स्वाध्याय स्वमन्थन स्वावलम्बन

UTTAR PRADESH RAJARSHI TANDON OPEN UNIVERSITY

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UGMM-05
Analytical Geometry

FIRST BLOCK
Conics

INDIRA GANDHI NATIONAL OPEN UNIVERSITY

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Block

1

CONICS

ANALYTICAL GEOMETRY

This is a short course on two and three-dimensional coordinate geometry in which we shall only deal with **conics and conicoids.**

As you know from your earlier study of mathematics, analytical geometry uses the ideas of a coordinate system. Such a system was invented by René Descartes, and published by him in 1637 in 'La Geometrie'. This was the first major step in the development of analytical geometry. His procedure in this work was to begin with a geometric problem, convert it into the language of algebraic equations, simplify the equations, and then solve the equations geometrically. Thus, he was the first person to formally present a one-to-one correspondence between algebra and geometry. Of course, long before this the Arab mathematician al-Khwarizmi (approximately 825 AD) used geometric figures as aids for solving problems in algebra. The ancient Indian mathematicians like Bhaskara (approx. 1150 AD) also did the same.

In 'La Geometrie' Descartes also did a detailed examination of the general equation of a conic passing through the origin. He was not the first to do so. Many centuries before him, the ancient Greek mathematician Apollonius (c. 225 BC) had written his treatise, 'Conics'. In this book he defined a conic, or conic section, to be any curve that is obtained when a plane cuts a cone. Before him Menaechmus had studied the properties of these curves in detail. But the Greeks used geometric methods to study conics. What was so refreshingly different about Descartes' approach was that he studied conics from an algebraic point of view. He indicated conditions on the coefficients under which the conic would be an ellipse, a parabola or a hyperbola. But his presentation wasn't easy to follow.

It was only in the late 18th century that mathematicians like Hachette, Biot and Monge made analytical geometry accessible to students of mathematics. In Block 1 of this course we will expose you to their elementary approach to the study of conics from an analytical point of view.

We start with conics in standard form, which you may be familiar with. We obtain their equations and elaborate on some of their properties. Then we generalise this study and show you that any second degree equation in x and y represents a conic section. We also define a tangent and normal to a conic and obtain their equations.

Some properties of conics have very useful applications in astronomy, geology, architecture, physics, electronics, engineering, military science, and other areas. We will also discuss these properties and their uses.

In Block 2 we start our discussion of three-dimensional analytical geometry. We specifically deal with spheres, cones and cylinders. In it you will study that a circle can be obtained by taking a planar section of a sphere. You will also see how Apollonius's definition and the modern definition of conics lead us to the same curves.

In Block 3 of this course we focus on conicoids, which are surfaces that are represented by a second degree equation in 3 variables. We will introduce you to their standard forms, and discuss their elementary properties. You will see that a planar section of a conicoid is a conic, and the type of conic depends on the conicoid that you start with.

We have prepared this course with two assumptions in mind. Firstly, we assume that you have already studied the material covered in our course Elementary Algebra (MTE-04). The second assumption is that you are familiar with some elementary two-dimensional analytical geometry. This includes equations of lines and properties of a circle in $\mathbf{R}^2$. Since we will be referring to these equations and related properties throughout Block 1, we have briefly covered what we need in Unit 1.

Now, a word about the way we have presented this course. In each of the three blocks we have first introduced you to the block. Then we have presented the units of the block. In each unit you will find plenty of exercises interspersed with the text. Please try them as and when you come to them. They are meant to help you check whether you've understood the material that is being discussed. We have also given our solutions to the exercises in a section at the end of the unit.

At the end of each block we have given a set of **miscellaneous exercises** covering the contents of the block. Doing them will give you a better grasp of the concepts given in the course, though it is not necessary for you to do them.

While you go through the course, you will notice that each unit is divided into sections. These sections are often further divided into sub-sections. The sections/sub-sections of a unit are numbered sequentially, as are the exercises, theorems and important equations in it. Since the material in the different units is heavily interlinked, we do a lot of cross-referencing. For this purpose we use the notation **Sec. x. y to mean Section y of Unit x.**

Another compulsory component of this course is an assignment, which you should attempt after studying all the blocks of the course. Your counsellor will evaluate it and return it to you with detailed comments. Thus, the assignment is a teaching as well as an assessment aid.

The course material that we have sent you is self-sufficient. If you have a problem in understanding any portion of it, please ask your counsellor for help. Also, if you feel like studying any topic in greater depth, you may consult :

1) "A Textbook of Coordinate Geometry" by Ramesh Kumar, Konark Publishers, 1991.

3) "Analytical Solid Geometry" by Shanti Narayan, S. Chand.

3) "Mathematics. A Textbook for Class XI, Part II", NCERT.

These books will be available at your Study Centre.

We hope you will enjoy this course!

BLOCK 1 CONICS

With this block we start our discussion on analytical geometry. In the three units of this block we will restrict ourselves to two dimensions. We start with a preliminary unit, in which we help you recall the various equations of a line. In it we also introduce you to rigid body motions, the concept of symmetry about a line or a point, and polar coordinates. You must be familiar with most of this material. But it will be used often in the other two units, and therefore we thought it necessary to include in the course.

In the next unit we introduce you to conics, and obtain their standard equations. We also discuss their geometrical properties and trace them.

In the final unit of this block, we prove that any second degree equation represents a conic. We discuss the conditions under which the equation represents an ellipse, a hyperbola, a parabola or a pair of lines. We also show you how to trace any conic and obtain its tangents. We end with a discussion on the curves obtained by intersecting two conics.

We end the block with a set of exercises that cover the contents of the whole block.

In the next block we shall go to three-dimensional space. But we shall often use what is covered in this block. So, before going to the next block, please ensure that you have achieved the objectives of the units in this block.

UNIT 1 PRELIMINARIES IN PLANE GEOMETRY

Structure

1.1 INTRODUCTION

In this short unit, our aim is to re-acquaint you with some two-dimensional geometry. We will briefly touch upon the distance formula and various ways of representing a line algebraically. Then we shall look at the polar representation of a point in the plane. Next, we will talk about what symmetry with respect to the origin or a coordinate axis is. Finally, we shall consider some ways in which a coordinate system can be transformed.

This collection of topics may seen random to you. But we have picked them according to our need. We will be using whatever we cover here, in the rest of the block. So, in later units we will often refer to a section, an equation or a formula from this unit.

You are probably familiar with the material covered in this unit. But please go through the following list of objectives and the exercises covered in the unit to make sure. Otherwise you may have some trouble in later units.

Objectives

After studying this unit, you should be able to

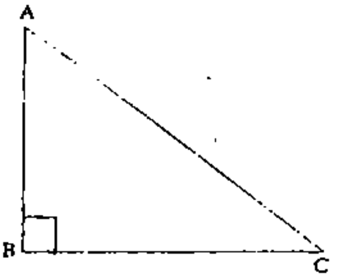
- find the distance between any two points, or a point and a line, in two-dimensional space;
- obtain the equation of a line in slope-intercept form, point-slope form, two-point form, intercept form or normal form;
- check if a curve is symmetric with respect to either coordinate axis or the origin;
- effect a change of coordinates with a shift in origin, or with a rotation of the axes;
- relate the polar coordinates and Cartesian coordinates of a point;
- obtain the polar form of an equation.

1.2 EQUATIONS OF A LINE

In this section we aim to refresh your memory about the ways of representing points and lines algebraically in two-dimensional space. Since we expect you to be familiar with the matter, we shall cover the ground quickly.

Firstly, as you know, two-dimensional space can be represented by the Cartesian coordinate system. This is because there is a 1-1 correspondence between the points in a plane and ordered pairs of real numbers. If a point P is represented by (x, y) under the correspondence, then x is called the **abscissa** (or **x-coordinate**) of P and y is called the **ordinate** (or **y-coordinate**) of P.

According to the Pythagoras Theorem, in the right-angled triangle ABC,



$(AB)^2 + (BC)^2 = (AC)^2$

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, then the distance between them is

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \ldots(1)$$

From Fig. 1, and by applying the Pythagoras theorem, you can see how we get (1).

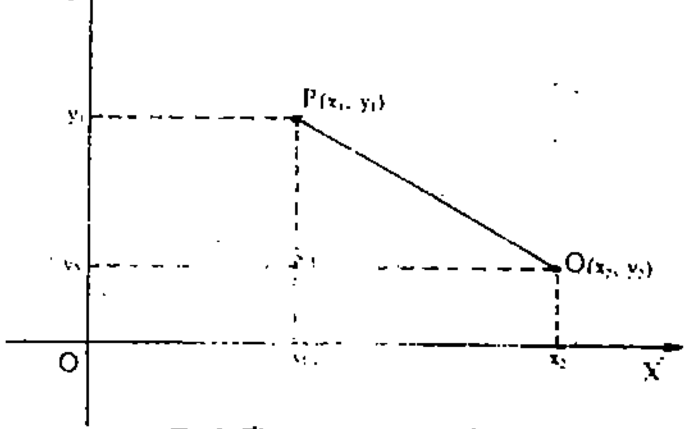


Fig. 1: Distance between two points

(1) is called the **distance formula.**

Another formula that you must be familiar with is the following:

if the point R(x, y) divides the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio m : n (see Fig. 2), then

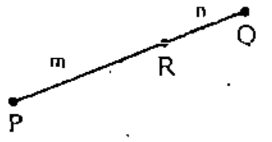


Fig. 2: R divides the segment PQ in the ratio m : n

$$x = \frac{nx_1 + mx_2}{m+n} \text{ and } y = \frac{ny_1 + my_2}{m+n} \quad \ldots(2)$$

(2) is called the **section formula**.

To regain practice in using (1) and (2), you can try the following exercises.

E1) What are the coordinates of the midpoint of the line segment with endpoints
   a) A(5, –4) and B(–3, 2) ?
   b) $A(a_1, a_2)$ and $B(b_1, b_2)$?

E2) Check if the triangle PQR, where P, Q and R are represented by (1, 0), (–2, 3) and (1, 3), is an equilateral triangle.

Let us now write down the various ways of representing a straight line algebraically. We start with lines parallel to either of the axes. A line parallel to the x-axis is given by the equation

$$y = a, \quad \ldots(3)$$

where a is some constant. This is because any point on the line will have the same ordinate (see Fig. 3).

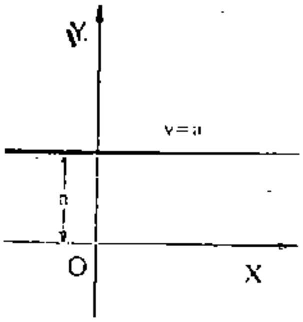


Fig. 3: y = a is parallel to the x-axis.

The plural of 'axis' is 'axes'.

What do you expect the equation of a line parallel to the y-axis to be? It will be

$$x = b, \qquad \ldots(4)$$

for some constant b.

Now let us obtain four forms of the equation of a line which is not parallel to either of the axes. Firstly, suppose we know that the line makes an angle α with the positive direction of the x-axis, and cuts the y-axis in (0, c). Then its equation will be

$$y = mx + c, \qquad \ldots(5)$$

where $m = \tan \alpha$. m is called its **slope** and c is its **intercept** on the y-axis. From Fig. 4 you should be able to derive (5), which is called the **slope-intercept form** of the equation of a line.

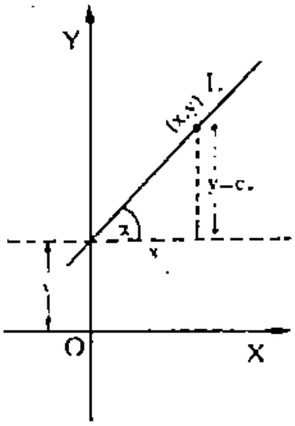


Fig. 4: L is given by $y = x \tan \alpha + c$.

Now, suppose we know the slope m of a line and that the point (x_1, y_1) lies on the line. Then, can we obtain the line's equation? We can use (5) to get the **point-slope form**,

$$y - y_1 = m(x - x_1), \qquad \ldots(6)$$

of the equation of the line.

We can also find the equation of a line that is not parallel to either axis if we know two distinct points lying on it. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the points on the line (see Fig. 5), then its equation in the **two-point form** will be

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \ldots(7)$$

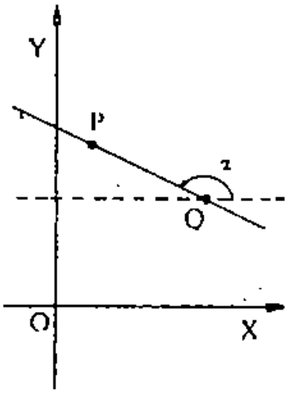


Fig. 5: The slope of PQ is $\tan \alpha$.

Note that both the terms in the equation are well-defined since the denominators are not zero.

Can you find the slope of the line given in (7)? If you rewrite it as

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + \left\{y_1 - x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\right\},$$

you can see that its slope is $\frac{y_2 - y_1}{x_2 - x_1}$, and its intercept on the y-axis is the constant term.

Why don't you try some exercises now?

E3) What are the equations of the coordinate axes?

E4) Find the equation of the line that cuts off an intercept of 1 from the negative direction of the y-axis, and is inclined at 120° to the x-axis.

E5) What is the equation of a line passing through the origin and making an angle θ with the x-axis?

E6) a) Suppose we know that the intercept of a line on the x-axis is 2 and on the y-axis is −3. Then show that its equation is

$$\frac{x}{2} - \frac{y}{3} = 1.$$

(Hint: See if you can use (7).)

b) More generally, if a line L cuts off an intercept a ($\neq 0$) on the x-axis and b($\neq 0$) on the y-axis (see Fig. 6), then show that its equation is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (8)$$

(8) is called the **intercept form** of the equation of L.

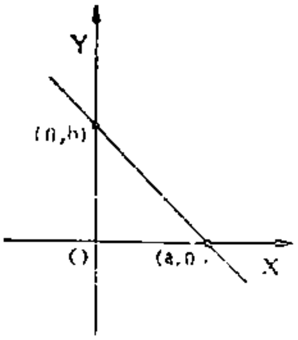


Fig. 6: L is given by $\frac{x}{a} + \frac{y}{b} = 1$.

We can obtain the equation of a line in yet another form. Suppose we know the length p of the perpendicular (or **the normal**) from the origin to a line L, and the angle α that the perpendicular makes with the x-axis (see Fig. 7).

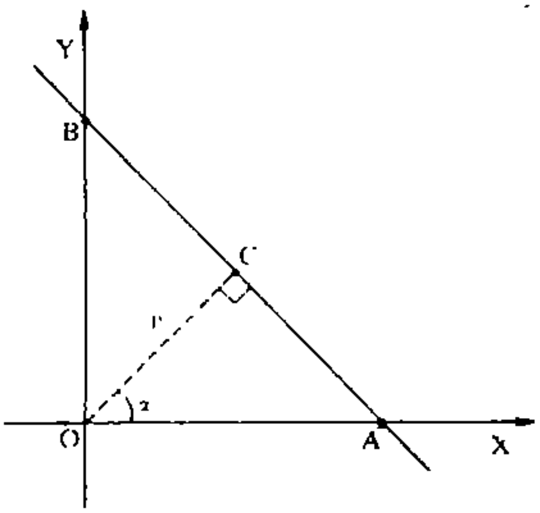


Fig. 7: $x \cos \alpha + y \sin \alpha = p$ is the normal form of AB.

Then, using (8) we can obtain the equation of L in the **normal form**

$$x \cos \alpha + y \sin \alpha = p \qquad \ldots(9)$$

For example, the line which is at a distance of 4 units from (0, 0), and for which $\alpha = 135°$, has equation $-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 4$, that is, $x - y + 4\sqrt{2} = 0$.

Here's a small remark about the form (9).

The distance of a line from a point is the length of the perpendicular from the point to the line.

Remark 1 : In (9) p is positive and the coefficients of x and y are "normalised", that is, the sum of their squares is 1. Using these facts we can easily find the distance of any line from the origin.

For example, let us find the distance of the origin from the line you got in E4. We rewrite its equation as $-\sqrt{3}x - y = 1$. Then we divide throughout by $\sqrt{(\sqrt{3})^2 + 1}$, to get $\frac{-\sqrt{3}}{2}x - \frac{1}{2}y = \frac{1}{2}$. This is in the form $ax + by = c$, where $a^2 + b^2 = 1$ and $c \geq 0$. Thus, the required distance is c, which is $\frac{1}{2}$.

Now, have you noticed a characteristic that is common to the equations (3) - (8)? They are all linear in two variables, that is, of the form $ax + by + c = 0$, where $a, b, c \in \mathbf{R}$ and at least one of a and b is non-zero. This is not a coincidence, as the following theorem tells us.

Theorem 1 : A linear equation in two variables represents a straight line in two-dimensional space. Conversely, the equation of a straight line in the plane is a linear equation in two variables.

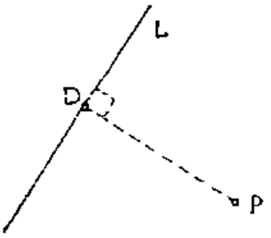


Fig. 8: PD is the distance from P to the line L.

So, for example, $2x + 3y - 1 = 0$ represents a line. What is its slope? We rewrite it as $y = -\frac{2}{3}x + \frac{1}{3}$, to find that its slope is $-\frac{2}{3}$. Do you agree that its intercepts on the x and y axes are $\frac{1}{2}$ and $\frac{1}{3}$, respectively? And what is its distance from the origin? To find this, we "normalise" the coefficients of x and y, that is, we divide the equation throughout by $\sqrt{2^2 + 3^2} = \sqrt{13}$. We get $\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{1}{\sqrt{13}}$, which is in the form (9). Thus, the required distance is $\frac{1}{\sqrt{13}}$.

In general the dist[illegible] of a point $P(x_1, y_1)$ from a line $ax + by + c = 0$ (see Fig. 8) is give[illegible]

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| \qquad \ldots(10)$$

You may like to try some exercises now.

E7) Find the distance of (1, 1) from the line which has slope -1 and intercept $\frac{1}{2}$ on the y-axis.

E8) What is the distance of
  a) $y = mx + c$ from (0, 0)?
  b) $x = 5$ from (1, 1)?
  c) $x \cos \alpha + y \sin \alpha = p$ from $(\cos \alpha, \sin \alpha)$?
  d) (0, 0) from $2x + 3y = 0$?

E9) Prove the equation (9).

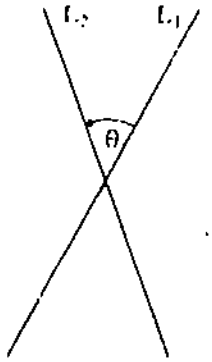


Fig. 9: θ is the angle between the lines L_1 and L_2.

Let us now see what the angle between two lines is. Suppose the slope-intercept forms of the lines are $y = m_1x + c_1$ and $y = m_2x + c_2$ (see Fig. 9).

Then the angle θ between them is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1m_2}. \qquad \ldots(11)$$

Tan θ can be positive or negative. If it is positive, θ is acute. If $\tan \theta < 0$, then θ is the obtuse angle between the lines (which would be $\pi - \theta$ in Fig. 9).

Note that the constant terms in the equations of the lines play no role in finding the angle between them.

Now, from (11) can you say when two lines are parallel or perpendicular? The conditions follow immediately if you remember what tan 0 and tan $\frac{\pi}{2}$ are. Thus, the lines $y = m_1x + c_1$ and $y = m_2x + c_2$

i) are parallel if $m_1 = m_2$, and ...(12)

ii) are perpendicular if $m_1m_2 = -1$. ...(13)

For example, $y = 2x + 3$ and $x + 2y = 5$ are perpendicular to each other, and $y = 2x + 3$ is parallel to $y = 2x + c \; \forall \; c \in \mathbf{R}$.

Why not try an exercise now?

E10) a) Find the equation of the line parallel to $y + x + 1 = 0$ and passing through (0, 0).
  b) What is the equation of the line perpendicular to the line obtained in (a), and passing through (2,1)?
  c) What is the angle between the line obtained in (b) and $2x - y$?

Let us now stop our discussion on lines, and move on to more general equations. We shall discuss a concept that will help us to trace the conics in the next unit.

1.3 SYMMETRY

While studying this block you will come across several equations in x and y. Their geometric representations are called curves. For example, a line is represented by the equation $ax + by + c = 0$, and a circle with radius a and centre (0, 0) is represented by the equation $x^2 + y^2 - a^2 = 0$.

Note that these equations are of the form $F(x, y) = 0$, where $F(x, y)$ denotes their left hand sides.

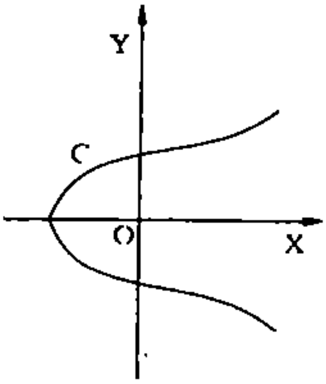


Fig. 10: The curve C is symmetric about the x-axis.

Now suppose the curve C, represented by an equation $F(x, y) = 0$, is such that when (x, y) lies on it, then so does $(x, -y)$.
Then $F(x, y) = 0 \Rightarrow F(x, -y) = 0$.
(For example, $x^2 + y^2 = a^2 \Rightarrow x^2 + (-y)^2 = a^2$.)
In this situation we say that C is **symmetric about the x-axis**. Similarly, C will be **symmetric about the y-axis** if $F(x, y) = 0 \Rightarrow F(-x, y) = 0$.

We say that C is **symmetric about the origin** (0, 0) if
$F(x, y) = 0 \Rightarrow F(-x, -y) = 0$.

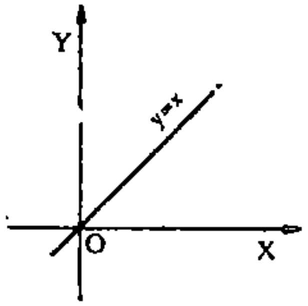


Fig. 11: The line y = x is symmetric about the origin.

Let us look at an example. The circle $x^2 + y^2 = 9$ is symmetric about both the axes and the origin. On the other hand, the line $y = x$ is not symmetric about any of the axes, but it is symmetric about the origin.

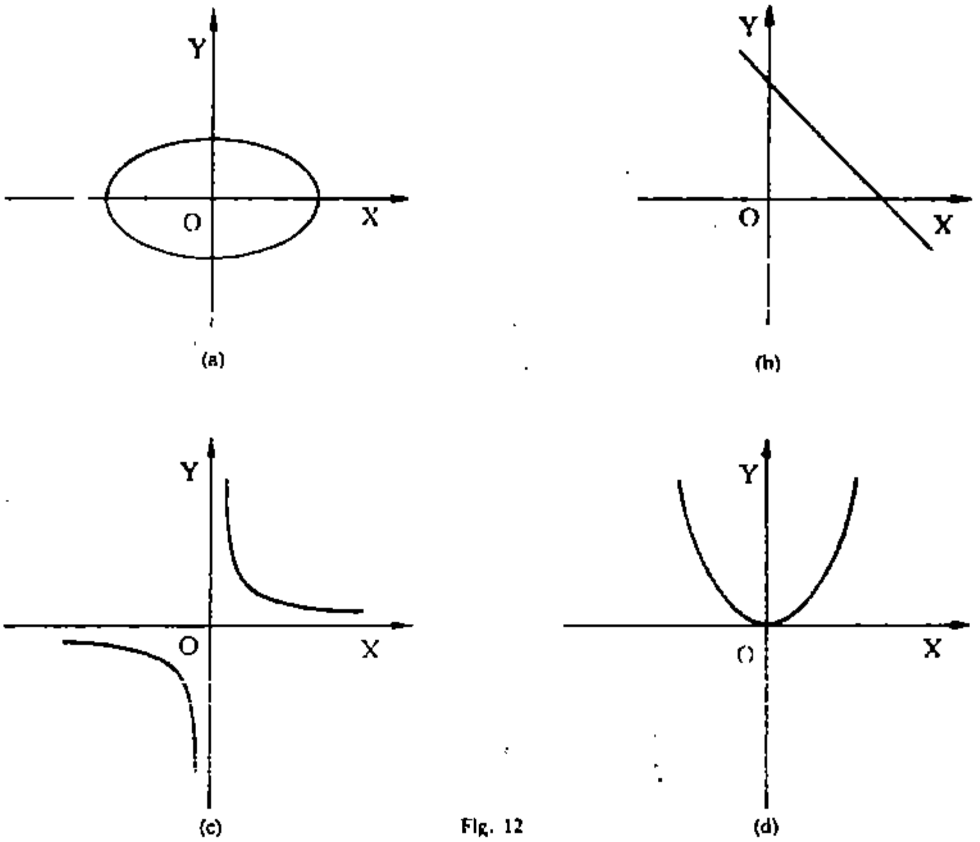
Geometrically, if a curve is symmetric about the x-axis, it means that the portion of the curve below the x-axis is the mirror image of the portion above the x-axis (see Fig. 10). A similar visual interpretation is true for symmetry about the y-axis. And what does symmetry about the origin mean geometrically? It means that the mirror image of the portion of the curve in the first quadrant is the portion in the third quadrant, and the mirror image of the portion in the second quadrant is the portion in the fourth quadrant (see Fig. 11).

Why don't you try some exercises on symmetry to see if you have grasped the concept?

E11) Which axis is the curve $y^2 = 2x$ symmetric about? Is it symmetric about the origin?

E12) Discuss the symmetries of the line $y = 2$.

E13) Which of the curves in Fig. 12 are symmetric about the x-axis? And which ones are symmetric with respect to the origin?



(a) (b)

(c) (d)

Fig. 12

E14) a) Show that if $F(x, y) = 0$ is symmetric about the x-axis, then $F(x, y) = 0$ iff $F(x, -y) = 0$.

b) Show that if $F(x, y) = 0$ is symmetric about both the axes, then it is symmetric about the origin. Is the converse true?

There is another concept that you will need while studying Units 2 and 3, which we shall now take up.

1.4 CHANGE OF AXES

In the next unit you will see that the general equation of a circle is $x^2 + y^2 + 2ux + 2vy + c = 0$. But we can always choose a coordinate system in which the equation simplifies to $x^2 + y^2 = r^2$, where r is the radius of the circle. To see why this happens, we need to see how to choose an appropriate set of coordinate axes. We also need to know how the coordinates of a point get affected by the transformations to a new set of axes. This is what we will discuss in this section.

There are several ways in which axes can be changed. We shall see how the coordinates of a point in a rectangular Cartesian coordinate system are affected by two types of changes, namely, translation and rotation.

1.4.1 Translating the Axes

The first type of change of axes that we consider is a shift in the origin without changing the direction of the axes.

Let XOY be a rectangular Cartesian coordinate system. Suppose a point O′ has the coordinates (a, b) in this system, what happens if we shift the origin to O′? Let O′X′, parallel to OX, be the new x-axis. Similarly let O′Y′, parallel to OY, be the new y-axis (see Fig. 13). Now, suppose a point P has the coordinates (x, y) and (x′, y′) with respect to the old and the new coordinate systems, respectively. How are they related? From Fig. 13 you can see that

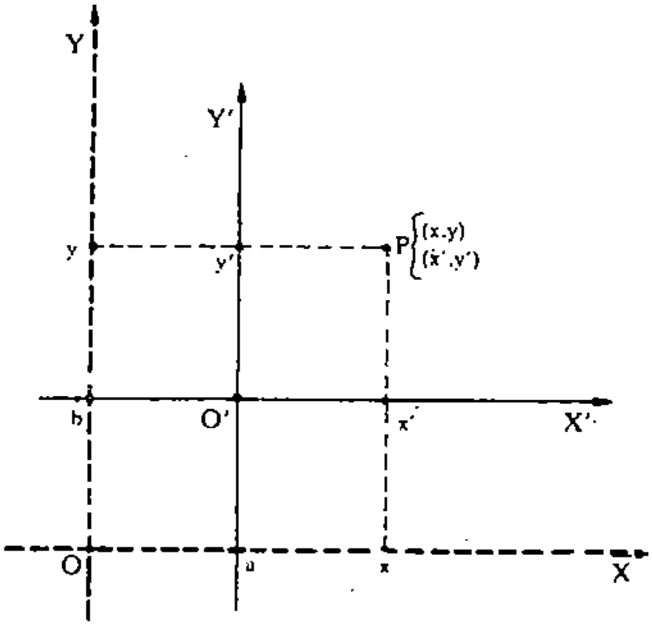


Fig. 13 : Translation of axes through (a, b).

$x = x' + a$ and $y = y' + b$. ...(14)

Thus, the new coordinates are given by

$x' = x - a$ and $y' = y - b$. ...(15)

For example, if we shift the origin to (-1, 2), the new (or current) coordinates (x′, y′) of a point P(x, y) will be given by $x' = x + '$ $y' = y - 2$.

When we shift the origin, keeping the axes parallel, we say that we are **translating the axes**. So, whenever we translate the axes to a point (a, b), we are transforming the coordinate system to a system with parallel axes through (a, b). We can write this briefly as **transforming to parallel axes through (a, b)**.

Now, if we translate the axes to a point (a, b), what will the resultant change in any equation be? Just replace x by $x' + a$ and y by $y' + b$ in the equation, and you get the new equation. For example, the straight line $x + 2y = 1$ becomes $(x' + a) + 2(y' + b) = 1$, that is, $x' + 2y' + a + 2b = 1$ in the new system.

Now for some exercises!

E15) If we translate the axes to (-1, 3), what are the new coordinates of the origin of the previous system? Check your answer with the help of a diagram.

E16) Transform the quadratic equation $5x^2 + 3y^2 + 20x - 12y + 17 = 0$ to parallel axes
a) through the point (-2, 2), and
b) through the point (1, 1).

If you've done E16, you would have realised how much simplification can be achieved by an appropriate shift of the origin.

Over here we would like to make an important observation.

Note : When you apply a translation of axes to a curve, the shape of the curve doesn't change. For example, a line remains a line and a circle remains a circle of the same radius. Such a transformation is called a **rigid body motion**.

Now let us consider another kind of change of axes.

1.4.2 Rotating the Axes

Let us now see what happens if we change the direction of the coordinate axes without shifting the origin. That is, we shall consider the transformation of coordinates when the rectangular Cartesian system is rotated about the origin through an angle θ. Let the coordinate system XOY be rotated through an angle θ in the anticlockwise direction about O in the XOY plane. Let OX′ and OY′ be the new axes (see Fig. 14). Let P be a point with coordinates (x, y) in the XOY system, and (x′, y′) in the X′OY′ system. Drop perpendiculars PA and PB from P to OX′ and OX, respectively. Also draw AC perpendicular to OX, and AD perpendicular to PB. Then $x = OB$, $y = PB$, $x' = OA$, $y' = PA$. Also $\angle DAO = \angle AOC = \theta$. Therefore, $\angle DPA = \theta$.

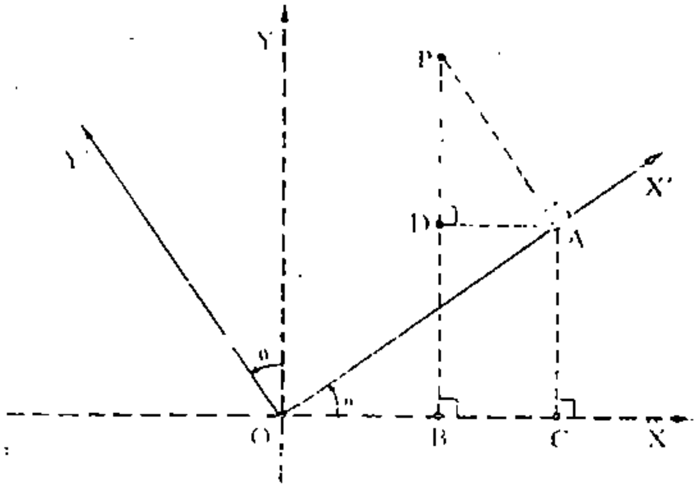


Fig. 14: The axes OX′ and OY′ are obtained by rotating the axes OX and OY through an angle θ.

$$\begin{aligned} \text{Thus, } x &= OB = OC - AD \\ &= OA \cos\theta - PA \sin\theta \\ &= x' \cos\theta - y' \sin\theta \qquad \ldots(16) \\ \text{and } y &= PB = PD + AC \\ &= x' \sin\theta + y' \cos\theta \qquad \ldots(17) \end{aligned}$$

(16) and (17) give us x and y in terms of the new coordinates x′ and y′.

Now, how can we get x′ and y′ in terms of x and y?

Note that the XY-system can be got from the x′y′-system by rotating through $(-\theta)$. Thus, if we substitute $-\theta$ for θ, x′ by x and y′ by y in (16) and (17), we get x′ and y′ in terms of x and y.

Thus, $$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ \text{and}\quad y' &= -x\sin\theta + y\cos\theta \end{aligned}\right| \qquad \ldots(18)$$

For example, the current coordinates of a point P(x, y), when the rectangular axes are rotated in the anticlockwise direction through 45°, are

$$x' = x\cos 45° + y\sin 45° = \frac{1}{\sqrt{2}}(x + y)$$

$$y' = -x\sin 45° + y\cos 45° = \frac{1}{\sqrt{2}}(y - x)$$

Now, what happens if we shift the origin and rotate the axes? We will need to apply all the transformations (14) – (17) to get the current coordinates.

For example, suppose we transform to axes inclined at 30° to the original axes, the equation $11x^2 + 2\sqrt{3}\,xy + 9y^2 = 12(x\sqrt{3} + y + 1)$, and then translate the system through $\left(\frac{1}{2}, 0\right)$, what do we get? We first apply (16) and (17), to get

$$11(x'\sqrt{3} - y')^2 + 2\sqrt{3}\,(x'\sqrt{3} - y')(x' + y'\sqrt{3}) + 9(x' + y'\sqrt{3})^2 =$$
$$12\{\sqrt{3}\,(x'\sqrt{3} - y') + (x' + y'\sqrt{3}) + 1\}, \text{ that is,}$$
$$6\left(x' - \frac{1}{2}\right)^2 + 4y'^2 = 3.$$

Now if we shift the origin to $\left(\frac{1}{2}, 0\right)$ and use (14), we find that the new coordinates (X, Y) are related to (x′, y′) by $x' = X + \frac{1}{2}$, $y' = Y + 0$.
Thus, the equation will become
$$6X^2 + 4Y^2 = 3.$$

Isn't this an easier equation to handle than the one we started with? In fact, both the translation and rotation have been carefully chosen so as to simplify the equation at each stage.

Note: The rotation of axes is a rigid body motion. Thus, when such a transformation is applied to a curve, its position may change but its shape remains the same.

Try these exercises now.

E17) Write the equation of the straight line $x + y = 1$ when the axes are rotated through 60°.

E18) a) Suppose the origin is shifted to (-2, 1) and the rectangular Cartesian axes are rotated through 45°. Find the resultant transformation of the equation $x^2 + y^2 + 4x - 2y + 4 = 0$.
  b) Now, first rotate the axes through 45° and then shift the origin to (-2, 1). What is the resulting transformation of the equation in (a)?
  c) From (a) and (b) what do you learn about interchanging the transformations of axes? (You can study more about this in our co 'Linear Algebra'.)

So far we have been working with Cartesian coordinates. But is there any other coordinate system that we can use? Let's see.

1.5 POLAR COORDINATES

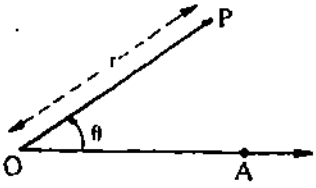


Fig. 15: Polar coordinates.

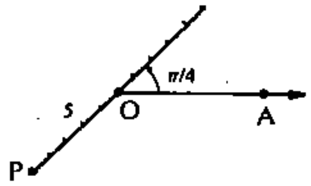


Fig. 16: P's polar coordinates are $\left(-5, \frac{\pi}{4}\right)$

A point has many different polar coordinates.

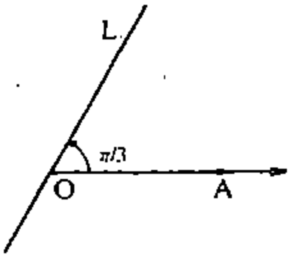


Fig. 17: The line L is given by $\theta = \pi/3$.

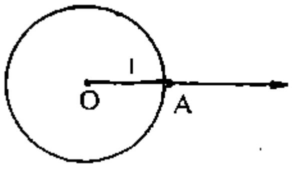


Fig. 18: The circle $r = 1$.

In the late 17th century the mathematician Bernoulli invented a coordinate system which is different from, but intimately related to, the Cartesian system. This is the **polar coordinate system**, and was used extensively by Newton. You will realise the utility of this system when you study conics in Unit 2. Now, let us see what polar coordinates are.

To define them, we first fix a pole O and a polar axis OA, as shown in Fig. 15. Then we can locate any point P in the plane, if we know the distance OP, say r, and the angle AOP, say θ radians. (Does this remind you of the geometric representation of complex numbers?) Thus, given a point P in the plane, we can represent it by a pair (r, θ), where r is the "directed distance" of P from O and θ is $\angle AOP$, measured in radians in the anticlockwise direction. We use the term "directed distance" because r can be negative also. For instance, the point P in Fig. 16 can be represented by $\left(5, \frac{5\pi}{4}\right)$ or $\left(-5, \frac{\pi}{4}\right)$. Note that by this method the point O corresponds to $(0, \theta)$, for any angle θ.

Thus, for any point P, we have a pair of real numbers (r, θ) that corresponds to it. They are called the **polar coordinates**.

Now, if we keep θ fixed, say $\theta = \alpha$, and let r take on all real values, we get the line OP (see Fig. 17), where $\angle AOP = \alpha$. Similarly, keeping r fixed, say $r = a$, and allowing θ to take all real values, the point $P(r, \theta)$ traces a circle of radius a, with centre at the pole (Fig. 18). Here note that a negative value of θ means that the angle has magnitude $|\theta|$, but is taken in the clockwise direction. Thus, for example, the point $\left(2, -\frac{\pi}{2}\right)$ is also represented by $\left(2, \frac{3\pi}{2}\right)$.

As you have probably guessed, the Cartesian and polar coordinates are very closely related. Can you find the relationship? From Fig. 19 you would agree that the relationship is

$$\left.\begin{array}{l} x = r\cos\theta,\ y = r\sin\theta, \text{ or} \\ r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x} \end{array}\right\} \qquad \ldots(19)$$

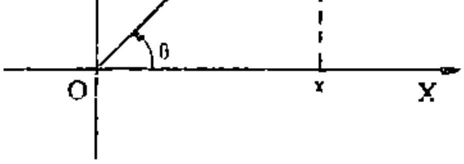


Fig. 19 : Polar and Cartesian coordinates.

Note that the origin and the pole are coinciding here. This is usually the situation.

We use this relationship often while dealing with equations. For example, the Cartesian equation of the circle $x^2 + y^2 = 25$, reduces to the simple polar form $r = 5$. So we may prefer to use this simpler form rather than the Cartesian one.

Doing the following exercises will help you get used to polar coordinates.

E19) From (9) and (19), show that the polar equation of the line AB in Fig. 7 is $r \cos(\theta - \alpha) = p$.

E20) Draw the graph of the curve $r \cos\left(\theta - \frac{\pi}{4}\right) = 0$, as r and θ vary.

E21) Find the Cartesian forms of the equations
a) $r^2 = 3r \sin\theta$
b) $r = a(1 - \cos\theta)$, where a is a constant.

Also see Unit 9 of MTE-01 (Calculus) for more about tracing of curves.

Apart from the polar coordinate system, we have another method of representing points on a curve. This is the representation in terms of a parameter. You will come across this simple method in the next unit, when we discuss each conic separately.

Let us now summarise that we have done in this unit.

1.6 SUMMARY

In this unit we have briefly run through certain elementary concepts of two-dimensional analytical geometry. In particular, we have covered the following points:

1) The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

2) The distance between (x_1, y_1) and the line $ax + by + c = 0$ is
$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|.$$

3) Any line parallel to the x-axis is $y = a$, and parallel to the y-axis is $x = b$, for some constants a and b.

4) The equation of a line in
   i) slope-intercept form is $y = mx + c$,
   ii) point-slope form is $y - y_1 = m(x - x_1)$,
   iii) two-point form is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$,
   iv) intercept form is $\frac{x}{a} + \frac{y}{b} = 1$,
   v) normal form is $x \cos\alpha + y \sin\alpha = p$.

5) The angle between two lines with slopes m_1 and m_2 is
$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right).$$
The lines are parallel if $m_1 = m_2$, and perpendicular if $m_1 m_2 = -1$.

6) Symmetry about the coordinate axes and the origin.

7) i) If we translate the axes to (a, b), keeping the directions of the axes unchanged, the new coordinates x′ and y′ are given by $x' = x - a$ and $y' = y - b$.
   ii) If we rotate the axes through an angle θ, keeping the origin unchanged, the new coordinates x′ and y′ are given by
   $x' = x \cos\theta + y \sin\theta$
   $y' = -x \sin\theta + y \cos\theta$.

8) **A point P in a plane can be represented by a pair of real numbers (r, θ), where r is the directed distance of P from the pole O and θ is the angle that OP makes with the polar axis, measured in radians in the anticlockwise direction. These are the polar coordinates of P.** They are related to the

Cartesian coordinates (x, y) of P by
$r^2 = x^2 + y^2$ and
$\theta = \tan^{-1} \frac{y}{x}$

In the next unit we shall start our study of ellipses and other conics. But before going to it, **please make sure that you have achieved the unit objectives listed in Sec. 1.1.** One way of checking is to ensure that you have done all the exercises in the unit. Our solutions to these exercises are given in the following section.

1.7 SOLUTIONS/ANSWERS

E1) a) $\left(\frac{5-3}{2}, \frac{-4+2}{2}\right) = (1, -1)$.

b) $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$

E2) $PQ = \sqrt{(1-(-2)^2+(0-3)^2} = \sqrt{18}$

$QR = \sqrt{(-2-1)^2+(3-3)^2} = 3$

$PR = \sqrt{(1-1)^2+(0-3)^2} = 3$

Thus, the sides of the triangle are not equal in length.
Hence, ΔPQR is not equilateral.

E3) The x and y-axis are $y = 0$ and $x = 0$, respectively.

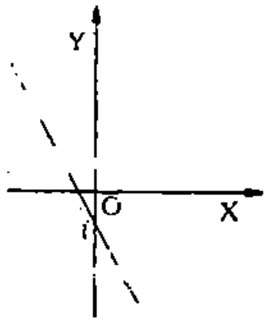


Fig. 20: $y = -(\sqrt{3}x + 1)$.

E4) In Fig. 20 we have drawn the line. Its equation is $y = mx + c$, where $c = -1$ and $m = \tan 120° = -\sqrt{3}$.
Thus, the required equation is
$y = -(\sqrt{3}\,x + 1)$.

E5) Here $c = 0$. Thus, the equation is $y = x \tan \theta$.

E6) a) (2, 0) and (0, -3) lie on the line. Thus, its two-point form is

$$\frac{y-0}{-3-0} = \frac{x-2}{0-2}, \text{ that is, } 2y = 3(x-2).$$

b) (a, 0) and (0, b) lie on the line. Thus, its equation is

$$\frac{y-0}{b-0} = \frac{x-a}{0-a} \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$

E7) The equation of the line is $y = -x + \frac{1}{2}$, that is, $2x + 2y - 1 = 0$.
The distance of (1, 1) from this line is

$$\left|\frac{2.1 + 2.1 - 1}{\sqrt{4+4}}\right| = \frac{3}{\sqrt{8}}$$

E8) a) $\left|\frac{m.0 - 0 + c}{\sqrt{m^2+1}}\right| = \frac{c}{\sqrt{m^2+1}}$.

b) $\left|\frac{1-5}{1}\right| = 4$.

c) $|1 - p|$.

d) 0

E9) Using the intercept form (8) and Fig. 7, we see that the equation of the line is

$$\frac{x}{OA} + \frac{y}{OB} = 1. \qquad \ldots(20)$$

Now, $\angle OAC = \frac{\pi}{2} - \alpha$ and $\angle OBC = \alpha$. Thus.

$OA = OC \operatorname{cosec}\left(\frac{\pi}{2} - \alpha\right) = p \sec \alpha = \frac{p}{\cos \alpha}$, and

$OB = OC \operatorname{cosec} \alpha = \frac{p}{\sin \alpha}$.

Thus, (20) $\Rightarrow \frac{x \cos \alpha}{p} + \frac{y \sin \alpha}{p} = 1$

$\Rightarrow x \cos \alpha + y \sin \alpha = p$.

E10) a) Any line parallel to $y + x + 1 = 0$ is of the form $y + x + c = 0$, where $c \in \mathbf{R}$. Since (0, 0) lies on it, $0 + 0 + c = 0$, that is, $c = 0$. Thus, the required line is $y + x = 0$.

b) The slope of the line $y + x - 0$ is -1. Thus, the slope of any line perpendicular to it is 1, by (13). Thus, the equation of the required line is of the form $y = x + c$, where $c \in \mathbf{R}$. Since (2, 1) lies on it, $1 = 2 + c \Rightarrow c = -1$. Thus, the required line is $y = x - 1$.

c) In this case $m_1 = 1$, $m_2 = 2$. Thus, the angle between the lines is

$$\theta = \tan^{-1}\left(\frac{1 - 2}{1 + 1.\times 2}\right) = \tan^{-1}\left(-\frac{1}{3}\right) = -\tan^{-1}\frac{1}{3}.$$

Note that both $-\tan^{-1}\left(-\frac{1}{3}\right)$ and $\tan^{-1}\left(-\frac{1}{3}\right)$ are angles between the given lines.

E11) If we substitute y by (-y) in the given equation, it remains unchanged. Thus, the curve is symmetric about the x-axis. If we substitute x by (-x), the curve changes to $y^2 = -2x$. Thus, it is not symmetric about the y-axis.

If we substitute (-x) and (-y) for x and y, respectively, in the equation, it changes to $y^2 = -2x$. Thus, it is not symmetric about the origin.

E12) It is not symmetric about either axis or the origin.

E13) (a) is symmetric with respect to the x-axis.
(a) and (d) are symmetric with respect to the y-axis.
(a) and (c) are symmetric with respect to the origin.

E14) a) The curve is symmetric about the x-axis. Thus,
$F(x, y) = 0 \Rightarrow F(x, -y) = 0 \;\forall\; x, y \in \mathbf{R}$.
$\therefore F(x, -y) = 0 \Rightarrow F(x, -(-y)) = 0 \Rightarrow F(x, y) = 0 \;\forall\; x, y \in \mathbf{R}$.
Hence, the equivalence.

b) The curve is symmetric about both the axes.
Now $F(x, y) = 0$
$\Rightarrow F(x, -y) = 0$, because of symmetry about the x-axis.
$\Rightarrow F(-x, -y) = 0$, because of symmetry about the y-axis.
$\Rightarrow$ F is symmetric about the origin.
The converse is clearly not true, as you can see from Fig. 11.

E15) In Fig. 21 we show the new and old systems.

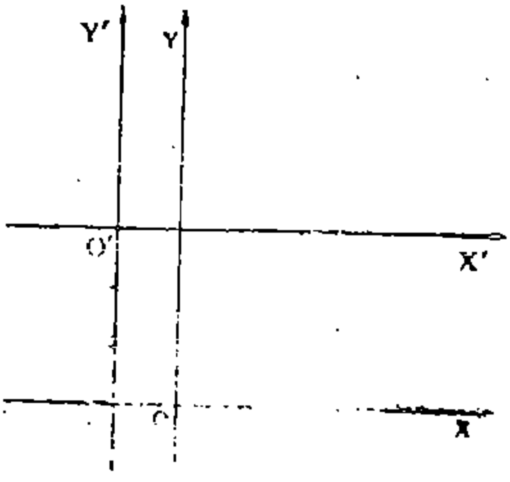


Fig. 21: The coordinates of O are [illegible] with respect to X' O'Y'.

E16) a) If the new coordinates are x' and y', then $x = x'-2$, $y = y'+2$. Thus, the equation becomes

$$5(x'-2)^2 + 3(y'+2)^2 + 20(x'-2) - 12(y'+2) + 17 = 0$$

$$\Rightarrow 5x'^2 + 3y'^2 - 15 = 0$$

$$\Rightarrow \frac{x'^2}{3} + \frac{y'^2}{5} = 1.$$

b) The equation becomes

$$5(x'+1)^2 + 3(y'+1)^2 + 20(x'+1) - 12(y'+1) + 17 = 0$$

$$\Rightarrow 5x'^2 + 3y'^2 + 30x' - 6y' + 33 = 0.$$

E17) Here $x = \frac{x'}{2} - \frac{y'\sqrt{3}}{2}$ and $y = \frac{x'\sqrt{3}}{2} + \frac{y'}{2}$

Thus, $x + y = 1$ becomes

$$\left(\frac{x'}{2} - \frac{y'\sqrt{3}}{2}\right) + \left(\frac{x'\sqrt{3}}{2} + \frac{y'}{2}\right) = 1, \text{ that is,}$$

$$x'(1 + \sqrt{3}) + y'(1 - \sqrt{3}) = 2.$$

E18) a) By shifting the origin, the new coordinates x' and y' are related to x and y by

$x = x' - 2$, $y = y' + 1$.

Thus, the equation becomes

$$(x' - 2)^2 + (y' + 1)^2 + 4(x' - 2) - 2(y' + 1) + 4 = 0$$

$$\Rightarrow x'^2 + y'^2 = 1 \qquad \text{...(21)}$$

Now, rotating the axes through 45°, we get new coordinates x and y given by,

$$x' = \frac{X - Y}{\sqrt{2}} \text{ and } y' = \frac{X + Y}{\sqrt{2}}$$

Thus, (21) becomes

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 + \left(\frac{X + Y}{\sqrt{2}}\right)^2 = 1$$

$$\Rightarrow X^2 - 2XY + Y^2 + X^2 + 2XY + Y^2 = 2.$$

$$\Rightarrow X^2 + Y^2 = 1$$

b) If we first rotate the axes, the given equation becomes

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 + 4\left(\frac{x' - y'}{\sqrt{2}}\right) - 2\left(\frac{x' + y'}{\sqrt{2}}\right) + 4 = 0$$

$\Rightarrow$ [illegible] $+ y'^2 + 2\sqrt{x}' - 3\sqrt{2}y' + 4 = 0$ ...(22)

Now applying the shift in origin to (-2, 1), the equation (22) becomes

$$X^2 + Y^2 + X(\sqrt{2} - 4) + Y(2 - 3\sqrt{2}) + 9 - 5\sqrt{2} = 0.$$

c) From (a) and (b) you can see that a change in the order of transformations makes a difference. That is, if T_1 and T_2 are two transformations, then T_1 followed by T_2 need not be the same as T_2 followed by T_1. Diagrammatically, the circles C_1 and C_2 in Fig. 22 correspond to the final equations in (a) and (b), respectively.

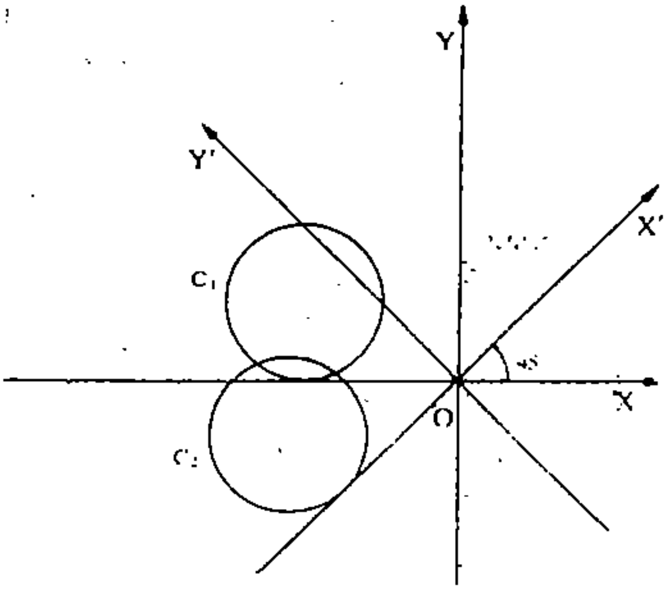


Fig. 22

E19) From (9), the equation of L is
$x \cos \alpha + y \sin \alpha = p$.
Usihg (19), this becomes
$r (\cos \theta \cos \alpha + \sin \theta \sin \alpha) = p$.
$\Rightarrow r \cos (\theta - \alpha) = p$.

E20)

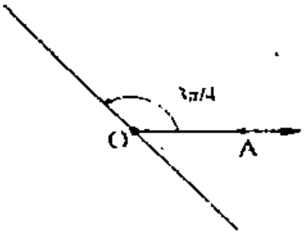


Fig. 23 : The line $r \cos \left(\theta - \frac{\pi}{4}\right) = 0$.

E21) a) Since $r^2 = x^2 + y^2$ and $y = r \sin \theta$, the equation becomes
$x^2 + y^2 = 3y$.

b) The equation becomes

$$\sqrt{x^2 + y^2} = a \left(1 - \frac{x}{\sqrt{x^2 + y^2}}\right)$$

$$\Rightarrow x^2 + y^2 + a (x + \sqrt{x^2 + y^2}) = 0.$$

UNIT 2 THE STANDARD CONICS

Structure

2.1 INTRODUCTION

In this unit you will be studying some curves which may be familiar to you. They were first studied systematically by the Greek astronomer Apollonius (approximately 225 B.C.). These curves are the parabola, ellipse and hyperbola. They are called conics (or conic sections) because, as you will see in this course, they can be formed by taking the intersection of a plane and a cone.

We start this unit by defining conics as curves that satisfy the 'focus-directrix' property. From this definition, we will come to the particular cases of standard forms of a parabola, ellipse and a hyperbola. The standard forms are so called because any conic can be reduced to one of these forms, and then the various properties of the conic under consideration can be studied easily. We will trace the standard forms and look at their tangents and normals. We will also discuss some other characteristics, along with some of their applications in astronomy, military science, physics, etc.

In the next unit we will discuss conics in general. And then, what you study in this unit will certainly be of help. If you achieve the following unit objectives, then you can be sure that you have grasped the contents of this unit.

Objectives

After studying this unit, you should be able to

- obtain the equation of a conic if you know its focus and directrix;
- obtain the standard forms of the Cartesian and polar equations of a parabola, an ellipse or a hyperbola;
- prove and apply the string property of an ellipse or a hyperbola;
- obtain the tangent or normal to a standard conic at a given point lying on it;
- check whether a given line is a tangent to a given standard conic or not;
- find the asymptotes of a hyperbola in standard form.

Let us now start our discussion on conics.

2.2 FOCUS-DIRECTRIX PROPERTY

Suppose you toss a ball to your friend. What path will the ball trace? It will be similar to the curve in Fig. 1, which is a parabola. With this section, we begin to take a close look at curves like a parabola, an ellipse or a hyperbola. Such curves are called conic sections, or conics. These curves satisfy a geometric property, which other curve satisfies. We treat this property as the definition of a conic section.

Fig. 1: The ball, when thrown, traces a parabola.

Definition: A **conic section**, or a **conic**, is the set of all those points in two-dimensional space for which the distance from a fixed point F is a constant (say, e) times the distance from a fixed straight line L (see Fig. 2).

The fixed point F is called a **focus** of the conic. The line L is known as a **directrix** of the conic. The number e is called the **eccentricity** of the conic.

Since there are infinitely many lines and points in a plane, you may think that there are infinitely many types of conics. This is not so. In the rest of this block we will list the types of conics that there are and discuss them in detail. As a first step in this direction, let us see what the definition means in algebraic terms.

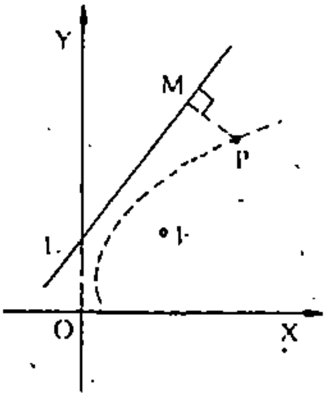


Fig. 2: The set of all points P, where PF = ePM is a conic.

We will obtain the equation of a conic section in the Cartesian coordinate system. Let F(a,b) be a focus of the conic, and $px+qy+r = 0$ be the directrix L (see Fig. 2). Let e be the eccentricity of the conic. Then a point P(x,y) lies on the conic iff

$$\sqrt{(x - a)^2 + (y - b)^2} = e\left|\frac{px + qy + r}{\sqrt{p^2 + q^2}}\right|$$, by Formulae (1) and (10) of Unit 1.

$$\Leftrightarrow [(x - a)^2 + (y - b)^2]\,(p^2 + q^2) = e^2(px + qy + r)^2 \qquad \ldots(1)$$

Thus, (1) is the equation of the conic with a focus at (a, b), a directrix $px + qy + r = 0$ and eccentricity e.

For example, the equation of the conic with eccentricity 1/2, a focus at (1, 1) and a directrix $x + y = 1$ is

$$(x - 1)^2 + (y - 1)^2 = \frac{1}{4} \cdot \frac{(x + y - 1)^2}{2}.$$

Why don't you try an exercise now?

E1) Find the equation of the conic section with
- a) eccentricity 1, (2, 0) as its focus and $x = y$ as its directrix,
- b) eccentricity 1/2, $2x+y = 1$ as its directrix and (0, 1) as its focus. (Note that in this case the focus lies on the directrix).

In E1 you have seen the two different possibilities that the focus may or may not lie on the directrix. Let us first consider the case when the focus does not lie on the directrix. In this case the conics we get are called **non-degenerate conics**. There are three types of such conics, depending on whether $e < 1$, $e = 1$ or $e > 1$.

When $e < 1$, the conic is an ellipse; when $e = 1$, we get a parabola; and when $e > 1$, we get a hyperbola. We shall discuss each of these conics in detail in the following sections.

Let us start with the non-degenerate conics with eccentricity 1.

2.3 PARABOLA

In this section we will discuss the equation and properties of a parabola. Let us first define a parabola.

Definition: A parabola is the set of all those points in two-dimensional space that are equidistant from a line L and a point F not on L. L is its directrix and F is its focus.

Let us use (1) to obtain the equation of a parabola. To start with let us assume F is (0, 0) and L is the straight line $x + c = 0$, where $c > 0$. (Thus, L is parallel to the y-axis and lies to the left of F.) Then, using (1), we see that the equation of the parabola is

$x^2 + y^2 = (x + c)^2$, that is,

$$y^2 = c(2x + c). \qquad \ldots(2)$$

Now, to simplify the equation let us shift the origin to $\left(-\frac{c}{2}, 0\right)$. If we put $\frac{c}{2} = a$, then we are shifting the origin to (-a, 0). From Sec. 1.4.1 you know that the new coordinates x' and y' are given by

$x = x' - a$ and $y = y'$

Thus, (2) becomes

$y'^2 = 4ax'$.

This parabola has a focus at (a, 0) (in the X'Y'-system) and the equation of the directrix is

$x' + a = 0$.

So, what we have found is that the equation

$$y^2 = 4ax \qquad \ldots(3)$$

represents a parabola with $x + a = 0$ as its directrix and (a, 0) as its focus.

This is one of the standard forms of the equation of a parabola.

There are three other standard forms of the equation of a parabola. They are

$$x^2 = 4ay, \qquad \ldots(4)$$

$$y^2 = -4ax, \text{ and} \qquad \ldots(5)$$

$$x^2 = -4ay, \qquad \ldots(6)$$

where $a > 0$.

These equations are called standard forms because, as you will see in Unit 3, we can transform the equation of any parabola into one of these forms. The transformations that we use are the rigid body motions given in Sec. 1.4. So they do not affect the geometric properties of the curve that is being transformed. And, as you will see in the following sub-sections, the geometry of the standard forms are very easy to study. So, once we have the equation of a parabola, we transform it to a standard form and study its properties. And these properties will be the same as the properties of the parabola we started with.

Now let us see what the standard forms look like.

2.3.1 Description of Standard Forms

Let us now see what a parabola looks like. We start with tracing (3). For this, let us see what information we can get from the equation. Firstly, the curve intersects each of the axes in (0, 0) only.

Next, we find that for the points (x, y) of the parabola, $x \geq 0$, since $y^2 \geq 0$. Thus, the curve lies in the first and fourth quadrants.

Further, as x increases, y also increases in magnitude.

And finally, the parabola (3) is symmetric about the x-axis, but not about the y-axis or the origin (see Sec. 1.3). Thus, the portions of the curve in the first quadrant and the fourth quadrant are mirror images of each other.

Using all this information about the curve $y^2 = 4ax$, we trace it in Fig. 3

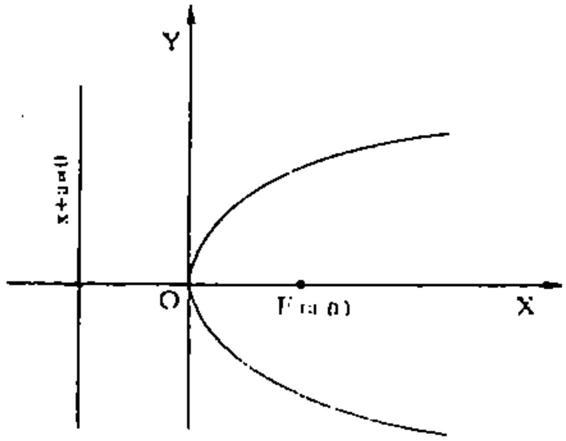


Fig. 3: $y^2 = 4ax$, a > 0.

The line through the focus and perpendicular to the directrix is called the axis of the parabola. Thus, in this case the x-axis is the axis of the parabola.

The point at which the parabola cuts its axis is called its **vertex**. Thus, (0, 0) is the **vertex** (plural 'vertices') of the parabola in Fig. 3.

Now, what happens if we interchange x and y in (3)? We will get (4). This is also a parabola. Its focus is at (0, a), and directrix is $y + a = 0$. If we study the symmetry and other geometrical aspects of the curve, we find that its geometrical representation is as in Fig. 4. Its vertex is also at (0, 0) but its axis is not the same as that of (3). Its axis is $x = 0$, that is, the y-axis.

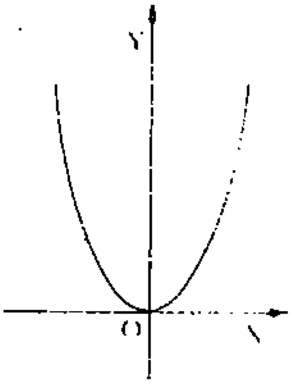


Fig. 4: $x^2 = 4ay$, a > 0.

'Foci' is the plural of 'focus'.

Why don't you trace some parabolas yourself now?

E2) Trace the standard forms (5) and (6) of a parabola. Explicitly state the coordinates of their vertices and foci.

So far we have considered parabolas whose vertices are at (0, 0) and foci lie on one of the coordinate axes. In Unit 3 you will see that by applying the changes in axes that we have discussed in Sec. 1.4, we can always obtain the equations of a parabola in one of these standard forms. In this section we shall keep our discussion to parabolas in standard form.

Now let us look at a simple mechanical method of tracing a parabola. On a sheet of paper draw a straight line L and fix a point F not on L. Then, as in Fig. 5, fix one end of a piece of string with a drawing pin to the vertex A of a set-square. The length of the string should be the length of the side AD of the set square.

Fix the other end of the string with a drawing pin at the point F. Now slide the other leg of the set-square along a ruler placed on the line L (as in Fig. 5), and keep a pencil point P pressed to the side AD so that the string stays taut. Then PD = PF. Thus, as P moves, the curve that you draw will be part of a parabola with focus F and directrix L.

Why don't you try this method for yourself? Instead of a set-square you could simply cut out a right-angled triangle from a piece of **cardboard**.

E3) Use the mechanical method to trace the parabola $x^2 + 8y = 0$

So far we have expressed any point on a parabola in terms of its Cartesian

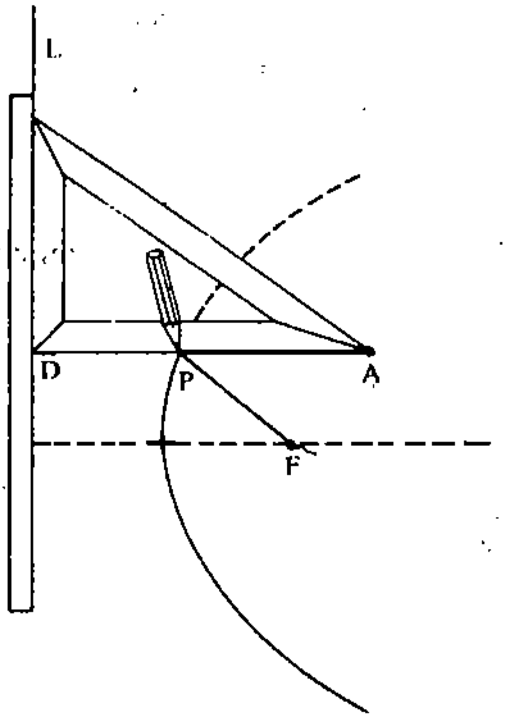


Fig. 5: Mechanical method for tracing a parabola.

coordinates x and y. But sometimes it is convenient to express it in terms of only one variable, or parameter, say t. You can check that the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$, for all $t \in \mathbf{R}$. Further, any point (x, y) on this parabola is of the form $(at^2, 2at)$ where $t = y/2a \in \mathbf{R}$. Thus, a point lies on the parabola $y^2 = 4ax$ iff it can be represented by $(at^2, 2at)$ for some $t \in \mathbf{R}$. In other words,

the parametric representation of any point on the parabola $y^2 = 4ax$ is $x = at^2$, $y = 2at$, where $t \in \mathbf{R}$.

