Uttar Pradesh Rajarshi Tandon
Open University

UGPHS-101

Vector, Mechanics and General Physics

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[AZURE-01]

UGPHS-101/2

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Vector, Mechanics and General Physics

Uttar Pradesh Rajarshi Tandon Open University

BLOCK

1

VECTOR AND MECHANICS

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UNIT : 1

VECTOR ANALYSIS

Structure:

1.1 INTRODUCTION

In physics different types of quantities are discussed. some quantities possess magnitude only, while other quantities possess both magnitude and direction. All measurable physical quantities are usually divided into two classes: Scalars and Vectors. In this unit, we wish to study vectors, representation and notation of a vector. Further we also introduce the concept of Tensor, define line, Surface and Volume integrals and consider their transformation through Gauss, Green and stokes theorem's with statement. In this unit, we also derive the expression for the gradient of a scalar function. However, we shall confine ourselves to the study of vectors and scalars in the field of physics.

1.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand the concept of Vector and Scalar
- Define some fundamental definition of Vectors
- Concept of Tensor
- Explain the concept of DOT Product and Cross Product.

- ❖ Define Line, Surface, Volume Integral
- ❖ Statement of Gauss, Stokes and Greens theorem

1.3 SCALARS AND VECTORS

1.3.1 Scalars

Scalar quantities, or scalars, are quantities that have only magnitude and can be completely specified if their magnitudes are given. These quantities obey the rules of ordinary algebra, and they can be added, subtracted, multiplied and divided accordingly. For example, if the mass of a body is 2 kg and that of another body is 3 kg, the total mass of the system comprising these two bodies is 2 kg + 3 kg = 5 kg. some examples of scalar quantities: mass, length, time, volume, speed, density, electric current, electric potential, gravitational potential, kinetic energy and magnetic potential energy.

1.3.2 Vectors

Vector quantities, or vectors, are those quantities which have both magnitude and direction, and can be completely specified only if both magnitude and direction are given. These quantities do not obey the rules of algebra; they obey the triangle law, which will be explained later. Some examples of vectors: displacement, velocity, momentum, force, intensity of electric field, intensity of magnetic field and intensity.

1.3.3 Free and Localized Vector

(a) **Free Vector:** A free vector is a vector which is not associated with any particular point in space, e.g. the translation of a rigid body.

(b) **Localized Vector:** A localized vector is a vector which occupies a definite position in space, e.g. the particle velocity of a moving fluid.

1.3.4 Representation and Notation of a Vector

Graphically, a vector is represented by an arrow (or a directed line segment) whose length is equal to the magnitude of the vector and whose direction is the direction of the vector.

Thus, the arrow OP represents a vector whose magnitude is the length OP, say a, and whose direction is that from O to P. The point O is called the origin or initial point, while the point P is called the terminal point or terminus of the vector OP.

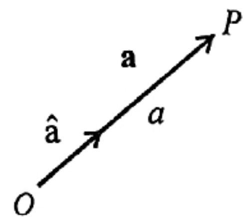


Figure - 1

Analytically, in printing, a vector is denoted by bold-faced type as a, and in writing, it is denoted by $\overrightarrow{OP}, \vec{a}$ or $\bar{a}$.

1.4 SOME FUNDAMENTAL DEFINITIONS OF VECTORS

1.4.1 Equal Vectors

Two vectors are said to be equal if they represent the same physical quantity, have the same magnitude and the same direction (are parallel). A vector, therefore, remains unchanged by a parallel translation. Figure (2) shows some equal vectors.

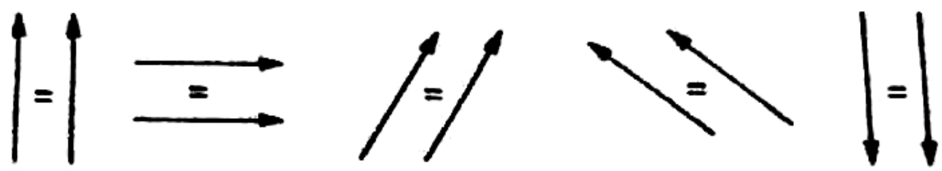


Figure - 2

1.4.2 Null Vectors or Zero Vectors

A vector having zero magnitude is called a null vector or zero vector.

A zero vector is represented by $\vec{0}$ (arrow over the number zero).

1.4.3 Unit Vectors

A vector having a magnitude equal to unity is known as a unit vector. Unit vectors are represented by special symbols, like $\hat{a}, \hat{n}, \hat{\imath}$, etc. Note that $|\hat{a}| = |\hat{n}| = |\hat{\imath}| = 1$.

Consider a vector $\vec{a}$ having a magnitude -------- as shown in figure below. What do we get if we multiply $\vec{a}$ with number $\frac{1}{3}\left(=\frac{1}{|\vec{a}|}\right)$?

Obviously, we get a vector of magnitude I in the direction of $\vec{a}$. This is a unit vector along $\vec{a}$. In general,

$$\hat{a} = \frac{1}{|\vec{a}|} . \vec{a}$$

Where ----- is a unit vector along $\vec{a}$.

Many a time, it is convenient to express a vector $\vec{a}$ as the product of its magnitude and a unit vector $\hat{n}$ having a direction corresponding to $\vec{a}$. A unit vector has no dimension.

In the Cartesian coordinate system, unit vectors along positive x, y and z directions have been assigned special symbols. Unit vectors along x, y and z directions are $\hat{\imath}, \hat{\jmath},$ and $\hat{k}$ respectively.

A vector of magnitude 10 in x direction is written as $10\hat{\imath}$ A vector of magnitude 10 in negative z direction can be written as $-10\hat{k}$.

1.4.4 Polar Vector

The vectors having defined terminating point and starting point is called polar vector.

Example: Position vector

1.4.5 Axial Vector

The vectors having no starting and terminating point are called axial vectors.

Example: Angular velocity, Torque, Angular momentum, Angular displacement, angular acceleration.

Note: Axial vector always lie along the axis of rotation.

1.5 ANGLE BETWEEN TWO VECTORS

An Angle between two vectors is the angle between their respective direction. In all the figures given below, the angle between $\vec{a}$ and $\vec{b}$ is 60^o. Please take a note of the third figure where the head of one vector is placed over the tail of the other.

The angle between two vectors is always from 0^o to 180^o. Therefore, angle between two vectors can never be greater than 180^o.

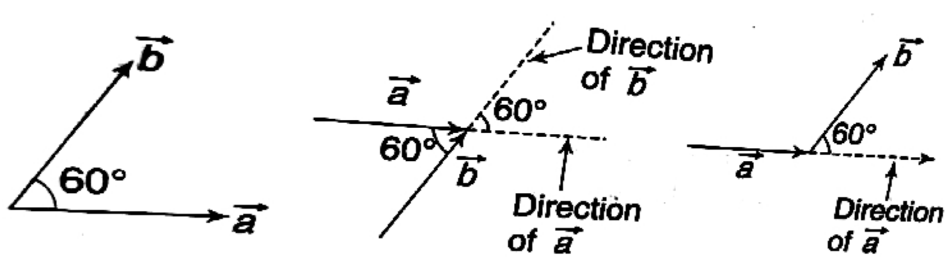


Figure - 3

1.6 CONCEPT OF TENSOR

Tensor are natural generalization of the concept of vectors. Vectors are physical quantities have a magnitude and a direction. They are described geometrically as an arrow as directed line segment. Both its magnitude and direction may be specified by giving three numbers such as (v_x, v_y, z_z) which are called the components of the vector. While the vector is a geometrical entity, its components depend on the coordinate system needed to describe it. How the components depend on the transform one coordinate system to another depends on the two coordinate systems. For S coordinate system S', rotated about the z-axis through an angle θ, with respect to a fixed coordinate system S, the components (v_x, v_y, z_z) and $(v_{x'}, v_{y'}, z_{z'})$ are related by

$$\begin{pmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

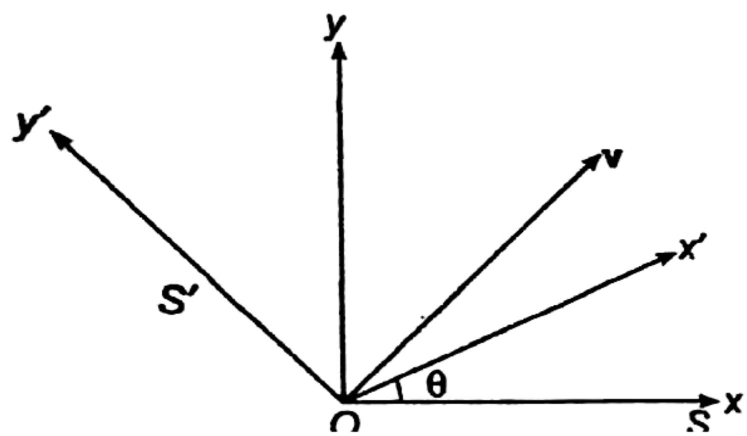


Figure – 4

1.7 SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT)

There are many physical quantities which are scalar but are defined using two vectors. For example, the physical quantity 'work' is defined using two vector quantities – displacement and force. The vector operation involved in such cases is known as scalar product.

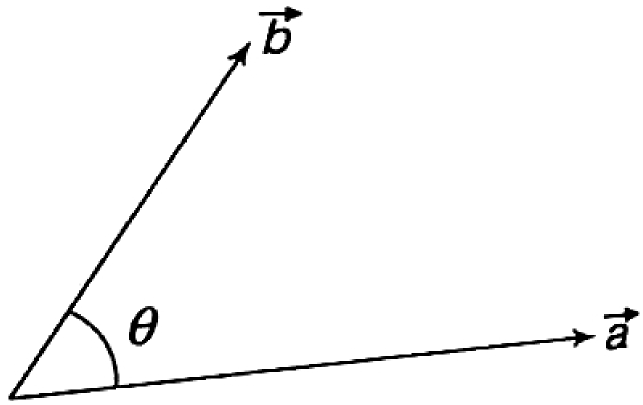


Figure - 5

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ (it is often known as dot product and is defined as

$$\vec{a}.\vec{b} = a.b.cos\theta.$$

where $a = |\vec{a}| = magnitude\ of\ \vec{a}$

$b = |\vec{b}| = magnitude\ of\ \vec{b}$

θ = angle between $\vec{a}$ and $\vec{b}$

The quantity ab cos θ is a scalar. We have defined a product of two vectors which gives a scalar. It must be noted that the magnitude of a vector is never negative (i.e. a and b are positive numbers), and hence, the sign of $\vec{a}.\vec{b}$ is decided by the sign of cos θ.

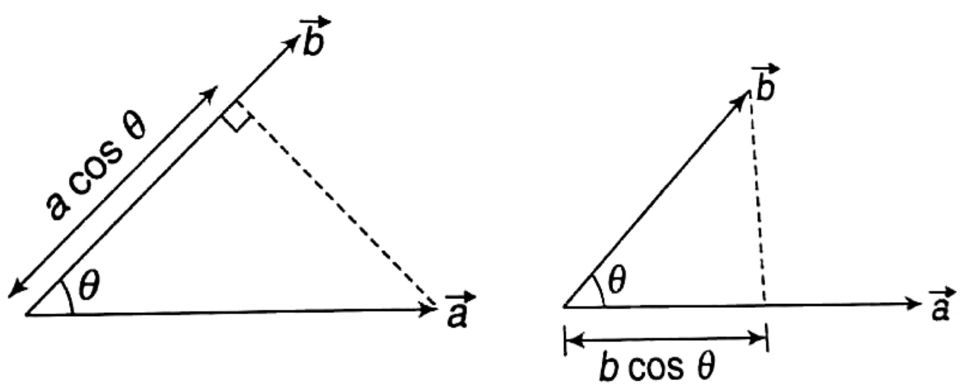


Figure - 6

$\vec{a}.\vec{b} > 0$ if θ is acute

$\vec{a}.\vec{b} = 0$ if θ is 90°

$\vec{a}.\vec{b} < 0$ if θ is obtuse.

We can give a geometric interpretation for the dot product as

$$\vec{a}.\vec{b} = (a\cos\theta)b$$

Here, acos θ = projection (i.e. component) of $\vec{a}$ in the direction of $\vec{b}$.

Therefore, $\vec{a}.\vec{b}$ is the product of magnitude of $\vec{b}$ and the component of $\vec{a}$ in the direction of $\vec{b}$.

We can also write the dot product as

$$\vec{a}.\vec{b} = a\ (b cos\theta)$$

= (magnitude of $\vec{a}$) × (projection of $\vec{b}$ in the direction of $\vec{a}$)

1.7.1 Properties of DOT Product

(1) Scalar product of a vector with itself is called the square of the vector.

$$(\vec{a})^2 = \vec{a}.\vec{b} = a.a cos 0^o = a^2$$

Angle between $\vec{a}$ and $\vec{a}$ is 0°.

(2) Scalar product is commutative.

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

(3) If $\vec{a}$ and $\vec{b}$ are two vectors and m is a scalar, then $m\vec{a}.\vec{b} = (m\vec{a}).\vec{b} = \vec{a}.(m\vec{b})$

(4) Scalar product is distributive with respect to addition.

$$\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$$

A corollary of this is

$$(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = a^2 + b^2 + 2\vec{a}.\vec{b}$$

1.7.2 DOT Product in Terms of Components in Cartesian System

$$\hat{\imath}.\hat{\imath} = 1.1.cos 0^o = 1$$

Similarly, $\hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

Also, $\hat{\imath}.\hat{\jmath} = 1.1.cos 90^o = 0$

Similarly, $\hat{\jmath}.\hat{k} = \hat{\jmath}.\hat{k} = 0$

Now consider two vectors $\vec{a}$ and $\vec{b}$ expressed in their component form as

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$$

$\vec{a}.\vec{b}$ is calculated as

$$\vec{a}.\vec{b} = \left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right).\left(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}\right)$$

$$= a_xb_x\,\hat{\imath}.\hat{\imath} + a_yb_y\,\hat{\imath}.\hat{\jmath} + a_zb_z\,\hat{\imath}.\hat{k}$$

$$+ a_ya_x\,\hat{\jmath}.\hat{\imath} + a_yb_y\,\hat{\jmath}.\hat{\jmath} + a_yb_z\,\hat{\jmath}.\hat{k}$$

$$+ a_zb_x\,\hat{k}.\hat{\imath} + a_zb_y\,\hat{k}.\hat{\jmath} + a_zb_z\,\hat{k}.\hat{k}$$

Out of the nine terms, six terms are zero as $\hat{\imath}.\hat{\jmath} = \hat{\imath}.\hat{k} = \hat{\jmath}.\hat{k} = 0$. In the remaining three terms, we should put $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$.

$$\therefore \qquad \vec{a}.\vec{b} = a_xb_x + a_yb_y + a_zb_z$$

1.7.3 Angle Between Two Vectors

Dot product is useful in finding angle between two vectors if the vectors are in component form.

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath}\ \ b_z\hat{k}$$

Magnitudes of the two vectors are

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \text{and} \quad b = \sqrt{b_x^2 + b_y^2 + b_z^2}$$

Dot product is $\vec{a}.\vec{b} = a_xb_x + a_yb_y + a_zb_z$.

Using the definition of dot product, we have

$$\vec{a}.\vec{b} = abcos\theta$$

$$\Rightarrow \qquad cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \frac{a_xb_x + a_yb_y + a_zb_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}.\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

Here, θ is the angle between the two vectors.

1.8 VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

This is second type of vector multiplication in which two vectors are multiplied to give a new vector.

Vector product of $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ (hence the name cross product) and is defined as:

$\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ with its proper direction given by the right-hand thumb when fingers are curled from the direction of $\vec{a}$ to $\vec{b}$ and has a magnitude equal to ab sinθ where θ is the angle between $\vec{a}$ and $\vec{b}$.

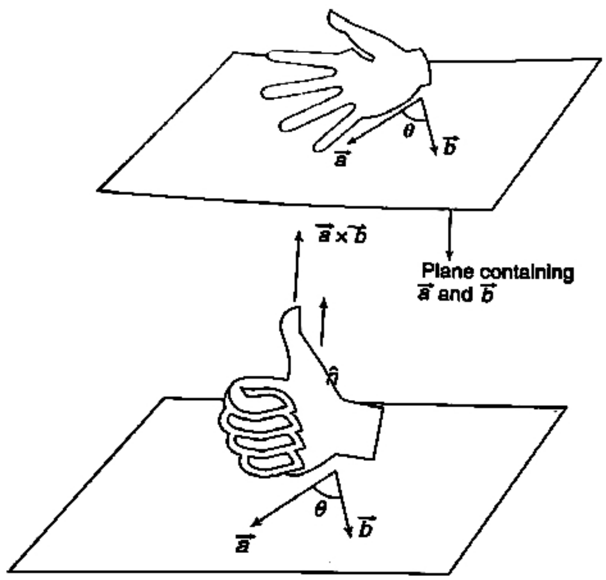


Figure - 7

$$|\vec{a} \times \vec{b}| = ab\, sin\theta$$

Let $\hat{n}$ be a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in the direction of right-hand thumb when fingers are curried from $\vec{a}$ to $\vec{b}$.

We can write

$$\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\hat{n}$$

$$= (ab\, sin\theta)\hat{n}$$

It is important to notice that $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ are vectors of same magnitude but have opposite directions. In fact,

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

1.8.1 Properties of Cross Product

Properties of Cross Product

(1) If $\vec{a}$ and $\vec{b}$ are vectors with non-zero magnitude and $\vec{a} \times \vec{b} = 0$, then $sin\theta = 0$.

It means either $\vec{a}$ and $\vec{b}$ are parallel or they are antiparallel.

(2) $\vec{a} \times \vec{a} = 0$ $[\because sin0^o = 0]$

(3) $\vec{a} \times \vec{b} \neq -\vec{b} \times \vec{a}$

It means cross-product is not commutative

(4) If m is a scalar (or a number)

$$m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$$

(5) Vector product is distributive with respect to addition

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(6) Vector product is not associative

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

1.8.2 Cross Product in Terms of Components in Cartesian System

Since $\vec{a} \times \vec{a} = 0$

$\therefore$ $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k}$

What is $\hat{\imath} \times \hat{\jmath}$? Obviously, magnitude of $\hat{\imath} \times \hat{\jmath}$ is $|\hat{\imath} \times \hat{\jmath}| = 1 \times 1 \times \sin 90^o = 1$.

The direction of $\hat{\imath} \times \hat{\jmath}$ is perpendicular to the xy plane. Once the x and y axes are chosen, there are two possible choices for positive z direction. We chose the positive z-axis in the direction of $\hat{\imath} \times \hat{\jmath}$, and therefore, we write

$$\hat{\imath} \times \hat{\jmath} = \hat{k}.$$

Such a coordinate system in which the right-hand's thumb gives the direction of positive z axis when fingers are curled from x to y direction is known as a right-handed Cartesian coordinate system.

In such a coordinate system, we have

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \quad \hat{\jmath} \times \hat{k} = \hat{\imath}, \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

and $\hat{j}\times\hat{\imath}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{\imath},\quad \hat{\imath}\times\hat{k}=-\hat{j}$

An easy way to remember this is the figure shown. If you wish to find $\hat{j}\times\hat{k}$, walk along the circle from $\hat{j}$ to $\hat{k}$. If you move clockwise, the result is positive, i.e. $\hat{j}\times\hat{k}=\hat{\imath}$.

In order to get $\hat{\imath}\times\hat{k}$, walk along the circle from $\hat{\imath}$ to $\hat{k}$. You are moving anticlockwise, hence the result is negative $\hat{\imath}\times\hat{k}=-\hat{j}$.

Thinking on similar lines, you can evaluate other products.

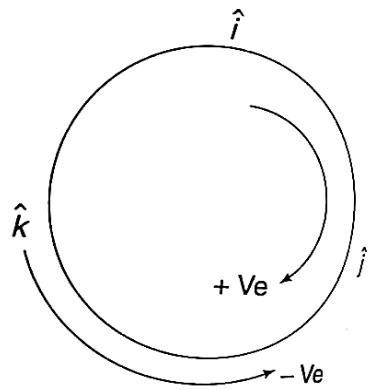


Figure - 8

Now consider two vectors expressed as

$$\vec{a}=a_x\hat{\imath}+a_y\hat{j}+a_z\hat{k}$$

$$\vec{b}=b_x\hat{\imath}+b_y\hat{j}+b_z\hat{k}$$

$$\vec{a}\times\vec{b}=a_x\hat{\imath}\times\left(b_x\hat{\imath}+b_y\hat{j}+b_z\hat{k}\right)$$

$$+a_y\hat{j}\times\left(b_x\hat{\imath}+b_y\hat{j}+b_z\hat{k}\right)$$

$$+a_z\hat{k}=\left(b_x\hat{\imath}+b_y\hat{j}+b_z\hat{k}\right)$$

Since $\hat{\imath}\times\hat{\imath}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

$$\therefore\qquad \vec{a}\times\vec{b}=a_xb_y(\hat{\imath}\times\hat{j})+a_xb_z(\hat{\imath}\times\hat{k})$$

$$+a_yb_x(\hat{j}\times\hat{\imath})+a_yb_z(\hat{j}\times\hat{k})$$

$$+a_zb_x(\hat{k}\times\hat{\imath})+a_zb_y(\hat{k}\times\hat{j})$$

Now, we put $\hat{j} \times \hat{k} = \hat{\imath}; \quad \hat{k} \times \hat{j} = -\hat{\imath}$

$$\hat{k} \times \hat{\imath} = \hat{j}; \quad \hat{\imath} \times \hat{k} = -\hat{j}$$

$$\hat{\imath} \times \hat{j} = \hat{k}; \quad \hat{j} \times \hat{\imath} = -\hat{k}$$

∴

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{\imath} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

Using determinants, we can write

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

1.8.3 Cross Product and Area Vector

In many applications in physics, we consider area to be a vector quantity. Direction of the area vector is taken perpendicular to the surface under consideration.

