be two solutions of the recurrence $u_n + a_1 u_{n-1} + a_2 u_{n-2} = 0$, where a_1 and a_2 are constants.

a) Show that $\{x_n y_n\}_{n\geq 0}$ satisfies a third order linear homogeneous recurrence with constant coefficients.

b) Show that $\{x_{2n}\}_{n\geq 0}$ satisfies a second order linear homogeneous recurrence with constant coefficients.

E26) Assume that for positive real numbers a, b and r, there exists $m \in N$ such that $(a + bn)r^n < n!$ for $n \geq m$.
Using this, prove that there does not exist any second order homogeneous linear recurrence with constant coefficients satisfied by the sequence $\{n!\}$.

With this we have come to the end of this unit and block on recurrences. Let us take a quick look at what we have covered in this unit.

9.5 SUMMARY

In this unit we have discussed the following points.

1. The solution of the linear homogeneous recurrence with constant coefficients,
$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k}, n \geq k,$
is $\sum_{j=1}^{m} \left[\sum_{i=0}^{t_j} b_{ij} C(i+n, i) \right] \alpha_j^n$
where $\alpha_1, \cdots, \alpha_m$ are the distinct characteristic roots of this recurrence with multiplicity $t_1, \cdots, t_m$, respectively.

2. The solution of a linear non-homogeneous recurrence is the sum of the general solution of its homogeneous part and a particular solution of the whole recurrence.

3. A particular solution of
$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k} + an^d, n \geq k$

is of the form

$n^m(A_0 + A_1 n + \cdots + A_d n^d)$

where $m \geq 0$ is the multiplicity of 1 as a characteristic root of the equation, and the A_is are constants.

4. A particular solution of

$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k} + ar^n$

is of the form

$An^m r^n$,

where $m \geq 0$ is the multiplicity of r as a characteristic root of the equation and A is a constant.

5. A particular solution of

$u_n = c_1 u_{n-1} + \cdots + c_k u_{n-k} + an^d r^n$

is of the form

$n^{m_1+m_2} r^n (A_0 + A_1 n + \cdots + A_d n^d)$

where $m_1 \geq 0$ and $m_2 \geq 0$ are the multiplicities of r and 1, respectively, as characteristic roots of the equation, and the A_is are constants.

6. The methods of inspection and telescoping sums for solving linear recurrences with constant coefficients.

7. The methods of iteration and substitution for solving linear recurrences with constant and non-constant coefficients.

9.6 SOLUTIONS/ANSWERS

E1) The characteristic equation is $z = 3$. So the characteristic root is 3, with multiplicity 1. Therefore, the solution is

$a_n = bC(0 + n, 0).3^n, b \in C,$

i.e.,$a_n = b3^n$.

E2) The recursion $u_n + c_1 u_{n-1} + c_2 u_{n-2} = 0$ has the characteristic equation $z^2 + c_1 z + c_2 = 0$.

We also know that the roots of this equation are $1 + i$ and $1 - i$. Therefore, from MTE-04 you know that c_1 equals the negative of the sum of its characteristic roots, i.e., -2, and c_2 is the product of these roots, i.e., 2.

E3) The characteristic equation of the given recurrence is

$z^3 - z^2 - z + 1 = (z-1)^2(z+1) = 0.$

So, $P_n^2 = (an + b) + c(-1)^n$, $n \geq 0$, for some constants a, b, c.

The initial conditions are $a + b - c = 0$, $2a + b + c = 1$ and $3a + b - c = 1$. Therefore,

$P_n^2 = \dfrac{2n - 1 + (-1)^n}{4}$, $n \geq 0$.

E4) Statements: 1) If $\{a_n\}$ and $\{b_n\}$ are two solution sequences of the non-homogeneous recurrence

$u_n = f_1(n)u_{n-1} + f_2(n)u_{n-2} + \cdots + f_k(n)u_{n-k} + g(n)$, (8)

then $\{c_n\}$ is a solution sequence of its associated homogeneous recurrence, where $c_n = a_n - b_n$.

2) Every solution of (8) is of the form $a_n + b_n$, where a_n is a particular solution of (8) and b_n is any solution of its associated homogeneous recurrence

$u_n = f_1(n)u_{n-1} + \cdots + f_k(n)u_{n-k}$ (9)

The proofs are exactly on the lines of the proofs for the 'constant coefficients case'.

E5) The characteristic root of the recurrence $T_n = 2T_{n-1}$ is $z = 2$. Therefore, the general solution of the homogeneous part is $T_n = a.2^n$, $n \geq 1$.
The particular solution to the non-homogeneous part is $T_n = b$, by Theorem 5. Plugging in this value of T_n into the recurrence gives $b = -1$.
Adding the solutions of the homogeneous and non-homogeneous parts, and using the initial condition $T_1 = 1$, we get $T_n = 2^n - 1$, $n \geq 1$.

E6) Here $a_{n+2} - a_{n+1} = 3(a_{n+1} - a_n) - 200, n \geq 0$,
i.e., $a_{n+2} - 4a_{n+1} + 3a_n = -200$.

The solution corresponding to the homogeneous part is $a.3^n + b(1)^n$, i.e., $a.3^n + b$, where $a, b \in C$.
Now, $-200 = (-200)(1)^n$, and 1 is a characteristic root.
So, by Theorem 5, a particular solution is An, A a constant.
Putting $a_n = An$ in the recurrence, we get
$A(n+2) - 4A(n+1) + 3An = -200 \Longrightarrow A = 100$.
$\therefore a_n = a.3^n + b + 100n$.
With $a_0 = 1000$ and $a_1 = 1500 - 200 = 1300$, we have
$a_n = 100(3)^n + 900 + 100n, n \geq 0$.

E7) The recurrence $u_n + c_1u_{n-1} + c_2u_{n-2} = 0$ has the characteristic equation $z^2 + c_1z + c_2 = 0$. From the given solution we see that its roots are 2 and 5.
Therefore, $c_1 = -(2+5) = -7$ and $c_2 = 2 \times 5 = 10$.
Now, setting the given particular solution $u_n = 3n - 5$ in the given equation, we get
$(3n-5) - 7(3n-8) + 10(3n-11) = an + b$.
Therefore, $a = 12$ and $b = -59$.

E8) The recurrence $v_n - 7v_{n-1} + 16v_{n-2} - 12v_{n-3} = 0$ has the characteristic equation $z^3 - 7z^2 + 16z - 12 = 0$, i.e., $(z-2)^2(z-3) = 0$.
So, $v_n = (an+b)2^n + c.3^n$, $n \geq 0$, for some a, b, c.
The particular solution is of the form $v_n = An^22^n + Bn3^n$.
Therefore, the recurrence reduces to

$$A.2^{n-1}\{2n^2 - 7(n-1)^2 + 8(n-2)^2 - 3(n-3)^2\}$$
$$+ B.3^{n-2}\{9n - 21(n-1) + 16(n-2) - 4(n-3)\} = 2^n + 3^n.$$

Solving this, we get $A = -1, B = 9$.
Therefore, $v_n = (-n^2 + an + b)2^n + (9n + c)3^n$, $n \geq 0$.
The initial conditions lead to the equations
$b + c = 1, 2(a + b - 1) + 3(c + 9) = 0$ and $4(2a + b - 4) + 9(c + 18) = 1$.
Solving these equations we get $a = 7, b = 42$ and $c = -41$.
Therefore, $v_n = (-n^2 + 7n + 42)2^n + (9n - 41)3^n$, $n \geq 0$.

E9) We know that 3 and -2 are the only characteristic roots with multiplicity 1 and 2, respectively. Therefore, the solution corresponding to the homogeneous part is

$A.3^n + (Bn + C)(-2)^n$.

The non-homogeneous part of the recurrence is

$a.2^n + bn(-1)^n + (cn^2 + dn + e), a, \cdots, e$ are constants.

Therefore, the solution corresponding to this part is

$D(2^n) + (-1)^n(E_0 + E_1n) + Fn^2 + Gn + E$.

The complete solution is the sum of the two solutions.

E10) Since the first few terms of the sequence are $0, 1, 16, 81, \ldots$, it is reasonable to guess that $b_n = n^4$ for $n \geq 0$.
Let's check this guess by induction.
Now, this guess is correct for $n = 0$ and $n = 1$.
Let's assume that it is true for $n - 1$. Now
$n^4 = (n-1)^4 + (4n^3 - 6n^2 + 4n - 1)$ for $n \geq 1$.
So, the principle of mathematical induction proves our guess.

E11) Summing both sides of the identity
$(j+1)^{k+1} - j^{k+1} = \sum_{r=0}^{k} C(k+1, r) j^r$ from $j = 1$ to $j = n$, we get the recurrence equation

$$\begin{aligned}(n+1)^{k+1} - 1 &= \sum_{j=1}^{n} \sum_{r=0}^{k} C(k+1, r) j^r \\ &= \sum_{r=0}^{k} \left\{ C(k+1, r) \sum_{j=1}^{n} j^r \right\} \\ &= \sum_{r=0}^{k} \{ C(k+1, r) \sigma_n^r \}\end{aligned}$$

In particular, $k = 4$ gives

$$(n+1)^5 - 1 = \sum_{r=0}^{4} \{C(5, r)\sigma_n^r\} = \sigma_n^0 + 5\sigma_n^1 + 10\sigma_n^2 + 10\sigma_n^3 + 5\sigma_n^4.$$

Therefore,

$$\begin{aligned}\sigma_n^4 &= \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 4n}{5} - \frac{n^2(n+1)^2}{2} \\ &\quad - \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.\end{aligned}$$

E12) Method 1: Since $C(2n, n) = \dfrac{(2n)!}{(n!)^2} = \dfrac{2(2n-1)}{n} \dfrac{(2n-2)!}{[(n-1)!]^2}$, it follows that $C(2n, n)$ is a solution of the given recurrence.
Method 2: $x_n = x_0 \prod_{k=1}^{n} x_k/x_{k-1} = x_0 \prod_{k=1}^{n} 2(2k-1)/k$
$= x_0\, 2^n [1.3.5 \cdots (2n-1)]/n! = x_0 (2n)!/(n!)^2$, $n \geq 0$.

E13) Let us apply the technique of Example 15. We can do so, since $n^3 \neq 0 \; \forall \; n \geq 1$.
From Example 15 we know that the solution to the homogeneous part is $u_n = u_0 (n!)^3$.
Suppose the solution of the given recurrence is
$a_n = u_n v_n$, where $u_0 v_0 = 1$.

$$\begin{aligned}\text{Then } u_n v_n &= n^3 u_{n-1} v_{n-1} + (n!)^2 \\ &= u_n v_{n-1} + (n!)^2\end{aligned}$$

$$\Rightarrow v_n = v_{n-1} + \frac{1}{u_0} \cdot \frac{1}{n!}$$

Now applying telescoping sums, we get

$$v_n = v_0 + \frac{1}{u_0} \left(\sum_{k=1}^{n} \frac{1}{k!} \right)$$

Then the solution of the recurrence is

$$a_n = (n!)^3\left[1+\sum_{k=1}^{n}\frac{1}{k!}\right]$$

E14) $\frac{a_n}{a_{n-1}} = n!n \; \forall \; n \geq 1$

$$\therefore \frac{a_n}{a_0} = \prod_{k=1}^{n}\frac{a_k}{a_{k-1}} = \prod_{k=1}^{n} k!k = 1!2!\cdots(n-1)!(n!)^2$$

$$\therefore a_n = 5\left[1!2!\cdots(n-1)!(n!)^2\right].$$

E15) Multiplying by $n-1$ reduces this recurrence to
$n(n-1)x_n - (n-1)(n-2)x_{n-1} = n-1, \; n \geq 1.$
Substituting $d_n = n(n-1)x_n$, we get
$d_n - d_{n-1} = n-1, d_0 = 0.$

$$\therefore d_n = \sum_{k=1}^{n}(k-1) = n(n-1)/2.$$

$$\therefore x_n = 1/2, \; n \geq 1.$$

E16) For convenience, let us define $\mathcal{F}_0 \doteq \mathcal{F}_2 - \mathcal{F}_1 = 0.$

a) $\sum_{k=1}^{n}\mathcal{F}_k = \sum_{k=1}^{n}(\mathcal{F}_{k+1} - \mathcal{F}_{k-1}) = (\mathcal{F}_{n+1}+\mathcal{F}_n) - (\mathcal{F}_1 + \mathcal{F}_0) = \mathcal{F}_{n+2} - 1.$

b) $\sum_{k=1}^{n}\mathcal{F}_{2k-1} = \sum_{k=1}^{n}(\mathcal{F}_{2k} - \mathcal{F}_{2k-2}) = \mathcal{F}_{2n} - \mathcal{F}_0 = \mathcal{F}_{2n}.$

c) $\sum_{k=1}^{n}\mathcal{F}_k^2 = \sum_{k=1}^{n}(\mathcal{F}_{k+1}\mathcal{F}_k - \mathcal{F}_k\mathcal{F}_{k-1}) = \mathcal{F}_{n+1}\mathcal{F}_n - \mathcal{F}_1\mathcal{F}_0 = \mathcal{F}_{n+1}\mathcal{F}_n.$

d) $\sum_{k=2}^{\infty}\left[\mathcal{F}_k/(\mathcal{F}_{k-1}\mathcal{F}_{k+1})\right] = \sum_{k=2}^{\infty}(\mathcal{F}_{k-1}^{-1} - \mathcal{F}_{k+1}^{-1})$
$= \lim_{n\to\infty}\{(\mathcal{F}_1^{-1} + \mathcal{F}_2^{-1}) - (\mathcal{F}_n^{-1} + \mathcal{F}_{n+1}^{-1})\} = 2.$

e) $\sum_{k=2}^{\infty}(\mathcal{F}_{k-1}\mathcal{F}_{k+1})^{-1} = \sum_{k=2}^{\infty}\{(\mathcal{F}_{k-1}\mathcal{F}_k)^{-1} - (\mathcal{F}_k\mathcal{F}_{k+1})^{-1}\}$
$= \lim_{n\to\infty}\{(\mathcal{F}_1\mathcal{F}_2)^{-1} - (\mathcal{F}_n\mathcal{F}_{n+1})^{-1}\} = 1.$

E17) Repeatedly replacing n by $n-1$, we get

$$\begin{aligned}
u_n &= \frac{1}{n}u_{n-1} + \frac{1}{n!}\\
&= \frac{1}{n}\left\{\frac{1}{n-1}u_{n-2} + \frac{1}{(n-1)!}\right\} + \frac{1}{n!} = \frac{1}{n(n-1)}u_{n-2} + \frac{2}{n!}\\
&= \frac{1}{n(n-1)}\left\{\frac{1}{n-2}u_{n-3} + \frac{1}{(n-2)!}\right\} + \frac{2}{n!}\\
&= \frac{1}{n(n-1)(n-2)}u_{n-3} + \frac{3}{n!}\\
&\;\vdots\\
&= \frac{1}{n!}u_0 + \frac{n}{n!} = \frac{n+1}{n!}, \; n \geq 0.
\end{aligned}$$

E18) By the iteration method, we have

$$\begin{aligned}
u_n &= u_{n-1} + n\\
&= u_{n-2} + (n-1) + n\\
&\;\vdots\\
&= u_1 + [n-(n-2)] + \cdots + (n-1) + n
\end{aligned}$$

$$\ldots + n$$
$$= \frac{n(n+1)}{2}$$

$$\therefore \sum_{k=1}^{n+2} u_k = \frac{1}{2}\sum_{k=1}^{n+2} k(k+1)$$
$$= \frac{1}{2}\left(\sum_{1}^{n+2} k^2 + \sum_{1}^{n+2} k\right)$$
$$= \frac{1}{2}\left[\frac{(n+2)(n+3)(2n+5)}{6} + \frac{(n+2)(n+3)}{2}\right]$$
$$= \frac{(n+2)(n+3)(n+4)}{6}$$

E19) Rewriting the recurrence in the form $ny_n - (n-1)y_{n-1} = 1$, suggests the substitution $x_n = ny_n$, $n \geq 1$. The recurrence then reduces to the form $x_n - x_{n-1} = 1$, and telescopes to

$$x_n - x_0 = \sum_{k=1}^{n} (x_k - x_{k-1}) = n.$$

Therefore, $x_n = n$ and $y_n = 1$ for $n \geq 1$.

E20) n has to be of the form 2^k for the recurrence to be valid.
Now, let us put $t_{2^k} = u_k$ in the recurrence. Then it reduces to

$$u_n = 3u_{n-1} + 2^{2n}, n \geq 1, u_0 = 2.$$

The solution of the homogeneous part is $u_n = A.3^n$.
The solution of the non-homogeneous part is $u_n = B.2^{2n} = B4^n$.
Putting this in the recurrence, we get $B = 4$.
$\therefore u_n = A.3^n + 4^{n+1}$.
Now, using the initial condition, we get $A = -2$.
$\therefore u_n = (-2)3^n + 4^{n+1}$.
$\therefore t_{2^n} = (-2)3^n + 2^{2(n+1)}$.

E21) Let $y_n = \sqrt{x_n}$. Then, $y_n - 5y_{n-1} + 6y_{n-2} = 0$ has the characteristic roots 2 and 3.
So $y_n = a.2^n + b.3^n$, for some a, b.
Since $y_0 = 2$ and $y_1 = 5$, $a = 1 = b$.
Therefore, $x_n = y_n^2 = (2^n + 3^n)^2$ for $n \geq 0$.

E22) The term n! of the non-homogeneous part provides us with the hint that we should divide both sides by n!. If we do, we get
$y_n - 4y_{n-2} = \frac{5}{9} \times 3^n$, $n \geq 2$, with $y_0 = 1, y_1 = -1$, where $y_n = x_n/n!$.
Since the homogeneous part of this has the characteristic polynomial $z^2 - 4 = 0$, it follows that
$y_n = a.2^n + b(-2)^n + c.3^n$, $n \geq 0$, for some a, b, c.

Inserting this value into the recurrence gives $c = 1$, while the initial conditions give rise to $a + b + c = a + b + 1 = 1$ and
$2a - 2b + 3c = 2a - 2b + 3 = -1$, solving which we get $a = 1, b = -1$.
Therefore, $x_n = \{2^n - (-2)^n + 3^n\}n!, n \geq 0$.

E23) Write u_n as a sum of its homogeneous solution, $u_n^{(h)}$, and particular solution, $u_n^{(p)}$. Then,
$u_n^{(h)} = a_1.2^n + a_2.3^n + (a_3 + a_4n)4^n, a_i \in C\ \forall\ i$.
Since g(n) is of the form $(A + Bn)2^n + C.3^n + (D + En)5^n$, the form that the particular solution takes is
$u_n^{(p)} = [A_0n + (b_0 + b_1n)n]2^n + C_0n.3^n + D_0.5^n + (E_0 + E_1n)5^n$

$= (A_1 n + B_1 n^2)2^n + C_0 n.3^n + D_0.5^n + (E_0 + E_1 n)5^n$, where the capital letters are constants.

Therefore, $u_n = u_n^{(h)} + u_n^{(p)}$.

E24) Let r_1, r_2 be the roots of the characteristic polynomial $z^2 + b_1 z + b_2 = 0$. Then

$$v_n = \begin{cases} a_1 r_1^n + a_2 r_2^n + cr^n & \text{if } r_1, r_2, r \text{ are all distinct,} \\ (a_1 + cn)r_1^n + a_2 r_2^n & \text{if } r = r_1 \neq r_2, \\ (a + bn)r_1^n + cr^n & \text{if } r_1 = r_2 \neq r, \\ (a + bn + cn^2)r^n & \text{if } r_1 = r_2 = r. \end{cases}$$

In any of these cases, the characteristic polynomial of the linear homogeneous recurrence with constant coefficients satisfied by $\{v_n\}$ has roots r_1, r_2 and r, not necessarily distinct. In other words, this polynomial is $(z - r_1)(z - r_2)(z - r) = (z^2 + b_1 z + b_2)(z - r)$.

E25) Let $z^2 + a_1 z + a_2 = (z - \alpha)(z - \beta)$.
If $\alpha \neq \beta$, $x_n = A\alpha^n + B\beta^n$ and $y_n = C\alpha^n + D\beta^n$ for some constants A, B, C, D and all $n \geq 0$.
If $\alpha = \beta$, $x_n = (A + Bn)\alpha^n$ and $y_n = (C + Dn)\alpha^n$ for some constants A, B, C, D and all $n \geq 0$.

a) So, if the roots are distinct,
$x_n y_n = AC(\alpha^2)^n + (AD + BC)(\alpha\beta)^n + BD(\beta^2)^n$, $n \geq 0$.
So, $\{x_n y_n\}$ satisfies a third order linear homogeneous recurrence with constant coefficients and distinct characteristic roots $\alpha^2, \alpha\beta$ and β^2.
More explicitly, the characteristic polynomial is
$(z - \alpha^2)(z - \alpha\beta)(z - \beta^2) = z^3 - (a_1^2 - a_2)z^2 + a_2(a_1^2 - a_2)z - a_2^3$,
and the recurrence relation is
$v_n - (a_1^2 - a_2)v_{n-1} + a_2(a_1^2 - a_2)v_{n-2} - a_2^3 v_{n-3} = 0$.
If the roots are equal,
$x_n y_n = AC(\alpha^2)^n + (AD + BC)n(\alpha^2)^n + BDn^2(\alpha^2)^n$, $n \geq 0$.
So $\{x_n y_n\}$ again satisfies a third order linear homogeneous recurrence with constant coefficients and the characteristic root α^2 of multiplicity three.
More explicitly, the characteristic polynomial is
$(z - \alpha^2)^3 = z^3 - 3a_2 z^2 + 3a_2^2 z - a_2^3$,
and the recurrence relation is
$v_n - 3a_2 v_{n-1} + 3a_2^2 v_{n-2} - a_2^3 v_{n-3} = 0$.

b) In this case, if the roots are distinct,
$x_{2n} = A(\alpha^2)^n + B(\beta^2)^n$, $n \geq 0$, and $\{x_{2n}\}$ satisfies a second order linear homogeneous recurrence with constant coefficients and distinct characteristic roots α^2 and β^2.
More explicitly, the characteristic polynomial is
$(z - \alpha^2)(z - \beta^2) = z^2 - (a_1^2 - 2a_2)z + a_2^2$, and the recurrence relation is
$w_n - (a_1^2 - 2a_2)w_{n-1} + a_2^2 w_{n-2} = 0$.
If the roots are equal,
$x_{2n} = (A + 2Bn)(\alpha^2)^n$, $n \geq 0$,
and $\{x_{2n}\}$ again satisfies a second order linear homogeneous recurrence with constant coefficients and the characteristic root α^2 of multiplicity two.
More explicitly, the characteristic polynomial is
$(z - \alpha^2)^2 = z^2 - 2a_2 z + a_2^2$,
and the recurrence relation is
$w_n - 2a_2 w_{n-1} + a_2^2 w_{n-2} = 0$.

E26) If the sequence $\{n!\}$ is to satisfy a second order homogeneous linear recurrence with constant coefficients, its nth term must be of the form $a_1r_1^n + a_2r_2^n$ for some a_1, a_2 provided $r_1 \neq r_2$, or of the form $(a + bn)r^n$ for some a, b.
If $r_1 \neq r_2$, $|a_1r_1^n + a_2r_2^n| \leq |a_1||r_1|^n + |a_2||r_2|^n \leq (|a_1| + |a_2|n)r^n$, where $r = \max(|r_1|, |r_2|)$.
So, in either case, $n! \leq (A + Bn)\alpha^n$, $n \geq 0$ for some positive A, B and α, and this is a contradiction to our supposition.

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UGMM-13
Discrete Mathematics

Block

4

GRAPH THEORY

BLOCK 4 GRAPH THEORY

Suppose you want to drive from Delhi to Calcutta. There are several routes for making the trip. How would you find the shortest route? Or, suppose you try to colour a map of India in such a way that neighbouring states are assigned different colours. How will you conclude that only four colours are necessary to do so, without actually doing the colouring?

These problems may seem very diverse at first sight; but they can be expressed as problems involving arrangements of certain objects and relationships between them. To study the arrangements, we view these objects as points in a plane, and the relationships as lines joining them. The branch of mathematics that deals with arrangement problems in this manner is known as **graph theory**.

The theory of graphs has a definite starting place in a paper published in 1736 by the Swiss mathematician Leonhard Euler (1707–1783). In this paper he solved a problem that was then known as the Königsberg bridge problem by formulating it in terms of graph theory. After this, there were a few sporadic efforts in the development of this theory for a century. In this century, this theory has again attracted attention for its utility. People found that this theory can be be very fruitfully used for solving problems related to the manufacture of integrated circuits, routing problems in transport network, and other important areas of industry and technology. As you study this block, we hope you will appreciate some of these applications.

The aim of this block is to introduce you to graph theory and its practical utility. We begin this in Unit 10.

In Unit 11 we introduce you to several special types of graphs. We also discuss 'trees', which were used in 1847 by Gustav Kirchoff (1824–1887) to model and study the working of electrical circuits. Arthur Cayley(1821-1895) also used trees to count the distinct isomers of saturated hydrocarbons in 1857.

In Unit 12 we go on to present Euler's path-breaking work in graph theory. We also discuss an equally important idea here, that of a Hamiltonian cycle. This concept, named after Hamilton(1805–1865), was initially used by him in a mathematical puzzle. This very concept has now been used to tackle practical problems like the travelling salesperson problem, which we discuss in the unit.

In the last unit of this block we discuss a famous problem in graph theory, that is, the 'four-colour problem'. Francis Guthrie communicated this problem to Augustus De Morgan in the 1850's through his brother. This was finally solved only in 1976 by Kenneth Appel and Wolfgang Haken, who presented a computer-aided proof of the four colour conjecture.. In this unit we also discuss the characterisation of planar graph which was given by Kasimir Kuratowski in 1930.

The subject you are going to study is a very exciting one, both in its underlying mathematical structure and in its applications in present day science and technology. We hope you will enjoy reading this block.

NOTATIONS AND SYMBOLS

$G = (V, E)$	graph G with vertex set V and edge set E.
K_n	complete graph with n vertices
$K_{m,n}$	complete bipartite graph with partite sets V_1 and V_2 when $\|V_1\| = m$ and $\|V_2\| = n$.
P_n	path involving n vertices
C_n	cycle on n vertices
$d_G(x)$	degree of vertex x in G.
$\delta(G)$	minimum vertex degree of G.
$\Delta(G)$	maximum vertex degree of G.
$\langle S\rangle_G$	subgraph of G induced by $S \subseteq V$
$\chi(G)$	Vertex chromatic number
$\chi'(G)$	Edge chromatic number

UNIT 10 BASIC PROPERTIES OF GRAPHS

Structure

10.1 INTRODUCTION

In our every day life, we come across various problems where we have to look at them as structures of objects and some family of subsets of those objects. For example, it may be an electric network where different gadgets are the objects and they are connected by electric wires. The lengths of these wires may not be important, but it is important to know how the wires are connected, that means it is important to know which gadgets are connected by the end points of the wires. Another example is that of the public transport in a city. Various places are the objects here and the bus routes are the connections and we want to know the places connected to the starting point. It may be the problem of establishing communication links between different centres. All these problems can be described by using the diagrams. They may be represented pictorially with a set of dots called vertices and a set of edges connecting various pairs of dots. Such representations are called graphs. The solutions to the given problems can be obtained by analysing their graphs. Ideas given by various mathematicians to solve such problems gave birth to a branch of mathematics called graph theory.

In this unit we shall begin with defining a graph and study some of its basic properties. In Sec. 1.2 and 1.3 we have defined various types of graphs. Throughout the sections, these graphs and their properties are illustrated with the help of examples. Finally, Sec. 1.4 is devoted to the study of subgraphs. In the following units of this block you would notice that how these simple basic ideas help us to solve many tough problems of the day today life. We can have graphs with vertices representing points in space, people, animal species, sports team etc. and edges might represent roads, telephone lines, communication channels etc.

Objectives

After reading this unit, you should be able to

- identify different ways of representing a graph;
- identify complete graphs, paths, cycles;
- obtain the union and complement of a graph;

- write the degree sequence of a graph and obtain the number of edges of a graph using the degrees of vertices;
- identify graphs isomorphic to a given graph;
- distinguish induced subgraphs from the given set of subgraphs;
- draw a graph on p vertices having the degree of regularity r, where p and r are integers with r < p, such that at least one of them is even.

10.2 GRAPHS

You must have used the term 'graph' while studying the calculus of real valued functions of a real variable. It is a set of the form $\{(x, f(x)) : x \in \text{the domain of the function } f\}$. Such a set helps us study the function f. The main difference between the object 'GRAPH' that we will define presently and the graph of a function is that our graph is the object of our study and not a tool to study something else. Before giving a formal definition of a graph let us look at some simple examples.

Example 1: Take two points x_1, x_2 in the plane and join them by any line. This line may be a straight line or an arc (see Fig.1).

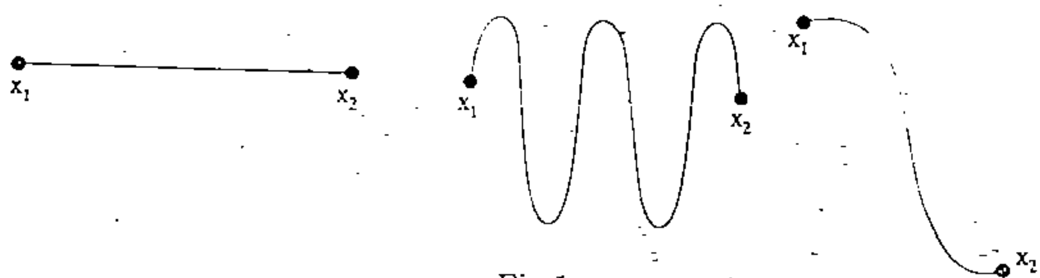

Fig.1

There can be many ways to join these points. Here we have shown three different ways.

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Similarly in Examples 2 and 3 we have shown different ways of joining 4 points.

Example 2: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_i to x_{i+1} by a line for $1 \leq i \leq 3$. Then join x_4 to x_1.

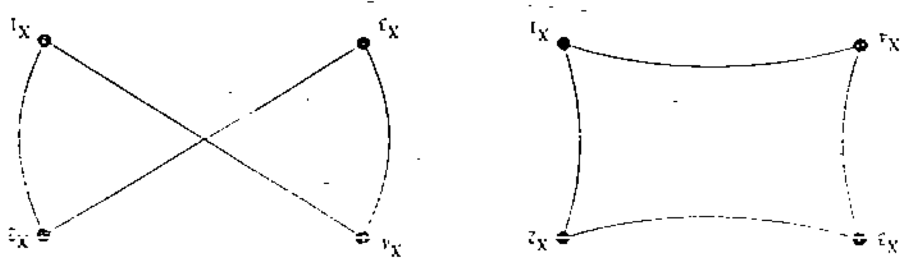

Fig.2

Here in Fig.2 we have given two different drawings. As far as our study in this block is concerned these drawings represent the same object.

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Example 3: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_1 to the three other points by lines.

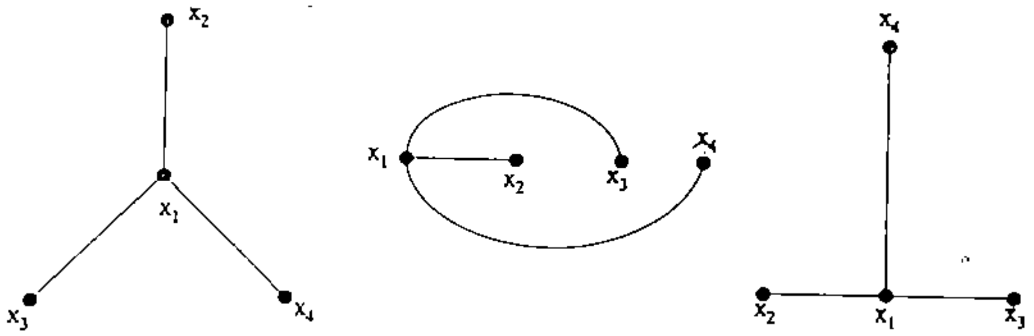

Fig.3

Again in Fig.3 above, the three drawings represent same object.

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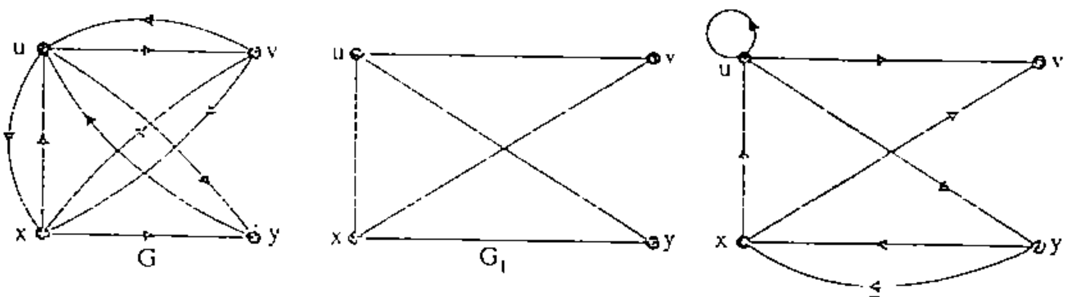
So you might have observed in above drawings the points are important as object. However, their positions are not important. Similarly, it is important to know which pairs are joined but the lines or the curves joining them are not important. You may notice that all the pictures in above examples share some common features. They are made up of a collection of points, collection of lines or curves joining some or all these points. We call these points vertices and we call the curve joining them edges. So to each drawing corresponds two sets - one of vertices say V and one of edges say E. If x_1 and x_2 are in V and are joined by an edge, the corresponding element of E would be the pair (x_1, x_2). Thus E is a subset of V × V. One natural question about edges may be bothering you. Is (x_1, x_2) the same as (x_2, x_1)? In other words do edges have a direction? The answer to this question is yes. This leads us to the following definitions.

Definition : A **simple graph** or an **undirected graph** G consists of a finite nonempty set V and a set E of 2 element subset of V. The set V is called the **vertex set** of G, the set E is called the **edge set** of G and we write $G = (V(G), E(G))$ to denote the graph G.

Definition : A **directed graph** or **digraph** G consists of a finite nonempty set V together with a subset A of the product set V × V. We call V the **vertex set** of G and A the **edge set** of G and we write $G = (V(G), A(G))$ to denote the digraph G.

Fig.4(b) shows a simple graph G_1 and 4(a) and 4(c) shows directed graphs G and G_2 respectively.



G (a)

G_1 (b)

G_2 (c)

Fig.4

For the graph $G_1 = (V(G_1), E(G_1))$ shown in Fig.4 (a), since E is a symmetric relation, there is always a pair of edges joining two vertices that are related To represent this relation E on the set V, we can just draw one edge between every two vertices with the arrowheads omitted as we have done in Fig.4(b). In a directed graph as show in Fig.4(a) and 4(c), directions are assigned to the edges. Thus, an undirected graph is a representation of a set and a symmetric binary relation on the set. In an undirected graph, an edge joining the vertices u and v can be denoted either by (u. v) and or by (v, u) as there is no need to make the distinction.

Note that a set and a symmetric relation on the set can be represented either as a directed (both ways) or as an undirected graph. However, an undirected graph can represent only a set and a symmetric relation on it. Sometimes, it may happen that in a graph there is a **loop**, i.e., an edge joins a vertex to itself as in Fig.4(c), where (u, u) shows a loop. It may also happen that there are two or more edges joining the same vertices just as in Fig.4(c) there are two edges joining y to x. Such edges are called **parallel** or **multiple edges** and such graphs are called **multigraph.** In case of simple graph both these situations are avoided.

For the discussion in this block we shall henceforth be concerning only simple graphs and shall refer to them as just graphs. Also, whenever there is no confusion, we shall write just V and E in place of V(G) and E(G).

For a graph $G = (V, E)$, each element v of V is called a **vertex** of G and each element e of E is called an **edge** of G. If $e = \{u, v\}$ is an edge, we denote it simply by $e = uv$ (or $e = vu$). In this case, u and v are referred to as **adjacent vertices**, u and v are said to be **incident** with e, and e is **incident** with u and v. Similarly, if distinct edges e_1 and e_2 of G have a vertex in common then e_1 and e_2 are called **adjacent edges.** For a graph $G = (V, E)$ the relation of 'adjacency' is **non-reflexive and symmetric relation on V.** If the set V is finite then we get **finite** graph. We call a graph having p vertices and q edges as a **(p, q)-graph.**

Example 4: For the graph G_1 of Fig.4, $G_1 = (V, E)$, where $V = \{u, v, x, y\}$ and $E = \{uv, ux, uy, vx, xy\}$. Thus the only non-adjacent vertices of G_1 are v and y. The edges uv and vx are adjacent since both are incident with the vertex v. The edges uv and xy are non-adjacent. Two alternate ways of drawing the graph G_1 are shown in Fig.5.

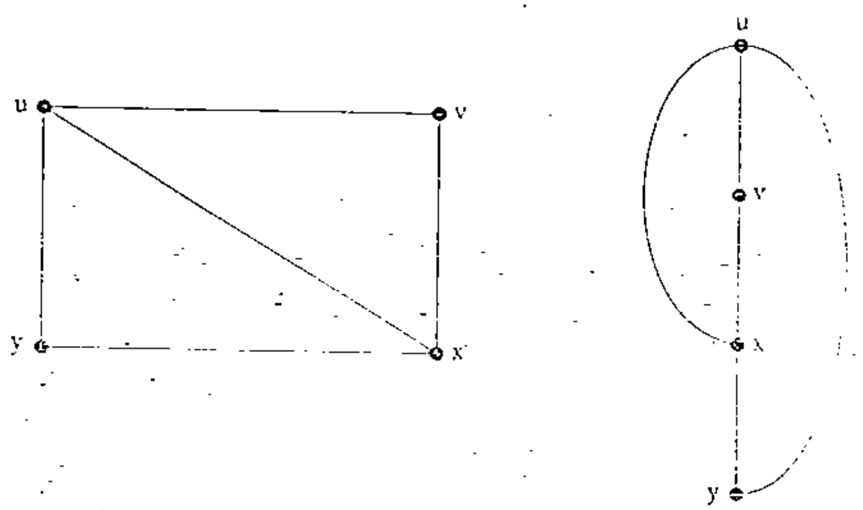


Fig.5

hus, there is no unique way of drawing a graph, the relative positions of the oints and curves have no special significance.

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ou may now try these exercises.

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1) Take three vertices x, y, z and draw all possible (3, 2)-graphs on these vertices.

2) There are four basic blood types: A,B, AB and O. Type O can donate to any of the four types. A and B can donate to AB as well as to their own types, but type AB can only donate to AB. Draw a digraph that presents this information.

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far, in our discussion we introduced you to the graphs that we are going discuss in this unit and the subsequent units of this block. We shall now miliarise you with types of graphs, but before that we give the following finitions.

efinition: Let $G = (V, E)$ be a graph. Take a set V of points on any rface S (like plane, sphere, etc.). Corresponding to every edge $x\,y \in E(G)$ aw a curve on the surface S joining x and y, such that this curve does not ss through any other point in V. Such a representation of the graph G is lled a **diagram** or a **drawing** of G on the surface S.

Examples 1,2 and 3 above you must have noted that on the surface S, you n have many different drawings of the same graph. There we did not give ese graphs any special name. We now define different types of graphs.

efinition : The **complete graph** K_n is a graph with n vertices, such that ery vertex is joined to every other vertex by an edge.

r example, in Fig.6, K_1 is just a single vertex, K_2 consists of two adjacent tices. The graph K_3 is often called a **triangle**.

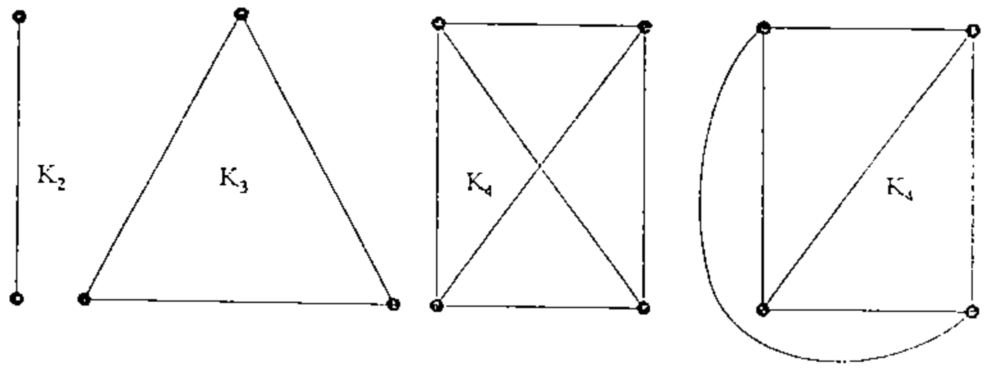


Fig.6

e last two figures in Fig. 6 represent two drawings of K_4 on the plane of paper.

a graph $G = (V, E)$ on n vertices $\{x_1, x_2, \ldots, x_n\}$, a $x_1 - x_n$ walk in G is nite alternating sequence of vertices and edges, $x_1 x_2, x_2, x_2 x_3, \ldots, x_{n-1} x_n, x_n$ begining with x_1 and ending with x_n, such t each edge in the sequence is incident with the vertices that immediately cede and follow it. The **length** of a walk is the number of edges it tains, with repeated edges counted. A $x_1 - x_n$ walk is **closed** if $x_1 = x_n$ l **open** if $x_1 \neq x_n$. A $x_1 - x_n$ walk in which no edge is repeated is a $- x_n$ **trail** and a closed trail is a **circuit**. Also, a trail with no repeated vertices is a path. Formally, we have the following definition.

Definition : The path P_n is a graph on n vertices $\{x_1, \ldots, x_n\}$ with the edge set given by $E(P_n) = \{x_i\, x_{i+1} : 1 \leq i \leq n-1\}$. For example, graph shown in Fig.7 is P_{13}.

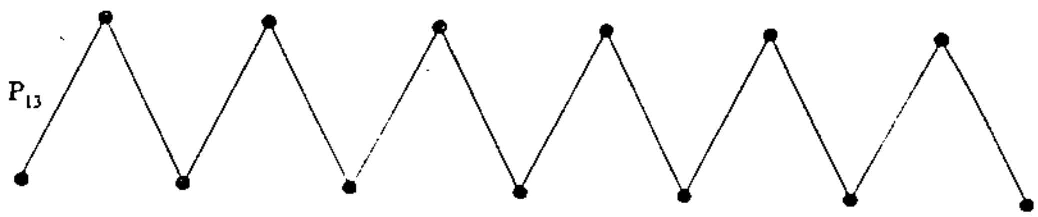


Fig.7

In a path no edge and no vertex is repeated. We shall talk about it in detail in the next unit. We now define a cycle.