And now let us look at the intersection of a line and a parabola.

2.3.2 Tangents and Normals

The line segment PQ is called a **chord** of the parabola.

Let us consider the parabola $y^2 = 4ax$. What is the equation of the line joining two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on it (see Fig. 6)?

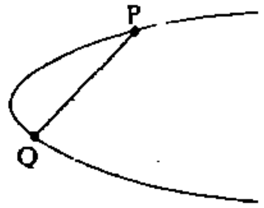


Fig. 6: PQ is a chord of the parabola.

From Unit 1 we know that the equation of PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Since P and Q lie on the parabola,

$y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$.

So we can write the equation of PQ as

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{\frac{1}{4a}(y_2^2 - y_1^2)}$$

$\Leftrightarrow (y - y_1)(y_1 + y_2) = 4a(x - x_1)$ (Note that $y_1 \neq y_2$, since P and Q are distinct.)

$\Leftrightarrow y(y_1 + y_2) - y_1y_2 = 4ax + y_1^2 - 4ax_1$

$\Leftrightarrow y(y_1 + y_2) = 4ax + y_1y_2$, since $y_1^2 = 4ax_1$. ...(7)

This is the equation of any line passing through two distinct points on the parabola.

In particular, the equation of the line joining A(a, 2a) and B(a, -2a) is $x = a$, which is parallel to the directrix of the parabola.

The chord AB has special significance. It is called the **latus rectum** of the parabola $y^2 = 4ax$, and its length is 4a. Note that the focus lies on the latus rectum. Thus, the latus rectum is the chord of the parabola which corresponds to the line through its focus and perpendicular to its axis (see Fig. 7).

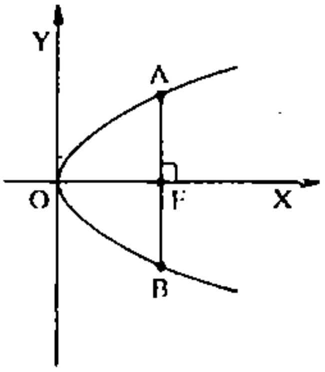


Fig. 7: AB is the latus rectum of the parabola.

Note that **the length of the latus rectum is the coefficient of** x in the equation of the parabola.

Similarly, the length of the latus rectum of $x^2 = 4ay$ is the coefficient of y.

Now for an exercise.

E4) Find the equation of the latus rectum of $x^2 + 2y = 0$.

Now, let us go back to Equation (7). Suppose we take the point Q closer and closer to P, that is, Q **tends** to P. Then x_2 tends to x_1 and y_2 to y_1. In this limiting case the line PQ is given a special name.

Definition: Let P and Q be any two points on a curve C which are close to each other. Then the line segment PQ is called a **secant** of C. The position of the line PQ when the point Q is taken closer and closer to P, and ultimately coincides with P, is called the **tangent to the curve C at P.** P is called the **point of contact** or **point of tangency.**

Thus, in Fig. 6, as Q moves along the curve towards P, the line PQ becomes a tangent to the parabola at the point $P(x_1, y_1)$ (see Fig. 8). So, from (7) we see that the equation of the tangent at P is

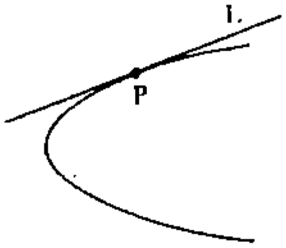


Fig. 8: The line L is the tangent at P to the parabola.

$$y \,.\, 2y_1 = 4ax + y_1^2$$
$$= 4a\,(x + x_1),\text{ since } y_1^2 = 4ax_1.$$
$$\Leftrightarrow yy_1 = 2a(x + x_1).$$

So, (8) is the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1).

For example, the tangent to (3) at its vertex will be the y-axis, $x = 0$.

And what will the equation of the tangent to $y^2 = x$ at (4, 2) be? It will be

$yy_1 = \left(\frac{x + x_1}{2}\right)$ where $x_1 = 4$ and $y_1 = 2$, that is, $4y = (x + 4)$.

Have you noticed how we obtained (8) from (3)? We give you a rule of thumb that we follow in the remark below.

Remark 1: To get the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1), we replace y^2 by yy_1 and x by $\frac{1}{2}(x + x_1)$. Similarly, the equation of the tangent to $x^2 = 4ay$ at a point (x_1, y_1) lying on it will be $xx_1 = 2a(y + y_1)$.

You may like to try the following exercise now.

E5) Find the equation of the tangent to

a) $x^2 + 2y = 0$ at its vertex, and

b) $y^2 + 4x = 0$ at the ends of its latus rectum.

E6) Give an example of a line that intersects $y^2 = 4ax$ in only one point, but is not a tangent to the parabola.

So, given a parabola and a point on it, you have seen how to find the tangent at that point. But, given a line, can we tell whether it is a tangent to a given parabola? Let us see under what conditions the line $y = mx + c$ is a tangent to $y^2 = 4ax$.

If $y = mx + c$ meets the parabola at (x_1, y_1), then

$y_1^2 = 4ax_1$ and $y_1 = mx_1 + c$.

So $(mx_1 + c)^2 = 4ax_1$, that is,

$$m^2x_1^2 + (2mc - 4a)\,x_1 + c^2 = 0. \qquad \ldots(9)$$

Now, there are two possibilities—$m = 0$ and $m \neq 0$. Can the first one arise? Can the line $y = c$ be a tangent to $y^2 = 4ax$? Suppose it is a tangent at a point (x_1, y_1). Then $y = c$ is the same as $yy_1 = 2a(x + x_1)$. This is not possible, since $a \neq 0$.

Thus, for $y = mx + c$ to be a tangent to $y^2 = 4ax$, we must have $m \neq 0$. Then (9) is a quadratic equation in x_1. So it has two roots. Corresponding to each root, we will get a point of intersection of the line and the parabola.

Thus, a line can intersect a parabola in at most two points. If the roots of (9) are real and distinct, the line and parabola have two distinct common points. If the roots of (9) are real and coincide, the line will meet the parabola in exactly one point. And if the roots of (9) are imaginary, the line will not intersect the parabola at all.

So, if $y = mx + c$ is a tangent to $y^2 = 4ax$, the discriminant of (9) must be zero, that is,

$(2mc - 4a)^2 = 4m^2c^2$

$\Rightarrow 4m^2c^2 - 16amc + 16a^2 = 4m^2c^2$

$\Rightarrow c = \frac{a}{m}$, since $m \neq 0$.

Thus,

> the straight line $y = mx + c$ is a tangent to $y^2 = 4ax$ if $m \neq 0$ and $c = \frac{a}{m}$.

And then, what will the point of contact be? Since (9) has coincident roots, we see that

$$x_1 = \frac{4a - 2mc}{2m^2} = \frac{4a - 2m\frac{a}{m}}{2m^2} = \frac{a}{m^2}; \text{ and then}$$

$$y_1 = mx_1 + c = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}.$$

Thus, $y = mx + \frac{a}{m}$ will be a tangent to $y^2 = 4ax$ at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Using the condition for tangency, we can say, for example, that the line $3x +$ [illegible] $= 5$ is a tangent to $y^2 + 15x = 0$, but not to $y^2 = 15x$.

And now for an exercise.

E 7) Under what conditions on m and c, will $y = mx + c$ be a tangent to $x^2 = 4ay$?

[illegible] parabola has several properties, but one of them in particular has many practical applications. This is the **reflecting property.** According to this, suppose a line L, parallel to the axis of a parabola, meets the parabola at a point P (see Fig. 9). Then the tangent to the parabola at P makes equal angles with L and with the focal radius PF. That is, $\alpha = \beta$ in Fig. 9.

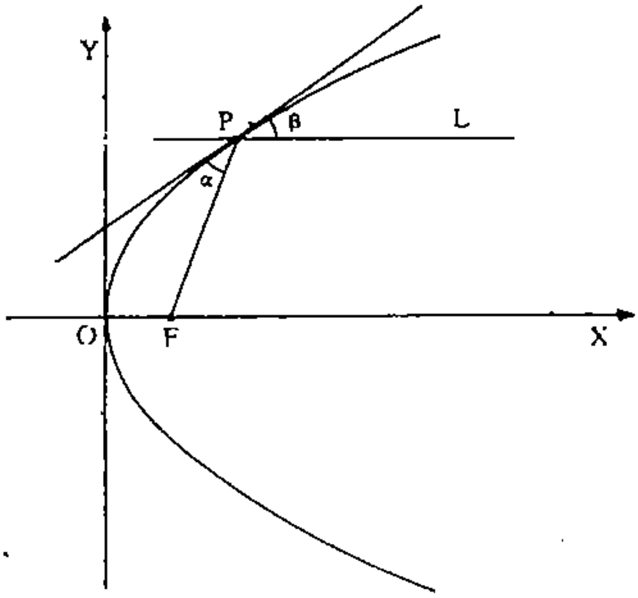


Fig. 9: Reflecting property of a parabola.

The **focal radius** of a point P on a parabola with focus F is the line segment PF.

The reason this property is called the reflecting property is the following application:

Take a mirror shaped like a parabola, that is, a parabolic mirror (see Fig. 10). If a ray of light parallel to the parabola's axis falls on the mirror, then the reflected ray will pass through the focus of the parabola. Thus, a beam of light, parallel to the axis converges to the focus, after reflection. Similarly, the rays of light that are emitted from a source at the focus will be reflected as a beam parallel to the axis. This is why parabolic mirrors are used in car headlights and searchlights.

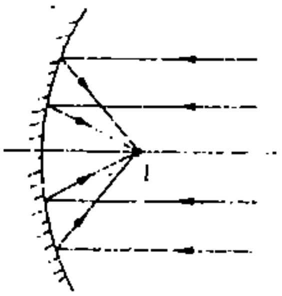


Fig. 10: A parabolic mirror.

'Focus' is Latin for 'fireplace'.

It is also because of this property that the ancient Greek mathematician Archimedes could use parabolic reflectors to set fire to enemy vessels in the harbour! How did he manage this? Archimedes ingeniously thought of applying the following fact:

If a parabolic reflector is turned towards the sun, then the rays of the sun will reflect and converge to the focus and create heat at this point.

This is also the basis of solar-energy collectors like solar cookers.

The reflecting property is also the basis for using parabolic radio and visual telescopes, radars, etc.

The following exercise is about the reflecting property.

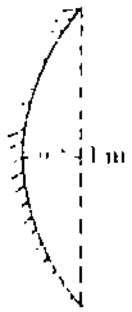


Fig. 11

E8) A parabolic mirror for a searchlight is to be constructed with width 1 metre and depth 0.2 metres. Where should the light source be placed? In Fig. 11 we have given a cross section of the mirror.

(Hint: The parabola is $y^2 = 4ax$, and (0.2, 0.5) lies on it.)

Now let us consider certain lines that are often spoken of along with tangent lines. These are the normals.

Definition: The **normal** to a curve at a point P on the curve is a straight line which is perpendicular to the tangent at P, and which passes through P (see Fig. 12).

Fig. 12: L is the normal to the parabola at P.

For example, a parabola's axis is the normal at its vertex.

Now, let $P(x_1, y_1)$ be a point on $y^2 = 4ax$. Then, you know that the equation of the tangent at P is

$yy_1 = 2a(x + x_1)$.

If $y_1 = 0$, then $x_1 = 0$ and the normal at (0, 0) is $y = 0$, the axis of the parabola.

On the other hand, if $y_1 \neq 0$, then the slope of the tangent at (x_1, y_1) is $\frac{2a}{y_1}$. So the slope of the normal will be $-\frac{y_1}{2a}$ (see Equation (13) of Unit 1). Then, from Unit 1 you know that the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1), \text{ that is,}$$

$$y = -\frac{y_1 x}{2a} + y_1 + \frac{y_1^3}{8a^2}, \qquad \ldots(10)$$

since $y_1^2 = 4ax_1$.

Note that (10) is valid even when $y_1 = 0$.

So, for example, what will the equation of the normal to $y^2 = x$ at (1, 1) be? Here $a = \frac{1}{4}$, $x_1 = y_1 = 1$. So, by (10) we find that the required equation is

$y = -2x + 1 + 2 = -2x + 3$.

We end this section with some easy exercises.

E9) Find the equation of the tangent and normal at (1, 1) to the parabola $x^2 = 4y$.

E10) What is the normal at the point of contact of the tangent $y = mx + \frac{a}{m}$ to the curve $y^2 = 4ax$?

With this we end our rather long discussion on the standard forms of parabolas. Now let us consider a conic whose focus doesn't lie on the directrix, and whose eccentricity is less than 1.

2.4 ELLIPSE

As the title of this section suggests, in it we shall study an ellipse and its properties. Let us start with a definition.

Definition: An **ellipse** is a set of points whose distance from a point F is e (< 1) times its distance from a line L which does not pass through F.

Let us find its Cartesian equation. For this we shall return to Equation (1) in Sec. 2.2. As in the case of a parabola, let us start by assuming that F is the origin and L is $x + c = 0$, for some constant c. Then (1) becomes

$x^2 + y^2 = e^2(x + c)^2$,

which is equivalent to

$$\left(x - \frac{c\,e^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2 c^2}{(1 - e^2)^2} \text{ (Check!).}$$

If we now shift the origin to $\left(\frac{c\,e^2}{1 - e^2}, 0\right)$, the equation in the new X'Y'-system becomes

$$x'^2 + \frac{y'^2}{1 - e^2} = \frac{e^2 c^2}{(1 - e^2)^2}.$$

This is of the form

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

where $a = \frac{e\,c}{(1 - e^2)}$ and $b^2 = \frac{(ec)^2}{1 - e^2} = a^2\,(1 - e^2)$.

In the X'Y'-systems, the focus is (-ae, 0) and the directrix is $x' + ac + c = 0$, that is, $x' + \frac{a}{e} = 0$.

Note that $b^2 = a^2\,(1 - e^2)$ and $e < 1$. Thus, $b^2 < a^2$.

So, if we simply retrace the steps we have taken above, and find the equation of an ellipse with focus (-ae, 0) and directrix

$x + \frac{a}{e} = 0$, we will get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{...(11)}$$

where $b^2 = a^2\,(1 - e^2)$.

(11) is the **standard form** of the equation of an ellipse. As in the case of a parabola, we can always rotate and translate the axes so that the equation of any ellipse can be put in this form, for some a and b. We call it the standard form because it is a convenient form for checking any geometrical properties of an ellipse, or for solving problems related to an ellipse.

Let us now study (11) carefully, and try to trace it.

2.4.1 Description of Standard Form

Let us start by studying the symmetry of the curve (see Sec. 1.3). Do you agree that the curve is symmetric with respect to the origin, as well as both the coordinate axes? Because of this, it is enough to draw the ellipse in the first quadrant. Why is this so? Well, the portion in the second quadrant will then be its reflection in the y-axis; and the rest of the curve will be the reflection in the x-axis of the portion in these two quadrants.

Next, let us see where (11) intersects the coordinate axes. Putting $y = 0$ in (11), we get $x = \pm\,a$; and putting $x = 0$, we get $y = \pm\,b$. So, (11) cuts the axes in the four points (a, 0), (-a, 0), (0, b), (0, -b).

Thirdly, let us see in which area of the plane, the ellipse (11) is defined. You can see that if $|x| > a$, y is imaginary. Thus, the ellipse must lie between $x = -a$ and $x = a$. Similarly, it must lie between $y = b$ and $y = -b$.

This information helps us to trace the curve, which we have given in Fig. 13.

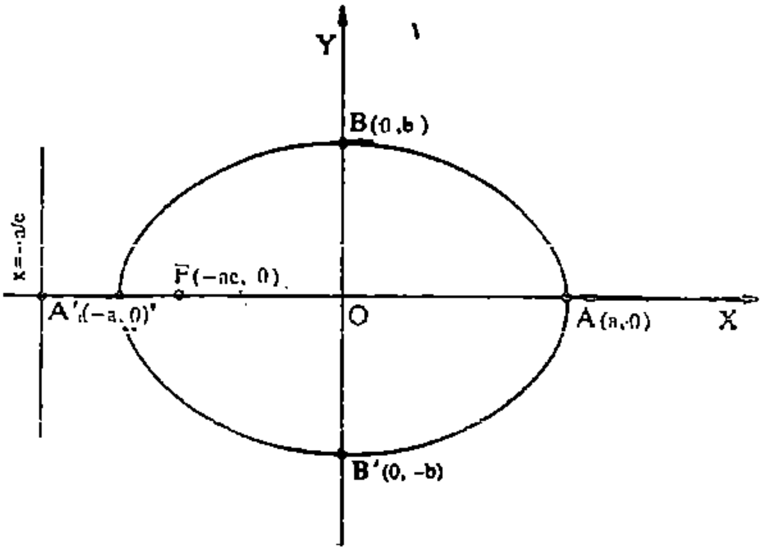


Fig. 13: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Looking at the symmetry of the curve, do you expect (ae, 0) to be a focus also? If you do, then you are on the right track. (11) has another focus at F'(ae, 0), with corresponding directrix $x = \frac{a}{e}$. Thus, (11) has two foci, namely, F(-ae,0) and F'(ae, 0) : and it has two directrices (plural of 'directrix'), namely,

$x = -\frac{a}{e}$ and $x = \frac{a}{e}$.

The chord of an ellipse which passes through the foci is called the **major axis** of the ellipse. The end points of the major axis are the **vertices** of the ellipse. Thus, in Fig. 13, A and A' are the vertices and the chord A'A is the **major axis.** Its length is 2a.

The midpoint of the major axis is called the **centre of the ellipse.** You can see that the centre of the ellipse (11) is (0, 0).

The chord of an ellipse which passes through its centre and is perpendicular to its major axis is called the **minor axis** of the ellipse. In Fig. 13, the minor axis is the line segment B'B. Its length is 2b.

Let us look at an example.

Example 1: Find the eccentricity, foci and centre of the ellipse $2x^2 + 3y^2 = 1$.

Solution: The given equation is $\frac{x^2}{\frac{1}{2}} + \frac{y^2}{\frac{1}{3}} = 1$.

Comparing with (11), we get $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{3}}$. Since $b^2 = a^2(1 - e^2)$, $\frac{1}{3} = \frac{1}{2}(1 - e^2)$, that is, $e = \frac{1}{\sqrt{3}}$.

The foci are given by $(\pm ae, 0) = \left(\pm \frac{1}{\sqrt{6}}, 0\right)$.

And of course, the centre is (0, 0).

Now let us sketch an ellipse whose major axis is along the y-axis. In this case a and b in (11) get interchanged.

Example 2: Sketch the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

Solution: This ellipse intersects the x-axis in $(\pm 3, 0)$, and the y-axis in $(0, \pm 5)$. Thus, its major axis lies along the y-axis, and the minor axis lies along the x-axis. Thus, a and b of (11) have become interchanged. Note that (0, 0) is the centre of this ellipse too.

Also, if e is the eccentricity of this ellipse, then $9 = 25(1-e^2)$. Therefore, $e = \frac{4}{5}$. Thus, the foci lie at (0, 4) and (0, -4). (Remember that in this case the major axis lies along the y-axis.) The directrices of this ellipse are $y = \pm \frac{25}{4}$.

We sketch the ellipse in Fig. 14.

Here are some exercises now.

E11) Find the length of the major and minor axes, the eccentricity, the coordinates of the vertices and the foci of $3x^2 + 4y^2 = 12$.

Hence sketch it.

E12) Find the equation of the ellipse with centre (0, 0), vertices at $(\pm a, 0)$ and eccentricity 0. Sketch this ellipse. Does the figure you obtain have another name?

E13) The astronomer Johann Kepler discovered in 1609 that the earth and other

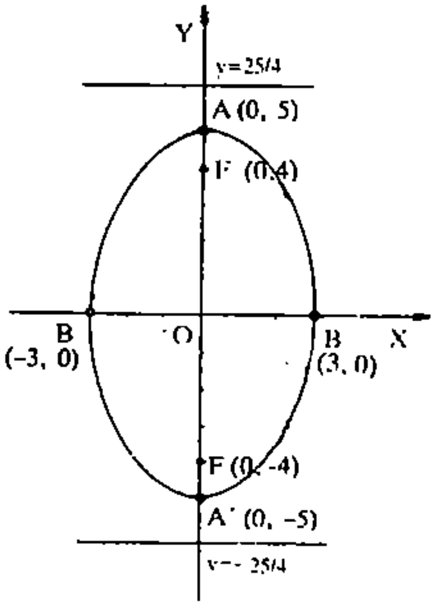


Fig. 14: The ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

planets travel in approximately elliptical orbits with the sun at one focus. If the ratio of the shortest to the longest distance of the earth from the sun is 29 to 30, find the eccentricity of the earth's orbit.

E14) Consider the ellipse $\frac{x^2}{4} + \frac{y^2}{4(1 - e^2)} = 1$, where e is its eccentricity. Sketch the ellipses that you get when $e = \frac{1}{4}$, $e = \frac{1}{2}$ and $e = \frac{3}{4}$. Can you find a relationship between the magnitude of e and the flatness of the ellipse?

What E 12 shows you is that **a circle is a particular case of an ellipse**, and the **equation of a circle** with centre (0, 0) and radius a is

$$x^2 + y^2 = a^2, \qquad \ldots(12)$$

You may be wondering about the directrices of a circle. In the following note we make an observation about them.

Note: As the eccentricity of an ellipse gets smaller and smaller its directrices get farther and farther away from the centre. Ultimately, when $e \to 0$, the directrices become lines at infinity.

At this point let us mention the **parametric representation** of an ellipse. As in the case of a parabola, we can express any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of a parameter t. In this case, you can check that any point (x, y) on the ellipse is given by $x = a \cos t$, $y = b \sin t$, where $0 \leq t < 2\pi$. Note that the vertices will correspond to $t = 0$ and $t = \pi$.

Let us now look at some important properties of an ellipse.

2.4.2 String Property

In this section we derive a property that characterises an ellipse. Let us go back to Equation (11). Its foci are F(ae, 0) and F'(-ae, 0). Now, take any point P(x, y) on the ellipse. The **focal distances** of P are PF and PF'. What is their sum? If you apply the distance formula, you will find that PF + PF' = 2a, which is a constant, and is the length of the major axis. This property is true for any ellipse. Let us state it formally.

Theorem 1 a) The sum of the focal distances of any point P on an ellipse is the length of the major axis of the ellipse.

b) Conversely, the set of all points P in a plane such that the sum of distances of P from two fixed points F and F' in the plane is a constant, is an ellipse.

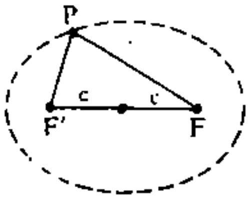


Fig. 15

Proof: We have already proved (a). Let us prove (b). We can rotate and translate our coordinate system so that F and F' lie on the x-axis and (0, 0) is the midpoint of the line segment F'F. Then if F has coordinates (c, 0), F' will be given by (-c, 0). Let P(x, y) be an arbitrary point, such that PF + PF' = 2a, where a is a constant (see Fig. 15). Then, by the distance formula we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

On squaring and simplifying we get

$$(a^2 - c^2)\, x^2 + a^2 y^2 = a^2(a^2 - c^2).$$

Now, since PFF' forms a triangle, FF' < PF + PF'. Therefore, 2c < 2a, that is, c < a. So we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b = \sqrt{a^2 - c^2}.$$

Comparing this with (11), we see that the set of all points (x, y) that satisfy the given condition is an ellipse with foci F and F', and major axis of length 2a.

Mathematicians often use Theorem 1 as the definition of an ellipse. That is,

an **ellipse** is the set of all points in a plane for which the sum of the distances from two fixed points in the plane is constant.

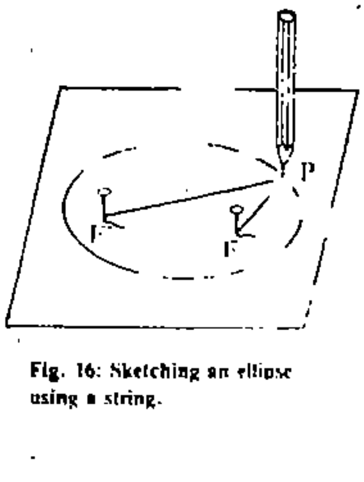


Fig. 16: Sketching an ellipse using a string.

This property of an ellipse is also called the **string property**, because it is the basis for the following construction of an ellipse.

A mechanical method for drawing an ellipse

Take a piece of string of length 2a and fix its ends at the points F and F' (where FF' < 2a) on a sheet of paper (see Fig. 16). Then, with the point of a pencil P, stretch the string into two segments. Now, rotate the pencil point all around on the paper while sliding it along the string. Make sure that the string is taut all the time. By doing this the point P will trace an ellipse with foci F and F' and major axis of length 2a.

Why don't you try this method now?

E15) Use the method we have just given to draw an ellipse with eccentricity $\frac{1}{2}$ and a string of length 4 units. What will the coordinates of its vertices and foci be?

There is another property of an ellipse which makes it very useful in engineering. We shall tell you about it in our discussion on tangents.

2.4.3 Tangents and Normals

In Sec. 2.3.2 you studied about a tangent of a parabola. We will discuss the tangents of an ellipse in the same manner.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $x_1 = x_2 = c$, say, then the equation of PQ is

$x = c$. ...(13)

Similarly, if $y_1 = y_2 = d$, say, then the equation of PQ is

$y = d$. ...(14)

If $x_1 \neq x_2$ and $y_1 \neq y_2$, then the equation of PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{(y - y_1)(y_1 + y_2)}{y_2^2 - y_1^2} = \frac{(x - x_1)(x_1 + x_2)}{x_2^2 - x_1^2}$$

$$\Rightarrow \frac{(y - y_1)(y_1 + y_2)}{b^2} = -\frac{(x - x_1)(x_1 + x_2)}{a^2}, \text{ since } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2}.$$

$$\Rightarrow \frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + 1, \qquad \ldots(15)$$

since (x_1, y_1) lies on the ellipse.

So, (13), (14) and (15) are the various possibilities for the equation of the line joining P and Q.

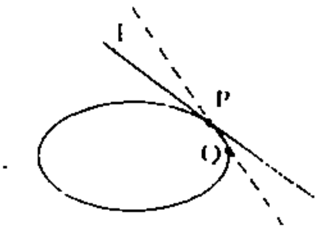


Fig. 17: L is tangent to the ellipse at P.

Now, to get the equation of the tangent at P, we see what happens to the equation of PQ as Q tends to P (see Fig. 17). In this case, can you see from Fig. 13 that we need to consider (15) only? This is because as Q nears P, the line PQ can't be parallel to either axis. Now, as x_2 tends to x_1 and y_2 approaches y_1, (15) becomes, in the limiting case,

$$2\left(\frac{x x_1}{a^2} + \frac{y y_1}{b^2}\right) = 2, \text{ that is,}$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1. \qquad \ldots(16)$$

Thus, (16) is the equation of the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Remark 1 may have already suggested this equation to you. The same rule of thumb works here too, that is, replace x^2 by xx_1 and y^2 by yy_1.

For example, the tangent to the ellipse in Example 1 at $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ is

$$2x\left(\frac{1}{2}\right) + 3y\left(\frac{1}{\sqrt{6}}\right) = 1, \text{ that is, } x + \sqrt{\frac{3}{2}}\, y = 1.$$

Now try this exercise on tangents.

E16) Find the equations of the tangents at the vertices and ends of the minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let us now find the equation of the normal to (11) at any point (x_1, y_1). If $y_1 = 0$, from E15 you know that the slope of the tangent will be $\pi/2$; and hence, at these points the normal is just the x-axis, that is, $y = 0$. Similarly, you can see that the normal at the points at which $x_1 = 0$ is the y-axis.

Now suppose $x_1 \neq 0$, $y_1 \neq 0$. What is the slope of the normal at (x_1, y_1)? By (16) you know that the slope of the tangent is $-\frac{b^2 x_1}{a^2 y_1}$. Thus, the slope of the normal at (x_1, y_1) to the ellipse is $\frac{a^2 y_1}{b^2 x_1}$ (see (13) of Unit 1). Thus, the equation of the normal at $(x_1 + y_1)$ is,

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1), \text{ that is,}$$

$$\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}. \qquad \ldots(17)$$

Why don't you try these exercises now?

A diameter of an ellipse is a chord that passes through the centre

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E17) Find the equation of the tangent and normal at (2, 1) to $x^2 + 4y^2 = 8$.

E18) Show that the tangents at the extremities of a diameter of an ellipse are parallel.

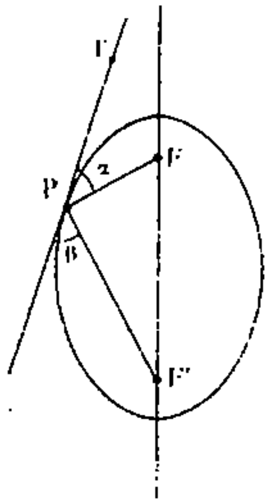


Fig. 18: $\alpha = \beta$

Fig. 19 : Reflected wave property.

Now, in Sec. 2.3.2, we discussed the reflecting property of a parabola. Do you expect it to hold for an ellipse too? The same property is not satisfied, but something like that is.

Reflecting property: The tangent to an ellipse at a point makes equal angles with the focal radii from that point.

That is, if you take the tangent PT at a point P on an ellipse (see Fig. 18), then it makes equal angles with the lines PF and PF'.

We shall not prove this here. We leave the proof to you as an exercise (see Miscellaneous Exercises).

Because of the reflection property, a ray of light (or sound, or any other type of wave) that is emitted from one focus of a polished elliptical surface is reflected back to the other focus (see Fig. 19). One of the applications of this fact is its use for making whispering galleries.

Why don't you try and apply this property now?

E19) An elliptic reflector is to be designed so as to concentrate all the light radiated from a point source on to another point 6 metres away. If the width of the reflector is 10 metres, how high should it be?

Let us now see under what conditions a given line will be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the line be $y = mx + c$.

Substituting for y in the equation of the ellipse we get

$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$, that is,

$x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0.$

$y = mx + c$ will be a tangent to the ellipse if this quadratic equation in x has equal roots. This will happen if its discriminant is zero, that is,

$4m^2c^2a^4 = 4(b^2 + a^2m^2)\, a^2(c^2 - b^2)$

$\Rightarrow c^2 = a^2m^2 + b^2,$ ...(18)

So (18) is the condition that $y = mx + c$ is a tangent to

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Here's an opportunity for you to use this condition.

E20) Check whether $y = x + 5$ touches the ellipse $2x^2 + 3y^2 = 1$.

We shall now stop our discussion on ellipses, and shift our focus to another standard conic.

2.5 HYPERBOLA

Let us now consider the conic we get if $e > 1$ in Equation (1), namely, a hyperbola. Let us define it explicitly.

Definition: A **hyperbola** is the set of points whose distance from a fixed point F is e (>1) times its distance from a fixed line L which doesn't pass through F.

There is a similarity between the derivation of the standard equation of an ellipse and that of a hyperbola. In the following exercise we ask you to obtain this equation for a hyperbola.

E21) a) Show that the equation of a conic with focus at (0, 0), directrix $x + c = 0$ and eccentricity $e > 1$ is

$$\left(x + \frac{c e^2}{e^2 - 1}\right)^2 - \frac{y^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}.$$

b) Shift the origin suitably so as to get the equation

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

where $a = \frac{ec}{e^2 - 1}$, $b = a\sqrt{e^2 - 1}$.

c) What are the coordinates of the focus and the equation of the directrix in the X'Y'-system?

E22) What is the equation of the conic with a focus at (-ae, 0) and directrix $x = -\frac{a}{e}$, where e (>1) is the eccentricity?

As in the case of the other conics, we can always translate and rotate our coordinate axes so as to get the focus of any given hyperbola as (-ae, 0), and its directrix as $x = -\frac{a}{e}$. Thus, we can always reduce, the equation of any hyperbola to the equation that you obtained in E21, namely,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \ldots(19)$$

where $b^2 = a^2(e^2 - 1)$.

So (19) is the **standard form** of the equation of a hyperbola.

Let us now trace this curve.

2.5.1 Description of Standard Form

Let us study (19) for symmetry and other properties.

Firstly, if $-a < x < a$, then there is no real value of y which satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Thus, no part of the curve lies between the lines $x = -a$ and $x = a$.

Secondly, it is symmetric about both the axes, as well as (0, 0). So it is enough to trace it in the first quadrant.

Thirdly, the points ($\pm a$, 0) lie on it, and it does not intersect the y-axis.

Finally, since $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$, as x increases so does y. Thus, the hyperbola extends to infinity in both the x and y directions. We have sketched this conic in Fig. 20. You can see that [illegible] disjoint branches, unlike the other conics

Looking at the curve's symmetry, do you feel that it has another focus and directrix? You can check that (ac, 0) is another focus with corresponding directrix $x = \frac{a}{e}$.

A hyperbola intersects the line joining its foci in two points. These points are called its **vertices**. The line segment joining its vertices is called its **transverse axis**. (Some people call the line joining the vertices the transverse axis) Thus, the points

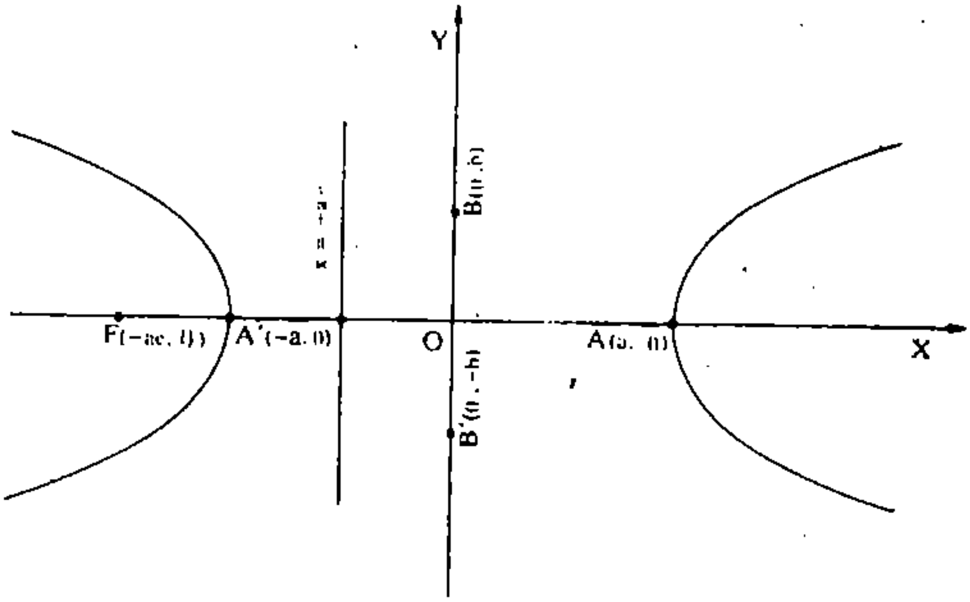


Fig. 20: The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

A' (-a, 0) and A(a, 0) are the vertices of the hyperbola in Fig. 20 and the line segment AA' is its transverse axis. The length of this transverse axis is 2a.

The midpoint of the transverse axis is the **centre** of the hyperbola. In Fig. 20, the line segment BB', where B is (0, b) and B' is (0, -b) is called the **conjugate axis** of the given hyperbola.

Note that it is perpendicular to the transverse axis and its midpoint is the centre of the hyperbola. The reason it is called the conjugate axis is because it becomes the transverse axis of the **conjugate hyperbola** of (19), $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. (We shall not discuss conjugate hyperbolas in this course. If you would like to know more about them, you can look up 'A Textbook of Coordinate Geometry' by Ramesh Kumar).

Let us consider an example of a hyperbola.

Example 3: For the hyperbola $4x^2 - 9y^2 = 36$ find the vertices, eccentricity, foci and the axes.

Solution: We can write the equation in the standard form as $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Comparing this with (19), we find that a = 3, b = 2.
Therefore, the vertices are (±3, 0).

Now, since $b^2 = a^2 (e^2 - 1)$, we find that $e^2 = \frac{13}{9}$. Thus, the eccentricity is $\frac{\sqrt{13}}{3}$. Then the foci are (±ae, 0), that is, (±$\sqrt{13}$, 0). The transverse axis is the line segment joining (3,0) and (-3,0), and the conjugate axis is the line segment joining (0, 2) and (0, -2).

Why don't you try some exercises now?

E23) Find the standard equation of the hyperbola with eccentricity $\sqrt{2}$. (Such a hyperbola is called a **rectangular hyperbola.**)

E24) Find the equation of the hyperbola with centre (0, 0), axes along the coordinate axes, and for which

a) a vertex is at (0, 3) and the transverse axis is twice the length of the conjugate axis;

b) a vertex is at (2, 0) and focus at F ($\sqrt{13}$, 0);

E25) a) Show that the lengths of the focal radii from any point P(x, y) on the hyperbola (19) are $|ex + a|$ and $|ex - a|$.

b) What is the analogue of (a) for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

E26) The more eccentric a hyperbola, the more its branches open out from its transverse axis. True or false? Why?

As in the case of the other conics, we can give a parametric representation of any point on a hyperbola. What do you expect it to be? Does the equation $\sec^2 t - \tan^2 t = 1, \forall\ t \in \mathbf{R}$ help? Using this, we can give the parametric form of any point on (19) by $x = a \sec t$, $y = b \tan t$, for $t \in \mathbf{R}$ such that $0 \leq t < 2\pi$.

Let us now look at some properties of a hyperbola.

2.5.2 String Property

In Theorem 1 you saw that an ellipse is the path traced by a point, the sum of the distances of which from two fixed points is a constant. A similar property is true of a hyperbola. Only, in this case, we look at the difference of the distances.

Theorem 2: a) The difference of the focal distances of any point on a hyperbola is equal to the length of its transverse axis.

b) Conversely, the set of points P such that $|PF_1 - PF_2| = 2a$, where F_1 and F_2 are two fixed points, a is a constant and $F_1F_2 > 2a$, is a hyperbola.

Proof: a) As you know, we can always assume that the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let P(x, y) be a point on it, and let its foci be F_1 and F_2. Further, let D_1 and D_2 be the feet of the perpendiculars from P on the two directrices (see Fig. 21). In the figure you can see both the cases – when P is on one branch of the hyperbola or the other.

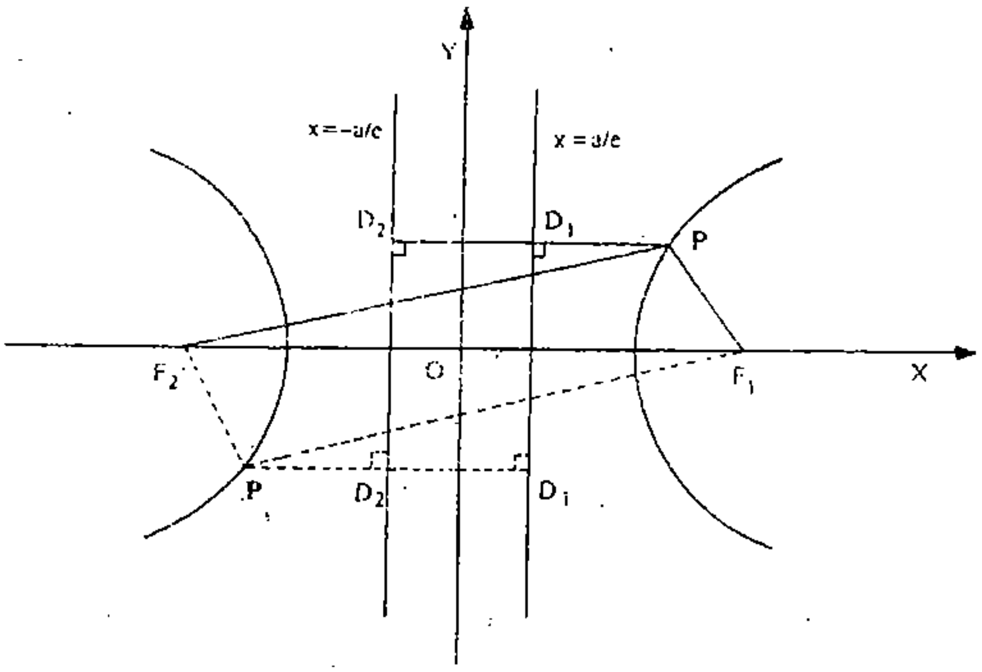


Fig. 21: $|PF_1 - PF_2|$ is constant.

Now, by definition,

$PF_1 = e\ PD_1$ and $PF_2 = e\ PD_2$. Therefore,

$|PF_1 - PF_2| = e\ |PD_1 - PD_2| = e\left(\frac{2a}{e}\right) = 2a$, the length of the transverse axis.

(You can also prove this by using E25(a).)

We ask you to prove (b) in the following exercise.

E27) Prove Theorem 2(b). Where is the condition $F_1F_2 > 2a$ used?

Theorem 2 is called the **string property**, for a reason that you may have guessed by now. We can use it to mechanically construct a hyperbola with a string. Since this construction is more elaborate than that of an ellipse, we shall not give it here.

The string property is also the basis for hyperbolic navigation – a system developed during the World Wars for range finding and navigation.

And now we shall see how to find the tangent to a hyperbola.

2.5.3 Tangents and Normals

You must have noticed the similarity between the characteristics of an ellipse and a hyperbola. The derivation of the equation of a chord joining two points on a hyperbola and of the equation of a tangent are also obtained as in Sec. 2.4.3. We shall not give the details here. Suffice it to say that all these equations can be obtained from the elliptic case by substituting $-b^2$ for b^2.

Thus, the equation of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point (x_1, y_1) lying on it is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \qquad \text{...(20)}$$

Also, the equation of the normal at a point (x_1, y_1) to

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$a^2\left(\frac{x - x_1}{x_1}\right) + b^2\left(\frac{y - y_1}{y_1}\right) = 0. \qquad \text{...(21)}$$

Similarly, the condition for the straight line $y = mx + c$ to be a tangent to

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$.

Now for a short exercise.

E28) a) Find the tangent and normal to $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at each of its vertices.

b) Is $3y = 2x$ a tangent to this hyperbola? If so, find the point of tangency.

We will now introduce you to some special tangents to a hyperbola. Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines $y = \pm \frac{b}{a} x$ (see Fig. 22). These lines satisfy the condition for tangency. They are the pair of tangents to the hyperbola which pass through its centre. Such tangents are called the **asymptotes** of the hyperbola.

Now, let P (x, y) be a point of the branch of the hyperbola in the first quadrant. Then its distance from $y = \frac{b}{a}x$ is

$$\frac{|ay - bx|}{\sqrt{a^2 + b^2}} = \frac{|a^2y^2 - b^2x^2|}{\sqrt{a^2 + b^2}\,(ay + bx)}$$

$$= \frac{a^2b^2}{\sqrt{a^2 + b^2}\,(bx + b\sqrt{x^2 - a^2})},$$

since $b^2x^2 - a^2y^2 = a^2b^2$ and $y = \frac{b}{a}\sqrt{x^2 - a^2}$.

$$= \frac{a^2b}{\sqrt{a^2 + b^2}\,(x + \sqrt{x^2 - a^2})}$$

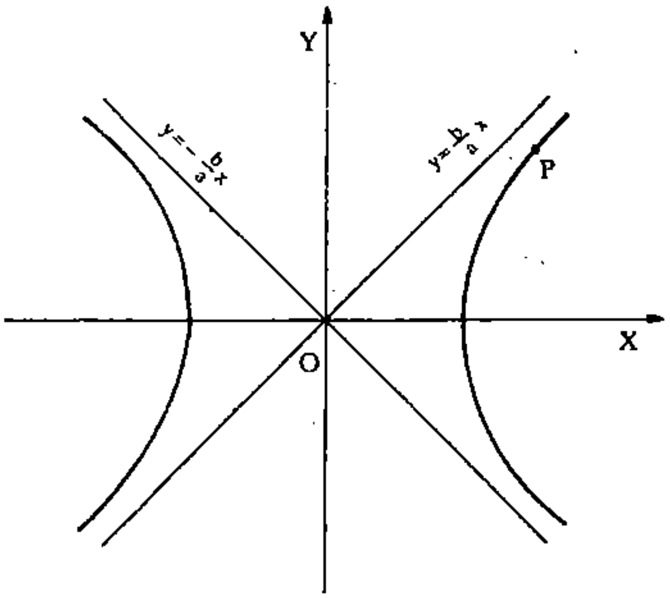


Fig. 22: $y = \pm \frac{b}{a} x$ are the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

As x increases, this distance gets smaller and smaller. Thus, as P tends to infinity along the branch $y = \frac{b}{a}\sqrt{x^2 - a^2}$ of the hyperbola, its distance from $y = \frac{b}{a}x$ tends to zero. But the asymptote never actually intersects the curve. So we say that its point of contact is at an infinite distance.

You can check that the same is true of $y = -\frac{b}{a}x$. In fact, tangents with their points of contact 'at infinity' are called asymptotes.

Asymptotes are discussed in the course MTE-01 in detail.

Try these exercises now.

E29) Find the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$. Are they the same for any value of a?

E30) Under what conditions on a and b will the asymptotes of the hyperbola (19) be perpendicular to each other?

So far we have discussed the Cartesian and parametric equations of the conics. But, in some applications the polar equation (see Sec. 1.5) of a conic is more useful. So let us see what this equation is.

2.6 POLAR EQUATION OF CONICS

Consider a conic with eccentricity e. Take a focus F as the pole. We can always rotate the conic so that the corresponding directrix L lies to the left of the pole, as in Fig. 23. Let the line FA, perpendicular to the directrix, be the polar axis and d the distance between F and L.

Let P(r, θ) be any point on the conic. Then, if D and E are the feet of the perpendiculars from P onto L and FA, we have

$$PF = ePD$$

$$\Rightarrow r = e(d - EF) = e(d - r\cos(\pi - \theta))$$

$$= e(d + r\cos\theta).$$

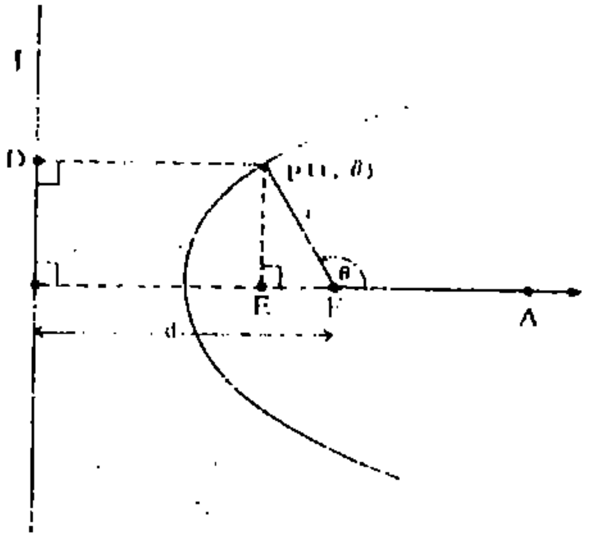


Fig. 23: Obtaining the polar equation of a conic.

$$\Rightarrow r = \frac{ed}{1 - e\cos\theta}, \qquad \ldots(22)$$

which is the polar equation of a conic.

Can you find the polar equations of the standard conics from this?

For instance, the polar equation of the parabola (3) is $r = \frac{d}{1 - \cos\theta}$, where d = 2a.

Now, suppose you try to derive the polar equation of a conic by taking the directrix L corresponding to a focus F, to the right of F. Will you get (22)? You can check that the equation will now be

$$r = \frac{ed}{1 + e\cos\theta}. \qquad \ldots(23)$$

Let us consider an application of the polar form.

Example 4: In Fig. 24 we show the elliptical orbit of the earth around the sun, which is at a focus F. The point A on the ellipse closest to the sun is called the perihelion; and the point A' farthest from the sun is called the aphelion.

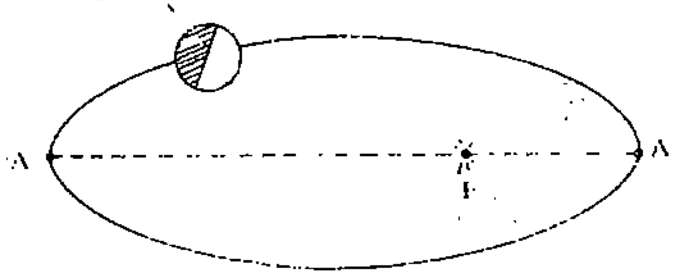


Fig. 24: Aphelion and perihelion on the orbit of the earth around the sun.

Show that the perihelion distance FA and the aphelion distance FA' are given by

$FA = \frac{ed}{1+e}$ and $FA' = \frac{ed}{1-e}$, where d is as given in (22) and (23).

Solution: The polar coordinates of A are (FA, 0) and of A' are (FA', π). Thus, from (23) we find that

$$FA = \frac{ed}{1+e} \text{ and } FA' = \frac{ed}{1-e}.$$

You can do the following exercises to see if you have understood what we have done in this section.

E31) Let 2a be the length of the major axis of the ellipse

$r = \frac{ed}{1 + e\cos\theta}$. Show that $a = \frac{ed}{1 - e^2}$.

E32) A comet is travelling in a parabolic course. A polar coordinate system is introduced in the plane of the parabola so that the sun lies at the focus and the polar axis is along the axis of the parabola, drawn in the direction in which the curve opens. When the comet is 3.0×10^7km from the centre of the sun, a ray from the sun to the comet makes an angle of $\pi/3$ with the polar axis. Find

a) an equation for this parabolic path,

b) the minimum distance of the comet from the sun,

c) the distance between the comet and the sun when $\theta = \frac{\pi}{2}$.

So in this unit you have seen the Cartesian, parametric and polar representations of the various conics for which the foci do not lie on the corresponding directrices. Such conics are called **non-degenerate conics.**

In case a focus of a conic lies on the directrix corresponding to it, the conic we get is called a **degenerate conic.** We will not go into details about them. But let us list the possible types that there are.

A degenerate conic can be of 5 types:

a point, a pair of intersecting lines, a pair of distinct parallel lines, a pair of coincident lines and the empty set.

Now let us do a brief run-through of what we have covered in this unit.

2.7 SUMMARY

In this unit we have discussed the following points:

1) The focus-directrix defining property of conics.

2) A standard form of a parabola is $y^2 = 4ax$. It's focus is at (a, 0), and directrix is $x = -a$. Its eccentricity is 1. The other standard forms are $x^2 = 4ay$, $x^2 = -4ay$ and $y^2 = -4ax$, $a > 0$.

3) The tangent to $y^2 = 4ax$ at a point (x_1, y_1) lying on it is $yy_1 = 2a(x + x_1)$.

4) $y = mx + c$ is a tangent to $y^2 = 4ax$ if $c = \frac{a}{m}$.

5) The normal at (x_1, y_1) to $y^2 = 4ax$ is $\frac{y_1}{2a}x + y = y_1 + \frac{y_1^3}{8a^2}$.

6) The standard form of the equation of an ellipse with eccentricity e (< 1) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$

Its foci are $(\pm ae, 0)$ and directrices are $x = \pm\frac{a}{e}$.

7) The sum of the focal distances of any point on an ellipse equals the length of the major axis of the ellipse.

8) The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

9) $y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ iff $c^2 = a^2m^2 + b^2$.

10) The normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{b^2}{y_1}(y - y_1) = \frac{a^2}{x_1}(x - x_1)$.

11) The standard form of the equation of a hyperbola with eccentricity e (> 1) is

ii) The standard form of the equation of a hyperbola with eccentricity $e(>1)$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$. Its foci are $(\pm ae, 0)$ and directrices are $x = \pm \frac{a}{e}$.

12) The difference of the focal distances of any point on a hyperbola equals the length of its transverse axis.

13) The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

14) $y = mx + c$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$.

15) The normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2}{x_1}(x - x_1) + \frac{b^2}{y_1}(y - y_1) = 0$.

16) The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm \frac{b}{a}x$.

17) The parametric representation of any point on

a) the parabola $y^2 = 4ax$ is $(at^2, 2at)$, where $t \in \mathbf{R}$;

b) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(a \cos\theta, b \sin\theta)$, where $0 \le \theta < 2\pi$;

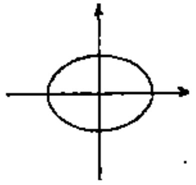
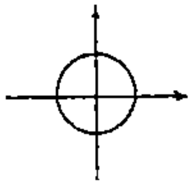
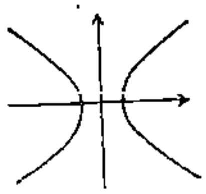
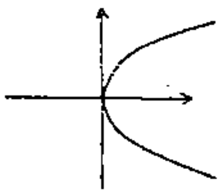
c) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(a \sec\theta, b \tan\theta)$, where $0 \le \theta < 2\pi$.

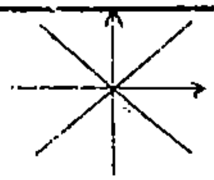
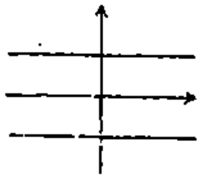
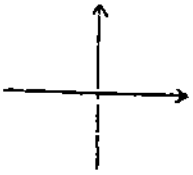
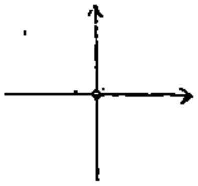
18) The polar equation of a conic with eccentricity e is $r = \frac{ed}{1 - e\cos\theta}$ or $r = \frac{ed}{1 + e\cos\theta}$, depending on whether the directrix being considered is to the left or to the right of the corresponding focus.

Here, d is the distance of the focus from the directix.

19) The list of possible conics is

Table 1: Standard Forms of Conics.

Conic	**Standard Equation**	**Sketch**
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a, b > 0$	
Circle	$x^2+y^2 = a^2,\ a \neq 0$	
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a, b > 0$	
Parabola	$y^2 = 4px,\ p > 0$	

Pair of intersecting lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, a, b ≠ 0	
Pair of parallel lines	$y^2 = a^2$, a > 0	
Pair of coincident lines	$y^2 = 0$	
Point conic	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, a, b ≠ 0	

And now you may like to check whether you have achieved the **objectives** of this unit (see Sec. 2.1). If you'd like to see our solutions to the exercises in this unit, we have given them in the following section.

2.8 SOLUTIONS/ANSWERS

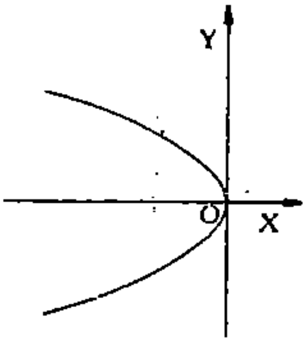


Fig. 25: $y^2 = -4ax$, a > 0.

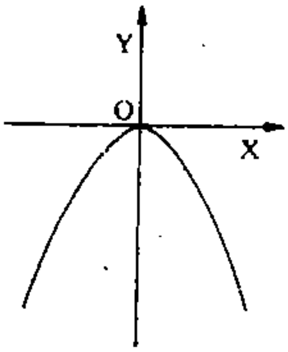


Fig. 26: $x^2 = -4ay$, a > 0.

E1) a) The required equation is $(x - 2)^2 + y^2 = 1^2 \frac{(x - y)^2}{2}$

$\Leftrightarrow x^2 + y^2 + 2xy - 8x + 8 = 0.$

b) The required equation is $x^2 + (y - 1)^2 = \frac{1}{4} \frac{(2x + y - 1)^2}{5}$

$\Leftrightarrow 16x^2 - 4xy + 19y^2 + 4x - 38y + 19 = 0.$

E2) In Fig. 25 we have traced the parabola $y^2 = -4ax$, a>0. Its vertex is (0, 0) and focus is (-a, 0). In Fig. 26 we have drawn $x^2 = -4ay$, a > 0. Its vertex is (0, 0) and focus is (0, -a)

E3) The parabola is similar to the one in Fig. 26.

E4) The parabola is $x^2 = -2y$. Thus, its focus is (0, - 1/2), and its latus rectum is $y = -\frac{1}{2}$.

E5) a) The equation is $xx_1 + 2\left(\frac{y + y_1}{2}\right) = 0$, where $x_1 = y_1 = 0$, that is, $y = 0$.

b) The ends of the latus rectum are (-1, 2) and (-1, -2). The tangents at these points are $x + y - 1 = 0$ and $x - y - 1 = 0$.

E6) The axis of the parabola intersects it at the vertex only, but it is not a tangent at the vertex.

E7) The first point to note is that no tangent line can be parallel to the axis of the parabola. For any other m, the line will be a tangent at (x_1, y_1) if $x_1^2 = 4ay_1$, $y_1 = mx_1 + c$, and $x_1^2 = 4a(mx_1 + c)$ has coincident roots. Thus, $y = mx + c$ will be a tangent if $m \neq \tan \pi/2$ and $c = -am^2$.

E8) We have to find the focus of the parabola. We know that (0.2, 0.5) lies on it. Therefore,
$0.25 = 4a(0.2) = 0.8a \Rightarrow a = 0.3125.$

E9) The tangent is $x = 2(y+1)$. Its slope is $\frac{1}{2}$.

Thus, the normal is $y-1 = -2(x-1)$.

E10) The point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The slope of the normal is $-\frac{1}{m}$.

Thus, its equation is

$$y - \frac{2a}{m} = -\frac{1}{m}\left(x - \frac{a}{m^2}\right)$$

$$\Leftrightarrow x + my = a\left(2 + \frac{1}{m^2}\right)$$

E11) The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

The major axis is of length 4 and lies along the x-axis.
The minor axis is of length $2\sqrt{3}$.

$\therefore (\sqrt{3})^2 = 2^2(1-e^2)$, where e is the eccentricity.

$$\Rightarrow e = \frac{1}{2}.$$

The vertices are $(\pm 2, 0)$ and the foci are $(\pm 1, 0)$. We trace the curve in Fig. 27.

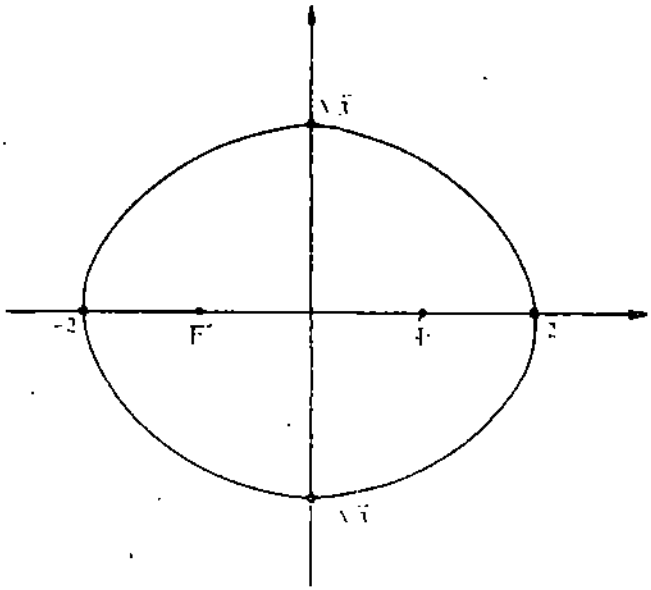


Fig. 27

E12) The equation is $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, that is, $x^2 + y^2 = a^2$. The foci coincide with the centre (0, 0). In this case the ellipse becomes a circle, given in Fig. 28.

Fig. 28: The circle $x^2 + y^2 = a^2$.

E13) The shortest and longest distances will be the distances of the vertices from the focus at which the sun lies.

So, suppose the orbit is $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ and the sun is at (ae, 0).

The vertices lie at (a, 0) and (-a, 0). Then

$$\frac{a - ae}{a + ae} = \frac{29}{30} \Rightarrow e = \frac{1}{59}.$$

E14) As e grows larger the minor axis becomes smaller, and the ellipse grows flatter. Thus, the eccentricity of an ellipse is a measure of its flatness

E15) Your ellipse should be similar to the one in Fig. 27. Its vertices will be $(\pm 2, 0)$. Its foci will be $(\pm 1, 0)$.

E16) The tangent at (a, 0) is $\frac{xa}{a^2} + \frac{y.0}{b^2} = 1 \Rightarrow x = a$. Similarly, the tangents at (-a, 0), (0, b) and (0, - b) are $x = -a$, $y = b$ and $y = -b$, respectively.

E17) The tangent is $2x + 4y = 8$, that is, $x + 2y = 4$. Therefore, the slope of the normal is 2. Thus, its equation is $y - 1 = 2(x - 2) \Rightarrow y = 2x - 3$.

E18) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The equation of any diameter will be $y = mx$, for some m, since it passes through (0, 0). Further, if one end of the diameter is (x_1, y_1), then $y_1 = mx_1$. Thus, $(-x_1, -y_1)$ also lies on the ellipse and the line $y = mx$. Thus, it is the other end of the diameter. So, we need to find the tangents at (x_1, y_1) and $(-x_1, -y_1)$. They are

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ and $-\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right) = 1$, respectively. Since both their slopes are $-\frac{b^4x_1}{a^2y_1}$, they are parallel.

Fig. 29

E19) We have shown a cross-section of the reflector in Fig. 29. The major axis of this ellipse is 10 metres and its foci lie at $(\pm 3, 0)$. Thus, its eccentricity, $e = \frac{3}{5}$.

So, if h is its height, then $h = 5\sqrt{1 - e^2} = 4$ metres.

E20) In this case $a^2 = \frac{1}{2}$, $b^2 = \frac{1}{3}$, $c = 5$ and $m = 1$.

$\therefore c^2 \neq a^2 m^2 + b^2$. So the line is not a tangent to the given ellipse.

E21) a) Using (1), we see that

$$x^2 + y^2 = e^2 (x + c)^2$$

$$\Rightarrow \left(x + \frac{ce^2}{e^2 - 1}\right)^2 - \frac{y^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}$$, as in Sec. 2.4.

b) Shifting the origin to $\left(-\frac{ce^2}{e^2 - 1}, 0\right)$, the equation in (a) becomes

$$x'^2 - \frac{y'^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}$$

$$\Rightarrow \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1, \text{ where } a = \frac{ec}{e^2 - 1} \text{ and } b = a\sqrt{e^2 - 1}.$$

c) In the X'Y'-system the focus is (-ae, 0) and directrix is $x + \frac{a}{e} = 0$.

E22) The required equation is

$$(x + ae)^2 + y^2 = e^2 \left(x + \frac{a}{e}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1$$

E23) The required equation is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2 (2 - 1)} = 1 \Leftrightarrow x^2 - y^2 = a^2.$$

E24) a) The transverse axis lies along the y-axis in this case. Thus, we interchange x and y in (19). Also, the length of the transverse axis is 6. So, the required equation is

$$\frac{y^2}{9} - \frac{x^2}{\left(\frac{3}{2}\right)^2} = 1 \Leftrightarrow y^2 - 4x^2 = 9.$$

b) Here, the transverse axis lies along the x-axis and $a = 2$ and $ae = \sqrt{13}$.

$\therefore e = \frac{1}{2}\sqrt{13}$.

Thus, the required equation is

$$\frac{x^2}{4} - \frac{y^2}{4\left(\frac{13}{4} - 1\right)} = 1 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

E25) a) The foci lie at $F(-ae, 0)$ and F' $(ae, 0)$. Thus

$PF = \sqrt{(x + ae)^2 + y^2}$

But, since P lies on the hyperbola,

$y^2 = (x^2 - a^2)(e^2 - 1)$.

$\therefore PF = \sqrt{(ex + a)^2} = |ex + a|$

Similarly, $PF' = |ex - a|$.

b) In this case $PF = ex + a$ and $PF' = a - ex$.

Remember, from Sec. 2.4.2, that $PF + PF' = 2a$.

E26) Suppose you fix the transverse axis and increase the eccentricity of a hyperbola. You will see that the lengths of its latera recta (plural of 'latus rectum') increase. Thus, the given statement is true.

A latus rectum of a hyperbola is the chord through a focus and perpendicular to its transverse axis.

E27) Let $F_1F_2 = 2c$ and let $(0, 0)$ bisect F_1F_2. Then the coordinates of F_1 will be $(-c, 0)$ and of F_2 will be $(c, 0)$. If P is given by (x, y), then $|PF_1 - PF_2| = 2a$

$\Rightarrow |\sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2}| = 2a$

$\Rightarrow (x^2 + c^2 + y^2 - 2a^2)^2 = \sqrt{(x^2 - c^2)^2 + 2y^2(c^2 + x^2) + y^2}$, on squaring and simplifying.

$\Rightarrow (c^2 - a^2)x^2 - a^2y^2 = a\ (c^2 - a^2)$, again squaring and simplifying.

Now, since $c > a$, we can rewrite this equation as

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1, \text{ which is a hyperbola.}$$

E28) a) The vertices are $(2, 0)$ and $(-2, 0)$.

The tangents at these points are $x = 2$ and $x = -2$, respectively. The normals at both these points is the x-axis.

b) Here $a^2 = 4$, $b^2 = 9$, $m = \frac{2}{3}$, $c = 0$.

$\therefore c^2 \neq a^2m^2 - b^2$. $\therefore 3y = 2x$ is not a tangent to the given hyperbola.

E29) $y = \pm x$, which are independent of a. Thus, these are the asymptotes of any rectangular hyperbola.

E30) $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ will be mutually perpendicular iff

$\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right) = -1$, that is, iff $a = b$, that is, iff the hyperbola is rectangular.