For example, think of a wire loop placed in a stream. You wish to find the rate of flow of water through the loop. Just knowing the area of loop and speed of flow will not surface. You must also know the orientation of the loop with respect to the direction of flow. If the loop is parallel to the flow, no water flows through it. If it is held normal to the direction of water flow, rate of water flow through it will be maximum. In such situations, we consider area to be a vector.

Consider a parallelogram formed by vectors $\vec{a}$ and $\vec{b}$ as adjacent sides. Area of the parallelogram is

A = base × height

$$= a.b \ sin\theta$$

$$= |\vec{a} \times \vec{b}|$$

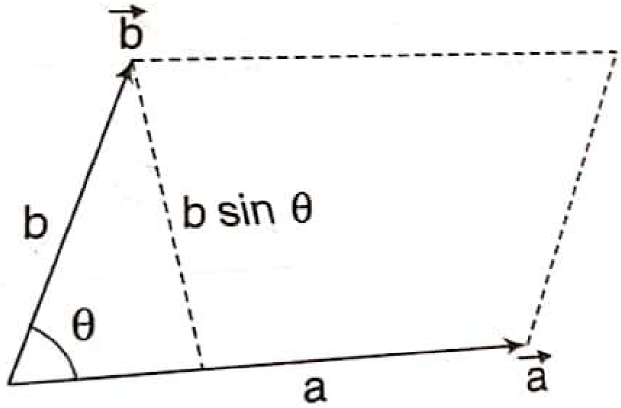


Figure - 9

If we consider area of parallelogram to be vector, its direction is perpendicular to the plane of the parallelogram. Hence, we can write

$$\vec{A} = \vec{a} \times \vec{b}.$$

To some extent direction of $\vec{A}$ is arbitrary. We could also write

$$\vec{A} = \vec{b} \times \vec{a}.$$

If the parallelogram is in the plane of the paper, its area vector ($\vec{A}$) is either coming towards you $(\vec{a} \times \vec{b})$ or it is going into the plane of the paper (i.e. along $\vec{b} \times \vec{a}$).

1.9 CURL OF VECTOR

Let V (x, y, z) be vector point function.

Then curl $\vec{V} = \nabla \times \vec{V}$ $\quad (\vec{V} = V_x^i + V_y^j + V_z^k)$

$$= \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right) \times \left(V_x^i + V_y^j + V_z^k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ V_x & V_y & V_z \end{vmatrix} = i\left(\frac{\delta V_z}{\delta y} - \frac{\delta V_y}{\delta z}\right) - j\left(\frac{\delta V_z}{\delta x} - \frac{\delta V_x}{\delta z}\right) + k\left(\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y}\right)$$

$$= i\left(\frac{\delta V_z}{\delta y} - \frac{\delta V_y}{\delta z}\right) + j\left(\frac{\delta V_x}{\delta z} - \frac{\delta V_z}{\delta x}\right) + k\left(\frac{\delta V_y}{\delta x} - \frac{\delta V_x}{\delta y}\right)$$

Curl $\vec{V}$ is a vector quantity.

1.9.1 Physical Interpretation of Curl

We know that $V = \omega \times \overline{r}$, where ω is the angular velocity. V is the linear velocity and $\overline{r}$ is the position vector of a point on the rotating body.

$$\omega = \omega_1^i + \omega_2^j + \omega_3^k$$

$$r = xi + yj + zk$$

Curl V = V × V

$$= \nabla \times (\omega \times \overline{r})$$

$$= \nabla \times \left[\left(\omega_1^i + \omega_2^j + \omega_3^k\right)\right] \times (xi + yj + zk)]$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\left(\omega_2^z - \omega_3^y\right)i - (\omega_1^z - \omega_3^x)j + \left(\omega_1^y - \omega_2^x\right)k\right]$$

$$= \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z}\right) \times \left[\left(\omega_2^z - \omega_3^y\right)i - (\omega_1^z - \omega_3^x)j + \left(\omega_1^y - \omega_2^x\right)k\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ \left(\omega_2^z - \omega_3^y\right) & (\omega_3^x - \omega_1^z) & \left(\omega_1^y - \omega_3^k\right) \end{vmatrix}$$

$$= (\omega_1 + \omega_1)i - (-\omega_2 - \omega_2)j + (\omega_3 + \omega_3)k$$

$$= 2\left(\omega_1^i + \omega_2^j + \omega_3^k\right) = 2\omega.$$

Curl $V = 2\omega$, which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

If curl $\vec{F} = 0$, the field F is termed irrotational.

1.10 SOLENOIDAL VECTOR

A vector point function f is said to be solenoidal vector if its divergent is equal to zero i.e., **div f = 0** at all points of the function. For such a vector, there is no loss or gain of fluid.

$$\nabla . f = \partial_x f_1 + \partial_y f_2 + \partial_z f_3 = 0$$

1.11 LAMELLAR VECTOR OR IRROTATIONAL VECTOR

A continuous vector function F is said to be irrotational in a simply connected region D, if its circulation along every closed curve in d vanishes, i.e., F is irrotational in D, if

$$\oint F . dr = 0 \qquad \text{.................. (1)}$$

1.12 LINE INTEGRAL

Consider an oriented curve C in space and assume that C is a simple curve in the sense that it has no points at which it intersects or touches itself. Let P_0 be the initial point and P' the terminal point of C in the chosen orientation. Further, let the parametric representation of C with the are length s as the parameter be

$$r = r(s) = x(s)\hat{\imath} + y(s)\hat{\jmath} + z(s)\hat{k} \qquad \text{.................. (2)}$$

such that the position vector r(s) is continuous and has a continuous first derivative, not equal to zero vector, for all s under consideration. Then C is a smooth curve in the sense that C has a unique tangent at each of its points.

Le F be a continuous vector point function defined at each point of C. Let C be subdivided into n portions in an arbitrary manner by a set of points P_0, P_1, P_2, ….. P_i, ….. P_n = P' whose position vectors are r_0, r_1, r_2, …… r_i, …… r_n and$r_i - r_{i-1} = \delta r_i$, i = 1 2, …….., n.

Let Q_1, Q_2,……, Q_i, ……. Q_n be arbitrary points chosen on the portions P_0P_1, P_1P_2, ….., $P_{i-1}P_i$, ….., Pn-1Pn and let $F(Q_i)$ be the value of F at Q_i, i =1, 2, …., n.

From the sum

$$I_n = \sum_{i=1}^{n} F(Q_i) . \delta r_i \qquad \text{.................. (3)}$$

and compute the limit of this sum as $n \to \infty$ and every $|\delta r_i| \to 0.$ Since F is continuous and C is smooth, this limit exists and is independent of the choice of the mode of subdivision and the point Q_i.

This limit is called the line integral of F along the curve C from P_0 to P' and is denoted by

$$\int_{P_0}^{P'} F.dr \quad or \quad \int_C F.dr \qquad \text{..................} (4)$$

If $F = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$, then (4) takes the form

$$\int_C F.dr = \int_C (F_1\,dx + F_2 dy + F_3\,dz) \text{...............} (5)$$

Further, since the unit tangent vector to C is $\hat{t} = dr/ds$, we can also write (4) as

$$\int_C F.dr = \int_C F.\frac{dr}{ds}ds = \int_C F.\hat{t}ds \quad \text{..................} (6)$$

Which implies that the line integral is the integral of the tangential component of F along the curve w.r.t. s. So, the line integral is also called the tangential line integral of F along C from P_0 to P'.

If the curve C is represented by $r = r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ in terms of some other parameter t, then (4) takes the form

$$\boxed{\int_C \boldsymbol{F.dr} = \int_C \boldsymbol{F.\frac{dr}{dt}dt}} \qquad \text{..................} (7)$$

If the vector field F is defined in a region along different curves joining P_0 and P', then the integral (4) will have generally different values along different curves although the end points of the curves are the same. Hence, in general, a line integral depends not only on the end points but also on the geometrical shape of the path of integration.

1.12.1 Physical Meaning of the Line Integral

Let a variable force F act on a particle which is displaced along a path C in space. Then the work done by F in this displacement is the line integral of F along C, i.e.

$$\boxed{\boldsymbol{W} = \int_C \boldsymbol{F.dr}} \qquad \text{..................} (8)$$

1.12.2 Line Integral Independent of Path

Under certain conditions, the value of the line integral (4) depends only on the end points P_0 and P' but does not depend on the path from P_0 to P'. In this section, we wish to obtain such condition.

Let a vector function F be defined and be continuous in a region D of space. Then the line integral (4) is said to be independent of path in D, if every pair of end point P_0 and P' in D the value of the integral is the same for all paths C in from P_0 to P'.

1.12.3 Line Integrals and Work

If the point of application of a force

$$F = iM(x, y, z) + jN(x, y, z) + kP(x, y, z) \qquad \text{............ (9)}$$

moves along a curve C from a point A(x, y, z) to a point B(x, y, z) then the work done by the force is

$$W = \int_C F.dr \qquad \text{..........(10)}$$

$$= \int_C (M\,dx + N\,dy + P\,dz). \qquad \text{..........(11)}$$

This integral is also called a line integral.

To evaluate the integral in (19) we might express the equation of curve C in terms of parameter t :

$$x = x(t), y = y(i), z = z(t) \qquad \text{..........(12)}$$

such that the curve is described from A to B as t varies from a value t_1 to a value t_2.

1.13 CIRCULATION AND IRROTATIONAL VECTOR

Circulation: The line integral of a continuous vector function F along a closed smooth curve C is called the circulation of F along C and is sometimes denoted by

$$\oint_C F.dr \left(instead\ of \int_C F.dr\right) \qquad \text{.................. (13)}$$

Simply Connected Region: A region (or domain) d is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

Irrotational Vector: A continuous vector function F is said to be irrotational in a simply connected region D, if its circulation along every closed curve in d vanishes, i.e., F is irrotational in D, if

$$\oint F.dr = 0 \qquad \text{.................. (14)}$$

1.14 SURFACE INTEGRAL

Consider a simple, smooth and orientable surface S. The surface S is simple in the sense that it has no points at which it intersects or touches itself, it is smooth in the sense that it has a unique normal at every point of it, and it is orientable in the sense that a chosen positive normal direction at any point of S can be continued in a unique and continuous manner to the entire surface. We consider here only two-sided surface which is orientable. Surfaces such as the Mobius strip or Klein bottle are excluded. The positive normal will be assumed conventionally to be in outward direction for a closed surface and in a right-handed sense for an open surface bounded by a closed curve C.

Let a continuous vector function F be defined at each point of S. Let S be arbitrarily subdivided into n parts S_1, S_2, ….., S_i, ……, S_n of areas $\delta S_1, \delta S_2, \dots, \delta S_i, \dots, \delta S_n$. Let $Q_1, Q_2, \dots, Q_i, \dots, Q_n$ be arbitrary points chosen in these parts, and let $F(Q_i)$ be the value of F at Q_i, i = 1, 2, …….., n and let $\hat{n}_i$ be the positive (outward) unit normal vector S_i at Q_i, i = 1, 2, ……. n.

Form the sum,

$$I_n = \sum_{i=1}^{n} F(Q_i).\hat{n}_i \delta S_i \qquad \text{………..(15)}$$

and find the limit of this sum as $n \to \infty$ and every $\delta S_i \to 0$. Since F is continuous and S is smooth, this limit exists and is independent of the mode of subdivision of S and the points Q_i.

This limit is called the surface integral of F over S and is denoted by

$$\iint_S F.\hat{n} dS \qquad \text{………………. (16)}$$

1.15 VOLUME INTEGRAL

Let scalar function ϕ (x, y, z) be defined and continuous in a bounded closed region V in space which is bounded by finitely many smooth surfaces. Subdivide V into n elementary portions enclosing volumes $\delta V_1, \delta V_2, \dots, \delta V_i, \dots, \delta V_n$. Let $Q_1, Q_2, \dots, Q_i, \dots, Q_n$ be arbitrarily chosen points in these portions and let $\phi(Q_i)$ be the value of ϕ at Q_i, i = 1, 2, ……, n.

Form the sum

$$I_n = \sum_{i=1}^{n} \phi(Q_i) \delta V_i \qquad \text{………..(17)}$$

and find the limit of this sum as $n \to \infty$ and every $\delta V_i \to 0$. Since ϕ is continuous, this limit axis's and is independent of the mode of subdivision of V and the points Q_i.

This limit is called the volume integral or space integral or triple integral of ϕ over the region V, and is denoted by

$$\iiint_V \phi dV \qquad \text{………………. (18)}$$

Similarly, we define the volume integral of a continuous vector function F by

$$\iiint_V \mathrm{F}\, dV \qquad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

1.16 CONSERVATIVE FIELD

In general, the line integral $\int_C F.dr$ depends on the function F, the curve C joining two points P and Q and also on the points P and Q themselves.

If the integral $\int_C F.dr$ depends on F and end points only but not on the curve joining the end points, then the vector field over the region is called conservative field.

Since $\int_C F.dr$ is also the work done by a force in moving a particle along a curve r(t), the field is conservative if the work done in moving the particle from one point to another is independent of the curve joining the points.

Theorem 1: The field F is conservative over a region if and only if $\oint F.dr = 0$ along any closed curve in the region.

Proof: Let PAQ and PBQ be two different paths joining the points P and Q in the region. Together, these paths make up a closed curve Γ, i.e., PAQBP.

If the field is conservative, we have

$$\int_{PAQ} F.dr = \int_{PBQ} F.dr.$$

Thus $\int_\Gamma F.dr = \int_{PAQ} F.dr + \int_{QBP} F.dr$

$$= \int_{PAQ} F.dr - \int_{PBQ} F.dr = 0.$$

Conversely, if $\oint F.dr$ over any closed curve Γ, say, PAQBP is zero, we have

$$\oint_\Gamma F.dr = \int_{PAQ} F.dr + \int_{QBP} F.dr = \int_{PAQ} F.dr - \int_{PBQ} F.dr = 0.$$

Therefore, $\int_{PAQ} F.dr = \int_{PBQ} F.dr.$

Theorem 2: The field F is conservative if there exists *a* single-valued differentiable scalar function ϕ such that F=∇ϕ, and conversely.

Proof: First, let F=∇ϕ. Then the work done in moving a particle from the point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ in the field is

$$\int_{P_1}^{P_2} F.dr = \int_{P_1}^{P_2} \nabla\phi dr = \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right).(dxi + dxj + dzk)$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

$$\phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1).$$

Thus, the work done is independent of the path joining the end points, i.e., the field F is a conservative.

Conversely, let the field $F = F_1 i + F_2 j + F_3 k$ be conservative. By hypothesis, $\int_C F.dr.$ is independent of the path joining any two points which we take as (x_1, y_1, z_1) and (x, y, z) respectively. Thus

$$\int_{(x,y,z)}^{(x_1,y_1,z_1)} F.dr = \phi(x,y,z).(say),$$

is independent of the path joining (x_1, y_1, z_1) and (x, y, z). Now

$$\phi(x+\delta x, y, z) - \phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x+\delta x,y,z)} F.dr - \int_{(x_1,y_1,z_1)}^{(x,y,z)} F.dr$$

$$\int_{(x,y,z)}^{(x+\delta x,y,z)} F.dr = \int_{(x,y,z)}^{(x+\delta x,y,z)} (F_1 dx + F_2 dy + F_3 dz).$$

Since the last integral is independent of the path joining (x, y, z) and $(x+\delta x, y, z)$, we choose the path to be a straight line joining these points so that dy and dz are zero. Then

$$\frac{\phi(x+\delta x,y,z) - \phi(x,y,z)}{\delta x} = \frac{1}{\delta x}\int_{(x,y,z)}^{(x+\delta x,y,z)} F_1 dx.$$

Taking the limit on both sides as $\delta x \to 0$, we have $\frac{\partial \phi}{\partial x} = F_1$.

Similarly, it follows that $\frac{\partial \phi}{\partial y} = F_2 \; and \; \frac{\partial \phi}{\partial z} = F_3$.

Thus, we have

$$F = F_1 i + F_2 j + F_3 k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = \nabla \phi. \quad \dots\dots\dots (20)$$

Note: The function ϕ obtained in the above theorem is called scalar potential of the field F.

Theorem 3: The necessary and sufficient condition that a field F be conservative is that

$$curl\, F = \nabla \times F = 0.$$

Proof: The condition is necessary. If the field F is conservation, then from Theorem 2, we have $F = \nabla \phi$. Thus

$$curl\, F = \nabla \times \nabla \phi = 0.$$

The condition is sufficient. Let

$$\phi(x, y, z) = \int_{x_0}^{x} F_x(x, y, z) dx + \int_{y_0}^{y} F_y(x_0, y, z) dy + \int_{z_0}^{z} F_z(x_0, y_0, z) dz,$$

where (x_0, y_0, z_0) is an arbitrary interior point of the region and $F = F_x i + F_y j + F_z k$. Then

$$\frac{\partial \phi}{\partial x} = F_x(x, y, z) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \int_{x_0}^{x} \frac{\partial F_x}{\partial y} dx + F_y(x_0, y, z).$$

Since $\nabla \times F = 0$, the component of $\nabla \times F$ along k is

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0. \qquad \text{Hence} \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}.$$

Therefore, $$\frac{\partial \phi}{\partial y} = \int_{x_0}^{x} \frac{\partial F_y}{\partial x} dx + F_y\,(x_0, y, z)$$

$$= F_y(x, y, z) - F_y(x_0, y, z) + F_y(x_0, y, z) = F_y(x, y, z),$$

and similarly, $$\frac{\partial \phi}{\partial z} = F_z(x, y, z).$$

Thus, we have $$F = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = \nabla \phi.$$

Hence, from Theorem 2, it follows that the field F is conservative.

Corollary: The necessary and sufficient condition that

$\int_C (Udx + Vdy + Wdz)$ is independent of the path C, is that

$$\nabla \times (Ui + Vj + Wk) = 0.$$

Remark: We can now characterize conservative field by the following statements:

Field F is conservative if and only if:

(i) $F = \nabla\phi$ (the choice of ϕ is not unique),

(ii) $\nabla \times F = 0,$

(iii) F is irrotational,

(iv) $\oint F.dr = 0,$ for every closed curve.

1.17 GAUSS'S DIVERGENCE THEOREM

Theorem : If V be a closed region in space whose boundary is a piecewise smooth orientable surface S and F (x, y, z) be a vector function which is continuous and has continuous first partial derivatives in some domain containing V, then the volume integral of the divergence of F taken over the volume V enclosed by the surface S is equal to the surface integral of the normal component of F taken of over the surface S, i.e.

$$\iiint_V div\, F.dV = \iint_S F.\hat{n}dS \qquad \text{..................} (21)$$

where $\hat{n}$ is the outward unit normal vector to S w.r.t. V.

1.18 STOKES'S THEOREM

Theorem : If S be a piecewise smooth oriented surface whose boundary is a piecewise smooth simple closed curve C and if F(x, y, z) be a vector function which is continuous and has continuous first partial derivatives in a region in space containing S, then the surface integral of the component of curl F normal to the surface S is equal to the line integral of F taken round the curve C, the normal to S being taken in the righthanded sense with respect to orientation of C, i.e.

$$\iint_S (curl\, F).\hat{n}dS = \oint_C F.dr \qquad \text{................} (22)$$

1.19 GREEN'S THEOREM

Gauss's divergence theorem, which is also called Green's theorem in space, leads to two forms of Green's theorem as given below.

Let ϕ and ψ be scalar functions defined in a region V enclosed by a surface S and possess continuous derivatives of second order at least. Let F = ϕ grad ψ, G = ψ grad ϕ and let V and S satisfy all assumptions of the divergence theorem. Then, we have

(a) First Form : In the divergence theorem (21), let $F = \phi\nabla\psi$. Then

$$\iiint_V div(\phi \text{ grad } \psi)dV = \iint_S \left((\phi \text{ grad } \psi)\right), \hat{n}dS$$

………………. (23)

or

$$\iiint_V (\phi \text{ div grad } \psi + \text{grad } \phi . \text{grad } \psi)dV =$$
$$\iint_S \phi(\hat{n}.grad\,\psi)dS$$

or $$\iiint_V (\phi\nabla^2\psi + \nabla\phi.\nabla\psi)dV = \iint_S \phi\frac{\partial\phi}{\partial n}dS$$

………………. (24) as grad $\psi = (\partial\psi/\partial n)\hat{n}$. The formula (24) is called Green's first formula or the first form of Green's theorem or Green's first identity.

(b) Second Form: Interchanging ϕ and ψ, we get

$$\iiint_V (\psi\nabla^2\phi + \nabla\psi.\nabla\phi)dV = \iint_S \psi\frac{\partial\phi}{\partial n}dS$$

We find that,

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS$$

This formula is called Green's second formula or the second form of Green's theorem or Green: second identity or Green's symmetrical theorem.

1.20 SUMMARY

In the present unit, we have studied about vector and Scalar. We have also studied some fundamental definitions of vectors.

- Scalar quanties are quantities with magnitudes only.
- Vector quantities are quantities with magnitude and direction both.

In this unit, we also introduced the concept of line, surface and volume integrals.

1.21 TERMINAL QUESTIONS

1. Define the following:
   (a) Unit Vector
   (b) Equal Vector
   (c) Null Vector
   (d) Polar Vector
   (e) Axial Vector
2. What do you understand by Notation of Vector.
3. How is a vector represented? Illustrate with an example.
4. What do you mean by Scalar and Vector quantities. Explain with example.
5. Define Line Integral. Explain physical meaning of Line Integral.
6. Write short notes on:
   (a) Solenoidal Vector
   (b) Stokes's Theorem
   (c) Green's Theorem
   (d) Gauss's Divergence Theorem
7. Explain the Concept of Tensor.
8. What do you understand by Conservative field.