Definition 5: A cycle C_n is a graph on n vertices $\{x_1, \ldots, x_n\}$ where $E(C_n) = \{x_i\, x_{i+1} : 1 \leq i \leq n-1\} \bigcup \{x_n\, x_1\}$. For instance, C_{16} is as shown in Fig.8.

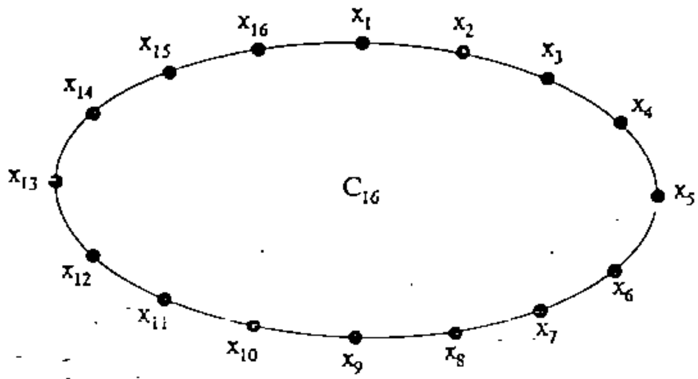
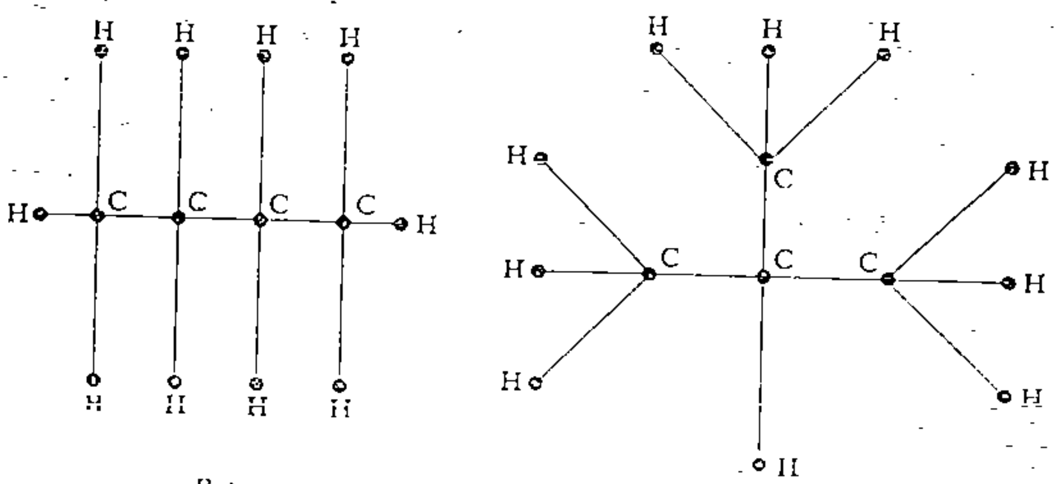


Fig.8

Note that the graph C_n is obtained by taking P_n and adding one more edge $x_n\, x_1$. **A cycle is a circuit in which the only repeated vertex is the first vertex, this being the same as the last vertex.** Also note that in practise we list only the vertices in a walk. There is no need to include the edges when listing the vertices and edges of a walk. For example a $x_1 - x_5$ walk in Fig.7 may be listed as the walk x_1, x_2, x_3, x_4, x_5.



Butane

Isobutane

Fig.9

It is interesting to know that the structure of molecules can also be represented by graphs.(see Fig.9) Various atoms are represented by the vertices and the structural bonds are represented by the edges. For example, butane as well as isobutane are both hydrocarbons $C_4 H_{10}$. The manner in which the bonds are present between the carbon and hydrogen atoms makes the difference. In both the compounds each carbon atom is attached to four other atoms. Unlike isobutane, in butane there is no carbon atom which is attached to all the other carbon atoms. Water molecule H_2O can be shown by path P_3 as in Fig.10.

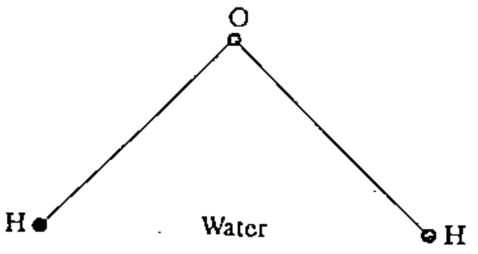


Fig.10

Let us now consider the following example.

Example 5: Suppose a, b, c, d, , u, v, x, y represent eight cities in India with the highways existing between certain pairs of cities. Fig.11 shows the graph of a roadmap of these cities where each city is represented as a vertex and two vertices are joined by an edge if the corresponding cities are linked by a highway. Find examples of the following walk in the graph given by Fig.11.

a) A u − v walk that is not a trail

b) A u − v trail that is not a path

c) A u − v path of length 5

d) A u − u circuit that is not a cycle

e) A u − u cycle of length 8.

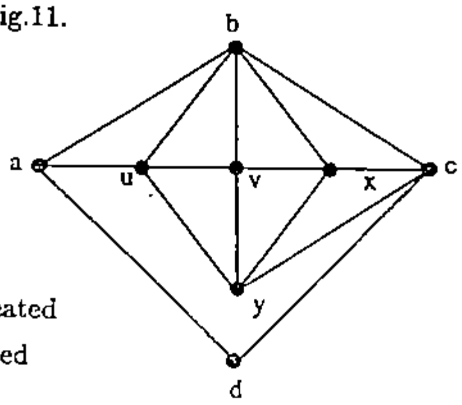


Fig. 11

Solution

a) The walk u, v, x, y, v, b, u, v is not a trail since the edge uv is repeated

b) The trail u, b, x, y, c, x, v is not a path since the vertex x is repeated

c) The path u, a, d, c, x, v has length 5

d) The circuit u, b, v, y, x, v, u is not a cycle since the vertex v is repeated

e) The cycle u, v, x, y, c, d, a, b, u has length 8.

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You may now try the following exercises.

E3) Write down the vertex set V and edge-set E of each graph in Examples 1), 2) and 3). Name these graphs if you can.

E4) Give an example, different from the one given above for each of the parts (a) - (c) in Example 5.

We now take up some more definitions.

Definition: Two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are disjoint if they have no vertex in common, i.e., $V(G) \cap V(H) = \phi$.

For example, the two graphs of Butane and Isobutane shown in Fig. 9 are disjoint graphs.

Note that when two graphs are disjoint then their edges are also disjoint. If not, then there would be an edge e in both G and H and then the ends of e would also be in both G and H.

Definition: The union of two graphs G and H is graph G, with vertex set consisting of all those vertices which are in either G or H (or both). and with edge set consisting of all those edges which are in either G or H (or both); symbolically.

$$V(G \cup H) = V(G) \cup V(H)$$
$$E(G \cup H) = E(G) \cup E(H)$$

For example, Fig.12 gives the two graphs G, H and their union G ∪ H.

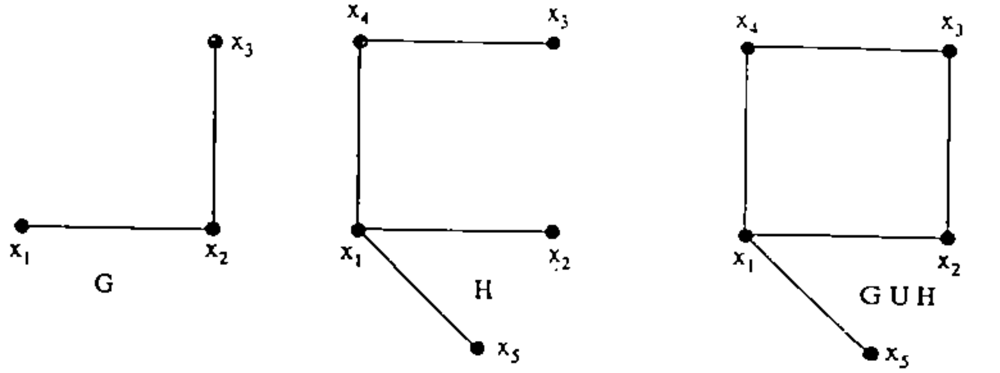

Fig.12

Example 6: Consider the following graph.

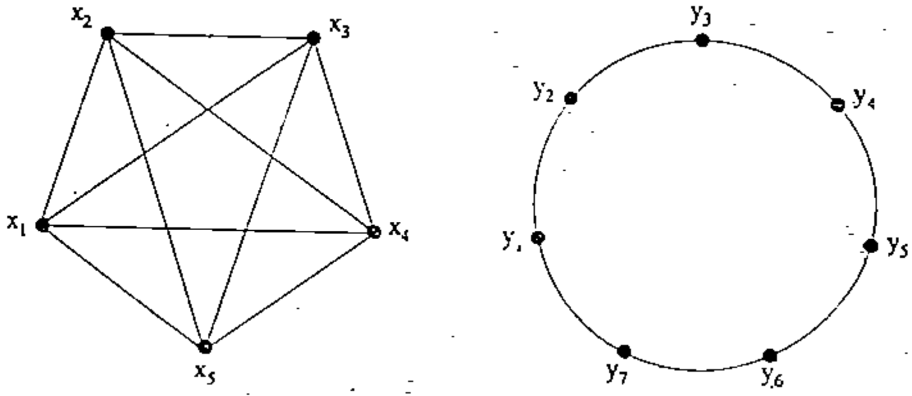

Fig.13

Clearly it is the union of K_5 and C_7.

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Example 7: Consider the following graph.

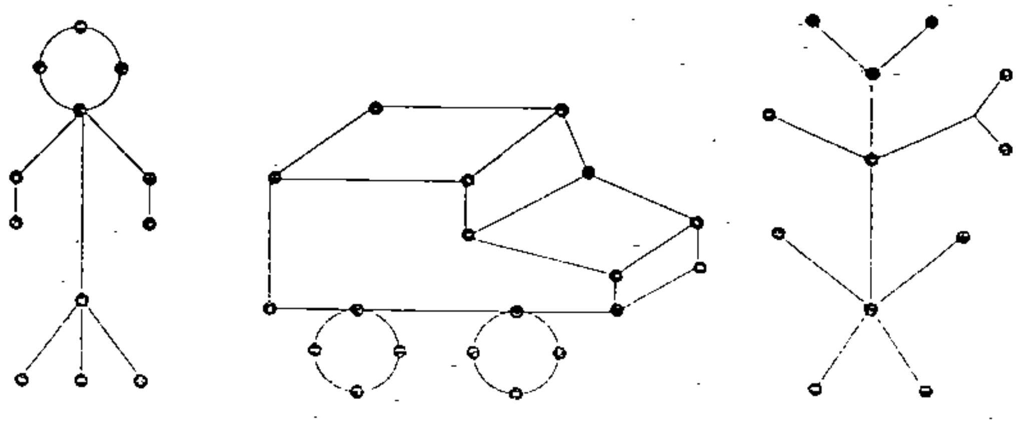

Fig.14

Clearly it is the union of three graphs.

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Example 8: The following graph is the union of five graphs. One of them is just an isolated point.

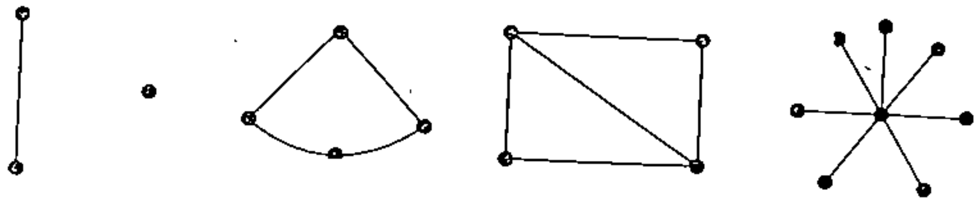

Fig.15

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Thus, we can go on taking unions of many graphs.
You may also come across situations where the two graphs have same set of vertices but there edge set are disjoint. Do we call such graphs by any special name? Let us consider the following definition.

Definition : Let $G = (V, E)$ be a (p, q)-graph. By the **complement** $\overline{G}$, we mean the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{x\,y : x\,y \notin E(G)\}$.
Clearly, $\overline{G}$ is a $(p, \overline{q})$-graph where $\overline{q} = \{$number of pairs of elements of $V\} - q$.

Since in a set V with p elements there can be $C(p, 2) = \frac{p(p-1)}{2}$ such pairs of elements, $\overline{q} = \frac{p(p-1)}{2} - q$.

Example 9: Fig. 16 shows C_5 and its complement.

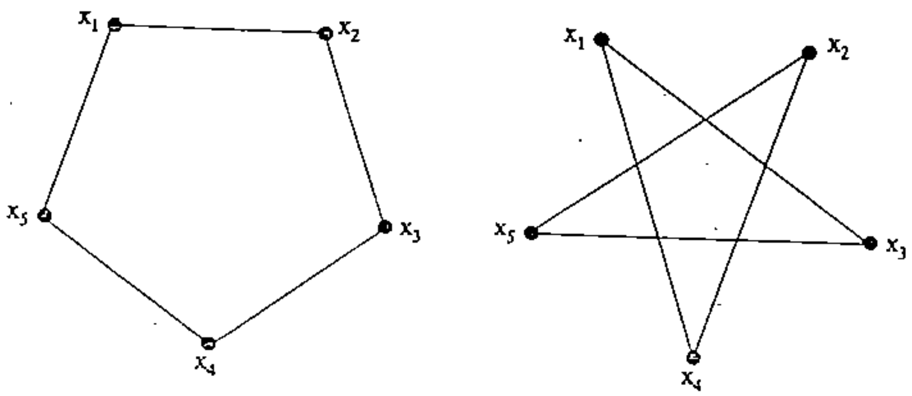

Fig.16

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Example 10: Consider the graph shown in Fig.17(a). Its complement breaks into two disjoint graphs.

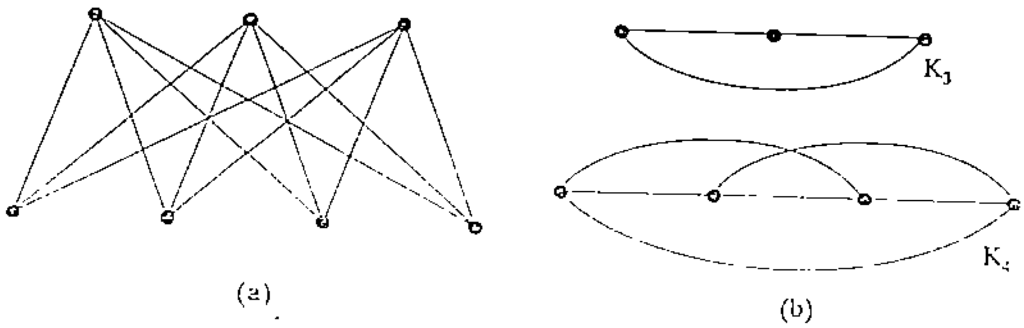

Fig.17

One is K_3 and the other is K_4. (see Fig.17(b)).

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Notice that in Example 9, G is a (5, 5) graph and $\overline{G}$ has 5 edges. In

Example 10, G is a (7, 12) graph and $\overline{G}$ has 9 edges. Do you see any relation between the vertices of G and edges of $\overline{G}$? You may try the following exercises and find out the answer.

E5) Three graphs G_1, G_2 and G_3 are listed below

$$G_1 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_5, u_1u_6, u_2u_3, u_2u_5, u_3u_4, u_4u_5\})$$

$$G_2 = (\{u_1, u_2, u_3, u_4, u_5\}, \{u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_4, u_2u_5, u_3u_4, u_3u_5\})$$

$$G_3 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_3u_6, u_5u_6\})$$

Find $\overline{G}_1, \overline{G}_2$ and $\overline{G}_3$.

E6) If G is a (p, q) graph then how many edges can $\overline{G}$ have?

Let us go back to the case of electric network. When a mechanic works on such a network, for his own safety it is best for him to know all the gadgets connected to a single point. Again, for using the public transport network effectively, it is necessary to know the places connected to the starting point. This starting point also changes from person to person. That means for efficient use of such a system it is best to keep in mind the places one can reach from any given point. We translate this problem in graph theory language.

10.3 REGULAR GRAPHS

You may recall that in the beginning we defined two vertices of a graph G to be adjacent if they are joined by an edge. Such vertices are also called neighbours. The set of all neighbours of a fixed vertex x of G is is called the neighbourhood set of x. Formally, we have the following definition.

Definition : Let G = (V, E) be a (p, q)-graph. For a vertex $x \in V$, by the **neighbourhood** $N_G(x)$ of x in G, we mean the set $\{y \in V : xy \in E\}$, that is, the set of all vertices adjacent to the vertex x. A vertex $y \in N_G(x)$ is called a **neighbour** of x in G.

Since our graphs are simple, there is a one-one correspondence between $N_G(x)$ and the set of all edges of G incident with the vertex x. By the **degree** $d_G(x)$ of the vertex x in G, we mean the number of edges incident with a vertex in G. Clearly $d_G(x) = |N_G(x)|$ where $|N_G(x)|$ denotes the number of elements of the set $N_G(x)$. Also since in a (p, q) graph the maximum number of edges incident with vertex x can be (p − 1), we have $0 \le d_G(x) \le (p-1)$ for every vertex x in G. Whenever there is no danger of confusion, we will simply write d(x) instead of $d_G(x)$. Also a vertex x of a graph G is called an even vertex if $d_G(x)$ is even, otherwise it is called an odd vertex. Now let us look at the following example.

Example 11: Consider the graph G shown in Fig. 18. First consider the vertex x_1. Clearly, three edges are incident with it and $d(x_1) = 3$. Likewise you may observe that

$$d(x_2) = 4, d(x_3) = 5, d(x_4) = 6 \text{ and } d(x_5) = 7$$

We can also write the above observations as $d(x_i) = i + 2$ for $1 \le i \le 5$.

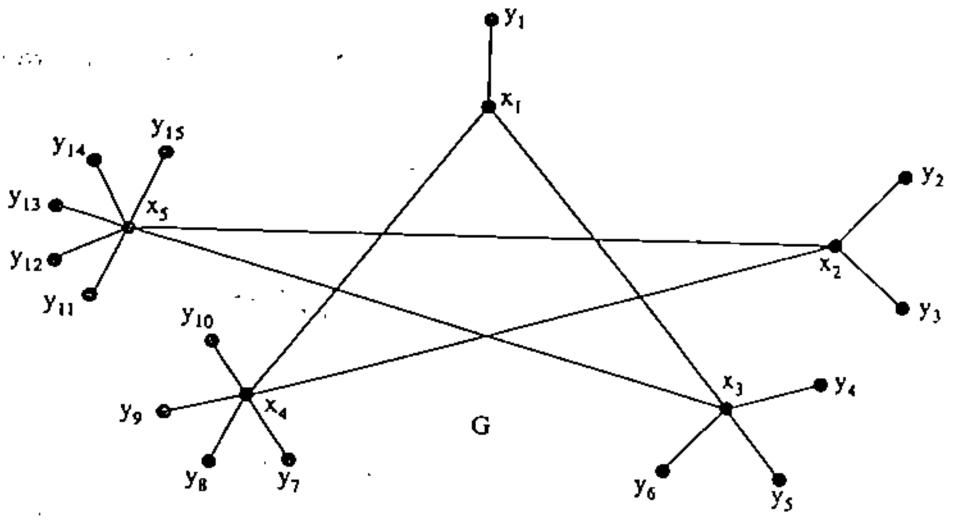

Fig.18

In the same manner we can write $d(y_j) = 1$ for $1 \leq j \leq 15$

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You may now try the following exercises.

E7) Write down degrees of all the vertices in the Examples 6 to 10.

E8) If G is a (p, q)-graph and x is a vertex in G, show that degree of x in $\overline{G}$ is $p - 1 - d_G(x)$.

Note that in Example 11

$$\begin{aligned} & d(x_1) + d(x_2) + \ldots + d(x_5) + d(y_1) + \ldots + d(y_{15}) \\ = \; & 40 \\ = \; & 2 \times 20 \\ = \; & 2 \times (\text{number of edges in G}) \end{aligned}$$

Same thing you would notice in Examples 6 to 10 which you have just solved in E7). This is no coincidence. This follows from the following theorem:

Theorem 1: If G is a (p, q)-graph with $V(G) = \{v_1, \ldots, v_p\}$ and if $d_i = d_G(v_i), 1 \leq i \leq p$, then $2q = \sum_{i=1}^{p} d_i$, that is, the sum of the degrees of vertices of G is even, or, in any graph, the sum of all vertex-degree is equal to twice the number of edges.

Proof: Consider the set $S = \{(x, e) : x \in V(G), e \in E(G), x \text{ is incident on } e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{1}^{p} d_i. \qquad (1)$$

Now choose an edge e in E(G). This can be done in q ways. This edge has precisely two end vertices and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q. \qquad (2)$$

This is because every edge is counted twice once for each vertex it contains. Equating (1) and (2) we get the required result.

Thus we can say that sum of the degrees of all the vertices of any graph is even. This result is also known as 'Hand Shaking Lemma'. Let us check it in the case of the following example.

Example 12: Consider the following graph:

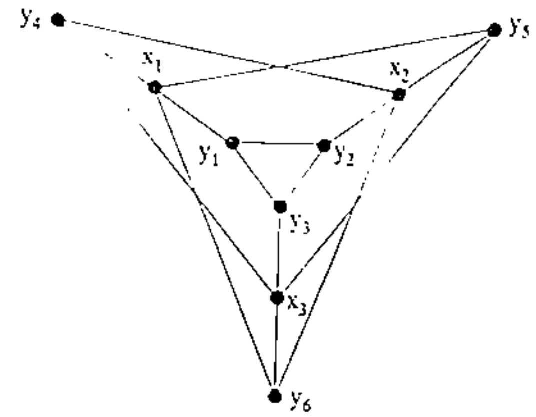


Fig.19

Clearly each $x_i, 1 \leq i \leq 3$, is an even vertex each having degree 4. All the remaining vertices $y_i, 1 \leq i \leq 6$ are odd vertices each having degree 3. What is the sum of the degrees of all the vertices? You may check that it is 30.

So far, in the discussion, you must have noticed that for a simple (p, q)-graph G the edge set E(G) is a subset of the set of all subsets of size 2 of elements of V(G). This means $q \leq \frac{p(p-1)}{2}$. But then you may wonder that is it always possible to go the other way round? That is, for any pair of positive integers (p, q) with $q \leq \frac{p(p-1)}{2}$, is it always possible to find a (p,q)-graph?

The answer to this question is given by Theorem 1. It gives us a necessary condition under which any (p, q) graph exists. It helps us to see that there does not always exist a graph with vertices having given degrees. Supposing you are asked to

Construct a graph on 12 vertices with 2 of them having degree 1, three having degree 3, and the remaining seven having degree 10.

Can you do it? No. This is because here the condition of Theorem-1 is not fulfilled. The sum of degrees of all vertices is
$1 + 1 + 3 + 3 + 3 + 10 + 10 + 10 + 10 + 10 + 10 + 10 = 81$, which is not even.

* * *

Thoerem-1 can be used to obtain another result which we discuss now.

Corollary: Any graph has even number of odd vertices.
Proof: Let G be a (p, q)−graph and let $\{x_1, \ldots, x_t\}$ be the set of all odd vertices and $\{x_{t+1}, \ldots, x_p\}$ be the set of all even vertices of G. Let $d_G(x_i) = 2c_i + 1, 1 \leq i \leq t$ and $d_G(x_i) = 2r_i, t+1 \leq i \leq p$. Then
$2q = \sum_{1}^{p} d_G(x_i)$ gives

$$\begin{aligned} 2q &= \sum_{i}^{t}(2c_i + 1) + \sum_{t+1}^{p}(2r_i) \\ &= 2(c_1 + c_2 + \ldots c_t) + t + 2(r_{t+1} + \ldots + r_p) \end{aligned}$$

Thus, $t = 2q - 2(c_1 + \ldots + c_t) - 2(r_{t+1} + \ldots + r_p)$. Which shows that t is even. That is, **there are even number of odd vertices.**

We now give another interesting result which follows from here.

Corollary: At any party, the number of people who have shaken the hands of an odd number of people is even.

Example 13: Consider K_{10}. All the vertices have degree 9. This means all the ten vertices are odd vertices. On the other hand in K_{11} all the vertices have degree 10, that is, all the eleven vertices are even vertices.

* * *

In Example 12 you may observe that the number of even vertices is odd. That does not mean this happens in every graph. The Graph C_{10} has 10 vertices and all of them have degree 2, that is, C_{10} has even number of 'even vertices'.

We now define the minimum and maximum vertex degree of a graph G.

Definition : If $G = (V, E)$ is a $(p, q)-$ graph then the integers $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is called the **minimum vertex degree of** G and the integer $\Delta(G) = \max\{d_G(x) : x \in V(G)\}$ is called the **maximum vertex degree of** G.
We can in fact number the vertices as $V(G) = \{v_1, \ldots, v_p\}$ with $d_i = d(v_i), 1 \leq i \leq p$ such that $d_1 \geq d_2 \geq \cdots \geq d_p$. This is called the **degree sequence** of the graph G.

For instance, the degree sequence of the graph G in Example 12 is 4,4,4, 3,3,3,3,3,3.

And now some exercises for you.

E9) Write down $\delta(G)$ and $\Delta(G)$ for all the graphs in Examples 1 to 3 and 9, 11,12.

E10) Each of the following parts gives a list of non-negative integers: Give an example each of a graph having that degree sequence or argue that no such graph exists.

(a) (3,2,2.2,1) (b) (3,2,2.2,1,1)
(c) (4,3,2.1.0) (d) (4,4,3.3.2,2) (e) (5.5,5.4,4.3,3)

E11) Let G be a (p, q) graph all of whose vertices have degree k or k + 1. If G has $p_k > 0$ vertices of degree k and p_{k+1} vertices of degree k + 1, then $p_k = (k+1)p - 2q$.

In Fig. 9 if you see the two graphs of butane and isobutane they look different. However, they have same degree sequence viz.,

$$4,4,4,4,1,1,1,1,1,1,1,1,1,1$$

It may also happen for some graphs that they have a constant degree sequence. That is, each of their vertices has the same degree. For example, if you look at C_5 and its complement, you would notice that $d(x_i) = 2, i \leq i \leq 5$. Thus, the degree sequence of C_5 and its complement is $2, 2, 2, 2, 2$, that is, it is a constant 2. Such graphs are very special. We define them as follows:

Definition: A graph G is said to be **regular** with the degree of regularity r if $d_G(x) = r$ for every vertex $x \in V(G)$. In this case we often say that G is an

r−regular graph. Clearly $0 \leq r \leq (p-1)$.
The graphs K_n, C_n, are regular with the degree of regularity $(n-1)$ and 2, respectively.

Of particular importance are the **cubic** graphs which are regular graphs of degree three and about which you shall be studying in the next unit. Well known example of a cubic graph is the **Petersen graph**. The two drawings of the Petersen graph are shown in Fig.20.

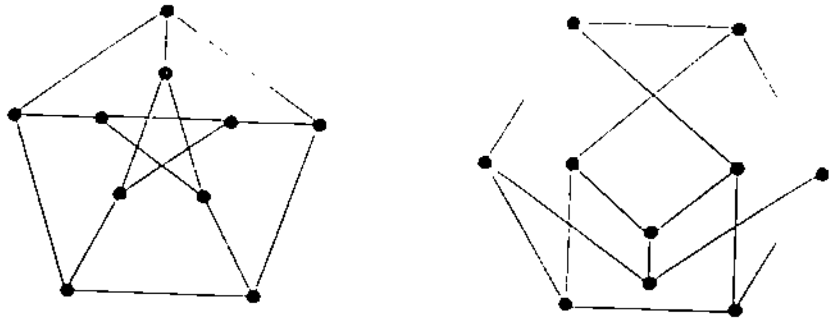


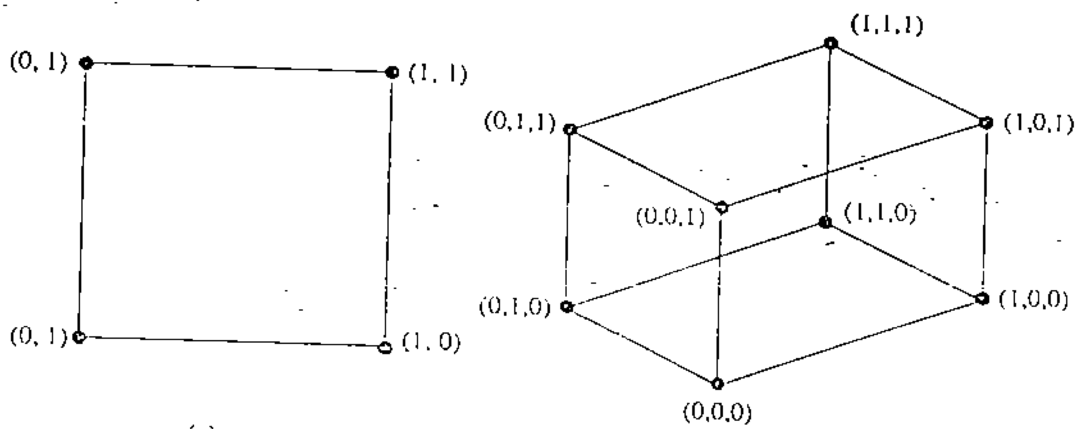
Fig.20

Note that it is a (10,15) graph. Let us now consider the following example of a regular graph.

Example 14: Hypercube Q_n: Let the vertex set consist of all n−tuples with entries 0, 1 only. The edge set is given by

$$E(Q_n) = \{\bar{a}\,\bar{b} : \bar{a} \text{ and } \bar{b} \text{ differ exactly at one co-ordinate}\}.$$

Here, by $\bar{a}$ we mean an n−tuple $(a_1, \ldots, a_n)$, where $a_i = 0$ or 1, for $1 \leq i \leq n$.



(a) Fig.21 (b)

Any vertex $\bar{a}$ is adjacent to precisely n other vertices. For example $(0, 0, \ldots, 0)$ is adjacent to $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, 0, \ldots 0)$, $\ldots$, $(0, 0, \ldots, 1)$. Hence the hypercube Q_n is n−regular.

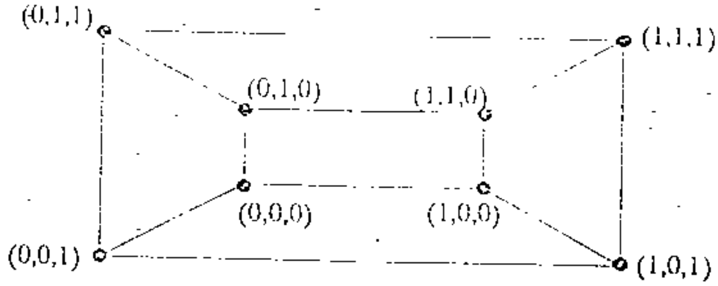


Fig.22

Fig. 21(a) shows the graph of Q_2 whereas Fig.21(b) and Fig.22 both show the graph of Q_3. You may check that n-regular hypercube Q_n has 2^n vertices and $n2^{n-1}$ edges.

* * *

If G is an $r-$regular graph on p vertices, then by Theorem 1, $2q = r \times p$. Thus 2 divides the product $p \times r$. This means at least one of p or r is even. This makes us ask the following question: Given a pair of integers $p, r, 0 \leq r \leq (p-1)$, where $p \times r$ is even, can we always construct an $r-$regular graph on p vertices? To get an answer to this question let us consider the following theorem.

Theorem 2: Given a pair of integers p, r, where at least one of them is even and $0 \leq r \leq (p-1)$, there always exists a regular graph on p vertices and with degree of regularity r.

Proof : The proof is constructive, i.e., we actually construct the graphs of the required degree. There are two cases:

Case 1: r is even. Write $r = 2s$, where s is some integer. Now construct a graph G as follows: $V(G) = \{v_1, \cdots, v_p\}$. Place them in circular manner as shown in Fig.23. Join each v_i by an edge to v_{i+j} for every $1 \leq i \leq p-s, 1 \leq j \leq s$. In addition join vertices in the following manner:

(i) Join the vertex v_{p-s+1} to the vertices $v_{p-s+2}, v_{p-s+3}, \cdots, v_1$.

(ii) Join the vertex v_{p-s+2} to the vertices $v_{p-s+3}, v_{p-s+4}, \cdots, v_2$.

(iii) Join the vertex v_p to the vertices $v_1, v_2, \cdots, v_s$.

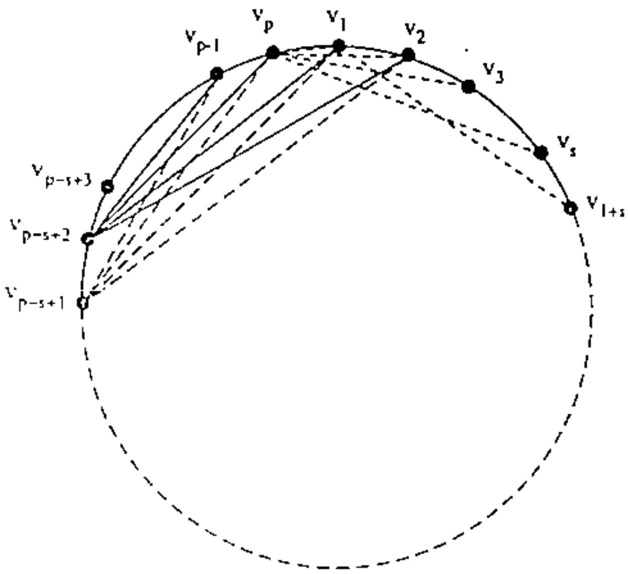


Fig.23

Note that as there are p vertices $v_1, v_2, \cdots v_p$ so that in (i) and (ii) above as soon as any of the subscript $p-s+2, p-s+3, p-s+4, \cdots$ etc. for any value of s exceeds p the cycle is repeated from $1, 2, \cdots$. That is, the corresponding vertices are $v_1, v_2, \cdots$ etc. You can check that this graph is $r-$regular. Although it is difficult to check it but the examples following this theorem would help you in doing so.

Case 2: r is odd. Then p must be even. Write $p = 2n$. Since r is odd, $(r-1)$ is even. Using Case 1, we can construct a graph on vertices $\{v_1, \cdots, v_p\}$ which is $(r-1)$regular. (see Fig.24)

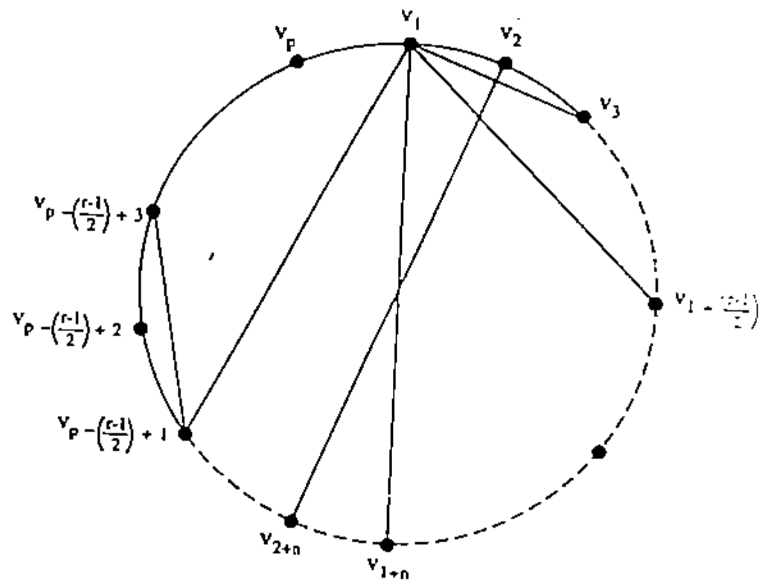


Fig.24

Now $\frac{(r-1)}{2} \le \frac{(p-2)}{2} < \frac{p}{2} = n$ (say). So, without fear of repetition we can add edges $v_i\, v_{i+n}, 1 \le i \le n$. Since one new edge is added at every vertex, the graph is r−regular.

We now illustrate the two cases with the help of the following examples.

Example 15: Suppose $p = 12, r = 4$. Take twelve vertices $\{x_1, \ldots, x_{12}\}$. Place them in a circular manner as shown in Fig.25.

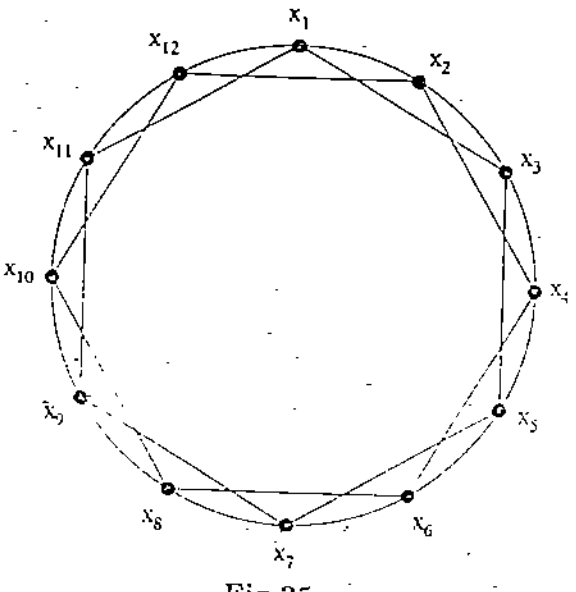


Fig.25

In this case $s = r/2 = 2$. Join x_i to x_{i+1} by an edge for every $1 \le i \le 11$. Join x_{12} to x_1 also by an edge. Now all the vertices have acquired degree 2. Now, join each x_i to x_{i+2} for every $1 \le i \le 10$. Finally join x_{11} to x_1 and x_{12} to x_2. You can see clearly that the resulting graph is 4−regular.

Example 16: Suppose $p = 12, r = 5$. Here we note that $\frac{r}{2}$ is not an integer. However, $\frac{(r-1)}{2}$ is an integer. Hence the construction of Example 15 can be repeated for 12 vertices and regularity $(r-1)$. Take the graph constructed in

Example 15. Now join each x_i to x_{i+6} for every $1 \leq i \leq 6$. Here $\frac{p}{2} = n = 6$.

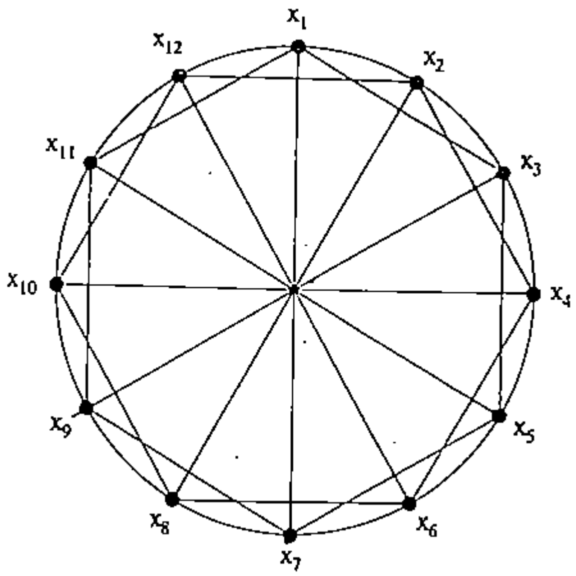


Fig.26

Again it is easy to see that the resulting graph is 5−regular (see Fig.26).

* * *

Example 17: Let us now construct 6−regular and 7−regular graphs on 12 vertices. Let us do this step by step. Let us first start with 6−regular graph. Here $s = 3$. Fig.27 shows the 12 vertices arranged in a circular manner, where each of $v_1, v_2, \cdots, v_9$ is joined to the next three vertices in an ascending manner. That is, v_1 is joined to v_2, v_3, v_4. v_2 is joined to v_3, v_4, v_5. v_3 is joined to v_4, v_5, v_6 etc.

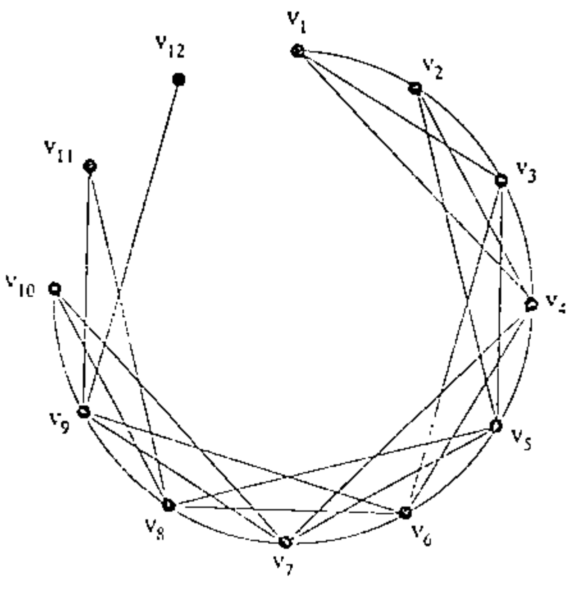


Fig.27

You may notice that at this stage $d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 5$, $d(v_i) = 6$ for $4 \leq i \leq 9$ $d(v_{10}) = 3$, $d(v_{11}) = 2$ and $d(v_{12}) = 1$.

Now, Fig.28 adds the remaining edges $\{v_{10}\, v_{11}, v_{10}\, v_{12}, v_{10}\, v_1\}$, $\{v_{11}\, v_{12}, v_{11}\, v_1, v_{11}\, v_2\}$, $\{v_{12}\, v_1, v_{12}\, v_2, v_{12}\, v_3.\}$

With the addition of these edges the degree of each of the vertices v_1 to v_{12} becomes 6. This gives 6−regular graph on 12 vertices (see Fig.28).

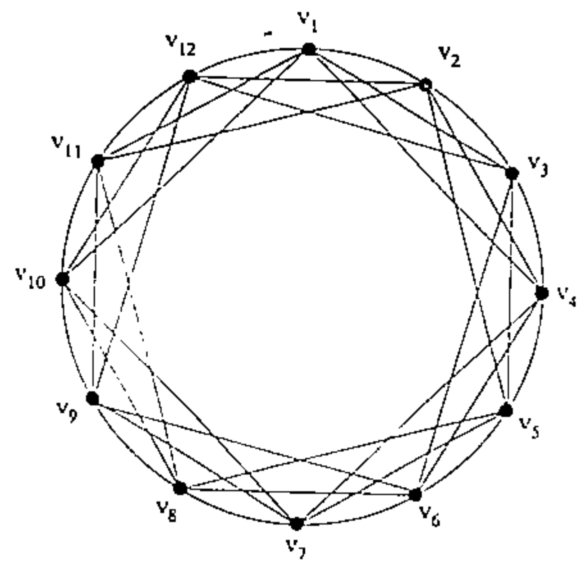


Fig.28

Further, Fig.29 shows that adding the edges v_i-v_{i+6}, $1 \leq i \leq 6$, gives a 7−regular graph on 12 vertices.