E31) As in Example 5, you can show that $FA = \frac{ed}{1 - e}$ and $FA' = \frac{ed}{1 + e}$,

where A and A′ are the vertices of the ellipse and F is a focus.

Then $2a = AA' = FA + FA' = \frac{2ed}{1 - e^2}$.

$\therefore a = \frac{ed}{1 - e^2}$.

E32) a) Since the curve is a parabola, its equation is $r = \frac{d}{1 - \cos\theta}$. We also know that $(3.0 \times 10^7, \pi/3)$ lies on it.

$\therefore d = 3.0 \times 10^7 \left(1 - \cos\frac{\pi}{3}\right) = 1.5 \times 10^7$ km.

Thus, the required equation is

$$r = \frac{1.5 \times 10^7}{1 - \cos\theta}$$

b) The minimum distance will be when the comet is at the vertex of the parabola, that is, when $\theta = \pi$.
Thus, the minimum distance

$$= \frac{1.5 \times 10^7}{2} \text{ km.}$$

c) The required distance is 1.5×10^7 km.

UNIT 3 GENERAL THEORY OF CONICS

Structure

3.1 INTRODUCTION

So far you have studied the standard equations of a parabola, an ellipse and a hyperbola. We defined these curves and other conics by the focus-directrix property of a conic. This defining property was discovered by Pappus (approx. 320 AD) long after the definition of conic sections by the ancient Greeks. In his book "Conics", the ancient Greek mathematician Apollonius defined these curves to be the intersection of a plane and a cone. You will study cones later, in Unit 6, but let us show you how conics are planar sections of a cone, with the help of diagrams (see Fig. 1).

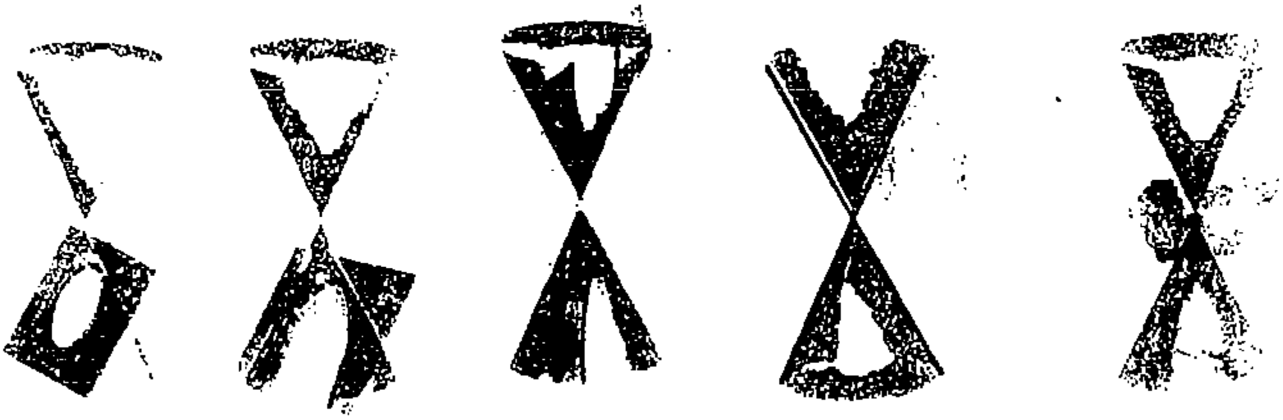


Fig. 1: A planar section of a cone can be (a) an ellipse, (b) a parabola, (c) a hyperbola (d) a pair of lines, (e) a point.

In this unit we will prove a result that may surprise you. According to this result, the general second degree equation $ax^2 + hxy + by^2 + gx + fy + c = 0$ always represents a conic section. You will see how to identify it with the various conics, depending on the conditions satisfied by the coefficients.

In Unit 2 you saw one way of classifying conics. There is another way of doing so, which you will study in Sec. 3.3. We shall discuss the geometric properties of the different types of conics, and see how to trace them. After that, we shall discuss the tangents of a conic. And finally, we shall see what curves can be obtained when two conics interesect.

With this unit we end our discussion on conics. But in the next two blocks you will be coming across them again. So, the rest of the course will be easier for you to grasp if you ensure that you have achieved the unit objectives given below.

Objectives

After studying this unit you should be able to

- identify the conic represented by a quadratic expression;
- find the centre (if it exists) and axes of a conic;
- trace any given conic;
- find the tangent and normal to a given conic at a given point;
- obtain the equations of conics which pass through the points of intersection of two given conics.

3.2 GENERAL SECOND DEGREE EQUATION

In Unit 2 you must have noticed that the standard equation of each conic is a second degree equation of the form

$$ax^2 + hxy + by^2 + gx + fy + c = 0,$$

for some a,b,c,f,g,h ϵ **R** and where at least one of a,h,b is non-zero.

We write the coefficients of xy, x and y as 2h, 2g, and 2f to have simpler expressions later on, as you will see.

In this section we will show you that the converse is also true. That is, we will prove that the general second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(1)$$

where at least one of a,h,b is non-zero, can be transformed into a standard equation of a conic. We achieve this by translating and rotating the coordinate axes. Let us see how.

We first get rid of the term containing xy by rotating the XY-system through a "suitable" angle θ about 0. You will see how we choose θ a little further on. Now, by (16) and (17) of Unit 1, we see that (1) becomes

$$\begin{aligned}
& a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\
& + b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta) + c = 0 \\
= \; & (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)x'^2 - 2\{(a - b)\sin\theta\cos\theta - \\
& h(\cos^2\theta - \sin^2\theta)\}x'y' + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 \\
& + (2g\cos\theta + 2f\sin\theta)x' + (2f\cos\theta - 2g\sin\theta)y' + c = 0.
\end{aligned}$$

The x'y' term will disappear if $(a - b)\sin\theta\cos\theta = h(\cos^2\theta - \sin^2\theta)$, that is,

$$\frac{1}{2}(a - b)\sin 2\theta = h\cos 2\theta.$$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

So, to get rid of the x'y' term, if a = b we can choose $\theta = \frac{\pi}{4}$; otherwise, we can choose $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2}{a - b}\right)$.

$\left(\text{We can always choose such a } \theta \text{ lying between } -\frac{\pi}{4} \text{ and } \frac{\pi}{4}.\right)$ For this choice of θ the x'y' term becomes zero.

So, if we rotate the axes through an angle $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a - b}\right)$, then (1) transforms into the second degree equation

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + C = 0, \qquad \ldots(2)$$

where $A = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$ and

$B = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$.

Thus, $A + B = a + b$.

Also, with a bit of computation, you can check that $ab - h^2 = AB$. Now various situations can arise.

Case 1 ($ab - h^2 = 0$): In this case we see that either $A = 0$ or $B = 0$. So, let us assume that $A = 0$. Then we claim that B must be non-zero. Do you agree? What would happen if $A = 0$ and $B = 0$? In this case we would get $a = 0$, $b = 0$ and $h = 0$, which contradicts our assumption that (1) is a quadratic equation.

So, let $A = 0$ and $B \neq 0$. Then (2) can be written as

$$B\left(y' + \frac{F}{B}\right)^2 = -2Gx' - C + \frac{F^2}{B} \qquad \ldots(3)$$

Now, if $G = 0$, then the above equation is

$$\left(y' + \frac{F}{B}\right)^2 = \frac{F^2 - BC}{B^2}, \text{ that is,}$$

$$y + \frac{F}{B} = \pm\sqrt{\frac{F^2 - BC}{B^2}}.$$

This represents a pair of parallel lines if $F^2 \geq BC$, and the empty set if $F^2 < BC$. On the other hand, if $G \neq 0$, then we write (3) as

$$\left(y' + \frac{F}{B}\right)^2 = \frac{-2G}{B}\left(x' + \frac{C}{2G} - \frac{F^2}{2BG}\right)$$

Now if we shift the origin to $\left(\frac{F^2}{2BG} - \frac{C}{2G}, \frac{-F}{B}\right)$, then the equation becomes

$$Y^2 = -\frac{2G}{B}X,$$

where X, Y are the current coordinates. From Sec. 2.3 you know that this represents a parabola with focus $\left(-\frac{G}{2B}, 0\right)$ and directrix $X = \frac{G}{2B}$.

Now let us look at the other case.

Case 2 ($ab - h^2 \neq 0$): Now both A and B are non-zero. We can write (2) as

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C, \text{ which is a constant K, say.}$$

Let us shift the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$. Then this equation becomes

$$AX^2 + BY^2 = K, \qquad \ldots(4)$$

where X and Y are the current coordinates.

Now, what happens if $K = 0$? Well, if both A and B have the same sign, that is, if $AB = ab - h^2 > 0$, then (4) represents the point (0, 0).

And, if $ab - h^2 < 0$, then (4) represents the pair of lines,

$$X = \pm\sqrt{-\frac{B}{A}}\,Y.$$

And, what happens if $K \neq 0$? Then we can write (4) as

$$\frac{X^2}{K/A} + \frac{Y^2}{K/B} = 1. \qquad \ldots(5)$$

Does this equation look familiar? From Sec. 2.4 you can see that this represents an ellipse if both $\frac{K}{A}$ and $\frac{K}{B}$ are positive, that is, if $K > 0$ and $AB = ab - h^2 > 0$.

But, what if $K/A < 0$ and $K/B < 0$? In this case $K < 0$ and $ab - h^2 > 0$. And then (5) represents the empty set.

And, if $\frac{K}{A}$ and $\frac{K}{B}$ are of opposite signs, that is, if $AB = ab - h^2 < 0$, then what will (5) represent? A hyperbola.

So we have covered all the possibilities for $ab - h^2$, and hence for (1). Thus, we have proved the following result.

Theorem 1: The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a conic.

While proving this theorem you must have noticed the importance we gave the expression $ab - h^2$. Let us tabulate the various types of non-degenerate and degenerate conics that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents, according to the way $ab - h^2$ behaves. (Recall from Unit 2 that a degenerate conic is a conic whose focus lies on the corresponding directix.)

Table 1: Classification of Conics.

Condition	Types of Conics	
	Non-degenerate	Degenerate
$ab - h^2 = 0$	parabola	pair of parallel lines, or empty set
$ab - h^2 > 0$	ellipse	point, or empty set
$ab - h^2 < 0$	hyperbola	pair of intersecting lines

Table 1 tells us about all the possible conics that exist. This is what the following exercise is about.

E1) a) Write down all the possible types of conics there are. Which of them are degenerate?

b) If (1) represents a circle, will $ab - h^2 = 0$?

Now let us use the procedure in the proof above in some examples.

Example 1: Find the conic represented by

$$9x^2 - 24xy + 16y^2 - 124x + 132y + 324 = 0.$$

Solution: The given equation is of the form (1), where $a = 9$, $b = 16$, $h = -12$. Now let us rotate the axes through an angle θ, where

$$\tan 2\theta = \frac{2h}{a - b} = \frac{24}{7}, \text{ that is, } \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{24}{7}, \text{ that is,}$$

$$12\tan^2\theta + 7\tan\theta - 12 = 0.$$

So we can take $\tan\theta = \frac{3}{4}$, and then $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

Then, in the new coordinate system the given equation becomes

$$25y'^2 - \frac{124}{5}(4x' - 3y') + \frac{132}{5}(3x' + 4y') + 324 = 0, \text{ that is,}$$

$$\left(y' + \frac{18}{5}\right)^2 = \frac{4}{5}x.$$

Now let us shift the origin to $\left(0, -\frac{18}{5}\right)$ Then the equation becomes

$$Y^2 = \frac{4}{5}X,$$

where X and Y are the current coordinates.

Can you recognise the conic represented by this equation? From Unit 2 you know that this is a parabola. Since the transformations we have applied do not alter the curve, the original equation also represents a parabola.

Example 2: Identify the conic $x^2 - 2xy$ [illegible] 2.

Solution: Over here, since a = b = 1, we choose $\theta = 45°$. So, let us rotate the axes through 45°. The new coordinates x′ and y′ are given by

$$x = \frac{1}{\sqrt{2}}(x' - y') \text{ and } y = \frac{1}{\sqrt{2}}(x' + y').$$

Then, in the new coordinate system,

$x^2 - 2xy + y^2 = 2$ transforms to

$y'^2 = 1$,

which represents the pair of straight lines $y' = 1$ and $y' = -1$.

You can do the following exercise on the same lines.

E2) Identify the conic

a) $x^2 - 2xy + y^2 + \sqrt{2}x = 2$.

b) $9x^2 - 6xy + y^2 - 40x - 20y + 75 = 0$.

So far you have seen that any second degree equation represents one of the following conics:

a parabola, an ellipse, a hyperbola, a pair of straight lines, a point, the empty set.

But, from Table 1 you can see that even if we know the value of $ab - h^2$, we can't immediately say what the conic is. So, each time we have to go through the whole procedure of Theorem 1 to identify the conic represented by a given equation. Is there a short cut? Yes, there is. We have a simple condition for (1) to represent a pair of lines. It can be obtained from the proof of Theorem 1 after some calculations, or independently. We shall only state it, and then see how to use it to cut short our method for identifying a given conic.

Recall the definition of a determinant from Unit 5 of MTE-04.

Theorem 2: The quadratic equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents a pair of straight lines if and only if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, that is, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Further, if the condition is satisfied, then the angle between the lines is

$$\tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right).$$

The 3×3 determinant given above is called the **discriminant** of the given conic. You can see that the discriminant looks neater if we take 2h, 2g and 2f as coefficients, instead of h, g and f.

Let us consider some examples of the use of Theorem 2.

Example 3: Show that $x^2 - 5xy + 6y^2 = 0$ represents a pair of straight lines. Find the angle between these lines.

Solution: With reference to Theorem 2, in this case $a = 1$, $h = -\frac{5}{2}$, $b = 6$, $g = 0 = f = c$. Thus, the related discriminant is

$$\begin{vmatrix} 1 & -\frac{5}{2} & 0 \\ -\frac{5}{2} & 6 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \text{ which is 0, as}$$

you know from our course 'Elementary Algebra'.
Thus, the given equation represents a pair of lines.

The angle between them is $\tan^{-1}\left(\frac{2}{7}\sqrt{\frac{25}{4} - 6}\right) = \tan^{-1}\frac{1}{7}$.

Example 4: Find the conic represented by $2x^2 + 5xy + y^2 = 1$.

Solution : In this case $ab - h^2 = -23 < 0$. So, from Table 1 we know that the equation represents a hyperbola or a pair of lines. Further, in this case the discriminant becomes.

$$\begin{vmatrix} 2 & \frac{5}{2} & 0 \\ \frac{5}{2} & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 1 \end{vmatrix} = \frac{17}{4} \neq 0.$$

So, by Theorem 2 we know that the given equation doesn't represent a pair of lines. Hence, it represents a hyperbola.

Why don't you do these exercises now?

E3) Check whether $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of lines.

E4) Show that the real quadratic equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines.

E5) Under what conditions on a, b and h, will the equation in Theorem 2 represent

a) a pair of parallel lines?

b) a pair of perpendicular lines?

So far we have studied all the conics in a unified manner. Now we will categorise them according to the property of centrality.

3.3 CENTRAL AND NON-CENTRAL CONICS

$+ c = 0$

In our discussion on the ellipse in Unit 2, we said that the midpoint of the major axis was the centre of the ellipse. The reason that this point is called the centre is because of a property that we ask you to prove in the following exercise.

---

that is,

E6) Consider the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let $P(x_1, y_1)$ be a point on this ellipse and O be (0, 0). Show that the line PO also meets the ellipse in $P'(-x_1, -y_1)$.

What you have just proved is that O(0, 0) bisects every chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that passes through it. Similarly, any chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ through O(0, 0) is bisected by O. Hence, according to the following definition, O is the centre of the ellipse and hyperbola given above.

Definition: The centre of a conic C is a point which bisects any chord of C that passes through it.

Not all conics have centres, as you will see. A conic that has a centre is called a

central conic. For example, an ellipse and a hyperbola are central conics.

Now, can a central conic have more than one centre? Suppose it has two centres C_1 and C_2. Then the chord of the conic intercepted by the line C_1C_2 must be bisected by both C_1 and C_2, which is not possible. Thus,

a central conic has a unique centre.

Let us see how we can locate this point.

Consider the conic (1). Suppose it is central with centre at the origin. Then we have the following result, which we will give without proof.

Theorem 3: A central conic with centre at (0, 0) is of the form $ax^2 + 2hxy + by^2 = 1$,

for some a, h, b in **R**.

This result is used to prove the following theorem about any central conic. We shall not prove the theorem in this course but we will apply it very often.

Theorem 4: Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a central conic. Then its centre is the intersection of the lines

$ax + hy + g = 0$ and $hx + by + f = 0$.

What this **theorem** tells us is that if $ax + hy + g = 0$ and $hx + by + f = 0$ intersect, then the conic is central; and the point of intersection of these straight lines is the centre of the conic.

But what if the lines don't intersect? Then the conic under consideration can't be central; that is, it is **non-central.** Thus, the conic is non-central if the slopes of these lines are equal, that is, if $ab = h^2$.

So, we have the following result:

The conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is i) central if $ab \neq h^2$, and ii) non-central if $ab = h^2$.

Does this result and Table 1 tell you which conics are non-central? You can immediately tell that a parabola doesn't have a centre.

Let us see how we can apply the above results on centres of conics.

Example 5: Is the conic $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ central? If it is, find its centre.

Solution: In this case $a = 17$, $b = 8$, $h = -6$, $g = 23$, $f = -14$.
So, $ab \neq h^2$. Hence, the conic is central. Its centre is the intersection of the lines $17x - 6y + 23 = 0$ and $3x - 4y + 7 = 0$, which is $(-1, 1)$.

Why don't you try some exercises now?

E7) Is the conic in E3 central? If yes, find its centre.

E8) Identify the conic $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$. If it is central, find its centre.

E9) Which degenerate conics are central, and which are not?

One point that has been made in this sub-section is that a parabola is a non-central conic, while an ellipse and a hyperbola are central conics. Now let us see if this fact helps us to trace a conic corresponding to a given quadratic equation.

3.4 TRACING A CONIC

Suppose you are given a quadratic equation. Can you get enough geometric information from it to be able to draw its geometric representation? You are now in a position to check whether it is a pair of lines or not. You can also tell whether it is a central conic or not. But there is still one piece of information that you would need before you could draw the required conic. You need to know the equation of its axis, or axes, as the case may be. So let us see how to find the axes. We shall consider the central and non-central cases separately.

3.4.1 Central Conics

Suppose we are given the equation of a central conic. By translating the axes, if necessary, we can assume that its centre lies at (0, 0). Then, by Theorem 3, its equation is

$$ax^2 + 2hxy + by^2 = 1 \qquad \ldots(6)$$

where $a, h, b \in \mathbf{R}$.

In Theorem 1 you saw that if we rotate the coordinate axes through an angle $\theta = \frac{1}{2} \tan^{-1} \frac{2h}{a - b}$, then the axes of the conic lie along the coordinate axes. Therefore, the axes of the conic are inclined at the angle θ to the coordinate axes. (Here if $a = b$, we take $\theta = 45°$.) Now,

$$\tan 2\theta = \frac{2h}{a - b}$$

$$\Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2h}{a - b}$$

$$\Rightarrow \tan^2 \theta + \left(\frac{a - b}{h}\right) \tan \theta - 1 = 0.$$

This is a quadratic equation in $\tan \theta$, and hence is satisfied by two values of θ, say θ_1 and θ_2. Then the slopes of the axes of the conic are $\tan \theta_1$ and $\tan \theta_2$. Note that the axes are mutually perpendicular, since $(\tan \theta_1)(\tan \theta_2) = -1$.

Now, to find the lengths of the axes of the conic, we write (6) in polar form (see Sec. 1.5). For this we substitute $x = r \cos \theta$, $y = r \sin \theta$ in (6). Then we get

$$r^2 (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) = 1$$

$$\Rightarrow r^2 = \frac{\cos^2 \theta + \sin^2 \theta}{a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta}, \text{ writing } 1 = \cos^2 \theta + \sin^2 \theta$$

$$= \frac{1 + \tan^2 \theta}{a + 2h \tan \theta + b \tan^2 \theta} \qquad \ldots(7)$$

If we substitute $\tan \theta_1$ and $\tan \theta_2$ in (7), we will get the corresponding values of r, which will give the lengths of the corresponding semi-axes.

A semi-axis is half the axis

Let us use what we have just done to trace the conic in Example 5. Since $ab - h^2 > 0$, from Theorem 1 we know that the conic is an ellipse. You have already seen that its centre lies at (-1, 1). Now, we need to shift the axes to the centre (-1, 1), to get the equation in the form (6). The equation becomes

$$\frac{17}{20} x'^2 - \frac{3}{5} x'y' + \frac{2}{5} y'^2 = 1.$$

Now we can obtain the directions of the axes from

$$\tan^2 \theta - \frac{3}{2} \tan \theta - 1 = 0.$$

This gives us $\tan \theta = 2, -\frac{1}{2}$

Therefore, we can take $\theta_1 = \tan^{-1} 2 \simeq 63.43°$ (approximately), and

$$\theta_2 = \frac{\pi}{2} + \tan^{-1} 2.$$

The lengths of the semi-axes, r_1 and r_2, are given by substituting these values in (7). So

$$r_1^2 = \frac{1 + 4}{\frac{17}{20} - \frac{6}{5} + \frac{8}{5}} = 4 \Rightarrow r_1 = 2, \text{ and}$$

$$r_2^2 = \frac{1 + \frac{1}{4}}{\frac{17}{20} + \frac{3}{10} + \frac{1}{10}} = 1 \Rightarrow r_2 = 1.$$

Thus, the length of the major axis is 4, and that of the minor axis is 2.

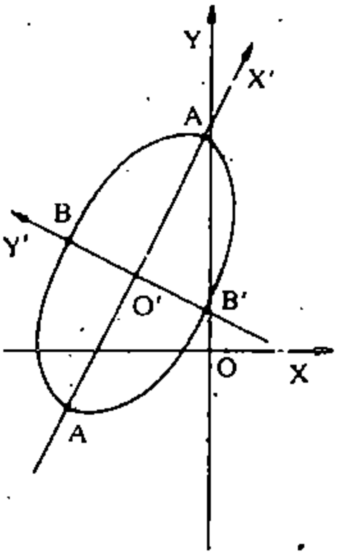


Fig. 2: The ellipse $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$.

So now we can trace the conic. We first draw a line O'X' through O'(-1, 1) at an angle of $\tan^{-1} 2$ to the x-axis (see Fig. 2). Then we draw O'Y' perpendicular to O'X'. Now we mark of A' and A on O'X' such that A'O' = 2 and O'A = 2. Similarly, we mark off B and B' on O'Y' such that O'B = 1 and O'B' = 1.

The required ellipse has AA' and BB' as its axes. For further help in tracing the curve, we can check where it cuts the x and y axes. It cuts the x-axis in (-.4, 0), (-2.2, 0), and the y-axis in (0, 2.7) and (0, .8). So the curve is what we have drawn in Fig. 2.

Now why don't you see if you've understood what has been done in this section?

E10) Trace the conic in E8.

E11) Under what conditions on the coefficients, will $x^2 + 2hxy + y^2 + 2fy = 0$ be central ? And then, find its centre and axes.

So far you have seen how to trace a central conic. But what about a non-central conic? Let us look at this case now.

3.4.2 Parabola

In this sub-section we shall look at a method for finding the axis of a parabola, and hence tracing it. We will use the fact that if (1) is a parabola it can be written in the form

$$\left(\frac{Ax + By + C}{\sqrt{A^2 + B^2}}\right)^2 = k\left(\frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}\right) \qquad \ldots(8)$$

where $Ax + By + C = 0$ is the axis of the parabola and $A'x + B'y + C' = 0$ is the tangent at the vertex, and hence they are perpendicular to each other.

The vertex (x_1, y_1) of this parabola is the intersection of $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, k is the length of its latus rectum, and

$F\left(x_1 + \frac{k}{4}\cos\theta, y_1 + \frac{k}{4}\sin\theta\right)$ is its focus, where $\tan\theta$ is the slope of the axis.

Let us see the method with the help of an illustration.

Example 6: Show that the conic $x^2 + 2xy + y^2 - 2x - 1 = 0$ is a parabola. Find its axis and trace it.

Solution: Here a = 1, b = 1, h = 1. $\therefore ab - h^2 = 0$.

Further, the discriminant of the conic is $\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -1 \neq 0.$

Hence, by Theorem 2, the equation does not represent a pair of straight lines. Thus, by Theorem 1, we know that the given conic is a parabola.

We can write the given equation as $(x + y)^2 = 2x + 1$.

Now we will introduce a constant c so that we can write the equation in the form (8). So, let us rewrite the equation as

$(x + y + c)^2 = 2x + 1 + 2cx + 2cy + c^2$, that is,

$$(x + y + c)^2 = 2(1 + c)x + 2cy + c^2 + 1. \qquad \text{...(9)}$$

We will choose c in such a way that the lines $x + y + c = 0$ and $2(1 + c)x + 2cy + c^2 + 1 = 0$ are perpendicular. From Equation (13) of Unit 1, you know that the condition is

$$(-1)\left[\frac{-2(1 + c)}{2c}\right] = -1 \Rightarrow c = -\frac{1}{2}.$$

Then (9) becomes

$$\left(x + y - \frac{1}{2}\right)^2 = x - y + \frac{5}{4}, \text{ that is,}$$

$$\left(\frac{x + y - \frac{1}{2}}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}}\left(\frac{x - y + \frac{5}{4}}{\sqrt{2}}\right).$$

This is in the form (8).

Thus, the axis of the parabola is $x + y - \frac{1}{2} = 0$, and the tangent at the vertex is $x - y + \frac{5}{4} = 0$.

The vertex is the intersection of these two lines, that is, $\left(-\frac{3}{8}, \frac{7}{8}\right)$.

The length of the latus rectum of the parabola is $\frac{1}{\sqrt{2}}$.

Thus, the focus is at $\left(-\frac{3}{8} + \frac{1}{4\sqrt{2}}\cos\theta, \frac{7}{8} + \frac{1}{4\sqrt{2}}\sin\theta\right)$, where θ is the angle that the axis makes with the x-axis, that is, $\theta = \tan^{-1}(-1)$.

$\therefore \sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}$.

Therefore, the focus is $F\left(-\frac{1}{4}, \frac{3}{4}\right)$.

What are the points of intersection of the parabola and the coordinate axes? They are $(1 + \sqrt{2}, 0)$, $(1 - \sqrt{2}, 0)$, $(0, 1)$, $(0, -1)$.

So, we can trace the parabola as in Fig. 3.

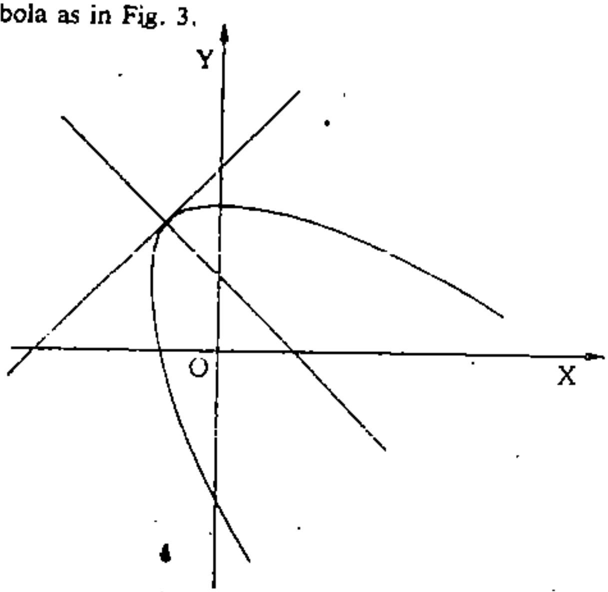


Fig. 3: The parabola $x^2 + 2xy + y^2 - 2x - 1 = 0$

Has the example helped you to understand the method for tracing a parabola? The following exercise will help you to find out.

E12) Trace the conic $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$.

Let us now see how to obtain the tangents of a general conic.

3.5 TANGENTS

In Unit 1 you studied the equations of tangents to the conics in standard form. Now we will discuss the equation of a tangent to the general conic (1).

So, consider two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
If $x_1 = x_2 = \alpha$, say, then the line PQ is $x = \alpha$.
Similarly, if $y_1 = y_2 = \alpha$, say, then the line PQ is $y = \alpha$.
Otherwise, the line PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{...(10)}$$

Since P and Q lie on the conic,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{...(11)}$$

and $ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0.$ ...(12)

Then (12) – (11)

$$\Rightarrow a(x_2^2 - x_1^2) + 2h(x_2y_2 - x_1y_1) + b(y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$\Rightarrow a(x_2^2 - x_1^2) + 2h(x_2y_2 - x_1y_2 + x_1y_2 - x_1y_1) + b(y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$\Rightarrow (x_2 - x_1)\,[a(x_1 + x_2) + 2hy_2 + 2g] + (y_2 - y_1)\,[(b(y_1 + y_2) + 2hx_1 + 2f) = 0.$$

$$\Rightarrow \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \frac{-\,[a(x_1 + x_2) + 2hy_2 + 2g]}{[b\,(y_1 + y_2) + 2hx_1 + 2f]}$$

Putting this in (10), we get

$$y - y_1 = -\left[\frac{a(x_1 + x_2) + 2hy_2 + 2g}{b(y_1 + y_2) + 2hx_1 + 2f}\right](x - x_1) \qquad \text{...(13)}$$

As (x_2, y_2) tends to (x_1, y_1), (13) gives us the equation of the tangent to the given conic at (x_1, y_1).
Thus, the equation of the tangent at $P(x_1, y_1)$ is

$$(y - y_1)\,(by_1 + hx_1 + f) + (x - x_1)\,(ax_1 + hy_1 + g) = 0$$

$$\Leftrightarrow x(ax_1 + hy_1 + g) + y(by_1 + hx_1 + f) + (gx_1 + fy_1 + c) = 0, \text{ using (11).}$$

$$\Leftrightarrow axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \qquad \text{...(14)}$$

Thus, (14) is the equation of the tangent to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) lying on the conic.
From (14) you can see that we can use the following rule of thumb to obtain the equation of a conic.

> In the equation of the conic, replace x^2 by xx_1, y^2 by yy_1, $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$ and $2xy$ by $(xy_1 + yx_1)$, to get the equation of the tangent at (x_1, y_1).

For instance, the tangent to the parabola $y^2 - 4ax = 0$, at a point (x_1, y_1) is $yy_1 - 2a\,(x + x_1) = 0$.
We have already derived this in Sec. 2.3.2.
In fact, the equations of tangents to the ellipse and hyperbola given in standard form are also special cases of (14), as you can verify from Unit 2.

Now you may like to try your hand at finding tangents at some points.

E13) Obtain the equations of the tangent and the normal to the conic in E8 at the points where it cuts the y-axis.

In Unit 2 you have seen that not every line can be a tangent to a given standard conic. Let us now see which lines qualify for being tangents to the general conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. With your experience in Unit 2, can you tell the conditions under which the line $px + qy + r = 0$ will be a tangent to this conic ?

Suppose it is a tangent at a point (x_1, y_1) to the conic. Now, either $p \neq 0$ or $q \neq 0$. Let us suppose $p \neq 0$. Then we can substitute $x = -\frac{(qy + r)}{p}$ in the equation of the conic, to get

$$\frac{a}{p^2}(qy + r)^2 - \frac{2h}{p}(qy + r)\,y + by^2 - \frac{2g}{p}(qy + r) + 2fy + c = 0.$$

$$\Rightarrow (aq^2 - 2hpq + bp^2)\,y^2 - 2y\,(prh + pqg - aqr - p^2f) + (ar^2 - 2gpr + cp^2) = 0$$

The roots of this quadratic equation in y give us the y-coordinates of the points of intersection of the given line and conic. The line will be a tangent if these points coincide, that is, if the quadratic equation has coincident roots, that is, if

$$(prh + pqg - aqr - p^2f)^2 = (aq^2 - 2hpq + bp^2)(ar^2 - 2gpr + cp^2). \qquad \ldots(15)$$

In terms of determinants (see MTE-04, Unit [illegible]), we can write this condition as

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & 0 \end{vmatrix} = 0 \qquad \ldots(16)$$

Thus, (15) or the determinant condition (16) tell us if $px + qy + r = 0$ is a tangent to the general conic or not.

For example, the line $y = mx + c$ will touch the parabola $y^2 = 4ax$, if

$$\begin{vmatrix} 0 & 0 & -2a & m \\ 0 & 1 & 0 & -1 \\ 2a & 0 & 0 & c \\ m & -1 & c & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-2a)\begin{vmatrix} 0 & 1 & -1 \\ -2a & 0 & c \\ m & -1 & 0 \end{vmatrix} - m\begin{vmatrix} 0 & 1 & 0 \\ -2a & 0 & 0 \\ m & -1 & c \end{vmatrix} = 0, \text{ expanding along the first row.}$$

$$\Rightarrow (-2a)(cm - 2a) - m(2ac) = 0$$

$$\Rightarrow c = \frac{a}{m}.$$

This is the same condition that we derived in Sec. 2.3.2.

Why don't you try these exercises now?

E14) Is $x + 4y = 0$ a tangent to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$? Find all the tangents to this conic that are parallel to the given line.

E15) a) Prove that the condition for $ax + by + 1 = 0$ to touch $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(ag + bf - 1)^2 = (a^2 + b^2)(g^2 + f^2 - c)$.

b) In particular, under what conditions on C will $y = Mx + C$ touch $x^2 + y^2 = A^2$?

In this section you saw that a line and a conic intersect in at most two points. Now let us see what we get when two conics intersect.

3.6 INTERSECTION OF CONICS

Consider the intersection of an ellipse and a circle (Fig. 4(a)) or of an ellipse and a hyperbola (Fig. 4(b)).

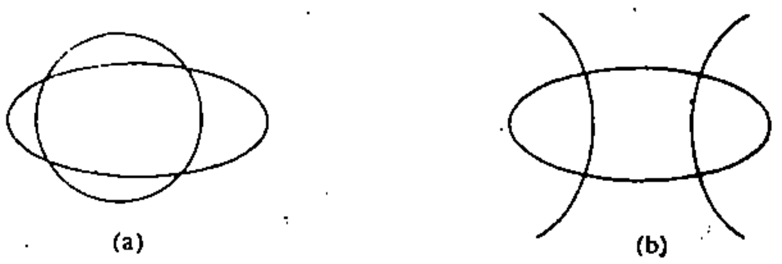

(a) (b)

Fig. 4: Intersecting conics.

You can see that these conics intersect in four points. But, do any two conics intersect in four points? The following result answers this question.

Theorem 5 : In general, two conics intersect in four points.

Proof : Let the equations of the two conics be

$ax^2 + 2(hy + g)x + by^2 + 2fy + c = 0$, and.

$a_1 x^2 + 2(h_1y + g_1)x + b_1y^2 + 2f_1y + c_1 = 0$.

These equations can be considered as quadratic equations in x. If we eliminate x from them, we will get a fourth degree equation in y. This will have four roots. Corresponding to each of these roots, we will get a root of x. So there are, in general, four points of intersection for the two conics.

Since a fourth degree equation with real coefficients may have two or four complex roots (see MTE-04, Unit 3), two conics can intersect in

i) four real points,
ii) two real and two imaginary points, or
iii) four imaginary points.

These points of intersection can be distinct, or some may coincide, or all of them may coincide.

Let us consider an example.

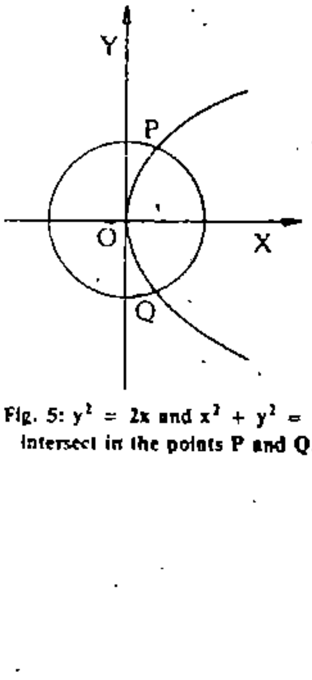


Fig. 5: $y^2 = 2x$ and $x^2 + y^2 = 1$ intersect in the points P and Q.

Example 7: Find the points of intersection of the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 1$ (see Fig. 5).

Solution: If (x_1, y_1) is a point of intersection, then $x_1^2 + y_1^2 = 1$ and $y_1^2 = 2x_1$. Eliminating y_1 from these equations, we get

$x_1^2 + 2x_1 = 1$, that is, $(x_1 + 1)^2 = 2$.

So $x_1 = -1 \pm \sqrt{2}$.

Then $y_1^2 = 2x_1$ gives us

$y_1 = \pm \sqrt{2}(\sqrt{2} - 1)^{1/2}$ if $x_1 = -1 + \sqrt{2}$, and

$y_1 = \pm \sqrt{2}i(\sqrt{2} + 1)^{1/2}$ if $x_1 = -1 - \sqrt{2}$.

Thus, there are only two real points of intersection, namely, $(\sqrt{2} - 1, \sqrt{2}(\sqrt{2} - 1)^{1/2})$ and $(\sqrt{2} - 1, -\sqrt{2}(\sqrt{2} - 1)^{1/2})$. This is why you see only two points of intersection in Fig. 4.

Here is an exercise for you now.

E16) Find the points of intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

You have seen that two conics intersect in four real or imaginary points. Now we will find the equation of any conic that passes through these points.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and
$a_1x^2 + 2h_1 xy + b_1y^2 + 2g_1 x + 2f_1 y + c_1 = 0$
be the equations of two conics.

Let us briefly denote them by $S = 0$ and $S_1 = 0$, respectively.

Then, for each $k \in R$, $S + kS_1 = 0$ is a second degree equation in x [illegible] it is a conic, for each value of k.

On the other hand, any point of intersection of the two conics satisfies both the equations $S = 0$ and $S_1 = 0$. Hence it satisfies $S + kS_1 = 0$. Thus, the conic $S + k S_1 = 0$ passes through all the points of intersection of $S = 0$ and $S_1 = 0$.

So we have proved

Theorem 6: The equation of any conic passing through the intersection of two conics $S = 0$ and $S_1 = 0$ is of the form $S + kS_1 = 0$, where $k \in \mathbf{R}$.

For different values of k, we get different conics passing through the points of intersection of $S = 0$ and $S_1 = 0$. But, will all these conics be of the same type? If you do the following exercises, you may answer this question.

E17) If $S = 0$ and $S_1 = 0$ are rectangular hyperbolas, then show that $S + kS_1 = 0$ is a rectangular hyperbola, for all real k.

(Hint: Recall that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a rectangular hyperbola if $a + b = 0$.)

E18) Let $S \equiv \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$ and $S_1 \equiv xy - 9 = 0$.

Under what conditions on k will $S + kS_1 = 0$ be
a) an ellipse ?
b) a parabola ?
c) a hyperbola ?

Now we have come to the end of our discussion on conics. Let us see what we have covered in this unit.

3.7 SUMMARY

In this unit we discussed the following points:

1) The general second degree equation
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
represents a conic. It is

i) a pair of straight lines iff $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$,

Further, if the condition is satisfied, then the angle between the lines is

$\tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right)$;

ii) a parabola if $ab - h^2 = 0$, and the determinant condition in (i) is not satisfied ,

iii) an ellipse if $ab - h^2 > 0$;

iv) a hyperbola if $ab - h^2 < 0$.

2) An ellipse and a hyperbola are central conics; a parabola is a non-central conic.

3) A central conic with centre at the origin is of the form $ax^2 + 2hxy + by^2 = 1$, where $a, h, b \in \mathbb{R}$.

4) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a central conic if $ax + hy + g = 0$ and $hx + by + f = 0$ intersect. And then, the centre of the conic is the point of intersection of these lines. The slopes of the axes of this conic are the roots of the equation

$$\tan^2 \theta + \left(\frac{a - b}{h}\right) \tan \theta - 1 = 0.$$

5) Tracing a conic.

6) The tangent to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is

$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Provided

$hy_1 + hx_1 + f \neq 0$

Further, a line $px + qy + r = 0$ is tangent to the given conic if

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & 0 \end{vmatrix} = 0$$

7) Two conics intersect in four points, which can be real or imaginary.

8) The equation of a conic passing through the four points of intersection of the conics $S = 0$ and $S_1 = 0$ is $S + kS_1 = 0$, where $k \in \mathbb{R}$.

3.8 SOLUTIONS/ANSWERS

E1) a) There are 3 types of non-degenerate conics: parabola, ellipse, hyperbola. There are 5 types of degenerate conics: point, pair of intersecting lines, pair of distinct parallel lines, pair of coincident lines, empty set.

b) A circle is a particular case of an ellipse. Thus, if (1) represents a circle then $ab - h^2 > 0$.

E2) a) $x^2 - 2xy + y^2 + \sqrt{2}\,x - 2 = 0$.

Here $a = 1 = b$, $h = -1$.

If we rotate the axes through $\pi/4$, then the new coordinates x' and y' are given by

$$x = \frac{1}{\sqrt{2}}(x' - y') \text{ and } y = \frac{1}{\sqrt{2}}(x' + y')$$

Thus, the given equation becomes

$2y'^2 + x' - y' = 2$.

$$\Leftrightarrow y'^2 - \frac{1}{2}y' + \frac{1}{2}x' = 1$$

$$\Leftrightarrow \left(y' - \frac{1}{4}\right)^2 = -\frac{1}{2}x' + \frac{17}{16} = -\frac{1}{2}\left(x' - \frac{17}{8}\right)$$

Now, if we shift the origin to $\left(\frac{17}{8}, \frac{1}{4}\right)$, the equation becomes the parabola

$Y^2 = -\frac{1}{2}X$, where X and Y are the new coordinates.

b) $9x^2 - 6xy + y^2 - 40x - 20y + 75 = 0$.

Here $a = 9$, $b = 1$, $h = -3$.

So, let us rotate the axes through θ, where

$\theta = \frac{1}{2}\tan^{-1}\left(-\frac{6}{8}\right)$. $\quad\therefore \tan 2\theta = -\frac{3}{4}$.

So we can take $\tan\theta = 3$, so that $\sin\theta = -\frac{3}{\sqrt{10}}$, $\cos\theta = -\frac{1}{\sqrt{10}}$.

Then the equation in the X'Y'-system becomes

$$\frac{59}{5}y'^2 - 10\sqrt{10}x' - 10\sqrt{10}y' + 75 = 0,$$

which can be transformed and seen to be the equation of a parabola.

E3) In this case $a = 3$, $b = 2$, $c = 2$, $f = \frac{5}{2} = g$, $h = \frac{7}{2}$.

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & \frac{7}{2} & \frac{5}{2} \\ \frac{7}{2} & 2 & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & 2 \end{vmatrix} = 0$$

Hence the given equation represents a pair of lines.

E4) Here the discriminant concerned is

$$\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Thus, the given equation represents a pair of lines.

E5) a) The lines will be parallel if $\sqrt{h^2 - ab} = 0$, that is, $ab - h^2 = 0$.

b) The lines will be perpendicular if $a + b = 0$.

E6) Since P lies on the ellipse, so does P'.

The equation of PO is $\frac{y - y_1}{-y_1} = \frac{x - x_1}{-x_1}$, that is, $x_1(y - y_1) = y_1(x - x_1)$.

P' also lies on this since $(-x_1, -y_1)$ satisfies this equation.

Hence, we have shown the result.

E7) In this case $ab \neq h^2$. So the conic is central. Its centre is the intersection of

$3x + \frac{7}{2}y + \frac{5}{2} = 0$ and $\frac{7}{2}x + 2y + \frac{5}{2} = 0$,

that is, $\left(-\frac{3}{5}, -\frac{1}{5}\right)$.

E8) In this case $a = 1 = b$, $h = -\frac{3}{2}$.

$\therefore ab - h^2 = -\frac{5}{4} < 0$.

So the given equation is central, and can be a hyperbola or a pair of intersecting lines.

$$\text{Since} \begin{vmatrix} 1 & -\frac{3}{2} & 5 \\ -\frac{3}{2} & 1 & -5 \\ 5 & -5 & 21 \end{vmatrix} \neq 0,$$

using Theorem 2 we can say that the equation represents a hyperbola. Its centre is the intersection of

$x - \frac{3}{2}y + 5 = 0$ and $-\frac{3}{2}x + y - 5 = 0$, that is, $(-2, 2)$.

E9) The central degenerate conics : point, pair of intersecting lines.

The non-central degenerate conics : pair of distinct parallel lines, pair of coincident lines. The empty set is both central and non-central.

E10) The equation represents a hyperbola with centre (-2, 2). If we shift the origin to (-2, 2), the equation becomes

$-x'^2 + 3x'y' - y'^2 = 1.$

Here $a = -1$, $b = -1$, $h = \frac{3}{2}$.

Thus, the axes of the conic are at an angle of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ to the x-axis.

So, putting these values of θ in (7), we get the lengths r_1 and r_2, of the semi-axes, on solving $r_1^2 = 2$ and $r_2^2 = -\frac{2}{5}$

Thus, $r_1 = \sqrt{2}$ and $r_2 = \sqrt{\frac{2}{5}}$.

Note that over here, though r_2^2 is negative, we only want its magnitude to compute the length of the axis.

Now, you know that if e is the eccentricity of the hyperbola then

$r_2 = r_1\sqrt{e^2 - 1}$, that is, $\sqrt{\frac{2}{5}} = \sqrt{2}\sqrt{e^2 - 1}$

$\Rightarrow e = \sqrt{\frac{6}{5}}$.

Now let us also see where the hyperbola cuts the x and y axes. Putting $y = 0$ in the given equation, we get

$x^2 + 10x + 21 = 0 \Rightarrow x = -3, -7.$

So, the hyperbola intersects the x-axis in (-3, 0) and (-7, 0). Similarly, putting $x = 0$ in the given equation and solving for y, we see that the hyperbola intersects the y-axis in (0, 3) and (0, 7).

With all this information the curve is as given in Fig. 6.

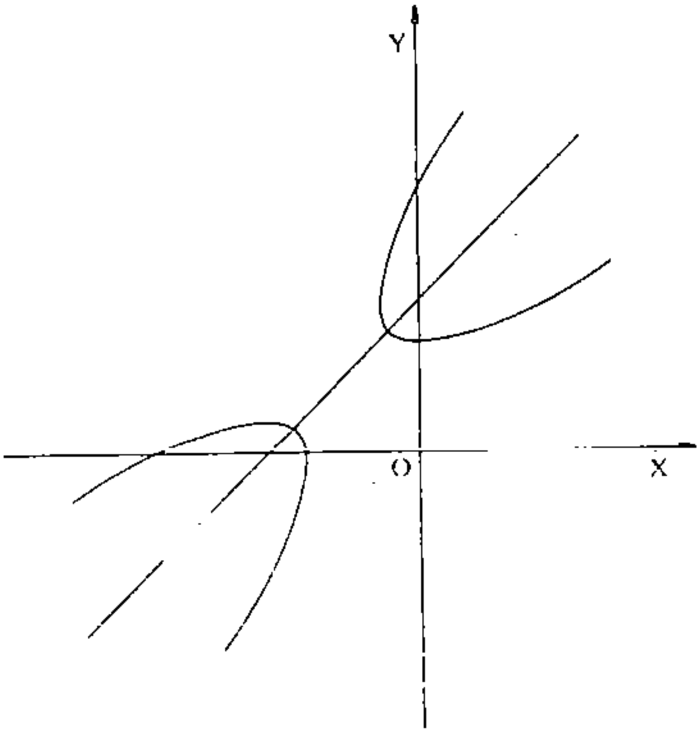


Fig. 6

E11) It will be central if $h^2 \neq 1$. And then its centre will be the intersection of $x + hy = 0$ and $hx + y + f = 0$, which is

$$\left(\frac{hf}{1 - h^2}, \frac{-f}{1 - h^2}\right).$$

If we shift the origin to this point, the given equation is transformed to

$$X^2 - 2hXY + Y^2 = \frac{f^2}{1 - h^2}$$

$$\Leftrightarrow \frac{1 - h^2}{f^2} X^2 - \frac{2h(1 - h^2)}{f^2} XY + \frac{1 - h^2}{f^2} Y^2 = 1.$$

This is in the standard form $AX^2 + 2HXY + BY^2 = 1$ of a central conic. Here $A = B = \frac{1 - h^2}{f^2}$. Therefore, the axes of the conic are at an angle of 45° and 135° to the x-axis. Since they pass through the centre, their equations are

$$y + \frac{f}{1 - h^2} = x - \frac{hf}{1 - h^2} \text{ and}$$

$$y + \frac{f}{1 - h^2} = -\left(x - \frac{hf}{1 - h^2}\right).$$

E12) The conic is a parabola since $ab = h^2$, and the determinant condition for it to represent a pair of lines is not satisfied.

We can rewrite the equation as

$(2x - y)^2 = 8x + 6y - 5.$

We introduce a constant c to the equation, to get

$(2x - y + c)^2 = 8x + 6y - 5 + 4cx - 2cy + c^2.$

$\Leftrightarrow (2x - y + c)^2 = 4(2 + c)x + 2(3 - c)y + c^2 - 5$

We choose c in such a way that

$$2\left(\frac{4(2 + c)}{2(c - 3)}\right) = -1 \Rightarrow c = -1.$$

Then the equation of the curve becomes

$(2x - y - 1)^2 = 4(x + 2y - 1)$

$$\Leftrightarrow \left(\frac{2x - y - 1}{\sqrt{5}}\right)^2 = \frac{4}{\sqrt{5}}\left(\frac{x + 2y - 1}{\sqrt{5}}\right).$$

The vertex of this parabola is the intersection of $2x - y - 1 = 0$ and $x + 2y - 1 = 0$, that is, $\left(\frac{3}{5}, \frac{1}{5}\right)$. The focus lies at

$$\left(\frac{3}{5} + \frac{1}{\sqrt{5}}\cos\theta, \frac{1}{5} + \frac{1}{\sqrt{5}}\sin\theta\right), \text{ where } \tan\theta = 2.$$

$$\therefore \sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}.$$

$\therefore$ The focus lies at $\left(\frac{4}{5}, \frac{3}{5}\right)$.

The curve intersects the y-axis in (0, 1) and (0, 5). It doesn't intersect the x-axis.

Thus, the shape of the parabola is as given in Fig. 7.

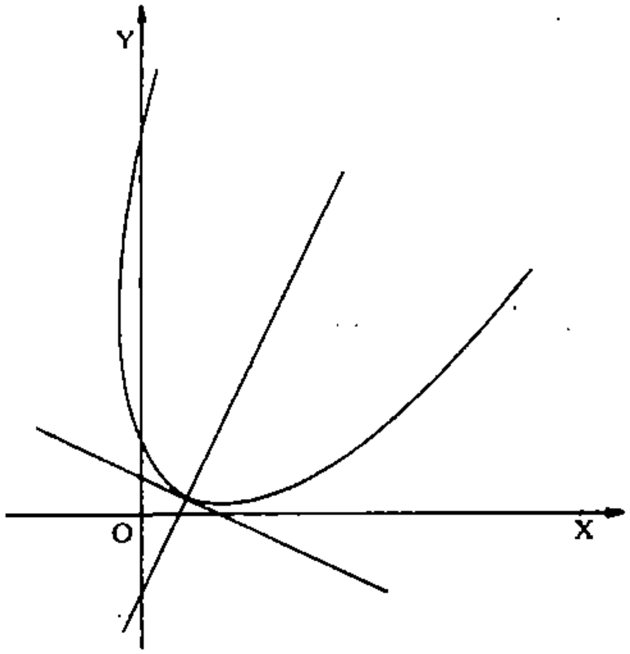


Fig. 7

E13) The conic's equation is

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0.$$

From E10 you know it intersects the axes in (-3, 0), (-7, 0), (0, 3), (0, 7).
The tangent at (-3, 0) is

$$-3x - \frac{3}{2}(x \cdot 0 - 3y) + y \cdot 0 + 5(x - 3) - 5(y - 0) + 21 = 0.$$

$$\Leftrightarrow 2x - \frac{1}{2}y + 6 = 0.$$

Its slope is 4.

Thus, the slope of the normal at (-3, 0) is $-\frac{1}{4}$. Hence, its equation is

$$y = -\frac{1}{4}(x + 3).$$

You can similarly check that the tangents at (-7, 0), (0, 3) and (0, 7) are respectively,

$4x - 11y + 28 = 0,$

$x - 4y + 12 = 0,$

$11x - 4y + 28 = 0.$

The normals at these points are respectively,

$$y = \frac{4}{11}(x + 7),$$

$$y - 3 = \frac{1}{4}x,$$

$$y - 7 = \frac{11}{4}x.$$

E14) $x + 4y = 0$ will be a tangent to the given conic if

$$\begin{vmatrix} 1 & 2 & -\frac{5}{2} & 1 \\ 2 & 3 & -3 & 4 \\ -\frac{5}{2} & -3 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow - \begin{vmatrix} 2 & 3 & -3 \\ -\frac{5}{2} & -3 & 3 \\ 1 & 4 & 0 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 & -\frac{5}{2} \\ -\frac{5}{2} & -3 & 3 \\ 1 & 4 & 0 \end{vmatrix} = 0$$

$\Leftrightarrow 40 = 0$, which is false.

Thus, the given line is not a tangent to the given conic.

Any line parallel to the given line is of the form $x + 4y + c = 0$. This will be a tangent to the given conic if (15) is satisfied, that is,

$(5c + 28)^2 = 3(3c^2 + 24c + 48)$

$\Leftrightarrow c = -5$ or -8.

Thus, the required tangents are

$x + 4y - 5 = 0$ and $x + 4y - 8 = 0$.

E15) a) Using (15), we see that the condition is

$$\begin{vmatrix} 1 & 0 & g & a \\ 0 & 1 & f & b \\ g & f & c & 1 \\ a & b & 1 & 0 \end{vmatrix} = 0$$

$\Leftrightarrow b(f - bc) - (1 - bf) + g\,[af + b(ag - af)] + a\,\{(ac - g) + f(bg - af)\} = 0$

$\Leftrightarrow b^2g^2 + a^2f^2 + 2bf - 2abfg - b^2c - 1 + 2ag - a^2c = 0.$

Adding a^2g^2 on both sides and simplifying, we get the given condition.

b) In (a) we put $g = 0 = f$, $c = -A^2$, $-\frac{a}{b} = M$, $-\frac{1}{b} = C$.

So the condition for $y = Mx + C$ to touch $x^2 + y^2 = A^2$ is

$C^2 = A^2(M^2 + 1)$

Thus, $C = A\sqrt{M^2 + 1}$.

E16) Substituting $x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$ in $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get

$$\frac{a^2}{b^2}\left(1 - \frac{y^2}{b^2}\right) + \frac{y^2}{a^2} = 1 \Leftrightarrow y^2 = \frac{a^2b^2}{a^2 + b^2}$$

$$\therefore y = \pm \frac{ab}{\sqrt{a^2 + b^2}}.$$

$$\text{Then } x^2 = a^2\left(1 - \frac{a^2}{a^2 + b^2}\right) = \frac{a^2b^2}{a^2 + b^2} \Rightarrow x = \pm \frac{ab}{\sqrt{a^2 + b^2}}$$

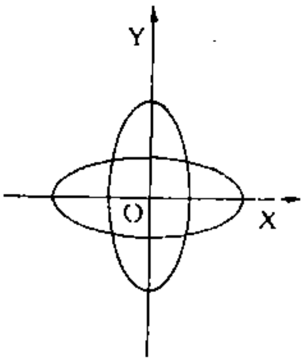


Fig. 8

Thus the 4 points of intersection are

$$\left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right), \left(\frac{-ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right), \left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$$

and $\left(\frac{-ab}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$.

We have drawn the situation in Fig. 8.

E17) Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

and $S_1 \equiv a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$

be rectangular hyperbolas. Then

$a + b = 0$ and $a_1 + b_1 = 0$.

$\therefore (a + b) + k(a_1 + b_1) = 0 \; \forall \; k \in \mathbf{R}$

$\Leftrightarrow (a + ka_1) + (b + kb_1) = 0 \; \forall \; k \in \mathbf{R}$

$\Leftrightarrow S + kS_1 = 0$ is a rectangular hyperbola $\forall \; k \in \mathbf{R}$.

E18) $S + kS_1 = 0$

$$\Leftrightarrow \frac{x^2}{9} - kxy + \frac{y^2}{4} - (1+9k) = 0.$$

a) This conic will be an ellipse if

$$\left(\frac{1}{9}\right)\left(\frac{1}{4}\right) - \frac{k^2}{4} > 0, \text{ that is, } k^2 < \frac{1}{9}.$$

b) The conic will be a parabola if

$$k^2 = \frac{1}{9} \text{ and } \begin{vmatrix} \frac{1}{9} & -\frac{k}{2} & 0 \\ -\frac{k}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -(1+9k) \end{vmatrix} \neq 0, \text{ that is,}$$

if $k = \pm \frac{1}{3}$ and $(1+9k)\left(\frac{1}{36} - \frac{k^2}{4}\right) \neq 0$.

But this can't be.

So the conic can't be a parabola.

But it will be a pair of lines if $k = \pm \frac{1}{3}$.

c) The conic will be a hyperbola if $k^2 > \frac{1}{9}$.

MISCELLANEOUS EXERCISES

(This section is optional)

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of conics. Our solutions to the questions follow the list of problems, in case you'd like to counter-check your answers.

1) Find the equation of the path traced by a point P, the sum of the squares of the distances from (1, 0) and (-1, 0) of which is 8.

A path traced by a moving point is called its locus.

2) Find the equation of the circle which passes through (1, 0), (0, -6) and (1, 4).

(**Hint:** The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$).

3) Prove the reflecting property for a parabola.

(**Hint:** Show that $\alpha = \beta$ in Fig. 9, Unit 2.)

4) Prove Theorem 2 of Unit 3.

5) A circle cuts the parabola $y^2 = 4ax$ in the points $(at_i^2, 2at_i)$ for $i = 1, 2, 3, 4$. Prove that $t_1 + t_2 + t_3 + t_4 = 0$.

(**Hint:** t_1, t_2, t_3, t_4 are the solutions of the quadric equation obtained by putting x = [illegible], y = 2at in the equation of a circle.)

6) Trace the curves $xy = 0$ and $xy - 4x - 5y + 20 = 0$.

7) What relations must hold between the coefficients of $ax^2 + by^2 + cx + cy = 0$ for it to represent a pair of straight lines?

8) Find the angle through which the axes should be rotated so that the equation $Ax + By + C = 0$ is reduced to the form $x =$ constant, and find the value of the constant.

9) Prove that $y^2 + 2Ax + 2By + C = 0$ represents a parabola whose axis is parallel to the x-axis. Find its vertex and the equation of its latus rectum.

10) Prove that the set of midpoints of all chords of $y^2 = 4ax$ which are drawn through its vertex is the parabola $y^2 = 2ax$.

11) a) Prove that $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ is negative, zero or positive, according as the point (x_1, y_1) lies inside, on or outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

b) Is the point (4, -3) inside or outside the ellipse $5x^2 + 7y^2 = 11$?

12) A line segment of fixed length a + b moves so that its ends are always on two fixed perpendicular lines (see Fig. 1). Prove that the path traced by a point which divides this segment in the ratio a : b is an ellipse.

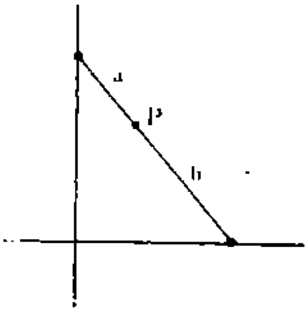


Fig. 1

13) Find the equation of the common tangent to the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

14) A normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x and y axes in M and N, respectively.

The lines through M and N drawn perpendicular to the x and y-axes, respectively, meet in the point P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

15) Consider the hyperbola in Fig. 20 of Unit 2. Through A and A′ draw parallels to the conjugate axis, and through B and B′ draw parallels to the transverse axis. Show that the diagonals of the rectangle so formed lie along the asymptotes of the hyperbola.

A method for drawing asymptotes of a hyperbola.

16) Which conics are represented by the following equations?

a) $(x-y)^2 +$ [illegible] $)^2 = 0$.

b) $r \sin^2\theta = 2a \cos \theta$,

c) $\frac{l}{r} = 1 + \cos\theta + \sqrt{3} \sin\theta$.

17) Trace the conics

a) $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$.

b) $xy - y^2 = a^2$

c) $(3x - 4y + 1)(4x + 3y + 1) = 1$.

18) Find the equation to the conic which passes through (1, 1) and the intersection of $x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ with the pair of straight lines $2x - y - 5 = 0$ and $3x + y - 11 = 0$.

Solutions

1) Let P be (x, y). Then
$[(x - 1)^2 + y^2] + [(x + 1)^2 + y^2] = 8$
$\Leftrightarrow 2x^2 + 2y^2 = 6$
$\Leftrightarrow x^2 + y^2 = 3$, which is a circle with centre (0, 0) and radius $\sqrt{3}$.

2) Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since (1, 0), (0, -6) and (3, 4) lie on it,
$1 + 2g + c = 0$,
$36 - 12f + c = 0$,
$9 + 16 + 6g + 8f + c = 0$.
Solving these three linear equations in g, f and c, we get

$$g = -\frac{71}{4},\ f = \frac{47}{8},\ c = \frac{69}{2}.$$

Thus the equation is

$$x^2 + y^2 - \frac{71}{2}x + \frac{47}{4}y + \frac{69}{2} = 0.$$

3) The parabola is $y^2 = 4ax$. The tangent T at a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

So $\tan\alpha = \frac{2a}{y_1}$.

The line PF, where F(a, 0) is the focus, is $\frac{y - y_1}{-y_1} = \frac{x - x_1}{a - x_1}$.

Its slope is $\frac{y_1}{x_1 - a}$.

$$\text{Thus, } \tan\beta = \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \frac{2a}{x_1 - a}}\text{, using (11) of Unit 1.}$$

$$= \frac{2a}{y_1}\text{, using the fact that } y_1^2 = 4ax_1.$$

Thus $\tan\alpha = \tan\beta$ and α and β are both less than or equal to 90°.

$\therefore \alpha = \beta$.

4) We want to show that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots(1)$$

can be written as a product of two linear factors iff its discriminant is 0.

If $a \neq 0$, we multiply (1) throughout by a and arrange it in decreasing powers of x. We get
$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac$.
On completing the square on the left hand, we get
$a^2x^2 + 2ax(hy + g) + (hy + g)^2 = y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac$

$\Leftrightarrow ax + hy + g = \pm \sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac}$

From this we can obtain x in terms of y, only involving the first degree iff the quantity under the square root sign is a perfect square, that is, iff $(gh - af)^2 = (h^2 - ab)(g^2 - ac)$,

$\Leftrightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

$$\Leftrightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

5) Let the circle's equation be

$x^2 + y^2 + 2gx + 2fy + c = 0.$

Substituting $x = at^2$, $y = 2at$ in this, we get

$a^2t^4 + 4a^2t^2 + 2agt^2 + 4aft + c = 0.$

We know that it has 4 roots t_1, t_2, t_3, t_4. So, from MTE-04 you know that the sum of the roots will be $\frac{1}{a^2}$ (coefficient of t^3) $= 0$.

$\therefore t_1 + t_2 + t_3 + t_4 = 0.$

6) $xy = 0$ is the pair of lines $x = 0$ and $y = 0$. We have traced it in Fig. 2.

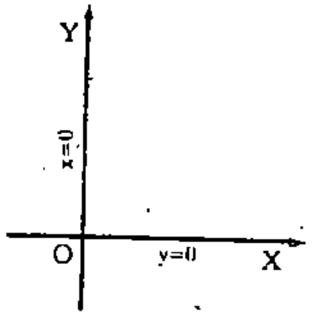


Fig. 2

$xy - 4x - 5y + 20 = 0$ is a pair of lines since its discriminant is 0. In fact, we can easily factorise it as

$(x - 5)(y - 4) = 0.$

Thus, it represents the pair of lines $x = 5$ and $y = 4$, which we have traced in Fig. 3.

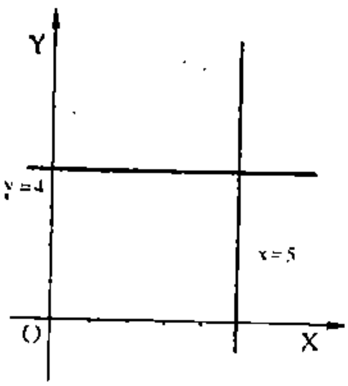


Fig. 3

7) $ax^2 + by^2 + cx + cy = 0$ represents a pair of lines iff

$$\begin{vmatrix} a & 0 & \frac{c}{2} \\ 0 & b & \frac{c}{2} \\ \frac{c}{2} & \frac{c}{2} & 0 \end{vmatrix} = 0 \Leftrightarrow (a + b)\frac{c^2}{4} = 0 \Leftrightarrow a = -b \text{ or } c = 0.$$

8) Let us rotate the axes through θ. Then the equation becomes

$A(x'\cos\theta - y'\sin\theta) + B(x'\sin\theta + y'\cos\theta) + C = 0.$

$\Leftrightarrow x'(A\cos\theta + B\sin\theta) + y'(B\cos\theta - A\sin\theta) + C = 0.$

This will reduce to the form $x' =$ constant iff $B\cos\theta = A\sin\theta$,

that is, $\theta = \tan^{-1}\frac{B}{A}$.