ANSWERS TERMINAL QUESTIONS

1. (a) Hint (Section 1.4.3)
   (b) Hint (Section 1.4.1)
   (c) Hint (Section 1.4.2)
   (d) Hint (Section 1.4.4)
   (e) Hint (Section 1.4.5)
2. Hint (Section 1.3.4)
3. Hint (Section 1.3.4)
4. Hint (Section 1.3)
5. Hint (Section 1.12)
6. (a) Hint (Section 1.10)

(b) Hint (Section 1.18)

(c) Hint (Section 1.19)

(d) Hint (Section 1.17)

7. Hint (Section 1.6)

8. Hint (Section 1.16)

1.22 SUGGESTED READINGS

1. Fundamental University Physics - I, M. Alonslo and E. Finn, Addition Wesley Publication.
2. Mathematics for Engineers – Volume - I S. N. Pandey, Paragon International Publication.
3. Concept of Physics – H. C. Verma, Bharti Bhawan, Patna.
4. College Physics: Hugh D. Young.

UNIT : 2

DYNAMICS OF A PARTICLE

Structure

2.1 INTRODUCTION

In the previous unit, we have studied about vector analysis. In the present unit, we will learn about dynamics of a Particle. We already learnt some basic concept of Mechanics and Dynamics of the Particles in the previous classes.

We used Kinematic quantities for describing motion without considering what might cause that motion. In order to understand this beauty, let us take a step forward by understanding "Force and Newton's laws of motion."

Newton's law of motion are heart and Soul of Physics. Though the laws are simple to state and involve little mathematical complexity.

Also, in this unit we shall study about work, Power and Energy. We, also understand the Elastic and Inelastic Collision.

2.2 OBJECTIVES

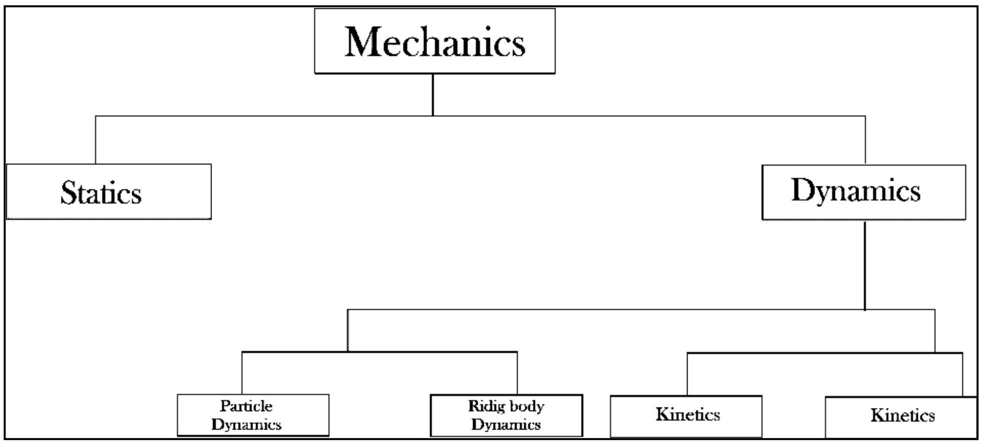
After studying this unit, you should be able to –

- ❖ Understand the Concept of Force, Momentum Impulse.
- ❖ Solve Problems based on Force, Momentum Impulse.
- ❖ Explain the Concept of Conservative and Non-Conservative Forces.

- ❖ Distinguish between Conservative and Non-Conservative Force.
- ❖ Apply Work – Energy Theorem.
- ❖ Compute Power in various Mechanical Systems.

2.3 WHAT IS MECHANICS

Mechanics is one of the main branches of Physics which deals with the study and behavior of Physical bodies when subjected to different types of forces or displacement and the subsequent effect of bodies on the environment.



2.3.1 Type of Mechanics

There are generally two main types of Mechanics -

1. Classical Mechanics
2. Quantum Mechanics

Classical Mechanics deals with the study of macroscopic objects while quantum mechanics deals with the study of microscopic object.

2.3.2 Branches of Mechanics

Statics: The Branch of mechanics that treats objects which are stationary (usually) or at constant velocity.

Dynamics: The effect of force when the body is in motion (combination of kinematics and kinetics)

Kinematics: Which is the description of motion without regard to force. We calculated accelerations, but never asked what forces are needed to product these acceleration (force which caused the motion are not considered)

Kinetics: Analysis of forces and torques that cause motion (Newton's second law)

Force which caused the motion are considered

All the principles of dynamics can be summarized in a neat package containing three statement called newton's laws of motion. These laws, that are corner stone of mechanics, are based on they cannot be deduced or proved from any other principles.

2.4 FORCE

Force is defined as a push or pull which tries to change or changes the state of rest or uniform motion of body.

Let us consider some examples, to open a door someone has to pull it. To throw a ball upwards, one has to give it an upward push. A boat moves in a flowing river without anyone rowing it. To roll a ball lying on a floor, we have to push it. From these examples it is clear that some external agency is needed to provide force to change the motion of the body.

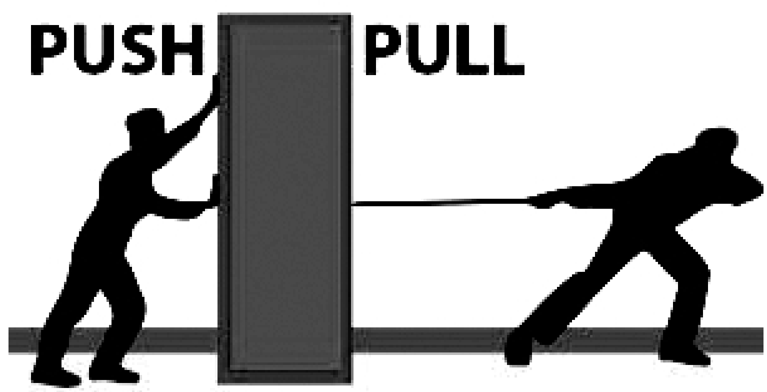


Figure - 1

What force can do?

a. Force can start a body means it can set a body in a motion.

b. it can stop a body i.e. it can continuously reduce the reduce velocity of the tat body and it can make the body to stop also.

c. It can change the direction of a body without applying force we can never change the direction of a body.

2.4.1 Types of Force

Forces is a physical cause that can change the state of motion of the dimension of an object.

Broadly we can divide force into two types

(a) Contact Forces

(b) Non-contact Force (field forces)

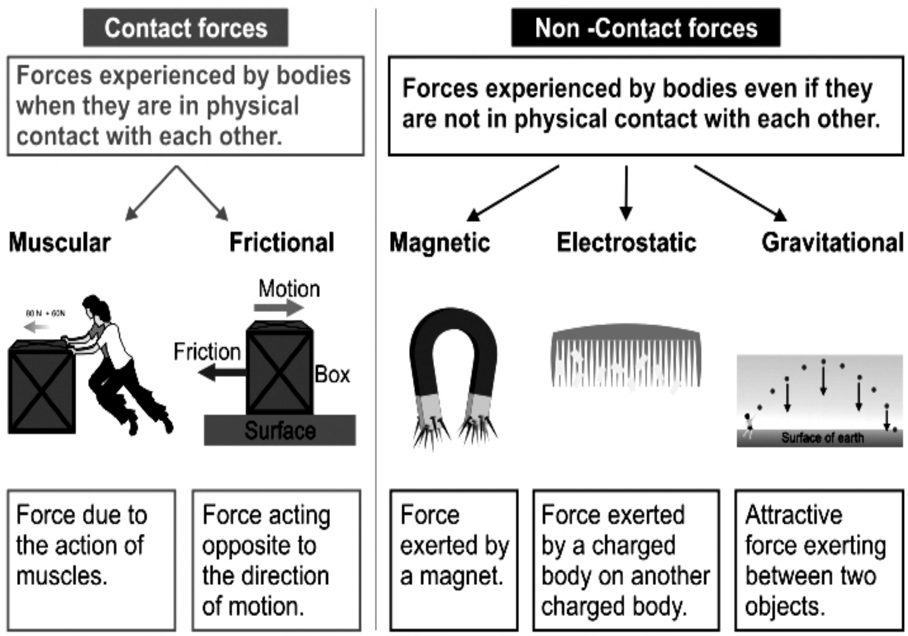


Figure - 2

(a) Contact Forces: Force which action a body either directly or through medium are called contact forces. If the contact is frictionless the contact force is perpendicular to the common surface and known as normal reaction.

If, however, the objects are in rough contact and move (or have a tendency to move) relative to each other without losing contact then frictional force arise which oppose such motion. Again, each object exerts a frictional force on the other and the two forces are equal and opposite. This force is perpendicular to normal reaction. Thus, the contact force (F) between two objects is made up of two forces.

(i) Normal reaction (N)

(ii) Force of friction (f)

and since these two forces are mutually perpendicular.

$$\boxed{F = \sqrt{N^2 + f^2}} \qquad \text{............... (1)}$$

Consider two wooden blocks A and B being rubbed against each other.

In the diagram, A is being moved to the right while B is being moved leftward. In order to see more clearly which forces, act on A and which on B, a second diagram is drawn showing a space between the blocks but they are still supposed to be in contact.

In Figure (3 & 4) the two normal reactions each of magnitude N are perpendicular to the surface of contact between the blocks and the two frictional forces each of magnitude f act along that surface, each in a direction opposing the motion of the block upon which it acts.

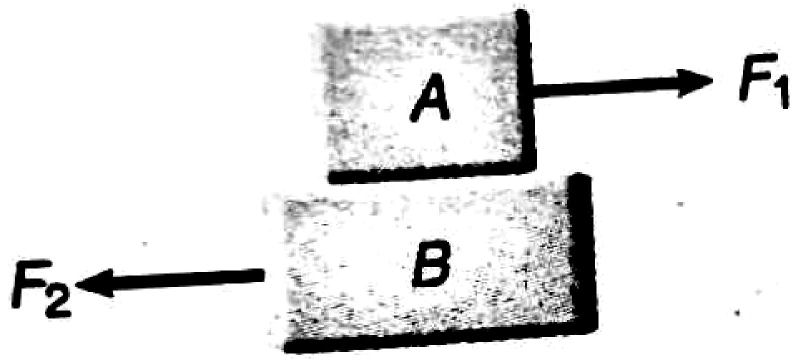


Figure – 3

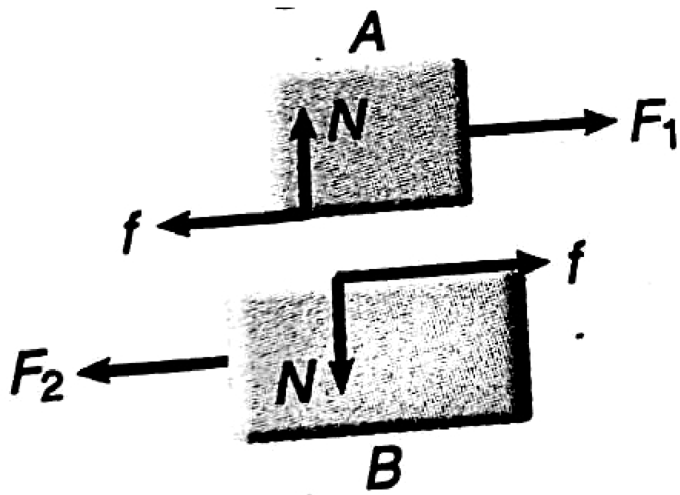


Figure – 4

(b) Non- Contact Forces or Field Forces : A force that one object can apply to another object without touching it. These are the forces in which contact between two objects is not necessary. Gravitational force between two bodies and electrostatic force between two charges are two examples of field forces. Weight (W = mg) of a body comes in this category.

2.4.2 Characteristics of Force

Force is a vector quantity therefore has magnitude as well as direction when forces act on a solid body, they usually deform the body.

To predict how a force affects motion of a body we must know its magnitude, direction and point on the body where the force is applied. This point is known a point of application of the force. the direction and the point of application of a force both decide line of action of the force.

Units of force:

Absolute Units:

a. Newton (S.I.)

b. Dyne (C.G.S.)

Gravitational Units

a. Kilogram force (M.K.S.)

b. Gram Force (C.G.S)

Newton: One Newton is that force which produces an acceleration of m/s^2 in a body mass 1 kilogram

$\therefore$ 1 Newton = 1kg m/s^2

Dyne: One dyne is that force which action on body of mass 1 kg produces in it an acceleration of 1 cms^{-2} in its own directions.

Relation between newton and dyne

We have $1N = 1\ Kg \times 1ms^{-2} = 1000\ g \times 100\ cms^{-2}$

$1N = 10^5\ gms^{-2} = 10^5$ dyne

$\therefore$ 1N = 105dyne

Kilogram- force: It is that force which produces an acceleration of 9.8 m/s2 in a body of mss 1 kg

$\therefore$ 1kg –f = 9.81 newton

Gram – Force: It is that force which produces an acceleration of 980cm/s in a body of mass 1g

$\therefore$ 1 gm –f = 980 dyne

2.4.3 Newton's First Law

State of rest and state of uniform motion are natural states of a body. A body continues to remain in its natural state of rest or uniform motion unless it is acted upon by an **unbalanced external** force.

> **"Everybody continues to be in it state of rest or of uniform motion in a straight line unless compelled by some external force to act otherwise."**

This is quite counter-intuitive as our everyday experience shows that a body stops moving if it is not continuously pushed or pulled. This is because we often forget about forces like friction and air resistance. A body stops moving not due to absence of forces, rather due to presence of forces. If we can somehow eliminate all resistive forces, a moving body will keep moving forever. Practically it is not possible to eliminate friction completely.

2.4.4 Concept of Inertia

The Concept of inertia was introduced and developed both in terms of objects of rest and objects in motion.

Inertia is a property of a body by virtue of which it stays in its state of rest of state of uniform motion in absence of an external unbalanced force. How much acceleration will be produced in a body when an external force is applied on it is decided by its inertia. Our daily life experience tells us that more massive a body is, higher is its tendency to resist any change in its state. It is far more difficult to push a truck into motion compared to a small box. In fact, the physical property mas is defined as a measure of inertia.

2.4.5 Types of Inertia

There are three types of inertia:

(i) Inertia of Rest

(ii) Inertia of Motion

(iii) Inertia of Direction

(i) **Inertia of Rest:** It is inability of a body by virtue of which it cannot move by itself. A body at rest remains at rest and cannot start moving on its own due to inertia of ret.

Applications:

- ❖ When horse starts suddenly, the rider falls backward due to inertia of rest.
- ❖ When we shake the branches of a fruit tree, the fruits fall down due to inertia of rest.

Figure: 5

(ii) **Inertia of Motion:** It is inability of a body in motion to stop by itself. A body in uniform motion can neither get accelerated nor get retarded on its own. It also cannot come to rest on its own.

Applications:

- ❖ A man jumping from moving bus falls forward due to inertia of motion. As his feet touch the ground lower part of the body comes to rest, while the remaining parts of the body keep on moving. As a result, he falls down in the direction of motion of the bus.

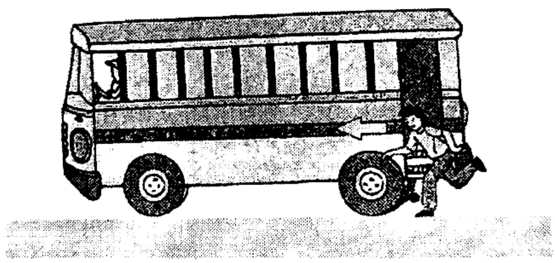


Figure: 6

- ❖ An athlete runs some distance, before taking a long jump due to inertia of motion, since length of jump depends upon his velocity at the instant of jump.

(iii) Inertia of Direction: It is inability of a body by virtue of which it cannot change its direction of motion by itself.

Applications:

- ❖ The sparks coming out of a grinding stone are tangential to the rotating stone due to directional inertia.
- ❖ The mud from the wheels of a moving vehicle files off tangentially.

2.4.6 Inertial and Non-Inertial Reference Frames

An observer in a reference frame S_1 finds that a body A is at rest. Let us assume that the observer measures the net external force acting on A to be zero. For her, no force means no acceleration and this confirms the validity of the first law of motion. There is another reference frame S_2 which is moving with a constant velocity (v) relative to the frame S_1. An observer in this frame also finds Newton's first law to be valid as he sees that in absence of any force, the body A continues to move uniformly. Now assume that there is a third frame S_3 which is accelerated with respect to S_1 (or S_2). An observer in this frame finds that the body A is accelerated. But the observers in S_1 and S_2 have confirmed that net force on A is zero. Observer in reference frame S_3 finds A to be accelerated in absence of a real force. Yes, Newton's first law fails, in usual sense, in a frame like S_3.

The reference frames in which Newton's first law (also known as law of inertia) holds are said to be inertial and those in which it fails are known as non-inertial frames.

A reference frame attached to the Earth is nearly inertial for most practical applications. All other frames which are at rest or moving uniformly with respect to the Earth are also inertial. Any frame that is accelerated with respect to the Earth is non-inertial. For now, we will avoid using Newton's laws in such frames.

It is important t note that the reference frame of the Earth is not inertial in a strict sense and in case of large-scale motions like those of ballistic missiles, ocean current, etc., we must account for the non-inertial character of the Earth's frame.

Example: 1

Why is it advised to tie our luggage kept on the roof of a bus with a rope?

Solution:

Because it may slide and fall due to following reasons:

(i) Initially, if the bus is in the state of rest, the luggage is also in the same state of rest. When the bus starts suddenly the luggage tends to remain in the state of rest due to inertia of rest. As a result, the luggage can be thrown in the backward direction and it may fall.

(ii) If the bus is in the state of motion, the luggage is also in the same state of motion due to inertia of motion. So, when the brakes are applied it may fall in the forward direction.

(iii) If the bus takes a sharp turn on a road, the luggage will resist any change of its state of direction due to inertia. As a result, the luggage can be thrown sideways and may fall. Therefore, it is advised to tie the luggage kept on the roof of the bus to prevent it from falling.

Self-Assessment Questions (SAQs)

1. Why does an athlete run some distance, before taking a long jump?

2.5 MOMENTUM

It is defined as the quantity of motion contained in a body. It is measured as the product of mass of the body and its velocity and has the same direction as that of the velocity. It is a vector quantity. It is represented by p.

Momentum (p) = Mass (m) × Velocity (v)

or $\boxed{p = mv}$

……. (2)

In vector form, we may write as

$$\vec{p} = m\vec{v}$$

……. (3)

Its unit is $kgms^{-1}$ which is same as Ns. It is a vector and any change in direction of motion implies a change in momentum.

Momentum of an extended body or a collection of particles is the sum of momenta of individual particles. As studied in last chapter, momentum for a collection of particles is given by product of total mass and velocity of COM.

$$\vec{P} = \sum m_i \vec{v}_i = M\vec{v}_{CM}$$

……. (4)

2.5.1 Newton's Second Law

According to Newton's second law of motion, the rate of change of linear momentum of a body is directly proportional to the external force applied on the body, and this change takes place always in the direction of the applied force.

Suppose two bodies of different masses are initially at rest, and a fixed force is applied on them for a certain interval of time. To start with, the lighter body picks up a greater speed than the heavier body. However, at the end of the time interval observations show that each body acquires the same linear momentum. It means that the same force applied for the same time causes the same change in linear momentum in bodies of different masses.

The law implies that when a bigger force is applied on a body of given mass, its linear momentum changes faster and vice-versa. The momentum will change in the direction of the applied force.

Mathematical Formulation of Newton's of Second Law

Consider a body of mass m moving with some initial velocity $\vec{v}$. If an unbalanced force $\vec{F}$ is applied, the velocity will change from $\vec{v}$ to

$\vec{v} = +\Delta\vec{v}$. The change in momentum will be $\Delta\vec{p} = m\Delta\,\vec{v}$ which changes from $\vec{p}$ (initial momentum) to $\vec{p} + \Delta\vec{p}$ (final momentum).

According to the second law.

$\vec{F} \propto \frac{\Delta\vec{p}}{\Delta t}$ or $\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$

.... (5)

Where k is a constant of proportionality. Taking the limit $\Delta t \to 0$, the term $\frac{\Delta\vec{p}}{\Delta t}$ becomes the derivative or differential coefficient of $\vec{p}$ w.r.t. t, denoted by $\frac{\Delta\vec{p}}{\Delta t}$.

□ $\vec{F} = k\frac{d\vec{p}}{dt}$

as $\vec{p} = m\vec{v}$

□ $\vec{F} = k\frac{d(m\vec{v})}{dt}$

□ $\vec{F} = km\frac{d\vec{v}}{dt}$ (Here mass remains the same)

□ $\vec{F} = km\vec{a}$ $\left(\because\ \vec{a} = \frac{d\vec{v}}{dt}\right)$

The value of constant of proportionality k is considered as 1 for simplicity, so that in both the SI and CGS system of units, it may be selected in simple manner.

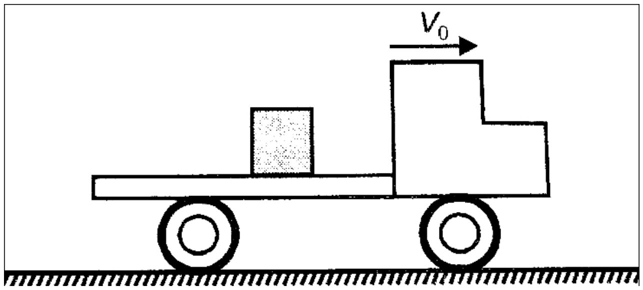
By taking k = 1, we get

$$\boxed{\vec{F} = m\vec{a}}$$

.... (6)

Example: 2

An iron block of mas m = 500 kg is kept at the back of a truck moving at a speed $v_0 = 90\ km\ h^{-1}$. The driver applies the brakes and slows down to a speed of $v = 54\ km\ h^{-1}$in 10 s. What constant force acts on the block during this time if the block does not slide on the truck-bed?