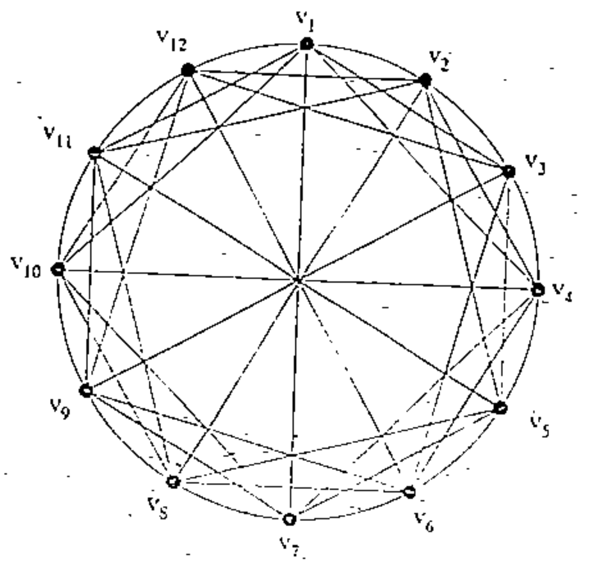


Fig.29

You may now test your knowledge of the construction of regular graphs by doing the following exercise.

E12) Construct 5,6 and 7 regular graphs on 10 vertices.

We cannot always 'see' the graphs and say that they are 'different.' They may differ in the way their vertices and edges are labelled or only in the way they are represented geometrically. But, still they may have some common features or may have some resemblance. This resemblance has a special name. We shall now define this formally.

You have noticed in Example 9 that the complement of C_5 is again a copy of C_5. We can make things more precise.

Let us redraw the two graphs as shown in Fig.30.

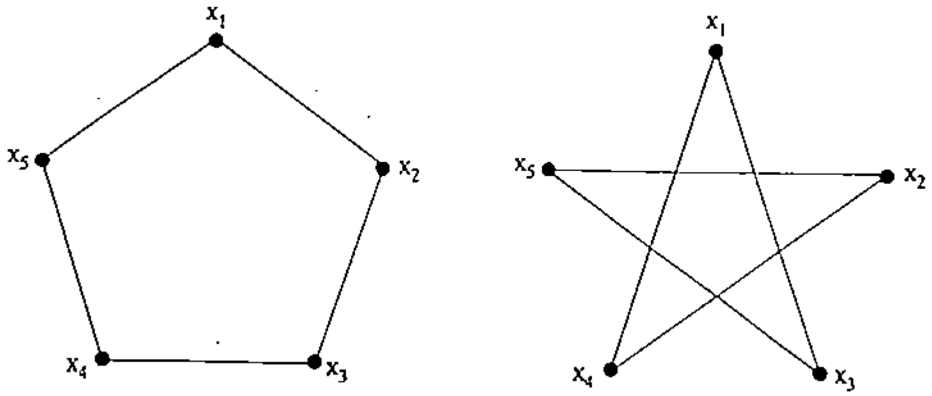


Fig.30

We can define a map $f : V(C_5) \longrightarrow V(\overline{C_5})$ as follows:
$f(x_1) = x_1, f(x_2) = x_3, f(x_3) = x_5, f(x_4) = x_2, f(x_5) = x_4$. What do we observe? Whenever $x_i\, x_j \in E(C_5), f(x_i)\, f(x_j) \in E(\overline{C_5})$. In other words the graph structure is preserved by the map f. Let us consider one more example.

Example 18: Consider following two graphs G and H as shown in Fig.31.

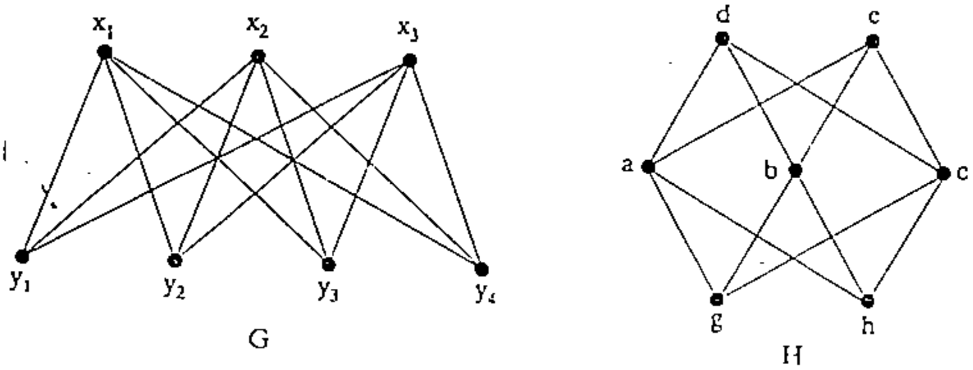


Fig.31

Define a map $f : V(G) \longrightarrow V(H)$ as follows:
$f(x_1) = a, f(x_2) = b, f(x_3) = c, f(y_1) = d, f(y_2) = e, f(y_3) = g, f(y_4) = h$. You may observe that $uv \in E(G)$ if and only if $f(u)\, f(v) \in E(H)$. Many properties of a vertex $u \in V(G)$ are shared by its image in $V(H)$. For example one can check that $d_G(u) = d_H(f(u))$ for every $u \in V(G)$.

* * *

Such graphs G and H are isomorphic. There is a one-one correspondence between the vertices of G and H. You may also notice that the number of edges joining any two vertices of G is equal to the number of edges joining the corresponding vertices of H. In other words, two graphs G and H are isomorphic if there is a one-one correspondence between V(G) and V(H) that preserves adjacencies and non-adjacencies. The two graphs shown in Fig.31 are isomorphic under the correspondence
$x_1 \leftrightarrow a, x_2 \leftrightarrow b, x_3 \leftrightarrow c, y_1 \leftrightarrow d, y_2 \leftrightarrow e, y_3 \leftrightarrow g, y_4 \leftrightarrow h$,

This leads us to the following definition:

Definition : Let $G = (V(G), E(G)), H = (V(H), E(H))$ be two graphs. By an isomorphism f from the graph G to the graph H, we mean a map $f : V(G) \longrightarrow V(H)$ such that
(1) f is both one-one and onto.

(2) $x\,y \in E(G)$ if and only if $f(x)f(y) \in E(H)$.
In this case we say that G and H are **isomorphic**. Otherwise they are called **non-isomorphic**.

In order to show that two graphs are isomorphic, it is enough to produce one isomorphism from one of them to the other. However, given two graphs, it is not easy to show that there does not exits any isomorphism between them. Then how do we go about showing that two given graphs are not isomorphic? The following six properties help us in this task. If two graphs are isomorphic each of these properties must be satisfied. Thus to show that two graphs are not isomorphic it is enough to show that any one of these properties does not hold. We shall now state them here.

Properties: Let f be an isomorphism from a graph G to a graph H. Then following holds:

1) If G is a $(p, q)-$ graph then H is also a $(p, q)-$graph.
2) The inverse map f^{-1} is an isomorphism from the graph H to the graph G.
3) If g is an isomorphism from the graph H to a graph K, then the composite map $g \circ f$ is an isomorphism from the graph G to the graph K.
4) f induces a bijective map $\bar{f} : E(G) \longrightarrow E(H)$, given by $\bar{f}(x\,y) = f(x)\,f(y)$.
5) For every $x \in V(G)$ a vertex y belongs to $N_G(x)$ if and only if $f(y)$ belongs to $N_H(f(x))$. Which means that $d_G(x) = d_H(f(x))$, for every $x \in V(G)$. Thus that the degree sequence of the graph G is same as the degree sequence of the graph H.
6) If G has a set of vertices $\{x_1, \ldots, x_n\}$ such that $x_n\,x_1$ and $x_i\,x_{i+1}$ are in E(G) for every $1 \leq i \leq (n-1)$, then the vertices $\{f(x_1), \ldots, f(x_n)\}$ in V(H) are such that $f(x_n)\,f(x_1)$ as well as $f(x_i)\,f(x_{i+1})$ are edges in E(H) for every $1 \leq i \leq (n-1)$. Thus, for every positive integer $n \geq 3$. the number of copies of C_n in G is equal to the number copies of C_n in H.

Let us now consider the following examples where we have used these properties to show non-isomorphism of the two graphs.

Example 19: Consider the two graphs as shown in Fig.32.

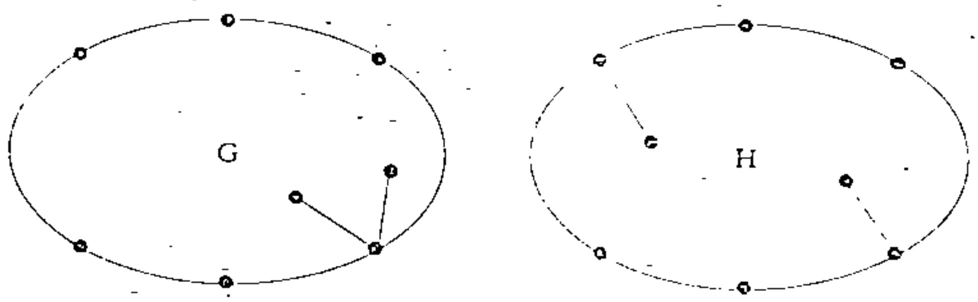


Fig.32

Both are $(8, 8)-$ graphs and have a copy of C_6 inside them. However they are not isomorphic. This can be seen easily by writing down their degree sequences. The degree sequence of the graph G is 4, 2, 2, 2, 2, 2, 1, 1 and that of the graph H is 3, 3, 2, 2, 2, 2, 1, 1.

* * *

Example 20: Consider the following two graphs G and H.

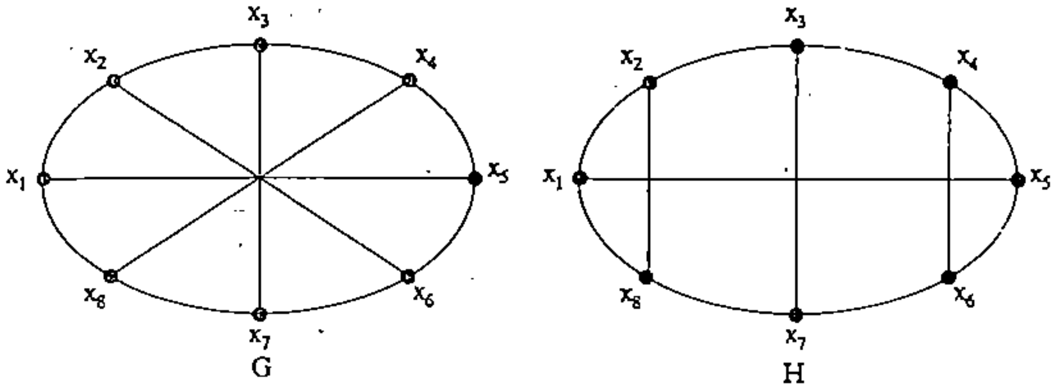


Fig.33

Both are (8, 12)−graphs and have a copy of C_8 inside them. Moreover both have degree sequence 3, 3, 3, 3, 3, 3, 3, 3. They are still not isomorphic. This can be seen by observing that the graph G has no copy of a triangle inside it and the graph H has two triangles $\{x_1, x_2, x_8\}$ and $\{x_4, x_5, x_6\}$. (see Fig.33)

* * *

Example 21: Consider the following two graphs G and H as shown in Fig. 34.

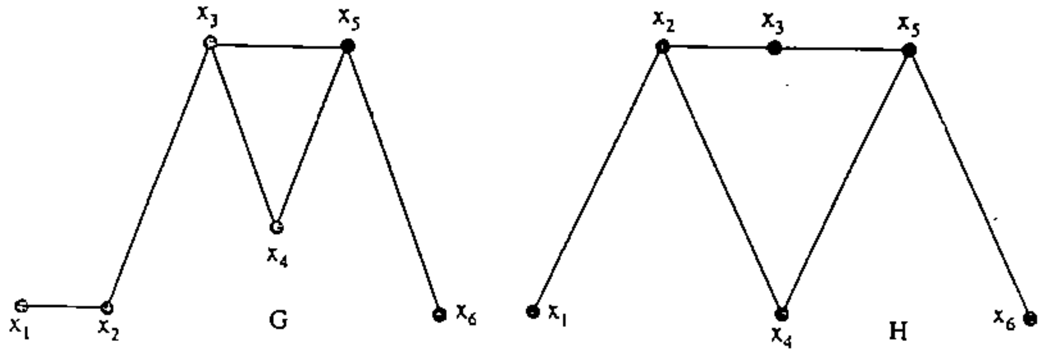


Fig.34

Both are (6, 6)−graphs having 3, 3, 2, 2, 1, 1 as their degree sequence. However, they are not isomorphic. In the graph G the two vertices x_3, x_5 having degree 3 are adjacent. Under any isomorphism (if it exists) they should be mapped to two adjacent vertices of degrees 3. We observe that in the graph H the two vertices of degrees 3 are not adjacent.

* * *

Notice that the two graphs shown in Fig.9, corresponding to butane and isobutane are not isomorphic. Unlike isobutane, no carbon atom is attached to all the other carbon atoms of butane.

And now the following exercises for you to try.

E13) Draw at least 6 non-isomorphic graphs on four vertices.

E14) A graph G is said to be self complementary if it is isomorphic to its complement $\overline{G}$. Show that for a self complementary (p, q)−graph G either p or (p − 1) is divisible by 4.

It is often the case that a graph under study is contained within some larger graph also being investigated. When we talk of an electric circuit, it is often described in terms of various sub-circuits. Transport in a country is always divided into various sections, for example, the railway transport in India is divided into Central railway, Western railway, ...etc. That is, whenever we study any system, it is important to study its subsystems. Likewise here in the next section we study subgraphs.

10.4 SUBGRAPHS.

We shall now formally define a subgraph of a given graph and study various types of subgraphs. But before we do that let us look at the following example.

Example 22: Consider the graph $G = (V(G), E(G))$ as shown in Fig.35.

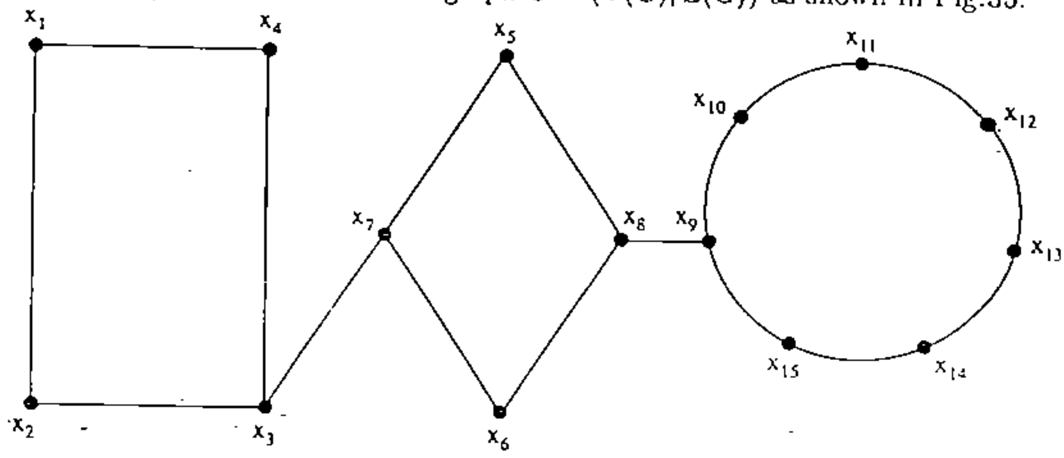


Fig.35

What if we just take a part of this graph G? Is this a graph? Yes. Consider the following.

Let $V(G_1) = \{x_1, x_2, x_3, x_4\}, E(G_1) = \{x_i\, x_{i+1} : 1 \leq i \leq 3\} \bigcup \{x_4 x_1\}$

You will note that G_1 is isomorphic to C_4.

If $V(G_2) = \{x_8, x_9\}, E(G_2) = \{x_8\, x_9\}$.

then G_2 is isomorphic to K_2. Also the graph

$$V(G_3) = \{x_9, \ldots, x_{15}\}, E(G_3) = \{x_{15}\, x_9\} \bigcup \{x_i\, x_{i+1} : 9 \leq i \leq 14\}$$

is isomorphic to C_7. (see Fig.36).

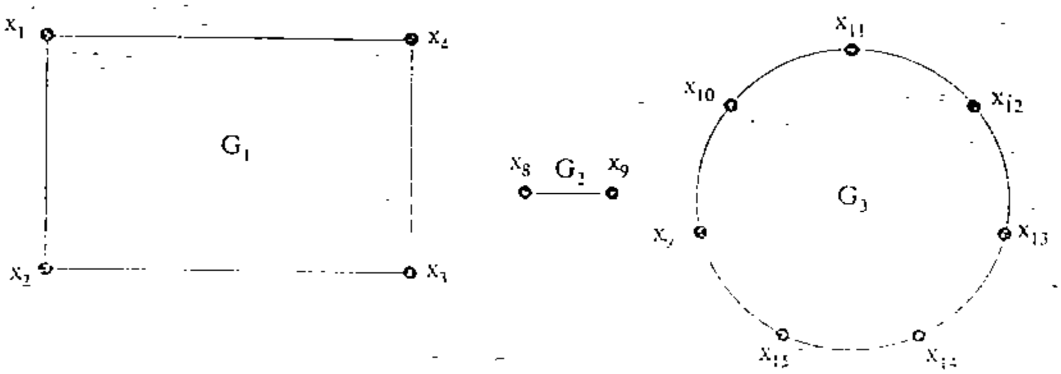


Fig.36

Note that all these graphs have one thing in common. Their vertex sets are subsets of $V(G)$ and edge sets are subsets of $E(G)$. In this sense, all these graphs are 'subgraphs' of the graph G. Formally, we have the following definition.

Definition 12: Let $G = (V(G), E(G))$ be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also. That is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Moreover, if H is a subgraph of a graph G, such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$, that is, H and G have exactly the same vertex set then H is called a **spanning subgraph** of G.

Example 23: In Example 22, the graph H, with

$$V(H) = V(G_3), E(H) = E(G_3) \bigcup \{x_9\, x_{12}\}$$

is not a subgraphs of the graph G. (why?). Clearly edge x_9x_{12} is not in E(G).

* * *

Example 24: Consider $G = K_4$ on four vertices x_1, x_2, x_3, x_4 as shown in Fig.37.

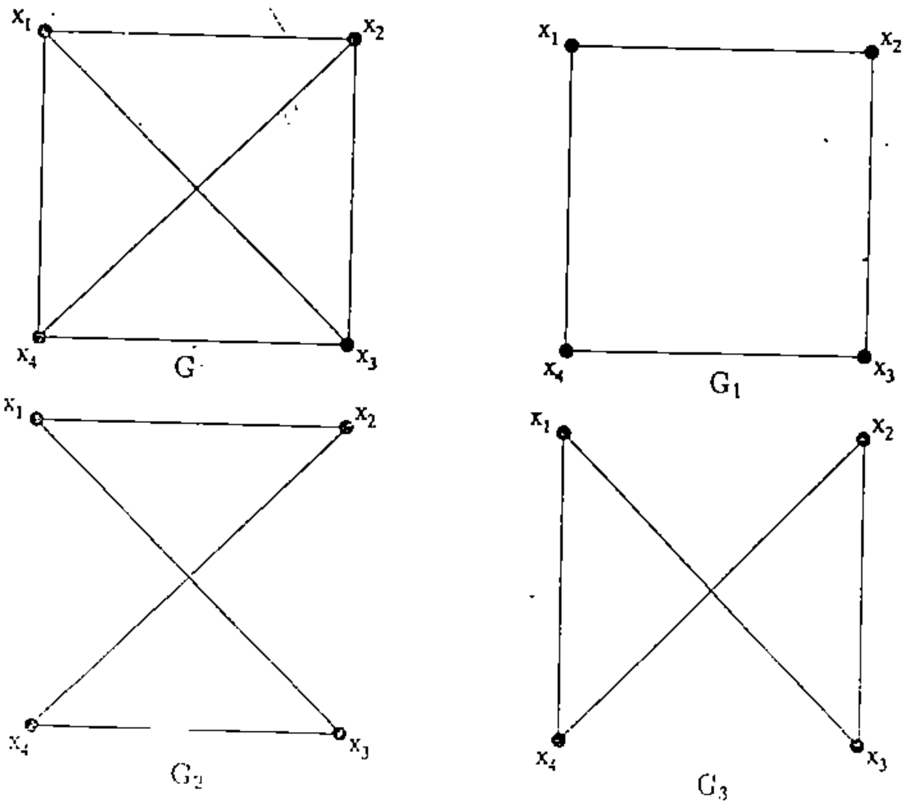


Fig.37

This has following three copies G_1, G_2 and G_3 of C_4 given by

$$\begin{aligned}
V(G_1) &= V(G), \text{ and} \\
E(G_1) &= \{x_1\, x_2, x_2\, x_3, x_3\, x_1, x_4\, x_1\} \\
V(G_2) &= V(G) \text{ and} \\
E(G_2) &= \{x_1\, x_2, x_2\, x_4, x_4\, x_3, x_3\, x_1\} \\
V(G_3) &= V(G) \text{ and} \\
E(G_3) &= \{x_1\, x_3, x_3\, x_2, x_2\, x_4, x_4\, x_1\}
\end{aligned}$$

Thus, in this case G_1, G_2 and G_3 are spanning subgraphs of the graph G.

* * *

Example 25: Consider the Petersen graph G, with the vertex set

$\{x_i : 1 \leq i \leq 5\} \bigcup \{y_j : 1 \leq l \leq 5\}$.(see Fig. 38).

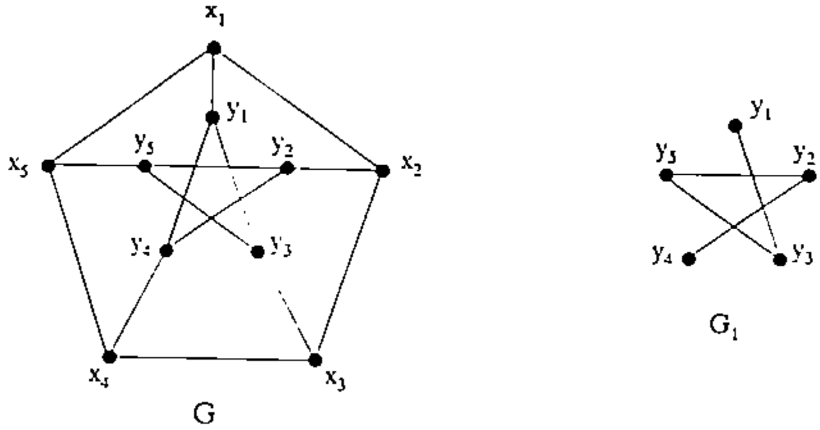


Fig.38

Consider the graph G_1, where
$V(G_1) = \{y_j : 1 \leq j \leq 5\}, E(G_1) = \{y_1y_3, y_3y_5, y_5y_2, y_2y_4\}$.
Here every edge of G_1 is an edge in G. On the other hand $y_4\, y_1$ is an edge in G not in G_1. Thus G_1 is a subgraph of G.

Now consider the graph G_2 as shown below in Fig.39 given by .

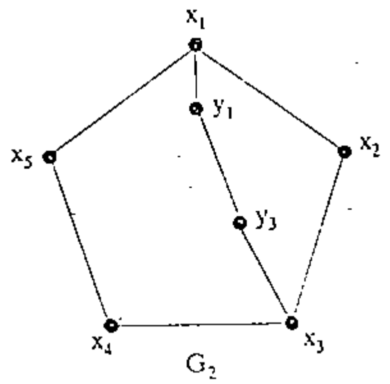


Fig.39

$$V(G_2) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_3\},$$
$$E(G_2) = \{x_1\, y_1, x_1\, x_2, x_1\, x_5, x_2\, x_3, x_3\, x_4, x_4\, x_5, x_3\, y_3, y_1\, y_3\}$$

Clearly G_2 is a subgraph of the graph G. Further, you may note that whenever two vertices of G_2 are joined by an edge in G, that edge belongs to E(G).

This peculiarity of the subgraph G_2 lead to the following definition:

Definition: Let G be a graph and let $S \subseteq V(G)$. By **the vertex induced subgraph** of the graph G, on the set S, we mean the subgraph with vertex set S and the edge set consisting of those edges of G which are joining the vertices of S. That is, edge set $= \{x\,y : x \neq y, x \in S, y \in S, x\,y \in E(G)\}$. We denote this subgraph by $< S >_G$. **It is the subgraph of G induced by S. The two points of S are adjacent in $< S >_G$ if and only if they are adjacent in G.**

The subgraph G_2 in Example 25 is a vertex induced subgraph of the graph G, whereas subgraph G_1 is not.

Note that every graph G is a subgraph of itself, i.e. G is a subgraph of G. Also, for any $v \in V(G)$, $\{v\}$ is a subgraph of G. Further, note that for a vertex $v \in V(G)$, by $G - v$ we mean the subgraph $< V(G) - \{v\} >_G$, which means a subgraph of G consisting of all points of G except v and all lines incident with v. For a subset S of V(G) the subgraph $< V(G) - S >_G$ is often written as $G - S$.

We now illustrate various types of subgraphs.

Example 26:. Consider the graph G as shown in Fig.40 (a). You may note the following subgraphs of the graph G here.
Fig.40(b) shows a subgraph H_1
Fig.40 (c) gives a vertex induced subgraph H_2 with $V(H_2) = V(H_1)$.
Fig. 40(d) shows $H_3 = G - v_4$ and
Fig. 40(e) gives the spanning subgraph H_4.

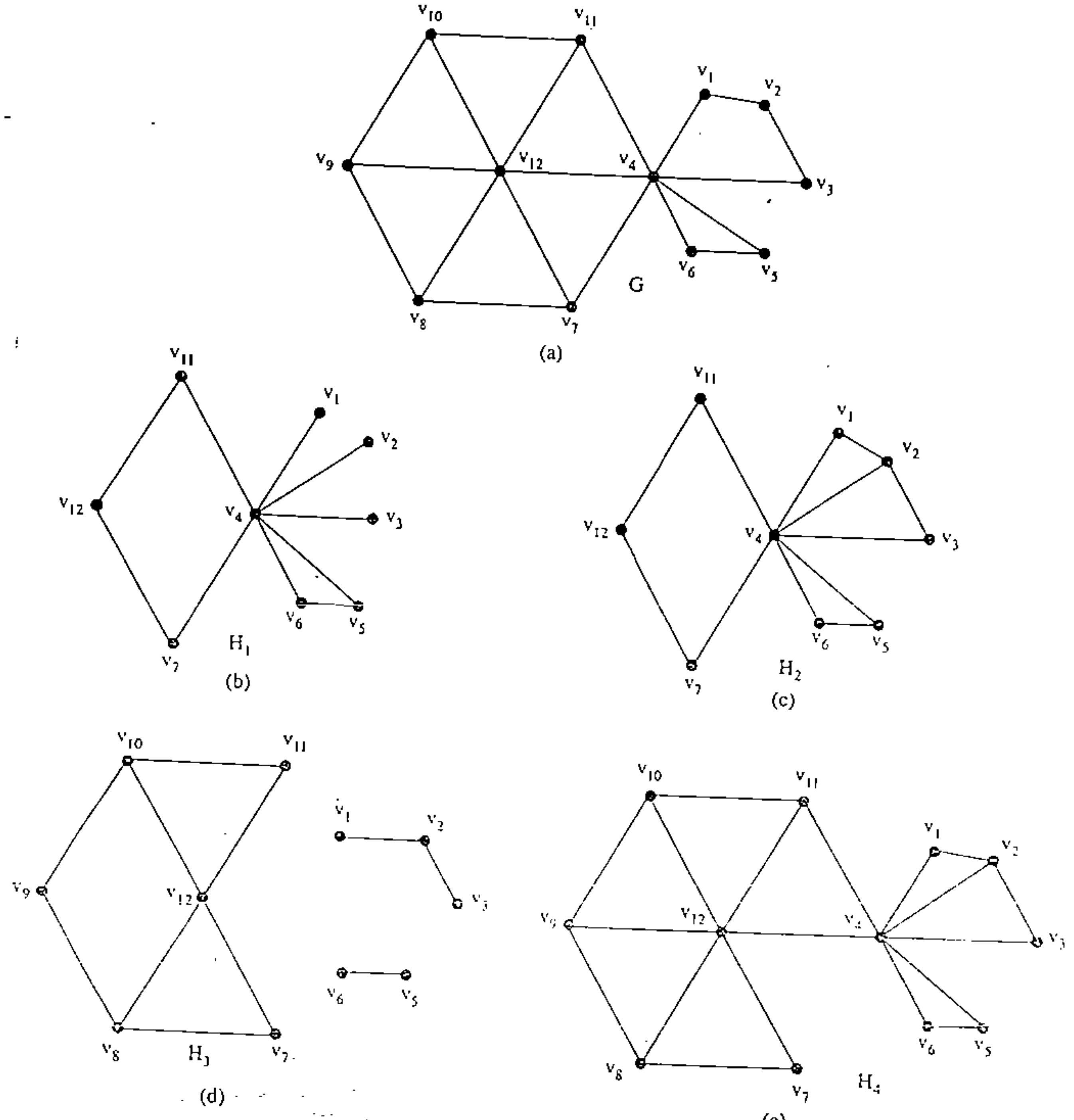


Fig.40

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Example 27: Consider the graph G and a subgraph H of G as in Fig.41.

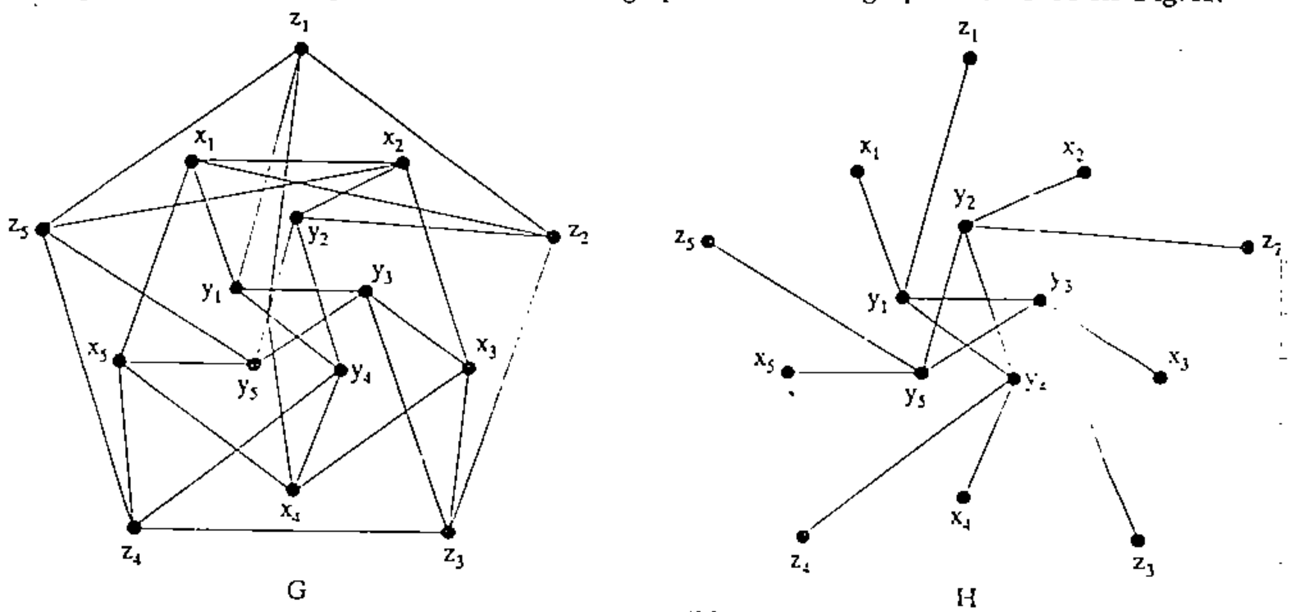


Fig.41

G is a regular graph with degree of regularity 4. But the subgraph H is not regular. However, you may note that $V(H) = V(G)$. Here $\delta(H) = 1 < 4 = \delta(G) = \Delta(G) = \Delta(H)$. Thus, it is clear from this example that a subgraph of a regular graph may or may not be regular.

* * *

Now, a few exercises for you.

E15) Show that for a subgraph H of a graph G, $\Delta(H) \leq \Delta(G)$.

E16) Give an example of a subgraph H of a graph G with $\delta(G) < \delta(H)$ and $\Delta(H) < \Delta(G)$.

E17) Give an example of a subgraph H of a graph G with $\delta(H) < \delta(G)$.

E18) Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k. How many vertices and edges have $G - v$?

We now end this unit by giving a summary of what we have covered here.

10.5 SUMMARY

1) A simple graph G consists of a finite nonempty set V of points together with a prescribed set E of 2 element subests of V.

2) The complete graph K_n is a graph with n vertices such that every vertex is joined to every other vertex by an edge.

3) The path P_n is a graph on n vertices $\{x_1, x_2, \ldots, x_n\}$ in which any two consecutive edges are adjacent and where no edge and no vertex is repeated.

4) Cycle is a circuit in which the only repeated vertex is the first vertex which being the same as the last vertex.

5) Complement of the (p, q) graph G is a $(p, \overline{q})$ graph $\overline{G}$ where $\overline{q}$ = number of pairs of elements of $V - q$.

6) The number of edges incident with a vertex in a graph G gives the degree of the vertex and a graph having the same degree of all its vertices is regular. Also in any graph the sum of the degrees of all its vertices is even.

7) There always exist an r regular graph on p vertices where p, r are integers and at least one of them is even.

8) For a graph $G = (V(G), E(G))$, a graph $H = (V(H), E(H))$ is a subgraph of G whenever $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

9) The subgraphs of a regular graph may or may not be regular.

10.6 SOLUTIONS / ANSWERS.

E1)

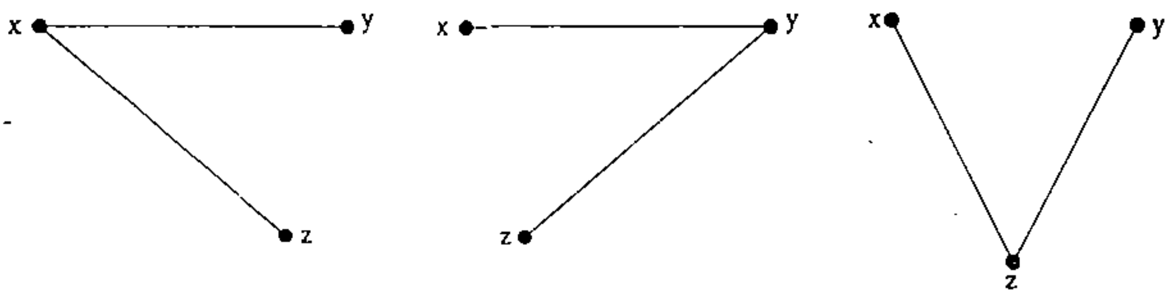


Fig.42

E2)

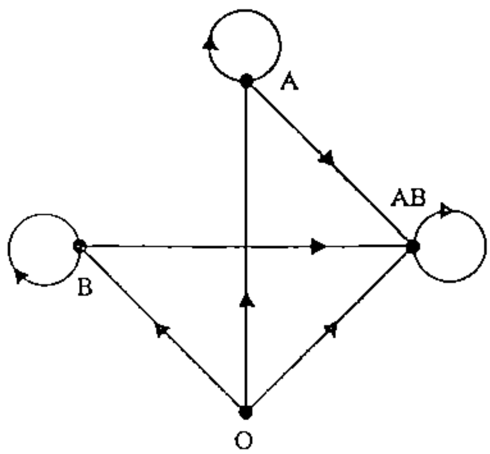


Fig.43

E3) Example 1. $V = \{x_1, x_2\}, E = \{x_1x_2\}$ Path, complete graph.
Example 2. $V = \{x_1, x_2, x_3, x_4\}, E = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ cycle
Example 3. $V = \{x_1, x_2, x_3, x_4\}, E = \{x_1x_2, x_1x_3, x_1x_4\}$

E4) a) The walk u, v, b, c, y, u, v is not a trail.

b) The trail u, b, a, u, v is not a path.

c) The path u, a, b, c, x, v has length 5.

d) The circuit u, y, v, b, x, v, u is not a cycle.

e) The cycle u, b, a, d, c, x, v, y, u has length 8.

E5) $E(\overline{G_1}) = \{u_1u_3, u_1u_4, u_2u_4, u_2u_6, u_3u_5, u_3u_6, u_4u_6, u_5u_6\}$
$E(\overline{G_2}) = \{u_2u_3, u_4u_5\}$
$E(\overline{G_3}) = \{u_1u_3, u_1u_6, u_2u_4, u_2u_5, u_2u_6, u_3u_5, u_4u_5, u_4u_6.\}$

E6) $\overline{G}$ can have $\frac{p(p-1)}{2} - q$ edges.

E7) Example 6, $d(x_i) = 4, 1 \le i \le 5, d(y_i) = 2, 1 \le i \le 7$.
Example 9, $d(x_i) = 2, 1 \le i \le 5$.
Similarly do others.

E8)
$$\begin{aligned} d_{\overline{G}}(x) &= |N_{\overline{G}}(x)| \\ &= |\{y \in V(G) : x\,y \notin E(G)\}| \\ &= |V(G)| - 1 - |N_G(x)| \\ &= p - 1 - d_G(x). \end{aligned}$$

E9) (1) 1, 1. (9) 2, 2

(2) 2, 2 (11) 1, 7

(3) 1, 3. 12) 3, 4.

E10) b) Graph has 3 vertices of odd degree, contradiction to corollary-1 of Theorem 1.

e) Sum of degree of all vertices of a graph is odd contradiction to theorem 1.

E11) $kp_k + (k+1)p_{k+1} = 2q$ (Using Theorem 1)

$$\begin{aligned} \text{Also, } p_k + p_{k+1} &= p \\ \text{Therefore, } kp_k + (k+1)(p - p_k) &= 2q \\ \text{or } p_k &= (k+1)p - 2q \end{aligned}$$

E12) Here $p = 10, r = 5$. So $\frac{r-1}{2}$ is an integer. Take 10 vertices $\{x_1, x_2, \ldots, x_{10}\}$. Join x_i to x_{i+1} for $1 \le i \le 9$. Join x_{10} to x_1. Now all the vertices have acquired degree $\frac{r-1}{2} = 2$. Join x_i to x_{i+2} or $1 \le i \le 8$. Join x_9 to x_1 and x_{10} to x_2. We now have 4 regular graph.
Here $\frac{p}{2} = n = 5$.
Thus to obtain 5 regular graph join
x_i to x_{i+5} for $1 \le i \le 5$ (see Fig. 44).

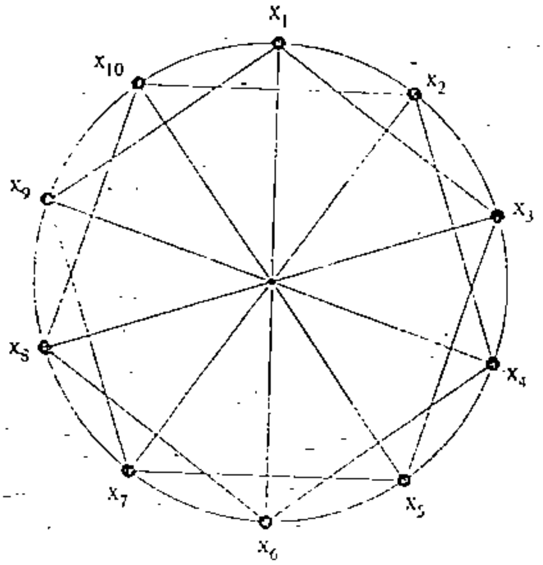


Fig.44

When $r = 6, s = 3$. Arrange 10 vertices in a circular manner. Join each of the vertices $v_1, v_2, \ldots, v_7$ to the next 3 vertices in an ascending manner. Also add the edges $\{v_8v_9, v_{810}, v_8v_1\}$, $\{v_9v_{10}, v_9v_1, v_9v_2\}$,

$\{v_{10}v_1, v_{10}v_2, v_{10}v_3\}$. You get 6-regular graph as shown in Fig.45.

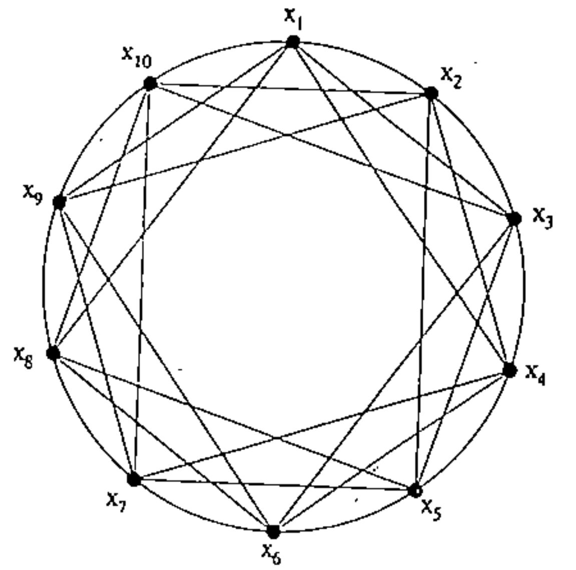


Fig.45

In Fig.45 if you add the edge $v_iv_{i+5}, 1 \leq i \leq 5$. You get a 7-regular graph on 10 vertices.

E13) $p = 4$ then $q = 4_{c_2} = 6$. So we want $(4, q)$-graph, with $0 \leq q \leq 6$. We are giving here in Fig. 46 all possible non-isomorphic graphs on four vertices.