And then the equation becomes

$$x'\left(A\frac{A}{\sqrt{A^2 + B^2}} + B\frac{B}{\sqrt{A^2 + B^2}}\right) + C = 0$$

$$\Leftrightarrow x' = \frac{-C}{\sqrt{A^2 + B^2}}$$

Thus, the constant is $\frac{C}{\sqrt{A^2 + B^2}}$.

9) We rewrite the given equation as

$y^2 \quad -(2Ax + 2By + C)$

$\Leftrightarrow (y + k)^2 = -2Ax - 2By - C + 2ky + k^2$, where k is a constant.

$\Leftrightarrow (y + k)^2 = -2Ax + 2(k - B)y + k^2 - C$

We choose k so that

$Ax + (B - k)y + \frac{C-k^2}{2} = 0$ is parallel to the y-axis, that is, $k = B$. Then the equation becomes

$$(y + B)^2 = -2A\left(x + \frac{C-B^2}{2A}\right).$$

Its axis is $y + B = 0$, vertex is $\left(\frac{B^2 - C}{2A}, -B\right)$ and the equation of its latus rectum is $x = \frac{B^2 - A^2 - C}{2A}$.

10) The midpoint of any chord through $P(x_1, y_1)$ and $O(0, 0)$ is $Q\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$. Since $y_1^2 = 4ax_1$, $\left(\frac{y_1}{2}\right)^2 = 2a\left(\frac{x_1}{2}\right)$.

Thus, the set of all such Q is $y^2 = 2ax$.

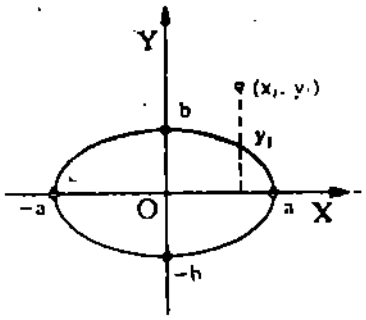


Fig. 4

11) a) Firstly, if (x_1, y_1) lies on the ellipse, then clearly

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Now, if (x_1, y_1) lies outside the ellipse (see Fig. 4) then either $|x_1| > a$ or $|y_1| > b$.

$\therefore x_1^2 > a^2$ or $y_1^2 > b^2$

$$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1.$$

Similarly, you can show that if (x_1, y_1) lies inside the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1.$$

b) Since $5(16) + 7(9) = 143 > 11$, the point lies outside the ellipse.

12) Let the perpendicular lines be the coordinate axes. Let the segment intersect the axes in (x,0) and (0,y). Then the coordinates of the point P are

$$(X, Y) = \left(\frac{ax}{a+b}, \frac{by}{a+b}\right).$$

Now, since $x^2 + y^2 = (a+b)^2$

$$\Rightarrow \frac{x^2}{(a+b)^2} + \frac{y^2}{(a+b)^2} = 1$$

$$\Rightarrow \left(\frac{ax}{a+b}\right)^2 \frac{1}{a^2} + \left(\frac{by}{a+b}\right)^2 \frac{1}{b^2} = 1$$

$$\Rightarrow \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

Thus, the path traced by P is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

13) Any tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx + \sqrt{a^2m^2-b^2}$, and any tangent to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is $x = m_1 y + \sqrt{a^2m_1{}^2 - b^2}$.

For these two lines to be the same, we must have $\frac{1}{m_1} = m$ and

$$\sqrt{a^2m^2 - b^2} = -\frac{1}{m_1}\sqrt{a^2m_1^2 - b^2}$$

$$\Leftrightarrow a^2m^2 - b^2 = a^2 - m^2 b^2$$

$\Leftrightarrow m^2 = 1$, provided $a^2 \neq b^2$.

Thus the common tangents are $y = x + \sqrt{a^2 - b^2}$ and $y = -x + \sqrt{a^2 - b^2}$.

14) See Fig. 5 for a diagram of the situation. The normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2}{x_1}(x - x_1) + \frac{b^2}{y_1}(y - y_1) = 0.$$

Thus, M is $\left(\frac{(a^2 + b^2)\, x_1}{a^2}, 0\right)$ and N is $\left(0, \frac{(a^2 + b^2)\, y_1}{b^2}\right)$.

Thus, the coordinates of P are

$$\left(\left(\frac{a^2 + b^2}{a^2}\right) x_1, \left(\frac{a^2 + b^2}{b^2}\right) y_1\right).$$

Now, since (x_1, y_1) lies on the hyperbola,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\Leftrightarrow \frac{a^2}{(a^2 + b^2)^2}\left(\frac{a^2 + b^2}{a^2}\right)^2 x_1^2 - \frac{b^2}{(a^2 + b^2)^2}\left(\frac{a^2 + b^2}{b^2}\right)^2 y_1^2 = 1$$

$\Leftrightarrow a^2x^2 - b^2y^2 = (a^2 + b^2)^2$, where $X = \frac{a^2 + b^2}{a^2} x_1$ and $Y = \frac{a^2 + b^2}{b^2} y_1$.

Now, as P varies, X and Y vary; but always satisfy the relationship $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$. Thus, this is the locus of the point P.

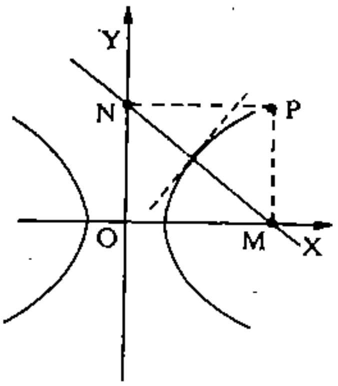


Fig. 5

15) The lines meet in (a, b), (a, -b), (-a, b) and (-a, -b). Thus the diagonals of the rectangles lie along $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, which are the asymptotes of the hyperbola.

16) a) $(x - y)^2 + (x - a)^2 = 0$

$\Leftrightarrow 2x^2 - 2xy + y^2 - 2ax + a^2 = 0.$

Here $a = 2, b = 1, h = -1, g = -a, f = 0, c = a^2$.

$\therefore ab - h^2 > 0$. Thus, the conic is an ellipse.

b) $r \sin^2\theta = 2a \cos\theta$.

Changing to Cartesian coordinates, this equation is $y^2 = 2ax$, a parabola.

c) $\frac{1}{r} = 1 + \cos\theta + \sqrt{3}\sin\theta$

$\Leftrightarrow 1 = \sqrt{x^2 + y^2} + x + \sqrt{3}y$, since $x = r\cos\theta$, $y = r\sin\theta$.

$\Leftrightarrow 2y^2 + 2\sqrt{3}xy + 2x + 2\sqrt{3}y + 1 = 0$

Here $ab - h^2 < 0$ and its discriminant is

$$\begin{vmatrix} 0 & \sqrt{3} & 1 \\ \sqrt{3} & 2 & \sqrt{3} \\ 1 & \sqrt{3} & 1 \end{vmatrix} = 1 \neq 0.$$

Thus, the curve represents a hyperbola.

17) a) You can check that $ab - h^2 = 0$ and the discriminant is non-zero. Thus, the equation represents a parabola. We can write it as

$(3x - 4y)^2 = 18x + 101y - 19.$

$\Leftrightarrow (3x - 4y + c)^2 = (6c + 18)x + y(101 - 8c) + c^2 - 19$, where we choose the constant c so that

$$3(6c + 18) - 4(101 - 8c) = 0$$

$$\Rightarrow c = 7.$$

Then the given equation becomes

$$(3x - 4y + 7)^2 = 15(4x + 3y + 2)$$

$$\Leftrightarrow \left(\frac{3x - 4y + 7}{5}\right)^2 = 3\left(\frac{4x + 3y + 2}{5}\right).$$

Thus, the axis of the parabola is $4x + 3y + 2 = 0$. The vertex is the intersection of $3x - 4y + 7 = 0$ and $4x + 3y + 2 = 0$;

that is, $\left(-\frac{29}{25}, \frac{22}{25}\right)$.

The length of its latus rectum is 3. Its focus F lies at $\left(-\frac{29}{25} + \frac{3}{4}\cos\theta, \frac{22}{25} + \frac{3}{4}\sin\theta\right)$, where $\tan\theta = -\frac{4}{3}$.

$\therefore$ F is (-0. 71, 0.28).

The curve intersects the y-axis in $\frac{101 \pm \sqrt{(101)^2 - 64 \times 19}}{32}$,

that is, approximately, $\frac{49}{8}$ and $\frac{3}{16}$.

It doesn't intersect the x-axis.

We have traced it in Fig. 6.

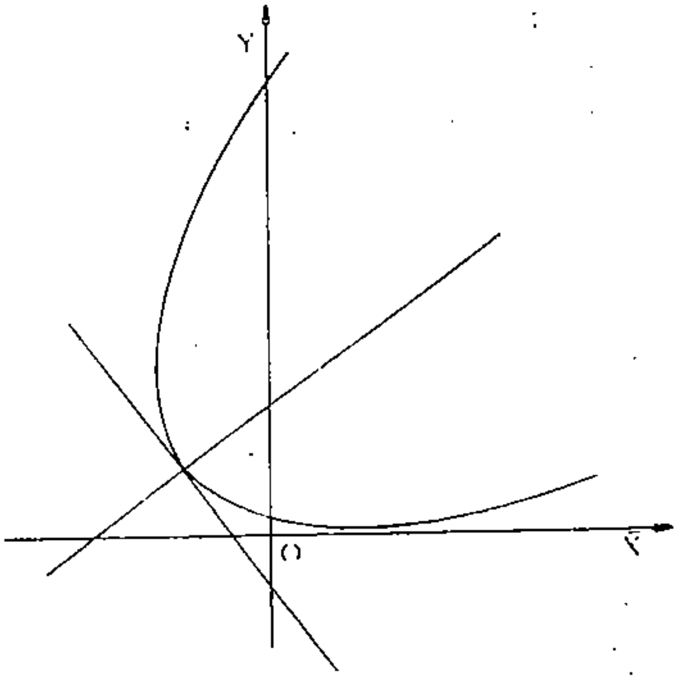


Fig. 6

b) $xy - y^2 = 1$

This is a hyperbola. Its centre is the intersection of $-\frac{1}{2}y$ [illegible] 0 and $-\frac{1}{2}x + y = 0$, that is, (0, 0). Its axes are inclined to the coordinate axes at an angle of θ, where tan [illegible] Thus, the slope of the transverse axis is $\theta_1 = \frac{\pi}{8}$, and [illegible] axis is $\theta_2 = \frac{5\pi}{8}$.

Since ~~$\tan\theta_1$~~ = .41, the length of the transverse axis, r_1, is given by

$$r_1^2 = \frac{1 + (.41)^2}{-(.41)+(.41)^2} = \frac{1.168}{.758} = 1.54.$$

$\therefore r_1 = 1.24$, approximately.

We similarly find $r_2 = .91$.

Thus its eccentricity is 1.24.

It doesn't interesect the axes.

With all this information, we have traced the curve in Fig. 7.

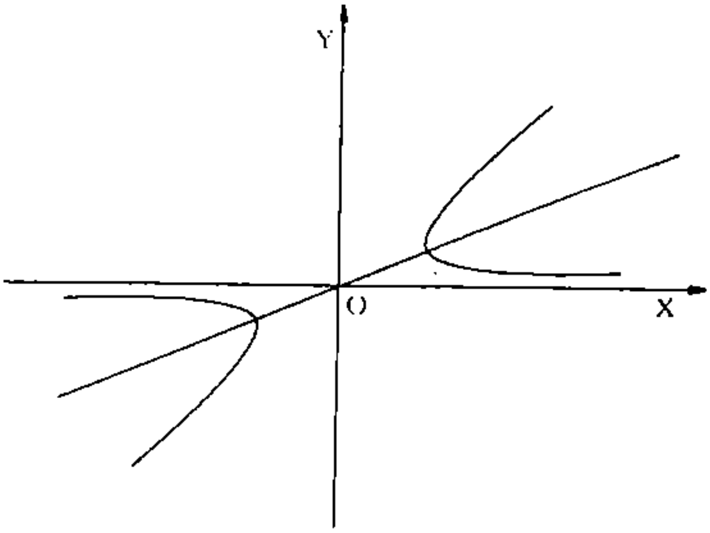


Fig. 7

The equation is a hyperbola whose centre is the intersection of

$3x - 4y + 1 = 0$ and $4x + 3y + 1 = 0$, that is, $\left(-\frac{7}{25}, \frac{1}{25}\right)$.

Here $a = 12$, $b = -12$, $h = -\frac{7}{2}$.

Thus, its axes are inclined at angles θ_1 and θ_2 to the coordinate axes, where $\tan\theta_1$ and $\tan\theta_2$ are roots of the equation

$$\tan^2\theta + \frac{a-b}{h}\tan\theta - 1 = 0$$

$$\Leftrightarrow \tan^2\theta - \frac{48}{7}\tan\theta - 1 = 0$$

$\Leftrightarrow \tan\theta_1 = 7$ and $\tan\theta_2 = -\frac{1}{7} \Rightarrow \theta_1 = 81.9°$ (approx.) and

$\theta_2 = 171.9°$ (approx.).

The length of its axes are r_1 and r_2, where

$$r_1^2 = \frac{1 + 49}{12 - 7\times 7 - 12\times 49} = -\frac{2}{25} \Rightarrow r_1 = .28,$$

$$r_2^2 = \frac{1 + \frac{1}{49}}{12 - 7\left(-\frac{1}{7}\right) - \frac{12}{49}} = \frac{2}{25} \Rightarrow r_2 = .28.$$

The curve intersects the coordinate axes in (0, 0), $\left(-\frac{7}{12}, 0\right)$, $\left(0, -\frac{1}{12}\right)$.

So, its curve is as given in Fig. 8.

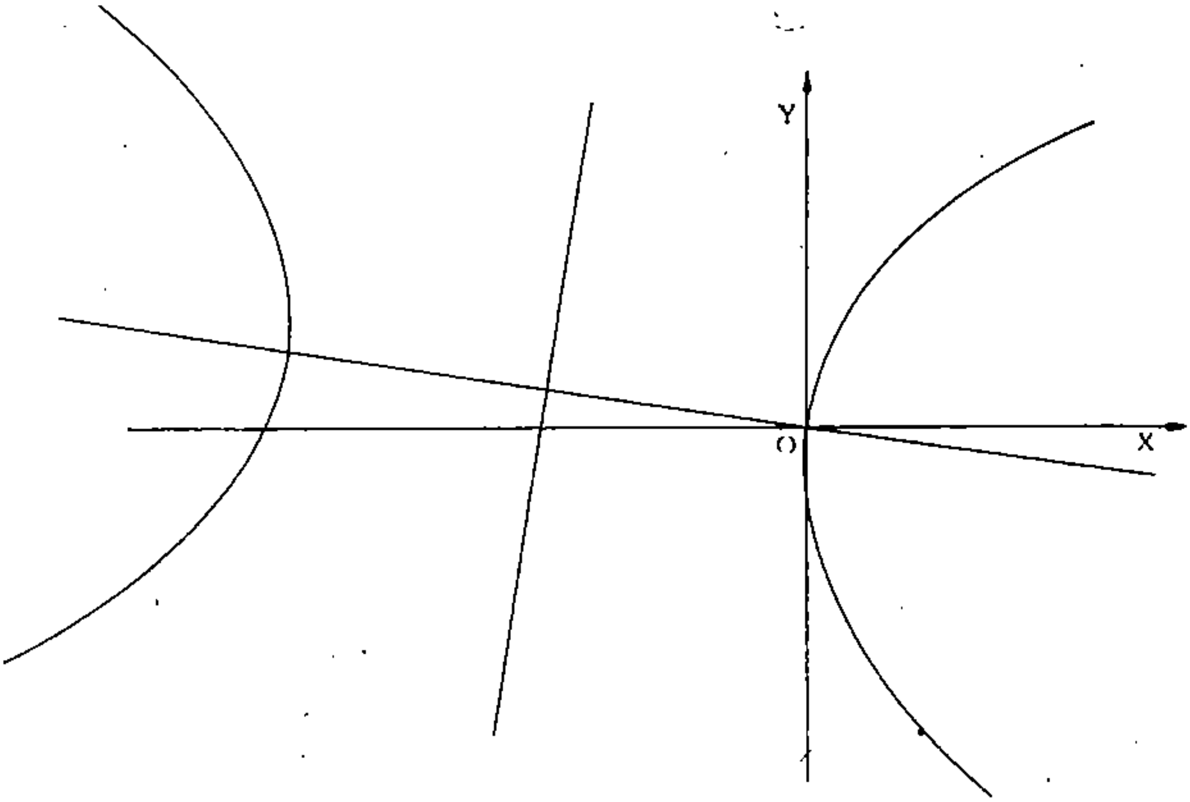


Fig. 8

18) Let $S_1 \equiv x^2 + 2xy + 5y^2 - 7x - 8y + 6 = 0$ and

$S_2 \equiv (2x - y - 5)(3x + y - 11) = 0.$

Then the required conic is $S_1 + kS_2 = 0$, where we choose k so that (1, 1) lies on the curve. Thus, the curve is

$(1+6k)x^2 + (2-k)xy + (5-k)y^2 - (7+37k)x - (8-6k)y + (6+55k) = 0.$

Since (1, 1) lies on it, $k = \frac{1}{28}$.

Thus, the conic is

$34x^2 + 55xy + 139y^2 - 233x - 218x - 218y + 223 = 0.$

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UGMM-05
Analytical Geometry

Block

2

THE SPHERE, CONE AND CYLINDER

BLOCK 2 THE SPHERE, CONE AND CYLINDER

In this block we shall introduce you to some basic solid geometry. We will start with a unit on lines and planes in three-dimensional space. In this unit you will see that a plane in 3-space is represented by a single linear equation in 3 variables, while a line is represented by a pair of such equations. We shall also discuss the intersection of lines and planes.

In the next unit of the block you will study spheres from an analytical point of view. This study will include finding the equation of a sphere and of a tangent plane to a sphere. We will also show you what the intersection of a plane and a sphere is, as well as the intersection of two or more spheres.

In the last unit of the block we will focus our attention on cones and cylinders. You will see that they are ruled surfaces, that is, sets of lines that satisfy certain conditions. We will also find the equation of a cone, of a tangent plane to a cone and of a particular type of cylinder.

At the end of this block, as in the last one, we have given a set of **miscellaneous exercises**. These exercises cover the contents of the block as a whole. Doing them will help you to improve your understanding of these contents.

A sphere, cone and cylinder are particular cases of a conicoid, the focal concept of the next block. We could have combined the two blocks, and dealt with these surfaces as specific instances of the general theory of conicoids. But we feel that doing them separately will prepare you to grasp the general theory more easily. This is why we have presented these surfaces and their geometrical properties separately in this block. So, if you ensure that you have achieved the objectives of the units in this block, you will find the next block easy to understand.

UNIT 4 PRELIMINARIES IN THREE-DIMENSIONAL GEOMETRY

Structure

4.1 INTRODUCTION

With this unit we start our discussion of analytical geometry in three-dimensional space, or 3-space. The aim of this unit is to acquaint you with some basic facts about points, lines and planes in 3-space. We start with a short introduction to the Cartesian coordinate system. Then we discuss various ways of representing a line and a plane algebraically. We also discuss angles between lines, between planes and between a line and a plane.

The facts covered in this unit will be used constantly in the rest of the course. Therefore, we suggest that you do all the exercises in the unit as you come to them. Further, please do not go to the next unit till you are sure that you have achieved the following objectives.

Objectives

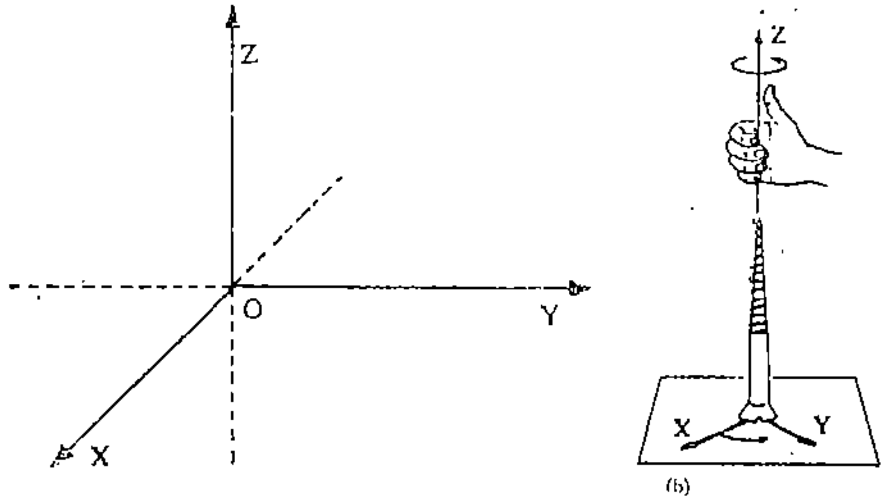
After studying this unit you should be able to

- find the distance between any two points in 3-space;
- obtain the direction cosines and direction ratios of a line;
- obtain the equations of a line in canonical form or in two-point form;
- obtain the equation of a plane in three-point form, in intercept form or in normal form;
- find the distance between a point and a plane;
- find the angle between two lines, or between two planes, or between a line and a plane;
- find the point (or points) of intersection of two lines or of a line and a plane.

Now let us start our discussion on points in 3-space.

4.2 POINTS

Let us start by generalising the two-dimensional coordinate system to three dimensions. We know that any point in two-dimensional space is given by two real numbers. To locate the position of a point in three-dimensional space, we have to give three numbers. To do this, we take three mutually perpendicular lines (axes) in space which intersect in a point O (see Fig. 1 (a)). O is called the **origin**. The positive directions OX, OY and OZ on these lines are so chosen that if a right-handed screw (Fig. 1(b)) placed at O is rotated from OX to OY, it moves in the direction of OZ.



(b)

Fig. 1 : The Cartesian coordinate axes in three dimensions

To find the coordinates of any point P in space, we take the foot of the perpendicular from P on the plane XOY (see Fig. 2). Call it M. Let the coordinates of M in the plane XOY be (x,y) and the length of MP be |z|. Then the coordinates of P are (x, y, z). z is positive or negative according as MP is in the positive direction of OZ or not.

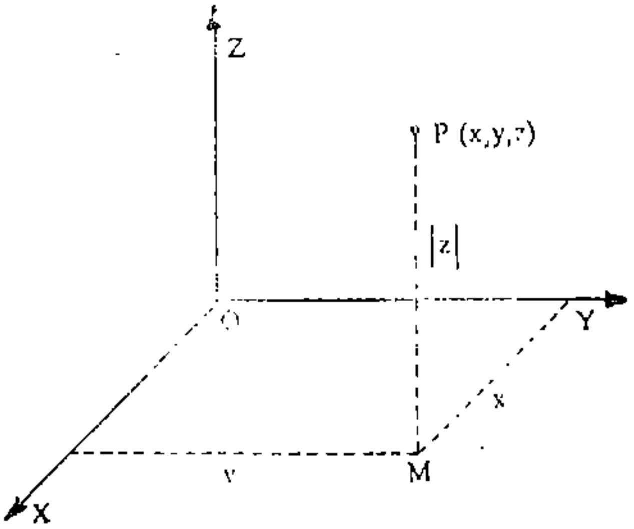


Fig. 2

So, for each point P in space, there is an ordered triple (x,y,z) of real numbers, i.e., an element of $\mathbf{R}^3$. Conversely, given an ordered triple of real numbers, we can easily find a point P in space whose coordinates are the given triple. So there is a one-one corrspondence between the space and the set $\mathbf{R}^3$. For this reason, three-dimensional space is often denoted by the symbol $\mathbf{R}^3$. For a similar reason a plane is denoted by $\mathbf{R}^2$, and a line by $\mathbf{R}$.

Now, in two-dimensional space, the distance of any point P (x,y) from the origin is $\sqrt{x^2+y^2}$. Using Fig. 2, can you extend this expression to three dimensions? By Pythagoras's theorem, we see that

$$OP^2 = OM^2+MP^2$$
$$= (x^2+y^2)+z^2$$

$\therefore OP = \sqrt{x^2 + y^2 + z^2}$

Thus, the distance of P (x, y, om the origin is $\sqrt{x^2 + y^2 + z^2}$.

And then, what will the distance between any two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be? **This is the distance formula**

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \qquad \text{......(1)}$$

as you may expect from Equation (1) of Unit 1.

Using (1), we can obtain the coordinates of a point R (x, y, z) that divides the join of P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) in the ratio m:n. They are

$$x = \frac{nx_1 + mx_2}{m + n}, \quad y = \frac{ny_1 + my_2}{m + n}, \quad z = \frac{nz_1 + mz_2}{m + n}. \qquad \text{.. ..(2)}$$

For example, to obtain a point A that trisects the join of P (1, 0, 0) and Q (1, 1, 1), we take m = 1 and n = 2 in (2). Then the coordinates of A are $\left(1, \frac{1}{3}, \frac{1}{3}\right)$.

Note that if we had taken m = 2, n = 1 in (2), we would have the other point that trisects PQ, namely, $\left(1, \frac{2}{3}, \frac{2}{3}\right)$.

Why don't you try some exercises now?

E1) Find the distance between P (1, 1, –1) and Q (–1, 1, 1). What are the coordinates of the point R that divides PQ in the ratio 3:4?

E2) Find the midpoint of the join of P (a, b, c) and Q (r, s, t).

Now let us shift our attention to lines.

4.3 LINES

In Unit 1 we took a quick look at lines in 2-space. In this section we will show you how to represent lines in 3-space algebraically. You will see that in this case a line is determined by a set of two linear equations, and not one linear equation, as in 2-space.

Let us start by looking at a triplet of angles which uniquely determine the direction of a line in 3-space.

4.3.1 Direction Cosines

Let us consider a Cartesian coordinate system with O as the origin and OX, OY, OZ as the axes. Now take a directed line L in space, which passes through O (see Fig. 3). Let L make angles α, β and γ with the positive directions of the x, y and z-axes, respectively. Then we define $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ to be the **direction cosines** of L.

For example, the direction cosines of the x-axis are cos 0, cos $\pi/2$, cos $\pi/2$, that is, 1, 0, 0.

Note that the direction cosines depend on the frame of reference, or coordinate system, that we choose.

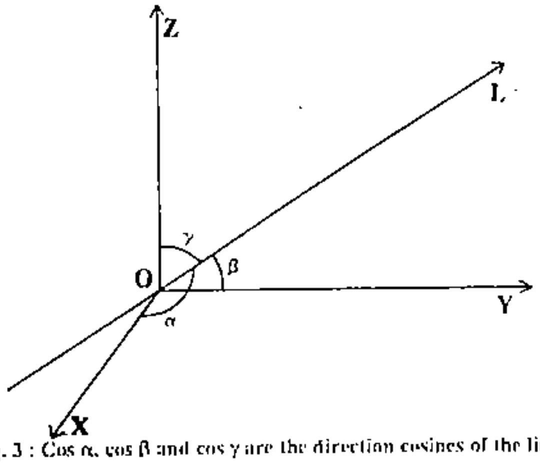


Fig. 3 : Cos α, cos β and cos γ are the direction cosines of the line L.

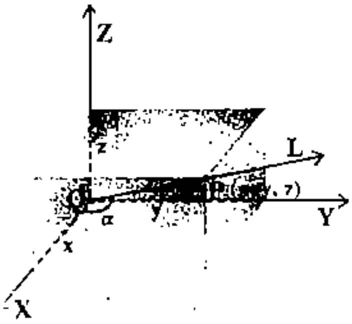


Fig. 4

Now take any directed line L in space. How can we find its direction cosines with respect to a given coordinate system? They will clearly be the direction cosines of the line through O which has the same direction as L. For example, the direction cosines of the line through (1,1,1) and parallel to the x-axis are 1,0, 0.

Now let us consider some simple properties satisfied by the direction cosines of a line. Let the direction cosines of a line L be cos α, cos β and cos γ with respect to a given coordinate system. We can assume that the origin O lies on L. Let P (x, y, z) be any point on L. Then you can see from Fig. 4 that

$x = OP \cos \alpha$, $y = OP \cos \beta$ and $z = OP \cos \gamma$.

Since $OP^2 = x^2 + y^2 + z^2 = OP^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$, we find that

$$\boxed{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad \ldots\ldots(3)}$$

This simple property of the direction cosines of a line is useful in several ways, as you'll see later in the course. Let us consider an example of its use.

Example 1 : If a line makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with the x and y axes, respectively, then what is the angle that it makes with the z-axis?

Solution : Put $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ in (3). Then, if γ is the angle that the line makes with the z-axis, we get

$$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1 \Rightarrow \cos \gamma = \pm \frac{1}{2} \Rightarrow \gamma = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}.$$

Thus, there will be two lines that satisfy our hypothesis. (Don't be surprised ! See Fig. 5.)

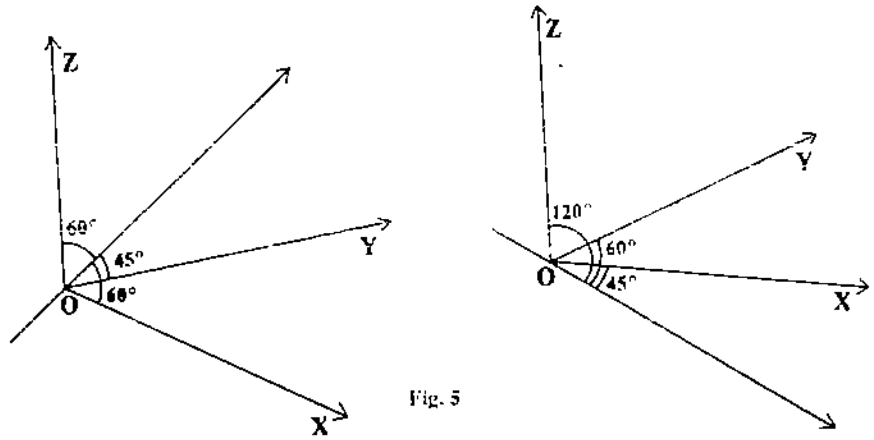


Fig. 5

They will make angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, respectively, with the z-axis.

There is another number triple that is related to the direction cosines of a line.

Definition : Three numbers a, b, c are called **direction ratios** of a line with direction cosines l, m and n, if $a = kl$, $b = km$, $c = kn$, for some $k \in \mathbf{R}$.

Thus, any triple that is proportional to the direction cosines of a line are its direction ratios.

For example, $\sqrt{2}$, 1, 1 are direction ratios of a line with direction cosines $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$.

You can try these exercises now.

E3) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line, show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

E4) Find the direction cosines of

a) the y and z axes,

b) the line $y = mx + c$ in the XY-plane.

E5) Let L be a line passing through the origin, and let P (a, b, c) be a point on it. Show that a, b, c are direction ratios of L.

E6) Suppose we change the direction of the line L in Fig. 3 to the opposite direction. What will the direction cosines of L be now?

Let us now see how the direction cosines or ratios can be used to find the equation of a line.

4.3.2 Equations of a Straight Line

We will now find the equations of a line in different forms. Let us assume that the direction cosines of a line are l, m and n, and that the point P (a, b, c) lies on it.

Then, if Q (x, y, z) is any other point on it, let us complete the cuboid with PQ as one of its diagonals (see Fig. 6).

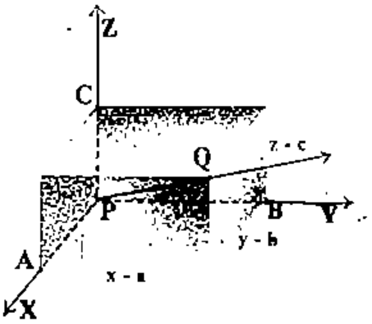


Fig. 6 : (x – a), (y–b) and (z –c) are direction ratios of PQ.

Then PA = x–a, PB = y–b and PC = z–c. Now, if PQ = r, you can see that

$\cos\alpha = \frac{x-a}{r}$, that is,

$l = \frac{x-a}{r}$. Similarly, $m = \frac{y-b}{r}$, $n = \frac{z-c}{r}$.

Thus, any point on the line satisfies the equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}. \qquad \text{......(4)}$$

Note that (4) consists of pairs of equations,

$$\frac{x-a}{l}=\frac{y-b}{m} \text{ and } \frac{y-b}{m}=\frac{z-c}{n}, \text{ or } \frac{x-a}{l}=\frac{z-c}{n} \text{ and } \frac{y-b}{m}=\frac{z-c}{n} \text{ or}$$

$$\frac{x-a}{l}=\frac{y-b}{m} \text{ and } \frac{x-a}{l}=\frac{z-c}{n}.$$

Conversely, any pair of equations of the form (4) represent a straight line passing through (a, b, c) and having direction ratios l, m and n.

(4) is called the **canonical form of the equations of a straight line.**

For example, the equations of the straight line passing through (1, 1, 1) and having direction cosines $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are

$$\frac{x-1}{1/\sqrt{3}}=\frac{y-1}{-1/\sqrt{3}}=\frac{z-1}{1/\sqrt{3}}, \text{ that is,}$$

$$\frac{x-1}{1}=\frac{y-1}{(-1)}=\frac{z-1}{1}.$$

Note that this is in the form (4), but 1, –1, 1 are **direction ratios** of this line, and not its direction cosines.

Remark 1 : By (4) we can see that the equations of the line passing through (a, b, c) and having direction cosines l, m, n are

$x = a+lr$, $y = b+mr$, $z = c+nr$, where $r \in$ R. .... (5)

This is a **one-parameter form** of the equations of a line, in terms of the parameter r.

Let us now use (4) to find another form of the equations of a line. Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on a line L. Then, if l, m and n are its directions cosines, (4) tells us that the equations of L are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \qquad \text{...... (6)}$$

Since Q lies on L, we get

$$\frac{x_2-x_1}{l}=\frac{y_2-y_1}{m}=\frac{z_2-z_1}{n}. \qquad \text{......(7)}$$

Then (6) and (7) give us

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \qquad \text{......(8)}$$

(8) is the generalisation of Equation (7) of Unit 1, and is called the **two-point form** of the equations of a line in 3-space.

For example, the equations of the line passing through (1, 2, 3) and (0, 1, 4) are

$x-1 = y-2 = -(z-3)$.

Note that, while obtaining (8) we have also shown that

> if P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on a line L, then x_2-x_1, y_2-y_1 and z_2-z_1 are direction ratios of L.

Now you can try some exercises.

E7) Find the equations to the line joining (–1,0, 1) and (1, 2, 3).

E8) Show that the equations of a line through (2,4, 3) and (–3, 5, 3) are $x + 5y = 22$, $z = 3$.

Now let us see when two lines are perpendicular.

4.3.3 Angle Between Two Lines

In Unit 1 you saw that the angle between two lines in a plane can be obtained in terms of their slopes. Now we will find the angle between two lines in 3-space in terms of their direction cosines.

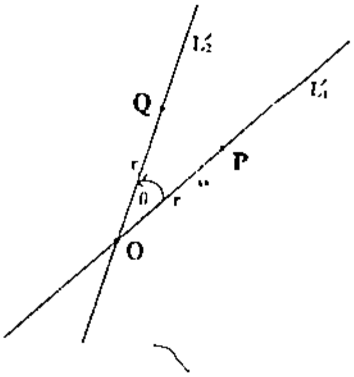


Fig. 7

Let the lines L_1 and L_2 have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively. Let θ be the angle between L_1 and L_2. Now let us draw straight lines L'_1 and L'_2 through the origin with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively. Then choose P and Q on L'_1 and L'_2, respectively, such that OP = OQ = r. Then the coordinates of P are $(l_1 r, m_1 r, n_1 r)$, and of Q are $(l_2 r, m_2 r, n_2 r)$. Also, θ is the angle between OP and OQ (see Fig. 7). Now

$$PQ^2 = (l_1 - l_2)^2 r^2 + (m_1 - m_2)^2 r^2 + (n_1 - n_2)^2 r^2$$
$$= 2(1 - l_1 l_2 - m_1 m_2 - n_1 n_2) r^2,$$

using $l_1^2 + m_1^2 + n_1^2 = 1$, $l_2^2 + m_2^2 + n_2^2 = 1$.

Also, from Fig. 7 and elementary trigonometry, we know that

$$PQ^2 = OP^2 + OQ^2 - 2OP.\ OQ \cos\theta$$
$$= 2(1 - \cos\theta) r^2.$$

Therefore, we find that θ is given by the relation

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 \qquad \ldots\ldots(9)$$

Using (9) can you say when two lines are perpendicular ? They will be perpendicular iff $\theta = \pi/2$, that is iff

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \qquad \ldots\ldots(10)$$

Now, suppose we consider direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 of L_1 and L_2, instead of their direction cosines. Then, is it true that L_1 and L_2 are perpendicular iff $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$?

If you just apply the definition of direction ratios, you will see that this is so.

And when are two lines parallel? Clearly, they are parallel if they have the same or opposite directions. Thus, the lines L_1 and L_2 (given above) will be parallel iff $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$ or $l_1 = -l_2$, $m_1 = -m_2$, $n_1 = -n_2$. In particular, this means that if a, b, c and a', b', c' are direction ratios of L_1 and L_2 respectively, then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} .$$

Let us just summarise what we have said.

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

(i) perpendicular iff $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$;

(ii) parallel iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

For example, the line $\frac{x}{2} = y = \frac{z}{3}$ is not parallel to the x-axis, since 2, 1, 3 are not proportional to 1, 0, 0. Further, $x = y = z$ is perpendicular to $x = -y$, $z = 0$, since 1, 1, 1 and 1, −1, 0 are the direction ratios of these two lines, and $1(1) + 1(-1) + 1(0) = 0$.

Why don't you try some exercises now?

E9) Find the angle between the lines with direction ratios 1, 1, 2 and $\sqrt{3}$, $-\sqrt{6}$, 4, respectively.

E10) If 3 lines have direction ratios 1, 2, 3; 1, −2, 1 and 4, 1, −2, respectively, show that they are mutually perpendicular.

In this section you saw that a line in 3-space is represented by a pair of linear equations. In the next section you will see that this means that a line is the intersection of two planes.

4.4 PLANES

In this section you will see that a linear equation represents a plane in 3-space. We will also discuss some aspects of intersecting planes, as well as the intersection of a line and a plane.

Let us first look at some algebraic representations of a plane.

4.4.1 Equations of a Plane

Consider the XY-plane in Fig. 1 (a). The z-coordinate of every point in this plane is 0. Conversely, any point whose z-coordinate is zero will be in the XY-plane. Thus, the equation $z = 0$ describes the XY-plane.

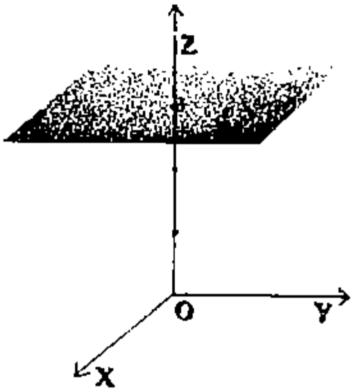


Fig. 8 : The plane z=3

Similarly, $z = 3$ describes the plane which is parallel to the XY-plane and which is placed 3 units above it (Fig. 8).

And what is the equation of the YZ-plane? Do you agree that it is $x = 0$?

Note that each of these planes satisfies the property that if any two points lie on it, then the line joining them also lies on it. This property is the defining property of a plane.

Definition : A **plane** is a set of points such that whenever P and Q belong to it then so does every point on the line joining P and Q.

Another point that you may have noticed about the planes mentioned above is that their equations are linear in x, y and z. This fact is true of any plane, according to the following theorem.

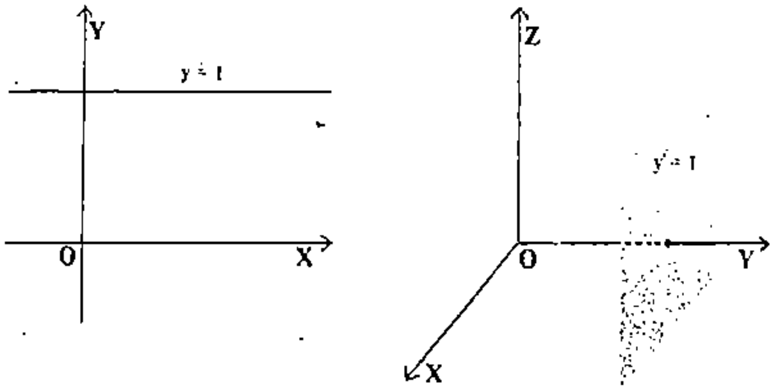
Theorem 1 : The general linear equation $Ax + By + Cz + D = 0$, where at least one of A, B, C is non-zero, represents a plane in three-dimensional space.

Further, the converse is also true.

We will not prove this result here, but will always use the fact that a plane is synonymous with a linear equation in 3 variables. Thus, for example, because of Theorem 1 we know that $2x + 5z = y$ represents a plane.

At this point we would like to make an important remark.

Remark 2 : In 2-space a linear equation represents a line, while in 3-space a linear equation represents a plane. For example, $y = 1$ is the line of Fig. 9 (a), as well as the plane of Fig. 9 (b).



(a) (b)

Fig. 9 : The same equation represents a line in 2-space and a plane in 3-space.

Let us now obtain the equation of a plane in different forms. To start with, we have the following result.

Theorem 2 : Three non-collinear points determine a plane. In fact, the unique plane passing through the non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by the determinant equation.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \qquad \text{......} (11)$$

We will not prove this result here, but we shall use it quite a bit.

As an example, let us find the equation of the plane which passes through the points (1, 1, 0), (–2, 2, –1) and (1, 2, 1). It will be

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow x \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 0 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 1 & 0 \\ -2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

$\Rightarrow 2x + 3y - 3z = 5.$

Why don't you try some exercises now?

E11) Show that the four points (0, −1, −1), (4, 5, 1), (3, 9, 4) and (−4, 4, 4) are coplanar, that is, lie on the same plane.
(**Hint** : Obtain the equation of the plane passing through any three of the points, and see if the fourth point lies on it.)

E12) Show that the equation of the plane which makes intercepts 2, −1, 5 on the three axes is $\frac{x}{2}+\frac{y}{(-1)}+\frac{z}{5}=1$.
(**Hint** : The plane makes an intercept 2 on the x-axis means that it intersects the x-axis at (2, 0, 0).)

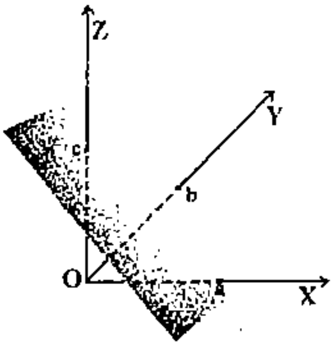


Fig. 10 : The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

In E12 did you notice the relationship between the intercepts and the coefficients of the equation?

In general, you can check that the equation of the plane making intercepts a, b and c on the coordinate axes (see Fig. 10) is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \ldots\ldots(12)$$

This is because (a, 0, 0), (0, b, 0) and (0, 0, c) lie on it.

(12) is called the **intercept form** of the equation of a plane.

Let us see how we can use this form.

Example 2 : Find the intercepts on the coordinate axes by the plane $2x-3y+5z=4$.

Solution : Rewriting the equation , we get

$$\frac{x}{2}-\frac{y}{\frac{4}{3}}+\frac{z}{\frac{4}{5}}=1.$$

Thus, the intercepts on the axes are $2, -\frac{4}{3}$ and $\frac{4}{5}$.

Now here is an exercise on the use of (12).

E13) Show that the planes ax+by+cz+d = 0 and Ax+By+Cz+D = 0 are the same iff a, b, c, d and A, B, C, D are proportional.
(**Hint** : Rewrite the equations in intercept form. The two planes will be the same iff their intercepts on the axes are equal.)

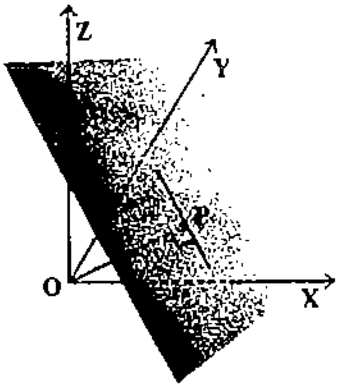


Fig. 11: Obtaining the normal form of the equation of the plane

Let us now consider another form of the equation of a plane. For this, let us drop a perpendicular from the origin O onto the given plane (see Fig. 11). Let it meet the plane in the point P. Let cos α, cos β and cos γ be the direction cosines of OP and p = |OP|. Further, let the plane make intercepts a, b and c on the x, y and z axes, respectively. Then

$$\cos\alpha=\frac{p}{a}, \cos\beta=\frac{p}{b} \text{ and } \cos\gamma=\frac{p}{c}. \qquad \ldots\ldots(13)$$

Now, from (12) we know that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Then, using (13), this equation becomes

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p \qquad \ldots\ldots(14)$$

This is called the **normal form** of the equation of the plane.

For example, let us find the normal form of the equation of the plane in Fig. 9 (b). The perpendicular from the origin onto it is of length 1 and lies along the x-axis. Thus, its direction cosines are 1, 0, 0. Thus, from (14) we get its equation as x = 1.

Note that (14) is of the form Ax+By+Cz = D, where $|A| \le 1$, $|B| \le 1$, $|C| \le 1$ and $D \ge 0$.

Now, suppose we are given a plane Ax + By + Cz + D = 0. From its equation, can we find the length of the normal from the origin to it? We will use E13 to do so.

Suppose the equation of the plane in the normal form is

$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0.$

Then this is the same as the given equation of the plane. So, by E13 we see that there is a constant k such that

$\cos\alpha = kA, \cos\beta = kB, \cos\gamma = kC, p = -kD.$

Then, by (3) we get

$k^2 (A^2+B^2+C^2) = 1$, that is, $k = \frac{\pm 1}{\sqrt{A^2+B^2+C^2}}$.

So, $p = -kD \frac{|D|}{\sqrt{A^2+B^2+C^2}}$, where we take the absolute value of D since $p \ge 0$.

Thus, the length of the perpendicular from the origin onto the plane Ax+By+Cz+D = 0 is

$$\frac{|D|}{\sqrt{A^2+B^2+C^2}}. \qquad \ldots\ldots(15)$$

For example, the length of the perpendicular from the origin onto x+y+z = 1 is $\frac{1}{\sqrt{3}}$.

Now let us go one step further. Let us find the distance between the point (a, b, c) and the plane Ax + By + Cz + D = 0, that is, the length of the perpendicular from the point to the plane. To obtain it we simply shift the origin to (a, b, c), without changing the direction of the axes. Then, just as in Sec.1. 4.1., if X, Y, Z are the current coordinates, we get x = X + a, y = Y + b, z = Z + c.

We have discussed the translation of axes in detail in Unit 7.

So the equation of the plane in current coordinates is A (X + a) + B (Y + b) + C (Z + c) + D = 0.

Thus, the length of the normal from (a, b, c) to the plane Ax + By + Cz + D = 0 is the same as the length of the normal from the current origin to

A (X+a) + B (Y+b) + C (Z + c) + D = 0, that is,

$$\frac{|Aa + Bb + Cc + D|}{\sqrt{A^2+B^2+C^2}}. \qquad \ldots\ldots(16)$$

For example, the length of the normal from (4, 3, 1) to 3x − 4y + 12z + 14 = 0 is

$$p = \frac{|12 - 12 + 12 + 14|}{\sqrt{3^2+4^2+12^2}} = \frac{26}{13} = 2.$$

Now you may like to do the following exercises.

E14) Find the distance of (2, 3, −5) from each of the coordinate planes, as well as from x + y + z = 1.

E15) Show that if the sum of the squares of the distances of (a, b, c) from the planes x + y + z = 0, x = z and x + z = 2y is 9, then $a^2 + b^2 + c^2 = 9$.

Now that you are familiar with the various equations of a plane let us talk of the intersection of planes.

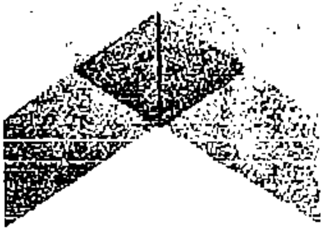


Fig. 12: A straight line is the intersection of two planes.

4.4.2 Intersecting Planes and Lines

In Sec. 4.3.2, you saw that a line is represented by two linear equations. Thus, it is the intersection of two planes represented by these equations (see Fig. 12).

In general, we have the following remark.

Remark 3 : A straight line is represented by a linear system of the form $ax + by + cz + d = 0$, $Ax + By + Cz + D = 0$. We write this in short as $ax + by + cz + d = 0 = Ax + By + Cz + D$. For example, $3x + 5y + z - 1 = 0 = 2x + 1$ represents the line obtained on intersecting the planes $3x + 5y + z = 1$ and $2x + 1 = 0$.

Now, suppose we are given a line

$ax + by + cz + d = 0 = Ax + By + Cz + D$.

This line clearly lies in both the planes $ax + by + cz + d = 0$ and $Ax + By + Cz + D = 0$. In fact, it lies in infinitely many planes given by

$$(ax + by + cz + d) + k\,(Ax + By + Cz + D) = 0, \qquad \ldots\ldots(17)$$

where $k \in \mathbf{R}$. This is because any point (x, y, z) lies on the line iff it lies on $ax+by+cz+d = 0$ as well as $Ax + By + Cz + D = 0$.

Let us see an example of the use of (17).

Example 3 : Find the equation of the plane passing through the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$.

Solution : The line is the intersection of $2(x + 1) = -3(y - 3)$ and $x + 1 = -3(z + 2)$, that is, $2x + 3y - 7 = 0 = x + 3z + 7$.

Thus, by (17), any plane passing through it is of the form

$(2x+3y-7) + k(x + 3z + 7) = 0$ for some $k \in \mathbf{R}$.

Since $(0, 7, -7)$ lies on it, we get

$21 - 7 + k(-21 + 7) = 0$, that is, $k = 1$.

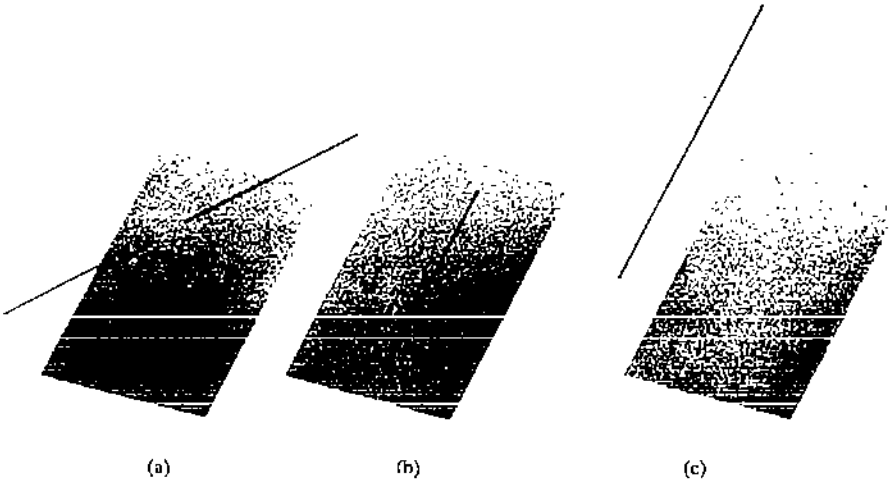
Thus, the required plane is

$3x + 3y + 3z = 0$, that is, $x + y + z = 0$.

You can do the following exercise on the same lines.

E16) Find the equation of the plane passing through $(1, 2, 0)$ and the line $x \cos\alpha + y \cos\beta + z \cos\gamma = 1$, $x+y = z$.

Now, given a line and a plane, will they always intersect? And, if so, what will their intersection look like? In Fig. 13 we show you the three possibilities.



(a) (b) (c)

Fig. 13 : (a) A line and a plane can either intersect in a point, or (b) the line can lie in the plane, or (c) they may not intersect at all.

Let us see some examples.

Example 4 : Check whether the plane x+y+z = 1 and the straight line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ intersect. If they do, then find their point (or points) of intersection.

Solution : By Remark 1 you know that any point on the line can be given by x = t, y = 1 + 2t and z = 2 + 3t, in terms of a parameter t. So if the line and plane intersect, then (t, 1 + 2t, 2 + 3t) must lie on the plane x+y+z = 1 for some t. Let us substitute these values in the equation. We get $t+1+2t+2+3t = 1$, that is , $6t = -2$, that is, $t = -\frac{1}{3}$. Thus, the line and plane intersect in a point, and the point of intersection is $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$.

Example 5 : Find the point (or points) of intersection of

(a) $\frac{x+2}{2} = \frac{y+3}{3} = \frac{z-4}{-2}$ and 3x+2y+6z = 12,

(b) $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and x+z = 1.

Solution : a) Any point on the line is of the form (2k–2, 3k–3, –2k+4), where k ∈ R.

Thus, if there is any point of intersection, it will be given by substituting this triple in 3x+2y+6z = 12.

So, we have

$3(2k-2) +2 (3k-3) +6 (-2k+4) = 12$

$\Rightarrow 0 = 0.$

This is true ∀ k ∈ **R**. Thus, for every k ∈ **R**, the triple (2k–2, 3k–3, –2k+4) lies in the plane. This means that the whole line lies in the plane.

b) Any point on the line is of the form (2t+1, t+2, -2t+3), where t ∈ **R**. This lies on x+z = 1 if, some t, (2t+1) + (–2t+3) = 1, that is, if 4 = 1, which is false. Thus, the line and plane do not intersect.

You can use the same method for finding the point (or points) of intersection of two lines. In the following exercises you can check if you've understood the method.

E17) Find the point of intersection of the line x = y = z and the plane x + 2y + 3z = 3.

E18) Show that the line $x-1 = \frac{1}{2}(y-3) = \frac{1}{3}(z-5)$ meets the line $\frac{1}{3}(x+1) = \frac{1}{5}(y-4) = \frac{1}{7}(z-9)$.

Now, consider any two planes. Can we find the angle between them? We can, once we have the following definition.

Definition : The **angle between two planes** is the angle between the normals to them from the origin.

So, now let us find the angle between two planes. Let the equations of the planes, in the normal form, be $l_1x+m_1y+n_1z = p_1$ and $l_2x + m_2y+n_2z = p_2$. Then the angle between the normals is $\cos^{-1}(l_1l_2+m_1m_2+n_1n_2)$.

Thus, the angle between the planes $l_1x +m_1y + n_1z = p_1$ and $l_2x+m_2y+n_2z = p_2$ is

$$\cos^{-1}(l_1l_2+m_1m_2+n_1n_2). \qquad \ldots\ldots (18)$$

$$\cos\theta = \frac{aA + bB + cC}{\sqrt{a^2 + b^2 + c^2}\,.\,\sqrt{A^2 + B^2 + C^2}} \qquad \ldots\ldots(19)$$

This is because a, b, c and A, B, C are direction ratios of the normals to the two planes, so that $\frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $\frac{b}{\sqrt{a^2 + b^2 + c^2}}$, $\frac{c}{\sqrt{a^2 + b^2 + c^2}}$ and $\frac{A}{\sqrt{A^2 + B^2 + C^2}}$, $\frac{B}{\sqrt{A^2 + B^2 + C^2}}$, $\frac{C}{\sqrt{A^2 + B^2 + C^2}}$ are their direction cosines.

Thus,

the planes $ax+by+cz+d = 0$ and $Ax+By+Cz+D = 0$

i) are parallel iff $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$, and

ii) are perpendicular iff $aA+bB+cC = 0$.

Let us consider an example of the utility of these conditions.

Example 6 : Find the equation of a plane passing through the line of intersection of the planes $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 13 = 0$, and which is perpendicular to the plane $2x - y - 2z + 5 = 0$.

Solution : The general equation of the plane through the line of intersection is given by

$7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$.

$\Rightarrow (7 + 4k)x + (3k - 4)y + (7 - 2k)z + 13k + 16 = 0$.

This will be perpendicular to $2x - y - 2z + 5 = 0$ if

$2(7 + 4k) - (3k - 4) - 2(7 - 2k) = 0$, that is, $k = -\frac{4}{9}$.

Thus, the required equation of the plane is $47x - 48y + 71z + 92 = 0$.

Try these exercises now.

E19) Find the equation of the plane through (1, 2, 3) and parallel to $3x + 4y - 5z = 0$.

E20) Find the angle between the planes $x + 2y + 2z = 5$ and $2x + 2y + 3 = 0$.

E21) Show that the angle between the line $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ and the plane

$Ax + By + Cz + D = 0$ is $\sin^{-1}\left(\frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2 + B^2 + C^2}\,.\,\sqrt{\alpha^2 + \beta^2 + \gamma^2}}\right)$.

(**Hint** : The required angle is the complement of the angle between the line and the normal to the plane.)

And now let us end the unit by summarising what we have done in it.

4.5 SUMMARY

In this unit we have covered the following points.

1) Distance formula : The distance between the points (x, y, z) and (a, b, c) is

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

2) The coordinates of a point that divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m:n are

$$\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right).$$

3) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line, then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

4) The canonical form of the equations of a line passing through the point (a, b, c) and having direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ is

$$\frac{x-a}{\cos\alpha}=\frac{y-b}{\cos\beta}=\frac{z-c}{\cos\gamma}.$$

5) The two-point form of the equations of a line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

6) The angle between two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right).$$

Thus, these lines are perpendicular iff $a_1a_2+b_1b_2+c_1c_2=0$, and parallel iff $a_1=ka_2$, $b_1=kb_2$, $c_1=kc_2$, for some $k\in\mathbf{R}$.

7) The equation of a plane is of the form $Ax+By+Cz+D=0$, where $A, B, C, D\in\mathbf{R}$ and not all of A, B, C are zero.

Conversely, such an equation always represents a plane.

8) The plane determined by the three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

9) The equation of the plane which makes intercepts a, b and c on the x,y and z-axes, respectively, is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

10) The normal form of the equation of a plane is $x\cos\alpha+y\cos\beta+z\cos\gamma=p$, where p is the length of the perpendicular from the origin onto the plane and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the perpendicular.

11) The length of the perpendicular from a point (a, b, c) onto the plane $Ax+By+Cz+D=0$ is

$$\frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

12) A line is the intersection of two planes.

13) The general equation of a plane passing through the line $ax + by + cz + d = 0 = Ax + By + Cz + D$ is $(ax + by + cz + d) + k\,(Ax + By + Cz + D) = 0$, where $k \in \mathbf{R}$.

14) The angle between the planes $ax + by + cz + d = 0$ and $Ax + By + Cz + D = 0$ is

$$\cos^{-1}\left(\frac{aA + bB + cC}{\sqrt{a^2 + b^2 + c^2}\,.\,\sqrt{A^2 + B^2 + C^2}}\right).$$

And now you may like to go back to Sec. 4.1, and see if you've achieved the **unit objectives** listed there. As you know by now, one way of checking this is to ensure that you have done all the exercises in the unit. You may like to see our solutions to the exercises. We have given them in the following section.

4.6 SOLUTIONS/ANSWERS

E1) $PQ = \sqrt{(1-(-1))^2 + (1-1)^2 + (-1-1)^2} = \sqrt{8}$

The coordinates of R are $\left(\frac{1}{7}, 1, -\frac{1}{7}\right)$.

E2) $\left(\frac{a+r}{2}, \frac{b+s}{2}, \frac{c+t}{2}\right)$.

E3) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2$.

E4) a) 0, 1, 0 and 0, 0, 1, respectively.

b) Any line in the XY-plane makes the angle $\pi/2$ with the z-axis. Now, if $m = \tan\theta$, then $y = mx+c$ makes an angle θ with the x-axis, and $\pi/2 - \theta$ with the y-axis. Thus, its direction cosines are $\cos\theta, \sin\theta, 0$.

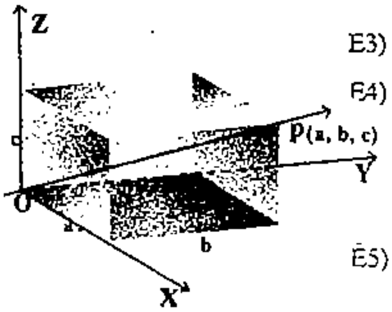


Fig. 14

E5) In Fig. 14 we have depicted the situation.

Let OP = r. Then you can see that the direction cosines of L are $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$.

Thus, a, b, c are direction ratios of L.

E6) Now, the line L makes angles of $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$ with the positive directions of the x, y and z-axes, respectively. Thus, its direction cosines are $-\cos\alpha$, $-\cos\beta$ and $-\cos\gamma$.

E7) $\frac{x+1}{2} = \frac{y}{2} = \frac{z-1}{2}$.

E8) The equations are

$\frac{x+3}{5} = \frac{y-5}{-1} = \frac{z-3}{0} = r$, say, that is,

$-(x+3) = 5\,(y-5)$ and $z = 3$, that is,

$x + 5y = 22$, $z = 3$.

E9) The direction cosines of the line with direction ratios 1, 1, 2 are $\frac{1}{\sqrt{1^2+1^2+2^2}}$,

$\frac{1}{\sqrt{1^2+1^2+2^2}}, \frac{2}{\sqrt{1^2+1^2+2^2}}$, that is, $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Similarly, the direction cosines of the other line are $\frac{\sqrt{3}}{5}, -\frac{\sqrt{6}}{5}, \frac{4}{5}$.

Thus, if θ is the angle between them,

$$\cos\theta = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{\sqrt{3}}{5}\right)+\left(\frac{1}{\sqrt{6}}\right)\left(\frac{-\sqrt{6}}{5}\right)+\left(\frac{2}{\sqrt{6}}\right)\left(\frac{4}{5}\right)$$

$$= \frac{1}{5\sqrt{6}}\left(8+\sqrt{3}-\sqrt{6}\right).$$

E10) Since $1(1)+2(-2)+3(1)=0$,
$1(4)-2(1)+1(-2)=0$, and
$1(4)+2(1)+3(-2)=0$,
the lines are mutually perpendicular.

E11) The equation of the plane passing through the first three points is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix} = 0$$

$\Rightarrow 5x - 7y + 11z + 4 = 0.$

Since $(-4, 4, 4)$ satisfies it, the 4 points are coplanar.

E12) The points $(2, 0, 0)$, $(0, -1, 0)$, $(0, 0, 5)$ lie on the plane.

Thus, its equation is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & 1 \end{vmatrix} = 0 \Rightarrow 5x - 10y + 2z = 10$$

$$\Rightarrow \frac{x}{2}+\frac{y}{(-1)}+\frac{z}{5}=1.$$

E13) The intercepts of the two planes on the axes are $-\frac{d}{a}, -\frac{d}{b}, -\frac{d}{c}$

and $-\frac{D}{A}, -\frac{D}{B}, -\frac{D}{C}$, respectively. Thus, the planes coincide

iff $-\frac{d}{a}=-\frac{D}{A}, -\frac{d}{b}=-\frac{D}{B}, -\frac{d}{c}=-\frac{D}{C}$, that is,

iff $\frac{a}{A}=\frac{b}{B}=\frac{c}{C}=\frac{d}{D}$, that is, iff a, b, c, d and A, B, C, D are proportional.

E14) The distance of $(2, 3, -5)$ from the XY-plane, $z = 0$, is $\frac{|-5|}{\sqrt{1^2}} = 5$.

Similarly, the distance of $(2, 3, -5)$ from $x = 0$ and $y = 0$ is 2 and 3, respectively. Its distance from $x+y+z = 1$ is

$$\frac{|2+3-5-1|}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}.$$

E15) We know that

$$\left(\frac{|a+b+c|}{\sqrt{3}}\right)^2+\left(\frac{|a-c|}{\sqrt{2}}\right)^2+\left(\frac{|a-2b+c|}{\sqrt{6}}\right)^2=9.$$

$\Rightarrow a^2 + b^2 + c^2 = 9.$

E16) The equation of the plane passing through the given line is
$(x \cos \alpha + y \cos \beta + z \cos \gamma - 1) + k(x + y - z) = 0,$ ......(20)
where $k \in \mathbf{R}$ is chosen so that (1, 2, 0) lies on the plane.

$\therefore (\cos \alpha + 2 \cos \beta - 1) + 3k = 0 \Rightarrow k = \frac{1}{3}(1 - \cos \alpha - 2 \cos \beta).$

Thus, the required equation is obtained by putting this value of k in (20).

E17) Any point on the line is (t, t, t). The line and plane will intersect if, for some $t \in \mathbf{R}$,

$t + 2t + 3t = 3 \Rightarrow t = \frac{1}{2}.$

Thus, the plane and line intersect in only one point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

E18) Any point on the first line is given by
$(t+1, 2t+3, 3t+5), t \in \mathbf{R}.$
Any point on the second line is given by
$(3k-1, 5k+4, 7k+9), k \in \mathbf{R}.$

The two lines will intersect if $t + 1 = 3k - 1$, $2t + 3 = 5k + 4$ and $3t + 5 = 7k + 9$ for some t and k in **R**.

On solving these equations we find that they are consistent, and k = 5 gives us the common point. Thus, the point of intersection is (14, 29, 44).

E19) Any plane parallel to $3x + 4y - 5z = 0$ is of the form
$3x + 4y - 5z + k = 0$, where $k \in \mathbf{R}$.
Since (1, 2, 3) lies on it, $3+8-15+k = 0 \Rightarrow k = 4$.
Thus, the required plane is $3x + 4y - 5z + 4 = 0$.