Solution:

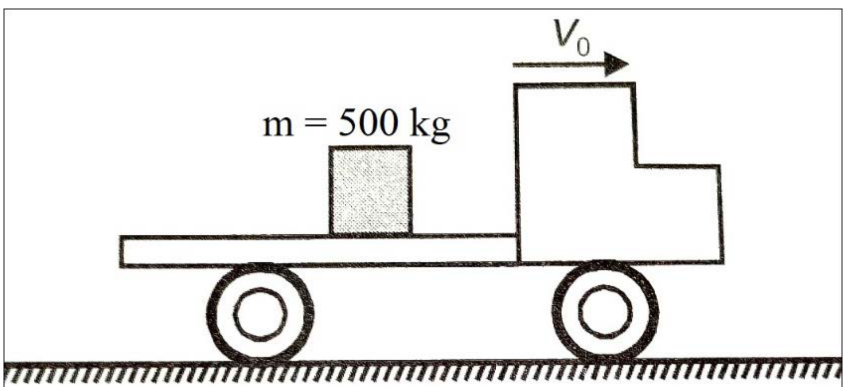
We know that,

The acceleration, $a = \frac{v - v_0}{t}$

$= \frac{54\ km\ h^{-1} - 90\ km\ h^{-1}}{10\ s}$

$= \frac{15\ ms^{-1} - 25\ ms^{-1}}{10\ s}$

$= -1 m/s^2$



or , Force F = ma

$= 500$ kg (-1 ms^{-2})

$= -\ 500$ N

-ve sign indicates that the force acts opposite to the velocity of the block. The magnitude of force is 500 N.

Self-Assessment Questions (SAQs)

2. Write short notes on

   (a) Momentum

   (b) Newton's Second Law

2.6 IMPULSIVE FORCE AND IMPULSE

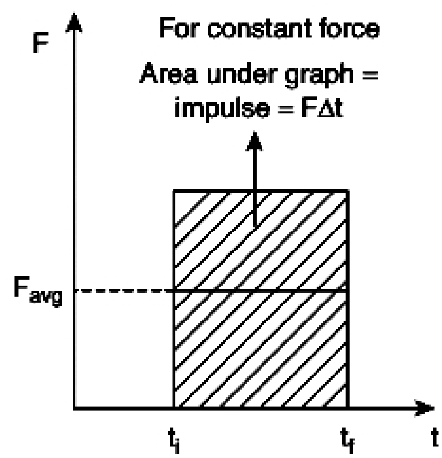
Impulsive forces: The forces which act on bodies for a short time are called impulsive forces. The examples of impulsive forces are

(i) A bat hitting the ball.

(ii) The collision of two billiard balls.

(iii) The firing of a gun.

The impulsive force is not constant but varies with time. For example. when a bat hits the ball. the impulsive force jumps from zero at the moment of contact to a very large value within a very short time and then abruptly returns to zero again as shown in Figure-7.



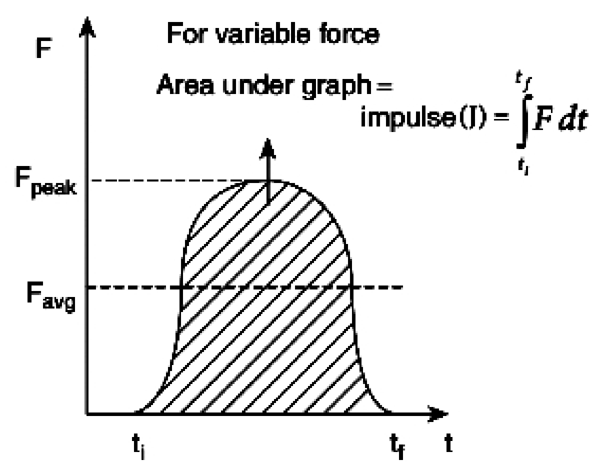


Figure - 7

Note that during the short time interval $\Delta t(= t_2 - t_1)$ the impulsive force is varying continuously. Therefore, it is not easy to measure the impulsive force. In such cases. we measure the total effect of the impulsive force, called *impulse.*

Impulse. If $\vec{F}_{av}$ is the average force [Sec Figure-7] exerted by the bat on the ball during the time interval Δr and the change in linear momentum during this time interval is $\vec{\Delta p}$, then according to Newton's second law of motion

$$\vec{F}_{av} = \frac{\vec{\Delta p}}{\Delta t} \quad \ldots\ldots. (7)$$

$$\vec{F}_{av}\Delta t = \vec{\Delta p} \text{ = **Change in linear Momentum**} \quad \ldots\ldots. (8)$$

The Product $\vec{F}_{av}\Delta t$ is called impulse of the force

hence impulse of a force is the product of average force during the impact and the time for which the impact lasts i.e.,

Impulse = $F_{av} \times \Delta t$ ....(in magnitude)

Mathematical Analysis. If F is the Force at any time t during the collision, then according to Newton's second law of the Motion,

$$\vec{F} = \frac{\vec{dp}}{dt} \text{ or } \vec{dp} = \vec{F}dt \quad \ldots\ldots. (9)$$

If we integrate it over the time interval $\Delta t = (t_1 - t_2)$ we get the total change in linear momentum $\Delta\vec{p}$ during that time interval

$$\Delta\vec{p} = \vec{F}_{av}\Delta t$$

……. (10)

From equation (9) and (10) $\int_{t_2}^{t_1} \vec{F}\,dt = \vec{F}_{av}\Delta t$ ......change in linear momentum

The quantity $\int_{t_2}^{t_1} \vec{F}\,dt$ is knows as impulse of the force $\vec{F}$ during the time interval t_1 to t_2 and is equal to change in the linear momentum of the body on which it acts (= area under the F –t graph during the time interval t_1 to t_2)

Units of impulse. The impulse has the same SI units as that of the liner momentum i.e. kg m/s or NS. Therefore, the dimensional formula of impulse is also the same as that of linear momentum i.e. $[MLT^{-1}]$. Impulse is a vector quantity and its direction is the same as the direction of change in linear momentum.

IMPULSE LINEAR MOMENTUM THEOREM

We know that,

Impulse = $\vec{F} \times t$

Impulse $= m(\vec{v} - \vec{u})$ ……. (11)

Impulse = Change in linear momentum

Thus, a given change in linear momentum can be produced by applying a larger force for a smaller time or by applying a smaller force for a larger time.

This is called impulse linear momentum theorem.

2.6.1 Applications of the Concept of Impulse

$$\text{Average impulsive force, } \vec{F}_{av} = \frac{Change\,in\,linear\,Momentum}{Time}$$

Therefore, whenever you wish the force of impact to be small, extend the time if impact. On the other Hand, If the time of impact is small, the impact of force will be large. There are a large number of practical applications where impulse linear momentum relationship plays an important role:

(a) While catching a ball, a cricket player extends his hand forward so that he has plenty of room to let his hands move backwards after making contact with the ball. This extends the time of impact and thus reduces the force of impact.

(b) A person is better of falling on a wooden floor than a concrete floor. The wooden floor allows for a longer time of impact and therefore a lesser force of impact than a concrete floor.

(c) A person jumping from an elevated position on a floor below balms his knees upon making contacts. This extends the time of impact. Therefore, the force of impact is reduced.

(d) China wares are wrapped in a paper or straw before packing to avoid breakage. This increase the time of impact between various articles during jerks. Thereby decreasing the force of impact on the articles.

(e) While catching a cricket ball, if you do not move your bands away upon contact. you may be hurt. it is because the time of impact will be small so that the impact force will be large.

2.6.2 Newton's Third Law

A force acting on a body is always the result of its interaction with another body, so forces always come in pairs. We cannot pull on a doorknob without the doorknob pulling back on us. When we kick a football, the forward force that our foot exerts on the ball launches it into its trajectory, but we also feel the force the ball exerts back on our foot. If we kick a wall, the pain that we feel is due to the force that the wall exerts on our foot.

In each of these cases, the force that we exert on the other body is in the opposite direction to the force that body exerts on us. Experiments show that whenever two bodies interact, the two forces that they exert on each other are always equal in magnitude and opposite in direction. This fact is started by Newton in his third law of motion as **"To every action, there is always an equal and opposite reaction."**

Note: These two forces act on two different bodies and we should always remember that forces always occur in pairs and these forces always act at the same instant of time.

Suppose a body A exerts a force on body B, then

$$\underset{(Force\ on\ A\ by\ B)}{\vec{F}_{AB}} = \underset{((Force\ on\ B\ by\ A)}{-\vec{F}_{BA}}$$

Action and Reaction

When there is a force exerted by body I on body II, there is also a force exerted by body II on body I. These forces are equal in magnitude and act in opposite directions. Such a pair of forces is called an **action-reaction pair**. Any of the two forces may be called the action, the other will be the reaction. Consider a book placed on a table.

The book pushes the table down with a force. The table pushes the book up with an equal force. If we call the downward force exerted by the book on the table as action, the upward force exerted by the table on the book is the reaction or vice-versa as shown in figure-8.

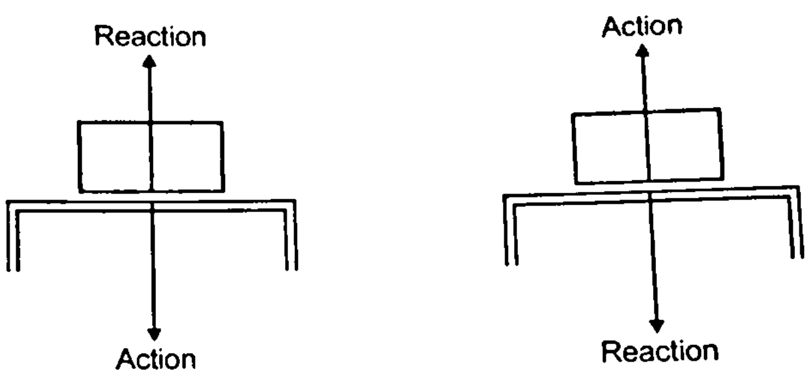


Figure - 8

Newton's third law can be rested as **'action and reaction are always equal and opposite'**.

Applications of Third Law

(a) **Recoiling of a gun:** When a bullet is fired from a gun, it exerts a forward force on the bullet and the bullet exerts an equal and opposite force on the gun. Due to high mass of the gun, it moves a little distance backward and gives a backward jerk to the shoulder of the gunman.

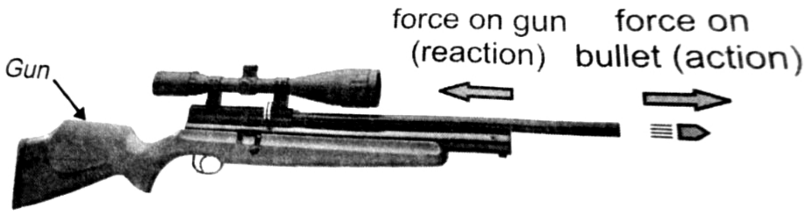


Figure - 9

(b) To walk, we press the ground in backward direction with foot: When we walk on the ground, our foot pushes the ground backward and in return the ground pushes our foot forward.

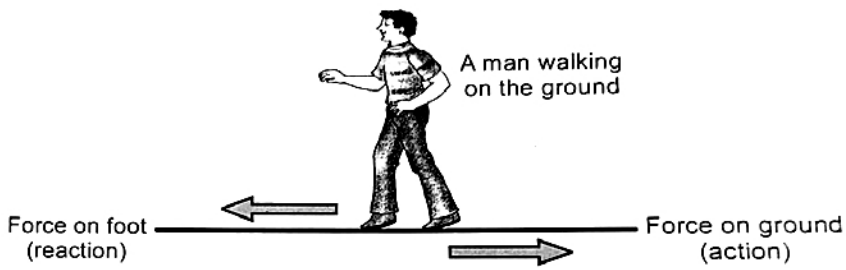


Figure - 10

(c) Jet Aeroplanes and Rockets: In jet engines and rockets, the fuel is burnt to produce a large quantity of hot gases. These hot gages come out of a nozzle with a great force (this is action). According to third law of motion, the equal and opposite reaction pushes the jet planes and rockets upward with a great speed (this is reaction).

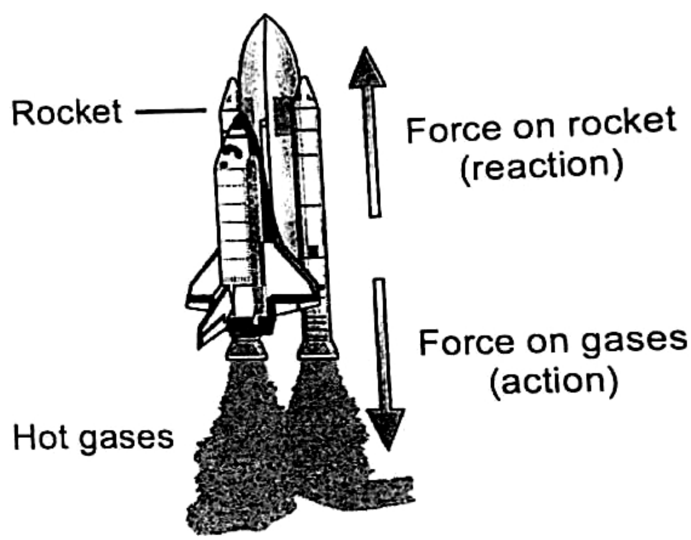


Figure - 11

Example: 3

Acceleration due to gravity near the surface of the Earth is g = 9.8 ms^{-2}. A ball of mass 10 kg is dropped from the top of a

buildings. Find the acceleration produced in the Earth due to the force exerted on it by the ball. Mas of the Earth is M = 6 × 10^{24} kg.

Solution:

The ball and the Earth are two interacting bodies. They exert equal and opposite force on one another.

Now,

Force exerted by the Earth on the ball is F = (mass) × (acceleration)

⇒ F = mg = 10 × 9.8 = 98 N (vertically down)

The ball pulls the Earth with an equal force towards itself.

For the Earth: Ma_e = 98 N

$$\Rightarrow \quad a_e = \frac{98\ N}{6 \times 10^{24} kg} = 1.65 \times 10^{-23} ms^{-2}$$

2.7 WORK

Work is said to be done by a force when the body is displaced actually through some distance in the direction of the applied force.

However, when the is no displacement in the direction of the applied force, no work is said to be done i.e. work done is zero, when displacement of the body in the direction of the force is zero.

Suppose a constant force $\vec{F}$ acting on a body produces a displacement $\vec{s}$ in the body along the positive x- direction figure – 12 & 13.

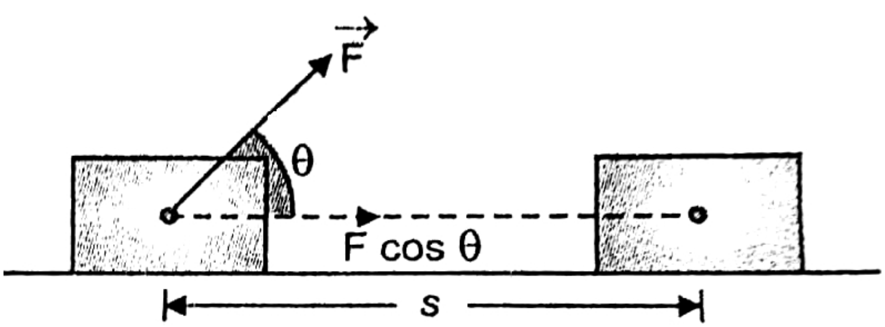


Figure – 12

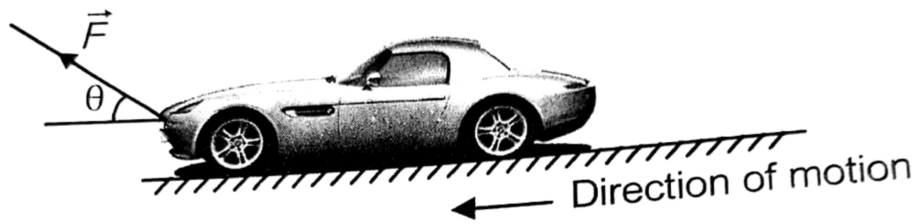


Figure - 13

If θ is the angle which $\vec{F}$ makes with the positive x-direction of the displacement, then the component of $\vec{F}$ in the direction of the displacement is (F cosθ). As work done by the force is the product of component of force in the direction of the displacement and the magnitude of the displacement,

□ $$\boxed{W = (F\cos\theta)s} \qquad \text{............... (12)}$$

if displacement is in the direction of force applied θ = 0° from (1), W = (F cos0°) s=Fs

Equation (12) can be rewritten as $\boxed{W = \vec{F}.\vec{s}}$ .... (13)

Thus, work done by a force is the dot product of force and displacement

In terms of rectangular components, $\vec{F}$ and $\vec{s}$ may be written as

$\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ and $\vec{s} = \hat{i}x + \hat{j}y + \hat{k}z$

From (8), $W = \vec{F}.\vec{s}$

W = $\left(\hat{i}F_x + \hat{j}F_y + \hat{k}F_z\right).\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$

$$\boxed{W = xF_x + yF_y + zF_z} \qquad \text{............... (14)}$$

Obviously, work is a scalar quantity, i.e. it has magnitude only and no direction. However, work done by a force can be positive or negative or zero.

2.7.1 Nature of Work Done

Although, work done is a scalar quantity, its value may be positive, negative or even zero:

(a) Positive work

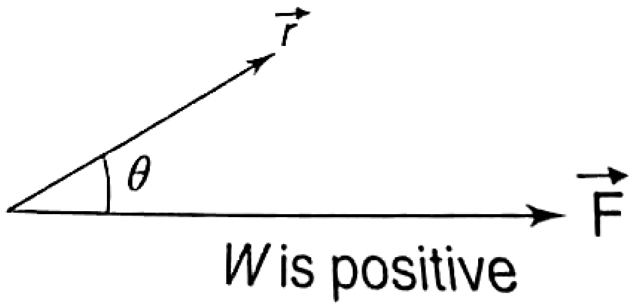


Figure - 14

$W = \vec{F}.\vec{s} = F\,s\cos\theta$

∴ when θ is acute (<90°), cosθ is positive. Hence, work done is positive.

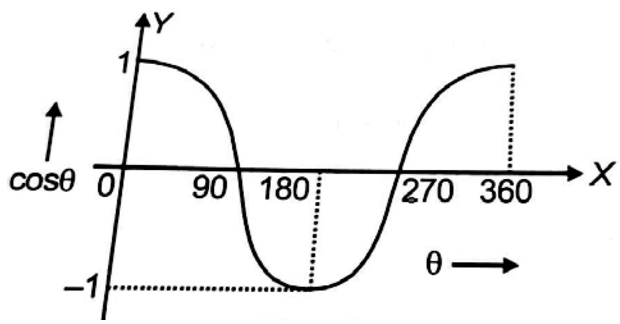


Figure – 15

For example:

(i) A body falls freely under the action of gravity θ = 0°, cosθ = +1. Work done by gravity on a body falling freely is positive. (Here, force of gravity and displacement are in the same direction).

(ii) A lawn roller is pulled by a force along the handle at an acute angle, work done by the force is positive.

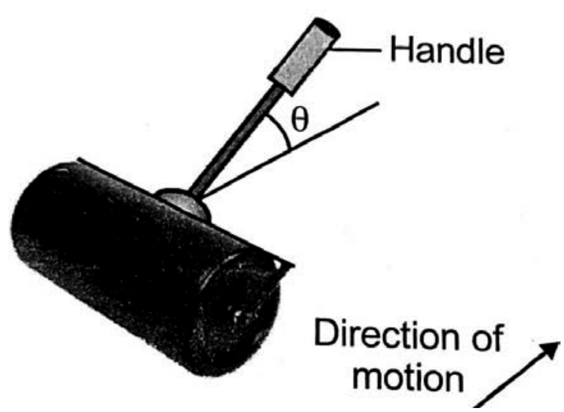


Figure - 16

(b) Negative work

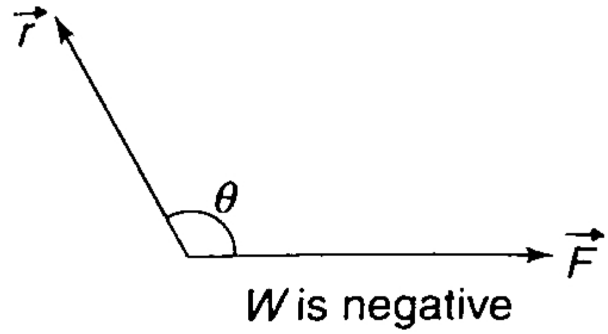


Figure - 17

$$As\ W = \vec{F}.\vec{s} = F\,s\cos\theta$$

∴ When θ is obtuse (>90°), cos θ is negative. Hence, work done is negative.

For example:

(i) Work done by the braking force on a moving vehicle is negative. Force is in a direction opposite to the direction of motion. $\theta = 180^o$, $\cos 180^o = -1$, W = -Fd (Refer to figure 15)

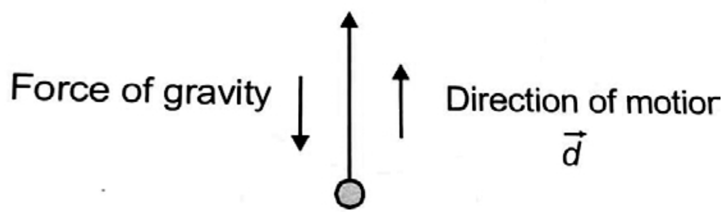


Figure - 18

(c) Zero work

When force applied $\vec{F}$ or the displacement $\vec{s}$ or both are zero, work done W=F s cos θ is zero. Again, when angle θ between $\vec{F}$ and $\vec{s}$ is 90°, cosθ = cos 90° = θ. Therefore, work done is zero.