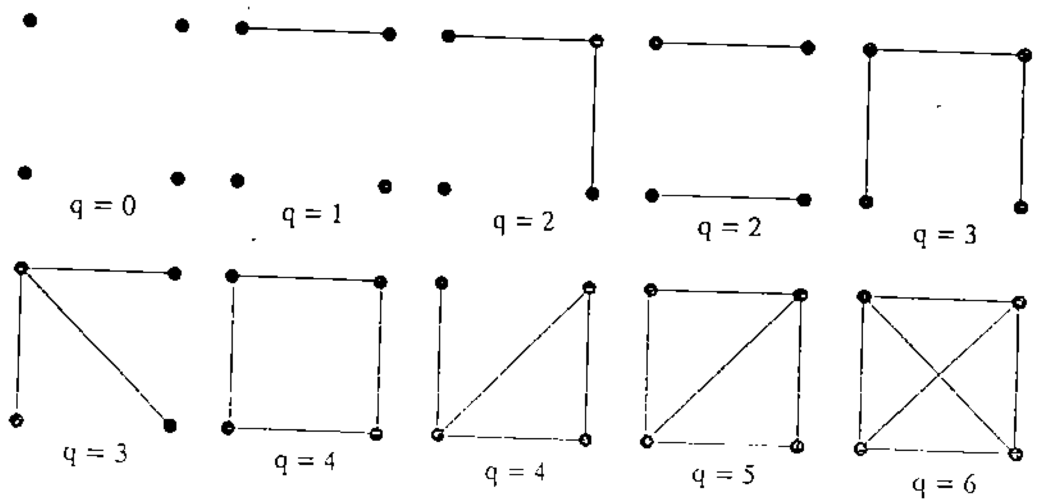


Fig.46

E14) Suppose G is a (p, q)−graph. Then

$$E(G) \bigcup E(\overline{G}) = \{\text{the set of all pairs of vertices in} V(G)\}.$$

Thus $q + \overline{q} = \dfrac{p(p-1)}{2}$. If the graph G is self complementary then $q = \overline{q}$. Thus, $p(p-1) = 2q + 2\overline{q} = 4q$, that is 4 divides $p(p-1)$. Since only one of p or $(p-1)$ is even, this means either p or $(p-1)$ is divisible by 4.

E15) Let $x \in V(H)$ such that $d_H(x) = \Delta(H)$. Then, $N_H(x) \subseteq N_G(x)$. Thus, $\Delta(H) = |N_H(x)| \leq |N_G(x)| \leq \Delta(G)$.

E16)

$$\begin{aligned} \delta(G) &= 1 < 2 = \delta(H) \\ \Delta(H) &= 2 < 3 = \Delta(G) \end{aligned}$$

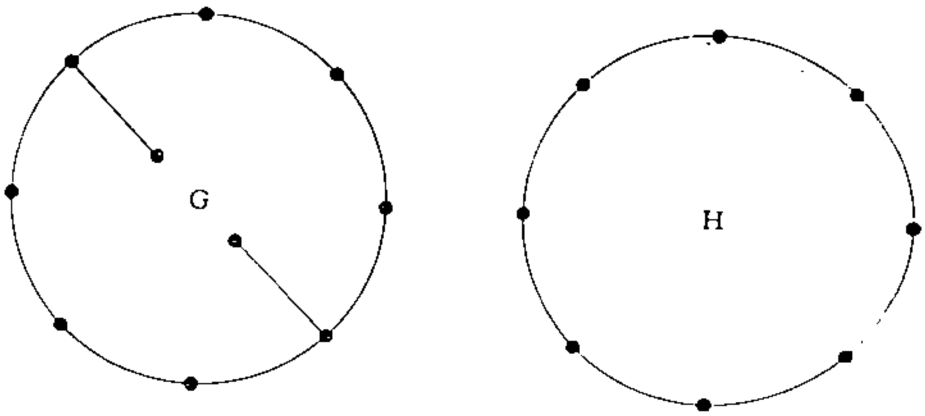


Fig.47

E17) $\delta(H) = 1 < 2 = \delta(G)$

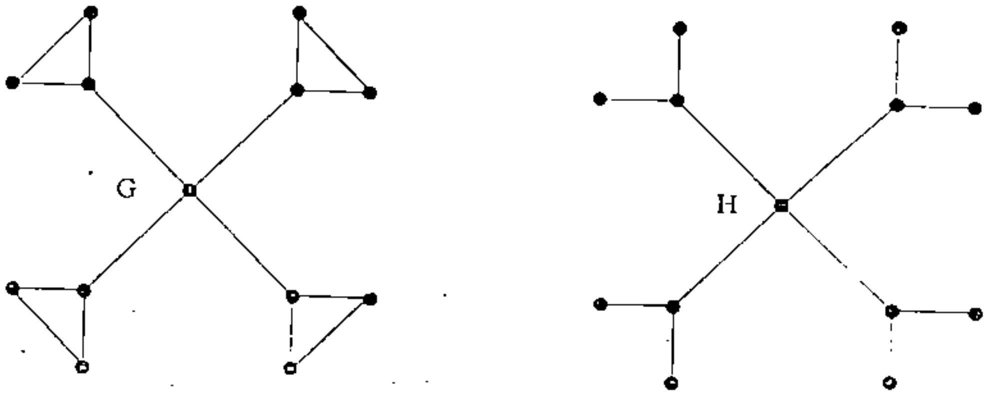


Fig.48

E18) $G - v$ will have $(n - 1)$ vertices and $m - k$ edges.

UNIT 11 SPECIAL GRAPHS

Structure **Page No.**

11.1 INTRODUCTION.

In the last unit you saw that graphs are often used to represent (that is, model) communication or transportation networks and several other systems such as representation of a molecule in a chemical compound and so on. In a transportation network, it is necessary to know which destinations are connected by a direct route. For example, if air travel is abolished then the people without any seaport cannot go to any other country unless their neighbours provide the initial road passage through their territory. When we use a graph to model this situation, it is important that there be a way to connect from any vertex to any other vertex. Such graphs are called connected graphs. In Sec.2.2 we will define connected graphs and we will show that any graph can be partitioned into connected graphs.

In Sec.2.3, we will familiarise you with a type of graph which is useful in electronics and other areas. These graphs are called bipartite graphs. Such graphs are very useful in studying real-life problems, for example in modelling neural network.

In Sec. 2.3 we have considered another type of graphs called 'Trees'. The graphs which represent chemical compounds butane and isobutane are trees. You are familiar with these graphs in Unit 10. Such graphs are of special interest to chemists. They want to find out whether any tree correspond to a chemical compound or not. Here we will show that a tree has got several interesting properties and these properties are used in studying some real-life situations.

Objectives

After reading this unit, you should be able to

- distinguish between walks, paths, circuits and cycles in a graph
- identify
  1) connected graphs
  2) bipartite graphs
  3) trees

11.2 CONNECTED GRAPHS

From Unit 10, you know that graphs are model for different real life situations especially situations involving routes; the vertices represent towns or junctions and each edge represents a road or some other form of communication link. This kind of a picture is very helpful in understanding connected graphs that we introduce in this section. To understand such graphs we need some definitions which describe ways of "going from one vertex to another". We shall first give these definitions in the following subsection.

11.2.1 Paths, Circuits And Cycles

Consider the graph in Fig.1. Imagine yourself walking along its the edges, going from vertex to vertex.

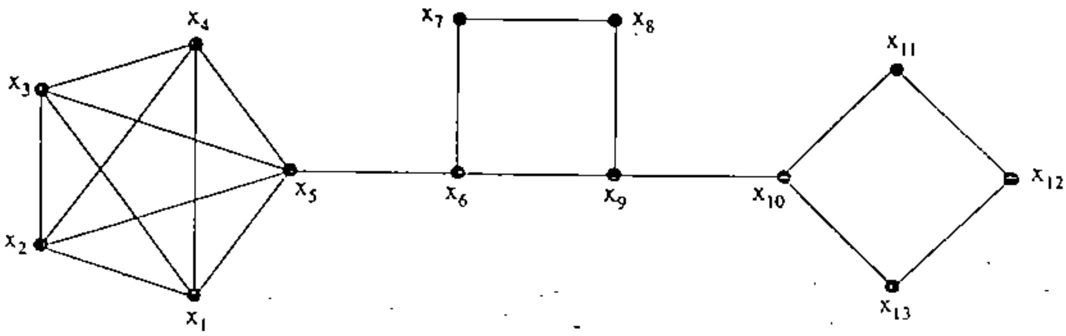


Fig.1

Suppose we want to start at the vertex x_1 and reach the vertex x_{12}. Is this possible? One possible way is to start from vertex x_1, walk along the edge x_1x_2, reach x_2, walk along the edge x_2x_3, reach x_3, walk along x_3x_4 reach x_4, continue this till we reach x_{12}. Suppose we denote the edge joining x_{i-1} and x_i as $(x_{i-1}x_i)$. Then we can describe this walk in an alternating sequence of vertices and edges as x_1, (x_1x_2), x_2, (x_2x_3), x_3, (x_3x_4), x_4, (x_4x_5), x_5, (x_5x_6), x_6, (x_6x_9), x_9, (x_9x_{10}), x_{10}, $(x_{10}\,x_{11})$, x_{11}, $(x_{11}x_{10})$, x_{10}, $(x_{10}x_{13})$, x_{13}, $(x_{13}x_{12})$, x_{12}. What does this represent? You recall from Unit 10 that this represents a walk. This is by no means the shortest way to reach x_{12} from x_1. We could have gone from x_1 to x_5 directly. Moreover, we passed through the vertex x_{10} twice. This is not necessary. So the above walk can be described as a leisurely walk. If we have more time at our disposal, we can trace and retrace more edges. For example, we could have gone from x_6 to x_9 and again back to x_6.

So what are we doing when choosing a walk? We are, in fact, choosing a sequence whose elements are vertices and edges, alternately.

Now we formally define a walk.

Definition: A walk in a graph G is a finite sequence $W = \{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$, where $v_0, v_1, \ldots, v_k$ are vertices and $e_1, e_2, \ldots, e_k$ are edges joining the vertices v_{i-1} and v_i, $1 < i < k$. Note that all the v_is or e_is may not be distinct. There may be repetition.

In this case we say that W is a walk from v_0 to v_k or W is a v_0-v_k walk or W is a walk joining v_0 and v_k. The vertex v_0 is called the **initial vertex** and the vertex v_k is called the **end vertex** of the walk W. The integer k which is the number of edges contained in a walk is called the length of the walk W and is denoted by l(W). Since the vertices as well as the edges can be repeated, the length can very well be greater than the number of the edges of the graph G.

Note: As you have seen, in a walk the vertices as well as edges can be repeated. So we cannot view this as a subgraph unless all the vertices as well as the edges in the walk are distinct.

Let's consider an example .

Example 1: Consider the graph on 5 vertices and 7 edges given in Fig. 2. Find a x_1-x_5 walk of length 8.

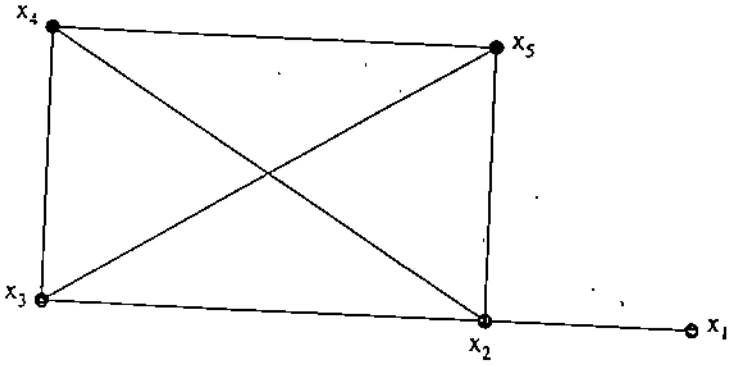


Fig.2

Solution: Consider the walk $W = \{x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_2, x_2, x_2 x_5, x_5, x_5 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5\}$. Then W is x_1-x_5 walk of length 8.

* * *

Another possible walk for the same graph could be $\{x_1, x_1x_2, x_2, x_2x_4, x_4, x_4x_3, x_3, x_3x_5\}$. Its length is $l(W) = 4$.

Why don't you try this exercise now?

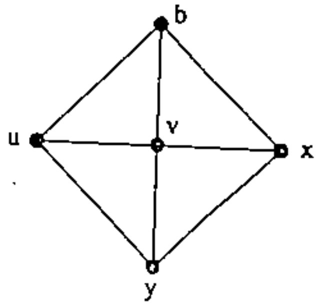


Fig.3

E1) For the graph given in Fig.3, find a u-v walk of length 7.

Since we are considering only graphs which do not have multiple edges or loops, we often write a walk W as $\{v_0, v_1 \ldots, v_k\}$. While doing so we assume that two consecutive vertices in a walk are joined by an edge in the graph and that edge is included in the walk. For example, the walk corresponding to Fig.1 can be written as

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{10}, x_{13}, x_{12}\}.$$

The concept of a walk is too general for our purposes, so we shall impose some further restrictions. Before that let us consider the graph given in Fig.4.

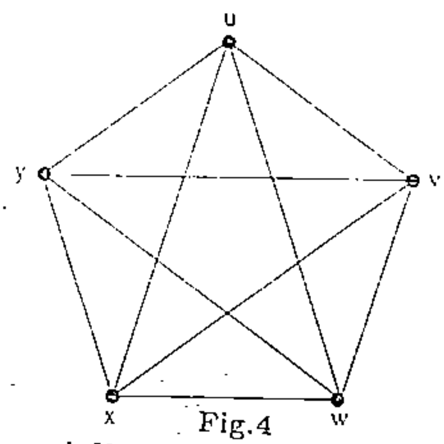


Fig.4

This is the complete graph K_5. Then

$W = \{u, v, x, w, v, x, y\}$, $W_1 = \{u, x, w, v, x, y\}$ and $W_2 = \{u, x, w, v, y\}$ are three walks joining u and y having lengths 6, 5 and 4, respectively. Also note that

i) in the walk W, vertices v and x as well as the edge v x are repeated,

ii) in the walk W_1, only the vertex x is repeated but no edge is repeated and

iii) in the walk W_2, neither a vertex nor an edge is repeated.

The walks W, W_1 and W_2, corresponding to Fig.4, are given special names according to the definition given below.

Definition: A walk is called a **trail** if all the edges in it are distinct. For example W_1 corresponding to Fig. 4 is a trail. Note that in a trail vertices can be repeated. A walk W is called a **path** if all the vertices are distinct. For example W_2 corresponding to Fig. 4 is a path.

If all the vertices in a walk are distinct, can edges repeat? Remember that an edge is traced only after tracing an end vertex. Therefore, all the edges of are also distinct. Hence, a path is always a trail. What about the converse? We leave it as an exercise for you. (see E2)

Next we shall give some more definitions.

Definitions: A walk u - v is **closed** if u = v and **open** if $u \neq v$.
A closed trail is called a **circuit**.
A circuit in which the only repeated vertex is the first vertex, this being the same as the last vertex is called a **cycle**.
Let us consider an example.

Example 2: Consider the following graph on 15 vertices.

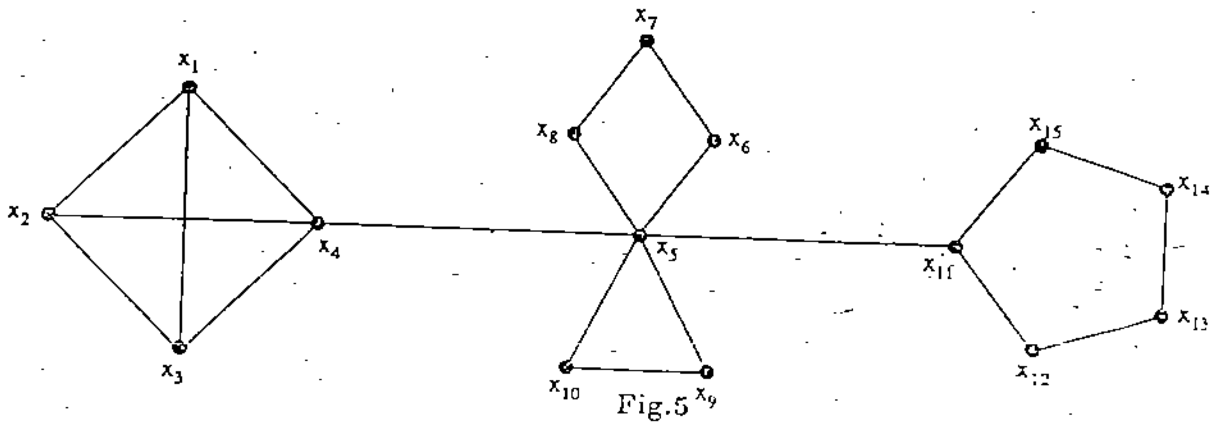


Fig.5

In this graph find the following:

i) a closed walk which is not a circuit,

ii) a circuit which is not a cycle,

iii) a cycle.

Solution: We shall find (i), (ii) and (iii) one by one.

i) There are several closed walks in it, which are not circuits.
$W = \{x_5, x_6, x_7, x_8, x_5, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{11}, x_5\}$ is a closed walk Here the edge $x_5\, x_{11}$ is repeated. Hence it is not a circuit.

ii) $W_0 = \{x_5, x_6, x_7, x_8, x_5, x_9, x_{10}, x_5\}$ is a circuit. Here the vertex x_5 is repeated three times. Thus, this is not a cycle.

iii) $W' = \{x_5, x_6, x_7, x_8, x_5\}$ is a cycle.

* * *

Try these exercises now.

E2) i) Is every trail a path? Give reasons for your answer.

ii) If all edges are distinct, then all vertices are distinct. True or false? Why?

E3) Is a cycle a path? Give reasons for your answer.

E4) Let $G = (V, E)$ be a graph where
$V = \{t, u, v, w, x, y, z\}$ and
$E = \{tu, tv, tw, ux, vw, vy, uz, wx, wz, xy, xz\}$. In G, find

i) a u-v trail that is not a path,

ii) a (u-u) circuit that is not a cycle,

iii) a (v-v) cycle of minimum length.

E5) Let G be a graph such that $\delta(G) \geq k$. Use the principle of induction to to show that the graph G has a path of length k starting at any given vertex. (Recall that $\delta(G) = \min\{d_G(x) : x \in V(G)\}$.

Now let us go back to the graph given in Fig. 4 again. In this graph $W = \{u, v, x, w, v, x, y\}$ is a walk. Suppose we omit the part $\{w, v\}$ we get $P = \{u, v, x, y\}$. You know that this object is a path. In the next theorem, we will prove that this phenomenon is true in general.

Theorem 1: If W is a u-v walk joining two distinct vertices u and v, then there is a path joining u and v contained in the walk.

Proof: Let W be a u-v walk given by
$W = \{u = u_0, e_1, u_1, \cdots, e_k, u_k = v\}$
We will find a path joining u and v contained in W, using the principle of mathematical induction.

Let p(k) denote the statement that if W is a u-v walk of length k, then there exists a path joining u and v contained in W.

If $k = 1$, then p(1) is true since every walk of length 1 is a path.

Now we assume that the statement $p(k-1)$ is true for all walks of length $\leq k - 1$. In other words, we assume that given any x-y walk of length $\leq k - 1$, there exists a path joining x and y contained in the walk. Then we want to show that the statement p(k) is true for W.

If W is already a path, we are done. Otherwise, there is at least one vertex which is repeated. Suppose j is the smallest integer such that the vertex u_j is repeated. Then there is an integer $t > j$ such that $u_j = u_t$. Now consider the walk W_1 obtained by removing the part $\{e_{j+1}, \cdots, e_t\}$, that is $W_1 = \{u = u_0, e_1, \cdots, u_j = u_t, e_{t+1}, \cdots e_k, u_k = v\}$. Clearly W_1 is a u-v walk contained in the walk W and its length $l(W_1) = k - t + j < k$, since $j < t$. Hence, by induction, we can get a path P joining u and v contained in W_1. Since P is contained in the walk W_1 and W_1 is contained in the walk, the path P is contained in the walk W. Thus, p(k) is true for W.

Therefore by induction, p(n) is true for all n. Hence the result

The Theorem above says that, if there is a walk joining two vertices in a graph, then we can always find a path joining them.

Now in many of the practical situations it is very important to know which of the vertices in a graph can be joined by a walk, and hence by a path. For instance, in the graph G, obtained by taking union of K_6 and K_5 (see Fig.6).

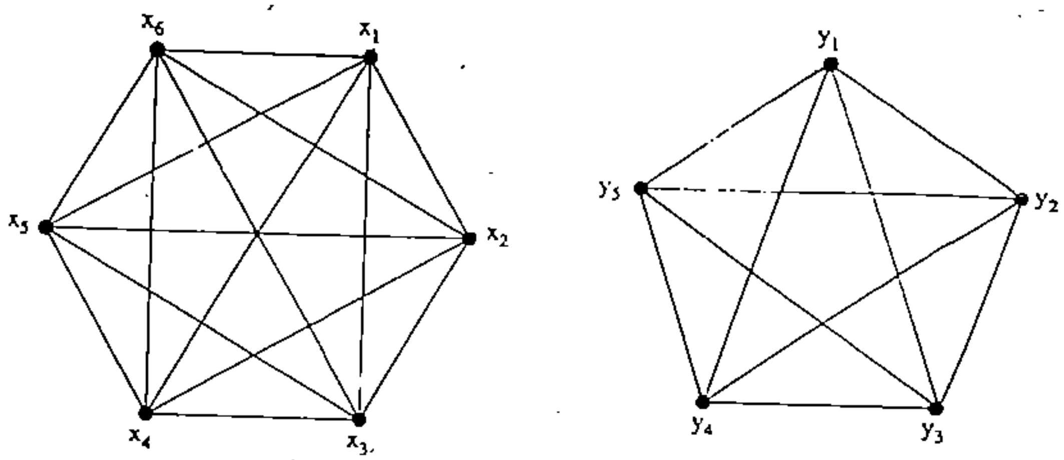


Fig.6

You can see that there is no (x_1-y_5) walk. Hence there is no way we can traverse from the vertex x_1 to the vertex y_5.

So, in the internal structure of a graph it sometimes matters whether two vertices are joined by a walk or not. This leads us to the definition of a connected graph which we will introduce in the next subsection.

11.2.2 Components

As you have noticed, almost all graphs we have discussed so far have been 'in one piece'. The exceptions are null graphs and the union of graphs each of which is in one-piece. We can formalise this difference by introducing the concept of connectedness which we shall define in this sub-section. We are not only interested in the main graphs being connected, but we are also interested in knowing the subgraphs which are connected, which are known as components. Here we shall discuss them in detail.

A graph, whose edge-set is empty is called a null graph.

Definition: A graph $G = (V, E)$ is called **connected** if for any two vertices $u, v \in V$, there exists a u-v walk in G. If G is not connected, then it is called **disconnected**.

This means that in a connected graph any two distinct vertices are joined by a walk. From Fig. 6 you can see that both the graphs K_6 and K_5 are connected, but their union is not connected since there is no walk connecting the vertices of K_6 to the vertices of K_5

Here are some exercises for you.

E6) Can a graph with one vertex be connected? Give reasons for your answer.

E7) Which of the graphs given in Fig.7 are connected?

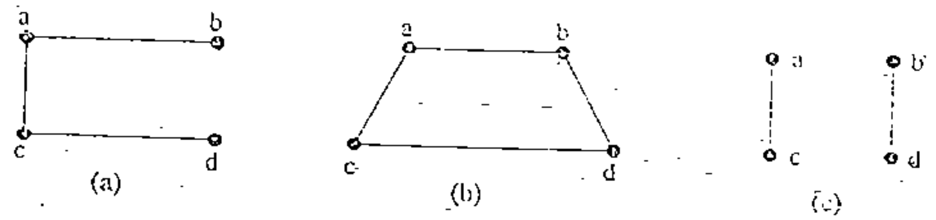


Fig.7

E8) If a graph G is connected, then all its subgraphs are connected. Prove or disprove this statement.

While solving E8, you would have realised that subgraphs of connected graphs need not be connected. But what about disconnected graphs? You can see that some subgraphs of such graphs are connected. Let us discuss them now.

Definition: Let $G = (V, E)$ be a graph A subgraph H of G is called a **component**.

i) if H is connected and it is not a subgraph of any other connected subgraph of G; and

ii) whenever K is a connected subgraph of G, and H is contained in K, then $H = K$.

Thus, a component is, in a sense, a 'maximal' connected subgraph of G. The number of components of G is denoted by $c(G)$.

Now, consider the graph G given by Fig.6. You can see that K_6 and K_5 are its components, and G is the union of these components.

Let us consider another example.

Example 3: Consider the graph G given by Fig.8. Find three components of this graph.

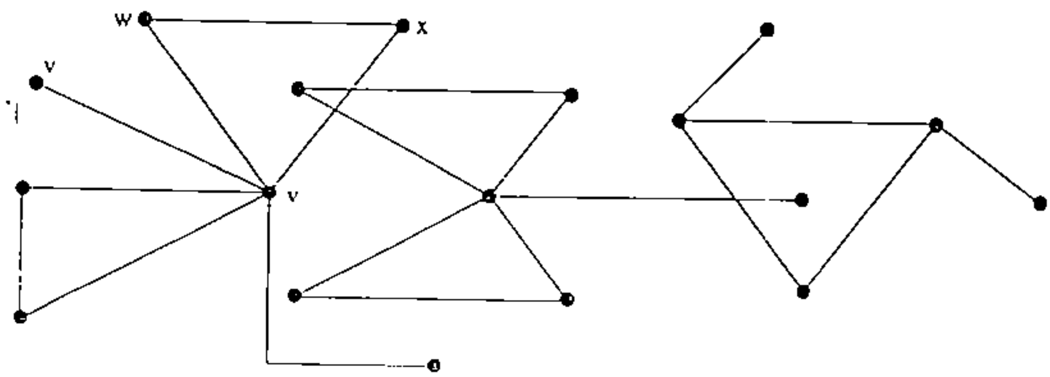


Fig.8

Solution: The three components of G are G_1, G_2 and G_3 (given in Fig.9(a), (b) and (c)).

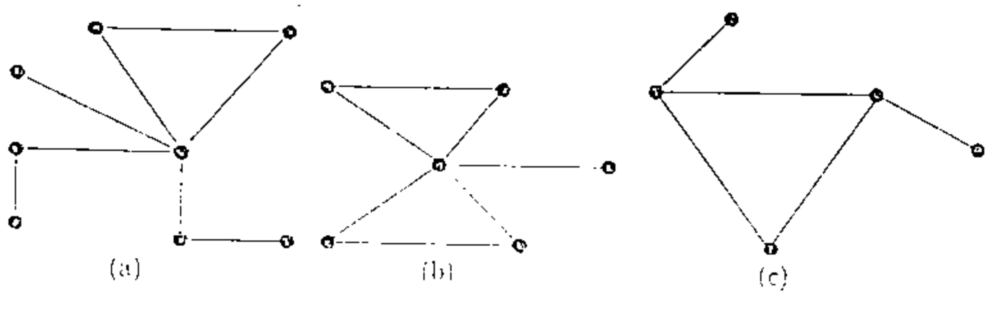


Fig.9

* * *

Here also G is the disjoint union of components G_1, G_2 and G_3.

You can now try this exercise.

E9) Consider the graph G given by Fig.10. Then find

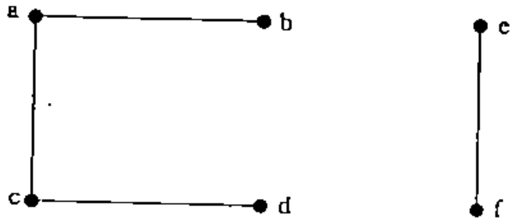
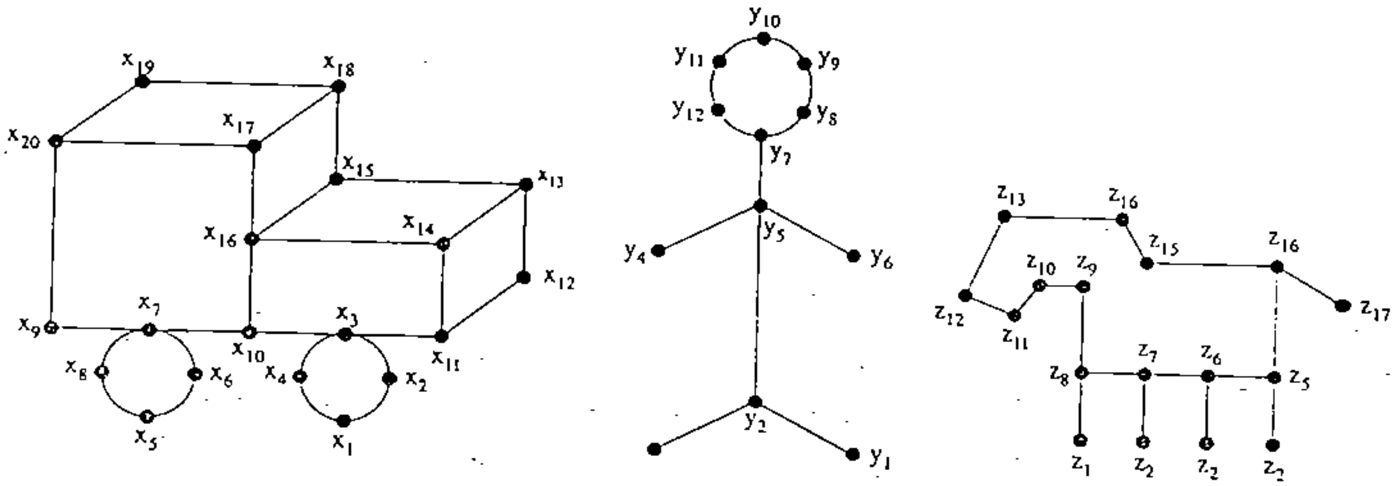


Fig.10

i) all the connected subgraphs of G.

ii) all the components of G. Are they disjoint? Give reasons for your answer.

E10) Consider the graph given in Fig.11. Show that the graph can be written as the disjoint union of its components.



(a) (b) (c)

Fig.11

In the case of Example 3 and E10, we saw that the graphs given in each case can be written as a disjoint union of the corresponding components. This phenomenon generalises to any graph as you will see in the following theorem. We shall only state the theorem; the proof of which, though it is not very difficult, is omitted.

Theorem 2: Every graph can be partitioned into components.

Now that we know each graph can be partitioned into components, we shall find something more about components in a graph. One direction of interest is to investigate bounds for the number of edges of a graph on n vertices with a given number of components. We shall now state a general result which gives the required bound as a special case.

Theorem 3: If G is a graph with n vertices and has k components,

$$n - k \leq n \leq \frac{1}{2}(n - k)(n - k + 1)$$

Note: If G is connected, then $k = 1$ and we get the bounds as

$$n - 1 < n \leq \frac{1}{2}n(n - 1)$$

Another approach used in the study of connected graphs is to ask the question 'how connected is a connected graph?'. One possible interpretation of this question is to ask how many edges or vertices must be removed from the graph in order to disconnect it. We shall discuss this in the next subsection.

11.2.3 Connectivity

Let us now consider the graph showing an electric circuit (see Fig.12). This graph is connected. Suppose we break the wire connecting d and e in the electric circuit. This means that in the graph showing the circuit, we are actually removing an edge corresponding to the wire. Now when we break this wire, the circuit becomes disconnected. This means that the removal of that edge in the graph makes the graph disconnected.

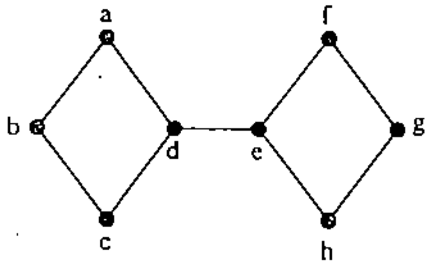


Fig.12

Note: Whenever we talk about removing an edge say xy, we mean, **removing only the connection between x and y, that is the edge not the incident vertices x and y.**

When we remove an edge uv from the graph, we denote the resulting graph by G − uv.

We just saw a situation in which the removal one edge disconnects the graph. But this is not always the case. For instance if we remove the edge ab in Fig. 12, the resulting graph is not disconnected. You can also see this situation in the graph given in Fig. 13, which represents the roads connecting the

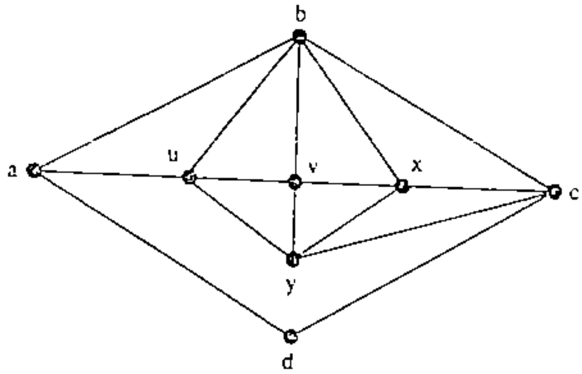


Fig.13

main towns in a state. In this case the removal of any single edge will not disconnect the graph since there always exist alternate connections

These types of edges lead us to the following definition

Definition An edge e of a graph G is called a **bridge** in G if the removal of e disconnects G.

For example, the edge uv in the graph given by Fig.12, is a bridge.

Here are some related exercises for you.

E11) Find the bridges in each of the graphs in Fig.14.

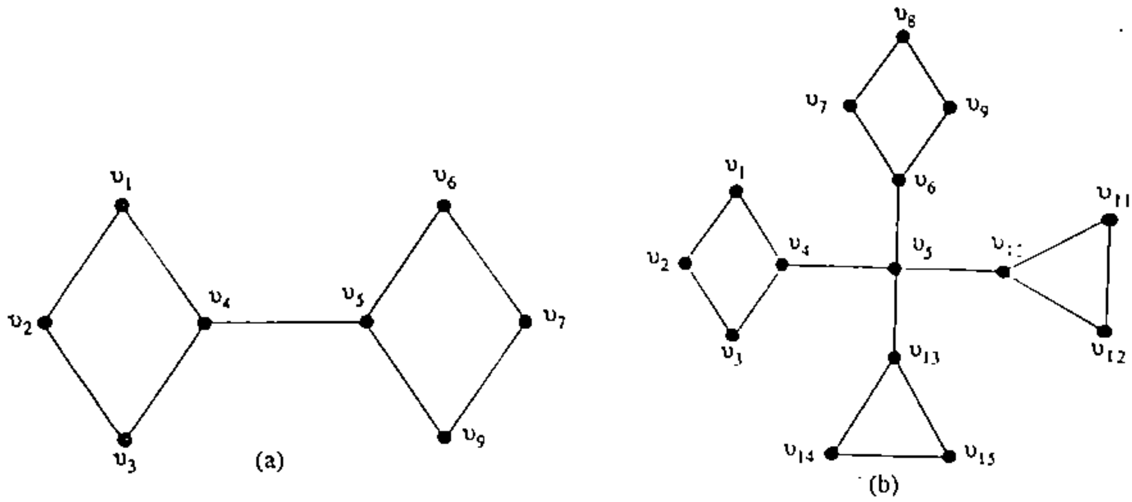


Fig.14

E12) Give an example of a graph without a bridge.

Let us consider another graph given by Fig.15.

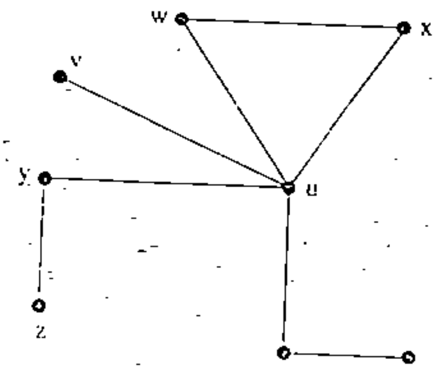


Fig.15

This graph is connected. Here, if we remove the edge uv, then the resulting graph gets disconnected, the components being $\{v\}$ and $G/\ \{v\}$. The number of components of the resulting graph $(G - uv)$ is 2. On the other hand, if we remove the edge uw, then the graph does not get disconnected. Note that the edge uw belongs to the cycle $\{u, w, x, u\}$, but the edge uv does not belong to any such cycle. The cycle seems to provide an alternate connection between the vertices u and w.

In fact it follows from the definition of a bridge that an edge e of a graph G is a bridge if and only if e does not belong to any cycle of G.

While doing the exercise E11, you must have obtained a graph which does not have a bridge. We cannot disconnect such a graph by removing just one edge; we need to remove more than one edge to disconnect it. Therefore, given a graph, it is natural to ask 'what is the minimum number of edges whose removal disconnects G?'. This number is given a special name according to the following definition.

Definition: The edge-connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G.

For example, the edge-connectivity of the graph given in Fig.14 is 1. In fact the edge-connectivity of any graph with a bridge is 1.

Let us consider an example.

Example 4: Find the edge-connectivity of the graph G given in Fig.16.

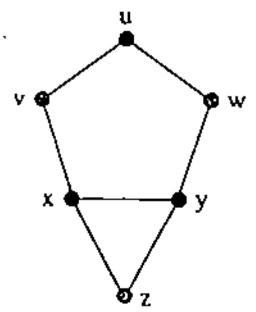


Fig.16

Solution: First note that this graph does not have any bridges. Therefore its edge-connectivity is more than 1. Now, if we remove the edges xz, zy, then the graph gets disconnected. Similarly there are other sets of two edges, namely {xv, vu}, and {uw, wy}, whose removal disconnects G. Therefore we get that the edge connectivity is 2.

* * *

Why don't you try this exercise now?

E13) Find the edge connectivity of

i) the graph given in Fig.15;

ii) the following graphs.

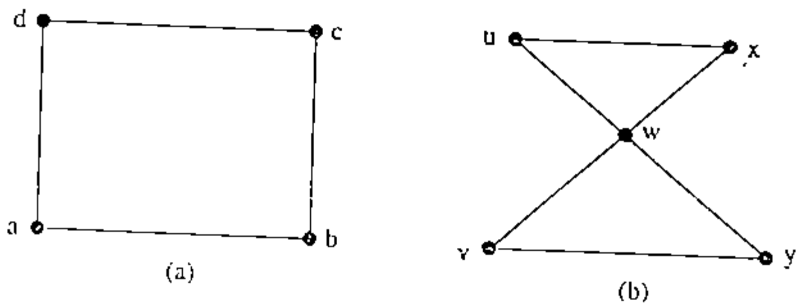


Fig.17

Let us now look at a set of edges of a connected graph.

Definition: A **cut set** S of a connected graph G is a set S of edges with the following properties:

i) the removal of all the edges in S disconnects G;

ii) the removal of any proper subset of S will not disconnect G.

For example consider the following graph given in Fig.18

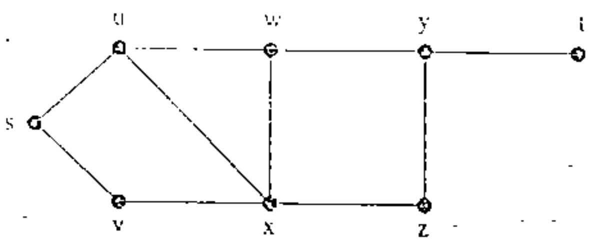


Fig.18

The set {uw, ux, vx} and {uw, wx, xz} are cut sets for this graph; whereas the set {uw, wx, xz, yz} is not a cut set since this set has a subset {uw, wx, xz} whose removal disconnects G.

Note that two cut sets of a graph need not have the same number of edges. For example, in the above graph in Fig.18, the sets {uw, ux, vx} and {wy, xz} are both cutsets.

Also note that the edge connectivity $\lambda(G)$ of a graph G is the size of the smallest cutset of G.

Try this exercises now.

E14) Which of the following sets of edges are cutsets of the graph given in Fig.19. and what is its edge-connectivity?

and

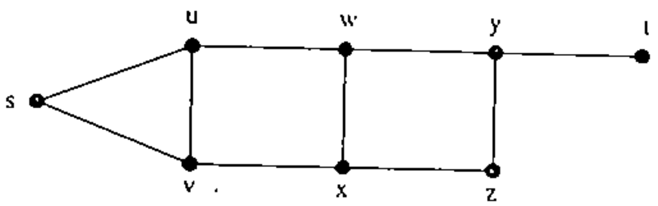


Fig.19

a) {su, sv}

b) {uv, wx, yz}

c) {ux, vx, wx, yz}

d) {yt}

e) {wx, xz, yz}

f) {uw, wx, wy}

We can also think of connectivity in terms of the minimum number of vertices which need to be removed in order to disconnect a graph. Note that when we remove a vertex, then if there is any edge incident with that vertex, that also gets removed. Let us see some examples. Let us consider the graphs given in Fig.17. Graph 17(b) can be disconnected by removing just one vertex w. But Graph 17(a) cannot be disconnected by removing one single vertex, but the removal of two non-adjacent vertices (such as a and c) disconnects it.

Now we can define vertex-connectivity and vertex-cut-set on similar lines as we have done for edges. Why don't you try it for yourself (see E 15).

E15) How would you define vertex-connectivity and cut vertex-set ?

E16) Find the vertex-connectivity and a cut vertex-set for the graph given in Fig.17 (b).

In the next section we shall introduce you to another type of graphs known as bipartite graphs.

11.3 BIPARTITE GRAPHS

In this section we shall define bipartite graphs and explain their importance through various problems.

Let us first start with the following problem.

Four workmen x_1, x_2, x_3 and x_4 are available to fill five jobs y_1, y_2, y_3, y_4 and y_5. x_1 is qualified for the jobs y_1 and y_2; x_2 is qualified for the jobs y_1 and y_3. x_3 is qualified for the job y_4; and x_4 is qualified for the jobs y_2, y_3 and y_5. The assignment problem is concerned with the following questions:

i) Can each person be assigned to a single job for which he is qualified?

ii) If so, how should the assignment be made?

iii) If not, at most, how many of them can be assigned?

The problem of the kind stated above is known as assignment problem. To solve this problem it is convenient to consider the following graph theoretic model of the situation.

The graph G has vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ and y_5 and edges defined in the following way: there is an edge joining x_i and y_j if x_i is qualified for the job y_j. The graph is shown in Fig.20.

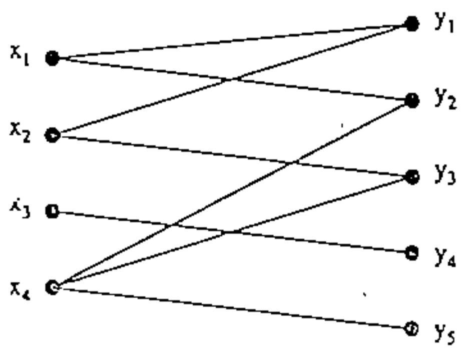


Fig.20

Then the problem of assigning people to jobs for which they are qualified is equivalent to the problem of selecting a subset of the set of edges such that each x will be connected to exactly one y by one of these edges.

Now, if you look at the graph given in Fig.20, you will see that the set of its vertices can be divided into two disjoint subsets such that no two vertices in a subset are adjacent. Let us formally define such graphs.

Definition : A graph G is said to be **bipartite** if $V(G) = X \cup Y$, where X and Y are non-empty subsets such that $X \cap Y = \phi$ and every edge in E(G) has one end vertex in the set X and the other end vertex in the set Y. The sets X, Y form a **partition** of the set V(G) and we often say that $X \cup Y$ is a **bipartition** of the graph G.