E20) If the angle is θ, then

$$\cos \theta = \frac{1(2) + 2(2) + 2(0)}{\sqrt{9}\,\sqrt{8}} = \frac{1}{\sqrt{2}}$$

$\therefore \theta = \pi/4.$

E21) If θ is the angle between the line and the plane, then $\frac{\pi}{2} - \theta$ is the angle between the line and the normal to the plane (see Fig. 15). Now, A, B, C are the direction ratios of the normal. Thus,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2 + B^2 + C^2}\,.\,\sqrt{\alpha^2 + \beta^2 + \gamma^2}}.$$

$$\Rightarrow \sin \theta = \frac{A\alpha + B\beta + C\gamma}{\sqrt{A^2 + B^2 + C^2}\,.\,\sqrt{\alpha^2 + \beta^2 + \gamma^2}}.$$

UNIT 5 THE SPHERE

Structure

5.1 INTRODUCTION

With this unit we start our discussion of three-dimensional objects. As the unit title suggests, we shall consider various aspects of a sphere here. A sphere is not new to you. When you were a child you must have played with balls. You must also have eaten several fruits like limes, oranges and watermelons. All these objects are spherical in shape. But all of them are not spheres from the point of view of analytical geometry.

In this unit you will see what a geometer calls a sphere. We shall also obtain the general equation of a sphere. Then we shall discuss linear and planar sections of a sphere. In particular, we shall consider the equations of tangent lines and planes to a sphere. Finally, you will see what the intersection of two spheres is and how many spheres can pass through a given circle.

Spheres are an integral part of the study of the structure of crystals of chemical compounds. You find their properties used by architects and engineers also. Thus, an analytical study of spheres is not merely to satisfy our mathematical curiosity.

A sphere is a particular case of an ellipsoid as you will see when you study Block 3. So, if you have grasped the contents of this unit, it will be of help to you while studying the next block. In other words, if you achieve the following objectives, it will be easier for you to understand the contents of Block 3.

Objectives

After studying this unit you should be able to

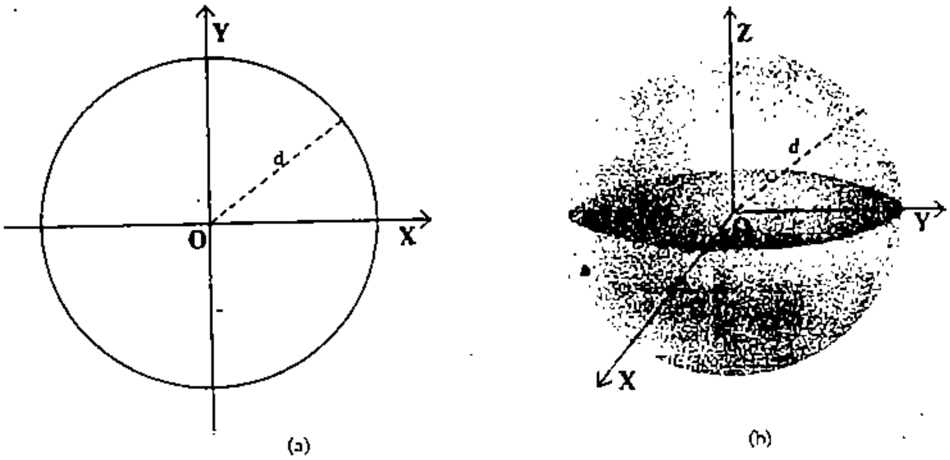
- obtain the equation of a sphere if you know its centre and radius ;
- check whether a given second degree equation in three variables represents a sphere;
- check whether a given line is tangent to a given sphere;
- obtain the tangent plane to a given point on a given sphere;
- obtain the angle of intersection of two intersecting spheres;
- find the family of spheres passing through a given circle.

Let us now see what a sphere is and how we can represent it algebraically.

5.2 EQUATIONS OF A SPHERE

In 2-space you know that the set of points that are at a fixed distance d from a fixed point is a circle. A sphere is a generalisation of this to 3-space (see Fig. 1).

Definition : The set of all those points in 3-space which are at a fixed distance d from a point C (a, b, c), is a **sphere** with **centre** C and **radius** d.



(a) (b)

Fig 1 : (a) A circle, (b) a sphere, with origin as centre and radius d.

Spheres are, of course, not new to you. A ball and a plum are spherical in shape. However, whenever we talk of a sphere in analytical geometry, we mean the surface of a sphere. Thus, for us a hollow ball is a sphere, and a solid cricket ball is not a sphere.

Let us find the equation of a sphere with radius d and centre C (a, b, c) now. If P (x, y, z) is any point on the sphere, then, by the distance formula ((1) of Unit 4), we get

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2, \qquad \ldots\ldots(1)$$

which is the required equation.

For example, the sphere with centre (0, 0, 0) and radius 1 unit is, $x^2+y^2+z^2=1$.

Now, if we expand (1), we get the second degree equation
$x^2+y^2+z^2-2ax-2by-2cz+a^2+b^2+c^2-d^2=0$.

Looking at this you could ask if every equation of the type

$$a(x^2+y^2+z^2)+2ux+2vy+2wz+d=0, \qquad \ldots\ldots(2)$$

where a, u, v, w, d ∈ **R**, represents a sphere.

It so happens that if a ≠ 0, then (2) represents a sphere. (What happens if a = 0? Unit 4 will give you the answer.)

Let us rewrite (2) as

$$x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z=-\frac{d}{a}.$$

Adding $\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}$ on either side of this equation, we obtain

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2}$$

Comparing this with (1), we see that this is a sphere with centre

$$\left(-\frac{u}{a},\ -\frac{v}{a},\ -\frac{w}{a}\right) \text{ and radius } \frac{\sqrt{u^2+v^2+w^2-ad}}{|a|}.$$

The following theorem summarises what we have said so far.

Theorem 1 : The general equation of a sphere is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$

Its centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$.

Note that the general equation given above will be a **real sphere** iff $u^2+v^2+w^2-d \geq 0$. Otherwise it will be an **imaginary sphere**, that is, a sphere with no real points on it.

So, what we have seen is that

> a second degree equation in x, y and z represents a sphere iff
>
> i) the coefficients of x^2, y^2 and z^2 are equal, and
>
> ii) the equation has no terms containing xy, yz or xz.

Why don't you see if you've taken in what has been said so far?

E1) Find the centre and radius of the sphere given by $x^2+y^2+z^2-2x+4y-6z=11$.

E2) Does $2x^2+1+2y^2+3+2z^2+5=0$ represent a sphere?

E3) Determine the centre and radius of the sphere $x^2+y^2+z^2=4z$.

Now, if you look at the general equation of a sphere, you will see that it has 4 arbitrary constants u, v, w, d. Thus, if we know 4 points lying on a sphere, then we can obtain its equation.

Let's consider an example.

Example 1 : Find the equation of the sphere through the points (0, 0, 0), (0, 1, −1), (−1, 2, 0) and (1, 2, 3).

Solution : Suppose the equation is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$

Since the 4 given points lie on it, their coordinates must satisfy this equation. So we get

$d=0$

$2+2v-2w+d=0$

$5-2u+4v+d=0$

$14+2u+4v+6w+d=0.$

Solving this system of simultaneous linear equations (see Block 2, MTE-04), we get

$$u=-\frac{15}{14},\ v=-\frac{25}{14},\ w=-\frac{11}{14},\ d=0.$$

Thus, the required sphere is

$7(x^2+y^2+z^2)-15x-25y-11z=0$

Note that it can happen that the system obtained by substituting the four points is inconsistent, that is, it does not have a solution. (Such a situation can occur if three of the points lie on one line.) In this case there will be no sphere passing through these points.

You can try some exercises now.

E4) Find the centre and radius of the sphere passing through (1, 0, 0), (0, 1, 0), (0, 0, 1) and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

E5) Is there a sphere passing through (4, 0, 1), (10, – 4, 9), (– 5, 6, – 11) and (1, 2, 3)? If so, find its equation.

A diameter of a sphere is a line segment through its centre and with end points on the sphere.

Now if, instead of four points on the sphere, we only know the coordinates of the two ends of one of its diameters, we can still determine its equation. Let us see how. Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be the ends of a diameter of a sphere (see Fig. 2). Then, if P (x, y, z) is any point on the sphere, PA and PB will be perpendicular to each other. Thus, from (10) of Unit 4, we see that

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0. \qquad \ldots\ldots(3)$$

This is satisfied by any point on the sphere, and hence is the equation of the sphere.

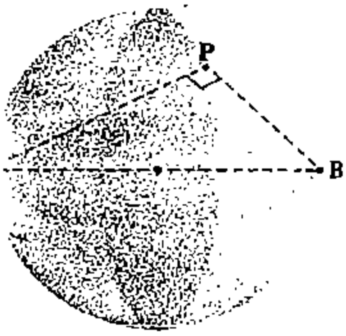


Fig. 2.

For example, the equation of the sphere having the points (–3, 5, 1) and (3, 1, 7) as the ends of a diameter is $(x + 3)(x - 3) + (y - 5)(y - 1) + (z - 1)(z - 7) = 0$,
that is, $x^2+y^2+z^2 = 6y + 8z - 3$.

You can obtain the diameter form of a sphere's equation in the following exercise.

E6) Find the equation of the sphere described on the join of (3, 4, 5) and (1, 2, 3).

By now you must be familiar with spheres. Let us now see when a line or a plane intersects a sphere.

5.3 TANGENT LINES AND PLANES

In this section we shall first see how many common points a line and a sphere can have. Then we shall do the same for a plane and a sphere.

5.3.1 Tangent Lines

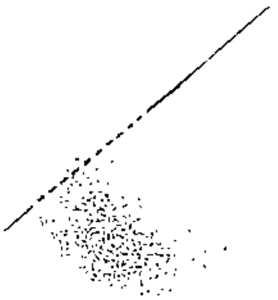


Fig. 3 : A line intersecting a sphere

Suppose you take a hollow ball and pierce it right through with a knitting needle. Then the ball and the needle will have two points in common (see Fig. 3). Do you think this is true of any line that intersects a sphere? See what the following theorem has to say about this.

Theorem 2: A line and a sphere can intersect in at most two points.

Proof: Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $\frac{x - a}{\alpha} = \frac{y - b}{\beta} = \frac{z - c}{\gamma} = t$ (say) be a given sphere and line, respectively. Then any point on the line is of the form $(\alpha t + a, \beta t + b, \gamma t + c)$, where $t \in R$. If this lies on the sphere, then

$$(\alpha t + a)^2 + (\beta t + b)^2 + (\gamma t + c)^2 + 2u(\alpha t + a) + 2v(\beta t + b) + 2w(\gamma t + c) + d = 0$$

$$\Rightarrow (\alpha^2 + \beta^2 + \gamma^2)t^2 + 2t(a\alpha + b\beta + c\gamma + u\alpha + v\beta + w\gamma) + (a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d) = 0 \qquad \ldots\ldots(4)$$

This is a quadratic in t. Thus, it gives two values of t. For each value of t, we will get a point of intersection. Thus, the line and sphere can intersect in at most two points.

Note that (4) can have real distinct roots, real coincident roots or distinct imaginary roots. Accordingly, the line will intersect the sphere in two points, in one point, or not at all. This leads us to the following definitions.

Definitions: If a line intersects a sphere in two distinct points, it is called a **secant line** to the sphere.

If a line intersects a sphere in one point P, it is called a **tangent** to the sphere at the point P; and P is called the **point of contact** of the tangent.

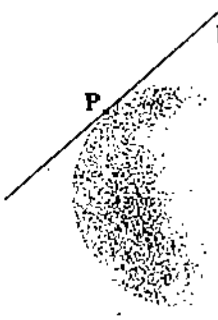


Fig. 4 : L intersects the sphere in only one point, P.

For example, the line L in Fig. 3 is a secant line to the sphere; and the line L in Fig. 4 is a tangent to the sphere at the point P.

Now, (4) will have coincident roots iff

$(a\alpha + b\beta + c\gamma + u\alpha + v\beta + w\gamma)^2 = (\alpha^2 + \beta^2 + \gamma^2)(a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d)$ ......(5)

Thus, (5) is the condition for $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ to be a tangent to

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$

Let us consider an example.

Example 2: Find the intercept made by the sphere $x^2 + y^2 + z^2 = 9$ on the line $x - 3 = y = z$.

Solution : Any point on the line is of the form $(t + 3, t, t)$, where $t \in$ **R**. This lies on the sphere if

$(t + 3)^2 + t^2 + t^2 = 9 \Rightarrow 3t^2 + 6t = 0 \Rightarrow t = 0, -2.$

Thus, the points of intersection are (3, 0, 0) and (1, –2, –2). Thus, the intercept is the distance between the two points, which is $\sqrt{4 + 4 + 4} = 2\sqrt{3}$.

You can try some exercises now.

E7) Check if $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z}{5}$ is a tangent to the sphere
$x^2 + y^2 + z^2 + 4x + 6y + 10z = 0.$

E8) If we extend the rule of thumb to find the tangent to a conic (see Unit 3) to a sphere, will we get the equation of a tangent line to a sphere? Why?

Let us now discuss the intersection of a plane and a sphere.

5.3.2 Tangent Planes

Consider a sphere and a plane that intersects it. What do you expect the intersection to be? The following result will give you the answer.

Theorem 3: A planar section of a sphere is a circle.

Proof: Let S be a sphere with radius r and centre O (see Fig. 5), and let the plane Π intersect it.

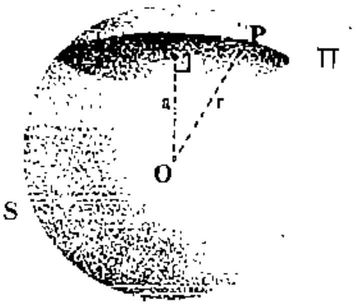


Fig. 5: A planar intersection of a sphere is a circle.

Drop a perpendicular ON from O onto Π, and let ON = a. Now let P be a point which belongs to Π as well as S. Then OP = r and $OP^2 = ON^2 + NP^2$.

Thus, $NP = \sqrt{r^2 - a^2}$, which is a constant.

Thus, the intersection of S and Π is the set of points in Π which are at a fixed distance from a fixed point N. Thus, it is a circle in the plane Π with centre N and radius $\sqrt{r^2 - a^2}$.

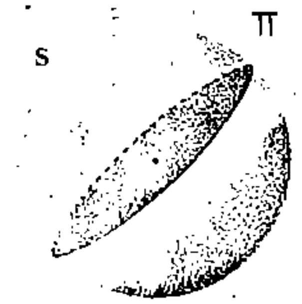


Fig. 6: The intersection of S and Π is a great circle of the sphere S.

If a = 0 in the proof above, the plane passes through the centre of the sphere. In this case the circle of intersection is of radius r and is called a **great circle** (see Fig. 6) of the sphere.

Note that a sphere has infinitely many great circles, one for each plane that passes through the centre of the sphere.

We have seen that the planar section of a sphere is a circle. Now let us find its equation. Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, and that of the plane intersecting it be $Ax + By + Cz + D = 0$. Then the equation of the planar section can be written as

$$\left.\begin{array}{l} x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 = Ax + By + Cz + D, \text{ or} \\ x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0, Ax + By + Cz + D = 0. \end{array}\right] \quad \ldots\ldots(6)$$

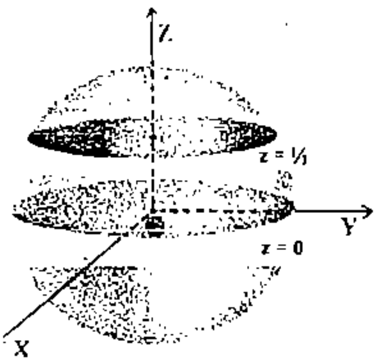


Fig. 7: Planar sections of $x^2 + y^2 + z^2 = 1$

For example, the equation of the planar section of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \frac{1}{2}$ (see Fig. 7) is $x^2 + y^2 + z^2 - 1 = z - \frac{1}{2} = 0$. This is the circle $x^2 + y^2 = \frac{3}{4}$, in the plane $z = \frac{1}{2}$.

Since the centre of the given sphere is (0, 0, 0), we can get a great circle of the sphere by intersecting it with $z = 0$. Thus, one great circle is $x^2 + y^2 = 1$ in the plane $z = 0$.

Let us consider an example of the use of Theorem 3.

Example 3: Find the centre and radius of the circle

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 2z = 3$.

Solution: The centre of the sphere is C(4, −2, −4) and its radius is

$r = \sqrt{16 + 4 + 16 + 45} = 9$.

The distance of the plane from the centre of the sphere is

$$d = \frac{|4 + 4 - 8 - 3|}{\sqrt{1 + 4 + 4}} = 1.$$

Thus, the radius of the circle $= \sqrt{r^2 - d^2} = 4\sqrt{5}$.

The centre of the circle is the foot of the perpendicular from C onto the plane. To find this, we first need to find the equations of the perpendicular. Its direction ratios are 1, −2, 2. Thus, its equations are

$$\frac{x - 4}{1} = \frac{y + 2}{-2} = \frac{z + 4}{2}.$$

Therefore, any point on the perpendicular is given by $(t + 4, -2t - 2, 2t - 4)$, where $t \in \mathbb{R}$. This point will be the required centre of the circle if it lies on the plane, that is, if

$$(t + 4) - 2(-2t - 2) + 2(2t - 4) = 3 \Rightarrow t = \frac{1}{3}.$$

Hence, the centre of the circle is $\left(\frac{13}{3}, -\frac{8}{3}, -\frac{10}{3}\right)$.

You can do the following exercise on the same lines.

E9) Find the centre and radius of the circle
$x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0 = 2x + 2y + z - 17.$

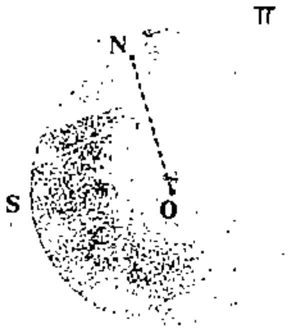


Fig. 8: The plane Π is tangen[illegible] the sphere S at the point N.

Now, if we take a = r in the proof of Theorem 3, then what happens to the circle of intersection? It reduces to a single point, that is, a **point circle**. And in this case the plane only touches the sphere (see Fig. 8).

Definition: A plane is **tangent** to a sphere at a point P if it intersects the sphere in P only. In this case we also say that the plane **touches** the sphere at P. P is called the **point of tangency**, or the **point of contact**, of the tangent plane.

Remark 1: If you go back to the proof of Theorem 3, you will see that the line joining the point of tangency to the centre of the sphere is perpendicular to the tangent plane. We will use this fact to obtain the equation of a tangent plane.

Let us find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point P (a, b, c).

As P lies on the sphere,

$a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d = 0.$ ......(7)

Also, the centre of the sphere is C (−u, −v, −w). Thus, the direction ratios of CP are a + u, b + v, c + w (see Equation (8) of Unit 4).

Now, the tangent plane passes through P (a, b, c). Thus, its equation will be

$f(x - a) + g(y - b) + h(z - c) = 0$, for some $f, g, h \in \mathbf{R}$. ......(8)

Now CP is perpendicular to (8), and hence, is parallel to the normal to (8). Further, f, g, h are direction ratios of the normal to the plane. Therefore, a + u, b + v, c + w and f, g, h are proportional.

$$\therefore \frac{f}{a+u} = \frac{g}{b+v} = \frac{h}{c+w} = t, \text{ say.}$$

Then (8) gives us

$(x - a)(a + u) + (y - b)(b + v) + (z - c)(c + w) = 0.$

$\Rightarrow xa + yb + zc + ux + vy + wz = a^2 + b^2 + c^2 + ua + vb + wc.$ ......(9)

Using (7) and (9), we get

$xa + yb + zc + ux + vy + wz = -ua - vb - wc - d.$

Thus,

> the equation of the tangent plane at the point (a, b, c) to the sphere
> $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is
> $xa + yb + zc + u(x + a) + v(y + b) + w(z + c) + d = 0.$

Is this the equation you got while doing E8? From the equation you may have realised that there is a similar thumb rule for the tangent plane (and not tangent line!) to a sphere.

Rule of Thumb: To obtain the equation of the tangent plane to a sphere at the point (a, b, c), simply substitute ax for x^2, by for y^2, cz for z^2; and in the linear terms substitute $\frac{x+a}{2}$ for x, $\frac{y+b}{2}$ for y, $\frac{z+c}{2}$ for z in the equation of the sphere.

For example, the equation of the tangent plane to $x^2 + y^2 + z^2 = a^2$ at (α, β, γ) is $x\alpha + y\beta + z\gamma = a^2$.

Why don't you try an exercise now?

E10) Find the equations of the tangent planes

a) to $x^2 + y^2 + z^2 + 2z = 29$ at $(2, 3, -4)$.

b) to $x^2 + y^2 + z^2 - 4y - 6z + 4 = 0$ at $(2, 3, 1)$.

So, what we have seen so far is that if a plane is at a distance d from the centre of a sphere with radius r, then

i) if $r < d$, the plane and sphere do not intersect;

ii) if $r = d$, the plane is tangent to the sphere; and

iii) if $r > d$, the plane intersects the sphere in a circle of radius $\sqrt{r^2 - d^2}$.

Now, if you are given the equations of a sphere and a plane, can you tell if the plane is tangent to the sphere? An obvious way would be to check what the distance of the centre of the sphere from the plane is. Let us use this method to derive the condition for the plane is. Let us use this method to derive the the condition for the plane $Ax + By + Cz + D = 0$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Now, the radius of the sphere is $\sqrt{u^2 + v^2 + w^2 - d}$.

The length of the perpendicular to the plane from the centre $(-u, -v, -w)$ of the sphere is

$$\frac{|Au + Bv + Cw + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

This distance must equal the sphere's radius since the plane is tangent to the sphere.

$$\therefore (Au + Bv + Cw + D)^2 = (A^2 + B^2 + C^2)(u^2 + v^2 + w^2 - d), \qquad \ldots\ldots(10)$$

which is the required condition.

Let us consider an example.

Example 4: Show that $2x - y - 2z = 16$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y + 2z - 3 = 0$, and find the point of contact.

Solution: The centre of the sphere is $(2, -1, -1)$ and its radius is $\sqrt{2^2 + 1^2 + 1^2 + 3} = 3$.

The length of the perpendicular from the centre to the plane $2x - y - 2z - 16 = 0$ is

$$\frac{|2.2 + 1 + 2 - 16|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{9}{3} = 3,$$ which is the same as the radius of the sphere.

So the plane touches the sphere.

Let (x_1, y_1, z_1) be the point of contact. Then the equation of the tangent plane is

$$xx_1 + yy_1 + zz_1 - 2(x + x_1) + (y + y_1) + (z + z_1) - 3 = 0$$

$$\Leftrightarrow (x_1 - 2)x + (y_1 + 1)y + (z_1 + 1)z - 2x_1 + y_1 + z_1 - 3 = 0.$$

But this should be the same as the given plane $2x - y - 2z - 16 = 0$.

So the coefficients of x, y, z and the constant term in both these equations must be proportional.

$$\therefore \frac{x_1-2}{2}=\frac{y_1+1}{-1}=\frac{z_1+1}{-2}=\frac{2x_1-y_1-z_1+3}{16}$$

$\Rightarrow x_1=-2y_1,\ z_1=1+2y_1$, and then

$$\frac{y_1+1}{-1}=\frac{2x_1-y_1-z_1+3}{16}=\frac{-7y_1+2}{16}\Rightarrow 9y_1=-18\Rightarrow y_1=-2.$$

$\therefore x_1=4$ and $z_1=-3$.

Thus, the point of contact is $(4, -2, -3)$.

Using the same method, if we are given a point and a plane, we can find the sphere with the point as the centre and the plane as a tangent plane. In the example below we illustrate this.

Example 5: Find the sphere with centre $(-1, 2, 3)$, and which touches the plane $2x - y + 2z = 6$.

Solution: The distance of the plane from the point is

$$\frac{|-2-2+6-6|}{\sqrt{2^2+1^2+2^2}}=\frac{4}{3}.$$

This should be the radius of the sphere.

So, since the centre of the sphere is $(-1, 2, 3)$, its equation will be

$$(x+1)^2+(y-2)^2+(z-3)^2=\frac{16}{9}.$$

Why don't you do some exercises now?

E11) Show that the plane $x + y + z = \sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 = 1$. Find the point of contact.

E12) Show that the equation of the sphere which lies in the octant OXYZ and touches the coordinate planes is $x^2 + y^2 + z^2 - 2k(x + y + z) + 2k^2 = 0$, for some $k \in \mathbb{R}$.

E13) Find the equation of the sphere with centre $(1, 0, 0)$, and which touches the plane $2x + y + z - 3 = 0$.

In this section we have seen what sets can be got by intersecting a line or a plane with a sphere. Now let us discuss what form the intersection of two or more spheres can take.

5.4 INTERSECTION OF SPHERES

In this section you will first see that the result of intersecting two spheres is the same as that obtained by intersecting a sphere and a plane, that is, a circle. And then you will see how to obtain infinitely many spheres whose intersection is a given circle.

5.4.1 Two Intersecting Spheres

Let us consider two spheres given by

$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$, and

$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$.

Then each point that satisfies $S_1 = 0$ as well as $S_2 = 0$ will also satisfy the equation $S_1 - S_2 = 0$, that is,

$$2(u_1-u_2)x + 2(v_1-v_2)y + 2(w_1-w_2)z + d_1 - d_2 = 0. \qquad \ldots\ldots(11)$$

But, from Theorem 1 of Unit 4 you know this is a plane. Thus, the points of intersection of the spheres $S_1 = 0$ and $S_2 = 0$ are the same as those of any one of the spheres and the plane (11). Since the intersection of a plane and a sphere is a circle, the intersection of $S_1 = 0$ and $S_2 = 0$ is a circle (see Fig. 9).

Fig. 9: Two spheres intersect in a circle which can be (a) a real circle, (b) a point circle, or (c) an imaginary circle.

In Fig. 9 (a), the circle of intersection has positive radius, while in Fig. 9 (b) the two spheres intersect in only a point. In Fig. 9 (c), they don't intersect at all. While studying the motion of rigid bodies, you may come across a situation in which two spheres just touch each other, as in Fig. 9 (b).

Remark 2: If $S_1 = 0$ and $S_2 = 0$ are any two spheres, which do not necessarily intersect, then $S_1 - S_2 = 0$ is a plane, and is called the **radical plane** of the spheres.

Now, when we discussed the intersection of lines or of planes, we spoke about the angle of intersection. Is it meaningful to talk about the angle of intersection of two spheres?

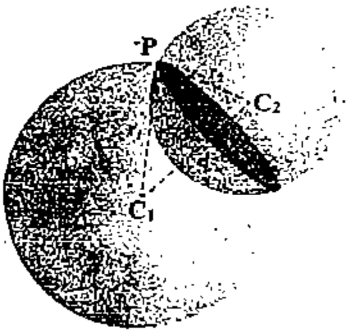


Fig. 10: θ is the angle between the two spheres.

Definition: The **angle of intersection of two spheres** is defined to be the angle between the tangent planes to these spheres at a point of contact.

You may wonder if the definition given above is 'proper'— the angle could vary from one point of contact to another. But, you will now see that the angle is independent of the point of contact.

Let the spheres be $S_1 = 0$ and $S_2 = 0$, where

$S_i \equiv x^2 + y^2 + z^2 + 2u_i x + 2v_i y + 2w_i z + d_i = 0$, where i = 1, 2.

Then, their radii are $r_1 = \sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}$ and $r_2 = \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}$, respectively.

Let d be the distance between their centres C_1 and C_2 (see Fig. 10). Let P be a point of intersection. Then the angle between the spheres is the angle between the tangent planes at P to each of the spheres. This, in turn, is the angle between the normals to these planes, which are PC_1 and PC_2. Thus, if θ is the required angle, then from elementary trigonometry you know that

$$2PC_1 . PC_2 \cos\theta - PC_1^2 + PC_2^2 \quad C_1C_2^2$$

$$\therefore \cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \qquad \text{......(12)}$$

Two spheres are called **orthogonal** if their angle of intersection is π/2.

Thus, in particular, the spheres will be orthogonal iff $r_1^2 + r_2^2 = d^2$

$$\Rightarrow (u_1^2 + v_1^2 + w_1^2 - d_1) + (u_2^2 + v_2^2 + w_2^2 - d_2) = (-u_1 + u_2)^2 + (-v_1 + v_2)^2 + (-w_1 + w_2)^2$$

$$\Rightarrow 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2 \qquad \text{......(13)}$$

Thus, the spheres $S_1 = 0$ and $S_2 = 0$ intersect at 90° iff (13) is satisfied.

Let us consider an example.

Example 6: Find the angle between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 - 2x = 0$.

Solution : Here $u_1 = 0$, $v_1 = 0$, $w_1 = 0$, $d_1 = -4$, $u_2 = -1$, $v_2 = 0$, $w_2 = 0$, $d_2 = 0$.

Thus, the centres of the two spheres are (0, 0, 0) and (1, 0, 0), their radii are 2 and 1, respectively, and the distance between their centres is 1.

Therefore, by (12), the angle between the two spheres is

$$\cos^{-1}\left(\frac{2^2 + 1^2 - 1^2}{2(2)\,(1)}\right) = \cos^{-1}(1) = 0.$$

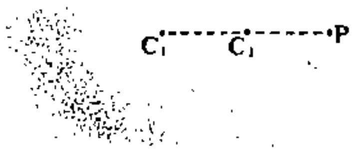


Fig. 11

You can see these spheres in Fig. 11. They intersect in only one point P, and the x-axis is the normal from the centres of the spheres to both tangent planes.

You can try some exercises on intersecting spheres now.

E14) Find the angle of intersection of the spheres $x^2 + y^2 + z^2 - 2x + 2y - 4z + 2 = 0$ and $x^2 + y^2 + z^2 = 4$.

E15) Find the equation of the sphere touching the plane $3x + 2y - z + 2 = 0$ at P (1, –2, 1) and cutting the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ orthogonally.

E16) a) Two spheres of radius r_1 and r_2 and with centres C_1 and C_2, respectively, will touch each other iff $r_1 + r_2 = C_1C_2$. True or false? Why?

b) Under what conditions on r_1, r_2 and C_1C_2 will the spheres not intersect?

E17) Show that the spheres $x^2 + y^2 + z^2 - 2x - 4y - 4z = 0$ and $x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$ touch each other. What is the point of contact?

So far you have seen that two spheres intersect in a circle. Now let us see whether, given a circle, we can find two or more spheres passing through it.

5.4.2 Spheres Through a Given Circle

Suppose we are given a circle. Can we find two distinct spheres whose intersection the circle is? In fact, we can construct many spheres passing through a given circle (see Fig. 12).

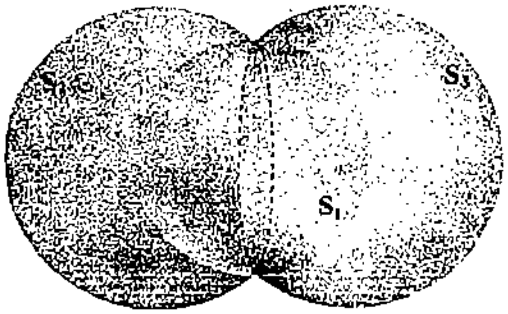


Fig. 12: Part of a family of spheres through a circle.

In Fig. 12, the circle is a great circle of the sphere S_1, but not of S_2, S_3, etc. Let us see what the method of construction of this kind of family is.

You know that a circle is the intersection of a sphere and a plane. So its equation is of the form $S = 0$, $\Pi = 0$, where

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$, and $\Pi \equiv Ax + By + Cz + D$.

Any sphere through this circle will be given by

$S + k\Pi = 0,$ ......(14)

where k is an arbitrary constant. Do you agree? Now, if you apply Theorem 1, you can see that (14) represents a sphere.

Further, every point that lies on the circle must satisfy (14). Thus, (14) represents a sphere through the given circle.

So, for each value of $k \in \mathbf{R}$ in (14) we get a distinct sphere passing through the given circle. Thus, we have infinitely many spheres that intersect in the given circle.

Now, a circle can also be represented as the intersection of two spheres $S_1 = 0$ and $S_2 = 0$. In this case what will the equation of any sphere containing it be? It will be $S_1 + kS_2 = 0$, where $k \in \mathbf{R}$. Thus, the infinite set $\{S_1 + kS_2 = 0 \mid k \in \mathbf{R}\}$ gives us the family of spheres passing through the given circle.

Let us consider some examples of the use of (14).

Example 7: Show that the circles $x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z - 2 = 0$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$, $2x - y + 4z - 1 = 0$ lie on a sphere, and find its equation.

Solution: The equation of any sphere through the first circle is

$x^2 + y^2 + z^2 - y + 2z + k(x - y + z - 2) = 0$, that is,

$x^2 + y^2 + z^2 + kx - (k + 1)y + (k + 2)z - 2k = 0,$ ......(15)

for some $k \in \mathbf{R}$.

Similarly, the equation of any sphere through the second circle is

$x^2 + y^2 + z^2 + (2k_1 + 1)x - (k_1 + 3)y + (4k_1 + 1)z - (k_1 + 5) = 0,$ ......(16)

for some $k_1 \in \mathbf{R}$.

To get a common sphere containing both circles, we must see if (15) and (16) coincide for some k and k_1 in R. Comparing the coefficients of x, y and z, and the constant terms in (15) and (16), we get

$k = 2k_1 + 1$, $k + 1 = k_1 + 3$, $k + 2 = 4k_1 + 1$, $2k = k_1 + 5$.

These equations are satisfied for $k = 3$ and $k_1 = 1$.

Thus, there is a sphere passing through both the circles and its equation is

$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0.$

Example 8: Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the origin.

Solution: Let the equation of the sphere be $x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0$, where $k \in \mathbf{R}$.

Since, it passes through (0, 0, 0), we get $-9 - 5k = 0$, that is, $k = -\frac{9}{5}$.

Thus, the required equation is

$5(x^2 + y^2 + z^2) = 9(2x + 3y + 4z).$

Example 9: Find the path traced by the centre of a sphere which touches the lines $y = x$, $z = 1$ and $y = -x$, $z = -1$.

Solution: Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation of a sphere that touches the two lines. Since $y = x$, $z = 1$ touches it, the intersection of the line and the sphere must be only one point. Any point on the line is $(t, t, 1)$, where $t \in \mathbf{R}$. It lies on the sphere if

$t^2 + t^2 + 1 + 2ut + 2vt + 2w + d = 0.$

This equation has equal roots if

$(u + v)^2 = 2(1 + 2w + d)$.

Similarly, since $y = -x$, $z = -1$ touches the sphere, we get

$(u - v)^2 = 2 (1 - 2w + d)$.

Subtracting these two conditions, we get

$4uv = 4w (1 + 1)$, that is, $uv = 2w$.

Thus, the centre of the sphere, $(-u, -v, -w)$, satisfies the equation $xy + 2z = 0$.

This is true for any sphere satisfying the given conditions. Thus, the required path is $xy + 2z = 0$.

Now, why don't you check if you've understood what we have done in this section so far?

E18) Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0$ and

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, x + 2y - 7z = 0$

lie on the same sphere. Find its equation also.

E19) Find the equations of the spheres that pass through $x^2 + y^2 + z^2 = 5$, $2x + y + 3z = 3$ and touch the plane $3x + 4y = 15$.

E20) Find the equation of the sphere for which the circle $2x - 3y + 4z = 8$, $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$ is a great circle.

We will stop our discussion on spheres for now, though we shall refer to them off and on in the next block. Let us now do a quick review of what we have covered in this unit.

5.5 SUMMARY

In this unit we have covered the following points.

1) The second degree equation
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$.
Conversely, the equation of any sphere is of this form.

2) A line intersects a sphere in at most two points. It is a tangent to the sphere if it intersects the sphere in only one point.

3) A plane intersects a sphere in a circle. When this circle reduces to a point circle P, then the plane is tangent to the sphere at P.

4) The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (x_1, y_1, z_1) is $xx_1 + yy_1 + zz_1 + u (x + x_1) + v (y + y_1) + w (z + z_1) + d = 0$. This is perpendicular to the line joining (x_1, y_1, z_1) to the centre of the sphere.

5) Two spheres intersect in a circle.

6) The angle of intersection of the two intersecting spheres $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ is $\cos^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$,

where r_1 and r_2 are their radii and d is the distance between their centres.

In particular, the two spheres are orthogonal if $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$.

7) There are infinitely many spheres that pass through a given circle.

You may now like to go back to Sec. 5.1 and go through the list of **unit objectives** to see if you have achieved them. If you want to see what our solutions to the exercises in the unit are, we have given them in the following section.

5.6 SOLUTIONS/ANSWERS

E1) Its centre is $\left(-\left(\frac{-2}{2}\right), -\left(\frac{4}{2}\right), -\left(\frac{-6}{2}\right)\right) = (1, -2, 3)$.

Its radius is $\sqrt{(-1)^2 + 2^2 + (-3)^2 - (-11)} = 5$.

E2) We can rewrite the equation as $x^2 + y^2 + z^2 + \frac{9}{2} = 0$.

This represents an imaginary sphere with centre at the origin.

E3) Its centre is (0, 0, 2) and radius is 4.

E4) Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. Then, since the given points lie on it, we get
$1 + 2u + d = 0$
$1 + 2v + d = 0$
$1 + 2w + d = 0$
$1 + \frac{2}{\sqrt{3}}(u + v + w) + d = 0$
On solving these equations, we find that $u = v = w = 0$, $d = -1$.
Thus, the centre of the sphere is (0, 0, 0) and radius is 1.

E5) Suppose such a sphere exists. Let its equation be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Since the given points lie on it, we get the linear system
$17 + 8u + 2w + d = 0$
$197 + 20u - 8v + 18w + d = 0$
$192 - 10u + 12v - 22w + d = 0$
$14 + 2u + 4v + 6w + d = 0$.
You can check that this system is inconsistent. Thus, the points do not lie on a sphere.

E6) The required equation is
$(x - 3)(x - 1) + (y - 4)(y - 2) + (z - 5)(z - 3) = 0$.
$\Leftrightarrow x^2 + y^2 + z^2 - 4x - 6y - 8z + 26 = 0$.

E7) Any point on the line is $(4t - 3, 3t - 4, 5t)$, where $t \in \mathbb{R}$. This will lie on the sphere if
$(4t - 3)^2 + (3t - 4)^2 + 25t^2 + 4(4t - 3) + 6(3t - 4) + 10(5t) = 0$.
$\Leftrightarrow 50t^2 + 36t - 11 = 0$

$$\Leftrightarrow t = \frac{-36 \pm \sqrt{(36)^2 + 2200}}{100}.$$

Since these are real distinct roots, the line will intersect the sphere in two distinct points. Hence, it will not be a tangent to the sphere.

E8) If we extend the rule of thumb to obtain the tangent at a point $P(x_1, y_1, z_1)$ on the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get
$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$.
This is a linear equation, and hence represents a plane, and not a line. Thus, it cannot represent the tangent line.

E9) The centre of the sphere is C (– 6, 6, 8).
Its radius is r = 5.
The distance of C from the plane is d = 3.
Thus, the radius of the circle = $\sqrt{r^2 - d^2} = 4$.

The equations of the perpendicular from C onto the plane are $\frac{x+6}{2} = \frac{y-6}{2} = z - 8$.

Thus, the centre of the circle is (– 4, 8, 9).

E10) a) $2x + 3y - 4z + (z - 4) = 29 \Leftrightarrow 2x + 3y - 3z - 33 = 0$
b) $2x + 3y + z - 2(y + 3) - 3(z + 1) + 4 = 0$
$\Leftrightarrow 2x + y - 2z - 5 = 0.$

E11) The radius of the sphere = 1.
The distance of the plane from the centre (0, 0, 0) of the sphere = 1.
Thus, the plane is tangent to the sphere.
If the point of contact is (a, b, c), then the equation of the plane is $ax + by + cz - 1 = 0$, as well as $x + y + z - \sqrt{3} = 0$.

$\therefore \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \frac{1}{\sqrt{3}}$.

Thus, the point of contact is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

E12) Let the equation be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Since the plane x = 0 is a tangent to it, the distance of (–u, –v, –w) from x = 0 must be $\sqrt{u^2 + v^2 + w^2 - d} = r$, say.
$\therefore -u = r$.
(Note that $|-u| = -u$, since the centre lies in the octant in which the x, y and z coordinates are all positive.)
Similarly, $-v = -w = r$.
Then $u^2 + v^2 + w^2 - d = r^2 \Rightarrow d = 2r^2$.
Thus, the equation of the sphere is
$x^2 + y^2 + z^2 - 2r(x + y + z) + 2r^2 = 0$.

E13) Its radius should be $\frac{|2-3|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$.

Thus, its equation is

$(x - 1)^2 + y^2 + z^2 = \frac{1}{6}$.

$\Leftrightarrow 6(x^2 + y^2 + z^2) - 12x + 5 = 0.$

E14) Their centres are C_1 (1, –1, 2) and C_2 (0, 0, 0), respectively.
Both their radii are 2, and $C_1C_2^2 = 6$.
Thus, the angle of intersection is

$$\cos^{-1}\left(\frac{4+4-6}{2(2)(2)}\right) = \cos^{-1}\left(\frac{1}{4}\right).$$

E15) Let the sphere be given by
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Then the plane $3x + 2y - z + 2 = 0$ is the same as
$x - 2y + z + u(x + 1) + v(y - 2) + w(z + 1) + d = 0$
$\Leftrightarrow x(1 + u) + y(v - 2) + z(w + 1) + u - 2v + w + d = 0$

$$\therefore \frac{1+u}{3} = \frac{v-2}{2} = \frac{w+1}{-1} = \frac{u-2v+w+d}{2}$$

$$\therefore\ v = \frac{2u+8}{3},\ w = \frac{-u-4}{3},\ d = \frac{4u+22}{3}.$$

Further, this sphere cuts
$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ orthogonally.
Thus, using (13) we see that
$-4u + 6v = d + 4$;

Substituting the values of v and d, we get $u = \frac{7}{2}$. And then $v = 5$, $w = -\frac{5}{2}$, $d = 12$.

Thus, the required sphere is $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$.

E16) a) This is true only if one sphere doesn't lie inside the other. Otherwise, as in Fig. 11, $C_1C_2 \neq r_1 + r_2$.

b) If one lies outside the other and $r_1 + r_2 > C_1C_2$, then they won't intersect.
If one lies inside the other and $|r_1 - r_2| > C_1C_2$, they won't intersect.

E17) Their centres are C_1 (1, 2, 2) and C_2 (–5, 0, –1).
$\therefore C_1C_2 = 7$ = sum of their radii.
Thus, they touch each other.
The plane $S_1 - S_2 = 0$ is the common tangent plane, where $S_1 = 0$ and $S_2 = 0$ are the two spheres.
This will be $6x + 2y + 3z + 5 = 0$.
The point of contact will be the intersection of the line C_1C_2 with this plane. Now, C_1C_2 is given by $\frac{x+5}{6} = \frac{y}{2} = \frac{z+1}{3}$. Any point on this is $(6t - 5, 2t, 3t - 1)$. This lies

on the tangent plane if $6(6t - 5) + 2(2t) + 3(3t - 1) + 5 = 0 \Rightarrow t = \frac{4}{7}$.

Thus, the point of contact is $\left(\frac{-11}{7}, \frac{8}{7}, \frac{5}{7}\right)$.

E18) Solving this on the lines of Example 7, you can check that they lie on the sphere
$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

E19) Any such sphere is given by $x^2 + y^2 + z^2 - 5 + k(2x + y + 3z - 3) = 0$, where $k \in \mathbf{R}$.

Its centre is $\left(-k, -\frac{k}{2}, -\frac{3k}{2}\right)$. Its distance from $3x + 4y = 15$ is the same as the radius of the sphere.

$$\therefore k^2 + \frac{k^2}{4} + \frac{9k^2}{4} + (3k + 5) = (k + 3)^2$$

$$\Rightarrow k = 2, -\frac{4}{5}.$$

Thus, the two spheres that satisfy the given conditions are
$x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0$ and $5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$.

E20) Any such sphere will be given by
$x^2 + y^2 + z^2 + 7y - 2z + 2 + k(2x - 3y + 4z - 8) = 0$, where $k \in \mathbf{R}$.
Since the given circle is a great circle of the sphere, the centre of the sphere must lie on the plane $2x - 3y + 4z = 8$.

$$\therefore 2(-k) - 3\left(\frac{3k-7}{3}\right) + 4(1 - 2k) = 8.$$

$$\Rightarrow k = \frac{13}{29}.$$

Thus, the equation of the sphere is

$$x^2 + y^2 + z^2 + \frac{26}{29}x + \frac{164}{29}y - \frac{6}{29}z - \frac{46}{29} = 0$$

$$\Rightarrow 29(x^2 + y^2 + z^2) + 26x + 164y - 6z - 46 = 0.$$

UNIT 6 CONES AND CYLINDERS

Structure

6.1 INTRODUCTION

In the previous unit we discussed a very commonly found three-dimensional object. In this unit we look at two more commonly found three-dimensional objects, namely, a cone and a cylinder. But, what you will see in this unit may surprise you—what people usually call a cone or a cylinder are only portions of very particular cases of what mathematicians refer to as a cone or a cylinder.

We shall start our discussion on cones by defining them, and deriving their equations. Then we shall concentrate on cones whose vertices are the origin. In particular, we will obtain the tangent planes of such cones.

The other surface that we will discuss in this unit is a cylinder. We shall define a general cylinder, and then focus on a right circular cylinder.

The contents in this unit are of mathematical interest, of course. But, they are also of interest to astronomers, physicists, engineers and architects, among others. This is because of the many applications that cones and cylinders have in various fields of science and engineering.

The surfaces that you will study in this unit are particular cases of conicoids, which you will study in the next block. So if you go through this unit carefully and ensure that you achieve the following objectives, you will find the next block easier to understand.

Objectives

After studying this unit you should be able to

- obtain the equation of a cone if you know its vertex and base curve;
- prove and use the fact that a second degree equation in 3 variables represents a cone with vertex at the origin if it is homogeneous;
- obtain the tangent planes to a cone;
- obtain the equation of a right circular cylinder if you know its axis and base curve.

Let us now see what a cone is.

6.2 CONES

When you see an ice-cream cone, do you ever think that it is a set of lines? That is exactly what it is, as you will see in this section.

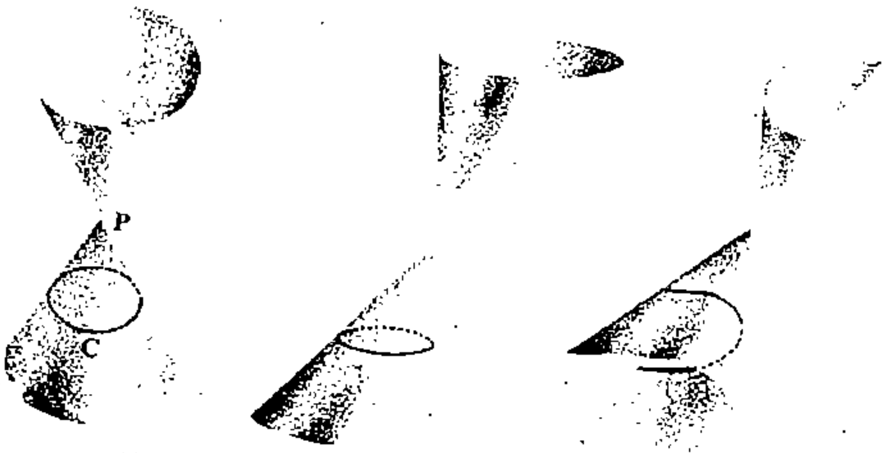
Definition: A cone is a set of lines that intersect a given curve and pass through a fixed point which is not in the plane of the curve. The fixed point is called the **vertex** of the cone, and the curve is called the **base curve** (or **directrix**) of the cone.

Each line that makes up a cone is called a **generator** of the cone.

Thus, we can also define a cone in the following way.

Definition: A cone is a surface generated by a line that intersects a given curve and passes through a fixed point which doesn't lie in the plane of the curve.

For example, in Fig. 1 (a), we give the cone generated by a line passing through the point P, and intersecting the circle C. The base curves of the cones in Fig. 1 (b) and Fig. 1 (c) are an ellipse and a parabola, respectively.



(a) (b) (c)

Fig. 1 : (a) A circular cone, (b) an elliptic cone, (c) a parabolic cone.

At this point we would like to make an important remark.

Remark 1: A cone is a set of lines passing through its vertex and base curve. Thus, it extends beyond the vertex and the base curve. So the cones in our diagrams are only a portion of the actual cones.

Now let us introduce some new terms.

Definitions: A cone whose base curve is a circle is called a **circular cone.** The line joining the vertex of a circular cone to the centre of its base curve is called the **axis** of the cone. If the axis of a circular cone is perpendicular to the plane of the base curve, then the cone is called a **right circular cone**.

Thus, the cone in Fig. 1 (a) is a right circular cone, while the one in Fig. 2 is not.

Fig. 2: A circular cone which is not right circular.

Cones were given great importance by the ancient Greeks who were studying the problem of doubling the cube. A teacher of Alexander the Great, Menaechmus, is supposed to have geometrically proved the following result. This result is the reason for the continuing importance of cones.

Theorem 1: Every planar section of a cone is a conic.

This theorem is the reason for an ellipse, parabola or hyperbola to be called a conic section (see Fig. 1 of Unit 3). It was proved by the Greek astronomer Appolonius (approximately 200 B.C.). We will not give the proof here.

Now, according to Theorem 1 would you call a pair of intersecting lines a conic? If you cut a right circular cone by a plane that contains its axis, what will the resulting curve be? See Fig. 3.

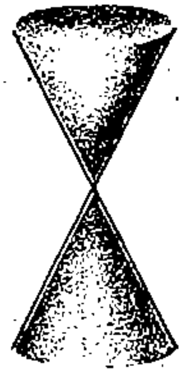


Fig. 3: A pair of intersecting lines is a conic section.

Let us now see how we can represent a cone algebraically. We shall first talk about a right circular cone, which we shall refer to as an **r.c. cone.**

So, let us take an r.c. cone. Let us assume that its vertex is at the origin, and its axis is the z-axis (see Fig. 4). Then the base curve, which is a circle of radius r (say), lies in a plane that is parallel to the XY-plane. Let this plane be z = k, where k is a constant. Then, any generator will intersect this curve in a point (a, b, k), for some a, b ∈ **R**. So the angle between the generator and the axis of the cone will be $\theta = \tan^{-1}\left(\frac{r}{k}\right)$, which is a constant.

This is true for any generator of the cone.

Thus, every line that makes up the cone makes a fixed angle θ with the axis of the cone. This angle is called the **semi-vertical angle** (or **generating angle**) of the cone.

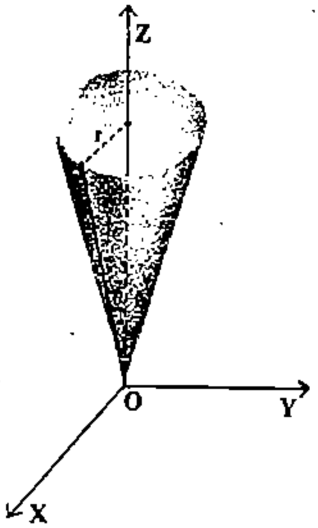


Fig. 4: A right circular cone with vertex at the origin and base curve $x^2 + y^2 = r^2$, z = k.

We can now define an r.c. cone in the following way.

Definition: A **right circular cone** is a surface generated by a line which passes through a fixed point (its vertex), and makes a constant angle with a fixed line through the fixed point.

Let us obtain the equation of an r.c. cone in terms of its semi-vertical angle. Let us assume that the vertex of the cone is O (0, 0, 0) and axis is the z-axis. (We can always choose our coordinate system in this manner.) Now take any point P (x, y, z) on the cone (see Fig. 5). Then, the direction ratios of OP are x, y, z, and of the cone's axis are 0, 0, 1. Thus, from Equation (9) of Unit 4, we get

$$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Thus, $x^2+y^2+z^2 = z^2 \sec^2\theta$

$\Rightarrow x^2+y^2 = z^2 \tan^2\theta.$

(1) is called **the standard form of the equation of a right circular cone.**

Now, why don't you try the following exercises?

E1) Show that the equation of the r.c. cone with vertex at (a, b, c), axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta}$ $= \frac{z-c}{\gamma}$ and semi-vertical angle θ is

$$[\alpha(x-a) + \beta(y-b) + \gamma(z-c)]^2 (\alpha^2+\beta^2+\gamma^2) \{(x-a)^2 + (y-b)^2 + (z-c)^2\} \cos^2\theta \ldots\ldots(2)$$

E2) Can you deduce (1) from (2) ?

E3) Find the equation of the r.c. cone whose axis is the x-axis, vertex is the origin and semi-vertical angle is $\frac{\pi}{3}$.

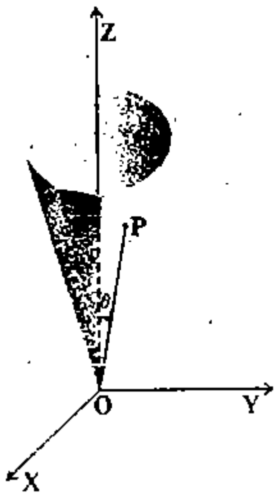


Fig. 5: $x^2 + y^2 = z^2 \tan^2\theta$

Let us now look at a cone whose vertex is the origin. In this situation we have the following result.

An equation is homogeneous of degree 2 if each of its terms is of degree 2.

Theorem 2: The equation of a cone whose base curve is a conic and whose vertex is (0, 0, 0) is a homogeneous equation of degree 2 in 3 variables.

Proof: Let us assume that the base curve is the conic
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = k.$

Any generator of the cone passes through (0, 0, 0). Thus, it is of the form

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}. \qquad \ldots\ldots(3)$$

This line intersects the plane $z = k$ at the point $\left(\frac{\alpha k}{\gamma}, \frac{\beta k}{\gamma}, k\right)$.

This point should lie on the conic. Thus,

$$\frac{k^2}{\gamma^2}\left(a\alpha^2 + 2h\alpha\beta + b\beta^2\right) + \frac{k}{\gamma}(2g\alpha + 2f\beta) + c = 0.$$

Eliminating α, β, γ from this equation and (3), we get

$$k^2\left(a\frac{x^2}{z^2} + 2h\frac{xy}{z^2} + b\frac{y^2}{z^2}\right) + k\left(2g\frac{x}{z} + 2f\frac{y}{z}\right) + c = 0.$$

$$\Rightarrow ax^2 + 2hxy + by^2 + 2gx\frac{z}{k} + 2fy\frac{z}{k} + \frac{cz^2}{k^2} = 0.$$

This is the equation of the cone. As you can see, it is homogeneous of degree 2 in the 3 variables x, y and z.

For example, the equation of the cone whose base curve is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the plane $z = 5$, and whose vertex is the origin, is

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{z^2}{25}.$$

Do you see a pattern in the way we obtain the equation of the cone from the equation of the base curve? The following remark is about this.

Remark 2: To find the equation of the cone with vertex at (0, 0, 0) and base curve in the plane $Ax + By + Cz = D$, $D \neq 0$, we simply **homogenise** the equation of the curve, as follows. We multiply the linear terms by $\frac{Ax + By + Cz}{D}$, and the constant term by $\left(\frac{Ax + By + Cz}{D}\right)^2$; and we leave the quadratic terms as they are. The equation that we get by this process is a homogeneous equation of degree 2, and is the equation of the cone.

Let us look at a few examples of cones with their vertices at the origin.

Example 1: Show that the equation of the cone with the axes as generators is
$fyz + gzx + hxy = 0$, where $f, g, h \in \mathbf{R}$.

Solution: By Theorem 2, the equation of the cone is
$ax^2+by^2+cz^2+2fxy + 2gzx + 2hxy = 0$, for some $a, b, c, f, g, h \in \mathbf{R}$.

Since the x-axis is a generator, (1, 0, 0) lies on it. Therefore, $a = 0$. Similarly, as it passes through (0, 1, 0) and (0, 0, 1), $b = c = 0$. So the equation becomes $fyz + gzx + hxy = 0$.

Example 2: Find the equation of the cone with vertex at the origin, and whose base curve is the circle $x^2+y^2+z^2 = 16$, $x+2y+2z = 9$.

Solution : On homogenising the equation of the sphere, we get

$$x^2+y^2+z^2 = 16\left(\frac{x+2y+2z}{9}\right)^2.$$

This is a second degree homogeneous equation in x, y, z and passes through the circle. Hence, it is the required equation of the cone.

The following exercises will give you some practice in homogenising equations.

E4) Find the equation of the cone with vertex at the origin and base curve

a) the parabola $y^2 = 4ax$, $z = 3$.

b) the ellipse $\frac{y^2}{3} + \frac{z^2}{5} = 1$, $x = -2$.

E5) Find the equation of the cone passing through $2x^2 + 3y^2 + 4z^2 = 1$ and $x + y + z = 1$.

Let us go back to Theorem 2 now. Do you think its converse is true? Consider the following result.

Theorem 3: A homogeneous equation of the second degree in 3 variables represents a cone whose vertex is at the origin.

Proof: Let the given equation be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \qquad \text{......(4)}$$

Let P (α, β, γ) be a point on this surface and O the origin. Then OP is given by the equations

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = r \text{ (say)}.$$

So any point on OP is (rα, rβ, rγ). Since P lies on (4),

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0. \qquad \text{......(5)}$$

Multiplying (5) throughout by r^2, we get

$$a(r\alpha)^2 + b(r\beta)^2 + c(r\gamma)^2 + 2f(r\beta)(r\gamma) + 2g(r\gamma)(r\alpha) + 2h(r\alpha)(r\beta) = 0.$$

Thus, (rα, rβ, rγ) also lies on (4), for any r ∈ **R**. In particular, O lies on (4). So, the line OP lies on the surface given by (4). In other words, OP is a generator of (4). Thus, the surface (4) is generated by lines through the origin. Each of these lines will also pass through any curve obtained by intersecting (4) by a plane, and any of these curves can be treated as a base curve. Thus, (4) represents a cone with the origin as vertex.

So, from what you have seen so far in this section; whenever you come across a homogeneous equation in 3 variables of degree 2, you know that it represents a cone.

Remark 3: If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, then (4) can be written as a product of two linear expressions. Thus, in this case (4) represents a pair of planes containing the origin. We shall consider this case as a **degenerate cone**, and any point on the line of intersection of the two planes can be considered as its vertex.

Using Theorem 3, we can show that if α, β, γ ∈ **R**, then a homogeneous equation in x – α, y – β, z – γ represents a cone with vertex at (α, β, γ). (We shall discuss this kind of shifting in detail in Unit 7.)

Why don't you try some exercises now?

E6) If $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ is a generator of the cone given by the homogeneous equation (4), then show that (α, β, γ) lies on (4).

E7) Which of the following equations represents a cone?

$3x + 4y + 5z = 0$; $x^2 + y^2 + z^2 = 9$; $3(x^2 + y^2 + z^2) = xy$; $xyz = yz + zx + xy$.

E8) If $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$, show that $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone.

Let us now go back to Example 1. This is an example of a cone with three mutually perpendicular generators. Its equation has no term containing x^2, y^2 or z^2. Does this mean that whenever a cone has three mutually perpendicular generators, its equation must have no term with x^2, y^2 or z^2? The following theorem helps us to answer this question.

Theorem 4: If the cone $ax^2 + by^2 + cz^2 + 2fyx + 2gzx + 2hxy = 0$ has 3 mutually perpendicular generators, then $a + b + c = 0$.

Proof: Let the direction cosines of the three mutually perpendicular generators be l_i, m_i, n_i, where $i = 1, 2, 3$. Since they are mutually perpendicular, we can rotate our coordinate system so that these lines become the new coordinate axes.

Then the direction cosines of the previous coordinate axes with respect to the new axes are l_1, l_2, l_3; m_1, m_2, m_3 and n_1, n_2, n_3, respectively.

So Unit 4 (Equations (3) and (10)) tell us that

$$\left.\begin{aligned} & l_1^2 + l_2^2 + l_3^2 = 1 \\ & m_1^2 + m_2^2 + m_3^2 = 1 \\ & n_1^2 + n_2^2 + n_3^2 = 1 \\ & l_1m_1 + l_2m_2 + l_3m_3 = 0 \\ & m_1n_1 + m_2n_2 + m_3n_3 = 0 \\ & n_1l_1 + n_2l_2 + n_3l_3 = 0 \end{aligned}\right\} \qquad \ldots\ldots(6)$$

Further, since the perpendicular lines are generators of the cone, using E6 we get

$$al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1 = 0$$
$$al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2 = 0$$
$$al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3l_3 + 2hl_3m_3 = 0$$

Adding these equations, and using (6), we get $a + b + c = 0$.

Actually, the converse of this result is also true. The proof uses a fact that you have already seen in Fig. 3 in the case of an r.c. cone, namely,

> any plane through the vertex of a cone intersects the cone in two lines, which may or may not be distinct.

If the lines of intersection of a cone and a plane through the cone's vertex are imaginary, the intersection reduces to a single point, namely, the cone's vertex (as in Fig. 1(e) of Unit 3)

The following result, which we shall not prove, tells us about the angle between the lines of intersection.

Theorem 5: The angle between the two lines in which the plane $ux + vy + wz = 0$ intersects the cone $C(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is

$$\tan^{-1}\left|\frac{2P\sqrt{u^2+v^2+w^2}}{(a+b+c)\,(u^2+v^2+w^2)-C(u,\,v,\,w)}\right|, \qquad \ldots\ldots(7)$$

$$\text{where } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

Looking at (7), can you give the condition under which the angle will be $\pi/2$?

The lines of intersection of the plane and the cone will be perpendicular iff

$C(u, v, w) = (a+b+c)\,(u^2+v^2+w^2)$. ......(8)

Let us use (8) to solve the following example, which includes the converse of Theorem 4.

Example 3: Show that if $a + b + c = 0$, then the cone

$C(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

has infinitely many sets of three mutually perpendicular generators.

Solution: Let $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ be any generator of the cone. Then, by E6, we know that

$C(\alpha, \beta, \gamma) = 0$. Therefore, using the fact that $a+b+c = 0$ and (8), we see that the plane $\alpha x + \beta y + \gamma z = 0$ intersects the cone in two mutually perpendicular generators, say L and L′.

Now $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ is normal to the plane $\alpha x + \beta y + \gamma z = 0$. Thus, it is perpendicular to both L and L′. Thus, these three lines form a set of three mutually perpendicular generators of the cone.

Note that we chose $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ arbitrarily. Thus, for each generator chosen we get a set of three mutually perpendicular generators. Hence, the cone has infinitely many such sets of generators.

Why don't you try some exercises now?

E9) Find the angle between the lines of intersection of $3x + y + 5z = 0$ and $6yz - 2zx + 5xy = 0$.

E10) Prove that $ax + by + cz = 0$, where $abc \neq 0$, cuts the cone $yz + zx + xy = 0$ in perpendicular lines iff $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

E11) Prove that if a right circular cone has three mutually perpendicular generators, its semi-vertical angle is $\tan^{-1}\sqrt{2}$.

Let us now discuss the intersection of a line and a cone.

6.3 TANGENT PLANES

In the previous unit you saw that a line can intersect a sphere in at most two points. What do you expect in the case of the intersection of a line and a cone? Let's see.

Let the equation of the cone be (4), that is, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Note that, by shifting the origin if necessary, we can always assume this equation as the cone's equation.

For convenience, we will write $C(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$.

Now consider the line $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$. Any point on this line is given by $(x_1 + r\alpha, y_1 + r\beta, z_1 + r\gamma)$, for some $r \in \mathbf{R}$. Thus, the line will intersect the cone, if this point lies on the cone for some $r \in \mathbf{R}$.

This happens if

$$a(x_1 + r\alpha)^2 + b(y_1 + r\beta)^2 + c(z_1 + r\gamma)^2 + 2f(y_1 + r\beta)(z_1 + r\gamma) + 2g(z_1 + r\gamma)(x_1 + r\alpha) + 2h(x_1 + r\alpha)(y_1 + r\beta) = 0.$$

$$\Leftrightarrow r^2C(\alpha, \beta, \gamma) + 2r\{\alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1)\} + C(x_1, y_1, z_1) = 0. \quad \ldots\ldots(9)$$

Now, if (x_1, y_1, z_1) doesn't lie on the cone, then (9) is a quadratic in r, and hence has two roots. Each root corresponds to a point of intersection of the line and the cone. Thus, we have just proved the following result.

Theorem 6: A straight line, passing through a point not on cone, meets the cone in at most two points.

Now suppose that the line $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$ is a tangent to the cone (4) at (x_1, y_1, z_1).

Then, since (x_1, y_1, z_1) lies on the cone, $C(x_1, y_1, z_1) = 0$. So (9) becomes

$$r^2C(\alpha, \beta, \gamma) + 2r\{\alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1)\} = 0.$$

This equation must have coincident roots, since the line is a tangent to the cone at (x_1, y_1, z_1). The condition for this is

$$\alpha(ax_1 + hy_1 + gz_1) + \beta(hx_1 + by_1 + fz_1) + \gamma(gx_1 + fy_1 + cz_1) = 0. \quad \ldots\ldots(10)$$

So, (10) is the condition for $\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma}$ to be tangent to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Note that (10) is satisfied by infinitely many values of α, β, γ. Thus,

at each point of a cone we can draw infinitely many tangents to the cone.

Now, from Sec. 4.3.3, you know that (10) tells us that each of these lines is perpendicular to the line with direction ratios

$ax_1 + hy_1 + gz_1, hx_1 + by_1 + fz_1, gx_1 + fy_1 + cz_1$.