For example:

(i) A body moving on a smooth horizontal surface is not acted upon by a horizontal force (as there is no friction), but may undergo displacement. So, W = 0 as F = 0 even though $d \neq 0$ and $\cos\theta \neq 0$.

2.7.2 Work Done by a Variable Force

(a) Graphical Method

(b) Mathematical Treatment

Now, we discuss it one by one:

(a) Graphical Method: A constant force is rare. It is the variable force which is encountered more commonly. We can their force learn to calculate work done by a variable force. Let us consider a force acting along the fixed direction say x-axis but having a variable. Magnitude, as shown in figure 19.

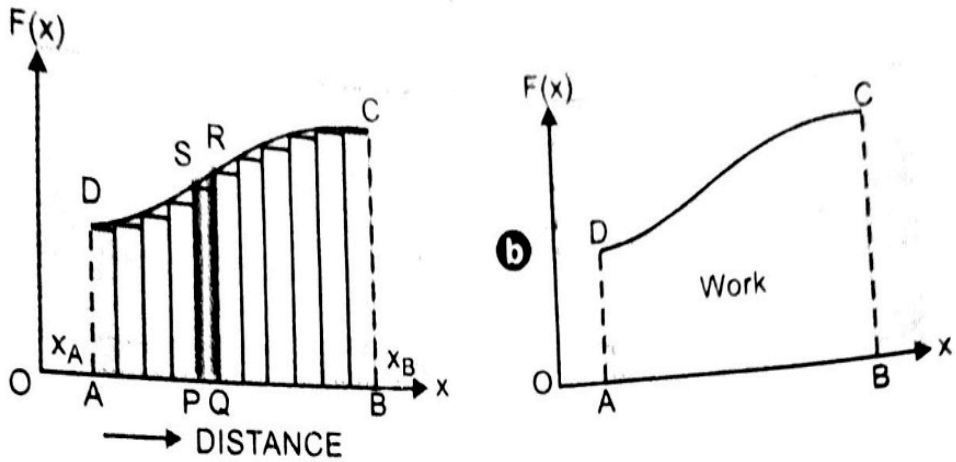


Figure - 19

Now,

We have to calculate work done in moving the body from A or B under the action of this variable force.

To do this, we assume the entire displacement from A to B is made up of a large number in infinitesimal displacements. One such displacement show in figure is from P to Q

Let displacement PQ = dx is infinitesimally small, we consider that all this displacement force is constant in magnitude (=PS) as well as in direction.

$\therefore$ Small amount of work done in moving the body form P to Q is

dW = F × dx = (PS)(PQ) = area of strip PQRS

Total work done in moving the body from A to B is

$$W = \sum dW$$

$$W = \sum F \times dx$$

If the displacement are allowed to approach zero, then the number of terms in the sum increases without limit. And the sum approaches a definite value equal to the area under the curve CD as shown in figure19.

$\therefore$ Hence, we may rewrite $W = \underset{dx \to 0}{Limit} \sum F(dx)$

From integral calculus, we may write it as

$W = \int_{x_A}^{x_B} F(dx),$ where x_A = OA and x_B = OB

$$W = \int_{x_A}^{x_B} \text{area of the strip PQRS}$$

= total area under the curve between F and x-axis from $x = x_A$ to $x = x_B$

$$\boxed{W = \text{Area ABCDA}} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

Hence, Work done by a variable force is numerically equal to area under the force curve and the displacement axis.

(b) Mathematical Treatment:

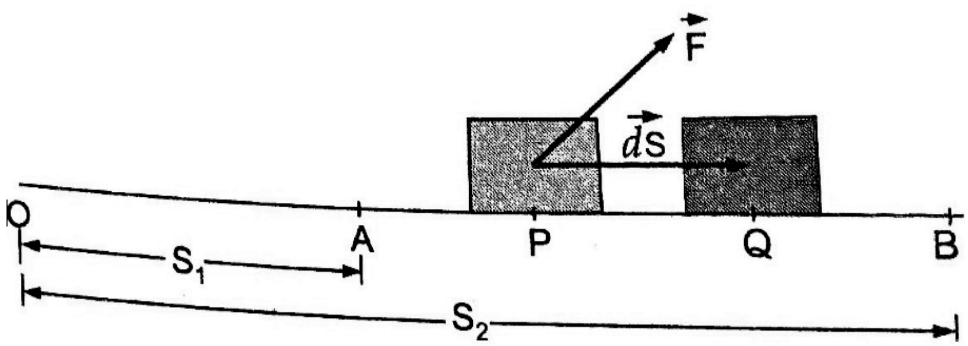


Figure - 20

From the figure 20, small amount of work done in moving the body from P to Q is

$$\boxed{dW = \vec{F}.d\vec{s}} \quad \ldots\ldots\ldots\ldots\ldots (16)$$

When $d\vec{s} \rightarrow 0$, total work done in moving the body from A to B can be obtained by integrating the above expression between S_A and S_B

$$\boxed{W = \int_{S_A}^{S_B} \vec{F}.d\vec{s}} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

Example: 4

A force F = (10+0.50x) act on a particle in x direction, where F is in newton and x is metre. Find the work done by the force during a displacement from x = 0 to x = 2 m.

Solution:

Hence, $F = (10 + 0.50x)$

Small amount of work done in moving the particle thought a small distance dx is

$$dW = \vec{F}.d\vec{x} = (10 + 0.5x)dx$$

Total Work done, $W = \int_{x=0}^{x=2} (10 + 0.5x) dx$

$$W = \left[10x + 0.5 \frac{x^2}{2} \right]_0^2 = 10(2-0) + \frac{0.5}{2}(2^2 - 0) = 20 + 1 = 21 joule.$$

Self-Assessment Questions (SAQs)

3. A gas filled in a cylinder with a movable piston is allowed to expand. What is the nature of the work done by the gas?

Solution:

Positive, as force due to gaseous pressure and displacement of piston are in the same direction.

2.8 POWER

Often, we say a person is physically fit and powerful, if he not only climbs up four floors of a tall building, but also climbs them fast.

Power of a person or machine is defined as the time rate at which work is done by it.

OR

The rate of which a force perform & work is known s its Power.

i.e. $\text{Power} = \text{Rate of during work} = \frac{\text{work done}}{time\,taken}$

This, is the average power P_{av}

Thus, power of a body measure how fast it can do the work. When a body takes lesser time to do a particular amount of work, its power is said to be greater and vice versa.

The power at a particular instant of time t is the radio of small work (dW) to small time interval (dt) around t, i.e. $P = \frac{dW}{dt}$ …………… (18)

Now, $dW = \vec{F}.d\vec{s}$, where $\vec{F}$ the force is applied and $d\vec{s}$ is the small displacement.

$$\therefore \quad P = \frac{\vec{F}.d\vec{s}}{dt}$$

But $\frac{d\vec{s}}{dt} = \vec{v}$, the instantaneous velocity.

$\therefore\ P = \vec{F}.\vec{v}$
…………… (19)

Thus, *power can be expressed as the dot product of force and velocity*

if θ *is angle between* $\vec{F}$ and $\vec{v}$ then

$P = \vec{F}.\vec{v} = Fv\cos\theta$. However, when $\vec{v}$ is along $\vec{F}$, θ=0°, ∴ P = Fv Cos° F*v*

As Power is the ratio of two scalar quantities W and t, therefore, **Power is scalar.** That is why it is expressed as dot product of $\vec{F}$ and $\vec{v}$

Dimension of Power can be deduced as : $P = \frac{W}{t} = \frac{M^1L^2T^{-2}}{T^1} = \left[ML^2T^{-3}\right]$

Unit of power

The absolute unit of power is SI is watt. Which is denoted by W.

From P = W/t

$1\,watt = \frac{1\,joule}{1\sec}, i.e.$

1 W = 1J s^{-1} Hence,

Power of an agent is said to be one watt, if it can do one joule of work in one second.

The bigger units of power are 1 kilo watt = 1000 watt, i.e. 1 kW = 10^3W

and 1 megawatt = 1,000,000 watt, i.e. 1MW = 10^6W

The absolute unit of power in cgs system is 1 erg s^{-1}

We know that,

1 W = 1 Js^{-1} = 10^7 erg s^{-1}

The gravitational unit of power in SI is **(kg f) m s^{-1}** and is cgs system, it is **(g f) cm s^{-1}**.

Another popular unit of power (used mostly in engineering) is **horse power** (h.p.)

$$\boxed{1\ h.p. = 746\ W}$$

We come across the unit watt while dealing with electrical goods like bulb, tube lights etc. e.g., power of a bulb is said to be 60 W when it consumes 60 J of energy in one second.

Let us calculate the energy used by a 100 W bulb when it is on for 10 hours.

$$E = P \times t = \frac{100}{1000} kW \times 10h = 1kWh = 3.6 \times 10^6 J$$

Our electricity bills show the energy consumption in units (kWh).

Remember: kWh is commercial unit of energy. It is not a unit of power.

This unit is still used to describe the output of automobile, motorbike etc.

Example: 5

An elevator weighing 500 Kg is to be lifted up at a constant velocity of 0-4 m/s. What should be the minimum horse power of the motor to be used?

Solution:

Here m = 500 Kg, v = 0.4m/s, P = ?

P = Fv = (mg) ×v = 500 × 9.8 × 0.4 = 1960 watt.

If we assume that there is no loss against friction etc. In the motor then minimum horse power motor $P = \frac{1960}{746} h.p. = 2.62\, h.p.$

2.9 ENERGY

The amount of work that a body or a system can perform is known as its energy. A well-fed young man can do more work than an old man, before both of them get completely exhausted. We say that the young man has more energy. Energy has many forms, viz. kinetic energy, thermal energy, chemical energy, nuclear energy, light energy, etc. Energy can neither be created nor destroyed. However, it can be concerted from one form to another.

Unit of energy is joule (J).

2.9.1 The Concept of Kinetic Energy

The kinetic energy of a moving body is measured by the amount of work which has been done in bringing the body from the rest position to its present position, or which the body can do in going from its present position to the rest position.

Let a body of mass m be in the rest position. When we apply a constant force F on the body, it starts moving under an acceleration. If a be the acceleration, then by Newton's second law, we have

$$A = F / m.$$

Suppose the body acquired a velocity v in moving a distance s. According to the reaction $v^2 = u^2 + 2as$ (u = 0, since the body was initially at rest), we have

$$v^2 = 2as = 2 \times (F / m) \times s$$

or $$F \times s = \frac{1}{2} m v^2.$$

But F × s (force × distance) is the work W which the force F has done on the body in moving it a distance s. It is due to this work that the body has itself acquired the capacity of doing work. This is the measure of the kinetic energy of the body. Hence if we represent kinetic energy of a body by K, then

$$\boxed{K = W = \frac{1}{2} m v^2} \qquad \text{…………… (20)}$$

$$kinetic\ energy = \frac{1}{2} \times mass \times speed^2.$$

Thus, **the kinetic energy of a moving body is equal to half the product of mass (m) of the body and the square of its speed (v^2)**.

Note:

(a) The expression $K.E. = \frac{1}{2} mv^2$ holds even when the force applied varies in magnitude or in direction or in both. Thus, the expression is valid irrespective of how the body acquires the velocity v.

(b) Kinetic energy of a body is always positive. It can never be negative.

(c) Kinetic energy of a body depends upon the frame of reference. For example, K.E. of a person of mass m sitting in a train moving with vel. v is $\left(\frac{1}{2} mv^2\right)$ in the frame of earth, and K.E. of the same person = 0, in the frame of the train.

2.9.2 The Concept of Potential Energy

The word potential suggests capacity or possibility for action. This term potential energy brings to one's mind 'stored energy'. A wound-up spring or a stretched bow string possesses potential energy.

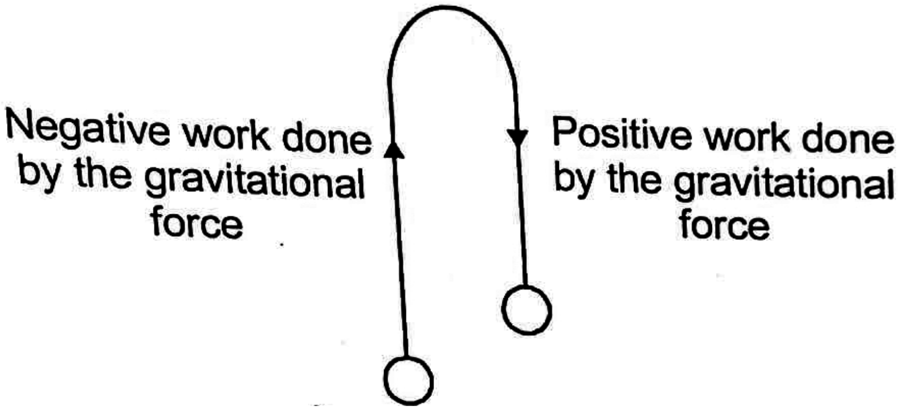


Figure – 21

The potential energy of a body is defined as the energy possessed by the body by virtue of its position or configuration in some field.

Thus, potential energy is the energy that can be associated with the configuration (or arrangement) of a system of objects that exert forces on one another. Obviously, if configuration of the system changes, then its potential energy changes.

2.9.3 Gravitational Potential Energy

Gravitational potential energy of a body is the energy possessed by the body by virtue of its position above the surface of the earth.

We know that,

Work done = force × distance

W= F × h = mgh

Note that we have taken the upward direction to be positive. Therefore, work done by applied force = + mgh. However, work done by gravitational force = - mgh.

This work gets stored as potential energy. The gravitational potential energy of a body, as a function of height (h) iş denoted by V (h), and it is negative of work done by the gravitational force in raising the body to that height.

$$\boxed{\textbf{Gravitational P.E.} = \mathbf{V\ (h)\ =\ mgh}} \quad \ldots\ldots\ldots\ldots\ldots (21)$$

If h is taken as a variable, then

$$-\frac{d}{dh}V(h) = -\frac{d}{dh}(mgh) = -mg = F,$$

Where F is the gravitational force on the body. The negative sign indicates that gravitational force is downwards.

Thus, gravitational force F equals the negative of the derivative of V (h), w.r.t. h, i.e.,

$$\boxed{F(x) = -\frac{dV}{dx} V(h)}$$

…………… (22)

Mathematically, the potential energy V (x) is defined if the force F (x) can be written as

$$F(x) = -\frac{dV}{dx}$$

This implies that $\int_{x_i}^{x_f} F(x) dx = -\int_{V_i}^{V_f} dV = V_i - V_f$

i.e., work done by a conservative force like gravity in taking the body from initial position (x_f) to final position (x_i) is equal to difference between initial and final P.E. of the body.

When the body is released from height h, it comes down with an increasing speed. The velocity v with which the body hits the ground is calculated from the fact that the gravitational P.E. of the body at height h manifests itself as K.E. of the body on reaching the ground, i.e.,

$$\boxed{\frac{1}{2} mv^2 = mgh}$$

or

$$\boxed{v = \sqrt{2gh}} \qquad \text{…………… (23)}$$

2.10 CONSERVATIVE AND NON-CONSERVATIVE FORCES

Conservative Forces:

A force is said to be conservative if the work done by the force (or against the force) in moving a body depends only upon the initial and final positions of the body and is independent of the path followed between the initial and final positions.

A Central Force is a Conservative Force:

A force acting upon a particle is a 'central' force if it is always directed towards or away from a fixed point, and its magnitude depends only on the distance of the particle from that point. Gravitational force between two masses, electrostatic force between two charges and magnetic force between two magnetic poles are examples of central forces and hence of conservative forces.

Non-Conservative Force:

A force is said to be non-conservative, if the work done by the force, or against the force, in moving a body from one position to another, depends upon the path followed between the two positions.

For example, frictional forces are non-conservative forces.

1. Conservative force as a negative gradient of potential energy

$$\vec{F} = -gradU = -\nabla U \quad \text{............... (24)}$$

We know that,

The potential energy of the particle under a conservative force.

$$U = -\int_{\infty}^{r} \vec{F}.d\vec{r} \quad \text{............. (25)}$$

Here, negative sign indicate that force and potential energy are in opposite direction.

$d\vec{r}$= differential displacement.

If motion is in three dimensions than, force act on a particle is 3D.

$$\vec{F} = \hat{\imath}F_x + \hat{\jmath}F_y + \hat{k}F_z$$

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

Putting the value of $\vec{F}$ and $d\vec{r}$ in Equation (25)

We get,

$$U = -\int_{-\infty}^{r} (\hat{\imath}F_x + \hat{\jmath}F_y + \hat{k}F_z).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$= -\int_{-\infty}^{\infty} (F_x dx + F_y dy + F_z dz)$$

$$U = -\int F_x dx - \int F_y dy - \int F_z dz$$

.... (26)

Now, partially differentiating equation (26) w.r.t. x and putting y and z as constant.

We get.

$$\frac{\partial U}{\partial x} = -F_x - 0 - 0$$

$$\boxed{F_x = -\frac{\partial U}{\partial x}} \quad \text{............... (27)}$$

Similarly, w.r.t. y and, we get

$$\boxed{F_y = -\frac{\partial U}{\partial y}} \qquad \text{and} \qquad \boxed{F_z = -\frac{\partial U}{\partial z}} \qquad \ldots\ldots\ldots\ldots\ldots (28)$$

Hence,

$$\vec{F} = \hat{\imath}F_x + \hat{\jmath}F_y + \hat{k}F_z$$

$$\vec{F} = -\hat{\imath}\frac{\partial U}{\partial x} + \hat{\jmath}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}$$

or,

$$\vec{F} = -\left(\hat{\imath}\frac{\partial U}{\partial x} + \hat{\jmath}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)U$$

$$\vec{F} = -\nabla U = -grad\, U$$

Thus,

$$\boxed{\vec{F} = -\nabla U = -grad\, U} \qquad \ldots\ldots\ldots\ldots\ldots (29)$$

When a particle moves origin to another point then we take as negative sign.

2. Work done by a conservative Force along closed path is zero.

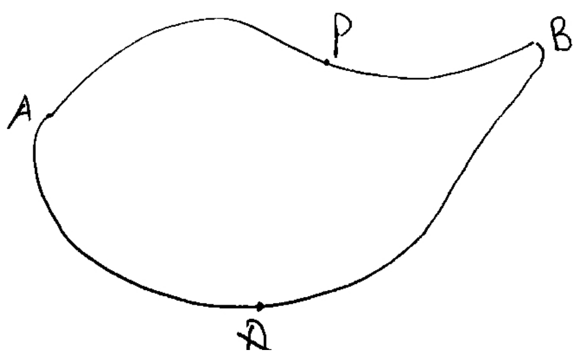


Figure - 22

For Conservative Force,

$W_{APB} = w_{AQB}$

Path APB = Path AQB

$$\int_A^B \vec{P}.d\vec{r} = \qquad \int_A^B \vec{F}.d\vec{r}$$

(Path APB) (Path AQB)

$$W_{APB} = -W_{BQA}$$

In Complete cycle,

$$W = W_{aPB} + W_{BQA} = 0$$

$$\boxed{W = 0}$$

$$\oint \vec{F}.d\vec{r} = 0$$

$\oint$ *line integral along closed path.*

Conservation force is the force under which if a particle moves in a closed path, there is no change in K.E. of the particle.

$$\boxed{\Delta K = 0}$$

According to work energy theorem.

$$\boxed{W = \Delta K}$$

for complete cycle/closed path $\boxed{W = 0}$

$$\therefore \quad \boxed{\Delta K = 0}$$

Self-Assessment Questions (SAQs)

4. What is conservative force? Explain with example.
5. Define conservative force. Prove that the curl of conservative force is zero.

2.11 WORK ENERGY THEOREM OR WORK ENERGY PRINCIPLE

Work Energy Theorem states that "the work done by the net force acting on a body is equal to the change in kinetic energy of a body" i.e.

Work = gain in kinetic energy

$$= \frac{1}{2}mv_2^{\ 2} - \frac{1}{2}mv_1^{\ 2}$$

Let us consider a body of mass m acted upon a net force F along x axis. If body moves from a position x_1 to position x_2 Along the x axis its velocity increase from v_1 and v_2 the work done

$$W = \int_{x_1}^{x_2} F dx$$

By Newton's second law, we know

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \quad \text{(putting } \frac{dx}{dt} = v\text{)}$$

Therefore,

$$W = \int_{x_1}^{x_2} mv\frac{dv}{dx}dx$$

$$= m\int_{x_1}^{x_2} v\frac{dv}{dx}dx = \int_{x_1}^{x_2} vdv$$

$$m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{1}{2}mv_2^{\ 2} = \frac{1}{2}mv_1^{\ 2}$$

$$\text{or } W = \frac{1}{2}mv^2{}_2 - \frac{1}{2}mv_1^{\ 2}$$

or $W = K_2 - K_1$

Where K_1 and K_2 are the initial and final kinetic energies of the body. Thus if ΔK represents the change kinetic energy, $\Delta K = K_2 - K_1$ then, we have

$$\mathbf{W = \Delta K} \qquad \text{...............} (30)$$

This is mathematical statement of Work Energy Theorem or Work Energy Principle.

2.12 LAW OF CONSERVATION OF MOMENTUM

According to conservation of linear momentum. "The total momentum of an isolated system of interacting particles is conserved". In words, "If there is no net external force acting on the system, the total momentum remains conserved." In words, "**for an isolated system the initial momentum of the system is equal to the final momentum of the system**".

Consider two objects A and B of masses m_1 and m_2 moving along the same direction at different velocities $\vec{u}_1$ and $\vec{u}_2$ respectively.