An alternative way of thinking of a bipartite graph is in terms of colouring its vertices with two colours, say red and blue - a graph is bipartite if we can colour each vertex red or blue in such a way that every edge has a red end and a blue end.

Bipartite graphs are useful in studying various real-life problems as for example, for modelling neural networks. Many different kinds of models have been formed for studying the neural networks. One such model that emulates the essential working of the network using graph theory is given in Fig.21. As you can see that this is a bipartite graph and the properties of bipartite graphs are used in studying this model.

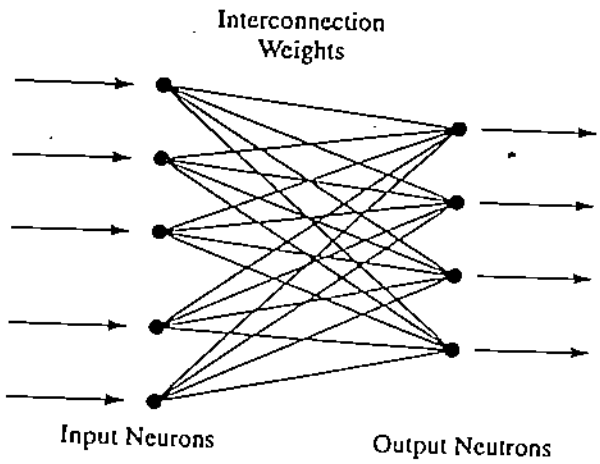


Fig.21

Given a bipartite graph, you may wonder if the bipartition is unique. The following example will give you an answer to this question.

Example 5: Consider the graph given in Fig.22. Find two different partitions that make G bipartite.

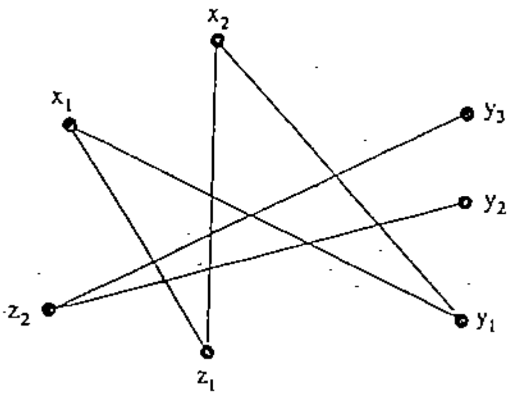


Fig.22

Solution: The vertex set is $\{x_1, x_2, y_1, y_2, y_3, z_1, z_2\}$. One way of partitioning this can be by taking $X = \{x_1, x_2, z_2\}$, $Y = \{z_1, y_1, y_2, y_3\}$. Another way can be $X_1 = \{x_1, x_2, y_3\}$, $Y_1 = \{z_2, z_1, y_1, y_2\}$. Both these partitions make G bipartite.

* * *

We shall now state a theorem which gives a characterisation for bipartite graphs. Before giving the statement let us just note the following.

Note: When a graph G is a cycle on n vertices, we often say that G is an n-cycle. A cycle C_n is said to be an **even cycle** if n is a positive even integer and it is called an **odd cycle** if n is a positive odd integer. The positive integer n is called the **length of the cycle**.

Now we state the theorem without giving its proof.

Theorem 4: A graph G is bipartite if and only if G does not contain any odd cycles as subgraphs.

You can try some exercises now.

E17) Check whether the following graphs are bipartite or not.

i) complete graph K_3 (see Sec. 10.1, Unit 1)

ii) hypercubes Q_2 and Q_3 (see Sec. 10.2, Unit 2)

Q_n is bipartite

E18) Show that the subgraph of a bipartite graph is bipartite.

E19) Show that if $G_1, \ldots, G_n$ are bipartite, then $\bigcup_{i=1}^{n} G_i$ is bipartite.

Let us now go back to the assignment problem. In that problem we are interested in finding special subgraphs of bipartite which gives a solution to the problem. We have defined such graphs below.

Definition: In a bipartite graph G, let X and Y denote the two disjoint subsets of vertices. A **matching** in G is a set of edges such that no two edges in the set are incident with the same vertex (in X or in Y). In other words a matching defines a one-to-one correspondence between the vertices in a subset of X and the vertices in a subset of Y.

For example, the following figure shows a bipartite graph and one of its matchings. Fig. 23(b) gives the matching.

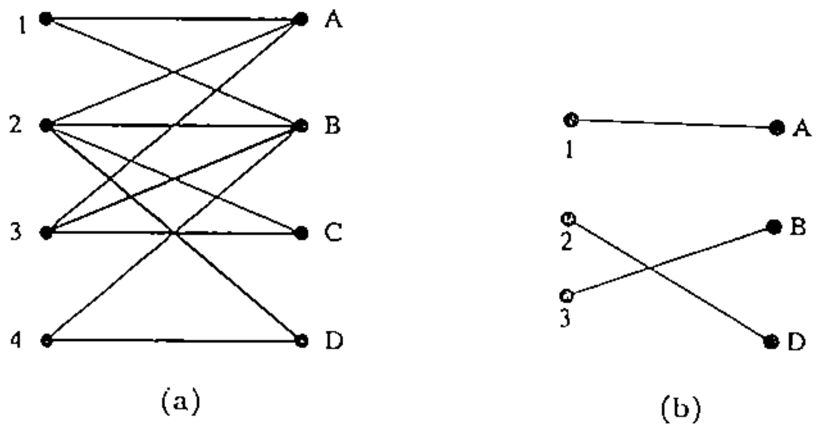


Fig.23

Can you find any other matching? We leave this as an exercise for you to check (see E 20 (i)).

Related to the concept of matching, we have another concept.

Definition: A matching of X into Y is called a **complete matching of X and Y** if, there is an edge incident with every vertex in X. In other words, a matching is complete if a one-to-one correspondence is defined between all the vertices in X and the vertices in a subset of Y.

Is the matching given in Fig.23(b) a complete matching? No, because in this matching, the vertex 4 is left out.

In graph-theoretic terminology, the assignment problem can be stated in the following way: if $G = G(X, Y)$ is a bipartite graph, when does there exist a complete matching from X to Y in G? So, for a given bipartite graph, we want to know whether there is a complete matching of the set of vertices in X into the set of vertices in Y. The following theorem gives a necessary and sufficient condition for the existence of a complete matching in a bipartite graph. As before we shall only state the theorem, omitting the proof.

Theorem 5: Let $G = G(X, Y)$ be a bipartite graph. A complete matching of X into Y exists in G if and only if $|A| \le |R(A)|$ for every subset A of X where $|A|$ denotes the number of elements in A (also called cardinality of A) and R(A) denotes the set of vertices in Y that are adjacent to the vertices in A.

Next we shall apply the above theorem to the assignment problem in the following example.

Example 6: Verify the conditions of Theorem 5 for the assignment problem given at the beginning of this section(See Fig.20).

Solution: To check the theorem we have to consider all subsets of the vertex set $X = \{x_1, x_2, x_3, x_4\}$, their cardinality, corresponding sets R(A) and their cardinality. The following table gives a list of all the possibilities.

Table 1

A	|A|	R(A)	|R(A)|
ϕ	0	ϕ	0
$\{x_1\}$	1	$\{y_1, y_2\}$	2
$\{x_2\}$	1	$\{y_1, y_3\}$	2
$\{x_3\}$	1	$\{y_4\}$	1
$\{x_4\}$	1	$\{y_2, y_3, y_5\}$	3
$\{x_1, x_2\}$	2	$\{y_1, y_2, y_3\}$	2
$\{x_2, x_3\}$	2	$\{y_1, y_3, y_4\}$	2
$\{x_3, x_4\}$	2	$\{y_2, y_3, y_4, y_5\}$	4
$\{x_1, x_4\}$	2	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_2, x_4\}$	2	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_1, x_3\}$	2	$\{y_1, y_2, y_4\}$	3
$\{x_1, x_2, x_3\}$	3	$\{y_1, y_2, y_3, y_4\}$	4
$\{x_2, x_3, x_4\}$	3	$\{y_1, y_2, y_3, y_4, y_5\}$	5
$\{x_1, x_3, x_4\}$	3	$\{y_1, y_2, y_3, y_4\}$	4
$\{x_1, x_2, x_4\}$	3	$\{y_1, y_2, y_3, y_5\}$	4
$\{x_1, x_2, x_3, x_4\}$	4	$\{y_1, y_2, y_3, y_4 y_5\}$	5

The table shows that the condition $|A| \leq |R(A)|$ is satisfied for all subsets A of X. Hence the conditions of Theorem 1, is satisfied.

* * *

The example above shows that there exists a complete matching from X into Y for the assignment problem. Therefore the assignment problem is solved. You can now try this exercise now.

E20) For the bipartite graph given in Fig.23, find a matching, apart from the one given in Fig.23. Does the graph given in Fig. 23(a) have a complete matching? Give reasons for your answer.

Let us now see another type of graph which has come into prominence because of its applications to electrical networks.

11.4 TREES

We are all familiar with the idea of a family tree. The concept of a tree in graph theory first arose in connection with work of a mathematician G. Kirchoff on electric networks in the 1840s, and with the work of another mathematician Cayley on the enumeration of chemical molecules in the 1870s. More recently, trees are used many areas, ranging from linguistics to computing.

For mathematicians, the interest and inportance of trees arises from the fact that, in many ways, a tree is a special type of graph, which has several interesting properties some of which we shall bring out in this section.

Let us first see what a tree means.

Consider the following graphs. Can you find any difference in their structures?

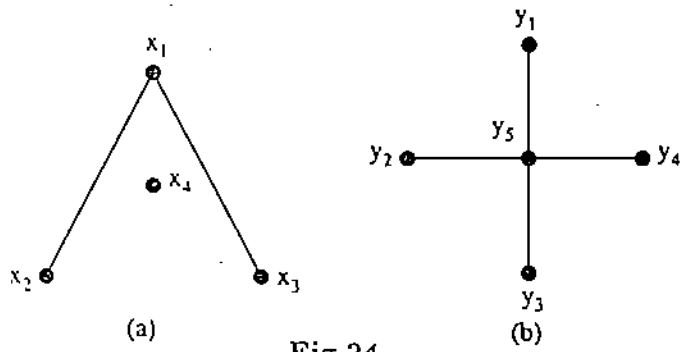


Fig.24

You might have noticed that (a) is disconnected. Also, (a) has no cycles. On the other hand, (b) is connected and has no cycles. From the following definition you will see that (b) is an example of a tree.

Definition: A graph with no cycles is called **acyclic**. A **tree** is a connected acyclic graph. A **forest** is a graph, each of whose components is a tree.

The following figure shows a forest with four components, (a), (b), (c) and (d) each of which is a tree.

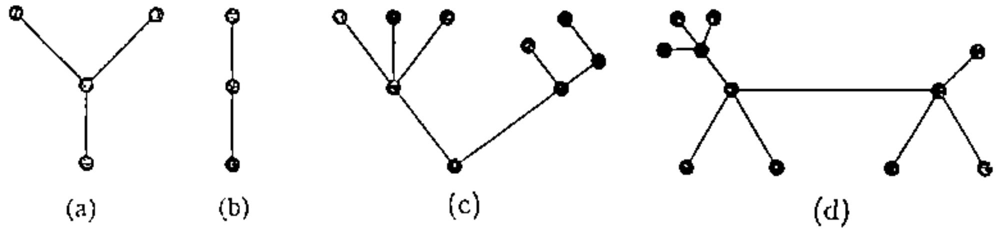


Fig.25

A tree has several interesting properties which we shall list in the following theorem.

Theorem 6: Let G be a graph with n vertices. Then the following statements are equivalent.

i) G is a tree.

ii) G is acyclic and has $(n-1)$ edges.

iii) G is connected, and has $(n-1)$ edges.

iv) G is connected, and every edge is a bridge.

v) any two vertices of G are connected by exactly one path.

Proof: If $n = 1$, all the five results are trivial. We shall, therefore, assume that $n \geq 2$. Now, from Unit 2 you know that if we prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) then all the statements are equivalent so let us do this. We shall prove the equivalent statements one by one.

(i) => (ii): By the definition, G does not have any cycles. Therefore it is acyclic. Now we will show that G has $(n-1)$ vertices. We will prove this by induction.

If $n = 1$ then the number of edges is 0. Therefore, the result is true for $n = 1$.

So, now let us assume that every tree on p vertices has $(p-1)$ edges for any positive integer p such that $1 < p < n$. Then we have to show that every tree on n vertices has $(n-1)$ edges. Now suppose we remove any edge. Since G is acyclic; the removal of any edge disconnects G into two graphs G_1 and G_2, such that G_1 and G_2 are connected and acyclic. Therefore G_1 and G_2 are trees and each has vertices less than n. Let n_1 and n_2 be the vertices in G_1 and G_2. Then $n_1 + n_2 = n$. Since n_1 and n_2 are less than n, by our induction assumption, the number of edges in G_1 and G_2 are $n_1 - 1$ and $n_2 - 1$ respectively. Therefore the total number of edges in both the graphs is $n_1 + n_2 - 2 = n - 2$. This together with the edge which is removed will give the total number of edges in the original graph. Therefore the total number is $n - 1$. Thus we got that every tree on n vertices has $n - 1$ edges. This is true for all n. Hence the result

(ii)=> (iii): Suppose that G is disconnected. Let $c(G) = t > 1$. Let $G_1, G_2, \ldots, G_t$ be components of G such that the number of vertices in each G_i is p_i for $i = 1, 2, \ldots t$, and the number of edges in each G_i is q_i, for $i = 1, 2, \ldots, t$. Then

$$p = p_1 + p_2 + \ldots + p_t, q = q_1 + \ldots + q_t$$

Now since every G_i is connected and acyclic, each G_i is a tree for $i = 1, 2, \ldots, t$. Therefore, by what we have shown while proving (i) ⇒ (ii), $q_i = p_i - 1, 1 \leq i \leq t$. Then

$$p - 1 = q = q_1 + \ldots + q_t = p - t$$

This is possible only if $t = 1$. This contradicts our assumption that $t > 1$. Therefore, G is connected.

(iii) => (iv): Suppose there is an edge which is not a bridge. Then the removal that edge will result in a graph with n vertices and $(n-2)$ edges. This not possible when G is connected by Theorem 3 in Sec. 11.2. Therefore every edge is a bridge.

(iv) => (v): Since T is connected each pair of vertices is connected by at least one path. If a given pair of vertices is connected by two paths, then they form a cycle, which contradicts the fact that every edge is a bridge. Therefore there is a unique path joining any two vertices.

(vi) => (i): We are assuming that any two vertices are connected by a unique path. So, the graph G is connected. It is also acyclic because if it contains a cycle $C = \{x_0, x_1, \ldots, x_n = x_0\}$, then we can find two distinct paths $P_1 = \{x_0, x_1\}$ and $P_2 = \{x_0, x_{n-1}, \ldots, x_2, x_1\}$ connecting the vertices x_0 and x_1, which contradicts our assumption. Therefore, G is a tree.

The theorem above tells us that a tree has got several nice properties which a general graph does not have. In fact the importance of trees in graph theory is that every connected graph contains a tree which has all the vertices of the original graph, as you will now see.

Let us consider a connected graph G. Consider a cycle in it and remove one of its edges, such that the resulting graph is connected. We repeat this procedure with one of the remaining cycles, continuing until there are no cycles left. The graph which remains is a connected subgraph of G which does not have any cycle. Therefore, it is a tree. Note that this tree has all the vertices of G. Such a graph is called a spanning tree, as you will realise from the following definition.

Definition: A spanning tree for a graph G is a connected acyclic subgraph which contains all the vertices of G.

The following figure shows a connected graph and one of its spanning trees.

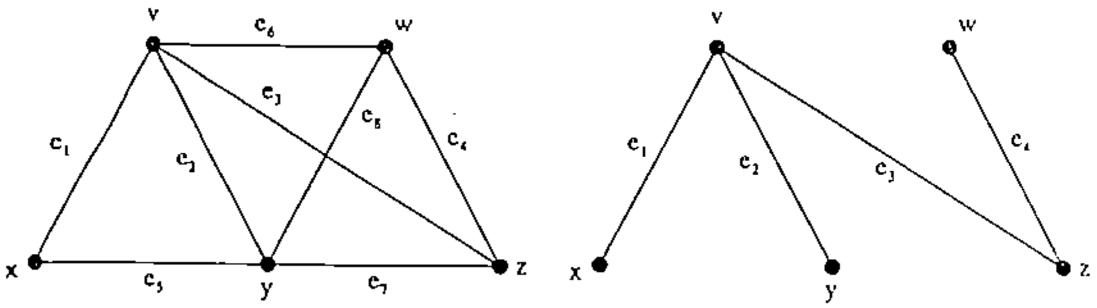


Fig.26

Does this graph have only one spanning tree? No, the graph in Fig. 27 gives another spanning tree for the graph.

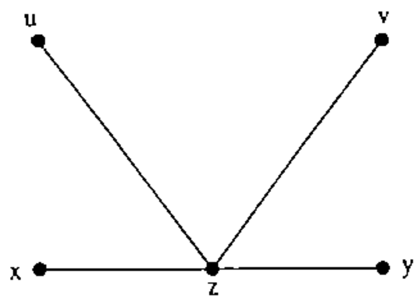


Fig.27

This shows that a connected graph can have several spanning trees. We shall now state the theorem, the proof of which is omitted.

Theorem 7: G is connected if and only if it has a spanning tree.

The theorem above tells that in a graph with k components, each component will have a spanning tree. Because of this result and because of the special structure of trees, in trying to prove a general result in graph theory, it is sometimes convenient to try to prove the corresponding result for a tree.

You can try some exercises now.

E21) Draw three spanning trees of the following graph .

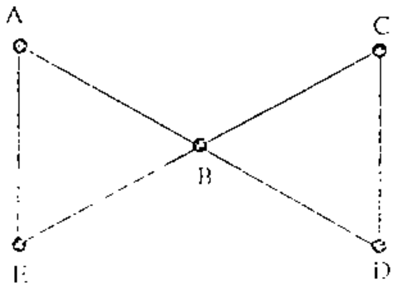


Fig.28

E22) Is every tree a bipartite graph? Give reasons for your answer.

So far we have seen three types of graphs: connected graphs, bipartite graphs and trees. You will see more types of graphs in the following units. Let us now summarise what we have covered in this unit.

E15) The vertex connectivity of a connected graph G is the smallest number of vertices whose removal disconnects G.

A cut vertex set H of a connected graph g is a set H of vertices with the following properties

i) the removal of all vertices in H disconnects G

ii) the removal of any proper subset of H will not disconnect G.

E16) Vertex connectivity is 1 and the set is $\{w\}$

E17) i) From the figure you can see K_3 has 3-cycles. So, by Theorem 4, it is not bipartite.

ii) Both Q_2 and Q_3 does not contain odd cycles, therefore, by Theorem 4, they are bipartite.

E18) Let G be a bipartite graph with a bipartition $X \cup Y$. Let H be a subgraph of G. If V(H) is disjoint from either X or Y, $E(H) = \phi$. You can take any portion of V(H) into two subsets. It will serve as a bipartition.

If, on the other hand, V(H) intersects both the subsets X, Y of V(G), then $V(H) = X' \cup Y'$, where $X' = X \cap V(H), Y' = Y \cap V(H)$, serves as a bipartition of H.

E19) Let $G_i, 1 \leq i \leq n$ be bipartite graphs with the bipartitions $V(G_i) = X_i \cup Y_i$, respectively. Let $G = \cup_{i=1}^{n} G_i$. Then, E(G) is the disjoint union $\cup_{i=1}^{n} E(G_i)$. Clearly, $V(G) = A \cup B$, where $A = \cup_{i=1}^{n} X_i, B = \cup_{i=1}^{n} Y_i$. is a bipartition of V(G). This can be seen as follows: Let e be an edge in E(G). Since E(G) is disjoint union $E(G_1), \ldots, E(G_n)$ and the edge e belongs to only one of them. Without loss of generality, suppose $e \in E(G_r)$. Since G_r is bipartite with a bipartition $X_r \cup Y_r$, this means e has one end vertex in X_r and the other in Y_r, that is, e has one end vertex in A and the other in B. Thus, G is bipartite with a bipartition $A \cup B$.

E20) Fig.31 gives another matching.

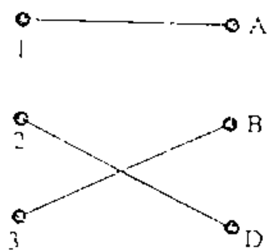


Fig.31

The following is a complete matching in the graph: {1A, 2B, 3C, 4D} (shown by thicker lines in Fig.23).

E21)

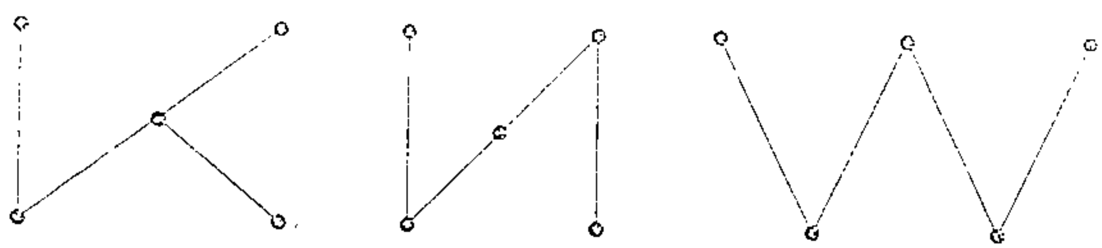


Fig.32

E22) Yes. Since a tree does not have cycles, by Theorem 4, it is a bipartite graph.

UNIT 12 EULERIAN AND HAMILTONIAN GRAPHS

Structure Page No.

12.1 INTRODUCTION

Suppose you go to a new city as a salesperson. You would naturally like to familiarise yourself with all the important routes. One way to do this is to buy a map of the city and go around the city. If you do this without proper planning you may pass through some of the routes more than once. To avoid this, you would need to sit down and plan your route. The most efficient way would use every route only once. But is it possible to find such a route ?

This question is so natural that you may not be surprised to know that a similar question was raised more than 250 years ago. Königsberg was a city in what was known as Prussia those days. The Pregel river flowed through this city forming two islands B and C in Fig.1.

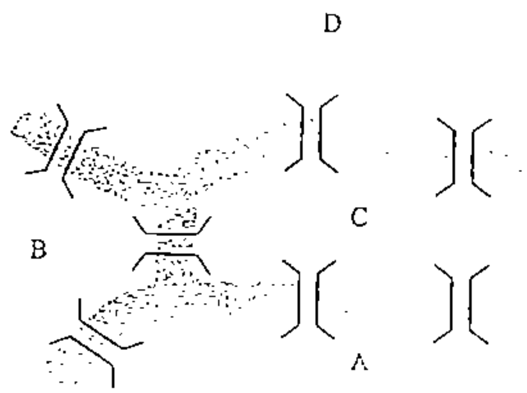


Fig.1: A schematic diagram of Königsberg.

The two islands and the rest of the city were connected to each other by seven bridges. Some of the citizens used to amuse themselves with the following question: Is it possible to go around the city using each bridge exactly once ?

In 1736, the great Swiss Mathematician Leonard Euler(pronounced as 'oiler') answered this question by converting this into a problem in graph theory. We will see this problem in Section 12.2(Sec.12.2 in brief), while discussing graphs named after Euler.

There is one more question similar to the Königsberg problem in recreational mathematics. Which figures can be drawn without lifting the pen from the paper and without going over any of the lines twice? This question is also answered in Sec.12.2.

While Euler's criterion tells you whether an efficient route for going round the city exists or not, Fleury's algorithm, discussed in Sec.12.3, will help you actually find the route.

A mathematical puzzle invented by Hamilton involves finding a cycle containing all the vertices of a certain graph. Motivated by this, we will discuss conditions for a graph to contain a cycle containing all the vertices of the graph. Such a graph is called a Hamiltonian graph in honour of Hamilton. In Sec.12.4 we will give some necessary and sufficient conditions for a graph to be Hamiltonian. Finally, in Sec.12.5 we discuss a related question, the travelling salesperson problem.

Objectives

After reading this unit, you should be able to

- check whether a given graph is Eulerian or not;
- apply Fleury's algorithm to find an Eulerian circuit in an Eulerian graph;
- check whether a given graph satisfies certain necessary conditions for a Hamiltonian graph;
- check whether a given graph satisfies certain sufficient conditions for a Hamiltonian graph;
- find a minimum-weight Hamiltonian cycle in a weighted complete graph.

12.2 EULERIAN GRAPHS

As we mentioned in the introduction, Euler solved the Königsberg problem by converting it into a problem in graph theory. He represented each land area by a vertex and each bridge by an edge(see Fig.2(a)).

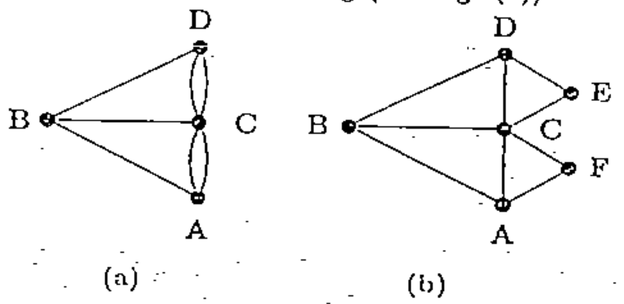


Fig.2

You might have noticed that the graph in Fig.2(a) is a multigraph. Here A and C are connected by two edges; So are C and D. Let us divide one of the edges connecting C and D by adding a new vertex E. Similarly, we divide one of the edges joining A and C by adding a vertex F. Then, we get the simple graph in Fig.2(b). If we can find a way of going around the graph in Fig.2(b) using each edge only once, then we can do so in the graph in Fig.2(a) also and vice-versa. This process of subdividing the vertices can be carried out for any multigraph. So, while looking for Eulerian circuits, we can still restrict ourselves to simple graphs. Then, the Königsberg bridge may be reformulated as follows:

Is there a circuit in the graph in Fig.2(b) containing each edge only once? (1)

Recall the definition of a trail from Unit 11. A **trail** is a walk in which no edge is repeated. A **closed trail**, also called a circuit, is a trail whose starting vertex and end vertex are the same. Related to these concepts, we have the following terms.

Definition : A trail containing all the edges of the graph is called an **Eulerian trail**. A graph is **Eulerian** if it contains an Eulerian circuit.

So, we can rephrase the question in (1) in the following way:

Is the graph in Fig.2(b) Eulerian? (2)

Before going further, we give a clarification of our definition of Eulerian graphs in the form of a remark.

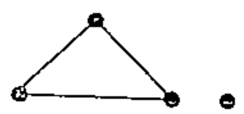


Fig.3

Remark: You might have noticed that we made connectedness a part of the definition of Eulerian graphs. This is to avoid examples like the one given in Fig.3. The graph has a circuit which contains all the edges of the graph. There is no edge through which we can reach the isolated vertex. Unless there is a very special reason, we will not bother about a place to which there is no access! So, such isolated vertices are of no interest to us. By making connectedness a part of the definition, such situations can be avoided.

Now, let us find some simple examples of Eulerian graphs. The simplest class of examples is cycles, for example C_6 in Fig.4(a). We can get another example by adding a cycle of length 3 to the graph in Fig.4(a) at v_1(see Fig.4(b)).

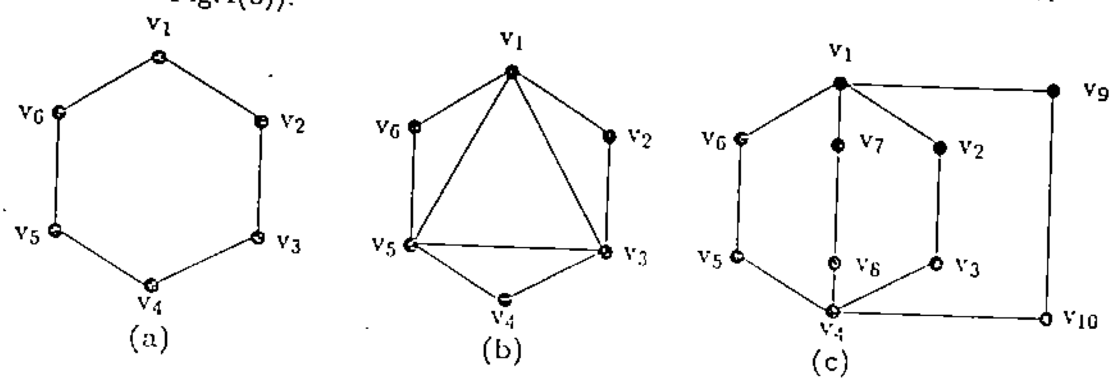


Fig.4

This is also Eulerian because we can start at the vertex v_1, traverse the inner triangle, come back to v_1 and traverse the outer cycle. We get another Eulerian graph by adding a cycle of length 6 at v_1 to Fig.4(a).(See Fig.4(c))

Now, you may like to verify whether you have understood the definition of an Eulerian circuit by attempting the following exercise.

E1) Prove that the graph given in Fig.4(c) is Eulerian by producing an Eulerian circuit in it.

You probably found Exercise 1 easy. In a simple example like this, you can easily prove that a graph is Eulerian by producing an Eulerian circuit by trial and error. This may not be possible in more complicated cases. It is impossible to prove that a graph is not Eulerian by trial and error; we may miss some clever way of tracing an Eulerian circuit. So, we need a necessary and sufficient condition for a graph to be Eulerian. The condition should also be easy to apply. The next theorem gives such a condition. Euler's proof of the necessary part of the theorem appeared in *Solutio problematis geometriam situs pertinentis*(The solution of a Problem relating to the Geometry of Position). Hierholzer proved the sufficiency part.

Theorem 1: A connected graph G is Eulerian if and only if the degree of each of its vertices is even.

Proof: Let the graph G be Eulerian and suppose T is an Eulerian circuit in G. Every time the circuit passes through a vertex, it uses two edges, one to reach the vertex and one to leave it. What about the vertex from which we start tracing the circuit? The edge with which we start the circuit is paired with the edge with which we end the circuit. Apart from this, every time we pass through the vertex in the intermediate stages we will use two edges incident at the vertex as before. Also, we use each of the edges only once. So, all the vertices of the graph have even degree.

To prove the converse, consider a connected graph in which each vertex has even degree. We will now prove that G contains an Eulerian circuit by induction on the number of edges in G. Suppose that the number of edges is 0; since we have assumed that the graph is connected, it consists of a single isolated point. Since the edge set is empty the statement that there is an Eulerian circuit containing all the edges is vacuously true. Assume that all the graphs with fewer edges than G contain an Eulerian circuit. All the vertices of G have even degree and G has no vertex of degree 0(isolated vertex) since it is connected. So, all the vertices have degree at least 2. We can start from an arbitrary point $u = u_1$ and trace a circuit C as follows:. We choose any edge u_0u_1 incident at u_0. Since u_0 has degree at least two, there is another edge incident at u_1, say u_1u_2. We go on tracing a circuit like this, always making sure that we enter and leave any vertex by different edges. During the course of tracing C, we may pass through u_0 several times. The process ends when we reach u_0 and find that there is no unused edge to leave u_0. If the circuit we have obtained contains all the edges, we are done. Otherwise, we remove this circuit from G and call the resulting (possibly disconnected) graph H. All the vertices in each of the components of H have even degree and all the components have fewer edges than G. So all the components are Eulerian. We now get an Eulerian circuit in G as follows: We start from any vertex v on the circuit C and traverse the edges of C till we come to a vertex that lies on one of the components of H. We then traverse the Eulerian circuit in that component, eventually returning to the circuit C. We continue along C in this fashion, taking Eulerian circuits of components of H as we come to them, finally returning to the vertex v we started with. We would have used each of the edges only once, that is, we have obtained an Eulerian circuit.

Note that, by connectedness of G, each component of H must contain a point of C.

Let us now see if we can solve the Königsberg bridge problem using Theorem 1.

Example 1: Check whether the Königsbergians can go round the city using each bridge only once.

Solution: You may recall that we have reduced the Königsberg bridge problem to finding an Eulerian circuit in Fig.2(b). According to the necessary part of the theorem, if a graph has an Eulerian circuit, it has no edges of odd degree. But, as you can see, all the vertices, except E and F, have odd degree. So, this graph does not have an Eulerian circuit. So, the Königsbergians cannot go around the city using each vertex only once.

Now, here are some exercises to test your understanding.

E2) After Euler proved his Theorem, much water has flowen under the bridges in Königsberg. In 1875, an extra bridge was built in Königsberg, joining the land areas A and D (See Fig.5). Is it possible now for the Königsbergians to go round the city, using each bridge only once?

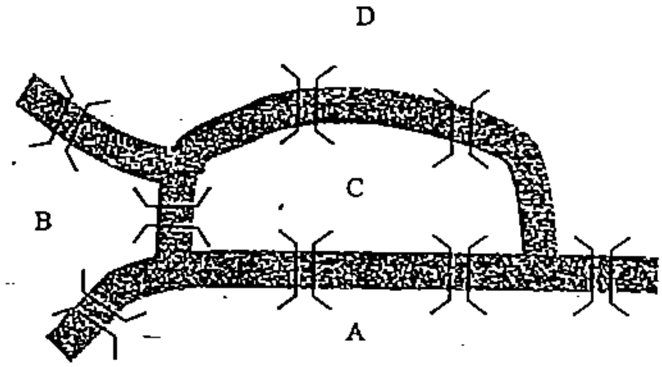


Fig.5

E3) By writing the degree sequences of the following graphs, check that they are Eulerian and write down some Eulerian circuits.

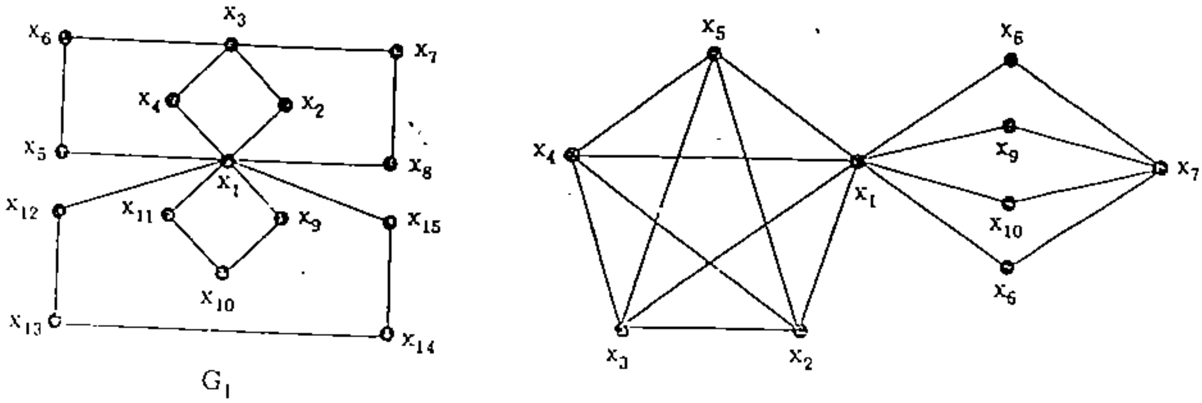


Fig.6

E4) a) For what values of n is K_n, the complete graph on n vertices, Eulerian ?

b) For what values of n, m is $K_{n,m}$ Eulerian ?

E5) Find out which one of Q_3, Q_4 is Eulerian and which one is not.

E6) Show that, in a connected Eulerian graph, an Eulerian circuit can be traced starting from any vertex.

Suppose now that the people of Königsberg will be happy if they can go around the city, still using all the bridges only once, but they do not mind ending their tour at a point different from their starting point. Is this possible? Let us now examine this question. We will convert this to a problem in graph theory. But, before that we need a couple of definitions that will be helpful in formulating our problem.

Definition : By an open trail we mean a trail in which the end vertices are distinct.

For example, $\{E, C, D, B, C, F\}$ is an open trail in the graph in Fig.2(b).

Definition : A graph G is edge traceable if G contains an open trail that contains all the edges of G.

Let us now look at an example of such a graph.

Example 2: Show that the graph in Fig.7 is edge traceable.

Solution: Consider the walk $\{v_5, v_1, v_2, v_5, v_4, v_3, v_2, v_4\}$. This contains all the seven edges of the graph and the end vertices are distinct. Since no edge is repeated, this walk is an open trail containing all the edges of the graph. So, the graph is edge traceable.

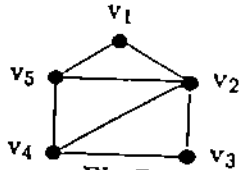


Fig.7

* * *

In view of the definition of an edge traceable graph, citizens of Königsberg will have to check whether the graph in Fig.2(b) is edge traceable. As an immediate consequence of Theorem 1, we get the following characterisation of edge traceable graphs.

Theorem 2: A connected graph G is edge traceable if and only if it has exactly two vertices of odd degree.

Proof: Suppose G is an edge traceable graph. Then, there is an open Eulerian trail T containing all the edges of G. Suppose x and y are the first and last vertices of T. We now add a new vertex a and join this to x and y. Let us call the new graph we obtain G′. This is illustrated in a particular case in Fig.8 below:

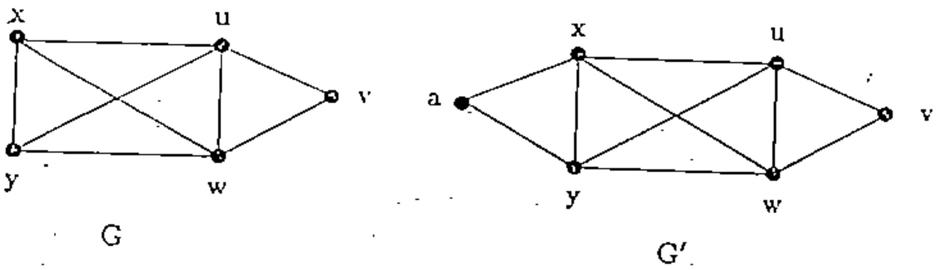


Fig.8

In the graph G′ in Fig.8, the Eulerian circuit is $\{a, x, u, v, w, y, u, w, x, y, a\}$

In the graph G′ we get an Eulerian circuit as follows: We start at a, trace the edge ax, trace the open Eulerian trail T, and trace the edge ya. So, by Theorem 1 all the edges of G′ have even degree. Except for x and y, the degrees of all the vertices are unaffected by the addition of the edges ax and ay. So, all of them must have even degree, considered as vertices in G. In the case of the vertices x and y, their degrees have become even after the edges ax and ay are added, i.e., after their degrees are increased by one. So, before the addition of the edges, their degrees must have been odd.

Conversely, suppose that exactly two vertices x and y have odd degree. Then, by adding a new vertex a and two new edges ax and ay, the degrees of all the vertices become even. So, we can find an Eulerian circuit starting at a. Let this Eulerian circuit be $\{v_0 = a, v_1, \ldots, v_n = a\}$. Since x and y are the only vertices to which a is adjacent, either $v_1 = x$ or $v_{n-1} = x$. If $v_1 = x$, we must have $v_{n-1} = y$ and $\{v_1 = x, v_2, \ldots, v_{n-1} = y\}$ is the open Eulerian trail. Similarly, if $v_1 = y$, we must have $v_{n-1} = x$, and $\{v_1 = y, v_2, \ldots, v_{n-1} = x\}$ is the open Eulerian trail.

Let us now look at the question that motivated us to prove the theorem above.

Example 3: Check whether it is possible for the Königsbergians to go around the city, still using each bridge only once, but ending the trip at a point different from the starting point.(See Fig.2(b))

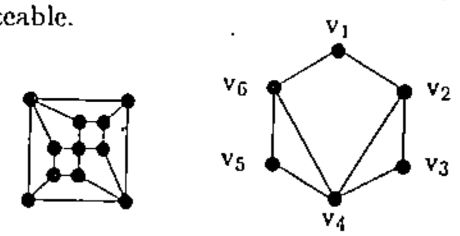
Solution: Referring to Fig.2(b), as we observed before, all the vertices except E and F have odd degree, i.e. there are four vertices of odd degree. So, it is not possible for Königsbergians to tour the city using each bridge only once, even if they are allowed to start and end the tour at two different points.

* * *

Here are some exercises for you to try.

E7) Consider the situation after the addition of a new bridge in 1875.(See Fig.5) Is it possible to tour the city using each bridge only once, if starting and ending the tour at two different points is permitted?

E8) By writing down the degree sequence, find out which of the following graphs are edge traceable.



(a) (b)

Fig.9

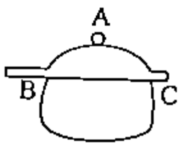


Fig.10(a)

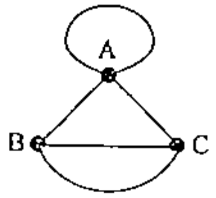


Fig.10(b)

We considered one more problem that we mentioned in the introduction to this unit. This asks for a method for determining whether a given figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice. There is such a method, which we shall now illustrate.

Example 4: Check whether the graph in Fig.10(a) can be drawn without lifting the pencil from the paper and without going over any of the lines twice.

Solution: The method involves 4 steps.

Step 1 (Add vertices at all the junctions where more than two lines meet.) In Fig.10(a) there are three such junctions A, B and C. So, add vertices at A, B and C to get the multigraph with loop in Fig.10(b).. Note that the curve joining A and B in Fig.10(a) is replaced by a straight edge in Fig.10(5). Similarly the curve joining A and C is represented by the edge AC.

Step 2 (If there are no loops go to step 3. If there are loops, eliminate the loops by adding two vertices of degree two.) If we add two vertices D and E of degree 2 to the earlier loop at A, we get the figure in Fig.10(c).

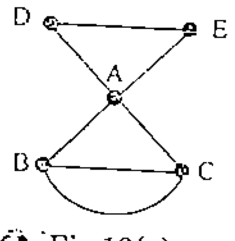


Fig.10(c)

Step 3 (If there are no multiple edges go to step 4. Otherwise, eliminate the multiple edges by adding vertices of degree 2.) In Fig.10(c) B and C are connected by two edges. We eliminate one of the multiple edges by adding a vertex F to one of the edges.

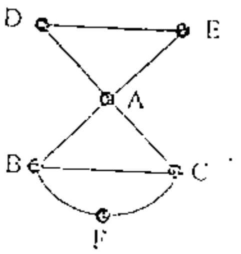


Fig.10(d)

Step 4 (Count the number of edges of odd degree in the resulting graph. If there are either two vertices of odd degree or no vertices of odd degree, the graph is edge traceable or Eulerian respectively. So, the graph can be drawn without lifting the pen from the paper. Therefore, the figure we started with can be traced without lifting the pen from the paper.) As you can see from Fig.10(d) there are exactly two edges, B and C, of odd degree. So, the figure can be traced without lifting the pencil from the paper.