Thus, the set of all the tangent lines at (x_1, y_1, z_1) is the plane

$$(x - x_1)(ax_1 + hy_1 + gz_1) + (y - y_1)(hx_1 + by_1 + fz_1) + (z - z_1)(gx_1 + fy_1 + cz_1) = 0$$

$$\Rightarrow x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0, \quad \ldots\ldots(11)$$

since $C(x_1, y_1, z_1) = 0$.

This plane is defined to be the tangent plane to the cone at (x_1, y_1, z_1).

Thus, (11) is the equation of the tangent plane at (x_1, y_1, z_1) to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

There is a very simple working rule for writing (11).

Rule of thumb: To write the equation of the tangent plane at any point (α, β, γ) on the cone

(4), replace x^2 by αx, y^2 by βy, z^2 by γz, $2yz$ by $\gamma y + \beta z$, $2zx$ by $\alpha z + \gamma x$ and $2xy$ by $\beta x + \alpha y$.

For example, the tangent plane to the cone $2x^2 + y^2 - 2xz = 0$ at $(1, 0, 1)$ is $2x(1) + y(0) - (x + z) = 0$, that is, $x = z$.

So far we have only found the tangent plane to a cone whose vertex is at the origin. What about a general cone? The following remark is about this.

Remark 4: We can find the tangent plane to a cone with vertex at (a, b, c) in the same manner as above. All we need to do is to shift the origin to (a, b, c) and find the tangent plane in the new coordinate system. And then we can shift back to the old coordinate system, making the required transformations in the equation of the plane. This will give us the required equation.

Now, if you look closely at (11), you will notice that (0, 0, 0) satisfies it. Thus, **the tangent plane to a cone passes through the vertex of the cone.**

Therefore, the tangent plan at $P(x_1, y_1, z_1)$ contains P as well as the vertex O of the cone. Hence, it contains the generator OP of the cone. Thus,

> the tangent plane to a cone touches the cone along the generator passing through the point of contact.

This generator is called the **generator of contact** of the plane. In Fig. 6 OP is the generator of contact of the tangent plane T.

You can try some exercises now.

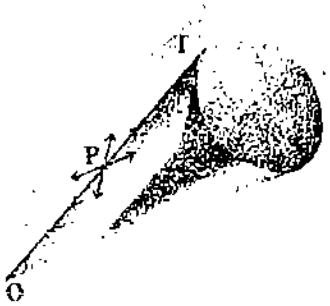


Fig. 6: T is the tangent plane to the cone at P.

E12) Find the equation of the tangent plane at the point $\left(\frac{1}{7}, \frac{1}{4}, 1\right)$ to the cone $5yz - 8zx - 3xy = 0$.

E13) Use Theorem 5 to obtain the condition under which a given plane is tangent to a cone.

If you've solved E13, you would have seen that the condition for $ux+vy+wz = 0$ to be tangent to the cone (4), that is, $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0, \text{ that is,}$$

$Au^2 + Bv^2 + Cw^2 + 2Huv + 2Fvw + 2Gwu = 0$,

where $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$, $F = gh - af$, $G = hf - bg$, $H = fg - ch$.

Thus, the line $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$, which is the normal at (0, 0, 0) to the tangent plane, is a generator of the cone $Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx = 0$. ......(12)

Thus, (12) is the cone generated by the normals to the tangent planes at the vertex (0, 0, 0) of the cone (4). Since it is homogeneous, its vertex is also (0, 0, 0).

Note that (12) is nothing but the determinant equation

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0.$$

Now, if we consider the surface generated by the normals at (0, 0, 0) to the tangent planes to (12), what do we get? On calculating, you will get a surprise! The cone is (4), because $BC - F^2 = a\Delta$, $CA - G^2 = b\Delta$, $AB - H^2 = c\Delta$, $GH - AF = f\Delta$, $HF - BG = g\Delta$, $FG - CH = h\Delta$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Because of this relationship between (4) and (12) we call them **reciprocal cones.**

Actually, the following example shows why the name is appropriate.

Example 4: Show that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal. (Here $abc \neq 0$.)

Solution: The reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ is given by the determinant equation

$$\begin{vmatrix} a & 0 & 0 & x \\ 0 & b & 0 & y \\ 0 & 0 & c & z \\ x & y & z & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow a \begin{vmatrix} b & 0 & y \\ 0 & c & z \\ y & z & 0 \end{vmatrix} - x \begin{vmatrix} 0 & b & 0 \\ 0 & 0 & c \\ x & y & z \end{vmatrix} = 0$$

$\Leftrightarrow x^2bc + y^2ac + z^2ab = 0$

$\Leftrightarrow \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$, dividing throughout by abc.

This is the required equation.

Now you can do the following exercises. This will help you to understand reciprocal cones better.

E14) Find the cone on which the perpendiculars drawn from the origin to tangent planes to the cone $19x^2 + 11y^2 + 3z^2 + 6yz - 10zx - 26xy = 0$ lie.

E15) Prove that the cone (4) has three mutually perpendicular tangent planes iff $bc + ca + ab = f^2 + g^2 + h^2$.

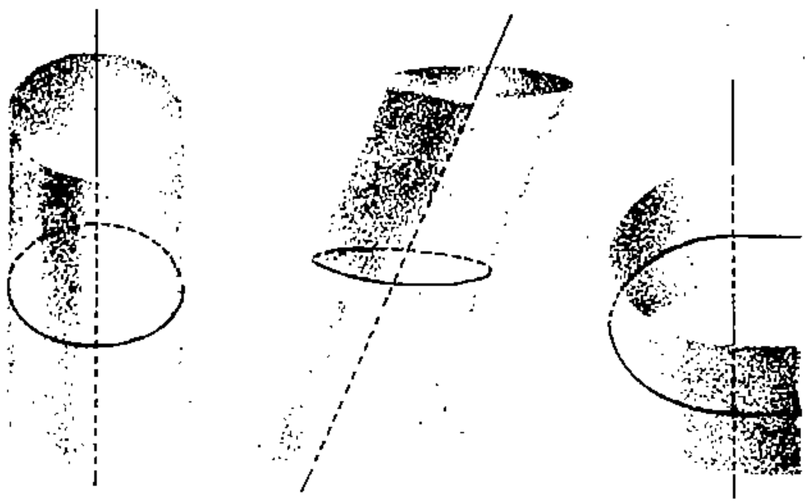
And now let us shift our attention to another surface that is generated by lines.

6.4 CYLINDERS

You must have come across several examples of the surface that we are going to discuss in this section. For instance, a drain pipe is cylindrical in shape, and so is a pencil. But for us, the pencil will not be a cylinder, only its surface will, according to the following definition.

Definition: A **cylinder** is the set of all lines which intersect a given curve and which are parallel to a fixed line which does not lie in the plane of the curve. The fixed line is called the **axis** of the cylinder and the curve is called the **base curve** (or **directrix**) of the cylinder.

All the figures in Fig. 7 represent portions of cylinders.



(a) (b) (c)

Fig. 7: a) A circular cylinder, b) an elliptic cylinder, c) a parabolic cylinder.

In Fig. 7 (b) the cylinder's base curve is an ellipse, while it is a parabola in Fig. 7 (c). Fig. 7 (a) is an example of a right circular cylinder according to the following definition.

Definition: A cylinder whose base curve is a circle, and whose axis passes through the centre of the base curve and is perpendicular to the plane of the base curve, is called a **right circular cylinder.**

As you can see, in common parlance when people talk of a cylinder, they mean a portion of a right circular cylinder.

Henceforth, in this section, by a cylinder we shall mean a right circular cylinder.

Let us now find the equation of a cylinder. We shall first assume that its axis is the z-axis, and its base curve is the circle $x^2 + y^2 = r^2$, $z = 0$ (see Fig. 8).

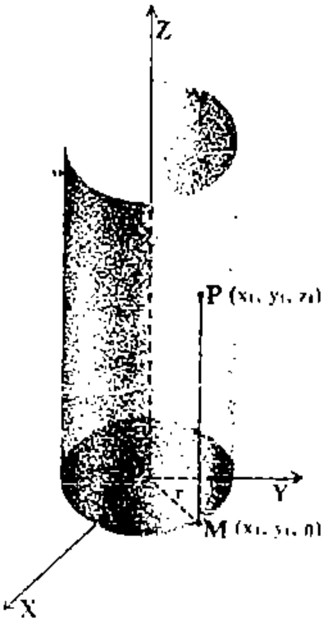


Fig. 8: The cylinder $x^2 + y^2 = r^2$.

Let P (x_1, y_1, z_1) be any point on the cylinder. Let the generator through P intersect the plane of the base curve (that is, the XY-plane) in M. Then the perpendicular distance of P from the axis is OM $= \sqrt{x_1^2 + y_1^2}$.

But this is also r. Thus,

$$r^2 = x_1^2 + y_1^2.$$

This equation is true for every point P (x_1, y_1, z_1) on the cylinder. Thus, the equation of the cylinder is

$$x^2 + y^2 = r^2. \qquad \ldots\ldots(13)$$

You may wonder why z is not figuring in the equation. This is because whatever value of z you take, the equation of the cylinder remains $x^2 + y^2 = r^2$.

What does this mean geometrically? It says that whatever plane parallel to the XY-plane you take, say $z = t$, and take its intersection with the cylinder, you will always get the circle $x^2 + y^2 = r^2$.

Thus, in a sense, the cylinder is made up of infinitely many circles, each piled up on the other!

The radius of a cylinder is the radius of its base curve.

Note that the pl $z = t$ is perpendicular to axis

Also note that the **length** of the perpendicu fro **to its axis is equal to its radius.**

Using this fact, let us find the equation of a cylinder of radius r and whose axis is

$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ (see Fig. 9).

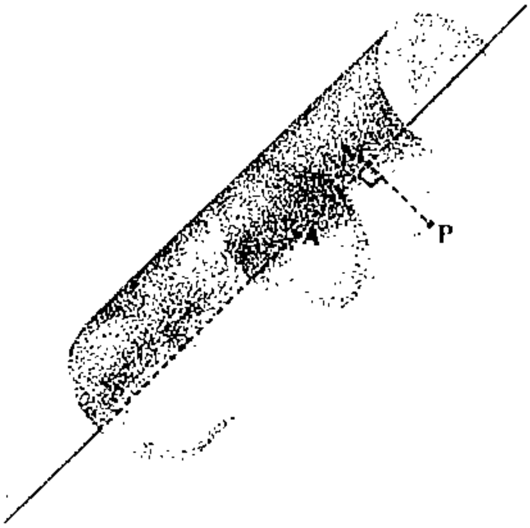


Fig. 9: A right circular cylinder with axis AM.

Let P (x_1, y_1, z_1) be any point on the cylinder. Let A be the point (a, b, c), which lies on the axis, and M be thefoot of the perpendicular from P onto the axis. Then PM = r.

Also, AM = AP cos θ, where θ is the angle between the lines AM and AP.

$$\therefore \ AM = \frac{(x_1-a)\alpha + (y_1-b)\beta + (z_1-c)\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}, \text{ using Equation (9) of Unit 4.}$$

Since AMP is a right-angled triangle, we get $AP^2 = AM^2 + MP^2$. Thus,

$$(x_1-a)^2 + (y_1-b)^2 + (z_1-c)^2 = \frac{\{(x_1-a)\alpha + (y_1-b)\beta + (z_1-c)\gamma\}^2}{\alpha^2+\beta^2+\gamma^2} + r^2.$$

This equation holds for any point (x_1, y_1, z_1) on the cylinder.

Thus, the equation of the right circular cylinder with radius r and axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta'} = \frac{z-c}{\gamma}$

is $\{(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2\}(\alpha^2+\beta^2+\gamma^2) = \{(x-a)\alpha + (y-b)\beta + (z-c)\gamma\}^2$. ...(14)

Let us look at an example.

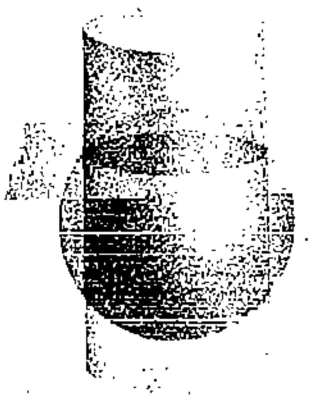


Fig. 10.

Example 5: Find the equation of the cylinder having for its base the circle $x^2+y^2+z^2 = 9$, $x - y + z - 3$.

Solution : The centre of the sphere is (0, 0, 0), and radius 3. The distance between (0, 0, 0) and the plane $x - y + z = 3$ is $\sqrt{3}$. So the radius of the base circle is $\sqrt{9-3} = \sqrt{6}$ (see Fig. 10).

The axis of the cylinder is perpendicular to the plane $x - y + z = 3$ and passes through (0, 0, 0). So its equations are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$.

Thus, using (14), we get the required equation as

$$3(x^2 + y^2 + z^2 - 6) = (x - y + z)^2$$

$$\Rightarrow x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0.$$

Why don't you try an exercise now?

E16) Find the equation of the cylinder

a) whose axis is $x = 2y = -z$ and radius is 4.

b) whose axis is $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$ and radius is 2.

We shall end our discussion on cylinders here. Let us now briefly review what we have covered in this unit.

6.5 SUMMARY

In this unit we have discussed the following points.

1) A cone is a surface generated by a line passing through a fixed point (its vertex) and intersecting a given curve (its base curve), such that the vertex does not lie in the plane of the base curve.

2) A cone whose base curve is a circle, and for which the line joining its vertex to the centre of the base curve is perpendicular to the plane of the base curve, is called a right circular cone.

3) A planar section of a cone is a conic.

4) The equation of a right circular cone with semi-vertical angle θ is $x^2+y^2 = z^2 \tan^2 \theta$.

5) A second degree equation in x, y, z represents a cone with vertex at the origin if it is homogeneous.

6) The cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ has 3 mutually perpendicular generators if $a+b+c = 0$.

7) Any plane through the vertex of a cone intersects the cone in two distinct or coincident lines. The angle between the lines obtained by intersecting $ux+vy+wz = 0$ with

$$C(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \text{ is}$$

$$\tan^{-1}\left|\frac{2P\sqrt{u^2 + v^2 + w^2}}{(a + b + c)(u^2 + v^2 + w^2) - C(u, v, w)}\right|,$$

$$\text{where } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

Thus, the plane is tangent to the cone iff $P^2 = 0$.

8) The equation of the tangent plane to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ at the point $P(x_1, y_1, z_1)$ is

$$(ax_1 + hy_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0.$$

This contains the line OP, where O is the vertex of the cone.

9) The cone formed by the normals to the tangent planes to a given cone at its vertex is the reciprocal of the given cone. The reciprocal cone of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix} = 0.$$

10) A cylinder is a surface generated by a line which is parallel to a fixed line (its axis) and which cuts a given curve (its base curve), such that the line and curve are not in the same plane.

11) A right circular cylinder is a cylinder whose base curve is a circle and axis is the line through the centre of the circle and perpendicular to its plane.

12) The equation of a right circular cylinder with base curve a circle of radius r and centre (0, 0, 0) in the plane $z = 0$ is $x^2 + y^2 = r^2$.

13) The equation of a right circular cylinder of radius r and axis $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ is

$$\{(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2\}\ (\alpha^2 + \beta^2 + \gamma^2) = \{(x-a)\,\alpha + (y-b)\,\beta + (z-c)\,\gamma\}^2.$$

And now, you may like to go back to Sec. 6.1 to see if you have **achieved the objectives listed there**. You must have solved the exercises as you came to them in the unit. In the next section we have given our answers to the exercises. You may like to have a look at them.

6.6 SOLUTIONS/ANSWERS

E1) Let P (x, y, z) be any point on the cone. Since V (a, b, c) is its vertex, the direction ratios of PV are $x - a$, $y - b$, $z - c$. Also, the direction ratios of the axis of the cone are α, β, γ.

$$\therefore \cos\theta = \frac{\alpha(x-a) + \beta(y-b) + \gamma(z-c)}{\sqrt{\alpha^2+\beta^2+\gamma^2}\ \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}.$$

Hence, we get (2).

E2) Yes. Just take $\alpha = 0 = \beta$, $\gamma = 1$, $a = b = c = 0$ in (2), and you will get (1).

E3) The direction ratios of the axis are 1, 0, 0, and the vertex is (0, 0, 0). If (x, y, z) is any point on the cone, then

$$\cos\frac{\pi}{3} = \frac{x.1 + y.0 + z.0}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow x^2 + y^2 + z^2 = 4x,$$

which is the required equation.

E4) a) $y^2 = 4ax\left(\frac{z}{3}\right) \Leftrightarrow 3y^2 - 4azx = 0.$

b) $\frac{x^2}{4} - \frac{y^2}{3} - \frac{z^2}{5} = 0.$

E5) $2x^2 + 3y^2 + 4z^2 = (x + y + z)^2$

$\Leftrightarrow x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2zx = 0.$

E6) Let $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = r$, say. Then putting $x = r\alpha$, $y = r\beta$, $z = r\gamma$ in (4), and dividing throughout by r^2, we get

$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$

$\therefore$ (α, β, γ) lies on (4).

E7) Only $3(x^2 + y^2 + z^2) = xy$.

E8) Substituting the value of d in the equation, we can write it as

$$a\left(x + \frac{u}{a}\right)^2 + b\left(y + \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = 0,$$

which is a homogeneous equation of degree 2 in $x + \frac{u}{a}, y + \frac{v}{b}, z + \frac{w}{c}$.

Thus, it is a cone with vertex at $\left(-\frac{u}{a}, -\frac{v}{b}, -\frac{w}{c}\right)$.

E9) The required angle is

$$\alpha = \tan^{-1}\left|\frac{2P\sqrt{3^2 + 1^2 + 5^2}}{0 - 6(1)(5) + 2(5)(3) - 5(3)(1)}\right|,$$

$$\text{where } P^2 = \begin{vmatrix} 0 & \frac{5}{2} & -1 & 3 \\ \frac{5}{2} & 0 & 3 & 1 \\ -1 & 3 & 0 & 5 \\ 3 & 1 & 5 & 0 \end{vmatrix} = \frac{225}{4}$$

$\therefore P = \frac{15}{2}$.

$\therefore \alpha = \tan^{-1}\sqrt{35}$.

E10) In this situation (8) tells us that the lines will be perpendicular iff
$bc + ca + ab = 0$.

$\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$

E11) Let its semi-vertical angle be θ.
Then the equation of the cone is (1), that is, $x^2 + y^2 = z^2 \tan^2\theta$.
Since this has three mutually perpendicular generators, Theorem 4 tells us that
$1 + 1 - \tan^2\theta = 0 \Rightarrow \theta = \tan^{-1}\sqrt{2}$.

E12) The required equation is

$$x\left\{-\frac{3}{2}\left(\frac{1}{4}\right) - 4(1)\right\} + y\left\{-\frac{3}{2}\left(\frac{1}{7}\right) + \frac{5}{2}(1)\right\} + z\left\{(-4)\left(\frac{1}{7}\right) - \frac{5}{2}\left(\frac{1}{4}\right)\right\} = 0.$$

$\Leftrightarrow -245x + 128y + 3z = 0.$

E13) A tangent plane must touch the cone along a generator. Thus, the two lines of intersection of the plane and the cone must coincide. Thus, the angle between these lines must be 0.
Thus, from Theorem 5, we see that $ux+vy+wz = 0$ is a tangent to the cone $C(x, y, z) = 0$
$\Leftrightarrow P = 0$ (since $u^2 + v^2 + w^2 \neq 0$.)

$$\Leftrightarrow \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0.$$

E14) The required cone is the reciprocal of the given cone. Thus its equation is

$$\begin{vmatrix} 19 & -13 & -5 & x \\ -13 & 11 & 3 & y \\ -5 & 3 & 3 & z \\ x & y & z & 0 \end{vmatrix} = 0$$

$\Leftrightarrow 3x^2 + 4y^2 + 5z^2 + 2yz + 4zx + 6xy = 0$

E15) The cone will have three mutually perpendicular tangent planes iff the reciprocal cone has three mutually perpendicular generators. Using Theorem 4 and its converse, we see that this happens iff in (12), $A + B + C = 0$, that is,
iff $(bc - f^2) + (ca - g^2) + (ab - h^2) = 0$, that is
iff $bc + ca + ab = f^2 + g^2 + h^2$.

E16) a) The equation is $(x^2 + y^2 + z^2 - 16)(1 + \frac{1}{4} + 1) = (x + \frac{y}{2} - z)^2$

$\Leftrightarrow 5x^2 + 8y^2 + 5z^2 - 4xy + 4yz + 8xy - 144 = 0.$

b) The required equation is

$14\ \{(x-1)^2 + y^2 + (z-3)^2 - 4) = \{2(x-1) + 3y + (z-3)\}^2$

$\Leftrightarrow 10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4xz - 8x + 30y - 74z + 59 = 0.$

MISCELLANEOUS EXERCISES

(This section is optional)

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of these contents. Our solutions to the questions follow the list of problems, in case you'd like to counter-check your answers.

1) Find the equations to the planes through the line $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-4}{5}$, which are parallel to the coordinate axes.

2) Find the equation of the plane passing through $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point (0, 7, -7). Also check if $x = \frac{2-y}{3} = \frac{z+2}{2}$ lies in this plane.

3) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in the points A, B and C. Find the equations determining the circumcircle of the triangle ABC (see Fig. 1).

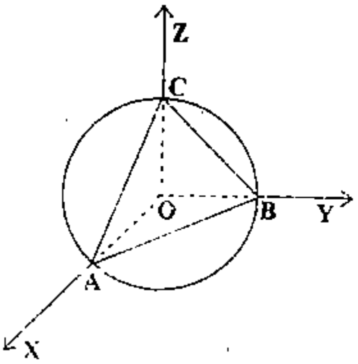


Fig. 1

4) Prove that if every planar section of a surface given by a quadratic equation is a circle, the surface must be a sphere.

5) Find an equation of the set of points which are twice as far from the origin as from (−1, 1, 1).

6) If the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ in a great circle, then show that $2(uu' + vv' + ww') - (d + d') = 2r'^2$, where r′ is the radius of the second sphere.

7) Find the equation of the sphere inscribed in the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ (see Fig. 2).

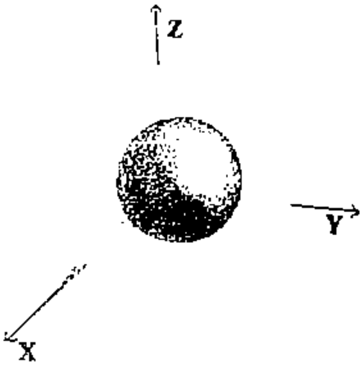


Fig. 2

8) Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant.
(**Hint**: Take the fixed point to be the origin.)

9) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.

10) If $x = \frac{y}{2} = z$ represents one out of a set of three mutually perpendicular generators of the cone $11yz + 6zx - 14xy = 0$, find the equations of the other two.
(**Hint**: Take a plane through the given line, and apply the condition for the lines of intersection of this plane and the cone to be perpendicular.)

11) Find the equation of the right circular cone with vertex (1, 1, 3), axis parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ and with one of its generators having direction ratios 2, 1, −1.

12) Find the equation of the cone which passes through the common generators of the cones $x^2 + 2y^2 + 3z^2 = 0$ and $5xy - yz + 5xz = 0$ and the line with direction ratios 1, 0, 1.
(**Hint**: The cone passing through the common generators of the cone $C_1 = 0$ and $C_2 = 0$ is $C_1 + kC_2 = 0$, where $k \in \mathbb{R}$.)

13) Find the equation of the right circular cylinder that is generated by lines which are parallel to $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, and which are tangent to the sphere $x^2+y^2+z^2 = r^2$ (see Fig. 3).
(Note: Such a cylinder is called the **enveloping cylinder** of the sphere.)

Fig. 3: The cylinder envelops the sphere.

14) The axis of a cylinder of radius 3 has equations $\frac{x}{2} = \frac{y+1}{2} = \frac{z-1}{1}$. Find the equation of the cylinder.

15) Prove that for a cylinder the tangent plane at any point is parallel to its axis (see Fig. 4).

Fig. 4: The plane is tangent to the cylinder.

SOLUTIONS

1) The equation to any plane through the given line is
$a(x-2) + b(y-3) + c(z-4) = 0$, where $2a + 4b + 5c = 0$
If this is parallel to the x-xis, we must have
$a(1) + b(0) + c(0) = 0 \Rightarrow a = 0$.
Thus, the equation of such a plane is $5y - 4z + 1 = 0$.
Similarly, you can check that the planes parallel to the y and z axes are $5x - 2z - 2 = 0$ and $2x - y - 1 = 0$, respectively.

2) The plane will be $a(x+1) + b(y-3) + c(z+2) = 0$, ...(1)
where $-3a + 2b + c = 0$. ......(2)
Since (0, 7, –7) lies on it, we have
$a + 4b - 5c = 0$. ......(3)
Eliminating a, b and c from (2) and (3), we get

$$\frac{a}{-10-4} = \frac{b}{1-15} = \frac{c}{-12-2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}.$$

∴ (1) becomes $1(x+1) + 1(y-3) + 1(z+2) = 0$
$\Rightarrow x + y + z = 0$.

The line $\frac{x}{1} = \frac{y-2}{-3} = \frac{z+2}{2}$ will lie on this plane, if it is parallel to the plane and any point on it lies on the plane. Since $1(1) + (-3)(1) + 2(1) = 0$, the line is parallel to the plane. Also, (0, 2, –2) is a common point. Thus, the line lies in the plane.

3) The circumcircle will be the intersection of the given plane with any sphere passing through A, B and C. So, let us take the sphere through O, A, B and C. The coordinates of these points are (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c). You can check that the equation is $x^2 + y^2 + z^2 - ax - by - cz = 0$.
Thus, the equations that give the circumcircle are

$$x^2 + y^2 + z^2 - ax - by - cz = 0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

4) Let the equation of the surface be
$ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz + 2ux + 2vy + 2wz + d = 0$.
It intersects $z = 0$ in the conic
$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$.
This will be a circle iff $a = b$ and $h = 0$.
Similarly, on intersecting with $x = 0$ and $y = 0$ we will get $a = b = c$ and $f = 0 = g = h$.
Thus, the equation of the surface reduces to
$a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0$,
which represents a sphere.

5) Let (x, y, z) be any point in the set. Then

$$\sqrt{x^2 + y^2 + z^2} = 2\sqrt{(x+1)^2 + (y-1)^2 + (z-1)^2}$$

$\Rightarrow 3(x^2 + y^2 + z^2) + 8x - 8y - 8z + 12 = 0$, which represents a sphere.

6) If the two spheres are $S = 0$ and $S_1 = 0$, then $(-u', -v', -w')$ lies on $S - S_1 = 0$, that is, on $2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0$.
$\therefore 2(u - u')u' + 2(v - v')v' + 2(w - w')w' = d - d'$;
$\Rightarrow 2(uu' + vv' + ww') - d + d' = 2(u'^2 + v'^2 + w'^2) = 2r'^2 + 2d'$
$\Rightarrow 2(uu' + vv' + ww) - (d + d') = 2r'^2$.

7) Let the equation be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Since the given planes are tangent to it, the distance of $(-u, -v, -w)$ from these planes is $r = \sqrt{u^2 + v^2 + w^2 - d}$. Thus, we see that

$u = v = w = -r$ and $|-u - v - w - 1| = \sqrt{3}r$

Solving these equations, we get $r = \dfrac{3 + \sqrt{3}}{6}$.

Thus, the equation of the sphere is

$x^2 + y^2 + z^2 - 2r(x + y + z) + 2r^2 = 0$, where $r = \dfrac{3 + \sqrt{3}}{6}$.

8) Let us assume that the fixed point is (0, 0, 0) and the three lines are the axes. Then let the sphere be given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Its intercept on the x – axis, that is, $y = 0 = z$, is $2\sqrt{u^2 - d}$.

Similarly, the other intercepts are $2\sqrt{v^2 - d}$ and $2\sqrt{w^2 - d}$.

Now, $\left(2\sqrt{u^2 - d}\right)^2 + \left(2\sqrt{v^2 - d}\right)^2 + \left(2\sqrt{w^2 - d}\right)^2 = 4(u^2 + v^2 + w^2 - 12d)$,

which is a constant, since the sphere is a given one.

9) Let a line of intersection be $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$. Then $2l + m - n = 0$ and $ul^2 - m^2 + 3n^2 = 0$.

Solving these equations, we get

$m = -2l, n = 0$ or $m = -4l, n = -2l$.

Thus, the two lines are

$\dfrac{x}{1} = \dfrac{y}{-2}, z = 0$ and $\dfrac{x}{1} = \dfrac{y}{-4} = \dfrac{z}{-2}$.

10) $2x - y + k(y - 2z) = 0, k \in \mathbb{R}$, gives any plane through the given line. This will cut the given cone in perpendicular lines if
$11(k - 1)(-2k) + 6(-2k)(2) - 14(2)(k - 1) = 0 \Rightarrow k = -2, 7/11$.
Thus, the planes are $2x - 3y + 4z = 0$ and $11x - 2y - 7z = 0$.

Now, $2x - 3y + 4z = 0$ intersects the cone in two perpendicular lines of which one is the given one which lies in the plane. Therefore, the other one has to be the normal to the plane at (0, 0,0). This is $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{3}{4}$. So this will be another of the required set of mutually perpendicular generators.

Similarly, the third generator will be the normal to $11x - 2y - 7z = 0$ at (0, 0, 0), that is, $\dfrac{x}{11} = \dfrac{y}{-2} = \dfrac{z}{-7}$.

11) If θ is its semi-vertical angle, then

$$\cos\theta = \frac{2 + 2 - 2}{3\sqrt{6}} = \frac{2}{3\sqrt{6}}.$$

Also, the axis is given by $\dfrac{x - 1}{1} = \dfrac{y - 1}{2} = \dfrac{z - 3}{2}$.

Thus, the equation of the r.c. cone is

$$[(x-1)+2(y-1)+2(z-3)]^2=9\,[(x-1)^2+(y-1)^2+(z-3)^2]\frac{4}{54}$$

$$\Leftrightarrow x^2+10y^2+10z^2+12xy+24yz+12xz-50x-104y-96z+221=0.$$

12) Let the cone be $(x^2+2y^2+3z^2)+k\,(5xy-yz+5xz)=0$, where $k \in \mathbf{R}$. Since the line with direction ratios 1, 0, 1 lies on it, (1, 0, 1) must satisfy it. This gives us $k=-\frac{4}{5}$.

Thus, the required cone is

$5\,(x^2+2y^2+3z^2)-4\,(5xy-yz+5xz)=0$.

13) Let (α, β, γ) be any point on the cylinder. A generator through this will be given by

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}.$$

This line intersects the sphere in $(ak+\alpha, bk+\beta, ck+\gamma)$, where k is given by

$(ak+\alpha)^2+(bk+\beta)^2+(ck+\gamma)^2=r^2$.

This quadratic equation in k gives two values of k, which correspond to two points of intersection. Thus, the generator will be a tangent to the sphere if these points coincide, that is, iff

$(a\alpha+b\beta+c\gamma)^2=(a^2+b^2+c^2)\,(\alpha^2+\beta^2+\gamma^2-r^2)$.

Thus, the locus of (α, β, γ), which is the equation of the enveloping cylinder, is

$(ax+by+cz)^2=(a^2+b^2+c^2)\,(x^2+y^2+z^2-r^2)$.

14) $3x^2+3y^2+8xy+4yz+4xz+4x+2y+4z+8=0$.

15) We can always assume the equation of the cylinder to be $x^2+y^2=r^2$.

Its axis is the z-axis, that is, $x=0$, $y=0$.

As in the case of a cone, you can show that its tangent plane at a point (x_1, y_1, z_1) is $xx_1+yy_1=r^2$.

This is parallel to the z- axis. Hence, the result.

NOTES

NOTES

UTTAR PRADESH
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UGMM-05
Analytical Geometry

Block

3

CONICOIDS

BLOCK 3 CONICOIDS

In the last block you were introudced to some basics of solid geometry. You also studied the geometrical properties of some commonly found surfaces like spheres, cones and cylinders. In this block you will see that those surfaces are particular cases of a more general class of surfaces called conicoids. Conicoids are surfaces, which satisfy a general second degree equation in three variables. While studying them, you will realise that these surfaces are analogous to conic sections.

Expository articles on conicoids appeared in the mid-eighteenth century. The first book was by Alexis Clairaut (1713-1765). But, a more extensive and systematic study can be found in the second volume of the book. "Introductio" by L.Euler.

We will start this block with a unit on general theory of conicoids. In this, we will first define conicoids and then classify them into two categories — central and non-central conicoids. We shall then introduce you to two transformations of the three-dimensional coordinate system — translation and rotation of axes. The purpose of these transformations is to reduce a general second degree equation to "Standard form".

In Unit 8, we shall concentrate on central conicoids. You will see that there are five types of such conicoids — cone, imaginary conicoid, ellipsoid, hyperboloid of one sheet and of two sheets. In this unit we will concentrate on the geometrical properties of only the last three central conicoids, since we have already studied cones in Block 2.

In the last unit of this block (and this course) we shall discuss non-central conicoids. These surfaces are broadly of two types — cylinders and paraboloids. As you have already studied cylinders in some detail in Block 2, we will restrict our attention to paraboloids. You will find that paraboloids can be further divided into elliptic and hyperbolic paraboloids. As in Unit 8, we shall study all the geometrical properties of these surfaces.

At the end of this block, we have given a set of miscellaneous exercises, which cover the contents of the block as a whole. Doing these exercises will help you to have an overall understanding of the contents in this block. After doing the exercises you can check your solutions with our solutions, provided at the end of the section.

While going through each unit of the block, please do try the exercises as and when you come to them. Also, do go through the unit objectives after studying a unit. This will help you to check if you've really grasped the contents of the unit.

After studying this block please attempt the assignment that is based on this course.

UNIT 7 GENERAL THEORY OF CONICOIDS

Structure

7.1 INTRODUCTION

You have seen in Block 1 that the general equation of second degree in two variables x and y represents a conic. In analogy with this we can ask: what will a general second degree equation in three variables represent? In Block 2 you have studied some particular forms of second degree equations in three variables, namely, those representing spheres, cones and cylinders. In this unit we study the most general form of a second degree equation in three variables. The surfaces generated by these equations are called quadrics or conicoids. This name is apt because, as you will see in Unit 8, they can be formed by revolving conic about a line called an axis.

Alexis Clairaut (1713-1765), a French mathematician, was one of the pioneers to study quadric surfaces. He specified that a surface, in general, can be represented by an equation in three variables. He presented his ideas in his book 'Recherche Sur Les Courbes a Double Courbure' in which he gave the equations of several conicoids like the sphere, cylinder, hyperboloid and ellipsoid.

We start this unit with a small section in which we define a conicoid. In the next section we discuss rigid body motions in a three-dimensional system. We shall consider two types of transformations: translation of axes and rotation of axes. You can see that a conicoid remains unchanged in shape and size under these transformations. Lastly, we shall discuss how to reduce the equation of conicoid into a more simple form.

Objectives

After studying this unit you should be able to

- define a general conicoid;
- obtain the new coordinates when a given coordinate system is translated or rotated;
- use the fact that translation and rotation of axes are rigid body motions;
- check whether a given conicoid has a centre or not;
- prove and apply the fact that if a conicoid has a centre, then it can be reduced to standard form.

7.2 WHAT IS A CONICOID?

In this section we shall define surfaces in a three-dimensional coordinate system which are analogous to conic sections in a two-dimensional system.

Let us start with a definition.

Definition: A general second degree equation in three variables is an equation of the form
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$, ... (1)
where a, b, c, d, f, g, h, u, v, w are real numbers and at least one of a, b, c, f, g, h is non-zero.

Note that if we put either $z = k$, a constant, $x = k$ or $y = k$, in (1), then the equation reduces to a general second degree equation in two variables, and therefore, represents a conic.

Now we shall see what a general second degree equation in three variables represents. Let us first consider some particular cases of (1).

Case 1 : Suppose we put $a = b = c = 1$ and $g = h = f = 0$ in (1). Then we get the equation
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .... (2)
Does this equation seem familiar to you? In Unit 6 you saw that if $u^2 + v^2 + w^2 - d > 0$, then (2) represents a sphere with centre $(-f, -g, -h)$ and radius $\sqrt{u^2+v^2+w^2-d}$.

Case 2 : Suppose we put $u = v = w = d = 0$ in (1), then we get
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.
What does this equation represent? You know from Unit 6 that this equation represents a cone.

Case 3 : If we put $a = b = 1$, $h = 0$ and $z = k$ in (1), then it reduces to
$x^2 + y^2 + 2ux + 2vy + d = 0, z=k$ .... (3)
This represents a right circular cylinder (see Unit 6, Sec.6.4).
Similarly you can see that if we put $x = k$ or $y = k$ and $a = b = 1$, $h = 0$, then again (3) represents a cylinder.

We will discuss the surfaces represented by (1) in detail in the next unit.

The particular cases 1, 2 and 3 suggest that the points whose coordinates satisfy (1) lie on a surface in the three-dimensional system. Such a surface is called a conicoid or a quadric. Algebraically, we define a conicoid as follows:

Definition : A conicoid (or quadric surface) in the XYZ-coordinate system is the set S of points $(x, y, z) \in R^3$ that satisfy a general second degree equation in three variables.
So, for example, if

$F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$
is the second degree equation, then
$S = \{(x,y,z) \in \mathbf{R}^3 \mid F(x,y,z)=0\}$
Note that S can be empty. For example, if $F(x,y,z) \equiv x^2+y^2+z^2+1=0$, then
$S = \{(x,y,z) \mid F(x,y,z) = 0\} = \phi$, the empty set.
In such cases we call the conicoid an imaginary conicoid.
Since the above expression is very lengthy, for convenience we often denote this conicoid by $F(x,y,z,) = 0$.

Note : In future, whenever we use the expression $F(x, y, z) = 0$, we will mean the equation (1).

In this unit you will see that we can always reduce $F(x, y, z)$ to a much smaller quadratic polynomial. To do this we need to transform the axes suitably. Let us see what this means.

7.3 CHANGE OF AXES

In Block 1, you saw that a general second degree equation can be transformed into the standard form using a suitable change of axes. You also saw that these standard forms are very useful for studying the geometrical properties of the conic concerned. Here we shall show that in the case of the three-dimensional system also we can transform an equation F (x, y, z) = 0 into the corresponding standard form by means of an appropriate change of coordinate axes. As in the case of the two-dimensional system, the transformations that we apply are of two types : change of origin and change of direction of axes. Let us consider these one by one in the following sub-sections.

7.3.1 Translation of Axes

Here we shall discuss how the coordinates of a point in the three-dimensional system get affected by shifting the origin from one point to another point, without changing the direction of the axes. The procedure is the same as in the two-dimensional case.

Let OX, OY, OZ be the coordinate axes of a three-dimensional system. What happens when we shift the origin from O to another point O′ (see Fig.1)?

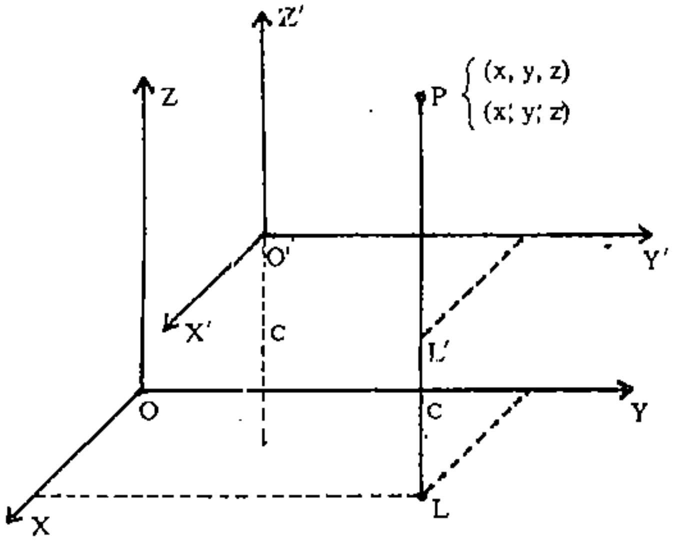


Fig.1 : Translation of axes through O′

Let the coordinates of O′ in the XYZ-system be (a, b, c). Let O′X′, O′Y′ and O′Z′ be the new axes which are paralllel to the OX, OY and OZ axes. Suppose P is a point as given in Fig.1. Let the coordinates of P in the XYZ-system be (x, y, z) and in the X′Y′Z′-system be (x′, y′, z′). We first find a relationship between z and z′. For this we draw a line through P, which is perpendicular to the XOY plane. Let it cut the planes XOY and X′O′Y′ in L and L′, respectively. Then we have

PL = z and PL′ = z′.

From Fig.1 we have

PL = PL′ + L′L

Now L′L is also the length of the perpendicular from the point O′ to the XOY plane. Therefore L′L = c

Thus we have z = z′ + c.

Similarly we get x = x′ + a and y = y′ + b.

Hence if we shift the origin from O(0, 0, 0) to another point O′ (a, b, c) without changing the direction of the axes, then the new coordinates of any point P (x, y, z) with respect to the origin O′ will be

x′ = x − a, y′ = y − b and z′ = z − c ......(4)

So, for example what will the new coordinates (x′, y′, z′) of a point P (x, y, z) be when we shift the origin to (2, −1, 1)? They will be x′ = x − 2, y′ = y + 1 and z′ = z − 1.

When we transform the axes in such a way, we say we have **shifted the origin** to (2, −1, 1) or **translated the axes through** (2, −1, 1).

Now, what will the effect of such a transformation be on any equation in x, y, z? If, in an equation in the XYZ-system we replace x, y, z, by x′+a, y′+b, z′+c, then we get the new equation in the X′Y′Z′-system. For example, when we shift the origin to (2, −1, 1), then the equation of the plane $3x+2y-z = 5$ will be transformed into $3x'+2y' - z' = 2$.

Note that the respective coefficients of x, y, z and of x′, y′, z′ remain unchanged under a shift in origin. Thus the direction ratios of the normal to a plane do not change when we shift the origin (Recall the definition of direction ratios from Unit 4.) Can you guess why this happens? This is obviously because we are not shifting the direction of the coordinate planes, we are only shifting the origin.

Now let us consider the effect of translation of axes on a general second degree equation.

Theorem 1 : Let the coordinates of a given surface S in a given coordinate system XYZ satisfy a second degree equation in three variables. Let us shift the origin from O to another point O′ giving rise to a new coordinate system X′Y′Z′. Then S is still represented by a general second degree equation in three variables in the new coordinate system.

In (5) the group of terms having degree 2, namely $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is called the **second degree part** of the equation and the group of the terms $2ux + 2vy + 2wz$ is called the **linear part** of the equation.

Proof : Let the given surface S satisfy the equation
$F(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$
For convenience we write the equation in the form

$$F(x,y,z) = \Sigma ax^2+\Sigma 2fyz+2\Sigma ux+d = 0 \qquad \ldots. (5)$$

Let (p, q, r) denote the coordinates of O′ in the XYZ-system. Consider now the new system of coordinate axes O′X′, O′Y′, O′Z′ parallel to the given system with origin O′. You know that the relation between the coordinates in the original and new systems are given by

$$\left.\begin{array}{l} x = x'+p \\ y = y'+q \\ z = z'+r \end{array}\right] \qquad \ldots. (6)$$

Substituting these expressions for x, y, z in (5), we get
$\Sigma a(x+p)^2+\Sigma 2f(y+q)(z+r)+\Sigma 2u(x+p)+d = 0.$
Now we expand the above expression and simplify by collecting like terms. We get

$$ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'+2u'x'+2v'y'+2w'z'+d = 0, \qquad \ldots. (7)$$

where

$$\left.\begin{array}{l} u' = (ap + hq + gr) + u \\ v' = (hp + bq + fr) + v \\ w' = (gp + fq + cr) + w \\ \text{and } d' = ap^2+bq^2+cr^2+2fqr+2grp+2hpq+2up+2vq+2wr+d \end{array}\right] \qquad \ldots. (8)$$

Let G (x′, y′, z′) denote the expression in the left hand side of (7). Then we see that any point (x′, y′, z′) belonging to S satisfies the equation G (x′, y′, z′) = 0, which is again a general second degree equation.

Hence we have proved the result.

If you compare (5) and (7), then you will see that the second degree part of the equation remain unchanged whereas the linear part changes. Hence, we can conclude that

> under the transformation of shifting the origin of a coordinate system, the second degree part of a general second degree equation does not change.

Why don't you try some exercises now?

E1) a) What will the new equation of a right circular cone, with vertex O, axis OZ and semi-vertical angle α be, when we shift the origin to (−1, 1, 0)?
b) What does the new equation represent? Sketch the surface.

E2) Obtain the transformed equation of the following equations when the origin is shifted to (1, −3, 2).

a) $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$.

b) $x^2 - 2y^2 - 3z = 0$.

Let us now consider the transformation in which the direction of the axes is changed. For this we need to understand the concept of a projection. So let us first see what a projection is.

7.3.2 Projection

In this section we shall talk about an important concept in geometry. This concept has even been used by artists through centuries for giving depth to their works of art. Let us define it.

Definition : Let A be a point in the XYZ coordinate system.
The **projection of A** on a line segment PQ is the foot of the perpendicular drawn from the point A to the line.

From Fig.2 you can see that the projection of A on PQ will be the point O where the plane through A and perpendicular to PQ meets the line PQ.

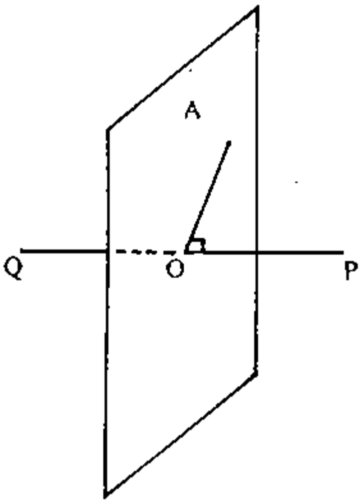


Fig. 2 : Projection of the point A on the line PQ is 0.

Definition : The projection of a line segment AB on a line PQ is the segment A′ B′ of the line PQ. where A′ and B′ are respectively the projections of the points A and B on the line PQ. (see Fig. 3)

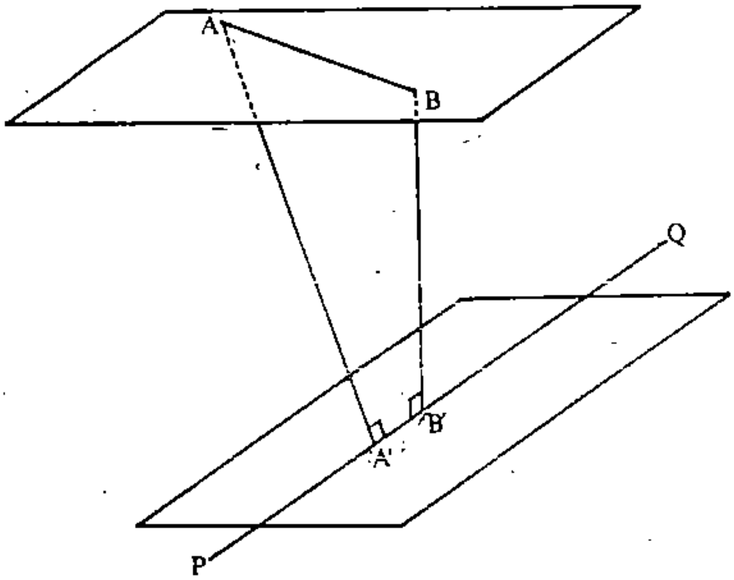


Fig. 3 : The Projection of the line segment AB on the line PQ is the line segment A′B′

Remark : From Fig.4 you can see that the length of A'B' = |AB| cosθ, where θ is the angle between AB and A'B' We also call this number the projection of AB on PQ.

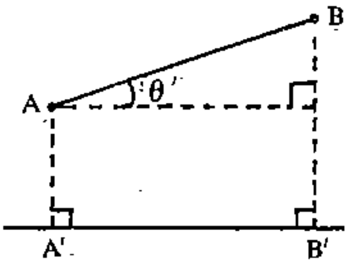


Fig. 4:

So what will the projection of BA be? It will be |BA| cos.(π + θ) that is − AB cosθ. This shows that the projection can be positive or negative depending upon the direction of the line segment.

Whenever you come across the term projection, from the context it will be clear whether we are referring to a line segment or a number.

We shall now state a simple result which we shall need while discussing rotation of axes.

Theorem 2: Suppose $A_1, A_2, \ldots\ldots, A_n$ are n points in space. Then the algebraic sum of the projections of $A_1A_2, A_2A_3, \ldots\ldots, A_{n-1}A_n$ on a line is equal to the projection of $A_1\ A_n$ on that line.

We will not prove this result in general here; but shall give you a proof in a particular case only. The proof, in any situation, is on the same lines.

Proof: Consider the situation in Fig.5 concerning 4 points A_1, A_2, A_3, A_4 and their projections A'_1, A'_2, A'_3, A'_4 on a given line L.

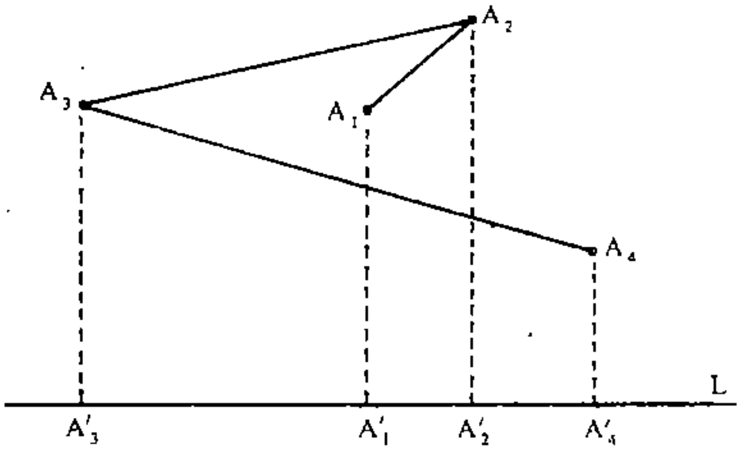


Fig. 5: Projections of the line segments A_1A_2, A_2A_3, A_3A_4 on a given line L.

Then

$$A'_1A'_2 + A'_2A'_3 + A'_3A'_4 = A'_1A'_2 - (A'_3A'_1A'_1A'_2) + (A'_3A'_1 + A'_1A'_2 + A'_2A'_4) = A'_1A'_4$$

This shows that the sum of the projections of the line-segments A_1A_2, A_2A_3, A_3A_4, is equal to the projection of A_1A_4. So, we have proved the result for the situation in Fig. 5.

Now consider another useful result involving projections.

Theorem 3 : Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points in the XYZ coordinate system. Then the projection of the line segment PQ on a line with direction cosines l, m, n is given by

$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

Proof : In Unit 4, Sec. 4.3.2, you have seen that the direction ratios of the line joining P and O are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$.

Let |PQ| denote the distance between P and Q, i.e.,

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Then the direction cosines of the line segment PQ are

$$\frac{x_2 - x_1}{|PQ|}, \quad \frac{y_2 - y_1}{|PQ|}, \quad \frac{z_2 - z_1}{|PQ|}$$

Let θ be the angle between the lines PQ and L. Then the projection of the line segment PQ on the line L is |PQ| cos θ. But, from Unit 4, Sec. 4.3.3, we have

$$\cos\theta = \frac{x_2-x_1}{|PQ|}l + \frac{y_2-y_1}{|PQ|}m + \frac{z_2-z_1}{|PQ|}n$$

Therefore, **the required projection** = $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$

For example, what will the projection of the line segment joining O (0, 0, 0,) to the point P (5, 2, 4) on the line having 2, −3, 6 as its direction ratios be ? We know that the direction cosines of the line with directionratios 2, −3, 6 is.

$$\frac{2}{\sqrt{(2)^2+(-3)^2+(6)^2}} \quad \frac{-3}{\sqrt{(2)^2+(-3)^2+(6)^2}}$$

$$\frac{6}{\sqrt{(2)^2+(-3)^2+(6)^2}}$$

i.e. $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

Thus, the projection of OP = $5 \times \frac{2}{7} + 2 \times (-\frac{3}{7}) + 4 \times \frac{6}{7} = 4$.

Now here is an exercise for you.

E3) Let P(6, 3, 2), Q(5, 1, 4), R(3, −4, 7) and S(0, 2, 5) be four points in space. Find the projection of the line segment PQ on RS.

Now we are in a position to discuss how the coordinates in space are affected by changing the direction of the axes without changing the origin.

7.3.3 Rotation of Axes

Let us now consider the transformation of coordinates when the coordinate system is rotated about the origin through an angle θ. Let the original system be OXYZ. Suppose we rotate the coordinate axes through an angle θ in the anti-clockwise direction. Let OX', OY', OZ' denote the new coordinate axes (see Fig. 6). Suppose the direction cosines of OX', OY' and OZ' be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, respectively.

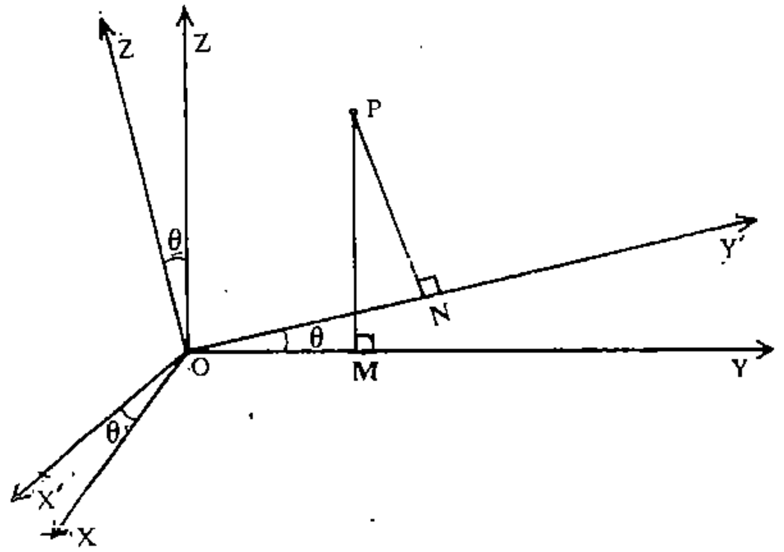


Fig.6 : The axes OX', OY' and OZ' are obtained by rotating the axes OX, OY and OZ through an angle θ

Let, P be any point in space having coordinates (x, y, z) and (x', y', z') with respect to the old and new coordinate systems. Let PN be the perpendicular from P on OY'. Then

ON = y'

The line segment ON is also the projection of OP on the line OY' with direction cosines l_2, m_2, n_2. Therefore, by Theorem 2, we have

ON = $(x-0)l_2 + (y-0)m_2 + (z-0)n_2$.

Hence we get

$y' = xl_2 + ym_2 + zn_2$ .... (9)

Similarly we can show that

$x' = xl_1 + ym_1 + zn_1$ .... (10)

and

$z' = xl_3 + ym_3 + zn_3$ .... (11)

Therefore given (x, y, z) and the direction cosines of the new coordinate axes, we can get the new coordinates (x', y', z') using equations (9), (10) and (11)

Now how can we find x, y, z in terms of x', y', z'? For this we draw PM perpendicular to OY (see Fig.6). Then

OM = y.

OM is also equal to the projection of OP on OY. Now let us see what the direction cosines of OY with respect to the new coordinate axes OX', OY', OZ' are. We know that the direction cosines of OX', OY', OZ' with respect to OY are m_1, m_2 and m_3. Do you agree that the direction cosines of OY with respect to OX', OY', OZ' are also m_1, m_2, m_3?(We leave this as an exercise for you to verify.) Therefore, by Theorem 2, we get

$Y = OM = (x' - 0)m_1 + (y' - 0)m_2 + (z' - 0)m_3$.

i.e., $y = m_1x' + m_2y' + m_3z'$ ...(12)

Similarly, we get

$x = l_1x' + l_2y' + l_3z'$ ...(13)

and

$z = n_1x' + n_2y' + n_3z'$ ...(14)

Hence (12), (13) and (14) give the coordinates of x, y, z in terms of x', y' and z'.

You may find that these equations are not easy to remember. For easy reference, we arrange the equations in a table, as shown in Table 1.

Table 1: Transformation of coordinates

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Note that for finding x, y, z we make use of the elements in the respective columns, and to find x', y' and z' we make use of the elements in the respective rows.

Let us consider an example.

Example 1 : Find the new equation of the conicoid

$3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 1$

when the coordinate system is transformed into a new system with the same origin and with the coordinate axes having direction ratios −1, 0, 1; 1, −1, 1; 1, 2, 1 with respect to the old system.

Solution : The given surface is

$3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy = 1$ ...(15)

Let OX' Y' Z' be the new coordinate system. Then the direction cosines of OX', OY' and OZ' with respect to the original axes are

$\frac{1}{\sqrt{2}}$, 0, $\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{3}}$; $\frac{-1}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{6}}$; $\frac{2}{\sqrt{6}}$; $\frac{1}{\sqrt{6}}$; respectively.

We form the transformation table:

	x	y	z
x'	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
y'	$\frac{1}{\sqrt{3}}$	$\frac{-1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
z'	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$

From the table we have

$$x = \frac{1}{\sqrt{2}} \times x' + \frac{1}{\sqrt{3}} \times y' + \frac{1}{\sqrt{6}} \times z'$$

$$= -\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}$$

$$\text{and } y = 0 \times x' + \left(\frac{-1}{\sqrt{3}}\right) \times y' + \left(\frac{-2}{\sqrt{6}}\right) \times z'$$

$$= -\frac{y'}{\sqrt{3}} - \frac{2y'}{\sqrt{6}}.$$

$$\text{and } z = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}$$

To find the new equation, we substitute the expressions of x, y, z in a (15). Then we get

$$3\left(-\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)^2 + 5\left(-\frac{y}{\sqrt{3}} + \frac{2z'}{\sqrt{6}}\right)^2 + 3\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)^2$$

$$+ 2\left(-\frac{y'}{\sqrt{3}} + \frac{2z'}{\sqrt{6}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)$$

$$\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right) + 2\left(-\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{3}} + \frac{z'}{\sqrt{6}}\right)\left(-\frac{y}{\sqrt{3}} + \frac{2z'}{\sqrt{6}}\right) = 1$$

Simplifying each term of the above expression and collecting the coefficients of like terms, we get
$2x'^2 + 3y'^2 + 6z'^2 = 1.$
This is the new equation of the conicoid.

You can do the following exercises on the same lines.

E4) Find the new equation of the following conicoids when the coordinates system is changed into a new system with the same origin and direction ratios 1, 2, 3; 1, −2, 1; 4, 1, −2; with respect to the old system.
a) $x^2 - 5y^2 + z^2 = 1.$
b) $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - y + z = 0$

E5) For the conicoid in Example 1 and E4, calculate the sum of coefficients of the square terms in the original equation and in the new equation. Can you infer anything from the outcome?

E6) What will the new equation of the plane x+y+z = 0 be when the coordinate system XYZ is transformed into another coordinate system X′Y′Z′ by the following equations?

$$x = \frac{x'}{\sqrt{6}} + \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}}$$

$$y = -\frac{2}{\sqrt{6}}x' + \frac{z'}{\sqrt{3}}$$

$$z = \frac{x'}{\sqrt{6}} - \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}}$$

E7) Does a cone remain a cone under rotation of axes? Give reasons for your answer.

Now let us consider the effect of rotation of axes on F(x,y,z) = 0.

Theorem 4 : Let S be a conicoid satisfying a second degree equation in a coordinate system XYZ is transformed into another coordinate system X′Y′Z′ by the direction of axes, without changing the origin, S will still be represented by a second degree equation.

Proof : Suppose S is represented by the second degree equation
$\Sigma ax^2 + 2\Sigma fyz + \Sigma\, 2ux + d = 0$ in the XYZ−systems.

Let $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the direction cosines of the new coordinate axes OX′ OY′, OZ′ respectively. Then you know that the coordinates (x, y, z), (x′, y′, z′) of a point in the old and new system respectively, satisfy the following relationship.

$x = l_1x' + l_2y' + l_3z'$
$y = m_1x' + m_2y' + m_3z'$
$z = n_1x' + n_2y' + n_3z'$

Let us substitute these expressions in the given equation. We consider the second degree parts and linear parts of the equation separately.

The second degree part is $\Sigma ax^2 + 2\Sigma fyz$. When we substitute the expressions for x,y,z in this part, we get

$\left\{(l_1x' + l_2y' + l_3z')^2 + 2f(m_1x' + m_2y' + m_3z')(n_1x' + n_2y' + n_3z')\right\}$.

The coefficient of x'^2 in the above expression is

$(al_1^2 + bm_1^2 + cn_1^2 + 2fm_2n_2 + 2gn_1l_1 + 2hl_1m_1)$.

Similarly the coefficient of y'^2 in the above expression is

$(al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2)$,

and that of z'^2 is

$(al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gh_3l_3 + 2hl_3m_3)$.

Similarly we collect the coefficients of y'z', z'x' and x'y' Then we get an expression of the form

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' \qquad \ldots(16)$$

You can also see that the linear part becomes

$$u'x + v'y + w'z \qquad \ldots.(17)$$

Where

$u' = ul_1 + vm_1 + wn_1$
$v' = ul_2 + vm_2 + wn_2$, and
$w' = ul_3 + vm_3 + wn_3$

From (16) and (17) we see that the transformed equation is a general second degree equation.

Now looking at expression (17), can you say anything about the change in the constant term? It remains unchanged under rotation of axes. Another interesting fact that you may have observed in the proof above is given in the following exercise.

E8) Suppose that the second degree part $\Sigma ax^2 + 2\Sigma fyz$ of a general second degree equation transforms into $\Sigma\, a'x'^2 + 2\Sigma f'\, y'z'$ under rotation of axes. Show that $a+b+c = a'+b'+c'$.

Well, let us see what we can gather from Theorems 1 and 4. They say that if a conicoid S is represented by a second degree equation $F(x,y,z) = 0$ in one coordinate system, then it is still represented by a second degree equation in any other coordinate system obtained by a translation or rotation of axes.

Thus

a conicoid remains a conicoid under translation or rotation of axes.

In fact, every geometrical figure remains unchanged in shape and size under translation or rotation of axes. Therefore these transformations are called **rigid body motions.**

In this section we have discussed two important transformations of a three-dimensional coordinate system. We also said that the importance of these transformations lies in the fact that we can use them to reduce any general second degree equation in variables into a simpler form. Let us see how this happens.

7.4 REDUCTION TO STANDARD FORM

In this section we shall show that by suitably applying the transformations that we have discussed in the previous section, we can write the general equation of a conicoid in a simpler form.

Let us consider a conicoid given by the equation

$F(x, y, z) = \Sigma ax^2 + 2\Sigma fyz + \Sigma 2ux + d = 0$

Let us assume that there exists a new Cartesian coordinate system, obtained by translating the origin, in which the linear part of $F(x,y,z) = 0$ vanishes. You will see that this is possible only for a particular type of conicoids.

Let the coordinates of O′ be (x_0, y_0, z_0) in the new system. Then we know that in the transformed equation, the second degree part is unchanged and the linear part becomes

$u'x' + v'y' + w'z'$

where u′, v′ and w′ are as in (8). We have assumed that the linear part vanishes. Therefore $u' = v' = w' = 0$. This means that we should have

$ax_0 + hy_0 + gz_0 + u = 0$

$hx_0 + by_0 + fz_0 + v = 0$

$gx_0 + fy_0 + cz_0 + w = 0$

In other words, (x_0, y_0, z_0) is a solution of the system of equations

$$\left.\begin{array}{l} ax + hy + gz + u = 0 \\ hx + by + fz + v = 0 \\ gx + fy + cz + w = 0 \end{array}\right] \quad \ldots.(18)$$

If the system of equations (18) has a solution for $(x_0, y_0, z_0) \in R^3$, then the point, (x_0, y_0, z_0), is called a centre of the given conicoid and we say that the conicoid has a centre at (x_0, y_0, z_0). You will understand why this is called a centre later in Unit 8.

Now let us assume that the given conicoid S has a centre. We transfer the origin to the centre (x_0, y_0, z_0). Then the transformed equation becomes

$\Sigma ax^2 + 2\Sigma fyz + 2u'x + 2v'y + 2w'z + d' = 0.$

(Recall that the second degree part does not change by shifting the origin.)

Since (x_0, y_0, z_0) is a solution to (18), we see that $u' = v' = w' = 0$. Therefore, the above equation reduces to

$(\Sigma ax^2 + 2\Sigma fyz) + d' = 0$

This equation does not have any linear part.

We have just proved the following result.

See Block 2, MTE-04 for simultaneous linear equations.

Theorem 5 : Suppose that S is a conicoid which is represented by a general second degree equation $F(x,y,z) = 0$ in a coordinate system XYZ. Suppose that S has a centre O′ (i.e. the system of equations (18) has a solution (x_0, y_0, z_0)). Then, by shifting the origin to the centre O′, the equation assumes the form

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0$

in the new coordinate system X′Y′Z′.

Let us consider an example.

Example 2 : Consider the conicoid given by the equation

$2x^2 + 3y^2 + 4z^2 - 4x - 12y - 24z + 49 = 0.$

Does it have a centre? If so, find it.