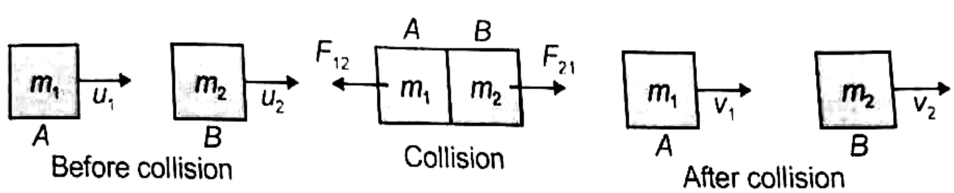


Figure - 23

If $|\vec{u}_1| > |\vec{u}_2|$, then they collide and during collision A exerts a force $\vec{F}_{21}$ on B and simultaneously B exerts a force $\vec{F}_{12}$ on, A. Let $\vec{v}_1$ and $\vec{v}_2$ are the velocities of two objects A and B after collision and they are moving along same straight line.

The momentum of B before collision,

$$\vec{p}_B = m_2\vec{u}_2$$

The momentum of B after collision,

$$\vec{p}'_B = m_2\vec{u}_2$$

The rate of chance of momentum of B is equal to equal to force by A on B (i.e. $\vec{F}_{12}$)

$$\therefore \qquad \vec{F}_{12} = \frac{m_1(\vec{v}_1 - \vec{u}_1)}{t}$$

According to Newton's third law of motion, the force $\vec{F}_{12}$ (action) must be equal opposite to $\vec{F}_{21}$ (reaction).

Therefore, $\vec{F}_{12} = -\vec{F}_{21}$

or $$\frac{m_1(\vec{v}_1 - \vec{u}_1)}{t} = -\frac{m_2(\vec{v}_2 - \vec{u}_2)}{t}$$

or $$\boxed{m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2} \qquad \text{……………} (31)$$

or Total momentum before collision = Total momentum after collision.

2.13 PRINCIPLE OF CONSERVATION OF ENERGY

According to this **principle, energy can neither be created nor can be destroyed, but can only be concerted from one form to another, the total amount of energy of the universe remaining constant.** In other words, whenever energy disappears in one form, an equal amount of energy appears in some other form. No violation of this principle has yet been observed.

2.14 COLLISION

Collision between two billiard balls or between two automobiles on a road are a few examples of collision from everyday life. Even gas atoms and molecules at room temperature keep on colliding against each other. For the collision to take place, physical contact is not necessary. In case of Rutherford's scattering experiment, the particles are suggested due to electrostatics interaction between the particle and the nucleus.

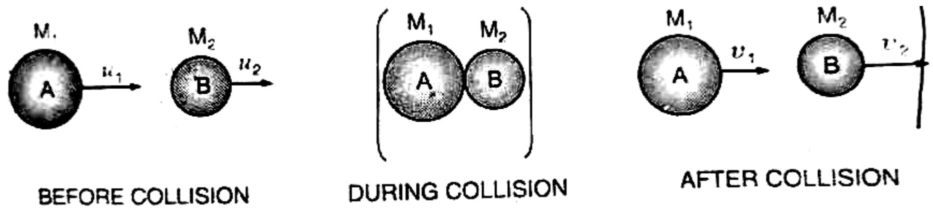


Figure - 24

Thus, in physics a collision is said to have occurred if the two bodies physically collide against each other or even when the path of the motion of one body is affected by other.

Types of Collision

(i) On the basis of the direction of colliding bodies

(a) Head on or one-dimensional collision

(b) Oblique collision

(ii) On the basis of conservation of kinetic energy

(a) Elastic collision

(b) Inelastic collision

(c) Perfectly inelastic collision

(d) Partially inelastic collision

2.14.1 Elastic collision

Those collision, in which both momentum and kinetic energy of the system are conserved are called elastic collisions.

The collision between two bodies and subatomic particles are elastic in nature. In daily life the collisions between two glass or preferably ivory balls may be taken as elastic collisions.

Characteristics of elastic collisions:

(a) The momentum is conserved.

(b) The total energy is conserved.

(c) The kinetic energy is conserved.

(d) The mechanical energy is not converted into any other form (sound, heat, light) of energy.

(e) Forces involved during the interaction are of conservative nature.

2.14.2 Inelastic Collision

Those collisions in which the momentum of the system is conserved but the kinetic energy is not conserved, are called inelastic collisions.

Most of the collisions in everyday life are inelastic collisions. The kinetic energy lost in an inelastic collision appears is some other form of energy, such as heat, sound etc.

2.14.3 Perfectly Inelastic Collision

Those collisions, in which the colliding bodies stick together after the collision and then move with a common velocity are called perfectly inelastic collision.

Mud throw on the wall an sticking to it, a bullet fired into wooden block and remaining embedded in it, are the examples of perfectly inelastic collision. In such collisions the momentum of the system conserved. But the loss of kinetic energy is maximum.

Characteristics of inelastic collisions:

(a) The momentum is conserved.

(b) The total energy is conserved.

(c) The kinetic energy is not conserved.

(d) A part or whole of the mechanical energy may be converted into other forms heat, light, sound) of energy.

(e) Some or all of the forces involved are non-conservative in nature.

It may be pointed out that in both the types of collisions,

(a) Momentum is conserved;

(b) Total energy is conserved and

(c) It is kinetic energy, which may or not be conserved.

What does momentum and energy conservation in collision means:

(a) Momentum Conservation: In a collision the effect of external forces such as gravity or friction are not taken into account as due to small duration of collision (st) average impulsive force responsible for collision is much larger than external force acting on the system and since this impulsive force is "Internal" therefore the total momentum of system always remains conserved.

(b) Energy Conservation: In a collision 'total energy' is also always conserved. Here total energy includes all forms of energy such as mechanical energy internal energy, excitation energy, radiant energy or even mass energy.

2.14.4 Elastic Collisions in One Dimention (Head on Collision)

The collision in which both the momentum and kinetic energy are conserved and the colliding bodies continue to move along the same straight line after the collisions, is called an elastic collision in one dimension.

Consider two perfectly elastic bodies A and B of masses M_1 and M_2 moving along the same straight line with velocities u_1 and u_2 respectively. The difference velocities i.e. (u_1-u_2) is called velocity of approach. Same straight line with velocities v_1 and v_2 in the same direction. The two bodies will separate after the collision only if

$v_2 > v_1$

The difference in final velocity i.e. (v_2-v_1) is called velocity of separation. As in an elastic collision, momentum is conserved we have

$$M_1u_1 + M_2u_2 = M_1v_1 + M_2v_2 \qquad \text{...............} (32)$$

Since kinetic energy is also conserved in an elastic collision we get,

$$\frac{1}{2}M_1u_1^2 + \frac{1}{2}M_2u_2^2 = \frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2 \qquad \text{...............} (33)$$

From the equation (32) we have

$$M_1(u_1 + v_1) = M_1(v_2 + u_2) \qquad \text{...............} (34)$$

form equations (33) we get

$$M_1(u_1^2 - v_1^2) = M_2(v_2^2 - u_2^2) \qquad \text{...............} (35)$$

Dividing the equation (35) by the equation (34) we get

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2}$$

or

$$u_1 + v_1 = v_2 + u_2$$

or

$$u_1 - u_2 = v_1 - v_2$$

From the equation (36), it follows that in one dimensional elastic collision, the laive velocity of approach $(u_1 - u_2)$ before collision is equal to the relative velocity of parathion $(v_1 - v_2)$ after collision.

Let us first find velocity of body A after collision. From equation we have

$v_2 = u_1 - u_2 + v_1$
…………… (36)

In the equation (32) substituting for v_2 we get

$$M_1 u_1 + M_2 u_2 = M_1 v_1 + M_2(u_1 - u_2 + v_1)$$

or

$$M_1 u_1 + M_2 u_2 = M_1 v_1 + M_2 u_1 - M_2 u_2 + M_2 v_1$$

$$(M_1 - M_2)u_1 + 2M_2 u_2 = (M_1 + M_2)v_1$$

or

$$v_1 = \frac{(M_1 - M_2)u_1 + 2M_2 u_2}{M_1 + M_2} \qquad \text{…………… (37)}$$

Again from (37) have

$$v_1 = v_2 - u_1 + u_2$$

In the equation (37) substituting for v_1 we get

$$M_1 u_1 + M_2 u_2 = M_1(v_2 - u_1 + u_2)M_2 v_2$$

or

$$M_1 u_1 + M_2 u_2 = M_1 v_2 - M_1 u_1 + M_1 u_1 + M_2 v_2$$

$$(M_2 - M_1)u_2 + 2M_1 u_1 = (M_1 + M_2)v_2$$

or

$$v_2 = \frac{(M_2 - M_1)u_2 + 2M_1 u_1}{M_1 + M_2} \qquad \text{…………… (38)}$$

Now,

We have to find the final velocities of the two bodies after collision in the following special cases:

1. **When the two bodies are of equal masses:** Let us consider that

$$M_1 = M_2 = M \text{ (say)}$$

From the equation (37) we have

$$v_1 = \frac{(M - M)u_1 + 2M_2 u_2}{M + M} = \frac{0 + 2M_2 u_2}{2M}$$

or

$$v_1 = u_2$$

Also, from the equation (38), we have

$$v_2 = \frac{(M - M)u_2 + 2Mu_2}{M + M} = \frac{0 + 2Mu_2}{2M}$$

or

$$v_2 = u_1$$

Thus, if two bodies of equal masses suffer elastic collision in one dimension then after the collision the bodies will exchange their velocities.

2. When the target body is at rest i.e. u_2=0: In the equation (37) and (38)

Thus, we have

$$v_1 = \frac{(M_1 - M_2)}{M + M} u_1 \qquad \text{...............} (39)$$

and

$$v_2 = \frac{2M_1}{M_1 + M_2} u_1 \qquad \text{...............} (40)$$

When the target body B is at rest let us find velocities of the two bodies in the following sub-cases;

(a) When the two bodies are of equal masses: Setting M_1=M_2=M in the equation (39) and (40), we get

$$v_1 = \frac{M - M}{M + M} u_1 = 0$$

and

$$v_2 = \frac{2M}{M + M} u_1 = u_1$$

Therefore, when the body A collides against the body B of equal mass at rest, the body A should come to rest and the body B should move on with the velocity of the body A.

Thus, is sometimes observed, when one of the two boys playing with glass ball, shoots a stationary glass ball with the help of his fingers: his own glass ball comes to rest while the stationary ball of the other boy starts moving with the same velocity.

In this case transfer of energy is hundred percent.

(b) **When the mass of body B is negligible as compared to that of A:** When $M_2 << M_1$ then in the equations (39) and (40) M_2 can be neglected as compared to M_1 i.e. $M_1 - M_2 \approx M_1$ and $M_1 + M_2 \approx M_1$. Therefore, we have

$$v_1 = \frac{M_1}{M_1} u_1 = u_1$$

and

$$v_2 = \frac{2M_1}{M_1} u_1 = 2u_1$$

Therefore, when a heavy body A collides against a light body B at rest the body A should keep on moving with the same velocity and the body B should start moving with a velocity double that of A.

Thus, in principle if a moving truck (heavy body) collides against a stationary drum then the truck would keep on moving with the same velocity while the drum would be set into motion with a velocity double the velocity of the truck.

(c) When the mass of body B is very large as compared to that of A: When $M_2 >> M_1$, then in the equations (39) and (40) M1 can be neglected in comparison to M_2 i.e., $M_1 - M_2 \approx -M_2$ and $M_1 + M_2 \approx M_2$. Therefore, we have

$$v_1 = \frac{-M_2}{M_2} u_1 = -u_1$$

and

$$v_2 = \frac{2M_1}{M_1} u_1 = 0 \qquad (M_2 \gg M_1)$$

Therefore, when a light body A collides against a heavy body B at rest the body A should start moving with equal velocity in opposite direction, while the body B should practically remain at rest.

This result is in according with the observation that when a rubber ball hits a stationary wall, the wall remains at rest, while the ball bounces back with the same speed.

Example:

8kg ball moving with velocity 4m/s collides with a 2kg ball moving with a velocity 8m/s in opposite direction. If the collision be perfectly elastic what are the velocities of the balls after the collision.

Solution: By conservation of momentum

$$M_1 u_1 - M_2 u_2 = M_1 v_2 - M_1 v_1 + M_2 v_2$$

$$8 \times 4 - 2 \times 8 = 8v_1 + 2v_2$$

As collision is elastic, so we have

$$\frac{1}{2} \times 8 \times 4^2 + \frac{1}{2} \times 2 \times 8^2 = \frac{1}{2} \times 8 \times v_1^2 + \frac{1}{2} \times 2 \times v_2^2$$

$$v_1 = -\frac{4}{5} m/s$$

$$v_2 = \frac{56}{5} m/s$$

2.15 SUMMARY

In the present unit, we have studied about various parameters related to dynamics of Particle, Force, Momentum, Impulse and also learnt about Newton's laws of Motion. In this unit, we also studied work, Energy and Power. Using these definitions, we formulated the laws to simplify many problems in dynamics; particularly those which require us to relate position and speed of particle.

2.16 TERMINAL QUESTIONS

1. Write answers to the following questions:

    (a) What is Mechanics? Explain the branches of Mechanics.

    (b) What is inertia? Why do we call the Newton's first law as the law of inertia?

    (c) What is force? Discuss the types of Forces.

2. State Newton's second law of Motion.
3. Discuss the elastic collision of two bodies in one dimension. Calculate the velocities of bodies after the collision.
4. What is an elastic collision? Calculate the velocities of the two bodies undergoing elastic collision in one dimension.
5. A body of mass 2 kg makes an elastic collision with another body at rest and continues to move in the original direction with a speed equal to one third of its original speed. Find the mass of the second body.
6. Two balls each of mass M moving in opposite directions with equal speed u undergo a head on collision. Calculate the velocity of the two balls after collision.
7. Define Kinetic energy. Derive an expression for the Kinetic Energy of a body moving with a Uniform Velocity.

8. Define Work and Power. What are their units in SI system?

9. What is a Conservative Force? Explain its various properties.

10. What are Conservative and Non-Conservative Forces? Explain.

11. Write short notes on: -

    (a) Applications of the Concept of Impulse.

    (b) Newton's Third Law.

    (c) Inertial and Non-Inertial Reference Frames.

    (d) Types of Inertia.

2.17 SOLUTION AND ANSWERS:

Self-Assessment Questions (SAQs):

1. Inertia of Motion

2. (a) Hint (Section 2.5)

   (b) Hint (Section 2.5.1)

3. Positive, as force due to gaseous pressure and displacement of piston are in the same direction.

4. Hint (Section 2.10)

5. Hint (Section 2.10)

ANSWERS TERMINAL QUESTIONS

1. (a) Hint (Section 2.3, 2.3.2)

   (b) Hint (Section 2.4.3, 2.4.3)

   (c) Hint (Section 2.4, 2.4.1)

   (d) Hint (Section 2.6)

2. Hint (Section 2.5.1)

3. Hint (Section 2.15, 2.15.1, 2.15.2)

4. Hint (Section 2.15.1)

5. Here,

$M_1 = 2kg, \quad u_2 = 0, \quad v_1 = \frac{u_1}{3}, \quad M_2 = ?$

In a elastic collision in one dimension,

$$v_1 = \frac{(M_1 - M_2)u_1 + 2M_2u_2}{M_1 + M_2}$$

$$\frac{u_1}{3} = \frac{(2 - M_2)u_1 + 2 \times M_2u_2}{2 + M_2}$$

$$\frac{u_1}{3} = \frac{(2 - M_2)u_1 + 0}{2 + M_2}$$

$$\frac{u_1}{3} = \frac{(2 - M_2)}{2 + M_2}u_1$$

$$\frac{1}{3} = \frac{(2 - M_2)}{2 + M_2}$$

$$2 + M_2 = 6 - 3M_2$$

$$4M_2 = 4$$

$$M_2 = \frac{4}{4} = 1,$$

$$M_2 = 1kg$$

6. **Solution:**

Here,

$$M_1 = M_2 = M; \quad u_1 = v, \;\; u_2 = -v$$

Now,

$$v_1 = \frac{(M_1 - M_2)u_1 + 2M_2u_2}{M_1 + M_2}$$

$$= \frac{(M - M)u + 2m(-v)}{M + M} = \frac{0 - \; 2Mv}{2M}$$

or

$$u_1 = \; -v$$

Also,

$$v_2 = \frac{(M_2 - M_1)u_2 + 2M_1u_1}{M_1 + M_2}$$

$$= \frac{(M - M)(-v) + 2Mv}{M + M}$$

$$= \frac{0 + 2MV}{2M}$$

or

$v_2 = v$

Thus, after the collision the two balls back with equal speeds v.

7. Hint (Section 2.9.1)
8. Hint (Section 2.7)
9. Hint (Section 2.10)
10. Hint (Section 2.10)
11. (a) Hint (Section 2.6.1)
    (b) Hint (Section 2.6.2)
    (c) Hint (Section 2.4.6)
    (d) Hint Section 2.4.5)

2.18 SUGGESTED READINGS:

1. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons.
2. Elementary Mechanics, IGNOU, New Delhi.
3. College Physics, Hugh D. Young.
4. An Introduction to Mechanics, Daniel Kleppner and Robert J. Kolenkow.

UNIT 3

ANGULAR AND ROTATIONAL MOTION

Structure:

3.1 INTRODUCTION

In the Previous Units, we have studied about some basic parameters of Kinematic quantities for describing dynamic of a Particle. We have also studied important conservation Principles such as conservation of momentum, conservation of energy. In the Present Unit, we shall study about the kinematics of rotational motion describes the relationships among angular velocity angular acceleration, Uniform Circle Motion and time. This, in turn, leads to the concept of angular momentum and the all – important conservation of angular momentum, which explains some surprising and seemingly counter intuitive phenomena involving rotating objects. In this unit, we have also studied and proved the general theorems on moment of inertia, i.e. theorem of Parallel axes and theorem of Perpendicular axes. In this unit, we shall also, study about the concept of rolling motion.

3.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand Rotational and translational motion.
- Solve Problems based on translational motion and Rotational motion.
- Apply Rotational Equation of Motion
- Define some Parameters of Rotational motion.
- Relate torque and moment of inertia.
- Apply theorems of Parallel and Perpendicular axes.
- Compute numerical related to angular momentum Torque, Rotational Kinetic Energy.
- Comparison Translational and Rotational motion.

3.3 MOTION OF A RIGID BODY

Consider that a rigid body is free to execute translational and rotational motion in space. To study the general motion of the rigid body, we should know its position and orientation at each instant of time. To fix its position at any instant, consider a point, say O' in the rigid body. (Figure-1)

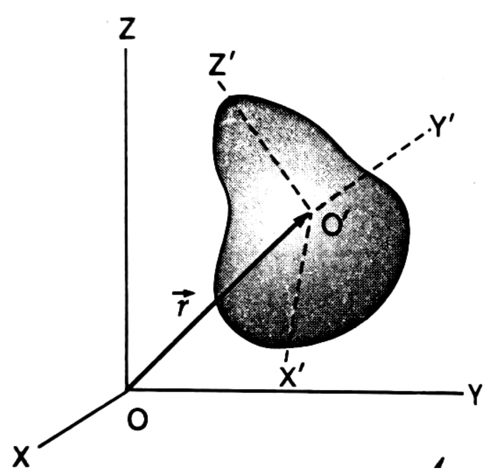


Figure - 1

Let $\vec{r} = (=\overrightarrow{OO'})$ be the position vector of the point O' in the body w.r.t. the origin O of a fixed (internal) frame of reference. Then, as the body moves in space (translational motion), its position vector $\vec{r}$ also changes and its value at any instant gives the position of the body at that instant.

The motion of the rigid body can be classified under the following type.

3.3.1 Transitional Motion

If a body moves such that its orientation does not change with respect to time then body is said to move in translational motion.

Example-1

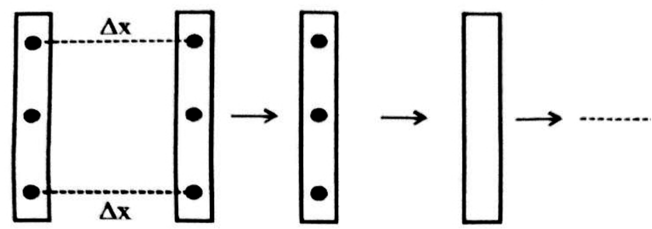


Figure - 2

Motion of a rod as shown.

Example-2

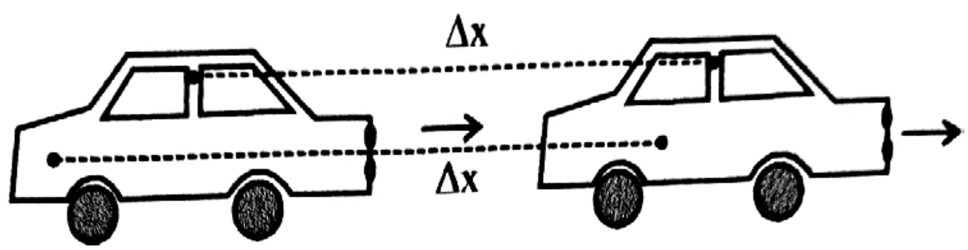


Figure - 3

Motion of body of car on a straight rod.

In both the above examples, velocity of all the particles is same as they all have equal displacements in equal intervals of time.

Properties of Translational Motion

(i) At any particular time, t = t, each point of rigid body will have the same velocity, if it performs translational motion.

(ii) Path of each point of rigid body in case of translational motion will be parallel to each other.

(iii) If one considers the frame of reference of any point of rigid body then we will observe that body is at rest.