* * *

If you go through the example above carefully, you may realise that there is a much easier method for deciding whether a figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice. In analogy with graphs, let us call the number of lines that meet in a junction, the degree of the junction for convenience. Note that, only those junctions where more than two lines meet can give rise to vertices of odd degree. All the other vertices that we added are of even degree. In view of this observation, we have the following result:

Theorem 3: A figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice if and only if the number of junctions whose degree is odd and at least 3 is either 2 or 0.

Here are some exercises for you to try.

E9) Which of the following figures can be drawn without lifting the pen from the paper and without covering any line segment more than once?

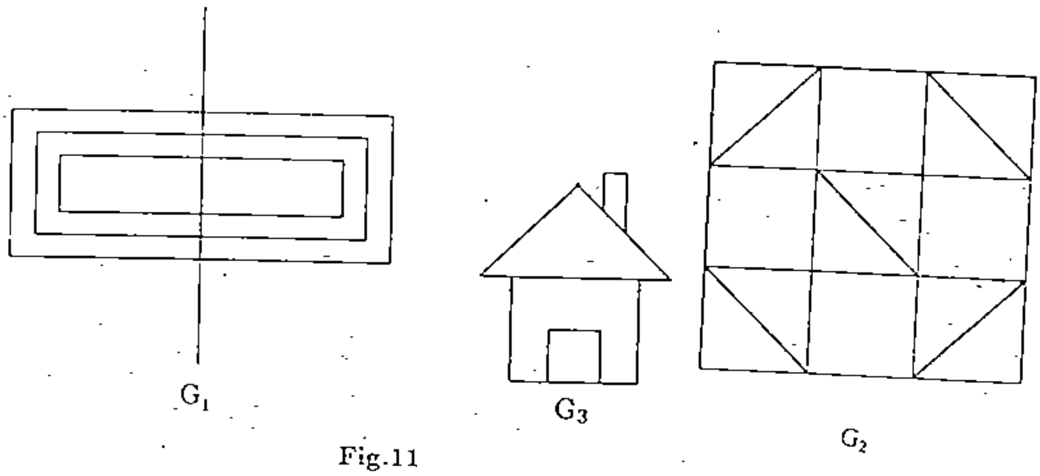


Fig.11

E10) Construct, if possible, Eulerian graphs with the following number of vertices and edges. When it is not possible, explain why it is not possible.

	a	b	c
Number of vertices	5	6	7
Number of edges	10	10	6

In this section we saw that if all the vertices of a graph have even degree, it is Eulerian. However, there are situations where we know that a graph is Eulerian, but we still may not be able to find an Eulerian circuit in it. The next section describes an algorithm due to Fleury that gives a method of finding an Eulerian circuit in an Eulerian graph.

12.3 FLEURY'S ALGORITHM

In the year 1962, Meigu Guan, a Chinese mathematician considered a problem which is known as the 'Chinese Postman Problem'. As a part of his daily routine, a postman picks up the mail at the post office, goes around the city, covering each street at least once and returns to the post office after delivering the mail. Naturally, he wishes to choose his route in such a way that he walks as little as possible. How should he go about choosing the route? Here, if we represent various streets by edges and find that the resulting graph is Eulerian, then the problem reduces to finding an Eulerian circuit C of the graph and taking the vertex representing the post office as the starting vertex. The Chinese postman problem is easily solved in this case since a good algorithm for determining Eulerian circuit is given by Fleury. This algorithm can be stated as follows:

Fleury's Algorithm: Choose any vertex and traverse the edges arbitrarily, except for the following conditions:

i) **At each stage, choose a bridge only if there is no alternative.**

ii) **At each stage, erase the edge after traversing it and also erase any isolated vertex that results from the removal of the edge.**

Let us look at a simple example to illustrate the algorithm.

Example 5: Find an Eulerian circuit in the graph in Fig.12 using Fleury's algorithm. Indicate the bridges you have chosen.

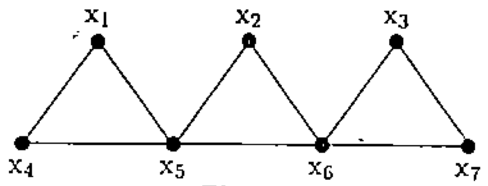


Fig.12

Solution: According to the algorithm, we can choose any vertex as the first vertex. Let us choose x_2.

Stage 1 There are no bridges to avoid at this stage. We choose the edge x_2x_5. After reaching x_5 we erase the edge x_2x_5 according to condition ii) of the algorithm. No isolated vertex results because of this erasure.

Stage 2 We are now at x_5. Note that x_5x_6 is a bridge. (See Fig.13 below)

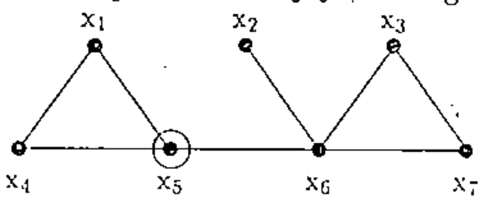


Fig.13

Since there are other alternatives, we must avoid this edge according to condition ii). We choose x_5x_1, which is not a bridge. After reaching x_1 we delete the edge x_5x_1. No isolated vertex results.

Stage 3 We choose x_1x_4 even though it is a bridge because we have no other choice. (See Fig.14.) After reaching x_4, we erase x_1x_4.

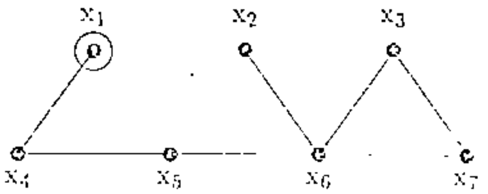


Fig.14

Now, x_1 becomes an isolated vertex. So, we erase it.

Stage 4 We choose the bridge x_4x_5 although it is a bridge because we have no other choice. After erasing x_4x_5, the vertex x_4 becomes isolated, so we remove it.

Stage 5 We choose the bridge x_5x_6, erase x_5x_6 after reaching x_6 and erase the vertex x_5 which becomes isolated.

Stage 6 We avoid the bridge x_6x_2, choose x_6x_7, erase x_6x_7. No isolated vertex results.

Stage 7 We choose the bridge x_7x_3, erase x_7x_3 after reaching x_3 and erase the resulting isolated vertex x_7.

Stage 8 We choose the bridge x_3x_6, erase x_3x_6 after reaching x_6 and erase the resulting isolated vertex x_3.

Stage 9 We choose the bridge x_6x_2, erase x_6x_2 after reaching x_2 and erase the isolated vertex x_2.

Stage 10 After reaching x_2, we find that there is no edge adjacent to x_2. The steps are complete.

So, the Eulerian circuit we have obtained is $\{x_2, x_5, x_1, x_4, x_5, x_6, x_7, x_3, x_6, x_2\}$. The bridges we have chosen are $\{x_1x_4, x_4x_5, x_5x_6, x_7x_3, x_3x_6, x_6x_2\}$

* * *

Remark: If G is an Eulerian graph with q edges, then Fleury's algorithm stops after exactly q steps. When it stops, we are back at the vertex u. So, we get an Eulerian circuit of the graph G. It can be proved that Fleury's algorithm always yields an Eulerian circuit. Due to the complexity of the proof, we omit it.

Here are some exercises to test your understanding of Fleury's algorithm.

E11) Find an Eulerian circuit in the graph in Fig.15. Indicate the bridges you have chosen.

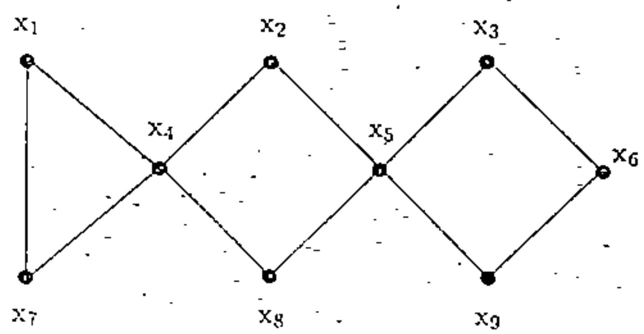


Fig.15

In this section we were interested in finding circuits in which all the edges of the graph occur exactly once. In the next section we are interested in finding cycles in which all the vertices occur exactly once.

12.4 HAMILTONIAN GRAPHS.

Suppose a transport company operates bus services between 10 different places. There are places with no direct bus service between them, but there is always a route between any two places that go through the other places. In this situation, the company wants to offer a round trip that passes through each of the cities exactly once. Is it possible?

Let us formulate this question as a problem in graph theory. Let us represent the places by vertices. Two vertices are adjacent if a direct bus connects the corresponding places. Since it is possible to go from one place to another, the graph we get is a connected graph. Now, the question is,

Is there a cycle in the graph in which each vertex occurs precisely once? (3)

A similar question was the basis of the mathematical game described by Hamilton. In this, he took a regular dodecahedron. Each of its 20 vertices is supposed to represent a city of the world. One of the players inserts 5 pins in 5 of the vertices. The other player is supposed to find a 'world tour' containing all the remaining 15 cities and come back to the starting vertex. This amounted to finding a cycle covering all the vertices of the regular dodecahedron. Fig.16 gives such a cycle.

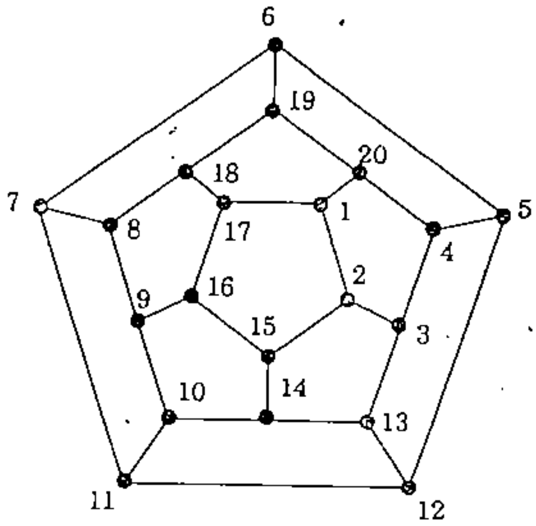


Fig.16

It is time now to give a name to such a cycle.

Definition : A cycle C in a graph G is called a **Hamiltonian cycle** if it contains all the vertices of G. A graph is called **Hamiltonian** if it contains a Hamiltonian cycle. A graph is called **non-Hamiltonian** if it does not contain any Hamiltonian cycle.

Can you think of examples of Hamiltonian graphs other than the one given in Fig.16? Is any cycle a Hamiltonian graph? Is any graph obtained by adding edges to a Hamiltonian graph also Hamiltonian? The answer to both the questions is 'yes'. For example, the graphs in Fig.17 are Hamiltonian.

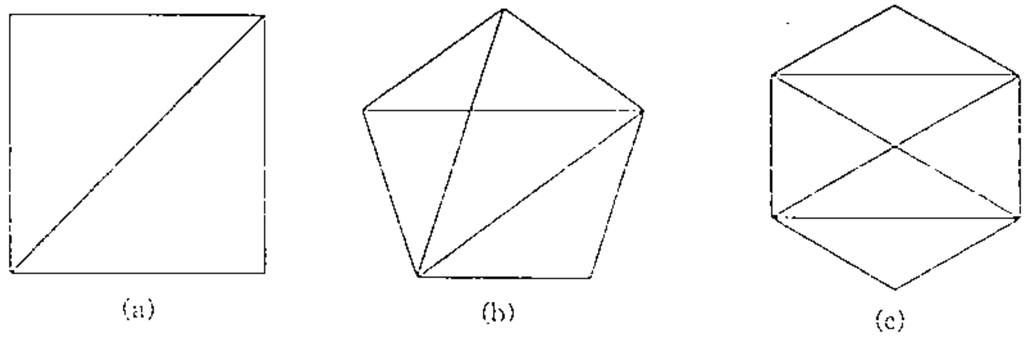


Fig.17

Are there any non-Hamiltonian graphs? Trees are obvious examples of non-Hamiltonian graphs; since they don't have any cycles, they cannot have a cycle containing all the vertices!

Note that, by definition, a Hamiltonian graph contains a cycle containing all the vertices. So, a Hamiltonian graph cannot have cut vertices or pendant vertices.(Recall that a pendant vertex is a vertex of degree 1.) This gives a simple method for constructing examples of non-Hamiltonian graphs. For example, the graph in Fig.18 given below is non-Hamiltonian because it has a cut vertex x.

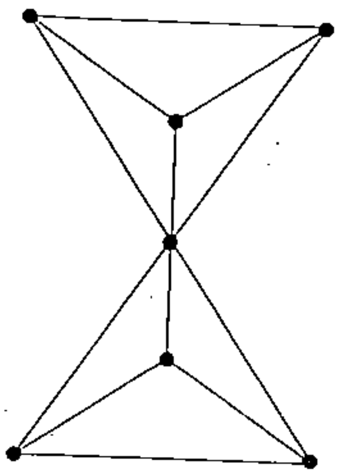


Fig.18

Here are some exercises to test your understanding of the discussion above.

E12) Construct a non-Hamiltonian graph on 5 vertices.

E13) Find a graph which is Hamiltonian but not Eulerian.

E14) Find a graph which is Eulerian but not Hamiltonian.

E15) Find a Hamiltonian cycle in the hypercube Q_3.

We have used the existence of a cut vertex to prove that the graph in Fig.18 is not Hamiltonian. However, this does not give us a foolproof method of identifying non-Hamiltonian graphs. For example, $K_{m,n}$, $m, n \geq 2$, has no cut vertices or pendant vertices, and it is not Hamiltonian when $m + n$ is odd, as we shall now show.

Example 6: Show that $K_{m,n}$ is not Hamiltonian when $m + n$ is odd.

Solution: Since $K_{m,n}$ is bipartite, it does not have cycles of odd length. On the other hand, it has an odd number of vertices. So, a Hamiltonian cycle in this graph, if it exists, must be of odd length. So, $K_{m,n}$ is not Hamiltonian when $m + n$ is odd.

* * *

From the previous example it is clear that we need some conditions for identifying non-Hamiltonian graphs which do not depend on the existence of cut vertex or pendent vertex. The following theorem gives a slightly better necessary condition for a graph to be Hamiltonian. We will omit the proof of this theorem in this course.

Recall that $c(G)$ denotes the number of components of G.

Theorem 4: If G is a Hamiltonian graph, then for every proper subset S of V(G), we must have

$$c(G - S) \leq |S|.$$

Let us now look at an example to illustrate the use of Theorem 4.

Example 7: Show that $K_{m,n}$ is not Hamiltonian if $m < n$.

Solution: Recall that the vertex set of $K_{m,n}$ can be partitioned into two disjoint subsets X and Y of cardinality m and n, respectively, in such a way that no two edges in the same subset are adjacent and every vertex in X is adjacent to every vertex in Y. Let us take X to be the set S in the Theorem. So, $|S| = n$ in this case. If we delete all the vertices in X, the graph becomes totally disconnected. so, there are m components in $G - S$, one corresponding to each vertex of Y. So, $C(G - S) < |S|$. So, by Theorem4, $K_{m,n}$ is non-Hamiltonian.

* * *

If the condition given in Theorem 4 is not satisfied, the graph is non-Hamiltonian. However, if the condition is satisfied, it does not mean that the graph is Hamiltonian. For example, consider the graph in Fig.19(a).

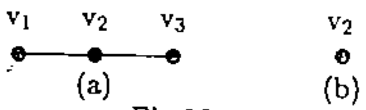


Fig.19

Let us remove the two end vertices, so that $S = \{v_1, v_3\}$ and $|S| = 2$. We will get a single isolated vertex(see Fig.19(b)), so $c(G - S) = 1$ and $c(G - S) < |S|$. So, the conditions of the Theorem are satisfied. This is a path of length 2. It does not contain any cycle so it is non-Hamiltonian.

Now for some exercises to check your understanding of the Theorem.

E16) Show that the following graph is non-Hamiltonian.
(**Hint:** Find a set $S \subset V(G)$ such that $c(G - S) > |S|$.)

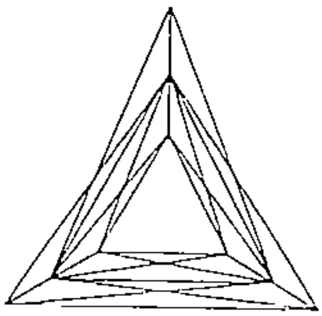


Fig.20

E17) Check whether the following graphs are Hamiltonian.

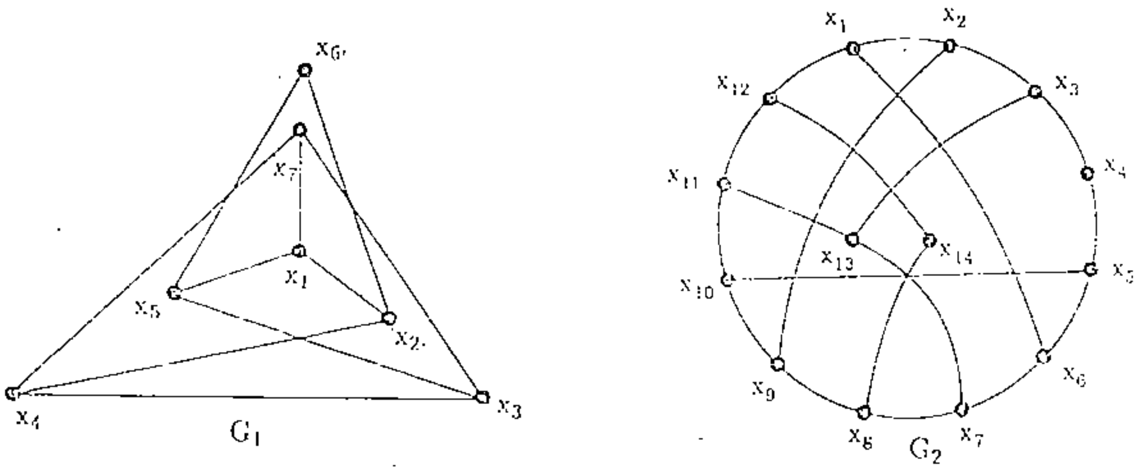


Fig.21.

E18) Show that the following graph is non-Hamiltonian.

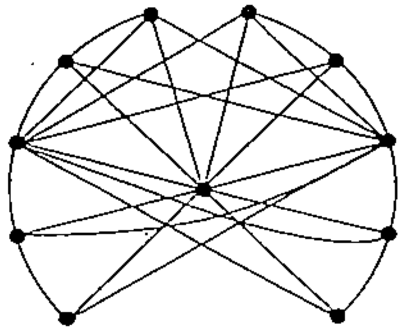


Fig.22

So far, we have seen some necessary conditions for a graph to be Hamiltonian. They are helpful if we want to show that a given graph is non-Hamiltonian. They are of no use if we want to show that a given graph is Hamiltonian. We need some sufficient conditions for this purpose. Since we are looking for a cycle covering all the vertices, it is reasonable to expect success whenever, at every vertex, there are enough choices of edges. This is confirmed by the following Theorems. Theorem 5 was proved by Dirac in 1952. This was generalised to Theorem 6 by Ore in 1960.

Theorem 5: If G is a simple graph on p vertices, $p \geq 3$, and if $\delta(G) \geq \frac{p}{2}$, then G is Hamiltonian.

Recall that, $\delta(G) = \min\{\deg_G(x) | x \in V(G)\}$

Theorem 6: Let G be a simple graph on p vertices, $p \geq 3$, satisfying the condition that

$$d(u) + d(v) \geq p \text{ for any two non-adjacent vertices u and v.} \quad (4)$$

Then G is Hamiltonian.

Can you see that Dirac's Theorem follows from Ore's Theorem? This is because if $\delta(G) \geq \frac{p}{2}$, then for any two vertices u and v, we have $d(u) + d(v) \geq 2\delta(G) \geq p$. So, the conditions of Ore's Theorem are satisfied whenever the conditions of Dirac's Theorem are satisfied. So, if we prove Ore's criterion, we will have also proved Dirac's criterion.

Proof of Ore's Theorem: We shall prove this result by contradiction(see Unit 20). Suppose the Theorem is false. Then, there are non-Hamiltonian graphs with more than 3 vertices satisfying (4). So, the following set is non-empty:

$$\mathcal{F} = \{G | |V(G)| = p, G \text{ is non-Hamiltonian and satisfies condition 4}\}$$

Choose a graph in $\mathcal{F}$ with the maximum number of edges among all such graphs. Let us denote this graph by G_M. As G_M is non-Hamiltonian, it cannot be complete. So, there are two vertices, call them u and v, which are not adjacent. So, adding the edge $e = uv$ to G_M, we get a new graph G'_M. The number of vertices in G'_M is still greater than 3 because we haven't removed any vertex. Since we haven't removed any edge, the degrees of all the vertices remain the same. So, condition (4) holds for any two vertices in G'_M also. But then, G'_M must be Hamiltonian. If it is not, it will be in $\mathcal{F}$. This is not possible because $|E(G'_M)| = |E(G_M)| + 1$ and G_M was chosen to be a graph in $\mathcal{F}$ with maximum possible edges.

Now, since G'_M is Hamiltonian, we can choose a Hamiltonian cycle C in G'_M. Since G is non-Hamiltonian the edge uv must lie on C.(Why?) Removing this edge, we get a path in G containing all the vertices. Let $P = \{u = u_1, u_2, \ldots, u_p = v\}$ be this path. Define $S = \{u_i : uu_{i+1} \in E(G_M)\}$, $T = \{u_j : u_jv \in E(G_M)\}$.
Clearly, $u_p = v \notin S \cup T$. (Why?)Hence, $|S \cup T| < p$. Now, if possible, suppose $S \cap T = ne\phi$ $u_i \in S \cap T$. Then, $\{u_1, \cdots, u_i, u_p, u_{p-1}, \cdots, u_{i+1}, u_1\}$ is a Hamiltonian cycle in the graph G.(See Fig.23) This contradicts the assumption that G is non-Hamiltonian. Hence, $S \cap T = \phi$, that is, $|S \cap T| = 0$.

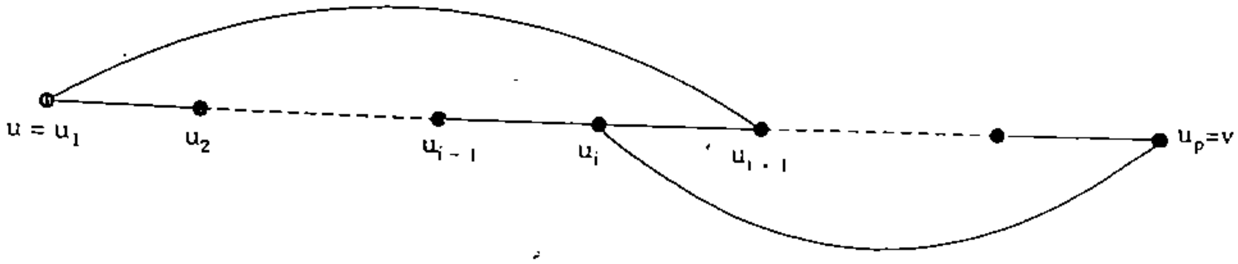


Fig.23

But then,

$$p \leq d_{G_M}(u) + d_{G_M}(v) = |S| + |T| = |S \cup T| < p. \text{ i.e. } p < p.$$

This is a contradiction. Thus, our assumption that the theorem is false, is wrong. In other words, every graph G on $p \geq 3$ vertices, satisfying (4) is Hamiltonian.

Remark: Note that Theorem 5 and Theorem 6 are just sufficient conditions. They are not at all necessary. For example, $C_n, n > 4$, is always Hamiltonian but C_n is a 2-regular graph, and therefore, $d(u) + d(v) = 4 < n$ always.

Here is an example to illustrate the use of the Theorems.

Example 8: To which of the graphs in Fig.24 does Dirac's criterion apply? To which does Ore's criterion apply?

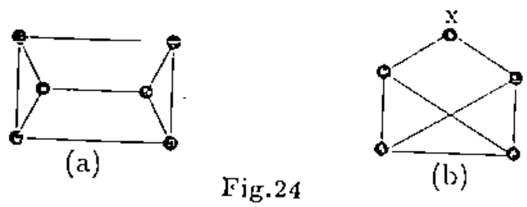


Fig.24

Solution: For the graph in Fig.24(a), $p = 6$ and $\deg(v) = 3$ for each vertex v. So, $\delta(G) = 3$. Thus, Dirac's criterion is satisfied for this graph.

For the graph in Fig.24(b), $p = 5$, but $\deg(x) = 2$. So, Dirac's criterion is not satisfied by this graph. However, $\deg(u) + \deg(v) \geq 2$ for all pairs of non-adjacent vertices u and v(in fact for all pairs u and v). So, Ore's criterion applies in this case.

* * *

Try the following exercise now to test your understanding of the example above.

E19) To which of the following graphs does Ore's Criterion apply ? To which of these does Dirac's criterion apply ?

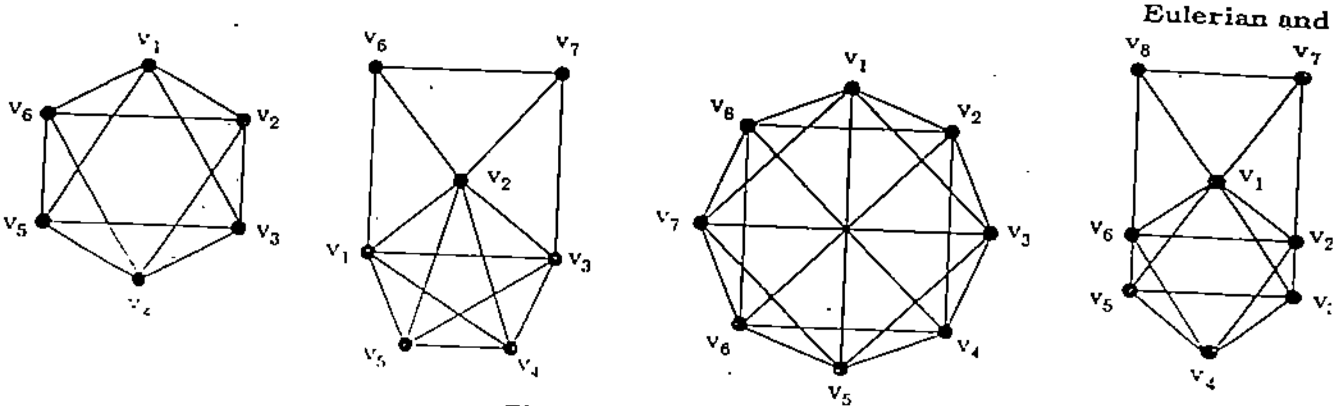


Fig.25

So far, we have seen a few necessary conditions and sufficient conditions for a graph to be Hamiltonian. Are there any conditions that are necessary and sufficient for a graph to be Hamiltonian? No! It is difficult to prove that a given graph is Hamiltonian. For example, the Petersen graph is not Hamiltonian. but it is not easy to show this. Indeed, so far no conditions have been found that are both necessary and sufficient for a graph to be Hamiltonian.

Now we have come to the end of our discussion the problem stated in the beginning of the section. In the next section we a related, but slightly different problem where we we assume that any two places are directly connected by a bus route. We are interested in finding a way of going around all the places. visiting each place only once, and doing so in the shortest possible time.

12.5 TRAVELLING SALESPERSON PROBLEM.

A travelling salesperson wants to visit a number of towns and return to the base. The travelling time between any two towns is known. How should he/she plan his/her journey so that he/she spends as short a time as possible but visits each town precisely once? This is known as the travelling salesperson problem. Here, one assumes that a direct route connects any two towns without passing through any of the other towns on the list. If we try to represent the towns by vertices and the direct route by edges. then we simply get a complete graph. How should we represent the time required to go from one town to the other? This question leads to the concept of a weighted graph.

Definition : A weighted graph is a pair (G, f), where G is a graph and f is a real valued function on the set $E(G)$.

In simple language, we associate some real number $f(e)$ with each edge e of the graph G. In the case of travelling salesperson problem, $f(e)$ is simply the time required to travel from one end vertex of e to the other end vertex.

Related to this we have another definition.

Definition : For a walk W in a weighted graph G, by the weight $f(W)$, of the walk W, we mean the sum of weights of all the edges in W.

So, our traveller's problem reduces to finding a Hamiltonian cycle of minimum weight in a weighted complete graph. One possible approach is to find a Hamiltonian cycle first and then search for edges having smaller weight and modify the cycle using them. The modification can be made as below:

Let $C = \{v_1, \cdots, v_p, v_1\}$ be a Hamiltonian cycle in a weighted complete graph. For a fixed i, first check whether there is a j such that

$$f(v_iv_j) + f(v_{i+1}v_{j+1}) < f(v_iv_{i+1}) + f(v_jv_{j+1}).$$

If this inequality holds, then replace the cycle C by

$$C_{i,j} = \{v_1, \cdots, v_i, v_j, v_{j-1}, \cdots, v_{i+1}, v_{j+1}, v_{j+2} \cdots, v_p, v_1\}.$$

See Fig.26(b).

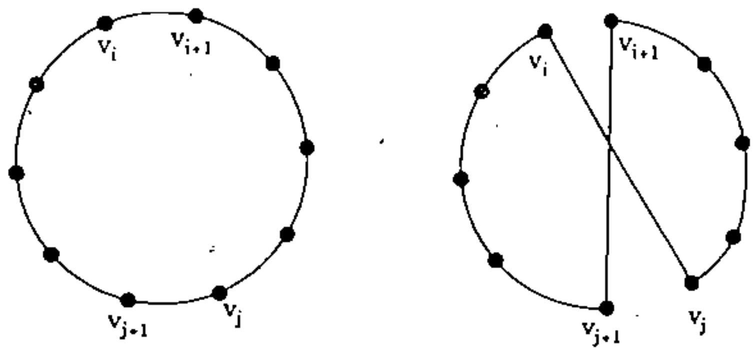


Fig.26

Clearly, the weight of the cycle $C_{i,j}$ is strictly less than that of the cycle C. After performing a sequence of such modifications, one is left with a cycle whose weight cannot be reduced further by this process. Of course, there is no guarantee that the resulting cycle will have the least possible weight. There may be other cycles with lower weight. But it will often be fairly good. Let us consider an example of this process.

Example 9: Consider the following copy of a weighted K_6. Starting with the cycle $\{L, M, N, O, P, T, L\}$, modify it to a cycle of lesser weight. The numbers on the edges indicate the weight assigned to them.

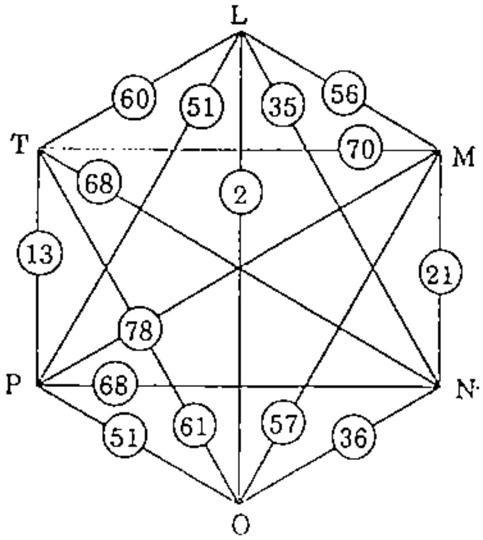


Fig.27

Solution: You can check that,

$$f(LO) + f(MP) = 80 < f(LM) + f(OP) = 107.$$

So, we modify the cycle to $\{L, O, N, M, P, T, L\}$.(see Fig.28(a). Now, $f(MT) + f(PL) = 121 < f(MP) + f(TL) = 138$. (See Fig.28)So, again we modify the cycle $\{L, O, N, M, P, T, L\}$ to $\{L, O, N, M, T, P, L\}$. Again,

$$f(OP) + f(NL) = 86 < f(ON) + f(PL) = 87. \text{ See Fig.28(c).}$$

Hence, replace the cycle $\{L, O, P, T, M, N, L\}$ by $\{L, O, N, M, T, P, L\}$. You can check that we can't decrease the weight of the cycle in the graph we have obtained in Fig.28(d).

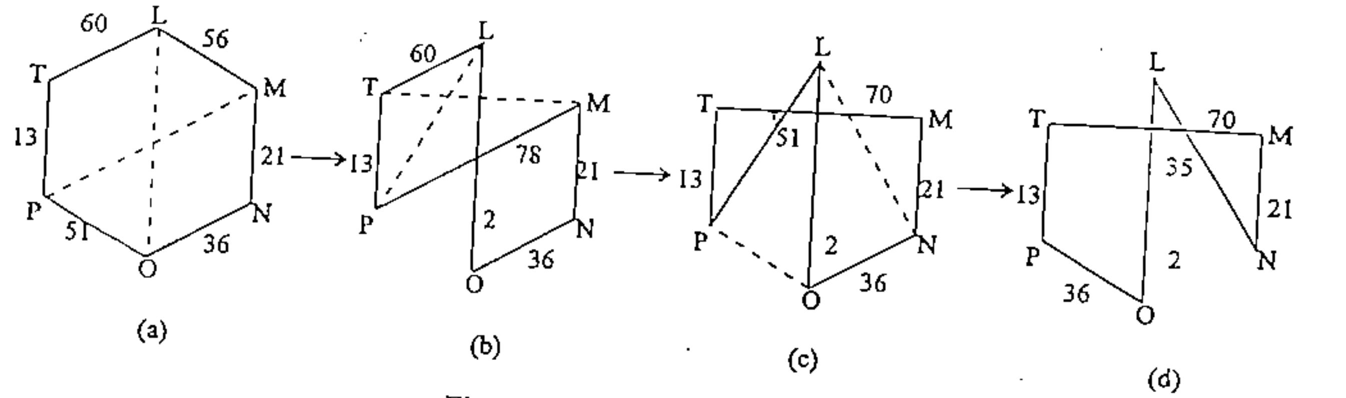


Fig.28

Hence, by this method we have reduced a cycle of weight 237 to a cycle of weight 192.

* * *

Here is a related exercise for you to try!

E20) Start with the cycle $\{v_1, v_2, v_3, v_4, v_5, v_1\}$ in the following weighted copy of K_5: Carry out the reduction step once to get a cycle of lesser weight.

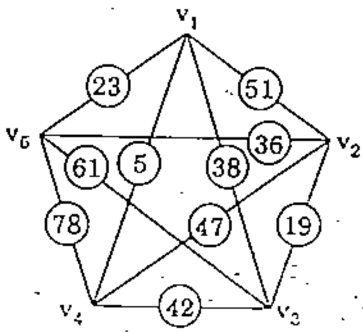


Fig.29

We have now reached the end of our unit. Let us briefly summarise what we have studied in this unit.

12.6 SUMMARY

In this unit we defined the following terms:

a) **Eulerian circuit:** A circuit in a graph is called Eulerian if each edge of the graph occurs exactly once in the circuit.

b) **Eulerian graph:** A connected graph is Eulerian if it contains an Eulerian circuit.

c) **Open trail:** A trail is open if the initial and end vertices of the trail are distinct.

d) **Edge traceable graphs:** A connected graph is edge traceable if it has an open trail.

e) **Hamiltonian cycle:** A cycle is called an Hamiltonian cycle if each vertex of the graph occurs exactly once in the cycle.

f) **Hamiltonian graphs:** A graph is called Hamiltonian if it contains a Hamiltonian cycle.

Also, in this unit, we discussed how to:

1) identify Eulerian graphs by considering the degree sequence.
2) identify which graphs are edge traceable by considering the degree sequence.
3) identify which figures can be drawn without lifting the pen from the paper and without going over any of the lines twice.
4) apply Fleury's algorithm to construct an Eulerian circuit.
5) apply some necessary conditions to show that a given graph is non-Hamiltonian.
6) apply sufficiency conditions due to Dirac and Ore to verify whether a given graph is Hamiltonian or not.
7) modify a given Hamiltonian cycle in a complete weighted graph to one of smaller weight.

12.7 SOLUTIONS/ANSWERS.

E1) Here is an Eulerian circuit:

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_1, v_7, v_8, v_4, v_{10}, v_9, v_1\}$$

Of course, there are many different Eulerian circuits and you might have come up with a different one.

E2) The situation will be as in Fig.30. After the addition of the new edge, both the vertices A and D have become even degree vertices. However, B and C still have odd degree. So, it is still not possible for the Königsbergians to go around the city using each bridge exactly once.

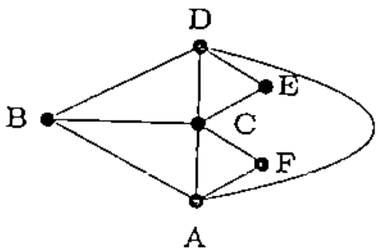


Fig.30

E3) The degree sequence of G_1 is $\{8, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$. All the vertices are even and hence the graph is Eulerian. You can check that the following gives an Eulerian circuit in it.

$$\{x_1, x_2, x_3, x_4, x_1, x_5, x_6, x_3, x_7, x_8, x_1, x_9, x_{10}, x_{11}, x_1, x_{12}, x_{13}, x_{14}, x_{15}, x_1\}.$$

The degree sequence of G_2 is $\{8, 4, 4, 4, 4, 4, 2, 2, 2, 2\}$. Since all the degrees are even. So, it is Eulerian. An Eulerian circuit in G_2 is as follows:

$$\{x_1, x_2, x_3, x_4, x_5, x_1, x_3, x_5, x_2, x_4, x_1, x_6, x_7, x_8, x_1, x_9, x_7, x_{10}, x_1\}.$$

E4) a) K_n is an $(n-1)$-regular graph. So, it is Eulerian when $n-1$ is even, (i.e.) n is odd.

b) $K_{n,m}$ has n vertices of degree $m-1$ and m vertices of degree $n-1$. So, it is Eulerian when n, m are odd.

E5) In Q_3, every vertex has degree 3 and hence it is a non-Eulerian graph. On the other hand, all the vertices of Q_4, have degree 4. Hence Q_4 is Eulerian.

E6) Suppose G is an Eulerian graph and $\{v_0, v_1, \ldots, v_n\}$ is an Eulerian trail in it. Let $x = v_i$ be any vertex in G. Then, the following is an Eulerian trail starting and ending at x:
$\{x = v_i, v_{i+1}, \ldots, v_n, v_0, v_1, \ldots, v_{i-1}\}$

E7) Refer to Fig.30. After the construction of the new bridge joining A and D all the vertices except B and C are even, i.e. there are two vertices of odd degree. So, it is possible to go round the city using each bridge only once, starting and ending the trip at two different points.

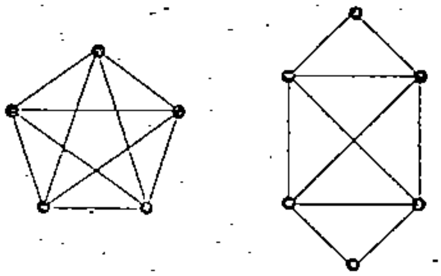
E8) a) Let us write down the degree sequence of the graph. It is $\{4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3\}$.
It has eight vertices of odd degree.
So, the graph in Fig.9(a) is not edge traceable.

b) The degree sequence of the graph in Fig.9(b) is $\{4, 3, 3, 2, 2, 2\}$. So, it has exactly two vertices of odd degree. So, the graph is edge traceable.

E9) Since G_1 has exactly two vertices of odd degree, it can be drawn without lifting the pencil from the paper and without going over any of the vertices twice.

Since G_2 has precisely two vertices of odd degree, this can also be traced without lifting the pen from the paper. Since G_3 has 6 vertices of odd degree,(degree 3), it cannot be traced without lifting the pen from the paper.

E10) The solutions for (a), and (b) are given below.



(a) Fig.31 (b)

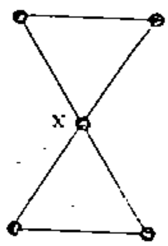


Fig.32

(c) Recall that any Eulerian graph is connected. Here, the number of vertices is one more than the number of edges. So, such a graph is a tree and therefore does not contain any cycle. Thus, there is no Eulerian graph with the given number of vertices and edges.

E11) You can check that one Eulerian circuit in the given graph is
$\{x_1, x_4, x_2, x_5, x_3, x_6, x_9, x_5, x_8, x_1, x_7, x_1\}$.
The bridges chosen are
$\{x_2x_5, x_3x_6, x_6x_9, x_9x_5, x_5x_8, x_8x_1, x_1x_7, x_7x_1, \}$

E12) For example, consider the graph in Fig.32 This is non-Hamiltonian because the vertex x is a cut vertex.

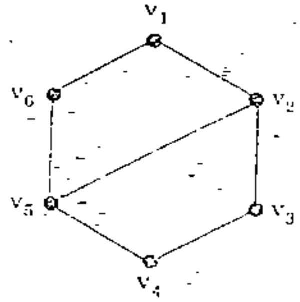


Fig.33

E13) See Fig.33. This has a Hamiltonian cycle $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$. But, it is not Eulerian because the vertices v_2 and v_5 have odd degrees.

E14) The graph given in Fig.32 is Eulerian because all its vertices have even degree. As, we have seen already, it is not Hamiltonian.

E15) A Hamiltonian cycle in Q_3 is $\{000, 100, 110, 010, 011, 111, 101, 001, 000\}$.

E16) If you remove the vertices marked x, y and z in Fig.34 from this graph, you will get four connected components, viz., one inner triangle and three isolated outer vertices.

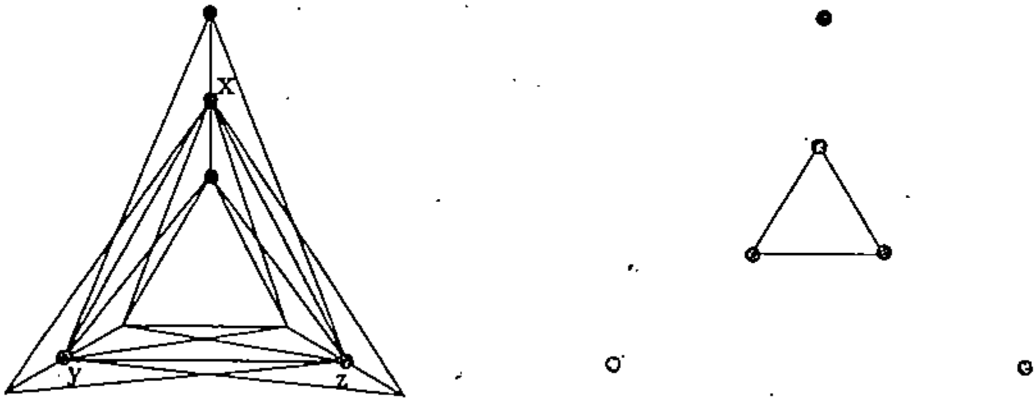


Fig.34

Hence, by Theorem 4, the given graph is non-Hamiltonian.