Solution : The given equation is

$F(x, y, z) = 2x^2 + 3y^2 + 4z^2 - 4x - 12y - 24z + 49 = 0$

We shall first check whether the given equation has a centre or not, that is, if the system of equations (18) has a solution or not. Here a = 2, b = 3, c = 4, u = −2, v = −6, w = −12. Then we have

$2x - 2 = 0$

$3y - 6 = 0$

$4z - 12 = 0$

This set of equations has a solution, namely, (1,2,3). Hence, (1,2,3) is a centre of S.

Note : Let us go back to (18) for a moment. There we saw that if the system of equations has a solution (x_0,y_0,z_0), then (x_0,y_0,z_0) is a centre of the conic. In Unit 5 of the course 'Elementary Algebra', you have seen that the system of equations has a solution if

$$\Delta = \begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

In fact if $\Delta \neq 0$, then there exists a unique solution.

Why don't you try some exercise now?

E9) Check whether the following conicoids have a centre or not

a) $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y + 4z + 1 = 0$

b) $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy - 2x + 2y - 2z - 3 = 0.$

c) $5x^2 + 6y^2 - 2x = 0$

E10) Find a centre of the conicoid
$14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy + 18x - 18y + 5 = 0$. What will its new equation be if the origin is shifted to this centre?

We shall now state a theorem without proving it. To prove this we need some advanced techniques in calculus which are beyond the scope of this course.

Theorem 6 : Suppose S is a given conicoid whose equation is given by $F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$
with respect to a given system of coordinates XYZ. Then there exists a new Cartesian coordinate system X'Y'Z' obtained by rotating the axes of the given system XYZ, without shifting the origin, such that the representation of S in the new system takes the form
$G(x',y',z') = a'x'^2 + b'y'^2 + c'z'^2 + 2u'x' + 2v'y' + 2w'z' + d' = 0.$
That is, the new equation does not contain the product terms. yz, zx and xy.

Combining Theorems 5 and 6 we have the following result.

Corollary 1 : Let S be a conicoid given by the equation $F(x,y,z) = 0$, which has a centre O' in a coordinate system XYZ. There exists a new coordinate system obtained by shifting the origin from O to O' and then rotating the system about O', in which the equation takes the simpler form

$$a'x'^2 + b'y'^2 + c'z'^2 + d' = 0 \qquad \text{....(19)}$$

If $d' \neq 0$ then we can divide throughout by d', and we get the form
$ax^2 + by^2 + cz^2 = 1$,
where $a = -\frac{a'}{d'}$, $b = -\frac{b'}{d'}$ and $c = -\frac{c'}{d'}$.

(19) is called the standard equation of a conicoid .

Recall that in the case of the two-dimensional system also we have seen that we can reduce any second degree equation into a simple form.

Let us now consider some examples.

Example 3 : Show that the conicoid given by $x^2 + 2yz - 4x + 6y + 2z = 0$ has a centre. Reduce it to standard form by shifting the origin to the centre, and then rotating the axes to get a new system in which the direction ratios of the new axes are given by

$$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; \; 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; \; 1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

with respect to the original coordinate system.

Solution : Here $a = 1, b = 0, c = 0, f = 1, g = 0, h = 0, u = -2, v = 3, w = 2.$

We first check whether the conicoid has a centre or not. Using (18) we see that (2,0,0) is a centre of the given conicoid.

Shifting the origin from (0,0,0) to (2,0,0), we get the new equation as
$x^2 + 2yz - 4 = 0$
Now we apply a rotation of axes to the new equation. We note that the direction cosines of the new axes are

$$0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; \; \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}; \; \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{4}}, -\frac{1}{\sqrt{4}}$$

From Sec.7.3.3 we have

$$x = -\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{2}}$$

$$y = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{4}}y' - \frac{1}{\sqrt{4}}z'$$

$$z = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{4}} y' - \frac{1}{\sqrt{4}} z'$$

Substituting these equations in the given equations of the conicoid, we see that

$$\left(\frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{2}} \right)^2 + 2 \left(- \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{4}} y' - \frac{1}{\sqrt{4}} z' \right) \left(\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{4}} y' - \frac{1}{\sqrt{4}} z' \right) - 4 = 0.$$

i.e. $\frac{y'^2}{2} + \frac{z'^2}{2} - x'^2 - 4 = 0$

i.e. $y'^2 + z'^2 - 2x'^2 = 8.$

which is in the standard form.

Now here is an exercise for you.

E11) Find the standard equation of the following conicoids.

a) $x^2+y^2+z^2-2x-2y-2z-1 = 0$, by shifting the origin to the centre.

b) $3x^2+5y^2+3z^2-2yz+2zx-2xy+2x+12y+10z+20 = 0$, by shifting the origin to the centre and then rotating the system so that the direction ratios of the new axes are
$-1,0,1;\ 1,1,1;\ 1,-2,1.$

We will stop our discussion on general theory of conicoids for now, though we shall refer to them off and on in the following units. In the next unit we will discuss the surfaces formed by (19).

Let us now do a quick review of what we have covered in this unit.

7.5 SUMMARY

In this unit we have covered the following points:

1) A general second degree equation in three variables
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$
represents a conicoid.

2) Translation of axes: the transformation of a coordinate system in which the origin is shifted to another point without changing the direction of the axes. The equations of transformations are given by
$x = x' + a$
$y = y' + b$
$z = z' + c$

3) Rotation of axes: the transformation of a coordinate system in which the direction of axes is changed without shifting the origin. The equations of transformation are given by the following table:

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

where $l_i, m_i, n_i, i = 1, 2, 3$ are the direction cosines of the axes.

4) A conicoid remains a conicoid under a translation or rotation of axes.

5) There is a Cartesian coordinate system in which the equation of a conicoid with a centre takes the standard form
$a'x'^2 + b'y'^2 + c'z'^2 + d' = 0.$

And now you may like to check whether you have achieved the objectives of this unit (see Sec.7.1). If you would like to see our solutions to the exercises in this unit, we have given them in the following section.

7.6 SOLUTIONS/ANSWERS

E1)a) In Unit 6 of Block 2 you have seen that the equation of a right circular cone with vertex O, axis OZ and semi-vertical angle α is

$x^2 + y^2 = z^2 \tan^2 \alpha$.

When we shift the origin to $(-1,1,0)$, then the coordinates in the new system are given by

$x' = x+1, \qquad y' = y-1, \qquad z' = z,$

i.e. $x = x' - 1 \qquad y = y'+1, \qquad z = z'$

Substituting for x,y,z in the given equation of the cone, we get

$(x'-1)^2+(y'+1)^2 = z'^2\tan^2\alpha$.

b) This equation represents a right circular cone with vertex at the point $(-1,1,0)$ axis along the line parallel to the z-axis through the vertex and semi-vertical angle α (see E1 in Sec.6.2 of Unit 6, Block 2).

E2) The equations of transformation are given by

$x = x'+1, \; y = y'-3, \; z = z'+2$

a) the given equation is

$x^2+y^2+z^2-4x+6y-2z+5 = 0$

Substituting for x,y,z, in the above equation, we get

$(x'+1)^2+(y'-3)^2+(z'+2)^2-4(x'+1)+6(y'-3)-2(z'+2) + 5 = 0$

Simplifying, we get

$x'^2+y'^2+z'^2-2x'+2z'-7 = 0$

which represents a sphere

b) The transformed equation is

$x'^2-2y'^2+2x'-4y'-3z'-7 = 0$.

E3) The direction ratios of RS are $-3, 6, -2$.

Therefore, the direction cosines of RS are $-\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}$.

Hence the projection of PQ on RS is

$$(5-6) \times (-\tfrac{3}{7}) + (1-3) \times \tfrac{6}{7} + (4-2) \times (-\tfrac{2}{7}) = \tfrac{3}{7} - \tfrac{12}{7} - \tfrac{4}{7} = -\tfrac{13}{7}.$$

E4) a) The given equation is

$x^2-5y^2 + z^2 = 1$

The direction ratios of the axes are given by

1, 2, 3; 1, −2; 1; 4, 1, −2.

Therefore the direction cosines of the axes are given by

$$\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}; \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, -\frac{2}{\sqrt{21}}.$$

The transformation Table is

	x	y	z
x'	$\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{3}{\sqrt{14}}$
y'	$\frac{1}{\sqrt{6}}$	$\frac{-2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$
z'	$\frac{4}{\sqrt{21}}$	$\frac{1}{\sqrt{21}}$	$\frac{-2}{\sqrt{21}}$

Then we have

$$x = \frac{1}{\sqrt{14}} x' + \frac{1}{\sqrt{6}} y' + \frac{3}{\sqrt{21}} z'$$

$$y = \frac{2}{\sqrt{14}} x' - \frac{2}{\sqrt{6}} y' - \frac{1}{\sqrt{21}} z'$$

$$z = \frac{3}{\sqrt{14}} x' + \frac{1}{\sqrt{6}} y' - \frac{2}{\sqrt{21}} z'$$

Substituting this in the given equation, we get

$$\left(\frac{1}{\sqrt{14}}x' + \frac{1}{\sqrt{6}}y' + \frac{4}{\sqrt{21}}z'\right)^2 \quad -5 \quad \left(\frac{2}{\sqrt{14}}x' - \frac{2}{\sqrt{6}}y' + \frac{1}{\sqrt{21}}z'\right)$$

$$+\left(\frac{3}{\sqrt{14}}x' + \frac{1}{\sqrt{6}}y' - \frac{2}{\sqrt{21}}z'\right)^2 = 1$$

Simplifying we get

$$\frac{10}{14}x'^2 - 3y'^2 + \frac{15}{21}z'^2 + \frac{48}{\sqrt{14}\sqrt{6}}x'y' + \frac{24}{\sqrt{6}\sqrt{21}}y'z' - \frac{24}{\sqrt{21}\sqrt{14}}x'z' = 1$$

b) The new equation is $x'^2 + y'^2 + z'^2 + \frac{8}{\sqrt{6}\ \sqrt{21}}y'z' - \frac{8}{\sqrt{21}\ \sqrt{14}}x'z' = 1.$

E5) From E4(a) we get
$a+b+c = -5+2 = -3$
and $a'+b'+c' = \frac{10}{4} - \frac{18}{6} + \frac{15}{11} = -3$
This show that under rotation the sum of coefficients of the square terms remains unaltered in value, i.e., $a+b+c = a'+b'+c'$.

Similarly you can observe the same for E4(b) and Example 1 also.

E6) From the given equations of transformations we see that the coordinate system is changed into another coordinate system with the same origin and the direction cosines of the new axes, with respect to the old system, are given by

$$\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

To get the new equation, we substitute the values of x,y,z in the equation $x+y+z = 0$. Then we get

$$\frac{x'}{\sqrt{6}} + \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}} - \frac{2}{\sqrt{6}} + \frac{z}{\sqrt{3}} + \frac{x'}{\sqrt{6}} - \frac{y'}{\sqrt{2}} + \frac{z'}{\sqrt{3}} = 0,$$

Therefore under the transformation the plane $x+y+z = 0$ becomes the plane $z' = 0$.

E7) Yes. This is because with vertex at the origin is represented by a homogeneous second degree equation.
$ax^2+by^2+cz^2+2hxy+2fyz+2gzx = 0.$
Now, from the proof of Theorem 2, we see that under rotation of axes, the above equation becomes
$a'x'^2+b'y'^2+c'z'^2+2h'x'y'+2f'y'z'+2g'z'x' = 0,$
which is again a homogeneous second degree equation.
Therefore, it represents a cone.

E8) From the proof of theorem 2, we have
$a' = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1 l_1 x + 2hl_1m_1$
$b' = al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2 l_2 + 2hl_2m_2$
$c' = al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3 l_3 + 2hl_3m_3$
then $a' + b' + c' = a(l_1^2 + l_2^2 + l_3^2) + b(m_1^2 + m_2^2 + m_3^2) + c(n_1^2 + n_2^2 + n_3^2) + 2f(m_1 n_1 + m_2 n_2 + m_3 n_3) + 2g(n_1l_1 + n_2l_2 + n_3l_3) + 2h(l_1m_1 + l_2m_2 + l_3m_3)$.
We know that

$$\sum_{i=1}^{3} l_i^2 = 1 = \sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2$$

(see Unit 4, Block 2).

Also, since the axes are mutually perpendicular, by condition for perpendicularity given in Unit 4, Block 2, we get

$$\sum_{i=1}^{3} m_i n_i = 0 = \sum_{i=1}^{3} n_i l_i = \sum_{i=1}^{3} l_i m_i$$

Therefore, we get $a' + b' + c' = a + b + c$

E9) a) We have to see whether the following system of linear equations is consistent or not.

$ax+hy+gz+u = 0$

$hx+by+fz+v = 0$

$gx+fy+cz+w = 0$

Here $a = 3$, $b = 7$, $c = 3$, $f = 5$, $g = -1$, $h = 5$, $u = 2$, $v = -6$. $w = -2$. So we have

$3x+5y-z+2 = 0$

$5x+7y+5z-6 = 0$

$-x+5y+3z-2 = 0$

On solving this system of equations, we find that it has a unique solution given by $x = \frac{1}{3}$, $y = -\frac{1}{3}$ and $z = \frac{4}{3}$. Therefore the given conicoid has a unique centre at $(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3})$

b) This conicoid has infinitely many centres.

c) This conicoid has no centre since the system of equations (18) is inconsisten in this case.

E10) The conicoid has a centre at $(-\frac{1}{2}, \frac{1}{2}, 0)$. Now we shift the origin from (0,0,0) to $(-\frac{1}{2}, \frac{1}{2}, 0)$. The equations of transformation are

$x = x' - \frac{1}{2}$, $y = y' + \frac{1}{2}$, $z = z'$

Substituting for x,y,z in the given equation, we get the new equation as

$14x^2+14y^2+8z^2-4yz-4zx-8xy = 4$

E11) a) We know that the given equation represents a sphere with centre at (1,1,1) (see Unit 5). Then by shifting the origin to the centre we get the standard equation as

$x'^2+y'^2+z'^2 = 4.$

b) The given equation is

$3x^2+5y^2+3z^2-2yz+2zx-2xy+2x+12y+10z+20 = 0.$

We first check whether the conicoid represented by this equation has a centre.

Here $a = 3$, $b = 5$, $c = 3$, $h = -1$, $g = 1$, $f = -1$, $u = 1$, $v = 6$, $w = 5$ and $d = 20$. The system of equations for transformations are given by

$3x-y+z+1 = 0$

$-x+5y-z+6 = 0$

$x-y+3z+5 = 0$

Solving these equations, we get $x = -\frac{1}{6}$, $y = -\frac{5}{3}$ and $z = -\frac{13}{6}$. Hence a centre is $(-\frac{1}{6}, -\frac{5}{3}, -\frac{13}{6})$.

Now we shift the origin to the centre. Then we get the new equation as

$3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + d' = 0,$

Where $d' = 3(-\frac{1}{6})^2 + 5(-\frac{5}{3})^2 + 3(-\frac{13}{6})^2 - 2(-\frac{5}{3})(-\frac{13}{6}) + 2(-\frac{13}{6})(-\frac{1}{6}) - 2(-\frac{1}{6})(-\frac{5}{3}) + 2(-\frac{1}{6}) + 12(-\frac{5}{3}) + 10(-\frac{13}{6}) + 20.$

Now we apply the rotation of axes. The equations of transformation are

$x = -x'+y'+z'$

$y = y'-2z'$

$z = x'+y'+z'$

Substituting for x,y,z in the given equation of the conicoid, we get

$3(-x'+y'+z')^2 + 5(y'-2z')^2 + 3(x'+y'+z')^2 - 2(y'-2z')(x'+y'+z') + 2(x'+y'+z')(-x'+y'+z') - 2(-x'+y'+z')(y'-2z') + d' = 0$

i.e., $4x'^2 + 9y'^2 + 36z'^2 + d' = 0$

This is the standard form of the given conicoid.

UNIT 8 CENTRAL CONICOIDS

Structure

8.1 INTRODUCTION

In the previous unit you were introduced to certain surfaces in a three-dimensional system, which are called conicoids or quadric surfaces. There we discussed some general theory of conicoids and showed that a conicoid remains a conicoid under translation and rotation of axes. You also saw that some conicoids possess a centre and some don't. Based on this the conicoids are classified into two types — central and non-central conicoids. In this unit we shall concentrate only on central conicoids.

We first observe that a central conicoid can be reduced to a simpler form by an appropriate change of axes. Then we use these simpler forms to discuss the geometrical properties of different types of central conicoids. You will see that there are four types of central conicoids — cone, imaginary ellipsoid, ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets. You have already studied cones in detail in Unit 6. The remaining three real conicoids are the three-dimensional versions of the central conics that you studied in Block 1, namely, ellipses and hyperbolas.

Ancient mathematicians like Euclid, Archimedes and Appolonius were familiar with the above mentioned geometrical objects, though they did not study them analytically. The analytical study of these objects started much later, with the application of the three-dimensional coordinate system to geometry. The first mathematician to suggest the extension of the two-dimensional coordinate system to three dimensions was the Swiss mathematician John Bernoulli (1667-1748). But the actual application of space coordinates to geometry was done by another Swiss mathematician Jacob Hermann (1678-1733). He applied them to obtain the equations of several types of quadric surfaces.

Even though we are only concerned with central conicoids in this unit, in the first section we shall consider some necessary general theory of conicoids which we have not covered in the previous unit. We shall define a centre of a conicoid and obtain a characterisation for central conicoids. Then we shall discuss the four different types of central conicoids in separate sections. We will end this unit with a discussion on sections obtained by intersecting a central conicoid by a line or a plane. In this connection we also discuss tangents to a central conicoid.

In the next unit we will tackle non-central conicoids. That will be easier for you to grasp if you ensure that you have achieved the objectives given below.

Objectives

After studying this unit you should be able to :

- check whether a conicoid is central or not if you are given its equation;
- obtain standard forms of an ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets;

- trace the standard forms of the above mentioned three conicoids;
- obtain tangent lines and tangent planes at a point to a central conicoid;
- check whether a plane is a tangent plane to a conicoid or not;
- use the fact that a planar section of a central conicoid is a conic.

8.2 A CONICOID'S CENTRE

In the last unit you saw that a point P is called a centre of a conicoid $F(x,y,z) = 0$ if its coordinates satisfy a system of linear equations (see Equations (18) of Unit 7). In this unit we define a centre geometrically and then see the relationship between the geometrical and analytical definitions. Let us start with a theorem. (We shall not prove it here, but we have discussed it in detail for a particular type of conicoid in Sec. 8.7.)

Theorem 1 : Any line intersects a conicoid in two points, which may be distinct real points, coincident real points or imaginary points.

Note : If a line L intersects a conicoid in imaginary points, then we say that L **does not intersect** S.

Let us consider an example.

Example 1: Consider the conicoid given by
$ax^2 + by^2 + cz^2 = 1$, $a,b,c, \neq 0$.
Let $P(x_1, y_1, z_1)$ be a point on this conicoid and O be (0,0,0).
Show that the other point of intersection of the line OP with the conicoid is $P'(-x_1, -y_1, -z_1)$.

Solution: You know from Unit 4 that the equation to the line passing through the points (0,0,0) and (x_1, y_1, z_1) is

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}.$$

Clearly, $(-x_1, -y_1, -z_1)$ lies on this line. You can easily verify that if (x_1, y_1, z_1) satisfies the equation $ax^2 + by^2 + cz^2 = 1$, then $(-x_1, -y_1, -z_1)$ also satisfies it. Hence, $P'(-x_1, -y_1, -z_1)$ is another point on the conicoid and line OP. But by Theorem 1, there are only two points of intersection. Therefore P and P' are the points of intersection.

The above example shows that any line through the origin O meets the conicoid $ax^2 + by^2 + cz^2 = 1$ in two points which are equidistant from O. Such a point O is called a centre of the conicoid, according to the following definition.

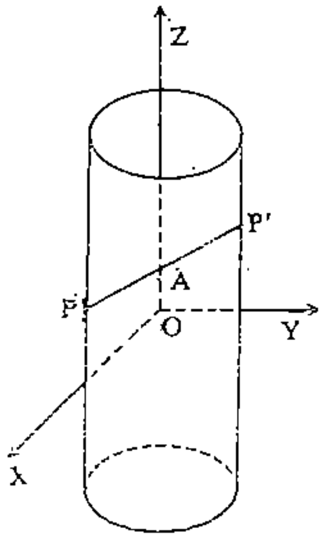
Definition : A point p is called a **centre of a conicoid** S if every line through P

i) intersects S in two points such that P is the midpoint of the line joining these two points, or

ii) does not intersect S at all.

Note that the definition is analogous to the definition of a centre in the two-dimensional system.

Using the above definition we can easily say that the origin O(0,0,0) is a centre of the sphere $x^2 + y^2 + z^2 = 1$. Does this sphere have any other point as centre? If you do E1, you'll be able to answer this.



ig. 1: A cylinder with z-axis as axis.

E1) Show that $(-u, -v, -w)$ is the only centre of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

E2) Show that every point on the z-axis is a centre of the cylinder $x^2 + y^2 - r^2 = 0$ (see Fig.1).

Next, we shall prove a result which tells us something about the equation of a conicoid with the origin as a centre.

Theorem 2 : The origin O is a centre of the conicoid
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ ....(1)
if and only if $u = v = w = 0$.

Proof : Suppose that $u = v = w = 0$. Then the given equation of the conicoid takes the form
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0$.
Suppose L is any line through O. Then, by Theorem 1, the line intersects the conicoid at two points, say P and Q. Now we have to show that O is the midpoint of PQ.

Let the coordinates of P be (x_1, y_1, z_1). Then we have
$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1 x_1 + 2hx_1y_1 + d = 0$.
We can rewrite this equation as
$a(-x_1)^2+b(-y_1)^2 + c(-z_1)^2+2f(-y_1)(-z_1)+2g(-z_1)(-x_1) + 2h(-x_1)(-y_1) + d = 0$.

This shows that $(-x_1, -y_1, -z_1)$ lies on the given conicoid. Further, you can see that the point $(-x_1, -y_1, -z_1)$ also lies on the line OP. Hence the point Q must be $(-x_1, -y_1, -z_1)$. This shows that O is the midpoint of PQ.
The above argument is true for any line L through O. Hence, O is the centre of the conicoid.

Conversely, suppose that O is a centre of the conicoid given by (1). Suppose P is any point (x_1, y_1, z_1) lying on the conicoid. Then the point P' $(-x_1, -y_1, -z_1)$ also lies on the conicoid, since O is the centre. Then we have
$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d = 0$ ....(2)
and
$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 - 2u_1x_1 - 2v_1y_1 - 2w_1z_1 + d = 0$ ....(3)
Subtracting (3) from (2), we get
$ux_1 + vy_1 + wz_1 = 0$
This shows that (x_1, y_1, z_1) lies on the plane $ux+vy+wz = 0$. This is true for any point (x_1, y_1, z_1) on the conicoid. But how can every point on a conicoid lie on the plane $ux + vy + wz = 0$? This can happen only if $u = v = w = 0$.
Hence the result.

Does Theorem 2 give you an inkling about why a centre is called a centre? You can see this fact in the following note.

Note : Suppose the origin is a centre of the conicoid S. Then we have seen that if a point $P(x_1, y_1, z_1)$ lies on S, then P' $(-x_1, -y_1, -z_1)$ also lies on S. This means that if a conicoid S has a centre at the origin, then S is symmetric about the centre. This is why such a point is called a centre (see Fig. 2).

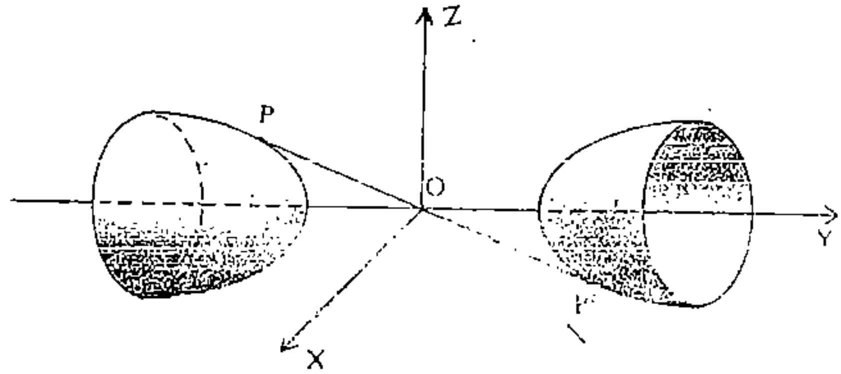


Fig. 2 : The Pair of points P and P' are symmetric about the centre O of the conicoid.

Now here is an exercise for you.

E3) Which of the following conicoids has a centre at the the origin?

a) $x^2 + y^2 + z^2 - 23x = 0.$

b) $2x^2 + 3y^2 - z^2 = 1.$

c) $14x^2 + 26y^2 + 2\sqrt{91}\,z^2 = 1.$

d) $41x^2 - 28y^2 = 0.$

In Unit 7 you saw that the existence of a centre is connected with the solvability of a system of equations. In the next theorem we establish this fact.

Theorem 3 : A conicoid S, given by the equation
$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$
has the point $P(x_0, y_0, z_0)$ as a centre if and only if

$$\left.\begin{array}{l} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{array}\right\} \qquad \ldots.(4)$$

Proof : Let us first assume that $P(x_0, y_0, z_0)$ is a centre of the given conicoid in the coordinate system XYZ. Now let us translate the origin from O to the centre P. Then from Unit 7 you know that the equation of the conicoid in the new coordinate system is given by

$$ax'^2 + by'^2 + cy'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2u'x' + 2v'y' + 2w'z' + d' = 0 \qquad \ldots.(5)$$

where

$u' = ax_0 + hy_0 + gz_0 + u$

$v' = hx_0 + by_0 + fz_0 + v$

$w' = gx_0 + fy_0 + cz_0 + w$

and

$d' = ax_0^2 + by_0^2 + cz_0^2 + 2fy_0z_0 + 2gz_0x_0 + 2hx_0y_0 + 2ux_0 + 2vy_0 + 2wz_0 + d$

Now, this conicoid has a centre at the origin. Therefore, by Theorem 2, we have $u' = v' = w' = 0$. This means that

$ax_0 + hy_0 + gz_0 + u = 0$

$hx_0 + by_0 + fz_0 + v = 0$

$gx_0 + fy_0 + cz_0 + w = 0$

Conversely, let us assume that (4) holds for some point $P(x_0, y_0, z_0)$. We shift the origin from O to P. Then we get an equation of the form (5). But since (4) holds for P, we see that

$u' = v' = w' = 0.$

Therefore the equation reduces to

$a'x'^2 + b'y'^2 + c'z'^2 + 2fy'z' + 2gz'x' + 2hx'y' + d' = 0.$

This equation does not have any first degree terms. Therefore, by Theorem 2, we see that the origin P is a centre.

Do you find any connection between Theorem 2 and Theorem 3? You might have noticed that Theorem 2 is a special case of Theorem 3 in the case when the point P is the origin.

Now let us go back to E1 and E2. From there you know that a conicoid may have a unique centre or infinitely many centres. In Unit 7 you have also seen examples of conicoids which have no centre. Mathematicians have divided all conicoids up into two types depending on whether they have a unique centre or not. We define these conicoids as follows.

Definition : A conicoid is called a **central conicoid** if it has a **unique centre**. A conicoid is called **non-central**, if it either has **no centre** or it has **infinitely many centres**.

Thus, a sphere is an example of a central conicoid and a cylinder is an example of a non-central conicoid.

Here is an exercise for you.

E4) Examine whether a cone is central or not.

We will look at non-central conicoids in the next unit. In this unit we will only discuss central conicoids.

Let us now start this discussion.

8.3 CLASSIFICATION OF CENTRAL CONICOIDS

In this section we shall obtain the different forms of central conicoids. We shall also study the shape of these conicoids.

Let us consider a central conicoid. Then, from Corollary 1 in Sec. 7.4 of Unit 7 you know that by first shifting the origin to the centre and then rotating the axes suitably about the centre, we can reduce the equation to its standard form

$$ax^2 + by^2 + cz^2 + d = 0. \qquad \ldots(6)$$

Now, let's go back for a moment to Theorem 3. Over there you saw that a conicoid $F(x,y, z) = 0$ has a unique centre iff the system of equations

$$ax + hy + gz + u = 0$$
$$hx + by + fz + v = 0$$
$$gx + fy + cz + w = 0$$

has a unique solution. This means that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$$

Now, if the conicoid is in the form (6), then the condition reduces to

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \neq 0, \text{ that is}$$

$abc \neq 0$.

This means that $a \neq 0$, $b \neq 0$, $c \neq 0$.

Then, depending upon the signs of a,b,c and d, we have the following five possibilities:

Case 1 ($d=0$) : In this case the equation reduces to
$ax^2 + by^2 + cz^2 = 0$.
You know from Unit 6 that this represents a cone, irrespective of the signs of a, b and c.

Case 2 ($d \neq 0$ and a, b, c, d are of the same sign) : In this case there are no real values of (x,y,z) which satisfy (6). This is because for any $(x,y,z) \in \mathbf{R}^3$, the left hand side is either positive or negative, never zero. We call such a conicoid an **imaginary conicoid**, Infact it represents an imaginary ellipsoid

Case 3 ($d \neq 0$ and the sign of the coefficients a,b,c are different from d): In this case we write (6) in the form
$ax^2 + by^2 + cz^2 = -d$,

$$\text{i.e.,} \quad \frac{x^2}{-\frac{d}{a}} + \frac{y^2}{-\frac{d}{b}} + \frac{z^2}{-\frac{d}{c}} = 1. \qquad \ldots(7)$$

Note that the numbers $-\frac{d}{a}$, $-\frac{d}{b}$ and $-\frac{d}{c}$ are positive.

Let $a_1 = \sqrt{-\frac{d}{a}}$, $b_1 = \sqrt{-\frac{d}{b}}$ and $c_1 = \sqrt{-\frac{d}{c}}$. Then (6) becomes

$$\frac{x^2}{{a_1}^2} + \frac{y^2}{{b_1}^2} + \frac{z^2}{{c_1}^2} = 1.$$

This equation is the three-dimensional analogue of the equation of an ellipse. We call the conicoid represented by this equation an **ellipsoid.**

Case 4 (d $\neq 0$ and two of the four coefficients a,b,c and d are of the same sign) Let us assume that $a > 0$, $b > 0$, $c < 0$ and $d < 0$.

Then $-\frac{d}{a}$, $-\frac{d}{b}$ and $\frac{d}{c}$ are positive. We put $a_2 = \sqrt{-\frac{d}{a}}$, $b_2 = \sqrt{-\frac{d}{b}}$, and $c_2 = \sqrt{\frac{d}{c}}$. Then (6) gives us

$$\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - \frac{z^2}{c_2^2} = 1.$$

The conicoid generated by this equation is called a **hyperboloid of one sheet**. (You will see why in Sec. 8.5.)

Similarly, we can obtain the equations of hyperboloids of one sheet in cases $a < 0$, $b < 0$, $c > 0$, $d > 0$, and so on.

Case 5 (d $\neq 0$ and two of a,b and c have the same sign as d) : As in the other case we assume that $a > 0$, $b < 0$, $c < 0$ and $d < 0$. Then $-\frac{d}{a} > 0$, $\frac{d}{b} > 0$, $\frac{d}{c} > 0$.

Put $a_3 = \sqrt{-\frac{d}{a}}$, $b_3 = \sqrt{\frac{d}{b}}$ and $c_3 = \sqrt{\frac{d}{c}}$. Then we have

$$\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} - \frac{z^2}{c_3^2} = 1.$$

The conicoid is called a **hyperboloid of two sheets.** (You will see why in Sec. 8.6.) The other forms of hyperboloids of two sheets can be similarly obtained.

Thus we saw that central conicoids can be classified into 5 types namely : cone, imaginary conicoid, ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets. We have tabulated this fact in Table 1.

Table 1: Standard Forms of Central Conicoids

Type	Standard form
Cone	$ax^2+by^2+cz^2=0$
Imaginary ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Hyperboloid of two sheets	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Why don't you do an exercise now?

E5) Identify the type of the conicoid from the following equations

a) $x^2+4y^2-z^2=1$

b) $16z^2=4x^2+y^2+16$

c) $\frac{x^2}{4}+y^2+\frac{z^2}{4}=1$

d) $z^2=3x^2+3y^2$

e) $x^2-y^2-z^2=9$

If you look closely at the cases where $d \neq 0$, you will see that the equations in these four cases can be given by a single equation
$ax^2+by^2+cz^2=1$

This will represent

i) an ellipsoid if a, b, c are all positive;

ii) a hyperboloid of one sheet if two of a, b, c are positive and the third is negative;

iii) a hyperboloid of two sheets if two out of a,b,c are negative and the third is positive;

iv) an imaginary conicoid if all the a,b,c are negative.

We shall study the shapes of the real conicoids listed above one by one. Let us start with ellipsoids.

8.4 ELLIPSOID

Let us consider the ellipsoid given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ where } a, b, c > 0 \qquad \dots (8)$$

Let S denote the surface generated by this equation.

From your experience of Unit 2, can you note some geometrical properties of S from the above equation? Of course, if a = b = c, then the equation represents a sphere. And you have studied the geometry of a sphere in detail in Block 2.

So, let us look at a more general case. From the following exercises you can get some idea of the geometrical aspects of the ellipsoid.

E6) Show that the surface represented by (8) is symmetric about the YZ-plane, ZX-plane and XY-plane.

E7) Do all the coordinate planes intersect the surface (8)? If so, find the sections obtained by the intersections.

E8) Check whether the coordinates axes intersect the surface (8).

If you have done the exercises, you will have noticed that the surface (8) intersects the coordinate axes in A(a,0,0) and A $(-a, 0, 0)$, B (0,b,0) and B′ $(0,-b,0)$, C(0,0,c) and C′ $(0,0,-c)$.

We say that a surface given by an equation $F(x, y, z) = 0$ is **symmetric with respect to the XY-plane**, if, when we replace z by $-z$ in $F(x, y, z)$, we get the same equation.
Symmetry with respect to the YZ-Plane and XZ-plane is similarly defined.

Now let us consider the intersections of the surface by planes parallel to the coordinate planes. Here we assume that $a \neq b$. Let us first consider a plane parallel to the XY-plane, say z = k, a constant.

Putting z = k in (8), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2} \qquad \dots (9)$$

If $\frac{k^2}{c^2}<1$, i.e., $-c<k<c$, the equation represents an ellipse.

This is true for all values of k such that $|k| \leq c$. If we put k = 0, we get an ellipse whose semi-axes are a and b.

Thus, the surface can be considered as a family of ellipses placed one on top of another lying between the planes $z = c$ and $z = -c$. Now what if $k > c$ in (9)? The equation only has imaginary roots. Therefore no portion of the surface lies beyond the plane $|z| = c$. Similarly, we can show that no part of the surface lies below $z = -c$. You can also check that the surface lies between the planes $y = -b$ and $y = b$, as well as between the planes $x = -a$ and $x = a$. Consequently, the surface is a bounded surface formed by ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},\ z=k, \text{ where } |k| \leq |c|.$$

Collecting all this information about (8), we get a figure as shown in Fig. 3.

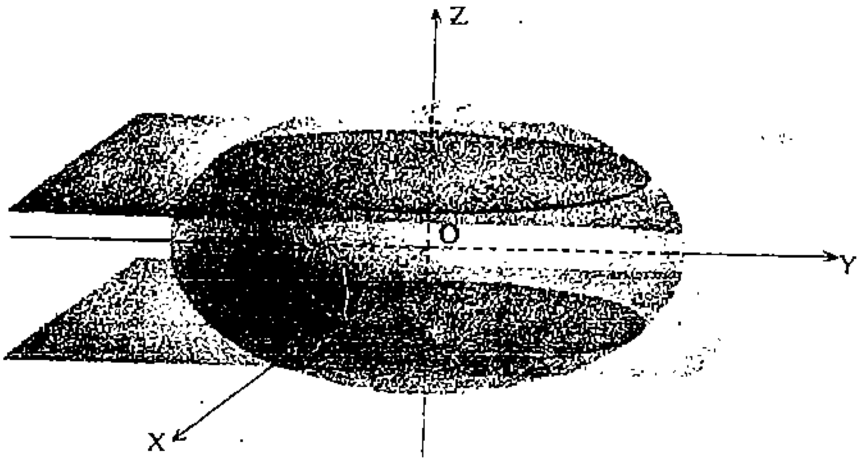


Fig. 3 : The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Its intersection with the planes $z = \frac{c}{2}$ and $z = -\frac{c}{2}$ are the ellipses E_1 and E_2

We shall make some remarks here.

Remark : Suppose we have $a > b = c$ in (8). Then (8) can be rewritten as

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1.$$

The intersection of this ellipsoid with the plane $z=0$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we revolve this ellipse about its major axis, i.e., the x-axis, then you can see that the surface formed is nothing but the given ellipsoid. This is the reason why the surface is known as an ellipsoid.

Now, what can you say about the ellipsoid in the case when $a = b > c$ in (8)? If $a = b > c$, the ellipsoid can be obtained by revolving the ellipse

$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,\ y = 0$ about its minor axis (i.e., the z-axis).

Let us consider an example now.

Example 3: Trace the conicoid $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$ ... (10

Solution : You know that the equation represents an ellipsoid. Let us try to trace it. We first consider the intersection of the surface with the coordinate axes. From (10) we see that the x-axis intersects the surface in points (3,0,0) and (−3,0,0), the y-axis intersects the surface in (0,4,0) and (0,−4,0) and the z-axis intersects the surface in two points (0,0,1) and (0,0,−1). Now let us consider the intersection of the ellipsoid with planes parallel to coordinate planes.

Consider the plane $z = h$, a constant. Putting $z = h$ in (10) we get

$$\frac{x^2}{9}+\frac{y^2}{16}=1-h^2 \qquad \ldots (11)$$

You know that this represents an ellipse for all h such that $|h| < 1$. If $h = 1$, we find that the intersection of surface with the plane $z = 1$ is the point (0,0,1). Similarly for $h = -1$, we get the point (0,0,−1).
If $|h| > 1$, then there is no (x,y) satisfying (11). This shows that no portion of the surface lies above the plane $z = 1$ and below the plane $z = -1$
Now we draw the ellipse corresponding to $z = 0$ (E_1 in Fig. 4).

Note that the major axis of the ellipse is 4 and the minor axis is 3. The figure shows some more ellipses corresponding to a few planes parallel to $z = 0$. Note that as h increases, the ellipses become smaller and smaller.

Likewise, if $x = h$, a constant, then again we get ellipses for $|h| < 3$. Since there is no (x,y,) satisfying (10) for $|h| > 3$, we see that no portion of the surface lies to the right of the plane $x = 3$ and left of the plane $x = -3$. For $x = 0$, we get the ellipse, as shown in Fig. 4.

Similarly, you can see that the intersection with the planes $y = h$ are also ellipses for $|h| \leq 4$, and no portion of the surface lies to the right of the plane $y = 4$ and left of the plane $y = -4$. The ellipse corresponding to $y = 0$ is shown in Fig. 4.

Having got the intersections with the coordinate planes and the coordinate axes, we obtain the ellipsoid as shown in Fig. 4.

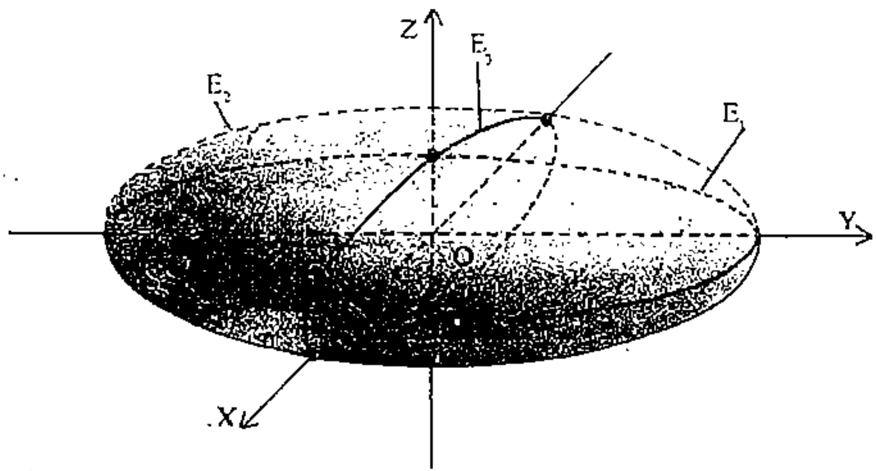


Fig. 4: The ellipses E_1, E_2 and E_3 are the intersections of ellipsoid $\frac{x^2}{9}+\frac{y^2}{16}+z^2=1$ with the coordinate planes, $z = 0$, $x = 0$ and $y = 0$ respectively.

Why don't you trace an ellipsoid now?

E9) a) Trace the ellipsoid $x^2+\frac{y^2}{4}+z^2=1$

b) Check whether the ellipsoid in (a) can be obtained by revolving an ellipse about any one of its axes.

So you have seen how to trace an ellipsoid in standard form. Actually, now you are in a position to trace any ellipsoid. How? Simply apply the transformations given in Sec 7.3 of Unit 7, and reduce the given equation of ellipsoid to standard form! But we shall not go into such details in this course.

Let us now consider an application of ellipsoids.

In Unit 2, you came across the reflecting property of an ellipse. This property is made use of in constructing whispering galleries. Whispering galleries are galleries with a rectangular base and ceiling in the form of an ellipsoidal surface. Because any vertical cross section of the ceiling is elliptical, the sound produced at one focus will be reflected at the other focus with little loss of intensity. This is called the **reflecting property** of an ellipsoid. This property is used by architects.

We shall now stop our discussion on ellipsoids and shift our attention to another central conicoid.

8.5 HYPERBOLOID OF ONE SHEET

In this section we shall study the shape of a hyperboloid of one sheet in detail and study some of its geometric properties.

As in the case of an ellipsoid, we shall restrict our attention to the standard forms.

Let us consider the hyperboloid of one sheet given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \ldots (12)$$

Do you agree that this equation is represented by the surface in Fig. 5?

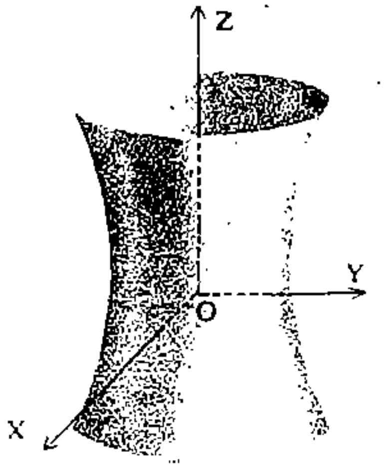


Fig. 5: The hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

You can obtain some geometrical properties of this surface by doing the following exercises.

E10) Show that the surface formed by (12) is symmetric about the coordinate planes.

E11) Check whether the coordinate axes intersect the surface formed by (12), if so, what are the intersections?

The next point to check is the intersection of (11) with the coordinate planes. Its intersection with the plane z = 0 is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Which is an ellipse. In fact, its intersection with any plane z = h will be an ellipse (see Fig. 6), and the size of the ellipses increase as h increases in both positive and negative directions. You may wonder why we don't call the surface an ellipsoid too.

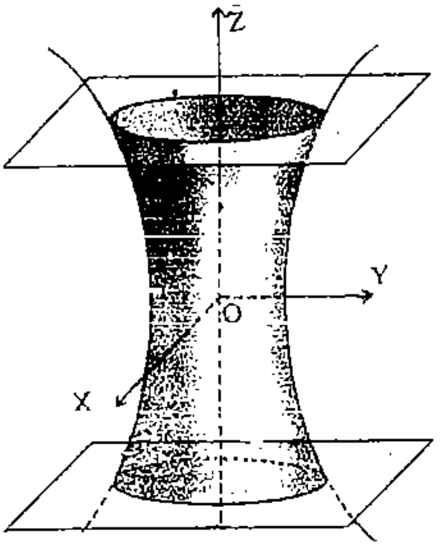


Fig. 6 : Sections of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ obtained by planes parallel to the XY-plane.

But, now look at what happens when we intersect the surface with the YZ-plane. The intersection is given by

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which represents a hyperbola (see Fig. 7).

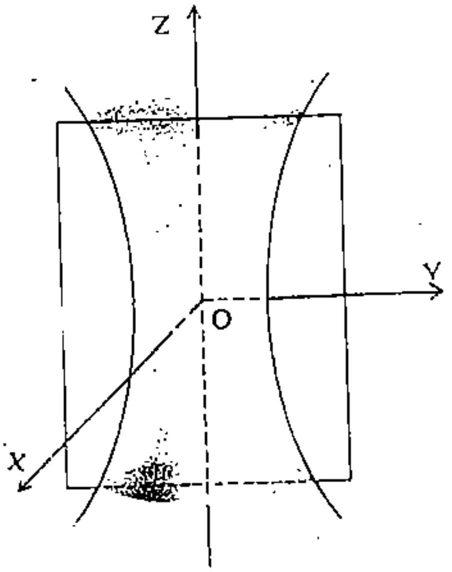


Fig. 7 : Intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ by the YZ-plane.

Similarly, you can see that the intersection of the surface with the plane y = 0 is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ (see Fig. 8).

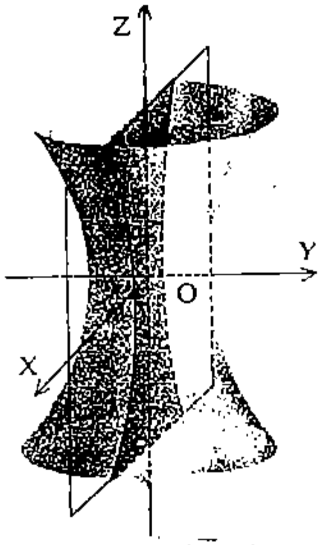


Fig.8 : Intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ by the ZX-plane.

From the above properties you surely agree with us that Fig.5 represents (12)

Sometimes we say that the surface (12) is generated by the variable ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{h^2}{c^2}, z=h,$$

which is parallel to the XY-plane and whose centre (0,0,h) moves along the z-axis. This is because, as you can see, it is made up of these ellipses piled one top of the other.

Now, why do you think (12) is called a hyperboloid of one sheet? Firstly, it is of one sheet because it is a connected surface. This means that it is possible to travel from one point on it to any other point on it without leaving the surface. In the next section you will see that we also come across hyperboloids of two sheets.

To see why it is called a hyperboloid see what happens if, for example, a = b in (12). Then the equation reduces to the form

$$\frac{x^2+y^2}{b^2} - \frac{z^2}{c^2} = 1. \qquad \ldots (13)$$

If we put x = 0 in this, then we get the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, x=0,$$

whose conjugate axis is the z-axis. If we revolve this hyperbola about its conjugate axis. then we get the hyperboloid (13). Similarly, we can obtain certain such hyperboloids by revolving a hyperbola about its transverse axes.

So far we have been discussing only one standard form of a hyperboloid of one sheet. You know, from Table 1, that there are two more types of hyperboloid of one sheet. The following exercises are about them and the standard form (12).

E12) Consider the hyperboloid of one sheet given by $x^2+y^2-z^2=1$.

a) What are its horizontal cross-sections for z = ±3, ±6?

b) What are the vertical cross-sections for x = 0 or y = 0?

Describe the sections in (a) and (b) geometrically.

E13) a) Sketch the surface defined by the equation

$$-\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{9} = 1$$

b) Sketch the curves of intersection of the surface given in (a) by the plane z=k, when |k| = 3, when |k| < 3 and when |k| > 3.

E14) a) Obtain the surface defined by the equation

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{9} = 1$$

b) What curve is formed by intersecting it with the plane x = 1?

Another interesting property of a hyperboloid of one sheet which makes it very useful in architecture is that it is a ruled surface. This means that the surface is composed of straight lines, and is therefore easy to make with a string (see Fig.9).

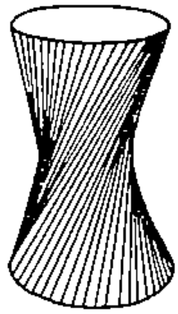


Fig 9 : A model of a hyperboloid of one sheet.

Let us now discuss another type of hyperboloid.

8.6 HYPERBOLOID OF TWO SHEETS

In this section we shall concentrate on the geometrical features of a hyperboloid of two sheets. Its analytical properties are very similar to those of a hyperboloid of one sheet or an ellipsoid. So it will be easy for you to bring out these properties yourself.

Let us start with a hyperboloid of two sheets given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \ldots (14)$$

Note that this equation has two negative coefficients while the equation of a hyperboloid of one sheet has only 1 negative coefficient.

So let us see what (14) looks like. To start with, why don't you try the following exercises concerning (14)?

E15) Discuss the symmetry of the surface obtained by (14) with respect to the coordinate planes.

E16) Do all the coordinate axes and coordinate planes intersect the surface? Give reasons for your answer.

In E16 you must have observed that the XZ-plane and XY-plane intersect the surface in hyperbolas. What about the YZ-plane? You must have observed that the YZ plane does not intersect the surface.

So, now you know why this surface is called a hyperboloid. But, you may wonder why this surface is called a hyperboloid of two sheets. This is because of the following property.

Let us consider the intersection of the surface and the plane x = h, a constant. You can see that the curve of intersection is the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{h^2}{a^2} - 1 \qquad \dots (15)$$

in the plane x = h.

This ellipse is real only if $\frac{h^2}{a^2} > 1$, i.e., $h > a$ or $h < -a$.

Therefore, it follows that those planes which are parallel to x = 0 and lie between the planes x = −a and x = a do not cut the surface. This means that no portion of the surface lies between the planes x = −a and x = a.

We note from (15) that the semi-axes of the ellipses are given by

$$b\sqrt{\frac{h^2}{a^2} - 1} \text{ and } c\sqrt{\frac{h^2}{a^2} - 1},$$

and the semi-axes increase as h increases. Hence we see that the surface has two branches: one on the left of the plane x = −a and one on the right of the plane x = a. Both these are generated by a variable ellipse. In fact, the shape of the surface is as shown in Fig. 10.

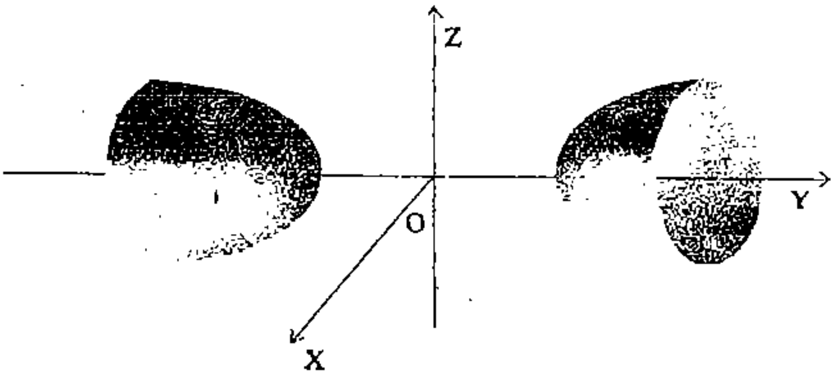


Fig. 10 : The hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

In the following exercise we ask you to trace the other two forms of hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ and}$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

E17) Check whether [illegible] coordinate planes intersect the surface

a) $z^2-\frac{x^2}{4}-y^2$

b) $\frac{y^2}{4}-\frac{x^2}{9}-z^2=1.$

If so, what are the curves of intersection?

E18) Sketch the surfaces given in E17.

So you have seen what the standard forms of the various central conicoids look like. Now let us see what their intersections with various lines and planes are.

8.7 INTERSECTION WITH A LINE OR A PLANE

In this section we shall first discuss the intersection of a line and a central conicoid. This will help us to derive conditions under which a line is a tangent to a central conicoid, and to obtain the tangent planes. Then we shall discuss the intersection of a plane and a central conicoid.

8.7.1 Line Intersection

Consider a central conicoid given by

$$ax^2+by^2+cz^2=1 \quad \ldots(16)$$

According to Theorem 1, the intersection of a given line with this conicoid is two points. Let us prove this fact.

Let L be a given line with direction ratios α, β, γ that passes through a given point (x_0, y_0, z_0). Then the equation of the line L is

$$\frac{x-x_0}{\alpha}=\frac{y-y_0}{\beta}=\frac{z-z_0}{\gamma} \quad \ldots(17)$$

If B(x,y,z) is another point on the line L, which is at a distance r from A, then the coordinates of B are given by $x=x_0+\alpha r$; $y=y_0+\beta r$, $z=z_0+\gamma r$.

If B is a point of intersection of L with the conicoid (16) then its coordinates must satisfy (17). This means that

$$r^2(a\alpha^2+b\beta^2+c\gamma^2)+2r(ax_0\alpha+by_0\beta+cz_0\gamma)+ax_0^2+by_0^2+cz_0^2-1=0 \quad \ldots(18)$$

(18) is a quadratic in r. So it will give us two values of r each of these values will give us a point of intersection of L with the conicoid (16).
Thus, L will meet (14) in two points, which may be real and distinct, coincident or imaginary.

This is true for any line. Hence Theorem 1 is true for a central conicoid.

Now let us suppose that the point A (x_0, y_0, z_0) lies on the conicoid (16) itself. Then (18) becomes

$$r^2(a\alpha^2+b\beta^2+c\gamma^2)+2r(ax_0\alpha+by_0\beta+cz_0\gamma)=0 \quad \ldots(19)$$

The line L will be tangent to the conicoid at A if the points of intersection coincide with the point A (x_0, y_0, z_0), i.e., if (19) has coincident roots. As you can see, the condition for this is

$$ax_0\alpha+by_0\beta+cz_0\gamma=0 \quad \ldots(20)$$

Thus, (20) gives the condition that the line

$$\frac{x-x_0}{\alpha}=\frac{y-y_0}{\beta}=\frac{z-z_0}{\gamma}$$

is a tangent to the conicoid $ax^2+by^2+cz^2=1$ at A (x_0, y_0, z_0).

For example, the line $x=z$, $y=4$ is tangent to the ellipsoid $\frac{x^2}{9}+\frac{y^2}{16}+z^2=1$ at (0,4,0) because $\alpha=1=\gamma$, $\beta=0$ You can also check that the line through the point (0,4,0) and parallel to the x-axis is also tangent to this ellipsoid. In fact, there are infinitely many lines that are tangent to this ellipsoid at the point (0,4,0).

This means that at each point of the ellipsoid we can draw infinitely many tangents to the conicoid. This is not only true for this ellipsoid. It is true for any conicoid. Let us see what the set of all tangents at a point of a conicoid look like.

Let us eliminate α, β, γ from (17) and (20) Then we get
$ax_0(x-x_0)+by_0(y-y_0)+cz_0(z-z_0) = 0.$
$<==> ax_0x+by_0y+cz_0z-ax_0^2-by_0^2 - cz_0^2 = 0$
$<==> axx_0+byy_0+czz_0 = 1,$ ... (21)
since (x_0, y_0, z_0) lies on (17).
(20) is the equation of a plane. Thus, the set of all tangent lines to (16) is the plane (21).

Definition : The set of all tangent lines to a conicoid at a point on the conicoid is called the **tangent plane.**
So let us assume that (17) is a tangent or (16) at (x_0, y_0, z_0)

For example, if the conicoid is an ellipsoid $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$, the equation of the tangent plane at any point (0,4,0) on it is $y = 4$.
Similarly, the equation of the tangent plane at the point (0,4,0) on the hyperboloid of one sheet $\frac{x^2}{9} + \frac{y^2}{16} - z^2 = 1$ is $y = 4$.

In Fig. 11 we have shown both these tangent planes.

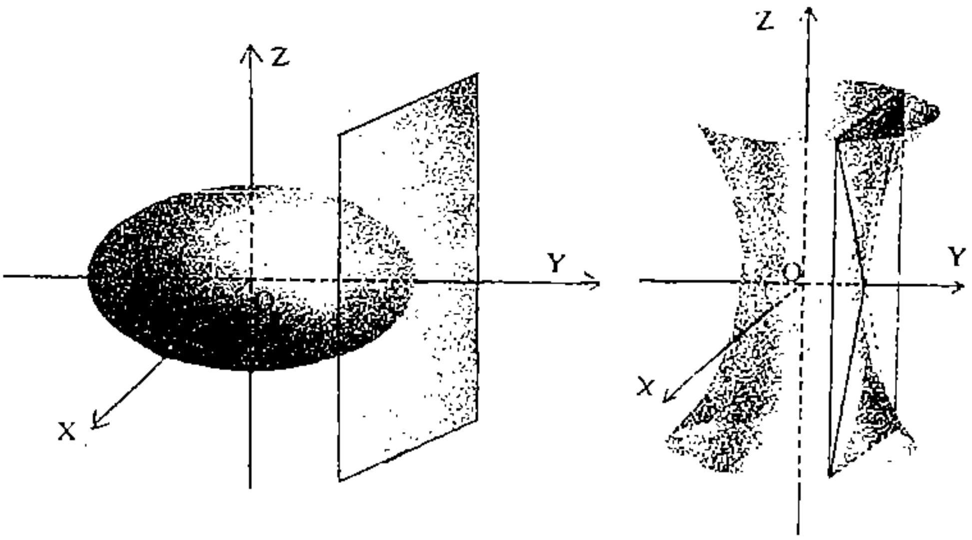


Fig. 11 : π is a tangent plane to (a) an ellipsoid (b) a hyperboloid.

Note that the tangent plane $y = 4$ intersects the given ellipsoid in only one point on the other hand, this plane intersects the given hyperboloid along the two tangent lines $x = \pm 3z$, $y = 4$ at (0,4,0). This should not be surprising, since the plane is built up of tangent lines.

Now, suppose we are given a plane and a conicoid. Can we say when the plane will be tangent to the conicoid? Let us see. Let us consider the plane given by
$ux + vy + wz = p$ ...(22)
and the conicoid given by
$ax^2+by^2+cz^2 = 1.$
You know from (20) that the plane $ux + vy + wz = p$ will be a tangent plane to the given conicoid at some point (x_0, y_0, z_0) if and only if its equation is of the form
$axx_0+byy_0+czz_0 = 1$ ...(23)
Therefore, if (22) represents a tangent plane, the coefficients of (22) and (23) must be proportional, i.e.,

$$\frac{ax_0}{u} = \frac{by_0}{v} = \frac{cz_0}{w} = \frac{1}{p}, p \neq 0$$

i.e., $x_0 = \frac{u}{ap}, y_0 = \frac{v}{bp}, z_0 = \frac{w}{cp}$ ... (24)
(Remember that $a \neq 0$, $b \neq 0$ and $c \neq 0$.)

Now, the point (x_0, y_0, z_0) lies on the given conicoid. Therefore

$$ax_0^2 + by_0^2 + cz_0^2 = 1 \quad => a\frac{u^2}{a^2p^2} + b\frac{v^2}{b^2p^2} + c\frac{w^2}{c^2p^2} = 1$$

$$=> \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = p^2, \qquad \dots (25)$$

which is the required condition that a plane $ux + vy + wz = p$ touches the conicoid $ax^2+by^2+cz^2=1$. Also note that the point of contact of the plane and the conicoid is given by (24).

Let us consider some examples.

Example 4: Show that the plane $3x+12y-6z = 17$ touches the hyperboloid $3x^2-6y^2+9z^2+17 = 0$, and find the point of contact.

Solution: We first rewrite the equation of the hyperboloid in the standard form as

$$\left(-\frac{3}{17}\right)x^2 + \frac{6}{17}y^2 + \left(-\frac{9}{17}\right)z^2 = 1$$

The condition that $3x+12y-6z = 17$ touches the conicoid is given by (25).

Here $a = \left(-\frac{3}{17}\right)$, $b = \frac{6}{17}$, $c = \left(-\frac{9}{17}\right)$, $u = 3$, $v = 12$, $w = -6$, $p = 17$. Then

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = \frac{9\times(-17)}{3} + \frac{144\times 17}{6} + \frac{36\times(-17)}{9}$$

$$= 17[-3+24-4] = 17^2=p^2$$

Thus the condition (25) is satisfied. Hence the plane touches the conicoid.
The point of contact is given by

$$x_0 = \frac{3\times(-17)}{3\times 17} = -1$$

$$y_0 = \frac{12\times 17}{6\times 17} = 2$$

$$z_0 = \frac{(-6)\times(-17)}{9\times 17} = \frac{2}{3}$$

Example 5 : Find equations of the tangent planes to the conicoid $7x^2+5y^2+3z^2=60$ which pass through the line $7x+10y-30 = 0$, $5y-3z = 0$.

Solution : From Block 1 you know that any plane through the given line is of the form

$7x + 10y - 30 + \lambda(5y - 3z) = 0$,

where λ is a real number. Since this plane is a tangent plane to the given conicoid, we see that the equation of the plane must be of the form

$$\frac{7xx_0}{60} + \frac{5yy_0}{60} + \frac{3yy_0}{60} = 1, \text{ for some } (x_0, y_0, z_0).$$

Comparing the coefficients, we get

$$\frac{7x_0}{7} = \frac{5y_0}{10+5\lambda} = \frac{3z_0}{-3\lambda} = \frac{60}{30}$$

i.e., $x_0=2$, $y_0=2\lambda+4$, $z_0=-2\lambda$

Since (x_0,y_0,z_0) lies on the given conicoid, we get

$7\times 4+5(2\lambda+4)^2+12\lambda^2 = 60$
$=> 28+20\lambda^2+80\lambda+80+12\lambda^2=60$
$=> 32\lambda^2+80\lambda+48=0$
$=> 2\lambda^2+5\lambda+3=0$.

This is a quadratic equation in λ. Its roots are -1 and $-\frac{3}{2}$. For each of these values of λ we get a tangent plane. Therefore, there are two tangent planes passing through the given line. The required equations of the planes are
$7x+3y+3z = 30$ and $14x+5y+9z = 60$.

You can now try some exercises.

E19) Find the equation of the tangent plane to the hyperboloid $x^2+3y^2-3z^2 = 1$ at the point $(1,-1,1)$

E20) Find the equations of the tangent planes to the conicoid $2x^2-6y^2+3z^2 = 5$ which pass through the line $x+9y-3z = 0$, $3x-3y+6z-5 = 0$.

So far we have been discussing the tangent planes to a central conicoid. You have found that sometimes such planes intersect the conicoid in a point, and sometimes in a pair of lines. Does this give you a clue to what π andS will be where π is a plane and S is a central conicoid? We discuss this now.

8.7.2 Planar Intersections

In this unit you have seen that the section of a standard ellipsoid or hyperboloid by a plane parallel to the coordinate planes is either an ellipse, a hyperbola or their degenerate cases, i.e., the section is a conic. What do you expect in the case of the section by any plane which is not parallel to the coordinate planes? Will it still be a conic? Let's see.

Let us consider a central conicoid given by
$ax^2+by^2+cz^2 = 1$, $abc \neq 0$.

We want to find the section of this conicoid by a plane $ux+vy+wz = p$. The following result tells us about this. (We shall not prove it here. If you are interested in knowing the proof, refer to the miscellaneous exercises at the end of the block.)

Theorem 4: The section of a central conicoid by a given plane is a conic section.

Further, if the conicoid is given by $ax^2+by^2+cz^2=1$ and the plane is given by $ux+vy+wz = p$, then the section will be a hyperbola, parabola or an ellipse according as
$bcu^2+cav^2+abw^2 < 0$, $bcu^2+cav^2+abw^2 = 0$ or $bcu^2+cav^2+abw^2 > 0$.
In case $abc>0$, the condition reduces to

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}<0,\ \frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0,\ \frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}>0.$$

This theorem is not difficult to prove. We can obtain the condition by eliminating either x, y or z from the equations $ax^2+by^2+cz^2 = 1$ and $ux+vy+wz = p$. Since it is lengthy, we have not included it here.

From Theorem 4 you know that a planar section of a central conicoid need not be a central conic. In Fig. 12 we illustrate this.

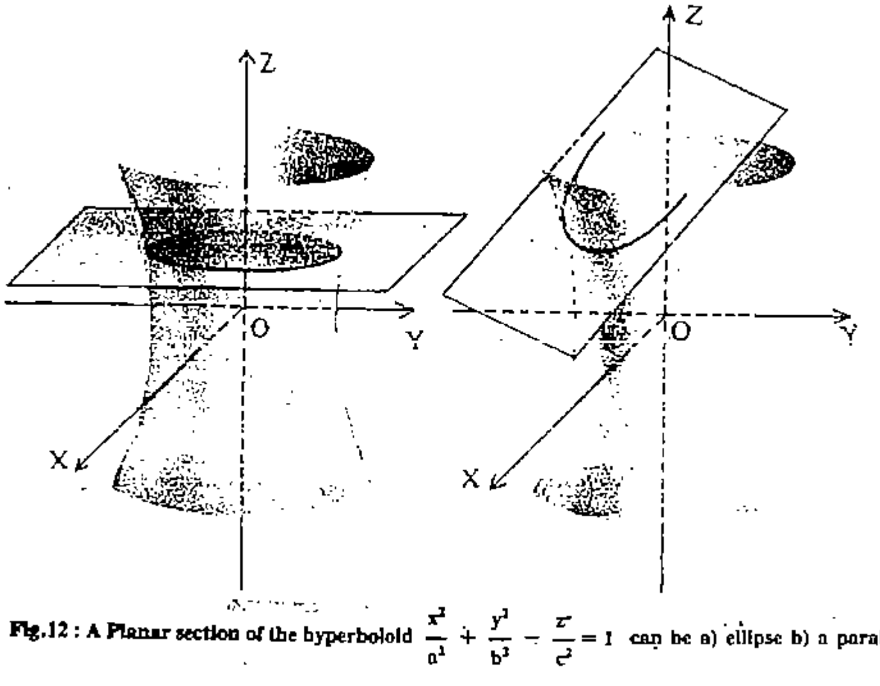


Fig.12 : A Planar section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ can be a) ellipse b) a parabola;

Let us consider an example.