3.3.2 ROTATION MOTION

If a body is rotating about the fixed axis of rotation then its motion is known as pure rotational motion or rotational motion

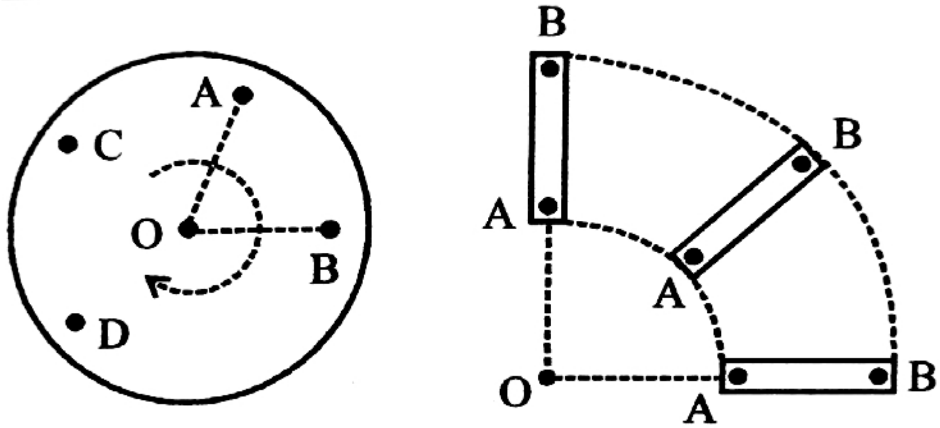


Figure - 4

In pure rotational motion.

Angular velocity of all the points is same about the fixed axis.

Properties of Pure Rotation

(i) Each point of body will rotate about the fixed center and center will lie on the axis of rotation.

(ii) Each point of rigid body will have fixed radius

3.3.3 GENERAL – MOTION

If a move such that its motion is neither pure rotational nor pure translational, then its motion is known as general motion.

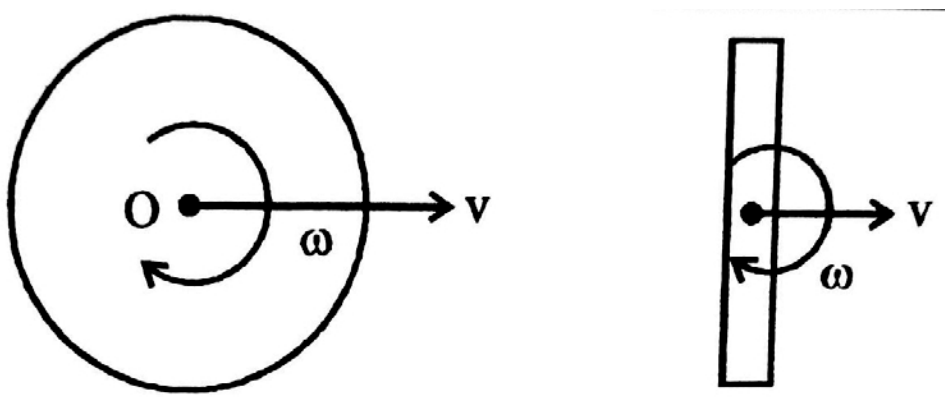


Figure - 5

Where, v = velocity of axis.

And ω = Angular velocity of system about O.

Property of general motion

If a body performs general motion then different points of body will have different velocity. If we consider the frame of reference of any point of body then we will observe that the body is purely rotating about our self.

3.3.4 Nature of Translational and Rotational Motion

Translational motion may be one, two or three dimensional, but the rotational motion of a rigid body can either be two or three dimensional. As the rotational motion in two dimensions is very simple as compared to that in three dimensions, we shall study the rotational motion of a rigid body in the former case only.

The rotational motion of a rigid body can be studied by applying Newton's laws of motion to the small parts of which the rigid body is made of. How really new principles are involved to study the rotational motion. However, it is found that the study of rotational motion becomes convenient by introducing a few physical quantities and concepts (such as angular velocity, angular acceleration, moment of inertia, torque, angular momentum, etc.) connected with rotational motion.

3.3.5 Comparison between Translational and Rotational Motion

Table -1

The comparison between translational and rotational motion are given in tabular form:

Translatory motion	Rotatory motion
1. All the constituent particles of the rigid body parallel to one another in straight lines	1. The particles move parallel to one another in circles of different radii about the given axis of rotation.
2. All the particles have same linear velocity.	2. All the particles have same angular velocity. As $v = r\omega$, the particles at different r have different linear velocities.
3. All the particles undergo same linear displacement.	3. All the particles undergo same angular

	displacement.
4. All the particles have same linear acceleration.	4. All the particles have same angular acceleration.
5. The position of the center of mass changes with time.	5. The distance of center of mass from the axis of rotation remains constant with respect to time.
6. Mass is analogous to moment of inertia. Mass depends on the quantity of matter in the body.	6. Moment of inertia (I) is analogous to mass. Moment of inertia (I) depends on distribution of mass about axis of rotation.
7. Kinetic energy of translation $=\frac{1}{2}mv^2$	7. Kinetic energy of rotation $=\frac{1}{2}I\omega^2$
8. Force produces the translator motion.	8. Torque produces the rotational motion.
9. Work done = W W = force × displacement	9. W = torque × θ
10. Force – mass × acceleration	10. Torque = I × angular acceleration
11. Linear momentum = p p = mass × linear velocity	11. Angular momentum = $I\omega$ where ω = angular velocity
12. Impulse = force × time	12. Angular impulse = torque × time
13. Power = force × velocity	13. Power = torque × ω

Table - 2

Translational Motion and Rotational Motion

There is a strong analogy between rotational motion and translational motion. Every law of physics governing rotational motion has a translational equivalent. The analogies between rotational and translational motion are summarized in table -2.

Translational motion term	**In symbols**	**Rotational motion term**	**In symbols**	**Magnitude of angular terms if *u*, *a* and *F* are perpendicular to *r*, the radius of the motion**
Displacement	$\underline{x}$	Angular displacement	$\underline{\theta}$	
Velocity	$\underline{v} = \frac{d\underline{x}}{dt}$	Angular velocity	$\underline{\omega} = \frac{d\underline{\theta}}{dt}$	$\omega = \frac{u}{r}$
Acceleration	$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2}{dt}$	Angular acceleration	$\underline{\alpha} = \frac{d\underline{\omega}}{dt} = \frac{d^2}{dt}$	$\alpha = \frac{a}{r}$
Momentum	$\underline{p} = m\underline{v}$	Angular momentum	$\underline{L} = I\underline{\omega}$	$L = mvr$
Kinetic energy	$K = \frac{1}{2}mu^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mu^2$
Force	$\underline{F}$	Torque (or moment)	$\underline{\tau}$	$\tau = Fr$
Newton's Second Law of motion	$\underline{F_{res}} = m\underline{a}$	Angular version of Newton II	$\underline{\tau_{res}} = I\underline{\alpha}$	$\tau = Fr = mr$
Mass	m	Moment of inertia	I	$I = mr^2$
Impulse	$F\Delta T = \Delta P$	Impulse	$\tau\Delta t = \Delta L$	
Work done	$W = Fs$	Work done	$W = \tau\theta$	
Power	$P = Fv$	Power	$P = \tau\omega$	

Self-Assessment Question (SAQ)

1. Classify types of motion for rigid body with suitable examples.
2. Explain the properties of Translational and Rotational motion.
3. Write a short note on General motion.

4. Give comparison/analogies between Translational and Rotational motion.

3.4 ROTATIONAL EQUATION OF MOTION

Three simple relations between rotational kinematic variables are:

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ and (iii) $\omega^2 - {\omega^2}_0 = 2\alpha\theta,$

where the symbols have their usual meaning. They correspond respectively to the three equations of linear motion

(i) $v = u + a\,t$ (ii) $s = u\,t + \frac{1}{2}at^2$ (iii) $v^2 - u^2 = 2\,a\,s$

We can deduce rotational kinematic equations as follow:

(a) $\omega = \omega_0 + \alpha\, t$

Suppose a rigid body is rotating about a given axis with a uniform angular acceleration α. We know that

$\alpha = \frac{d\omega}{dt}$ or $d\omega = \alpha\, dt$ .... (1)

At t = 0, let $\omega = \omega_0$

At t = t, let $\omega = \omega$

∴ Integrating equation 1 within proper limits, we get

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$$

$$[\omega]\omega_0 = \alpha[t]_0{}^t$$

$$\omega - \omega_0 = \alpha\,(t - 0) = \alpha\, t \quad \text{or}$$
$$\omega = \omega_0 + \alpha\, t$$

(b) $\theta = \omega_0 t + \frac{1}{2}\,\alpha\, t^2$

If ω is angular velocity of the rigid body at any time t, then we know that

$\omega = \frac{d\theta}{dt}$ or $d\theta = \omega dt$ .... (2)

At t = 0, let $\theta = 0$

At t = t, let $\theta = \theta$

∴ Integrating equation 2 within proper limits, we get

$$\int_0^\theta d\theta = \int_0^t \omega dt = \int_0^t (\omega_0 + \alpha\, t)dt$$

$$\int_0^\theta d\theta = \int_0^t \omega_0\, dt + \int_0^t \alpha\, t\, dt$$

$$[\theta]_0^\theta = \omega_0 [t]_0^t + \alpha \left[\frac{t^2}{2}\right]_0^t$$

$$\theta - 0 = \omega_0 (t - 0) + \frac{\alpha}{2}(t^2 - 0)$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha\, t^2$$

(c) $\omega^2 - \omega_0^2 = 2\,\alpha\,\theta$

We know that $\omega = \frac{d\theta}{dt}$ and $\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right) = \left(\frac{d\omega}{d\theta}\right)\omega$ or

$\omega\, d\omega = \alpha\, d\theta$ .... (3)

When $\theta = 0, \omega = \omega_0$, initial angular velocity and when $\theta = \theta, \omega = \omega$, final angular velocity

∴ Integrating equation 3 within proper limits, we get

$$\int_{\omega_0}^{\omega} d\omega = \int_0^\theta \alpha\, d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha\, [\theta]_0^\theta$$

$$\frac{\omega^2 - \omega^2{}_0}{2} = \alpha\,(\theta - 0) = \alpha\theta \qquad \text{or}$$

$\omega^2 - \omega_0^2 = 2\,\alpha\,\theta$ .... (4)

Example-1

The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 second.

(i) What is the angular acceleration, assuming the acceleration to be uniform?

(ii) How many revolutions does the wheel make during this time?

Solution:

Given, $\omega_0 = 2\pi n_0 = 2\pi \times \frac{1200}{60}$

$= 40\pi\ rad/s$

$\omega = 2\pi n = 2\pi \times \frac{3120}{60}$

$= 104\ \pi\ rad/s$

(i) Angular acceleration

$\alpha = \frac{\omega - \omega_0}{t} = \frac{104\pi - 40\pi}{16}$

$= 4\pi\ rad/s^2$

(ii) The angular displacement can be obtained as

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$= 40\pi \times 16 + \frac{1}{2} \times 4\pi \times (16)^2$

$= 1152\ \pi\ rad$

Number of revolutions in 16 s

$= \frac{0}{2\pi} = \frac{1152\pi}{2\pi} = 576$

Example-2

The angular acceleration of a flywheel is given by $\alpha = 12 - t$ where α is in rad/s^2 and t in second. If the angular velocity of the wheel is 60 rad/s at the end of 4 second, determine the angular velocity at the end of 6 second. How many revolutions take place in these 6 second?

Solution:

Given that $\alpha = 12 - t$

i.e. $\frac{d\omega}{dt} = 12 - t$

or $d\omega = (12 - t)dt$

Integrating above equation, we get

$\int d\omega = \int (12 - t)dt$

or $\omega = 12t - \frac{t^2}{2} + C$

At $t = 4\ s, \omega = 60\ rad/s$

$\therefore$ $60 = 12 \times 4 - \frac{4^2}{2} + C$

or $C = 20$

$\therefore$ $$\omega = 12t - \frac{t^2}{2} + 20 \qquad \text{............ (5)}$$

At $t = 6\ s$, $$\omega = 12 \times 6 - \frac{6^2}{2} + 20$$

$$= 74\ rad/s$$

Now we can write

$$\frac{d\theta}{dt} = 12t - \frac{t^2}{2} + 20$$

Integrating the above equation, we get

$$\int d\theta = \int \left(12t - \frac{t^2}{2} + 20\right) dt$$

or $$\theta = \frac{12t^2}{2} - \frac{t^3}{6} + 20t + C'$$

Let at $t = 0, 0 = \theta_0$ $\quad\therefore\quad$ $C' = \theta_0$

Thus, we have $$\theta = 6t^2 - \frac{t^3}{6} + 20t + \theta_0$$

or $$\theta - \theta_0 = 6t^2 - \frac{t^3}{6} + 20t \qquad \text{............ (6)}$$

Angular displacement during 6 s

$$\theta_6 - \theta_0 = 6 \times 6^2 - \frac{6^3}{6} + 20 \times 6$$

= 300 rad

$\therefore$ Number of revolutions $= \frac{300}{2\pi} = 47.8$

Self-Assessment Question (SAQ)

5. The motor of an engine is rotating about its axis with an angular velocity of 100 rpm. It comes to rest in 15 s, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

6. The motor of an engine is rotating about its axis with an angular velocity of 120 r.p.m. It comes to rest in 10 s, after switched off. Assuming constant deceleration, calculate the number of revolutions made by it before coming to rest.

3.5 SOME FUNDAMENTAL DEFINITIONS OF ROTATIONAL MOTION

System of particles can move in different ways as observed by us in daily life. To understand that we need to understand few parameters.

3.5.1 Angular Velocity

Angular velocity is the angle described by a rotating body per unit time. It is a vector quantity and is denoted by $\underset{\omega}{\rightarrow}$.

Consider a particle moving along a circular path in the anticlockwise direction. Let the rotating particle be at A at any instant (t = 0). Let the particle be at B after a time t [Fig. 6].

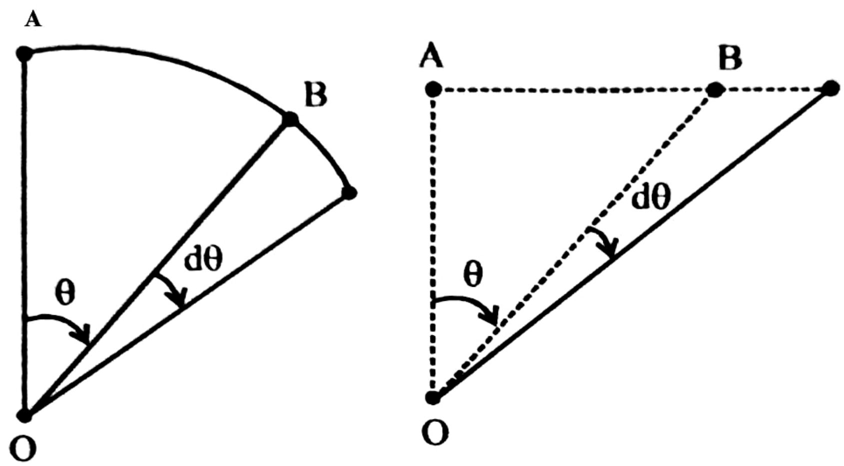


Figure - 6

Let the angle AOB described by the particle during this time be *θ radian. Then the magnitude of angular velocity is given by

$$\omega = \frac{\theta}{t}$$

If the particle describes one complete revolution, then

$\theta = 2x \ and \ t = T$ (time period)

In that case, $\omega = \frac{2x}{T}$

If the particle describes n revolution in one second, i.e., if n be the frequency of revolution, then

$$\omega = 2xn \qquad \text{.................. (7)}$$

The angular velocity is measured in radian per second.

Direction of angular velocity is given by right hand thumb rule.

According to right hand thumb rule, if we curl the fingers of right hand along with the body, then right-hand thumb gives us the direction of angular velocity.

It is always along the axis of the motion.

3.5.2 Uniform Angular Velocity

If the particle describes equal angles in equal intervals of time, then the angular velocity is said to be uniform.

Instantaneous angular velocity of a particle is given by

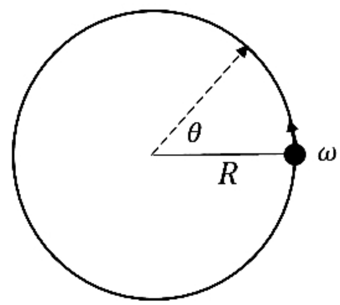


Figure - 7

$$\omega = \frac{d\theta}{dt}$$

Where dθ is the infinitesimally small angle covered in infinitesimally small time dt.

3.5.3 Angular Displacement

It is the angle described by the position vector $\underset{r}{\rightarrow}$ about the axis of rotation.

Angular displacement $(\theta) = \frac{Linear\ displaceme\ nt\ (s)}{Radius\ (r)}$

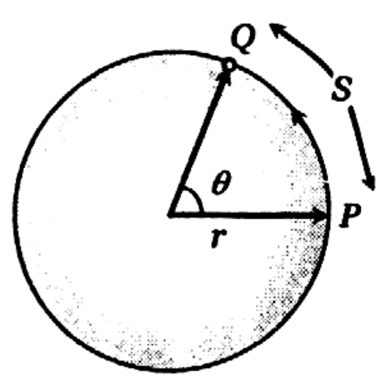


Figure - 8

Consider a particle moves from A to B in the following figures-Angle is the angular displacement of particle about O.

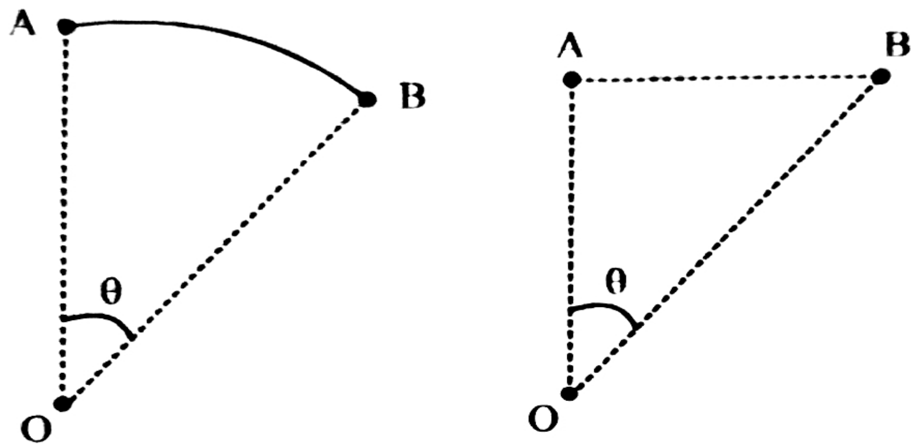


Figure - 9

- Unit: radian
- Dimension: $[M^0 L^0 T^0]$
- Vector form: $\vec{s} = \vec{\theta} \times \vec{r}$

i.e., angular displacement is a vector quantity whose direction is given by right hand rule. It is also known as axial vector. For anit-clockwise sense of rotation direction of θ is perpendicular to the plane, outward and along the axis of rotation and vice-versa.

- 2π radian = 360^o = 1 revolution.
- If a body rotates about a fixed axis then all the particles will have same angular displacement (although linear displacement will differ from particle to particle in accordance with the distance of particles from the axis of rotation).

3.5.4 Angular Acceleration

The rate of change of angular velocity is defined as angular acceleration.

If particle has angular velocity ω_1 at time t_1 and angular velocity ω_2 at time t_2 then,

Angular acceleration $\vec{\alpha} = \dfrac{\vec{\omega_2} - \vec{\omega_1}}{t_2 - t_1}$ .... (8)

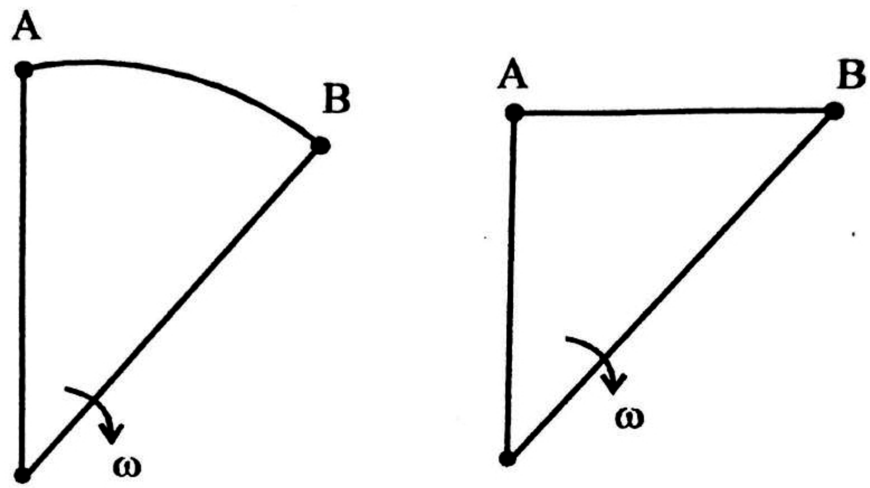


Figure - 10

$$\alpha = \frac{d\omega}{dt}$$

Instantaneous angular acceleration $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$. .... (9)

- ❖ Unit: rad/sec^2
- ❖ Dimension: $[M^0 L^0 T^2]$.
- ❖ If $\alpha = 0$, circular or rotational motion is said to be uniform.
- ❖ Average angular acceleration $\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_2}$.
- ❖ Relation between angular acceleration and linear acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$.
- ❖ It is an axial vector whose direction is along the change in direction of angular velocity i.e. normal to the rotational plane, outward or inward along the axis of rotation (depends upon the sense of rotation).

3.5.5 Uniform Circular Motion

A particle moving in a circular is said to perform uniform circular motion if its speed remains constant. Since $v = \omega r$, the angular speed (ω) is also constant in a uniform circular motion.