E17) A Hamiltonian cycle in the graph G_1 is $\{x_7, x_3, x_4, x_2, x_6, x_5, x_1, x_7\}$. Now check that the following cycle in the graph G_2 is a Hamiltonian cycle.

$$\{x_{12}, x_{14}, x_8, x_9, x_{10}, x_{11}, x_{13}, x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_{12}\}.$$

E18) There are three vertices of degree eight in this graph. If we remove them we get four connected components. Now apply Theorem 4.

E19) (a) This is a 4-regular graph. So, $\delta(G) = 4$. Here $p = 6$ and therefore the condition $\delta(G) \geq \frac{p}{2}$ is satisfied. So, Dirac's criterion(and therefore Ore's criterion) apply here.

(b) Here $p = 7$. The vertices v_6 and v_7 have degree $3 < \frac{7}{2}$. Therefore, Dirac's criterion does not apply. However, the only pair of non-adjacent vertices in this graph are

$$(v_6, v_4), (v_6, v_5), (v_6, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_1)$$

Ore's condition is satisfied for these pair of vertices. So, this graph is Hamiltonian.

c) Here $p = 8$ and the graph is 4-regular. So, Dirac's criterion is satisfied.

d) Here $p = 8$, but the vertices v_8 and v_4 have degree 3 which less than $\frac{p}{2} = 4$. So, Dirac's criterion is not satisfied. The only pairs of non adjacent vertices are (v_7, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_6), (v_8, v_2), (v_8, v_3), (v_8, v_4), (v_8, v_5). You can check that Ore's criterion is satisfied for these pair of vertices.

E20) Notice that

$$\phi(v_1v_2) + \phi(v_4v_5) = 51 + 78 = 129$$
$$\phi(v_1v_4) + \phi(v_2v_5) = 5 + 36 = 41$$

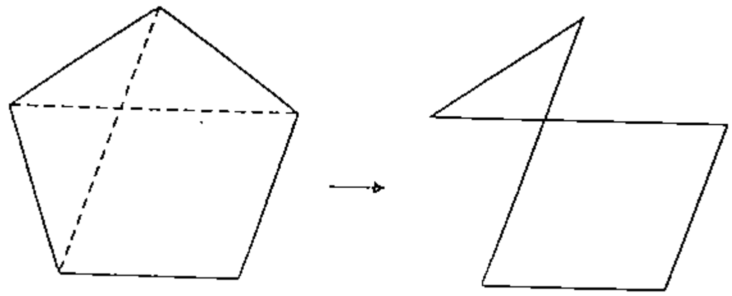


Fig.35

We can modify the given cycle to get the following cycle of smaller weight: $\{v_1, v_4, v_3, v_2, v_5, v_1,\}$

UNIT 13 GRAPH COLOURINGS AND PLANAR GRAPHS

Structure

Page No.

13.1 INTRODUCTION

You must have seen political maps of India with different states coloured differently to distinguish between them. Have you ever wondered what is the minimum number of colours required to colour the map so that any two states with a common boundary are given two different colours? This problem of finding the minimum number of colours needed to colour a given map is called the map colouring problem.

We can formulate the problem in terms of graph theory. We can construct a graph in such a way that each state of India corresponds to a vertex of India and if two states are adjacent, the corresponding vertices are also adjacent. So, we have to colour the vertices of the graph in such a way that any pair of adjacent vertices have different colours. In the map colouring problem, we ask for the minimum number of colours needed to carry out such a colouring.

Note that the construction mentioned above leads to a special class of graphs called planar graphs. If we are interested in map colouring problem alone, it is enough to restrict ourselves to such graphs. However, the general vertex colouring problem, which asks for the minimum number of colours needed to colour the vertices of a given graph, not necessarily planar, is interesting in itself. So, we start our unit by discussing this problem in Sec.13.2.

In Sec.13.3, as a preparation for our study of map colouring problem, we study planar graphs. In this section we will prove some basic results about planar graphs. We will also prove a characterisation of planar graphs due to Kuratowski.

In Sec.13.4, we study the map colouring problem. We give a brief history of the four colour theorem, which says that any map can be coloured with four colours. The proof of this theorem is beyond the scope of this course. However, we will prove the weaker result that any map can be coloured with five colours.

In Sec.13.5, we end our unit with a brief discussion of edge colourings. We restrict ourselves to the definition of edge colouring, some examples of edge colouring and statements of some of the well known results in this field.

Objectives

After reading this unit, you should be able to

- compute the chromatic number of some simple graphs;
- compute some upper and lower bounds for the vertex chromatic number $\chi(G)$ of a graph G;
- Verify whether a given graph is planar or not using Kuratowski's theorem in simple cases;
- give an edge-colouring with $\chi'(G)$ colours for some simple graphs, where $\chi'(G)$ is the edge chromatic number of a graph G.

13.2 VERTEX COLOURINGS

In this section we start our study of colourings with vertex colouring. In the Subsection 13.2.1, we define vertex colouring and give some examples. In the Subsection 13.2.2, we will prove some simple bounds on the minimum number of colours needed to colour the vertices of a given graph. Let us now start our study of vertex colouring with the definition and some examples of colourings.

13.2.1 Definition and Examples

Look at the graph in Fig.1. We have given a colouring of K_3 using three colours, namely red, green and blue.

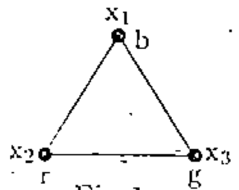


Fig.1

Why have we used three colours? It is because we want the adjacent vertices to have different colours. In K_3, any two vertices are adjacent so we need to colour each of the vertices with different colours. Keep this example in mind when you read the definition of vertex colouring given below.

Definition : A **k-vertex colouring** of a graph G is an assignment of k colours to each of the vertices of G in such a way that no two adjacent vertices have the same colour. A graph is **k-vertex colourable** if there is a k-vertex colouring. The **minimum number** of colours required to colour a graph G is called the **vertex chromatic number of G**, usually denoted as $\chi(G)$.

In this section, we will be discussing only vertex colouring. So, we will use the terms '**k-colouring**', '**k-colourable**' and '**chromatic number**', respectively. We will say that a graph is k-chromatic if it has chromatic number k.

In Fig.1, we were able to use the names of the colours, red, green and blue, because we needed only three colours. Suppose we need, say, 20 colours, can we still use the names to refer to the colours? We may not remember the names of so many colours and may probably decide to call them colour 1, colour 2, etc. This will do just as well, because the names of the colours are not important as long as you can distinguish between the different colours. We will also use $1, 2, 3, \ldots$ to denote our colours. However, to distinguish them from usual numbers, we will denote them as $\boxed{1}, \boxed{2}, \ldots$.

Let us now look at some examples.

Example 1: Colour the graphs in Fig.2 with the minimum possible number of colours. Also, find the chromatic numbers of the graphs.

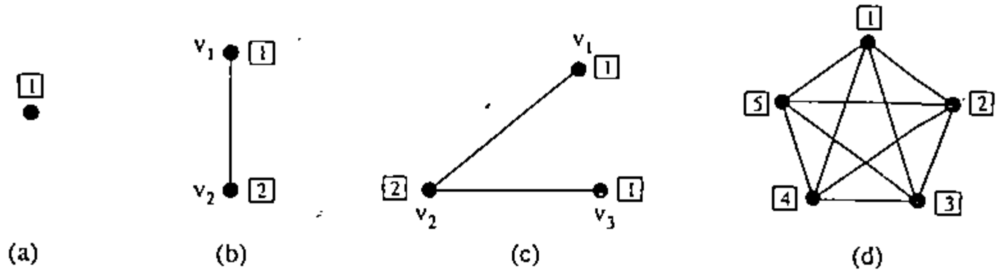


Fig.2 Some examples of colouring.

Solution: In Fig.2(a), K_1, has just one vertex. Let us colour this with $\boxed{1}$. Thus, this graph is 1-colourable and its chromatic number is 1.

In Fig.2(b), K_2 has two adjacent vertices. We assign $\boxed{1}$ to the vertex v_1 and $\boxed{2}$ to the vertex v_2. Thus, we have a 2-colouring. Is there a 1-colouring? No! The two vertices are adjacent and so we need at least two colours. In other words, the chromatic number $\chi(K_2) = 2$.

In Fig. 2(c), we have three vertices and we can colour them with three different colours. But, can we also have a two colouring? Notice that, v_1 and v_3 are not adjacent. So, we can colour them with the same colour, say, $\boxed{1}$. v_2 is adjacent to both v_1 and v_3. So, we cannot assign $\boxed{1}$ to this. Let us assign $\boxed{2}$ to v_2. So, we have a 2-colouring. As we cannot have a 1-colouring, this graph has chromatic number 2.

In Fig. 2(d), we have K_5. In this any two vertices are adjacent, so we need as many colours as there are vertices, that is, we need five colours. So, K_5 has chromatic number 5.

* * *

Remark: In the above example, we saw that the chromatic number of K_1 is 1. More generally, if a graph consists of isolated vertices, its chromatic number is 1. Conversely, if the chromatic number of a graph is 1, it consists of isolated vertices.

Also, we saw that the chromatic number of K_5 is 5. More generally, the chromatic number of K_n is n, because any pair of vertices are adjacent in K_n.

Example 2: Find the chromatic number of a bipartite graph with edge set non empty.

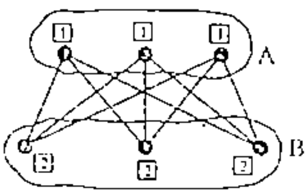


Fig.3

Solution: From unit 11, you may recall that a graph G is bipartite if the vertex set of G can be partitioned into two non empty disjoint subsets A and B such that any two vertices in a given set are non-adjacent. We get a 2-colouring of G by assigning $\boxed{1}$ to the vertices in A and $\boxed{2}$ to all the vertices in B.(This is illustrated in a particular case in Fig.3). Further, note that, since A and B are non empty and since the edge set of G is non empty, at least one vertex in A is adjacent to a vertex in B and these two vertices must have different colours. So, we cannot manage with less than two colours. So,

$\chi(G) = 2$ if G is a bipartite graph with non empty edge set.

* * *

Remark: We saw in example 2 that the chromatic number of a bipartite graph with non empty edge set is 2. The converse is also true. Given a graph G and a 2-colouring of G, we can partition the edge set of G into two non empty sets A and B defined as follows:

$$A = \{v \in V(G) \mid v \text{ is assigned the colour } \boxed{1}\}$$

$$B = \{v \in V(G) \mid v \text{ is assigned the colour } \boxed{2}\}$$

By the definition of colouring no two vertices in A are adjacent and similarly for B. Since A and B are disjoint, G is bipartite by definition.

Here are some exercises to test your understanding of the above examples.

E1) What is the chromatic number of a tree with at least two vertices?

E2) What is the chromatic number of an even cycle C_{2n}, $n \geq 2$?

E3) Is an odd cycle C_{2n+1}, $n \geq 1$, 2-colourable? What is its chromatic number?

If a graph is k-colourable, are all its subgraphs k-colourable? Let us see. Let G be a k-colourable graph and H be its subgraph. We assign to each vertex of H the same colour that we assigned to it, considered as a vertex of G. If two vertices are non-adjacent in G, they are non-adjacent in H and therefore this gives a colouring of H. In other words, $\chi(H) \leq k = \chi(G)$ for every subgraph H of G. We can also recast this statement in the following form. If **a graph G has a subgraph H with chromatic number k, the chromatic number of G must be at least k.** This fact helps us in finding the chromatic number of a graph sometimes. We illustrate this in the next example.

Example 3: Find the chromatic number of Grötzsch graph.(See Fig.4.)

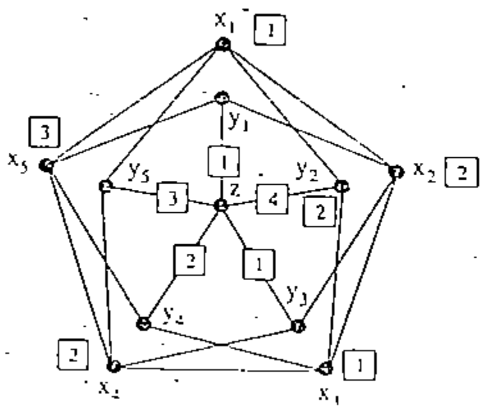


Fig.4 Grötzsch graph

Solution: The figure above gives a 4−colouring of this graph. Can this graph have a three colouring? Let us try to find one. Since the outer 5-cycle is an odd cycle, it needs three colours. So, we need at least three colours. Let us suppose the colours of $x_1, \cdots, x_5$ are as shown in Fig.4. Since y_1 is adjacent to x_2 and x_5 we have to give it a colour different from $\boxed{2}$ and $\boxed{3}$. So, we assign $\boxed{1}$ to it. Similarly, the colours of y_4 and y_5 must be $\boxed{2}$ and $\boxed{3}$ respectively. Since the vertex z is adjacent to vertices to which the colours $\boxed{1}$,

[2] and [3] have been allotted, we have to use a fourth colour for this vertex. So, this graph is not 3−colourable. Therefore, this has chromatic number 4.

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Try the exercises given below to test your understanding of the example given above.

E4) Show that the chromatic number of the Petersen graph, given in Fig. 5, is 3.

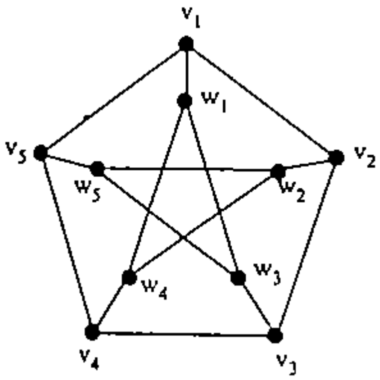


Fig.5 Petersen graph.

In the above examples and exercises, we saw that if a graph G has a subgraph H with chromatic number $\chi(H) = n$, $\chi(G) \geq n$. In particular, if a graph G has a subgraph H which is isomorphic to K_n(such a subgraph H is known as a **clique of size n**), the chromatic number of G is at least n. However, the converse is not true, i.e. if a graph has chromatic number $\geq n$, it need not have a clique of size n. Petersen graph provides a counter example for this. As we have seen, the chromatic number of Petersen graph is 3. Convince yourself—you need not prove it—that it does not contain a clique of size 3, i.e., a subgraph isomorphic to K_3. More generally, in 1955, Mycielski proved that, for any integer k, there exists a k-chromatic graph without triangles. The proof of this result is beyond the scope of this course. However, it is not difficult to prove the much weaker result that if the chromatic number of a connected graph is greater than 2, it contains an odd cycle. We leave this as an exercise for you, along with some more exercises, to test your understanding of the material we have covered so far.

E5) Show that if $\chi(G) \geq 3$ for a graph G, it contains an odd cycle.

E6) (a) Find a 3-colouring of the figure in Fig.6.

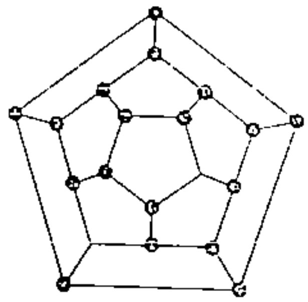


Fig.6

(b) What is the chromatic number of the graph in Fig.6?

E7) Find the chromatic number of the following graph.

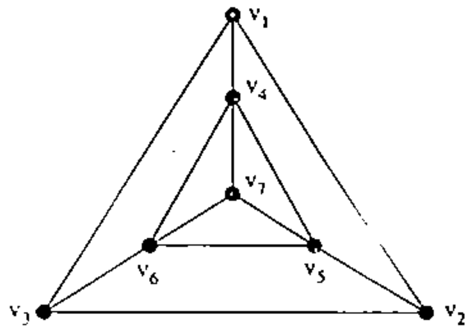


Fig.7

E8) Construct a graph with chromatic number 5.

Recall that, we have shown that any 2-colourable graph is bipartite. How was this done? We had put all the vertices having the same colour in a single set. There were two colours and so we got two subsets. They were disjoint because no vertex can be assigned two colours.

We are going to extend the ideas to n-colourable graphs. We do this through the concept of colour classes. First, let us define the colour classes of a colouring.

Definition : For a k-colouring of a graph G, consider the set $C_i = \{x \in V(G) \mid x \text{ is assigned the colour } \boxed{i}\}$, for $1 \le i \le k$. Clearly, $C_i \cap C_j = \phi$, for every $i \neq j$, and $V(G) = C_1 \cup \cdots \cup C_k$. If $\chi(G) = k$, each of the k colours is assigned to at least one vertex.(Why?) So none of these subsets is empty. Therefore, we get a partition of the vertex set V(G) into k mutually disjoint non empty subsets. The subsets $C_1, \cdots, C_k$ are called **the colour classes** of G given by the colouring.

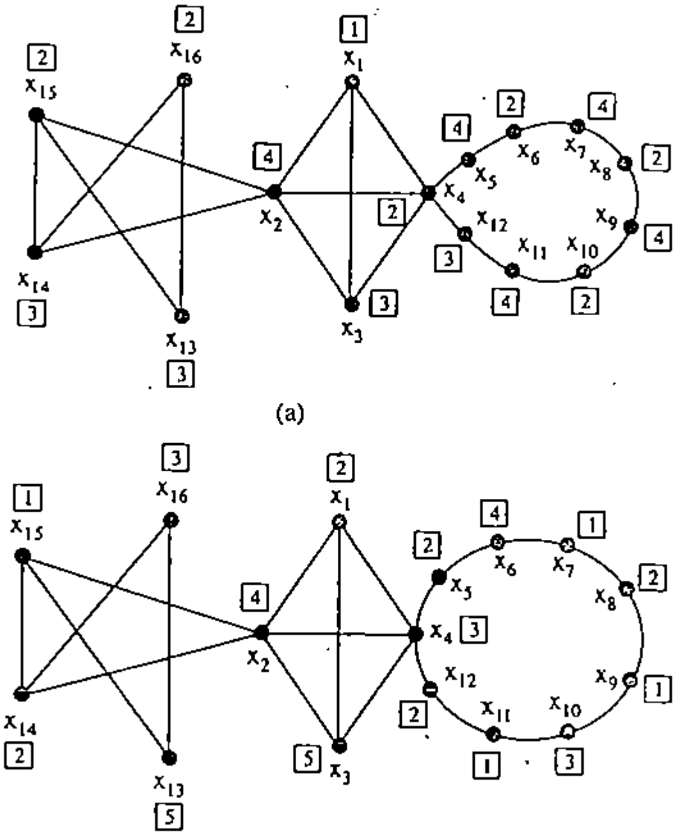
So, the colour classes of a 2-colourable graph gives a bipartition of the vertex set of the graph, making it bipartite.

Let us now look at some examples of colour classes.

The colour classes can be defined for any colouring of a graph G, not just for a $\chi(G)$-colouring.

Example 4: Find the colour classes in the two different colourings of the same graph. The colour classes given by the colouring in Fig.8(a) are $C_1 = \{x_1\}$, $C_2 = \{x_4, x_6, x_8, x_{10}, x_{15}, x_{16}\}$, $C_3 = \{x_3, x_{12}, x_{13}, x_{14}\}$ and $C_4 = \{x_2, x_5, x_7, x_9, x_{11}\}$.
You can check that $C_1 = \{x_7, x_9, x_{11}, x_{15}\}$, $C_2 = \{x_1, x_5, x_8, x_{12}, x_{14}\}$, $C_3 = \{x_4, x_{10}, x_{16}\}$, $C_4 = \{x_2, x_6\}$, and $C_5 = \{x_3, x_{13}\}$ are the colour classes corresponding to the colouring in Fig.8(b).

* * *



(a)

(b)

Fig.8

Try the next exercise to test your understanding of the above example.

E9) Colour the following graph in two different ways and give the colour classes in each of the cases.

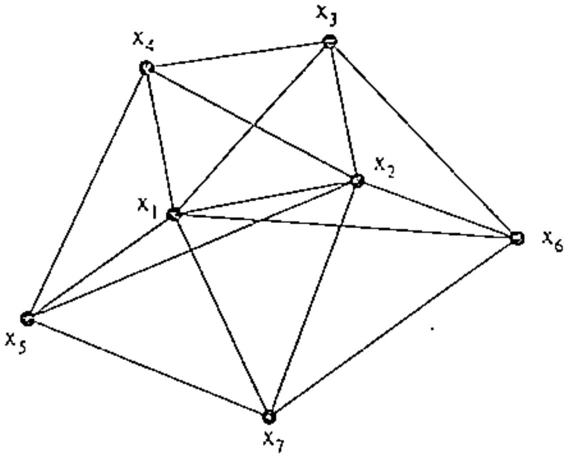


Fig.9

We have seen that any colouring of a graph G gives rise to the colour classes. You know that, if x, y are two vertices in a colour class C_i, then $xy \notin E(G)$. So, **each colour class consists of mutually non-adjacent vertices.** We now give a name to those subsets of the vertex set of a graph with this property.

Definition : A subset S of the vertex set V(G) of a graph G, is said to be an **independent set** if any two vertices in S are non-adjacent. An independent set is called **maximal** if it is not contained in any other independent set. The number of vertices in a largest independent set of G, is called the **independence number** of the graph G and it is denoted by $\alpha(G)$.

Example 5: Find three different maximal independent sets in the graph given Fig.10.

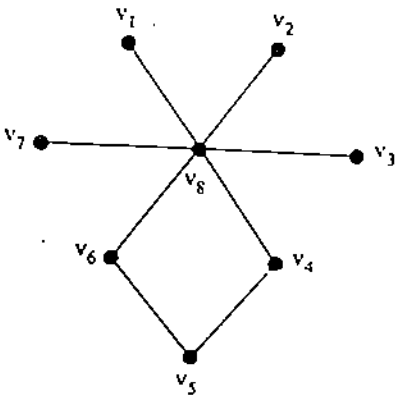


Fig.10

Solution: In Fig.10 we have the following maximal independence sets:

$\{v_8, v_5\}$, $\{v_5, v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$

We check that $\{v_8, v_5\}$ is a maximal independent set. This is easy to see because all the other remaining vertices are adjacent to one of these two vertices. So, if any more vertices are added, the resulting set will no longer be an independent set. You can check that the other two sets are also maximal independent sets in the same way.

* * *

Now test your understanding of independent set by trying the following exercise:

E10) Find an independent set of cardinality 4 in the graph given below:

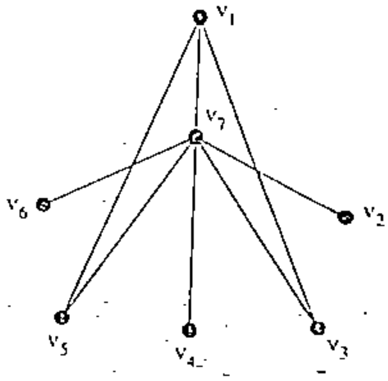


Fig.11

E11) Find out $\alpha(G)$ for the graphs given in Fig.7 and Fig.8

Remark: We saw that both a colour class of a colouring and independent sets have the property that any two vertices in it are linearly independent. However, while colour classes depend on a particular colouring, independent set does not. This is the difference between these two concepts.

13.2.2 Bounds for Chromatic Numbers

In this section we will prove some bounds for the chromatic number of a graph in terms of $\Delta(G)$. For this, we need the concept of k-critical graphs.

Recall that,
$\delta(G) = \min\{d_G(v) \mid v \in E(G)\}$
$\Delta(G) = \max\{d_G(v) \mid v \in E(G)\}$

We will introduce you to this concept through an example. Consider the graph K_4. If we remove a vertex, we get a graph isomorphic to the graph in Fig.12(a).

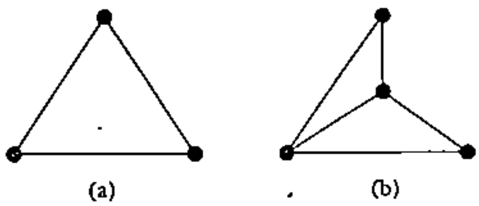


Fig.12 Graphs obtained by removing a vertex or an edge from K_4

If we remove an edge, we get a graph isomorphic to the graph Fig.12(b). Both the graphs have chromatic number three, as you can easily verify. Also, any other proper subgraph of K_4 is contained in one of these graphs. So, the chromatic number of any subgraph of K_4 is strictly less than that of K_4 which is 4. This shows that K_4 shows that K_4 is 4-critical, as the following definition shows.

Definition : A graph G is said to be **critical** or **k−critical** or **critically k−chromatic** if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of the graph G.

Thus, in the discussion before the definition of k-critical graphs, we have shown that K_4 is 4-critical. Let us look at one more example.

Example 6: Show that Grötzsch graph is 4-critical.

Solution: Refer to Fig.4 for Grötzsch graph. Let us remove a vertex of this graph. Depending on the vertex, we get a subgraph isomorphic to one of the three following graphs:

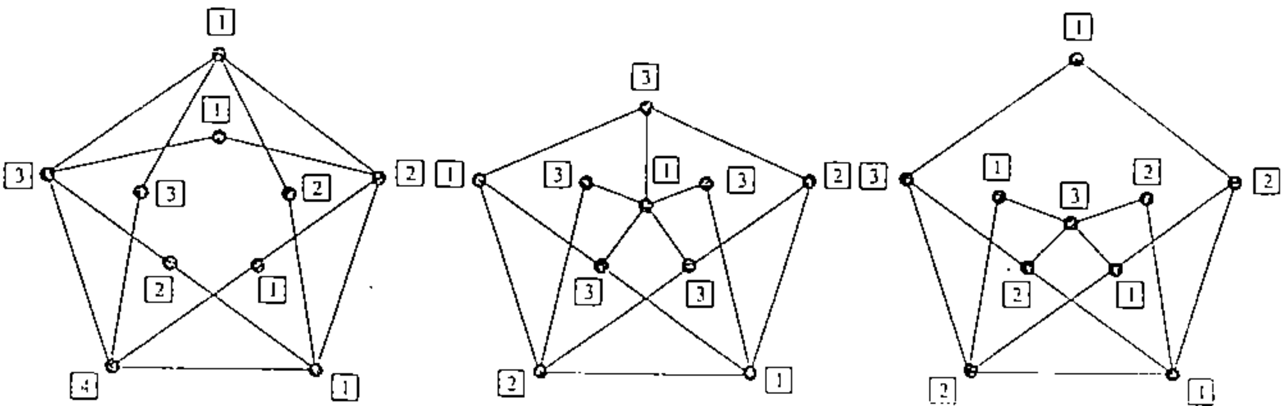


Fig.13 Graphs obtained by removing a vertex from Grötzsch graph

From the colouring given in these figures, it is clear that these graphs are 3−colourable. Moreover, all of them contain 5−cycles, that is, they are not 2−colourable. Thus, their chromatic number is $3 < 4 = \chi(G)$. This means that $\chi(G - v) < \chi(G)$, for every vertex v of the graph G.

Now, if we remove just one edge of G, without removing any vertex, we get a subgraph isomorphic to one of the graphs in Fig.14. The dotted lines indicate the edges we have deleted.

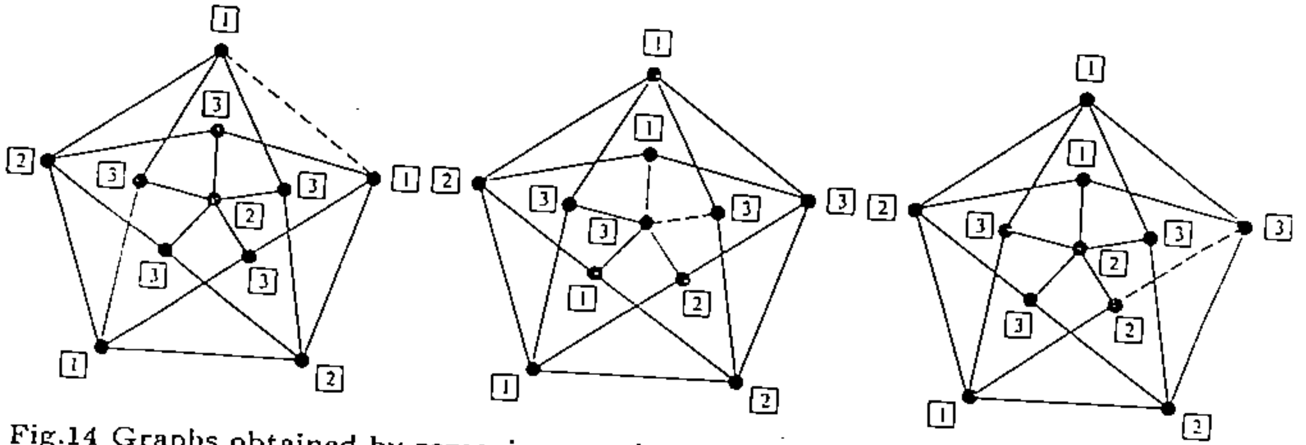


Fig.14 Graphs obtained by removing an edge from the Grötzsch graph.

The 3−colouring of the subgraphs that we get is also given in the figure. Moreover, these graphs also contain 5−cycles and hence have chromatic number 3. Thus, $\chi(G-e) < \chi(G)$, for every edge e of G. But then, every proper subgraph of G is in fact subgraph of one of the six graphs in Fig.13 and Fig.14. Thus, $\chi(H) < \chi(G)$, for every proper subgraph H of the graph G. So, the Grötzsch graph is 4-critical.

* * *

Here is an exercise to test your understanding of the definition of k-critical graphs.

E12) Show that K_n is an n-critical graph.

E13) Check whether the Petersen graph is 3-critical.

Now, consider a graph with chromatic number k. Need it be k-critical? The following example will help you answer this question.

Example 7: Show that the graph given in Fig.15 is 3-chromatic, but it is not 3-critical.

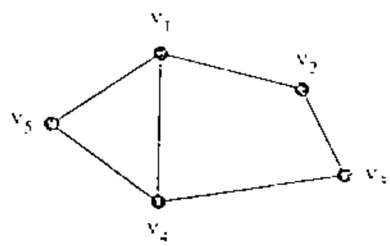


Fig.15

Solution: We can assign [1] to v_3 and v_5, [2] to v_2 and v_4, [3] to v_1. This gives us a three colouring of the graph. This graph is 3-chromatic because it contains a 3-cycle $\{v_1, v_4, v_5\}$. If we remove the vertex v_2, the resulting graph still has chromatic number 3 because it still contains the 3-cycle v_1, v_4, v_5. So, G has a subgraph which has the same chromatic number as G. So, it cannot be 3-critical.

* * *

Now that we know that a graph with chromatic number k needn't be

3-critical, you may wonder if it will contain a k-critical subgraph. The following result tells you that this is true.

Theorem 1: Let G be a graph with chromatic number k. Then, it has a subgraph which is k-critical.

Proof: Consider a graph G with $\chi(G) = k$. If it is k-critical, we are done. If it is not, it has a vertex v such that $\chi(G - v) = k$. If $G - v$ is k-critical, we are done. Otherwise, we can remove another vertex and get a subgraph with chromatic number k. We repeat the process. In the worst case, we will be left with a k-chromatic subgraph of G on k vertices. If we remove any vertex from this graph, we will get a graph on $k - 1$ vertices which is (k-1)-colourable. So, the k-chromatic subgraph on k vertices that we have obtained is k-chromatic.

We now discuss an example that illustrates Theorem 1.

Example 8: Find a 3-critical subgraph of the graph given in Fig.15.

Solution: On removing the vertices v_2 and v_3, we get a graph isomorphic to K_3. This is 3-critical.

* * *

Here is a related exercise for you to try!

E14) Find a 3-critical subgraph of the Petersen graph.

Let now make table of the values of $\chi(G)$ and $\Delta(G)$ for some of the examples we have discussed so far to see if we can find a relationship between these two quantities.

G	$\chi(G)$	$\Delta(G)$
Grötzsch graph.	4	5
Petersen graph	3	3
Dodecahedron	3	3
C_5	3	2
K_4	4	3
K_5	5	4

Observe that, except for C_5, K_5 and K_4, all the other graphs satisfy the relation $\chi(G) \leq \Delta(G)$. In 1941, R. I. Brooks proved the following result.

Theorem 2: Let G be a connected graph which is neither an odd cycle nor a complete graph. Then,

$$\chi(G) \leq \Delta(G)$$

We will not prove theorem 2 in this course. However, we will prove the following weaker result.

Theorem 3: For every k−chromatic graph G,

$$\chi(G) \leq \Delta(G) + 1$$

Let us now illustrate the application of Brooks' theorem through an example.

Example 9: Find the chromatic number of the graph in Fig.16.

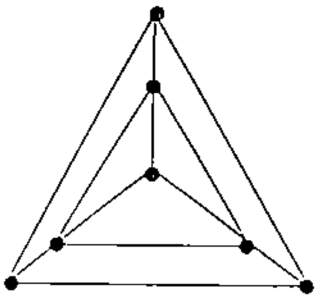


Fig.16

Solution: The maximum degree $\Delta(G)$ is 4 for this graph. So, by Brooks' theorem, the chromatic number at most 4. But, it has a subgraph isomorphic to K_4,(the subgraph formed by the inner triangle and the vertex in the centre.). So, its chromatic number is at least 4. Therefore its chromatic number is exactly 4.

* * *

Remark: The bound given by Brooks' theorem may not be as good as it was in example 9. For example, in the case of $K_{1,n}$, $\Delta(K_{1,n}) = n$, $\chi(K_{1,n})$ is 2. So the difference $\chi(K_{1,n}) - \Delta(K_{1,n}) = n - 2$, is large when n is large.

We now prove a lemma that will be used in the proof of theorem 2.

Lemma 1 If G is a k−critical graph with minimum degree $\delta(G)$, then $(k-1) \leq \delta(G)$.

Proof: If possible, let G be a k−critical graph with $\delta(G) < (k-1)$. Let $v \in V(G)$ such that $\delta(G) = d_G(v)$. Since G is k−critical, $\chi(G - v) \leq (k-1)$, that is $G - v$ has a $(k-1)$−colouring. Since $d_G(v) < (k-1)$, v is adjacent to fewer than $k-1$ vertices. So, there is at least one colour [i], among the $k-1$ colours, that is not assigned to any of the $k-1$ vertices adjacent to v. We can assign this colour to v to get a $k-1$ colouring of G. This contradicts the fact that $\chi(G) = k$. Thus our assumption is wrong, that is, $\delta(G) \geq (k-1)$.

Corollary 1 Every k−chromatic graph G has at least k vertices of degree $\geq (k-1)$.

Proof: Let G be a k−chromatic graph, that is, $\chi(G) = k$. Let H be a k−critical subgraph of G. Thus, $|V(H)| \geq k$ and $\delta(H) \geq (k-1)$. This means that every vertex x of H satisfy the property that $d_G(x) \geq d_H(x) \geq \delta(H) \geq (k-1)$. There are at least k such vertices. This proves the result.

We now prove theorem 2.

Proof of Theorem 2: Using Corollary 1 to Lemma 1, choose a vertex $x \in V(G)$, such that $d_G(x) \geq (k-1)$. But then, $\Delta(G) \geq d_G(x) \geq (k-1)$, that is $\chi(G) = k \leq \Delta(G) + 1$.

In the introduction, we mentioned that the map colouring problem can be reduced to finding the minimum colours needed to colour a special class of graphs called planar graphs. In the next section, we define planar graphs and prove some basic results that will be useful in the study of map colouring problem.

13.3 PLANAR GRAPHS

In transistor radios and television sets, you must have seen printed circuit boards. These boards have slots for various components and these slots are connected to each other. The connections between these slots must be made in such a way that no two connections cross each other. Given an electronic circuit, is it always possible to design a printed circuit board corresponding to it?

This can be formulated as a problem in graph theory. We replace the electronic components by vertices and the connections between them by edges. If the resulting graph can be drawn in such a way that no two of the edges cross each other except at the vertices, then we can design a printed circuit board for the given circuit. Graphs that can be drawn this way are called planar graphs.

We begin our study of planar graphs in this section. We begin this section by defining planar graphs. After giving some examples, we will prove some basic results on planar graphs like Euler's formula. From this, we will derive some necessary conditions for a graph to be planar. Using these conditions, we will show that $K_{3,3}$ and K_5 are non-planar.

Let's start by seeing what a planar graph is.

Definition : A graph G is called **planar** if it can be drawn on the plane in such a way that no two edges cross each other at any point except possibly at the common end vertex. Such a drawing is called a **plane drawing.**

Here are some examples of planar graphs. In the first row of Fig.17 we have given the five regular solids called Platonic solids. In the second row, we have given the corresponding graphs. In each of these graphs, the vertices correspond to the vertices of the corresponding solid and the edges correspond the edges of the solid.

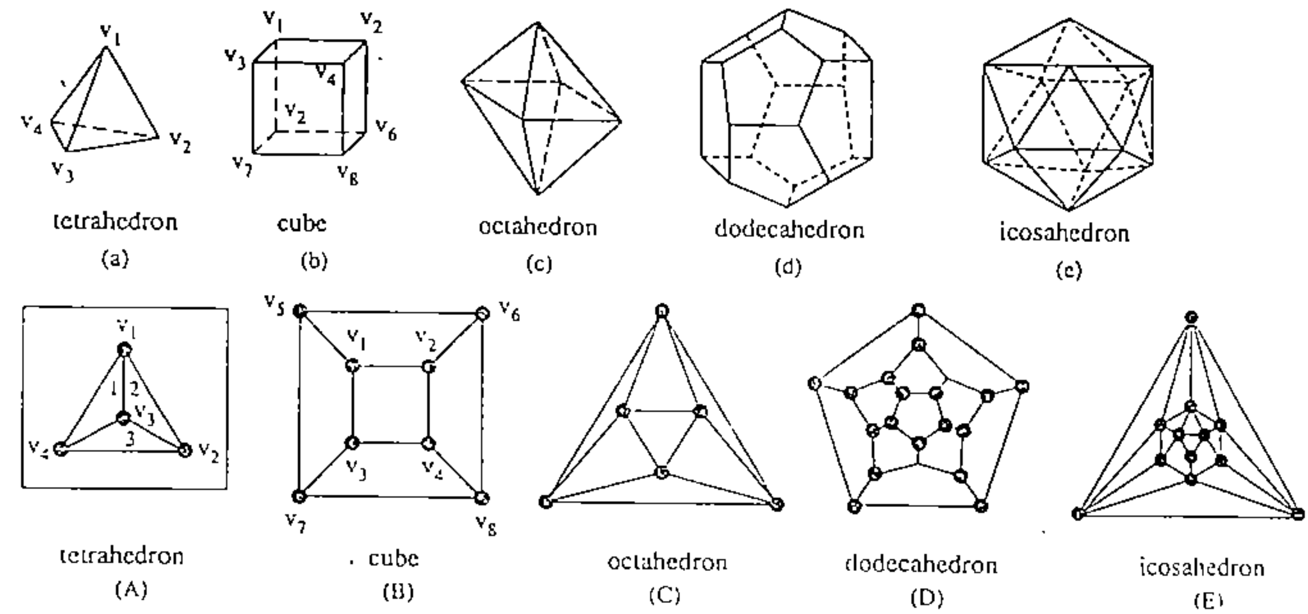


Fig.17 Regular solids and their graphs.

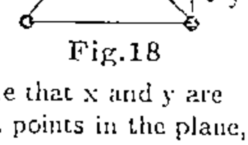


Fig.18

Note that x and y are just points in the plane, they are not vertices.

Next, we introduce the concept of a region. Look at the tetrahedron in Fig.17(a). It has four faces. The graph corresponding to it is given in Fig.17(A). It divides the plane into four faces or regions. Similarly, the cube given in Fig.17(b) divides the plane into six regions.

In all the above cases, it is very clear what the different regions are. But, look at the graph in Fig.18. Into how many regions does it divide the plane? Two or three? Do the points x and y lie in the same region or in different region? To avoid such confusion we need to define the concept of a region rigorously. Here is the definition of a region.

Definition : Given a plane drawing of a planar graph G, by a **region** or **face** of G, we mean a maximal portion of the plane for which any two points a, b in it can be joined by a simple curve in such a way that, neither does the curve have any point in common with the curve representing any edge nor does any vertex lies on that curve, that is, the curve lies completely in that portion of the plane. If R is a region of a planar graph G, by the **boundary** of R, we mean all those points x in the plane corresponding to the vertices and edges of G having the property that x can be joined to any point in that region by a simple curve all whose points, except x, are in that region. There is always one unbounded region and it is called the **exterior region** of G. Any other region is called an **interior** region.

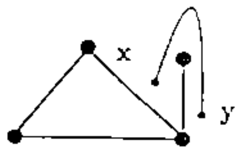
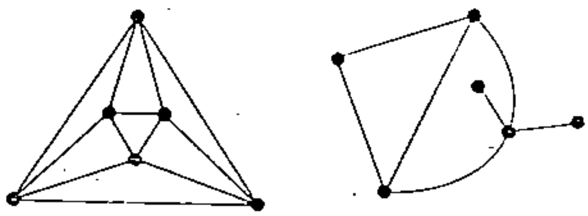


Fig.19

Let us go back to Fig.18 again. Armed with this definition, we can answer the question we raised. As you can see in Fig.19, the points x and y can be joined by a curve that does not cross any of the edges. So, there are only two regions, the region inside the triangle and the region outside it. Both the points lie in the exterior region of the triangle.

Let us now look at an example to understand these concepts better.

Example 10: Find the number of regions in the graphs given in 20, including the exterior region.



(a) (b)

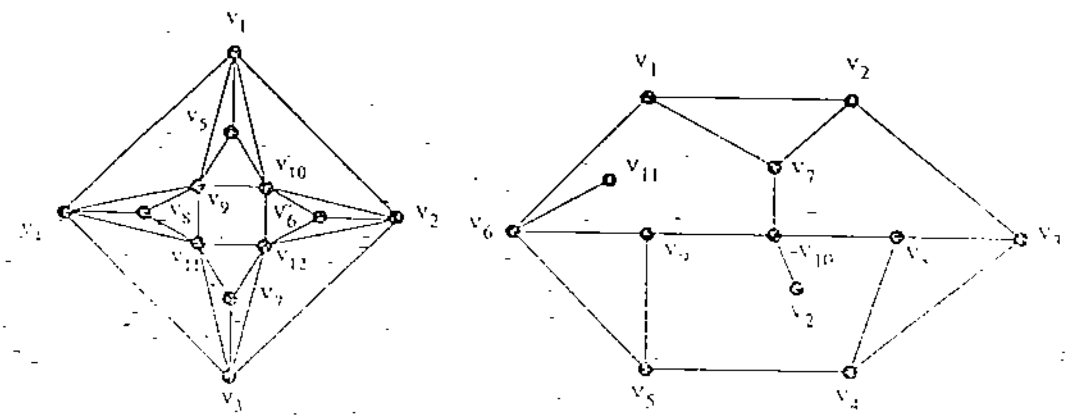
Fig.20

Solution: The graph in Fig.20(a) has 8 regions. In the graph in Fig.20(b), there are 3 regions.