Example 6: Show that the section of the hyperboloid $9x^2+6y^2-14z^2=3$ by the plane $x+y+z=1$ is a hyperbola.

Solution: Here $a=3$, $b=2$, $c=-\frac{14}{3}$ and $u=v=w=1$

Therefore, $bcu^2+cav^2+abw^2 = 2\times(-\frac{14}{3})-14+6 < 0$.

Hence by Theorem 4, the section is a hyperbola.
You can try some exercises now.

E21) Find the sections of the following conicoids by the plane given alongside.
  a) $2x^2+y^2-z^2=1$; $3x+4y+5z=0$.
  b) $3x^2+3y^2+6z^2=10$; $x+y+z=1$.

Let us now end our discussion on central conicoids by summarizing what we have covered in this unit.

8.8 SUMMARY

In this unit we have covered the following points :

1) The definition of a centre of a conicoid.

2) The necessary and sufficient condition for the conicoid to $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ have a centre is $\begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$.

3) Conicoids are divided into two groups — those with a unique centre (that is, central conicoids) and those which have either no centre or infinitely many centres (that is, non-central conicoids).

4) The standard form of a central conicoid is
$ax^2+by^2+cz^2+d=0$, $abc \neq 0$.
If $d \neq 0$, then there are four categories, as given in the table below ·

Table 1: Standard Form of central conicoids

Type	Standard form
Cone	$ax^2+by^2+cz^2=0$
Imaginary ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Hyperboloid of two sheets	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

If d = 0, then the equation represents a cone.

5) How to trace the standard forms of an ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets.

6) The condition for a line to be a tangent to the central conicoid $ax^2+by^2+cz^2+d = 0$ at (x_0, y_0, z_0) is $ax_0\alpha+by_0\beta+cz_0\gamma = 0$, where α, β, γ are the direction ratios of the line.

7) The equation of the tangent plane to a central conicoid $ax^2+by^2+cz^2 = 1$ at a point (x_0, y_0, z_0) is $axx_0 + byy_0 + czz_0 = 1$.

8) The condition that the plane $ux + vy + wz= p$ is a tangent to the central conicoid $ax^2 + by^2 + cz^2 = 1$ is
$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}= p^2.$$

9) A planar section of a central conicoid is a conic section.

Now you may like to go back to Sec. 8.1 to see if you have achieved the objectives listed there. You must have solved the exercises in the unit as you came to them. In the next section we have given our answers to the exercises. You may like to have a look at them.

8.9 SOLUTIONS/ANSWERS

E1) Let O be the point $(-u, -v, -w)$. Suppose that the sphere has another point O′ as its centre. Let the line joining O and O′ intersect the surface at P and P′. Then by definition, both O and O′ are the midpoints of the line segment P P′. This is possible only if O = O′. Hence O is the unique centre of the sphere.

E2) Let the equation of the cylinder be $x^2+y^2 = r^2$, $z = 0$.
Suppose $A(0,0,z_1)$ is a point on the z-axis.
Let a line through A meet the cylinder at two points P and P′. Let the coordinates of P be (x_2, y_2, z_2). Then the equation of the chord is
$$\frac{x}{x_2}=\frac{y}{y_2}=\frac{z-z_1}{z_2-z_1}$$
Now, consider the point $(-x_2, -y_2, 2z_1 -z_2)$. This point lies on the line as well as on the cylinder. Therefore, the point must be $(-x_2, -y_2, 2z_1 -z_2)$.
Also, we can see that A is the midpoint of the line segment P P′.
The above argument is true for all lines passing through A. Hence A is a centre of the cylinder. Similarly we can show that all points on the z-axis are centres of the cylinder.

E3) a) Origin is not a centre.
b) Origin is a centre.
c) Origin is a centre.
d) Origin is not a centre.

E4) A cone has a unique centre, and hence is central.

E5) a) Hyperboloid of one sheet.
b) Hyperboloid of one sheet.
c) Ellipsoid.
d) Cone.
e) Hyperboloid of two sheet.

E6) If we change x to −x, there is no change in the equation. This implies that the surface (8) is symmetric about the YZ-plane. Similarly, the surface is symmetric about the XZ-plane and XY-planes.

E7) Yes. All the coordinate planes intersect the surface.
Let the equation of the ellipsoid be
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The equation of the YZ-plane is x = 0. To find the intersection, we put x = 0 in the equation of the ellipsoid. Then we get

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which represents an ellipse (or a circle).
Similarly, substituting y = 0 or z = 0, we get an ellipse (or a circle) in the XZ-plane or the XY-plane.

E8) The equation of the ellipsoid is (8). We first check whether the x-axis intersects the surface. Any point on the x-axis is of the form (r,0,0). So, to find the intersection of the x-axis, we substitute (r,0,0) in (8). We get

$$\frac{r^2}{a^2}=1\text{, i.e., } r=\pm a.$$

Thus, the x-axis intersects the surface in the two points (a,0,0) and (−a,0,0). Similarly, the y-axis intersects it in two points (0,b,0) and (0,−b,0) and the z-axis intersects it in the points (0,0,c) and (0,0,−c).

E9) a) The given ellipsoid is $x^2+\frac{y^2}{4}+z^2=1$.

We first find the intersection of the ellipsoid with the coordinate plane. Let z = h, a constant. Then we get

$$x^2+\frac{y^2}{4}=1-h^2.$$

If $|h|<1$, this is an ellipse centred at the origin.
Likewise if x = h, a constant, we get

$$\frac{y^2}{4}+z^2=1-h^2.$$

This again represents an ellipse if $|h|<1$.
Putting y=h, we get circles given by

$$x^2+z^2=1-\frac{h^2}{4}$$

if $|h|\leq 4$.
Let us now draw the ellipses and the circle for z = 0, x = 0, y = 0 respectively (see Fig.13).

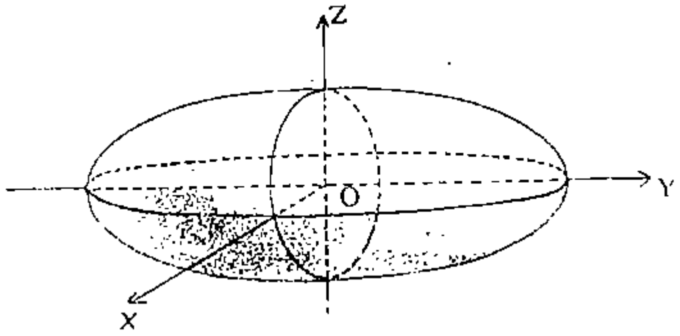


Fig.13: The ellipsoid $x^2+\frac{y^2}{4}+z^2=1$.

The shaded portion in Fig. 13 shows the ellipsoid.

b) Suppose we revolve the circle $x^2+z^2=4$ around the y-axis. We get the ellipsoid.

E10) Let the equation of the surface be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

Changing x to −x, y to −y and z to −z, there is no change in the equation. Therefore, the surface is symmetric about the XY, YZ and ZX planes.

E11) Let the equation of the surface be $\frac{x^2}{a^2} + \frac{y^2}{^2} - \frac{z^2}{c^2} = 1$.

Then we see that the x-axis intersects the surface at the points (a,0,0) and (−a,0,0).
Similarly the y-axis intersects the surface at the points(0,b,0) and (0,−b,0).
Next we put z = 0 in the given equation of the conicoid, to get $z^2 = -c^2$.

This shows that the points of intersection are imaginary.
That is, the z-axis does not intersect the surface.

E12) a) The given equation is $x^2 + y^2 - z^2 = 1$.
For $z = \pm 3, \pm 6$, the horizontal cross-sections are circles with centres on the z-axis and radius i and 2 respectively (see Fig.14).

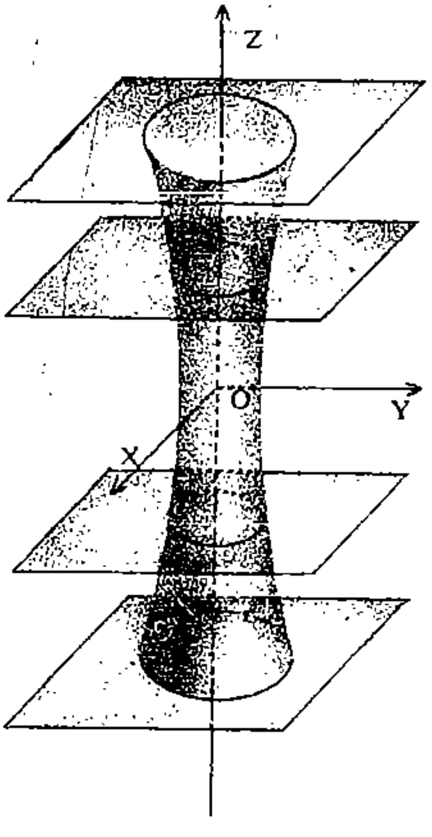


Fig.14: Circles obtained by intersecting the planes $z = \pm 1, \pm 2$ with the hyperboloid of one sheet $x^2+y^2+z^2 = 1$.

b) When $x=0$, the equation is $y^2-z^2=1$,
Which represents a hyperbola (similar to Fig. 7) in the plane $x = 0$.
When $y = 0$, the equation is $x^2 - z^2 = 1$ or $x = \pm z$, which again represent a hyperbola.

E13) a) By renaming the axes, we can obtain the surface as shown Fig.15

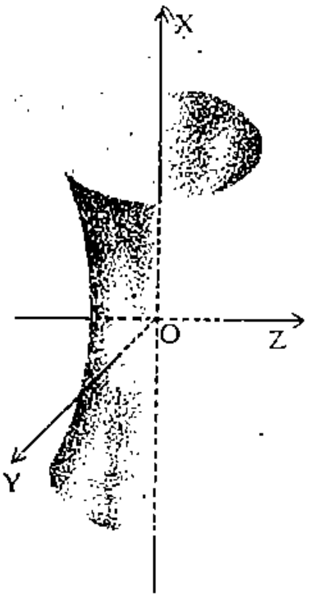


Fig. 15: The hyperboloid of one sheet $-\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{9}=1$.

b) When $|k| = 3$, we get

$$\frac{y^2}{9}-\frac{x^2}{16}=0$$

$$\Rightarrow \left(\frac{1}{3}y+\frac{1}{4}x\right)\left(\frac{1}{3}y-\frac{1}{4}x\right)=0$$

which represents a pair of lines.

When $|k| < 3$, we get

$$\frac{y^2}{9}-\frac{x^2}{16}=1-\frac{k^2}{9}$$

which represents a hyperbola with transverse axis parallel to the y-axis.

When $|k| > 3$, we get

$$\frac{y^2}{9}-\frac{x^2}{16}=1-\frac{k^2}{9}$$

i.e. $\frac{x^2}{16}-\frac{y^2}{9}=\frac{k^2}{9}-1$

which represents a hyperbola with transverse axis parallel to the x-axis.

The cross-sections are shown in Fig. 15.

E14) a)

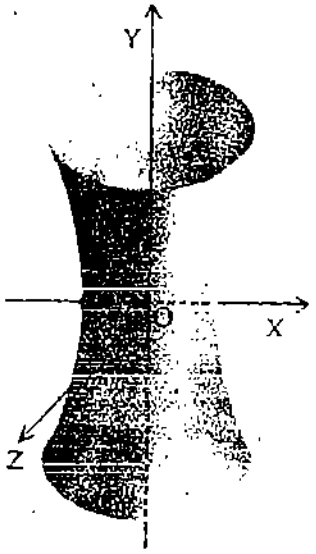


Fig. 16: The hyperboloid of one sheet $\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{9}=1$.

b) The hyperbola $\frac{z^2}{8} - \frac{y^2}{32/9} = 1$ in the YZ-plane.

E15) The surface is symmetric about the coordinate planes.

E16) The coordinate planes $y = 0$ and $z = 0$ intersects the surface in hyperbolas $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ respectively.

The plane $x = 0$ does not intersect the surface. The x-axis meets the surface in points $(a,0,0)$ and $(-a,0,0)$. The y-axis and the z-axis do not meet the surface.

E17) a) The XY-plane does not intersect the surface. The YZ-plane and the XZ-plane intersect the surface in hyperbolas

$$z^2-y^2=1 \text{ and } z^2- \frac{x^2}{4} = 1.$$

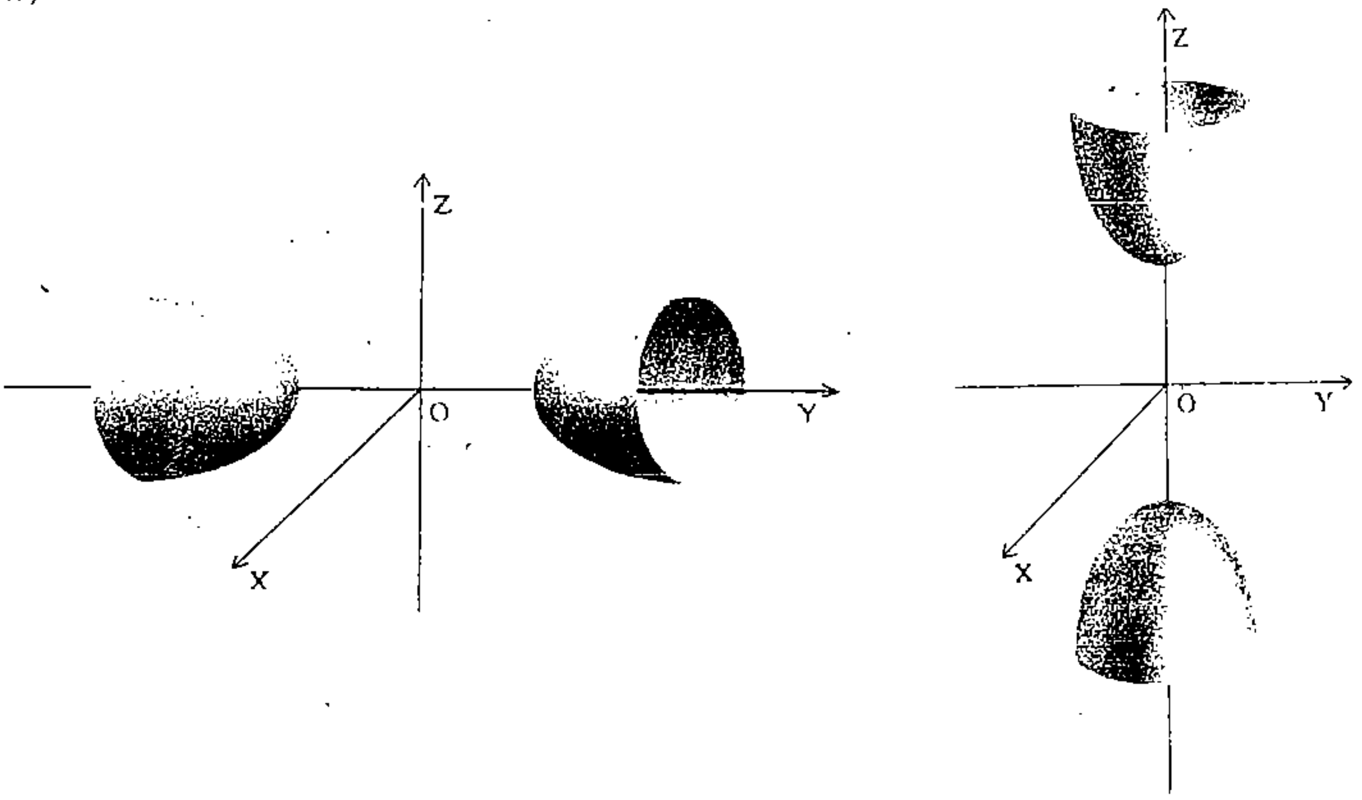
b) The XZ-plane does not intersect the surface. The XY-plane and the YZ-plane intersect the surface in hyperbolas

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

and

$$\frac{y^2}{4} - z^2 = 1.$$

E18)



(a) (b)

Fig. 17 : The hyperboloid of two sheets a) $z^2 - \frac{x^2}{4} - y^2 = 1$. b) $\frac{y^2}{4} - \frac{x^2}{9} - z^2 = 1$.

E19) The equation of the tangent plane at (x_0,y_0,z_0) is $axx_0 + byy_0 + czz_0 = 1$.
The given equation is $x^2 + 3y^2 - 3z^2 = 1$.
So, here $a = 1$, $b = 3$, $c = -3$, $x_0 = 1$, $y_0 = -1$ and $z_0 = 1$.
Thus the required plane is $x - 3y - 3z = 1$.

E20) Any plane through the line $x + 9y - 3z = 0$, $3x - 3y + 6z - 5 = 0$ is
$x + 9y - 3z + \lambda (3x-3y+6z-5)=0$,
where λ is a real number.

Now, we know that this plane touches the given conicoid.
Therefore, we must have

$\frac{2xx_0}{5} - \frac{6yy_0}{5} + \frac{3zz_0}{5} = 1$ for some (x_0,y_0,z_0). Then we get

$$\frac{2x_0}{1+3\lambda} = \frac{6y_0}{9-3\lambda} = \frac{3z_0}{-3+6\lambda} = \frac{1}{\lambda}$$

$$\Rightarrow x_0 = \frac{1+3\lambda}{2\lambda}, \; y_0 = \frac{9-3\lambda}{6\lambda}, \; z_0 = \frac{-3+6\lambda}{3\lambda}$$

$$\Rightarrow x_0 = \frac{1+3\lambda}{2\lambda}, \; y_0 = \frac{3-\lambda}{2\lambda}, \; z_0 = \frac{-1+2\lambda}{\lambda}$$

Substituting this in the equation $2x^2-6y^2+3z^2=5$, we see that $\lambda=1$ and $\lambda=-1$.
Hence, there are two tangent planes given by
$4x+6y+3z = 5$ and $2x-12y+9z=5$.

E21) a) Here $a=2$, $b=1$, $c=-1$, $u=3$, $v=4$, $w=5$.
Then $bcu^2+cav^2+abw^2=-9-32+50 > 0$.
Therefore the section is an ellipse.

b) The section is an ellipse.

UNIT 9 PARABOLOIDS

Structure

9.1 INTRODUCTION

In the previous unit we discussed central conicoids. In this unit, which is the last unit of this course, we look at non-central conicoids. You are already familiar with one type of non-central conicoid, namely, a cylinder. Here we discuss another such surface, called a paraboloid.

We shall start this unit with a discussion on the standard forms of a paraboloid. You will see that paraboloids can be divided into two types : elliptic and hyperbolic paraboloids. In Sec. 9.3, we discuss the shapes of the two types of paraboloids. The last section contains a brief discussion on the intersection of a paraboloid with a line and intersection with a plane.

Like central conicoids, paraboloids are also used in various fields. The most commonly found paraboloidal surfaces are dish antennas, which most of us are familiar with. You can see some more applications in the unit

The way this unit unfolds is the same as the previous one, except for one difference. In this unit we assume that you have enough experience by now to bring out many properties of the surface by yourself. Accordingly, you will find that we have left most results to you to prove.

Now please go through the following list of objectives. If you achieve these, then you can be sure that you have grasped the contents of this unit.

Objectives

After studying this unit you should be able to

- check whether a given equation of a conicoid represents an elliptical paraboloid or a hyperbolic paraboloid;
- trace the standard elliptic or hyperbolic paraboloid;
- obtain the tangent lines and tangent planes to a standard paraboloid.

9.2 STANDARD EQUATIONS OF A PARABOLOID

In this section we shall obtain the standard equations of a non-central conicoid. Then we shall define a paraboloid and discuss its standard equations.

To begin with, let us go back to Theorem 4 in Unit 7 for a moment. According to this theorem any second degree equation can be reduced to an equation of the form

$ax^2+by^2+cz^2+2ux+2vy+2wz+d = 0$ ... (1)

Now let us assume that (1) represents a non-central conicoid. Since the conicoid has no centre, by Theorem 3 in Unit 8 we find that either

i) exactly two of the a, b and c are zero, or

ii) only one of the a, b and c is zero.

Let us look at these cases separately.
We first consider the case (i). Let us assume that $a = 0$, $b = 0$ and $c \neq 0$. (We can deal with the cases $a, c = 0, b \neq 0$; $b, c = 0, a \neq 0$ similarly.) In this case (i) becomes
$cz^2 + 2ux + 2vy + 2wz + d = 0$

$$\Rightarrow c\left(z + \frac{w}{c}\right)^2 = -2ux-2vy-d+ \frac{w^2}{c}.$$

By shifting the origin to $\left(0, 0, -\frac{w}{c}\right)$, we see that the equation takes the form

$cZ^2 + 2uX + 2vY + d_0 = 0$,
where X, Y, Z, are the coordinates in the new system. What does this equation represent? Let's see.
If both u and v are zero, then the surface represents a pair of lines.
If one of the coefficients u and v is non-zero, say $v \neq 0$ and $u = 0$, then you can see that the surface is built up of a series of parabolas along a line parallel to the x-axis. Thus it is a parabolic cylinder. In fact, even if both u, v are non-zero, the surface is a parabolic cylinder.

Let's now go to case (ii)
Here we assume that $a = 0$ and $b, c \neq 0$. (We can deal with the other two cases $b = 0, c, a \neq 0$; and $c = 0, a, b \neq 0$ similarly.)
In this case (1) reduces to the form
$by^2+cz^2+2ux+2vy+2wz+d = 0$,

$$\text{i.e., } b\left(y+ \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2= -2ux-d +\frac{v^2}{b} + \frac{w^2}{c}$$

$$= -2ux+d_1, \text{ where } d_1 =\frac{v^2}{b} + \frac{w^2}{c}-d$$

By shifting the origin to $(0, -\frac{v}{b}, -\frac{w}{c})$, we see that the above equation takes the form

$$bY^2+cZ^2+2uX + d_1 = 0 \qquad \ldots(2)$$
where X,Y,Z are coordinates in the new system.

For example, $y^2+10z^2 = 2$ and $2y^2+z^2 = 12x$ represent non-central conicoids. But is there a difference in the type of conicoid represented by them ? Let's see.

Suppose that $u = 0$ in (2). In this case we get $by^2 + cz^2 + d = 0$. Do you recognize the surface given by this equation? It represents a cylinder or a pair of planes. We have already discussed these surfaces in detail in Block 2.

Now, let us assume that $u \neq 0$. Then we rewrite (2) in the form
$by^2 + cz^2 = -2ux - d$

i.e., $by^2 +cz^2 = 2u' \left(x - \frac{d}{2u'}\right)$, where $u' = -u$.

Now, by translating the origin to the point $\left(\frac{d}{2u'}, 0, 0\right)$, the equation reduces to

$$bY^2 + cZ^2 = 2u'X \qquad \ldots(3)$$
Do you agree that this equation is a three-dimensional version of the standard equation of a parabola? We call this surface a paraboloid.

So, for example, the equation $2y^2 + z^2 = 12x$ represents a paraboloid.

What are the other forms of an equation of a paraboloid? We leave this as an exercise for you (see E1).

E1) Discuss what happens to (1) in the following cases:
a) $b = 0$, $a, c \neq 0$.
b) $c = 0$, $a, b \neq 0$.

If you've done E1, you must have found that there are two more types of equations which represent a paraboloid, namely

$$ax^2+by^2 = 2wz, \text{ and } ax^2+cz^2 = 2vy \qquad (4)$$

Now let us look at the coefficients of these equations more closely, as we did in the case of central conicoids. Let us consider (4). We have the following two cases:

Case 1 (a and b are of the same sign): Suppose a and b are positive. Let $a_1 = \sqrt{a}$ and $b_1 = \sqrt{b}$ then (4) becomes

$$\frac{x^2}{1/a_1^2} + \frac{y^2}{1/b_1^2} = 2wz$$

Similarly, if a and b are negative, we can write (4) in the above form. thus, when a and b are of the same sign, (4) reduces to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wz \qquad \dots (5)$$

The paraboloid represented by this equation is called an **elliptic paraboloid.**

Case 2 (a and b are of opposite signs): In this case you can see that (4) reduces to the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz \qquad \dots (6)$$

The surface represented by this equation is called **hyperbolic paraboloid.**

If you do E2, you will see why the adjectives 'elliptic' and 'hyperbolic' are appropriate.

E2) Show that the intersection with any plane parallel to the XY-plane of the paraboloid

i) $x^2 + 2y^2 = 3z$ is an ellipse.

ii) $3x^2 - y^2 = 4x$ is a hyperbola.

E3) Check whether the following equations represent a paraboloid or not? for those that do, classify the paraboloids as elliptic or hyperbolic.

a) $4y^2-4z^2-2x- 14y-22z+33 = 0$

b) $x^2+y^2+z^2-2x+4y = 1$

c) $4x^2-y^2- z^2-8x-4y+8z-2 = 0$

d) $9x^2+4z^2-36 = 0$

e) $2x^2+20y^2+22x+6y-2z-2 = 0$

From E2 you may have already realised that the two types of parabola are geometrically different. Let us now see whether there are more differences.

9.3 TRACING PARABOLOIDS

In this section we shall discuss the geometry of the two types of paraboloids, and see how to trace their standard forms. We shall trace an elliptic paraboloid here and leave the tracing of a hyperbolic paraboloid as an exercise for you.

So, let us consider (5), the standard equation of an elliptic paraboloid. We can observe some geometrical properties, similar to the properties you have seen in Unit 8 for an ellipsoid or hyperboloid.
In E4 we have asked you to obtain them, using the knowledge you have gained in previous units.

E4) Check whether the surface (5) is symmetrical about the coordinate planes

E5) Do all the coordinate axes intersect the surface (5)? If so, what are their intersections?

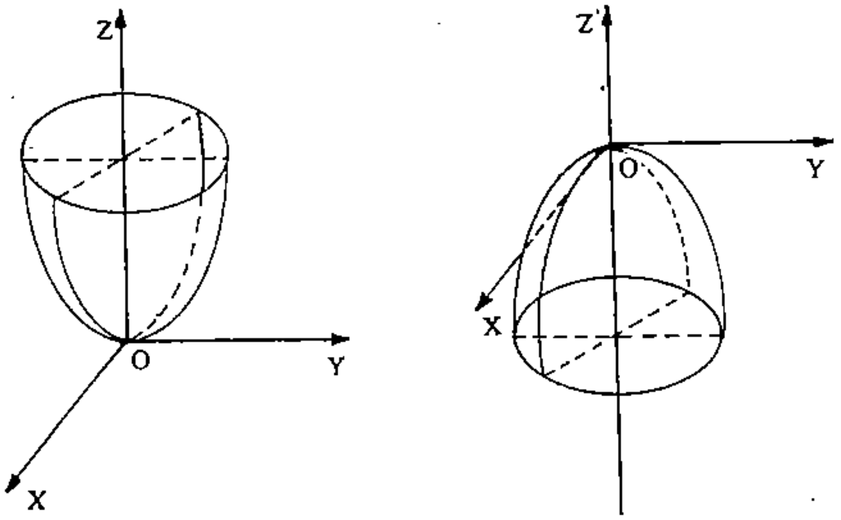
E6) Obtain the intersections of the surface (5) with the XZ- and YZ-planes.

If you've done E6, then you must have realized why this surface is called a paraboloid. You already know why the surface is called elliptic from E2. You may be more convinced about this fact if you look at the following property.

Let us look at sections of the surface by the plane z = k, where k is a constant. It is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wk \qquad \ldots (7)$$

The left hand side of (7) is positive for all values of x and y. Therefore, w and k must be of the same sign. So if $w > 0$, then $k > 0$. In this case (7) represents an ellipse (or circle if a = b) with centre at (0,0,k) on the positive direction of the z-axis. Note that the size of the ellipse increases as k increases.
If k = 0, the plane z = 0 just touches the surface at the point (0,0,0).
If $k < 0$, the plane z = k does not intersect the surface. Therefore, no portion of the surface (5) lies below the plane z = 0. We have drawn the surface in Fig. 1 (a).



(a) (b)

Fig.1 : The elliptic Paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2wk$ in case (a) w>0, (b) w< 0

Now what if $w < 0$ in (7)? Then k has to be negative. In this case we get an ellipse whose centre lie on the negative direction of the z-axis. As above, you can see that no portion of (5) lies above the plane z=0. We have drawn the surface in Fig.1 (b). Now you know geometrically why this surface is called an elliptic paraboloid.

Why don't you try to trace some paraboloids on your own now?

E7) Trace the paraboloid given by
  a) $x^2+y^2 = z$; and describe its sections obtained by the planes x = 0 and y = 0.
  b) $y^2+4z^2 = x$ and describe its sections by the planes y = 0 and z = 0.

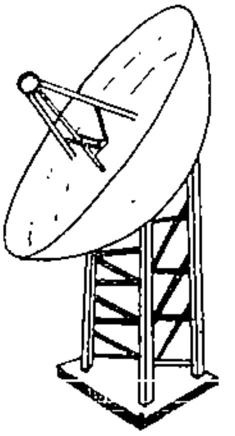


Fig.2: Antenna in the shape of a circular paraboloid.

The paraboloid you got in E7 (a) is called a **circular paraboloid.** There you can see that the plane section of the surface by the plane x = 0 is a parabola with focus at the point (0,0,1/4). When we revolve this parabola about the z-axis, we get the surface you have traced in E7 (a). Therefore we also call this surface a **paraboloid of revolution.**

Paraboloids of revolution have many applications. Circular paraboloids are used for dish antennas and antennas in radio telescopes (see Fig.2). This is because of the property that of all the paraboloids having the same area, a circular paraboloid has the largest reflecting surface.

Circular paraboloids are also used for satellite trackers and microwave radio links.

Now let us consider the hyperbolic paraboloid given by (6), that is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz.$$

As in the case of the elliptic paraboloid, we have two cases: $w < 0$ and $w > 0$. We shall restrict our discussion to $w < 0$. (Exactly similar properties hold for the case $w > 0$.) In this case we ask you to find out the properties of the following exercise.

E8) a) What are the properties of a hyperbolic paraboloid which are analogous to those obtained by you in E4 for an elliptic paraboloid?

b) What are the sections of the paraboloid (6) with $z = k$, $k < 0$, and $k > 0$.

In E8 (b) you must have observed that the section of a hyperbolic paraboloid by the plane $z = k$ $(k \neq 0)$ is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wk$. This hyperbola is real for all non-zero values of k $(\neq 0)$, positive or negative.

if $k > 0$, it will have its tranverse axis parallel to the x-axis, and if $k < 0$, its transverse axis will be parallel to the y-axis. You can see one branch of the hyperbola in Fig.3 (a).
In Fig.3 (b) you can see the parabolas which are sections of the paraboloid by the planes $x = 0$ and $y = 0$.

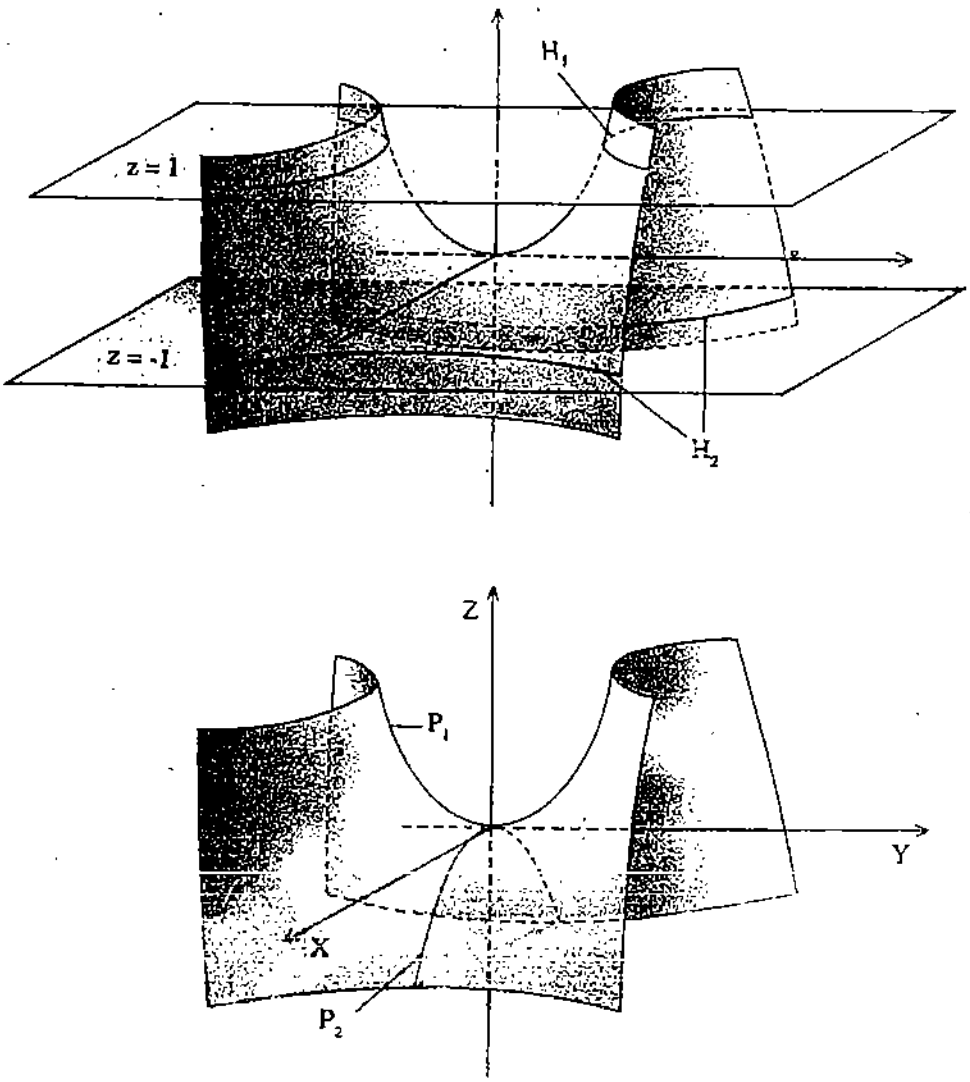


Fig. 3 : The planar section of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wz$ by the planes (a) $z = 1$ and $z = -1$, are the hyperbolas H_1 and H_2 (b) $x = 0$ and $y = 0$, are the parabolas P_1 and P_2

You can also observe that for $k > 0$, the length of the semi-transverse axis of the hyperbola is $\sqrt{2\ k\ a}$, which increases as k increases. Similarly for $k < 0$, the length increases as $|k|$ increases.

You can now try these exercises.

E9) Consider the hyperbolic paraboloid given by
$x^2-2y^2 = z$
a) What are its sections by the planes $x = 0$ and $y = 0$?
b) What are its sections by the planes $z = 0, \pm 1$?
c) Sketch the surface described by the given equation.

E10) Sketch the surface given by the equation $2z^2 - y^2 = x$.

So far we have discussed how to trace the standard paraboloids. For this purpose we considered their intersection with planes parallel to the coordinate planes. Now let us discuss the intersection of the paraboloid with a general plane, as well as with a line.

9.4 INTERSECTION WITH A LINE OR A PLANE

In this section we shall discuss some results for paraboloids similar to those obtained for central conicoids in Sec 8.7 of Unit 8. We shall present some of these results in the form of exercises for you to do.

Let us start with an exercise on the analogue of Theorem 1 in Unit 8 for the paraboloid given by

$$ax^2+by^2 = 2z. \qquad \ldots (8)$$

E11) Prove that a line intersects a paraboloid at two points which may be real or imaginary.

What can you say about the intersection of a line parallel to the z-axis with the elliptic paraboloid (8) (when a and b are of the same sign)? Let us go back for a moment to Fig.1. There you can see that there is only one real point of intersection on the paraboloid. Let us see what happens to the intersections with the x- and y-axes. Again from Fig.1 and E7, we observe that the lines just **touch** the paraboloid, that is, the points of intersection are coincident. You know that such lines are called **tangent lines.** As in the case of central conicoids, the set of all tangent lines at a point of the surface is a plane, called the **tangent plane.** You should be able to write the equation of the tangent plane from the corresponding results on central conicoids. In fact, this is what the following exercises are about.

E12) Prove that the condition for a line with direction ratios α, β, γ through the point (x_0,y_0,z_0) to be a tangent to the paraboloid (8) is $ax_0\alpha+by_0\beta-\gamma=0$.

E13) prove that the equation of the tangent plane at a point (x_0,y_0,z_0) on the paraboloid (8) is
$axx_0+byy_0 = (z+z_0)$.

E14) a) Prove that the plane $ux+vy+wz = p$ will be a tangent plane to the paraboloid (8) iff

$$\frac{u^2}{a}+\frac{v^2}{b}+2pw=0 \qquad \ldots (9)$$

b) Obtain the point of contact.

If you have done E14, you know that the point of contact in that situation will be

$$\left(\frac{-u}{aw}, \frac{-v}{bw}, \frac{-p}{w}\right)$$

Let us consider an example.

Example 1: Show that the plane $8x-6y-z-5 = 0$ touches the paraboloid $\frac{x^2}{2} - \frac{y^2}{3} = z$, and find the point of contact.

Solution: Let us check the condition given in (9). Here $a = 1$ $b = -\frac{2}{3}$, $u = 8$, $v = -6$, $w = -1$, $p = 5$.
Substituting in (9), we get
$64 - 54 - 10 = 0$, which is true.
Therefore the plane touches the paraboloid.
The point of contact is (8,9,5).

In the following exercises we ask you to apply what you have proved in E13 and E14.

E15) Find the equations of tangent plane to the given conicoid at the indicated point
a) $x^2+y^2 = 4z$, $(2,-4,5)$
b) $x^2-3y^2 = z$, $(3, 2, -3)$

E16) Show that the plane $2x-4y-z+3 = 0$ touches the paraboloid $x^2-2y^2 = 3z$, and find the point of contact.

Let us now see what the intersection of a paraboloid with a general plane is. Consider the following theorem, which is analogous to Theorem 4 in Unit 8. We will not be doing its proof here, but leave it as an exercise for you to do if you are interested in proving it. (See Miscellaneous Exercises.)

Theorem 4
a) The section of a paraboloid $ax^2+by^2 = 2z$ by a plane $ux+vy+wz = p$ is a conic.
b) If $w = 0$, the section is always a parabola.
c) If $w \neq 0$, then the section is
  i) a hyperbola if a and b are of opposite signs.
  ii) a parabola if at least one of a and b is zero.
  iii) an ellipse if a and b are of the same sign.

In Fig.4 we have diagrammatically illustrated some particular cases

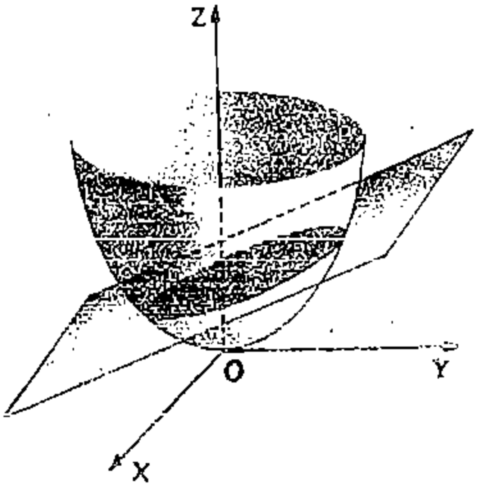


Fig.4 : Planar section of $ax^2+by^2 = 2z$ by the plane $ux+vy+wz = p$ when $u,v,w, p \neq 0$

In Fig.4 you can see the elliptical section

Why don't you try an exercise now?

E17) Sketch the sections of the following concoids by a plane perpendicular to the XY-plane.

a) $3x^2-y^2 = z$.

b) $2x^2+y^2 = z$.

We shall end this unit now with a brief review of what we have covered in it.

9.5 SUMMARY

In this unit we have discussed the following points.

1) The standard form of a non-central conicoid is $ax^2+by^2+2wz + d = 0$. If $w = 0$, the equation represents a cylinder or a pair of straight lines. If $w \neq 0$, the surface represented by the equation is called a paraboloid.

2) The standard equation of a paraboloid is $ax^2+by^2 = 2wz$, $w \neq 0$. There are two types of paraboloids. When a and b are of the same signs, we get an elliptic paraboloid. When a and b are of opposite signs, we get a hyperbolic paraboloid.

3) How to trace an elliptic paraboloid and a hyperbolic paraboloid.

4) The condition for a line with direction ratios α, β, γ to be a tangent to the central conicoid $ax^2+by^2 = 2z$ at (x_0, y_0, z_0) is $ax_0\alpha+by_0\beta=\gamma$.

5) The equation of the tangent plane to the paraboloid $ax^2+by^7=2z$ at a point (x_0,y_0,z_0) is $axx_0+byy_0 = (z+z_0)$.

6) The condition that the plane $ux+vy+wz = p$ is a tangent plane to the paraboloid $ax^2 + by^2 = 2z$ is
$$\frac{u^2}{a}+\frac{v^2}{b}+ 2wp=0$$

7) The planar section of a paraboloid is a conic section.

Now you may like to go back to Sec.9.1 to see if you've achieved the objectives listed there. You must have solved the exercises as you came to them in the unit. In the next section we have given our answers to the exercise. You may like to have a look at them.

9.6 SOLUTIONS/ANSWERS

E1) a) Putting $b = 0$, $a, c \neq 0$ in (1) we get

$$ax^2 + cz^2 + 2ux + 2vy + 2wz + d = 0,$$

i.e., $$a\left(x + \frac{u}{a}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = -2vy-d+\frac{u^2}{a}+\frac{w^2}{c}$$

$$= -2vy+d_2, \text{ where } d_2= \frac{u^2}{a}+\frac{w^2}{c}-d$$

By shifting the origin to $\left(-\frac{u}{a}, 0, -\frac{w}{c}\right)$, the above equation reduces to the form

$$aX^2 + cZ^2 + 2vY + d = 0,$$

where X,Y,Z denote the coordinates in the new system.

b) Similarly, putting $c = 0$, $a, b \neq 0$, in (1) we get that the equation reduces to the form

$$aX^2 + bY^2 + 2wZ + d = 0,$$

where X,Y,Z denote the coordinates in the system obtained by shifting the origin to $\left(-\frac{u}{a}, -\frac{v}{b}, 0\right)$

E2) Any plane parallel to the XY-plane is of the form z = k, where k is a constant, $k \neq 0$.

i) The given ellipsoid is $x^2 + 2y^2 = 3z$.
Putting $z = k$ in this equation, we get

$$\frac{x^2}{3k} + \frac{y^2}{2k} = 1,$$

which represents an ellipse.

ii) Similarly, putting $z = k$ in the equation of the hyperboloid, we get

$$\frac{x^2}{4/3k} - \frac{y^2}{4k} = 1,$$

which is a hyperbola.

E3) (a) and (e) represent a paraboloid. (b), (c) and (d) do not represent a paraboloid. (a) represents a hyperbolic paraboloid, whereas (e) represents an elliptic paraboloid.

E4) Surface (5) is symmetric about YZ and ZX-planes. It is not symmetric about the XY-plane.

E5) Yes. When we put $y = 0$ and $z = 0$ in (5), we see that $x = 0$. Thus, (0,0,0) is the only point of intersection of the x-axis with (5). Similarly, we can show. that (0,0,0) is the only point of intersection with the y and z-axes.

E6) a) Putting $z = 0$ in (5), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$$

The only point which satisfies the above equation is (0,0,0). Therefore the XY-plane intersects the surface in the point (0,0,0).

b) Putting $y = 0$ in (5) we get

$$\frac{x^2}{a^2} = 2wz,$$

i.e. $x^2 = 2a^2wz$,

which represents a parabola.

c) Similarly, the YZ-plane intersects the surface in a parabola.

E 7 a)

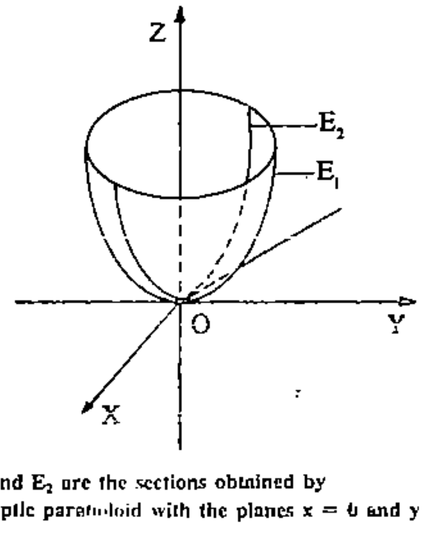


Fig. 5: The parabolas E_1 and E_2 are the sections obtained by intersecting the elliptic paraboloid with the planes x = 0 and y = 0, respectively.

b)

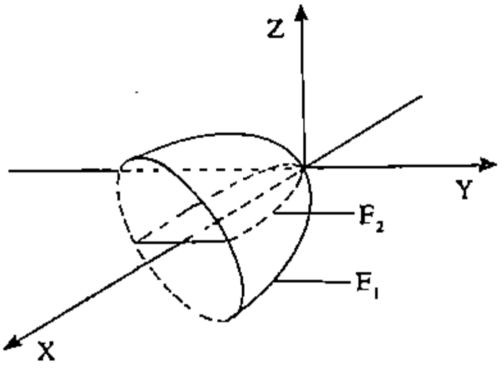


Fig.6 : F_1 and F_2 are the sections obtained by intersecting the elliptic paraboloid with the planes $y = 0$ and $z = 0$, respectively.

E8) The surface has the following properties.

i) It is symmetrical about the XZ-plane and the YZ-plane.

ii) The coordinate axes intersects the surface in the point (0,0,0).

iii) By putting $z = 0$ in (6), we see that the XY-plane intersects the surface in two lines

$$y = \pm \frac{b}{a}x.$$

Similarly, by putting $x = 0$ in (6), we get

$y_2 = -2b^2wz$,

which represents a parabola, i.e., the intersection with the YZ- plane is a parabola.

The interesection of (6) with the ZX-plane also is a parabola given by the equation

$x^2 = 2a^2wz$.

b) Putting $z = k$ in (6), we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2wk.$$

When $k < 0$ and $k > 0$, this represents a hyperbola.

When $k = 0$, this represents a pair of lines $y = \pm \frac{b}{a}x$.

E9) a) The section by the plane $x = 0$ is the parabola

$y^2 = -\frac{1}{2}z$ (see Fig. 7)

The section by the plane $y = 0$ is the parabola

$x^2 = z$ (see Fig. 7)

b) The section by the plane $z = 0$ is the pair of lines

$y = \pm\frac{1}{2}x$.

The section by the plane $z = 1$ is the hyperbola

$x^2 - 2y^2 = 1$

and the section by the plane $z = -1$ is the hyperbola

$2y^2 - x^2 = 1$.

This shows that the set of all tangent lines i.e.; tangent plane is given by the equation

$axx_0 + byy_0 = z + z_0$.

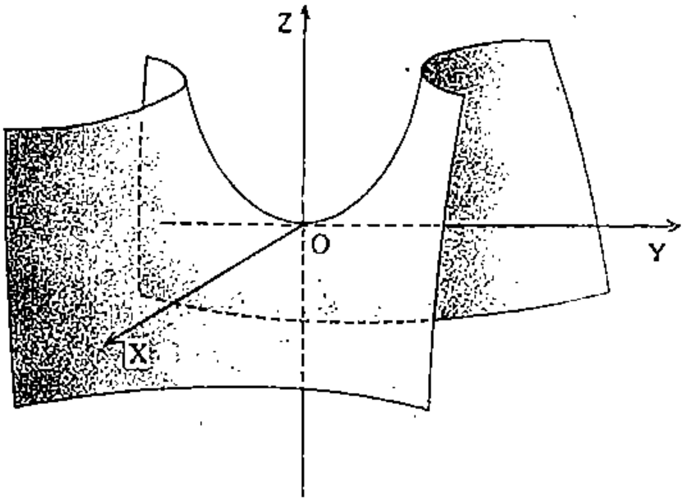


Fig.7: The hyperbolic paraboloid $x^2 - 2y^2 = z$.

E10) The figure is similar to the figure in E9 with a change of the coordinate axes.

E11) The equation of a line through (x_0, y_0, z_0) with direction cosines α, β, γ is

$$\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta} = \frac{z-z_0}{\gamma}$$

Any point on this line is of the form $(\alpha r+x_0, \beta r+y_0, \gamma r+z_0)$ for some r. When this line meets the paraboloid $ax^2+by^2=2z$, we have

$a(\alpha r+x_0)^2+b(\beta r+y_0)^2=2\alpha(\gamma r+z_0)$.

i.e. $(a\alpha^2+b\beta^2)r^2+2r(a\alpha x_0+b\beta y_0-\gamma) + ax_0^2+by_0^2-2z_0=0$ ... (10)

This is a quadratic equation in r which gives two values of r, which may be real or imaginary. Hence the result.

E12) The equation of the line L passing through (x_0, y_0, z_0), having direction ratios α, β, γ is

$$\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta} \quad \frac{z-z_0}{\gamma}$$

In E13 we saw that this line L meets the conicoid in two points, which may be real and distinct, real and coincident or imaginary.

If the line is a tanget to the conicoid at the point (x_0, y_0, z_0), then the points of intersection coincide. That is, (10), in (E11) has real coincident roots. The condition for this is

$a\alpha x_0+b\beta y_0-\gamma = 0$. ... (11)

Note that since $(x_0, y_0, z_0$ lies on the conicoid, we have $ax_0^2 + by_0^2 - 2z_0 = 0$.

E13) The conicoid is $ax^2 + by^2 = 2z$ ... (12)

We know that the tangent plane at (x_0, y_0, z_0) is the set of all tangent lines at (x_0, y_0, z_0). Let us assume that the line

$$\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta} \quad \frac{z-z_0}{\gamma}$$

where α, β, γ are the direction ratios of the line, is a tangent to the conicoid (11).

Eliminating α, β, γ between (11) and (12), we get

$ax_0(x-x_0) + by_0(y-y_0) - (z-z_0) = 0$

$axx_0 + byy_0 - (ax_0^2 + by_0^2) = z - z_0$

$axx_0 + byy_0 - 2z_0 = z - z_0$, since $ax_0^2 + by_0^2 = 2z_0$

$axx_0 + byy_0 = z + z_0$

E14) By E 13, the plane $ux + vy + wz = p$ will be a tangent plane to the paraboloid $ax^2 + by^2 = 2z$ if it is of the form
$axx_0 + byy_0 = z + z_0$
for some point (x_0, y_0, z_0) on the paraboloid. That means, we have

$$\frac{ax_0}{u} = \frac{by_0}{v} = \frac{-1}{w} = \frac{z_0}{p}$$

$$\Longrightarrow x_0 = -\frac{u}{aw}, y_0 = -\frac{v}{bw}, z_0 = -\frac{p}{w}.$$

Since (x_0, y_0, z_0) lies on the conicoid $ax^2 + by^2 = 2z$, we get

$$a\left(\frac{u^2}{a^2w}\right) + b\left(\frac{v^2}{b^2w}\right) = -\frac{2p}{w}$$

i.e., $\frac{u^2}{aw} + \frac{v^2}{bw} = -2p$

i.e., $\frac{u^2}{a} + \frac{v^2}{b} + 2pw = 0$ ... (13)

b) The point of contact is given by $\left(-\frac{u}{aw}, -\frac{v}{bw}, \frac{-p}{w}\right)$

E15) Equation of the tangent plane is $axx_0 + byy_0 = z+z_0$ ...(14)

a) The given equation can be written as
$\frac{x^2}{2} + \frac{y^2}{2} = 2z$
So in this case $a = \frac{1}{2} = b$ and $x_0 = 2$, $y_0 = -4$ and $z_0 = 5$.
Then we have from (14)
$\frac{1}{2} \times x \times 2 + \frac{1}{2} \times y \times (-4) = z+5$
i.e. $x - 2y - z = 5$.

b) $6x - 12y - z + 3 = 0$

E16) The given conicoid can be rewritten as
$\frac{2x^2}{3} - \frac{4y^2}{3} = 2z$.
The given plane is $2x - 4y - z = -3$. The condition that this plane touches the paraboloid is given by (13) in E14. In this case $u = 2$, $v = -4$, $w = -1$, $p = -3$, $a = \frac{2}{3}$, $b = \frac{4}{3}c$. Then we have
$\frac{u^2}{a} + \frac{v^2}{b} + 2pw = b-12+6=0$.
This shows that the plane touches the paraboloid.
The point of contact is given by $(3,3,-3)$.

E17) a)

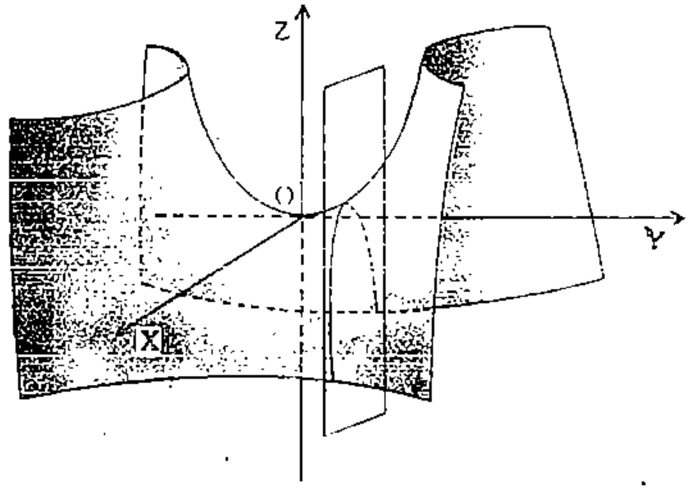


Fig. 8 : F is the section of the paraboloid $3x^2 - y^2 = z$ by a plane perpendicular to the XY-plane.

b)

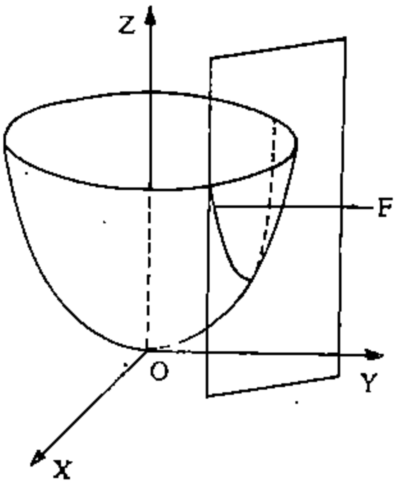


Fig. 9 : F is the section of the paraboloid $2x^2 + y^2 = z$ by a plane perpendicular to the XY-plane.

MISCELLANEOUS EXERCISES

(This section is optional)

In this section we have gathered some problems related to the contents of this block. You may like to do them to get a better understanding of these contents. Our solutions to the questions follow the list of problems, in case you'd like to counter-check your answers.

1) Which type of the Conicoids do the following equations represent?
   - i) $x^2 - 16z^2 = 4y^2$
   - ii) $5x^2 + 2y^2 - 6z^2 = 10$
   - iii) $4y^2 = x$
   - iv) $x^2 + 4y^2 + 16z^2 = 12$
   - v) $2y^2 + x^2 = 4z$
   - vi) $4x^2 - 3y^2 - 6z^2 = 10$
   - vii) $x^2 + y^2 + z^2 = 4$
   - viii) $2z^2 + x = y^2$
   - ix) $4y^2 + 9x^2 - 36z^2 + 36 = 0$
   - x) $25x^2 - 9y^2 = 225$

2) a) The hyperbola $\frac{z^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ is rotated about the z-axis. What is the surface formed in this situation? Obtain the equation of the surface.

   b) Find the value of x_0 such that the plane $x = x_0$ intersects the surface obtained in (a) in a pair of straight lines.

3) a) A normal at a point P of a conicoid is the line through P which is perpendicular to the tangent plane at that point. Find the equation of a normal to a central conicoid $ax^2 + by^2 + cz^2 = 1$ at a point (x_0, y_0, z_0).

   b) Using (a) obtain a normal at the point $(1, 1, \frac{1}{\sqrt{2}})$ to the ellipsoid

   $$\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1.$$

4) Find the equation of a normal to a paraboloid $ax^2 + by^2 = 2z$ at any point (x_0, y_0, z_0) on it.

5) Suppose that the XYZ-coordinate system is transformed into another coordinate system with the same origin and with the coordinate axes having direction cosines $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$ and $(0,0,1)$ with

respect to the old system. What does the equation $xy = z$ represents in the new system?

6) Find the equations to the tangent plane of the given surfaces at the indicated points

a) $x^2 + 2y^2 + 2z^2 = 5$: (1,1,1)
b) $9x^2 + 4y^2 - 36z = 0$: (2, − 3,2)
c) $x^2 + 4y^2 - 4z^2 - 4 = 0$ (2,1,1)

7) Prove that the section of a central conicoid by a given plane is a conic section. Further, if the conicoid is $ax^2+by^2+cz^2=1$ and the plane is $ux + vy + wz = p$, then prove that the section will be

i) an ellipse if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} > 0$.

ii) a hyperbola if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} < 0$.

iii) a parabola if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0$.

8) a) Prove that the section of a paraboloid $ax^2 + by^2 = 2z$ by a plane $ux + vy + wz = 0$ is a conic section.

b) If $w = 0$, show that the section is always a parabola

c) If $w \neq 0$, then show that the section is
  i) a hyperbola if a and b are of opposite signs
  ii) a parabola if at least one of a and b is zero
  iii) an ellipse if a and b are of the same sign

9) a) Show that the intersection of the plane $y = 2$ and the ellipsoid $\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ is an ellipse.

b) Find the lengths of the semi-major axis and semi-minor axis, the coordinates of the centre, and the coordinates of the focii of the ellipse obtained in (a).

10) A tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the coordinate axes in the points A,B,C. Prove that the centroid of the triangle ABC lies on the surface
$\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9$ (A centroid of a triangle is the point of intersection of the medians of the triangle.)

Answers

1) i) Cone ii) hyperboloid of one sheet iii) cylinder iv) ellipsoid v) elliptic paraboloid vi) hyperboloid of two sheets vii) sphere viii) hyperbolic paraboloid ix) hyperboloid of two sheets x) cylinder.

2) a) The surface formed is a hyperboloid of one sheet. It's equation is given by
$$\frac{z^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2}{b^2} = 1.$$
b) Putting $z = z_0$ in the above equation, we get
$$\frac{z_0^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2}{b^2} = 1.$$
The equation represents a pair of straight lines only if $z_0 = \pm a$.

3) a) The equation of the tangent plane at (x_0, y_0, z_0) is
$axx_0 + byy_0 + czz_0 = 1$
where ax_0, by_0 and cz_0 are the direction ratios of the normal to the plane. This means that the direction ratios of any line perpendicular to the plane are ax_0, by_0 and cz_0. Since a normal to a conicoid at (x_0, y_0, z_0) is a line perpendicular to the tangent plane and passes through (x_0, y_0, z_0), its equation is given by
$$\frac{x-x_0}{ax_0} = \frac{y-y_0}{by_0} = \frac{z-z_0}{cz}$$

b) The equation of the normal to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ at } (x_0, y_0, z_0) \text{ is}$$

$$\frac{x-x_0}{\frac{x_0}{a^2}}=\frac{y-y_0}{\frac{y_0}{b^2}}=\frac{z-z_0}{\frac{z_0}{c^2}}$$

Here $x_0=1$, $y_0=1$, $z_0=\frac{1}{\sqrt{2}}$, $a=2$, $b=2$, $c=1$.

Then we have

$$\frac{x-1}{1/4}=\frac{y-1}{1/4}=\frac{z-1/\sqrt{1}}{\frac{1}{\sqrt{2}}}$$

This is the equation of a normal at the point

$$\left(1, 1, \frac{1}{\sqrt{2}}\right)$$

4) $$\frac{x-x_0}{ax_0}=\frac{y-y_0}{by_0}=\frac{z-z_0}{-1}$$

5) The equations of transformations are

$$x=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$

$$y=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y$$

$$z=z'.$$

Substituting for x, y and z in the equation $xy = z$, we get

$$\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)=z'$$

$$\text{i.e. } \frac{x'^2}{2}-\frac{y'^2}{2}=z'$$

which represents a hyperbolic paraboloid.

6) i) The equation represents an ellipse. Therefore the tangent plane is
$x + 2y + 2z = 5$.

ii) The equation represents an elliptic paraboloid. The equation of the tangent plane is
$3x - 2y - 3z = 4$.

iii) The equation represents an hyperboloid of one sheet.
The equation of the tangent plane is
$x + 2y - 2z = 0$.

7) Let the equation of the central conicoid be
$ax^2 + by^2 + cz^2 = 1$, $abc \neq 0$.
Suppose that the plane $ux + vy + wx - p = 0$ ($u \neq 0$) intersects the conicoid.

$$ux+vy+wz-p=0 \Longrightarrow x=\frac{-vy-wz+p}{u}$$

Substituting for x in the given equation of the conicoid, we get

$$\frac{a}{u^2}(-vy-wz+p)^2+by^2+cz^2=1$$

$$\left(\frac{av^2}{u^2}+b\right)y^2+\left(\frac{aw^2}{u^2}+c\right)z^2+\frac{2avw}{u^2}yz-\frac{2avp}{u^2}y$$
$$-\frac{2awp}{u^2}z+\frac{a}{u^2}p^2-1=0.$$

This is a general second degree equation and therefore represents a conic section. Hence the result.

Let us now find the nature of the conic section,

i) You know from Block 1 Unit 3 that the above equation represents an ellipse if

$$\frac{av^2+bu^2}{u^2}\cdot\frac{aw^2+cu^2}{u^2}-\frac{a^2v^2w^2}{u^4}>0$$

i.e $(av^2+bu^2)(aw^2+cu^2) - a^2v^2w^2 > 0$

i.e. $a^2v^2w^2 + acv^2u^2 + abu^2w^2 + bcu^4 - a^2v^2w^2 > 0$

i.e $acv^2 + abw^2 + bcu^2 > 0$

since abc $\neq 0$, we can divide throughout the left hand side by abc. Then we get

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} > 0.$$

ii) Similarly we can show that the section will be an hyperbola if

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} < 0.$$

iii) The section will be a parabola if

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 0.$$

8) a) The given paraboloid is $ax^2 + by^2 = 2z$. The given plane is $ux + vy + wz = p$.
Now $ux + vy + wz = p \Longrightarrow wz = -4x - vy + p$.
Substituting for wz in the given equation of the conicoid, we get
$ax^2 + by^2 + 2(ux + vy - p) = 0$
i.e. $wax^2 + wby^2 + 2ux + 2vy - 2p = 0$.
This is a general second degree equation and therefore represents a conic section. The section will be a hyperbola, a parabola or an ellipse according as $w^2ab < 0$, $w^2ab = 0$, $w^2 ab > 0$ respectively.

b) If $w = 0$, we get $w^2ab = 0$ and therefore in this case the section is always a parabola.

c) If $w \neq 0$, then the condition in (a) reduces to
$ab < 0$, $ab = 0$ or $ab > 0$
Thus the section in this case will be
i) a hyperbola if $ab < 0$ i.e. a and b are of opposite signs.
ii) a parabola if $ab = 0$ i.e. at least one of a and b is zero.
iii) an ellipse if $ab > 0$ i.e. a and b are of the same sign.

9) a) Substituting for $y = 2$ in the given equation of the ellipsoid, we get

$$\frac{x^2}{9} + \frac{4}{9} + \frac{z^2}{16} = 1$$

$$\text{i.e. } \frac{x^2}{9} + \frac{z^2}{16} = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \frac{x^2}{5} + \frac{9}{80}z^2 = 1$$

$$\Rightarrow \frac{x^2}{5} + \frac{z^2}{80/9} = 1.$$

This represents an ellipse.

b) The length of the semi-major axis $= \sqrt{80/9}$.
The length of the semi-minor axis $= \sqrt{5}$
(0,0) is the centre.

The focii are given by $(0, \pm c)$ where $c = \sqrt{\left(\frac{80}{9}\right)^2 - 5^2} = \frac{25}{9}\sqrt{7}$.

i.e., $\left(0, \frac{25}{9}\sqrt{7}\right)$ and $\left(0, -\frac{25}{9}\sqrt{7}\right)$

10) Suppose $ux + vy + wz = p$ is a tangent plane to the given ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Then we have the relation

$$p^2 = a^2u^2 + b^2v^2 + c^2 \qquad \ldots(*)$$

Now we find the intersection of the plane and the coordinate axes.
The plane meets the x-axis, y-axis and z-axis at the point

$A\left(\frac{p}{u}, 0, 0\right)$, $B\left(0, \frac{p}{v}, 0\right)$, $C\left(0, 0, \frac{p}{w}\right)$ respectively.

Let (x_0, y_0, z_0) be the centroid of the Δ ABC. Then

$$x_0 = \frac{p/u+0+0}{3} \text{ i.e. } u = \frac{p}{3x_0}$$

$$y_0 = \frac{0+p/v+0}{3} \text{ i.e. } v = \frac{p}{3v_0}$$

$$z_0 = \frac{0+0+p/w}{3} \text{ i.e. } w = \frac{p}{3z_0}$$

Substituting for u, v and w in (*) we get

$$p^2 = \frac{a^2p^2}{9x_0^2} + \frac{b^2p^2}{9y_0^2} + \frac{c^2p^2}{9z_0^2} \quad \text{That is, } \frac{a^2}{x_0^2} + \frac{b^2}{y_0^2} + \frac{c^2}{z_0^2} = 9.$$

This shows that the centroid (x_0, y_0, z_0) lies on the surface represented by the equatioin $\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9$.

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