* * *

Check your understanding now. Try the following exercise.

E15) Find the number of regions in each of the graphs in Fig.21.



(a) (b)

Fig.21

Let us now calculate the quantity $p - q + r$ for all the planar graphs in Fig.17 and for the graph in Fig.20(b):

	p	q	r	p − q + r
Fig.20(b)	6	7	3	2
K_4	4	6	4	2
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2
Icosahedron	12	24	18	2

As you can see, $p - q + r$ is always 2 for all these planar graphs.

The following theorem, proved by Eulef in 1736, proves our observation.

Theorem 4: If G is a connected planar (p, q)−graph, then for any plane drawing of G, the number r of the regions of G is constant and

$$p - q + r = 2 \quad (1)$$

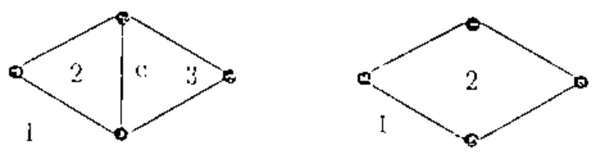
Proof: We apply induction(See Unit 2) on the number q of the edges of G. For our convenience let us write down the equation in words also.

$$\text{Number of vertices} - \text{Number of edges} + \text{Number of regions} = 2 \quad (2)$$

for any planar graph G.

If $q = 0$, then G just consists of p isolated vertices. Hence, $r = 1$ and the formula holds. Now, by induction, assume that the formula holds for plane drawing of a (p, t)−graph for every $t \leq (q - 1)$, and suppose G is a (p, q)−graph. If G is a tree, then $p = q + 1$ and $r = 1$ so that the formula holds. If G is not a tree, let e be an edge that lies on a cycle of G and consider the subgraph $G - e$ of G.

When we remove an edge e, we join exactly two regions to make one region out of them, that is $G - e$ has p vertices, $(q - 1)$ edges and $(r - 1)$ regions. This is illustrated in a particular case in Fig.22.



(a) (b)

Fig.22

In Fig.22(a), there are 4 vertices, 5 edges and 3 regions. After removing the edge labelled e, the regions 2 and 3 merge and become a single region. The new graph in Fig.22(b) has 4 vertices, 4 edges and 2 regions.

Now, by induction assumption, the relation in Eqn.(1) holds for this $G - e$. Using the form given in Eqn.(2) for $G - e$, we get,

$$\begin{aligned} 2 &= \text{Number of vertices} - \text{Number of edges} + \text{Number of regions} \\ &= p - (q - 1) + (r - 1) \\ &= p - q + r \end{aligned}$$

i.e., $p - q + r = 2$. But, p, q and r are, respectively, the number of vertices, the number of edges and the number of regions in G. This proves the result for the graph G.

From the formula in Eqn.(1), we have $r = q - p + 2$. Since p and q are fixed once we fix a graph, it also follows that the number of regions in a plane drawing of a planar graph is independent of the plane drawing.

Recall that a graph on p-vertices can have up to $\frac{p(p-1)}{2}$ edges. In the case of planar graphs, there is a much better bound. We give this bound(without proof) in the next theorem.

Theorem 5: If G is a planar (p, q)−graph, with $p \geq 3$, then $q \leq 3p - 6$. Further, if G is also bipartite, we have $q \leq 2p - 4$

So far, we have given many examples of planar graphs. But, we haven't given any example of non-planar graphs. We now make use of the bound given in theorem 5 in the next example to give such an example.

Example 11: Show that K_5 is planar.

Solution: Suppose that K_5 is planar. Then the number of edges and vertices in K_5 satisfy the relation $q \leq 3p - 6$ given in theorem 5. K_5 has 5 vertices and 10 edges, so $10 \leq 3 \times 5 - 6$, i.e. $10 \leq 9$, a contradiction.

* * *

Try the next exercise to check your understanding of theorem 5.

E16) Verify that $K_{3,3}$ is non-planar using theorem 5.

You must have noticed that, so far, we have given only necessary conditions for planarity. In the next subsection we will give a necessary and sufficient condition.

13.3.1 When is a graph planar?

We have already seen that K_5 and $K_{3,3}$ are not planar. To prove this we used a necessary condition derived from Euler's formula. However, the condition is not sufficient. For example, in Grötzsch graph, $p = 11$, $q = 20$ and $20 \leq 33 - 6 = 27$. So, the condition in Theorem 5 is satisfied. But, as we shall show later, Grötzsch graph is not planar. Is there a necessary and sufficient condition for a graph to be planar?

Yes! In 1930, K. Kuratowski, a Polish mathematician, proved a necessary and sufficient condition for a graph to be planar We will state this theorem and illustrate its application through an example. To understand the statement, let us first consider Fig.23 below.

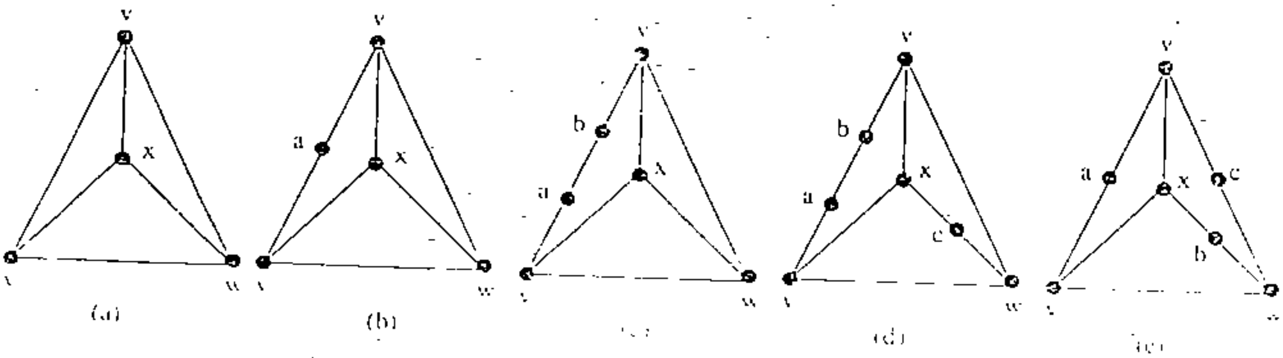


Fig.23 Subdivision of a graph

In this figure, we have started with K_4 and inserted vertices of degree 2 at some of the existing edges. For example, in Fig.23(b) we have removed the edge uv, added a new vertex a and two more new edges va and au. We have similarly altered the graphs in Fig.23(b), Fig.23(c), Fig.23(d) and Fig.23(e). In this way we have got **subdivisions** of the graph in Fig.23(a), as you shall now see.

Definition : A graph G′ is a subdivision of a graph G if it can be obtained by adding one or more new vertices of degree 2 on the existing edges of G.

In other words, we 'subdivide' some of the existing edges.

Note that, if a graph is planar, all its subgraphs are planar. Equivalently, if a subgraph of a graph is non-planar, the graph itself is non-planar. Also, if a graph G′ is the subdivision of a planar graph G, then G′ is also planar. If a graph G is non-planar, any subdivision of G is also non-planar. So, if a graph contains a non-planar subgraph or a subgraph which is a subdivision of a non-planar graph, it is non-planar. For example, the graph in Fig.24(a) is non-planar since it contains as a subgraph a subdivision of K_5, (shown by dotted lines)which is a non-planar graph.

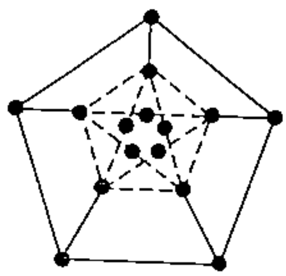


Fig.24

In proving the non-planarity of the graph in Fig.24, is it just a coincidence that it had a subdivision of K_5 as a subgraph? No! Kuratowski theorem(stated below) says that a non-planar graph has to contain a subgraph which is a subdivision of K_5 or $K_{3,3}$. So, we need to restrict our search for non-planar subgraphs(or their subdivisions) to only these two graphs.

We now state Kuratowski's theorem.

Theorem 6: A graph G is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Let us now look at an example to see how this theorem can be used to prove non-planarity.

Example 12: Show that the Grötzsch graph(See Fig.4) is non-planar.

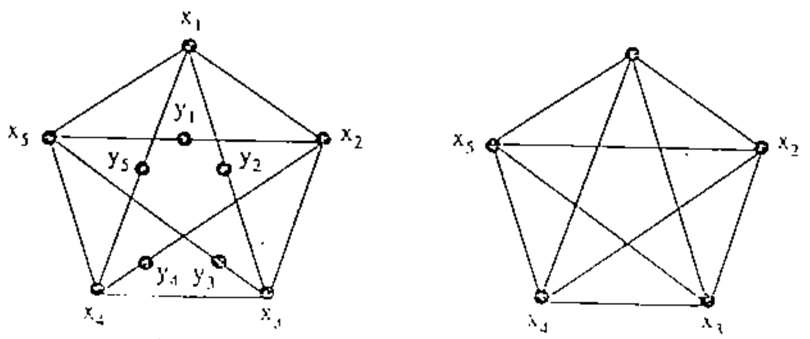


Fig.25 Non-planarity of Grötzsch graph.

Solution: From Kuratowski's theorem we know that we have to look for a subgraph which is a subdivision of K_5 or K_4. But, in this case, which of these two should we look for? Note that subdivision of a graph does not affect the degree of any of the vertices of a graph; it only introduces new vertices of degree 2.

So, if our graph contains a subdivision of K_5, it will contain at least 5 vertices of degree 4. If it contains a subdivision of $K_{3,3}$ it will have at least six vertices of degree 3. Let us first check if our graph contains a subdivision of $K_{3,3}$. But, the Grötzsch graph contains only five vertices of degree 3, namely y_1, y_2, y_3, y_4 and y_5. So, it cannot contain a subdivision of $K_{3,3}$. So, let us check if it contains a subdivision of K_5. K_5 contains 5 vertices of degree 4. In Grötzsch graph also there are vertices of degree 4, namely x_1, x_2, x_3, x_4 and x_5. Let us remove the middle vertex, labelled as z. We get the graph given in Fig.25(a). As you can see, it can be obtained from K_5 in Fig.25(b) by adding degree two vertices to x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5 and x_1x_5. So, it is non-planar.

* * *

Now, an exercise for you to try!

E17) Show that the Petersen graph is non-planar.
(Hint: Consider the graph obtained by removing the two horizontal edges.)

In the next section we will discuss the map colouring problem. We will show that this can be reduced to colouring of planar maps. We will also show that any planar map can be coloured with five colours.

13.4 MAP COLOURING PROBLEM

The four colour problem asks whether any map can be coloured with 4 colours. We begin this section with a brief discussion of the history of the four colour problem. We then show how to construct a planar graph corresponding to a given map in such a way that colouring the graph is equivalent to colouring the map. So, if we can prove that any planar map can be coloured with four colours, we would have proved that any map can coloured with four colours. Appel and Haken proved that four colours are enough to colour planar graphs in 1979, so the four colour problem is now solved. They used nearly 1200 hours of computer time on some of the fastest computers available at that time. This gives an idea about the complexity of the proof and we will not be giving the proof in this course. However, we will prove the weaker result that five colours are always enough to colour any planar graph. Now, for some history!

Francis Guthrie communicated the four colour problem to De Morgan through his brother Fredrick Guthrie, who was a student at the University College, London at that time. It appeared in print for the first time when Cayley published a paper on this problem in Royal Geographical Society in 1879. In this paper, he outlines where the difficulties lie in this problem. In the same year, A. B. Kempe published a proof of the theorem in American Journal of Mathematics. However, in 1890, P. J. Heawood pointed out a mistake in Kempe's proof. He also showed that the proof can be modified to show that five colours are enough to colour any map. Since then, many mathematicians, G. D. Birkhoff, Veblen, Ore, Franklin among others, contributed to the solution of the problem. Appel and Haken finally solved the problem in 1979.

We now show how to construct a planar graph corresponding to a given map in such a way that colouring vertices of the graph is equivalent to colouring the map.

Consider the map given in Fig.26(a) below. There are 10 regions in the map, A, B, C, D, E, F, G, H, I and J, including the exterior region. In this map we add a vertex corresponding to each region of the map. (See Fig.26(b).) Note that we have added a vertex corresponding to the exterior region, namely, J.

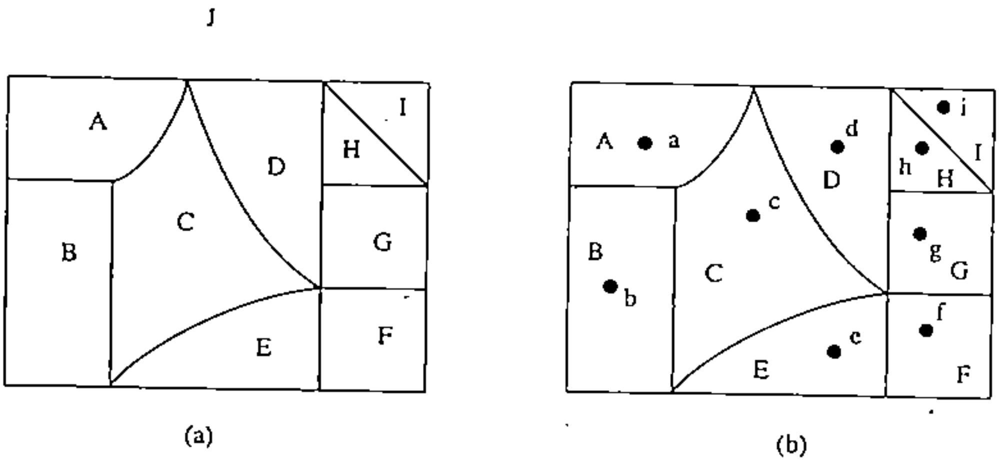


Fig.26

We join two vertices if the corresponding regions have an edge in common. For example, we have connected a and c because they have a common boundary(see Fig.27 below).

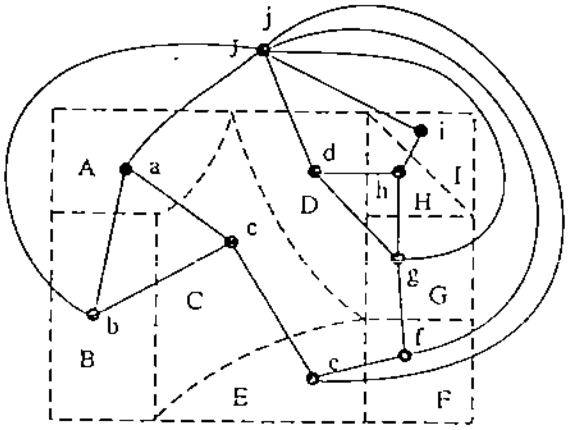


Fig.27

We have not connected the vertices a and e because they do not have a common boundary. We do not connect two vertices, if the corresponding regions share only a point and not a boundary. For example, we have not connected c and g by an edge for this reason. As you can see, we get a planar graph and colouring the graph is equivalent to colouring the map. (We assume that the exterior region of the map is coloured with a single colour.) So, the four colour problem can be stated as follows:
Is it possible to colour any planar graph with four colours? The following theorem answers this question.

Theorem 7 [Appel-Haken](1979): Any planar graph can be coloured with four colours.

As we mentioned in the introduction we will not be proving this theorem. Can we do with three colours always? No! As we have seen, K_4(it is the graph corresponding to a tetrahedron) is planar, but it cannot be coloured with three colours. So, we cannot improve the result in Theorem 7.

We now prove a result that will be used in the proof of the five colour theorem.

Theorem 8: For every planar graph G, the minimum degree $\delta(G)$ is at most 5.

Proof: If possible, let G be a planar graph such that $\delta(G) \geq 6$. But then, by Theorem 5,

$$6p \leq \sum_{x} d_G(x) = 2q \leq 6p - 12.$$

This is impossible. Hence, $\delta(G) \leq 5$.

We can prove the five colour theorem now.

Theorem 9: Every planar graph is 5-colourable.

Proof: Let G be a planar graph on p vertices. We prove the theorem by induction on p. If $p \leq 5$, then the theorem is clearly true. Now, assume that every planar graph with $(p-1)$ vertices, $p > 1$, is 5-colourable. By Theorem 8, $\delta(G) \leq 5$. Let v be a vertex of G such that $\delta(G) = d_G(v)$. Consider $G - v$. By induction this is 5-colourable. Let us take a 5-colouring of $G - v$. In this colouring all the vertices other than v have received some colour. We have to get a 5-colouring of the graph G by changing the colours assigned to the vertices other than v, if necessary, and assigning some colour to v.
If $d_G(v) < 5$, then there are at most four vertices adjacent to v in G. Hence, there is at least one colour $\boxed{i}$ not assigned to any of the neighbours of v. By assigning $\boxed{i}$ to v and retaining the same colours for the other vertices, we get a 5-colouring of G.
If $d_G(v) = 5$ but the neighbours of v in G utilise only four or less colours then as before we can complete a 5-colouring of G.
Now suppose $d_G(v) = 5$ and the neighbours of v in G utilise all the five colours. Renumbering the vertices if necessary, we can suppose that neighbours v_1, v_2, v_3, v_4, v_5 of the vertex v are numbered in such a way that the colour $\boxed{i}$ is assigned to the vertex v_i and they are arranged around v in a plane drawing of G as shown in Fig.28:

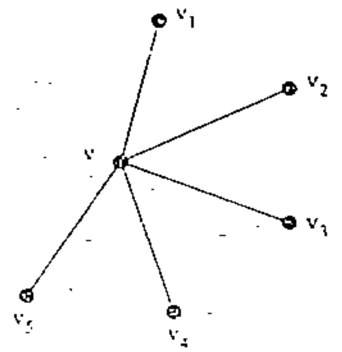


Fig.28

Let $S_i = \{x \in V(G) : \boxed{i} \text{ is assigned to } x\}$. Consider the vertex induced subgraph $H_{1,3}$ of G induced by $S_1 \bigcup S_3$.
Case 1: If the vertices v_1 and v_3 belong to two different components of $H_{1,3}$, then take the component(see unit 11 for the definition of component of a

graph) containing v_1.

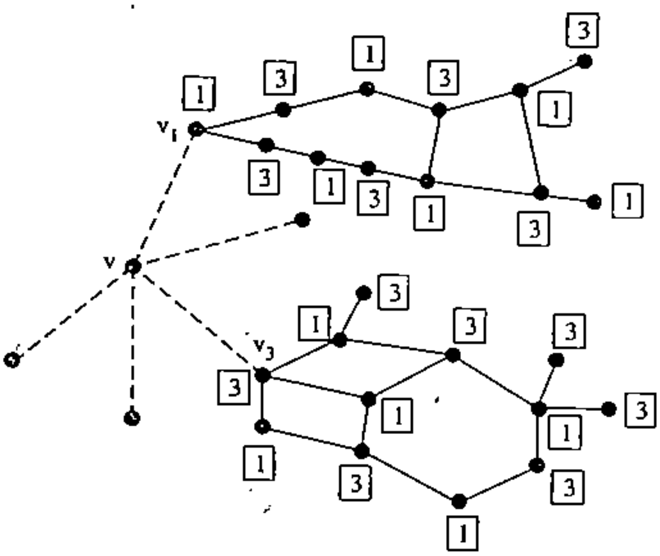


Fig.29

Interchange the colours only in this component. In other words, assign the colour [1] to all the vertices that are assigned [3] and assign [3] to all the vertices that are assigned [1]. In the modified colouring the vertices v_1, v_3 both receive the colour [3]. Now we can assign the colour [1] to the vertex v and get a 5−colouring of G.

Case 2: If v_1, v_3 belong to same component, then there is a path P joining them. Because of the colouring, the vertices of this path must have received colours [1] and [3] alternately, starting with colour 1 at v_1 and ending with colour [3] at v_3.

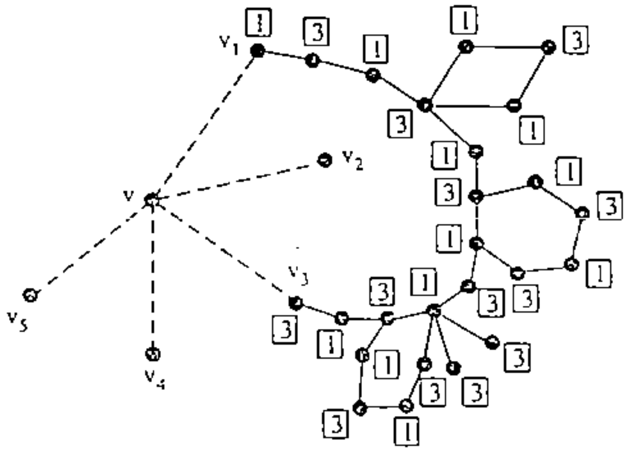


Fig.30

This means the union of P and $\{v_1v, v_3v\}$ is a cycle C (say). Moreover, the vertex v_2 belongs to the interior region created by this cycle and the vertex v_4 belongs to the exterior region created by this cycle. You must remember that the vertices of this cycle, other than v, have received colours [1],[3] only. Now, consider the vertex induced subgraph $H_{2,4}$ of G induced by $S_2 \bigcup S_4$. If there is a path joining v_2 and v_4 in this subgraph, then vertices on it will have colours [2], [4] only. But then it has to cross the barrier created by the cycle C. Where can it cross? All the vertices of C have colours 1, 3 only. So there cannot be a common vertex to use. This means there cannot be a path joining v_2 and v_4 in the subgraph $H_{2,4}$, that is the vertices v_2 and v_4 belong-

to different components of $H_{2,4}$. Instead of taking $H_{1,3}$, we take $H_{2,4}$ and go back to the Case 1 and complete the 5−colouring of G.
Thus G is 5−colourable.

If we can colour the vertices of a graph, why can't we colour the edges of a graph? Is it interesting? In the next chapter, we will answer this question.

13.5 EDGE COLOURINGS.

In this section, we consider the problem of colouring the edges of a graph in such a way that no two adjacent edges receive the same colour. We will not prove any of the important results in this subject although we will state some of them. The purpose of section is to give a brief introduction to edge colouring. We begin by defining edge colouring.

Definition : A **k-edge colouring** of a graph G is an assignment of k colours to each of the edges of G in such a way that no two edges incident with the same vertex have the same colour. A graph is **k-edge colourable** if there is a k-edge colouring. The **minimum** number of colours required to colour a graph is called the **edge chromatic number of G**, usually denoted as $\chi'(G)$.

Let us now look at some examples of edge colouring. The easiest case is the edge colouring of those graphs which have edge chromatic number 1.

Example 13: Find all the graphs that have edge chromatic number 1.

Solution: Suppose a graph G has edge chromatic number 1. Since the edge chromatic number is one, the graph is 1-edge colourable and no two edges share an end vertex, that is, the graph must be union of some isolated vertices and some mutually disjoint edges. Conversely, graph which are union of isolated vertices and mutually disjoint edges have edge chromatic number 1.

* * *

Example 14: Colour the edges of the graphs K_3, K_4, K_5.

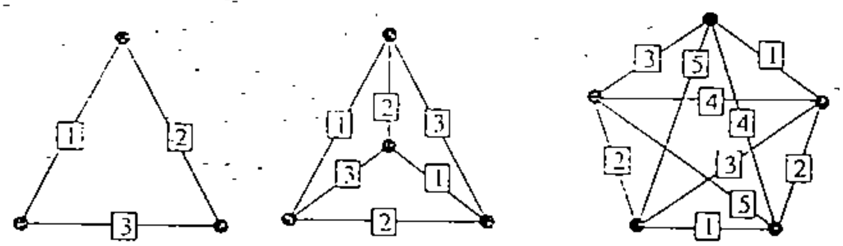


Fig.31

The colouring of K_3, K_4, K_5 is given in Fig.31. Here no two adjacent edges have received same colour. In all the cases, we have used least possible colours.

* * *

Example 15: Give a edge colouring of Petersen graph.

Solution: Fig.32 gives a 4-edge colouring of the Petersen graph.

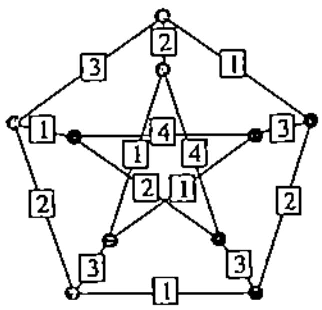


Fig.32

Again no two adjacent edges have received same colour. You can quickly check that three colours will not be enough.

* * *

Example 16: Give edge colourings of all the trees on 5 vertices.

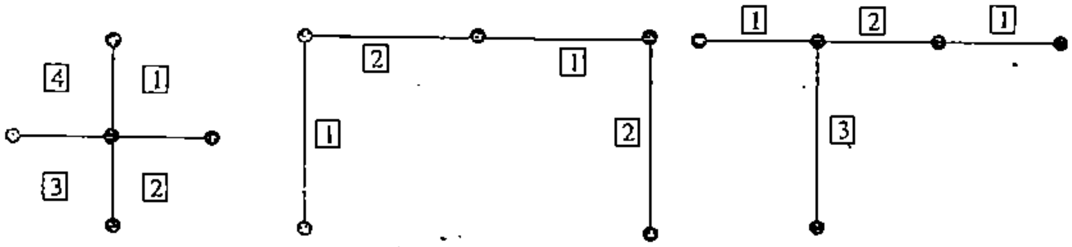


Fig.33

Again, we have used the least possible number of colours and no two adjacent vertices received same colour.

* * *

Example 17: Find the edge-chromatic number of C_n.

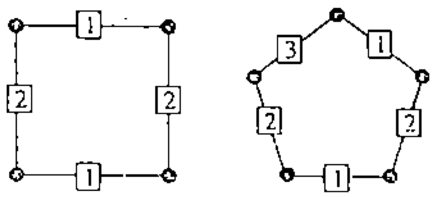


Fig.34

Solution: As in the case of vertex colouring, if n is even, the edge chromatic number is 2. We can colour the edges alternately with the two colours. If n is odd, the edge chromatic number is 3. We have illustrated this in the case of C_4 and C_5 in Fig.34.

* * *

If G is a graph and $v \in V(G)$ such that $d_G(v) = \Delta(G)$, then all the edges incident on v must receive different colours. Hence, any edge colouring of G will need at least $\Delta(G)$ colours, that is,

$$\Delta(G) \leq \chi'(G) \qquad (3)$$

Regarding an upper bound for $\chi'(G)$, in 1964, Vizing proved the following result.

Theorem 10: For any graph G, we have

$$\chi'(G) \leq \Delta(G) + 1 \quad (4)$$

From (3) and (4) it follows that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) \quad (5)$$

From (5), it follows that there are only two possibilities for the edge chromatic number of a graph G, either $\Delta(G)$ or $\Delta(G) + 1$. Thus, this result divides the set of all graphs into two classes. A graph G is said to belong to class 1, if $\chi'(G) = \Delta(G)$ and it is said to belong to class 2, if $\chi'(G) = \Delta(G) + 1$. We often say that G is a **class 1 graph** or G is a **class 2 graph**. The problem of determining the class of a graph is called the **classification problem**. We now discuss some of the results known in this direction.

Theorem 11: The edge chromatic number of K_n is n if it is odd($\neq 1$) and $n - 1$ if it is even.

Recall that, K_n is $n - 1$-regular. So, $\Delta(K_n) = n - 1$. So, K_n belongs to class 1 if n is even and it belongs to class 2 if it is odd.

Regarding bipartite graphs, in 1916, König proved that $\chi'(G) = \Delta(G)$, in other words, it is a class 1 graph.

In 1977, Erdös and Wilson proved that if p(n) is the probability that a graph on n vertices selected at random belongs to class 1, then $p(n) \longrightarrow 1$ as $n \longrightarrow \infty$, that is almost all graphs belong to class 1. However, large families of class 2 graphs are known.

E18) What is the edge chromatic number of $K_{m,n}$?

E19) Consider following tree T.

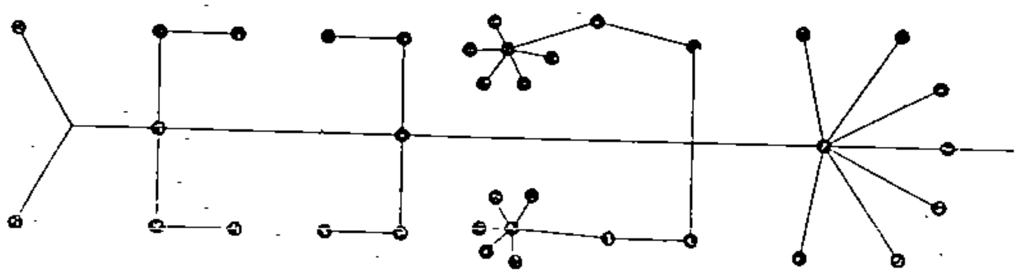


Fig.35

Give an explicit $\Delta(T)-$ colouring of T.

We have now reached the end of this unit. Let us now summarise briefly what we have learnt so far.

13.6 SUMMARY

In this unit we defined:

a) **Vertex colouring of a graph:** A vertex colouring of a graph is an assignment of colours to vertices in such a way that no two adjacent vertices receive the same colouring.

b) **Vertex chromatic number of a graph:** The chromatic number of a graph is the minimum number of colours required to colour the graph.

c) **A colour class of a colouring:** For each colour of a colouring, the set of all vertices that are coloured with that colour is the colour class of that colour.

d) **Independent set** A subset of the vertex set is independent of any two vertices in the set are non adjacent.

e) **Planar graph:** A graph is planar if there is a plane drawing in which no two edges cross each other, except at vertices.

d) **Subdivision of a graph:** A graph G_2 is a subdivision of another graph G_1 if it can be obtained by from G_1 by adding vertices of degree two at the existing edges.

e) **Edge colouring of a graph:** An edge colouring of a graph is an assignment of colours to edges in such a way that no two edges incident at the same vertex are given the same colour.

f) **Edge chromatic number of a graph:** The edge chromatic number of a graph is the minimum number of colours needed to colour the edges of graph.

g) **Class 1 and class 2 graphs:** A graph is of class 1 if its edge chromatic number is $\Delta(G)$; it is of class 2 if it has chromatic number $\Delta(G)+1$.

In this unit, we studied:

a) some upper bounds for the chromatic number of a graph.

b) Euler's formula for planar graphs, which states that
Number of vertices − Number of edges + Number of regions = 2
for any planar graph.

c) Kuratowski's characterisation of planar graphs which says that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

d) the four colour theorem(without is proof) which says that any planar graph can be coloured with four colours.

e) the five colour theorem(with proof) which says that any planar graph can be coloured with five colours.

f) the Vizing's bound for the edge chromatic number of a graph, namely $\chi'(G) \leq \Delta(G) + 1$.

13.7 SOLUTIONS / ANSWERS.

E1) Recall that bipartite graphs were characterised as graphs without odd cycles. Trees are acyclic graphs, i.e. they do not contain cycles as subgraphs and therefore they are bipartite. Since trees are connected and we have assumed it has atleast two vertices, it has chromatic number 2.

E2) Even cycles do not contain odd cycles as subgraphs. So, they are bipartite. Therefore, they have chromatic number 2.

E3) The chromatic number of an odd cycle is 3. Since it is not bipartite, its chromatic number is atleast 3. We get a 3-colouring of C_{2n+1} as follows: Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of C_{2n+1}. We assign $\boxed{1}$ to all the vertices in the set $\{v_i \in V(C_{2n+1}) \mid i \text{ odd}, 1 \leq i \leq 2n\}$ and $\boxed{2}$

to all the vertices in the set $\{v_i \mid i \text{ even}, 2 \leq i \leq 2n\}$. Now, v_{2n+1} is adjacent to both v_1 and v_{2n}. So, we cannot assign [1] or [2] to this vertex. Therefore, we assign the third colour [3] to v_{2n+1}.

E4) A three colouring of Petersen graph is given below:

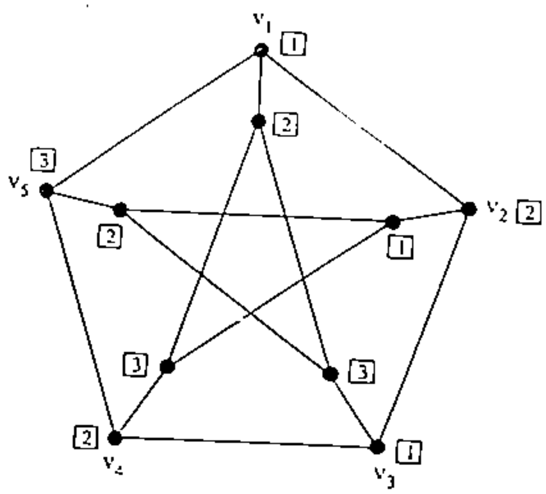


Fig.36

Further Petersen graph contains a 5−cycle which has chromatic number three. So, Petersen graph has chromatic number three.

E5) Since it has chromatic number greater than 2, it cannot be bipartite. So, it must contain an odd cycle.

E6) (a) A 3-colouring of the graph is given below:

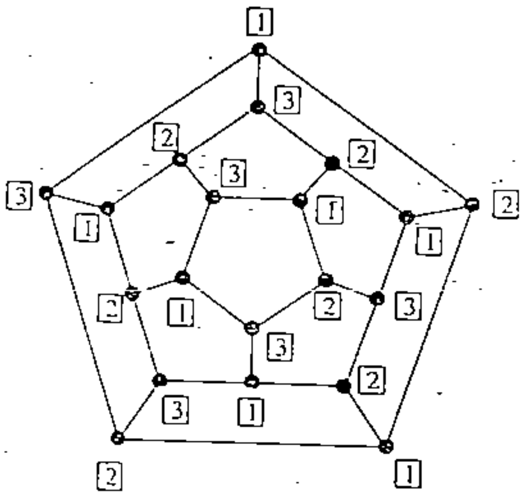


Fig.37

(b) The chromatic number of this graph is three. We have already seen that this graph has a 3-colouring. Further, it has cycles of length 5 as subgraphs and we have already seen that cycles of odd length have chromatic number 3.

E7) In the graph in Fig.7(a), the graph induced by v_4, v_5, v_6, v_7 is K_4. So, it has a clique of size 4 and therefore we need atleat 4 colours. We get a 4-colouring by assigning [1] to v_1, [2] to v_2, [3] to v_3, [2] to v_4, [3] to v_5, [1] to v_6 and [4] to v_7. So, the chromatic number is 4.

E8) The figure given in Fig.38 is 5-chromatic.

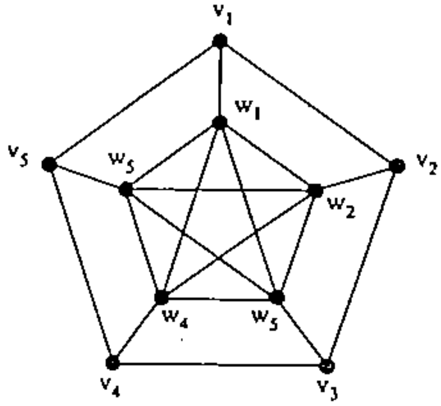


Fig.38

The graph in Fig.38 contains a clique of size 5, namely the subgraph induced by the vertices w_1, w_2, w_3, w_4, w_5. So, we need atleast 5 colours. We first give a 5-colouring to the subgraph isomorphic to K_5 by assigning $\boxed{i}$ to w_i, $1 \leq i \leq 5$. Next, we assign $\boxed{2}$ to v_1, $\boxed{3}$ to v_2,$\boxed{1}$ to v_3 $\boxed{2}$ to v_4, and $\boxed{1}$ to v_5.

E9) Two different colourings are given in Fig.39

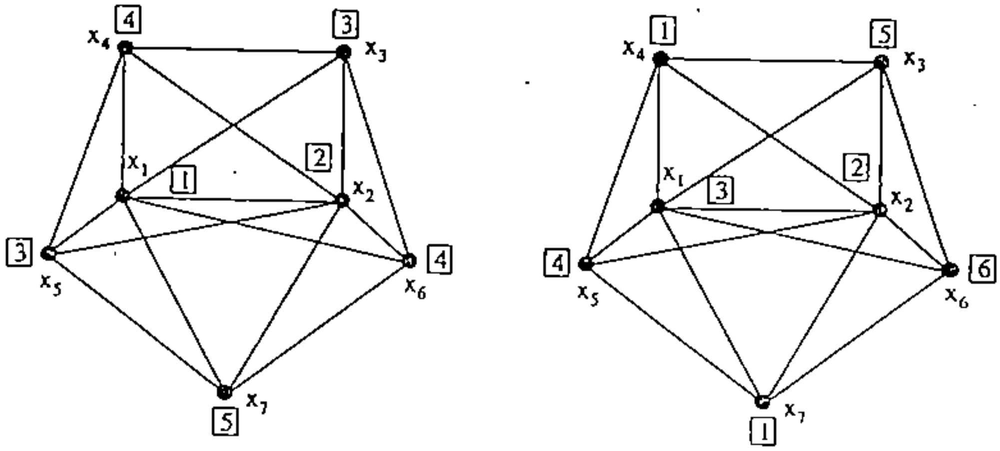


Fig.39

The colour classes for the colouring in Fig.39(a) are $\{x_1\}$, $\{x_2\}$, $\{x_7\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$. The colour classes for the colouring in Fig.39(b) are $\{x_1\}$, $\{x_2\}$, x_3, $\{x_4, x_7\}$ and $\{x_5, x_6\}$

E10) $\{v_1, v_2, v_4, v_5,\}$

E11) For Example 1, you can see that $\{x_{13}, x_{14}, x_3, x_5, x_7, x_9, x_{11}\}$ is an independent set and any other set has ≤ 7 elements. Thus $\alpha(G) = 7$.

For Example 2, you note that for every vertex x_i, there are precisely two vertices in G not adjacent to x_i. But those two are adjacent. Hence, $\alpha(G) = 2$.

E12) If we remove any vertex from K_n, we get K_{n-1} which has chromatic number $n - 1$. Let $\{v_1, v_2, v_3, v_4, v_5\}$ denote the vertex set of K_n. Let us remove an edge from K_n. Renumbering the edges if necessary, we can assume that the edge we have removed is v_1v_n. Then, we get an $n - 1$ colouring as follows: Assign $\boxed{1}$ to both v_1 and v_n. For each v_i, $2 \leq i \leq n - 1$, assign $\boxed{i}$.

E13) If we remove any of the vertices v_1, v_2, v_3, v_4, v_5. (See Fig.40 below)

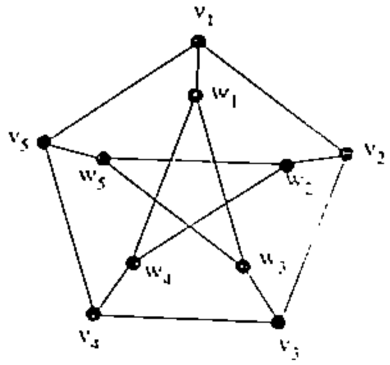


Fig.40

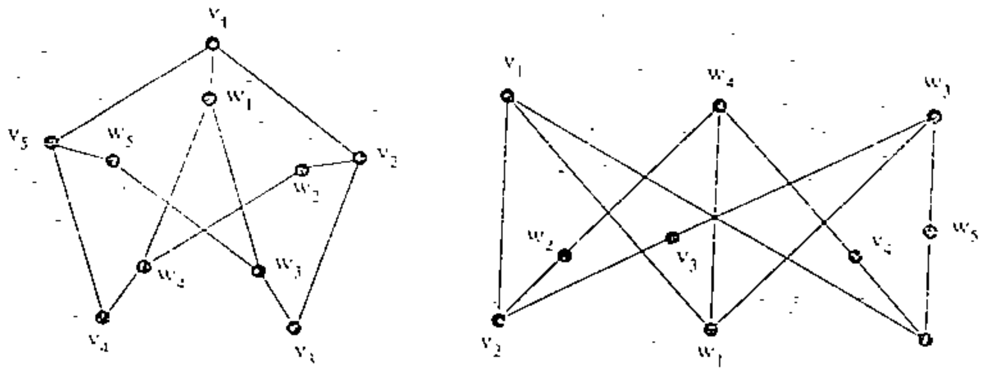
the odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$ is unaffected and the resulting subgraph has chromatic number 3. Similarly, the graph obtained by deleting any of the vertices v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, we will get a graph which has chromatic number three since it will contain the odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$.

E14) Refer to Fig.40. The odd cycle $\{w_1, w_2, w_3, w_4, w_5\}$ is a 3-critical subgraph.

E15) a) 18 b) 7

E16) Since $K_{3,3}$ is bipartite, we can apply Theorem 5. Here $p = 6$ and $q = 9$. But, $2p - 4 = 10 > 9 = q$. So, $K_{3,3}$ is not planar.

E17) The graph obtained by deleting the two horizontal edges is shown in Fig.41(a).



(a) (b)

Fig.41

We have redrawn Fig.41(a) in Fig.41(b) so that you can clearly see that it is a subdivision of $K_{3,3}$.

E18) Since $K_{m,n}$ is bipartite graph, by König's result, $\chi'(K_{m,n}) = \Delta(K_{m,n}) = \text{Min}(m, n)$.

E19) The required $\Delta(T)$ – colouring is given Fig.42. You must remember that it is not unique.

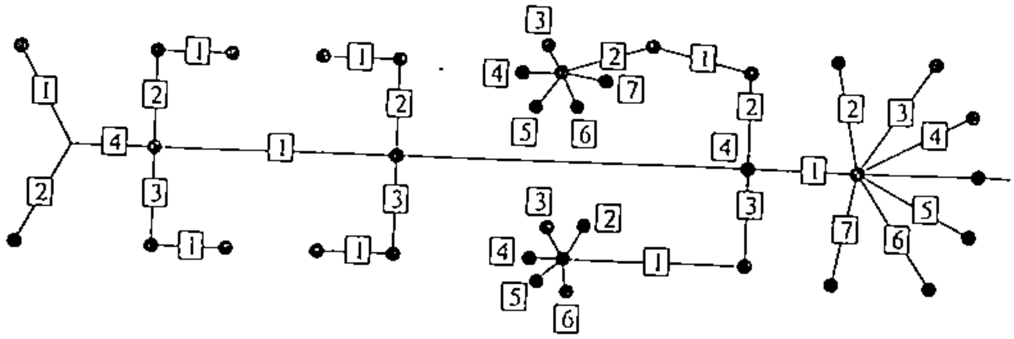


Fig.42