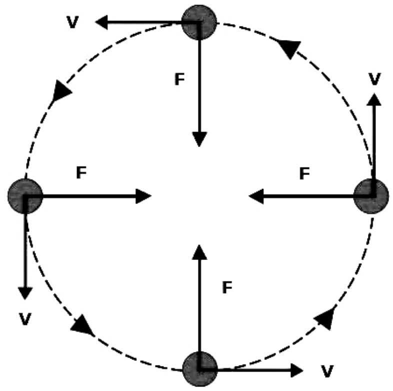


Figure - 11

For a particle moving uniformly in a circle, the time required to complete one rotation is called time period (T).

$$T = \frac{2\pi rad}{\omega rads^{-1}} = \frac{2\pi}{\omega} s$$

$$\Rightarrow \omega \frac{2\pi}{T} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Frequency (*f*) of rotation is defined as the number of completed rotations per unit time.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$\Rightarrow \omega = 2\pi f. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Unit of frequency is s^{-1} also known as hertz (Hz).

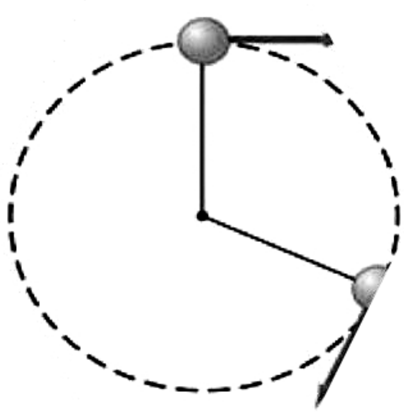


Figure - 12

If a particle moves in a circle with changing speed, its motion is said to be non-uniform circular motion.

3.5.6 Relation between Angular Velocity and Linear Velocity

Let us consider a body P moving along the circumference of a circle of radius r with linear velocity v and angular velocity w as shown in Figure 13. Let it move from P to Q in time dt and dθ be the angle swept by the radius vector.

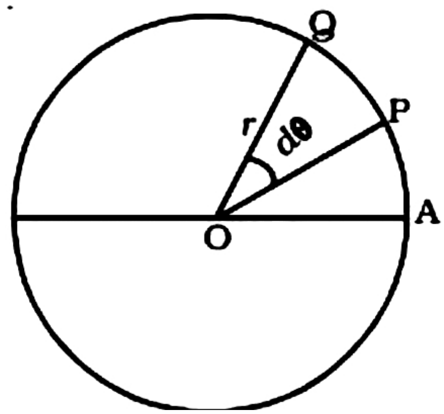


Figure - 13

Now, Suppose PQ = ds, be the arc length covered by the particle moving along the circle, then the angular displacement dθ is expressed as dθ = ds/r. But ds = vdt

dθ/dt = v/r

i. e. Angular vel. $\omega = \frac{v}{r}$ or $v = \omega r$ .... (12)

In vector notation

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad \text{.................. (13)}$$

3.5.7 Relation between Angular Acceleration and Liner Acceleration

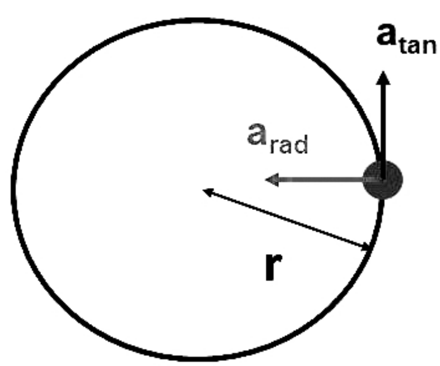


Figure-14

We know that,

Linear acceleration = Rate of change of linear velocity

$\Rightarrow a = \frac{dv}{dt}$

…….. (14)

Angular acceleration = Rate of change of angular velocity

$\Rightarrow \alpha = \frac{d\omega}{dt}$

…….. (15)

Using equation (14) and (15), we get

$$\frac{a}{\alpha} = \frac{dv}{d\omega} = \frac{d(r\omega)}{d\omega}$$

$$= r\frac{d\omega}{d\omega} \; [r \; is \; constant] = r$$

$$\Rightarrow \qquad a = \alpha \, r$$

In vector form $\qquad \vec{a} = \vec{\alpha} \times \vec{r}$

Example - 3

What is the ratio of the angular speeds of the minute hand and the hour hand of a watch?

Solution:

$\omega = \frac{2\pi}{T}$

For minute had: $\qquad \omega_1 = \frac{2\pi \; rad}{1 \; h}.$

For hour hand: $\qquad \omega_2 = \frac{2\pi \; rad}{12 \; h}$

$\therefore \qquad \frac{\omega_1}{\omega_2} = \frac{12}{1}$

Example - 4

A flywheel of diameter 2 m has an angular speed of 120 rpm. Find the linear speed of a point on its rim.

Solution:

$v = \omega r$

r = 1.0 m

$\omega = 120 \; rotation \; per \; minute$

$= \frac{120 \times 2\pi \; rad}{60 \; s} = 4\pi \; rads^{-1}$

$\therefore \qquad v = \omega r = 4\pi \; ms^{-1} = 12.6 \; ms^{-1}$

Example - 5

Angular displacement of a particle moving in a circle of radius r = 1 m is given by $\boldsymbol{\theta = t + 0.75t^2}$

(i) Find its angular velocity at time *t* = *2 s*.

(ii) Find its average angular velocity in the first 2 s of its motion.

Solution:

$\theta = t + 0.75t^2$

(i) $\omega = \frac{d\theta}{dt} = 1 + 1.5t = 1 + 1.5 \times 2 = 4\, rads^{-1}$

(ii) $\langle\omega\rangle = \frac{\Delta\theta}{\Delta t} = \frac{\theta_{t=2} - \theta_{t=0}}{2}$

$= \frac{2 + 0.75 \times 2^2 - 0}{2} = 2.5\, rads^{-1}$

Example - 6

Radius of the Earth is R = 6400 km. Find the speed of a man standing on the surface of the Earth at a latitude in 60^o.

Solution:

Man is rotating in a circle of radius r = R cos 60^o = 3200 km.

Angular speed of the Earth

$$\omega = \frac{2\pi}{T} = \frac{2\pi\, rad}{24\, h}$$

$$\therefore \quad v = \omega r = \frac{2\pi}{24} \times 3200\, \frac{km}{h} = 837.3\, \frac{kn}{h}$$

While calculating torque of a force about an axis, one just needs to learn the following three points:

1. If the given force is parallel to the axis, torque about the said axis is zero.
2. If the line of action of the force intersects the axis, there is no torque.
3. A force F has torque about an axis (say z) only when F is neither parallel to z-axis nor its line of action intersects z-axis. When a force is perpendicular to z-axis about it line of action does not intersect it (skewed lines), torque about z-axis is $\tau = Fd$, where d is the perpendicular distance between the line of force and z-axis.

Self-Assessment Question (SAQ)

7. Find the angular speeds of the second hand and the minute hand of a watch.

8. Find the speed of the tip of the hour hand of a watch. The tip of the hour is 1cm long.

3.6 ANGULAR MOMENTUM

The angular momentum of a particle about a fixed point is defined as the moment of its linear momentum about that point. It is measured by the vector product of linear momentum $\vec{p} = m\ \vec{v}$ of the particle and its vector distance $\vec{r}$ from the fixed point in the inertial frame.

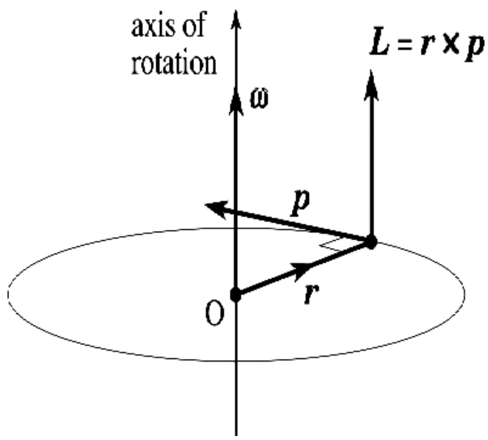


Figure-15

∴ Angular momentum $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

Angular momentum being the vector product of two vectors is obviously a vector quantity. Its direction is perpendicular both to $\vec{r}$ and $\vec{p}$ or $\vec{v}$ as given by the right-hand screw rule.

Note: We know the importance of linear momentum in dealing with the translational motion of a particle or a system of particles. Angular momentum is an analogous concept in dealing with rotational motion.

3.6.1 Angular Momentum of a Particle

Consider a particle moving with momentum $\vec{P}$. Position vector of the particle (at any instant) is $\vec{r}$ with respect to a point O. Angular momentum of the particle about O is defined as

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times (m\vec{v}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

where m and $\vec{v}$ are mass and velocity of the particle.

Consider the plane defined by $\vec{r}$ and $\vec{P}$ to be our xy-plane. Angular momentum ($\vec{L}$) is a vector directed perpendicular to $\vec{r}$ and $\vec{P}$ and it must be along z-direction. In the figure shown, $\vec{r}\times\vec{P}$ (by right hand rule) is along positive z-direction. If the particle were moving in exactly opposite direction. $\vec{r}\times\vec{P}$ would have been along negative z-direction.

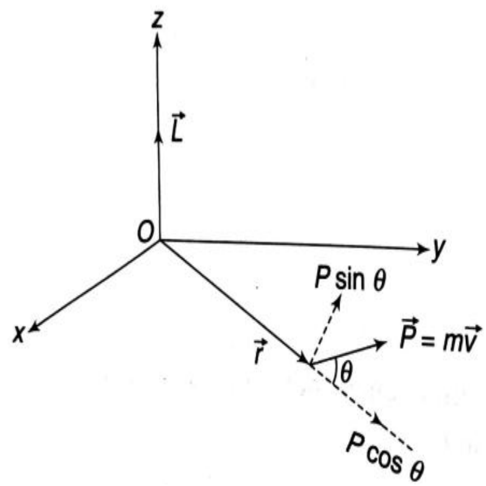


Figure - 16

Magnitude of angular momentum of the particle about O is

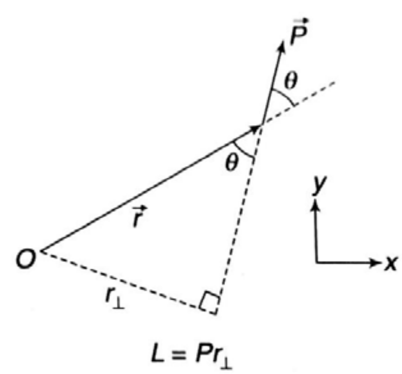
$$L = rP\sin\theta = r\,P_{\perp} \quad \text{.................. (17)}$$

where $P_{\perp}$ component of momentum perpendicular to $\vec{r}$. One can also interpret the above expression as

$$L = P(r\sin\theta) = P\,r_{\perp} \quad \text{................ (18)}$$

where $r_{\perp}$ is perpendicular distance of line of $\vec{P}$ from O.

Note: Unit of a angular momentum is kgm^2s^{-1}.



In this fig. $\vec{L}$ is directed out of the plane of the fig. Direction of $\vec{L}$ about O can also be conveyed simply by saying that it is anticlockwise

Figure - 17

However, if the motion is not restricted to xy plane, angular momentum of the particle about O can be in any direction. The component of $\vec{L} = \vec{r} \times \vec{P}$ parallel to z-axis is defined as angular momentum of the particle about z-axis (L_z)

Mathematically, $\vec{\tau} = \vec{r} \times \vec{F}$ and $\vec{L} = \vec{r} \times \vec{P}$

3.6.2 Angular momentum of a rigid body

The sum of the moments of the linear momentum of all the particles of a rotating rigid body about the axis of rotation is called its angular momentum.

When a body is free to rotate about an axis the angular velocity of all the particles, at whatever distance they may be, is the same. Since the distance of the various particles from the axis of rotation is not the same, their linear velocities will be different.

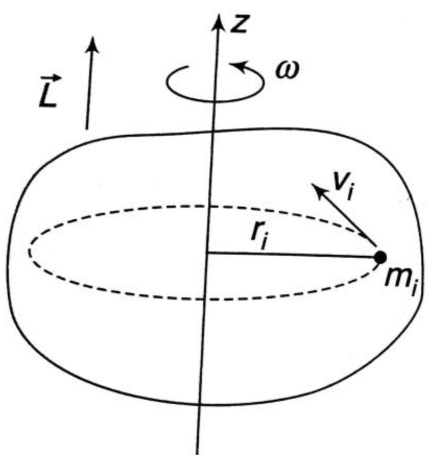


Figure - 18

Where,

$$m_i = m_1, m_2 \ldots\ldots$$
$$v_i = v_1, v_2 \ldots\ldots$$
$$r_i = r_1, r_2 \ldots\ldots$$

Consider the particles $m_1, m_2 \ldots$ of the rigid body lying at distances $r_1, r_2 \ldots$ from the axis of rotation XY and having magnitudes of linear velocities $v_1, v_2 \ldots$ respectively. If ω is the magnitude of angular velocity, then

Linear velocity of the particle $m_1 = m_1 = r_1\omega$

∴ Magnitude of linear momentum of the particle $m_1 = m_1 v_1 = m_1 r_1 \omega$.

Hence the magnitude of moment of linear momentum or angular momentum of the particle m_1 about the axis of rotation $= m_1 v_1 r_1 = m_1 r_2^1 \omega$.

Similarly, the moment of linear momentum or angular momentum of the particle m_2 about the axis of rotation $= m_2 v_2 r_2 = m_2 r_2^2 \omega$

∴ Sum of the magnitudes of moments of linear momentum or angular momentum of all the particles $m_1 r_2^1 \omega + m_2 r_2^2 \omega + \cdots$ or

$$L = \sum mr^2\omega = I\omega \quad \ldots\ldots. (19)$$

Where, $I = \sum mr^2$ = moment of inertia of the rigid body about the axis of rotation and L is the magnitude of the angular momentum.

Vectorially $\vec{L} = I\vec{\omega}$ ………... (20)

Example-7

A particle has $\vec{P} = P_x\hat{i} + P_y\hat{j}$ momentum and its position vector at an instant is $\vec{r} = x\hat{i} + y\hat{j}$. Find the angular momentum of the particle

(a) About the origin (b) about the z-axis.

Solution:

$\vec{L} = \vec{r} \times \vec{P}$

(a) $L = \vec{r} \times \vec{P} = (x\hat{i} + y\hat{j}) \times (P_x\hat{i} + P_y\hat{j})$

$= (xP_y - yP_x)\hat{k}$

(b) Angular momentum about z-axis - z component of $\vec{L}$

∴ $L_z = xP_y - yP_x$

Example-8

Calculate the angular momentum of the earth rotating about its own axis of rotation. Mass of the earth $= 6 \times 10^{24} kg$

Mean radius $= 6.4 \times 10^6 metre$.

Solution:

Angular momentum $L = I\omega$

Taking the earth to be a solid sphere, moment of inertia of the earth about its own axis (a diameter) $I = \frac{2}{5}Mr^2$.

As the earth rotates once in 24 hrs. about its own axis, the

Angular velocity $\omega = \frac{2\pi}{24\times60\times60} = 7.27\times10^{-5} rad/sec.$

Moment of inertia $I = \frac{2}{5}\times6\times10^{24}\times6.4\times10^6\times6.4\times10^6 = 9.83\times10^{36} kg\ m^2$

∴ Angular momentum $L = I\omega$

$$= 98.3\times10^{36}\times7.27\times10^{-5} = 714\times10^{31}$$

$$= 7.14\times10^{33} kg\ m^2/sec$$

Self-Assessment Questions (SAQs)

9. Calculate angular momentum of Neptune about the sum. Given, mass of Neptune = 10^{27} kg ; its distance from sun is 5×10^{12} m and period of revolution around the sun = 5×10^9 s.

10. A disc is rotating with angular speed ω. If a child sits on it, what is conserved?

11. Can a body in translatory motion have angular momentum? Explain.

12. Choose the right option: -

(i) Angular momentum is

(A) Axial vector (B) polar vector

(C) scalar (D) None of the above

(ii) The moment of momentum is called

(A) couple (B) torque

(C) impulse (D) angular momentum

(iii) Angular momentum of a body is defined as the product of

(A) mass and angular velocity

(B) centripetal force and radius

(C) linear velocity and angular velocity

(D) moment of inertia and angular velocity

(iv) The unit of angular momentum is

(A) N m (B) kg m^{-1} s^{-1}

(C) kg m^2 s^{-1} (D) kg^2 m^2 s^{-1}

3.7 TORQUE OR MOMENT OF FORCE

Torque is a physical quantity that measures the turning effect of a force.

When you switch on a fan, it begins to rotate with increasing speed. The fan is experiencing a torque (applied by electric motor). since there is no acceleration of the COM of the fan, we know that the sum of all force on it is zero.

This implies that forces may have zero resultant but non-zero turning effect.

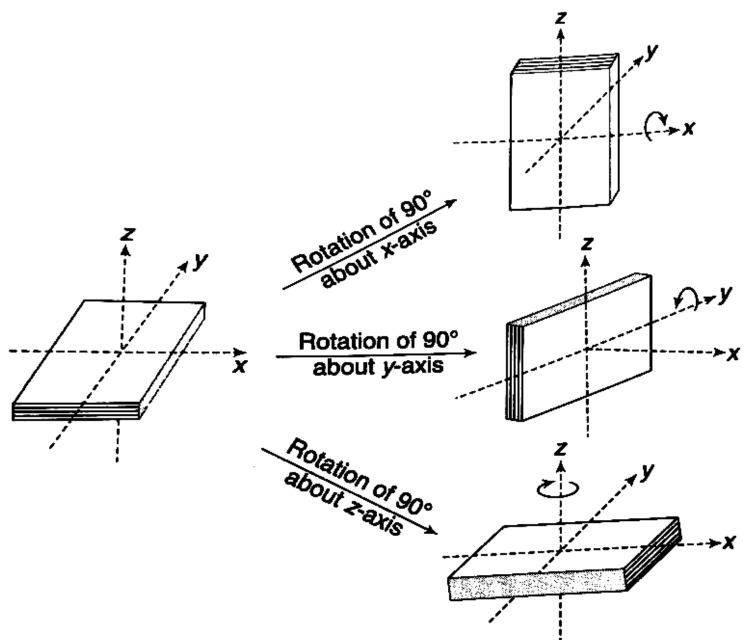


Figure-19

Torque. If a force $\vec{F}$ acts on a particle at a point P whose position with respect to the origin O of the inertial frame is given by the displacement vector $\vec{r}$, the torque $\vec{\tau}$ on the particle with respect to the origin O is defined as $(\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta\hat{n})$, where θ is smaller angle between $\vec{r}$ and $\vec{F}$; $\hat{n}$ is unit along $\vec{\tau}$

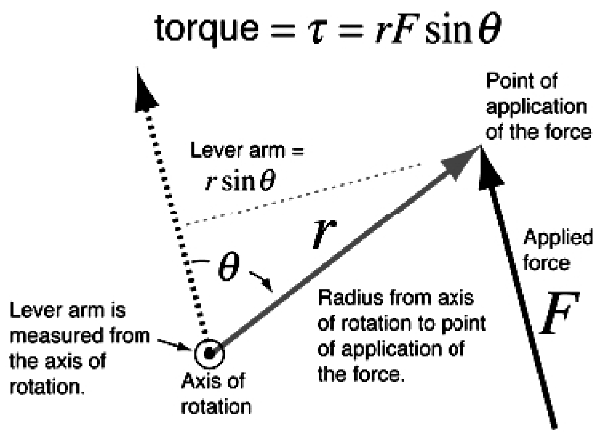


Figure-20

Torque is a vector quantity. Its direction is normal to the plane formed by $\vec{r}$ and $\vec{F}$ and the sense is given by the right-hand rule for the cross product of few vectors.

Units. The unit of torque is Newton metre (Nm).

3.7.1 Expression for Torque in Polar Co-ordinate

We know that, expression for torque in cartesian co-ordinate is $\tau = (xF_y - yF_x)$ .... (19)

Let, the line of action of force $\vec{F}$ makes an angle α with X-axis, Fig.21.

$\therefore \quad F_x = F\cos\alpha$ ……….(20)

$F_y = F\sin\alpha$ ……… (21)

If x, y are the co-ordinates of the point P,

where $\overrightarrow{OP} = \vec{r}$ and $\angle XOP = \theta,$

then $x = r\cos\theta$ ……… (22)

and $y = r\sin\theta$ ……… (23)

Substituting these values in Equation (19), we get

$$\tau = (r\cos\theta)F\sin\alpha - (r\sin\theta)(F\cos\alpha)$$

$$= rF[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$$

$$\tau = rF\sin(\alpha - \theta) \quad \text{……… (24)}$$

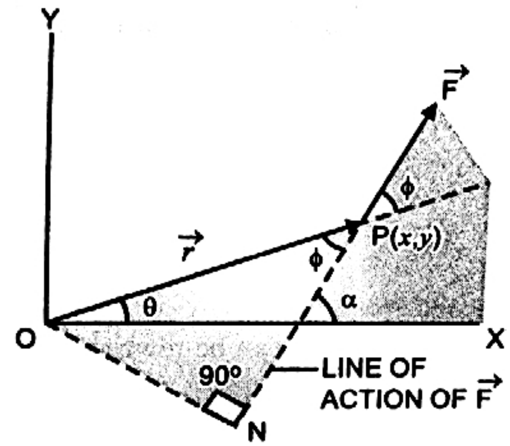


Figure - 22

Let ϕ be the angle which the line of action of $\vec{F}$ makes with the position vector $\overrightarrow{OP} = \vec{r}$.

As is clear from 21, θ+ϕ=α or ϕ+θ=θ

Putting the value in Equation (24), we get

$$\tau = rF \sin\phi \qquad \text{......... (25)}$$

Equation (25), is the expression for torque in polar co-ordinates.

3.7.2 Power Associated with Torque

We know that,

work done (dW) in rotating a particle through a small angle (dθ) as

$$dW = t\,(d\theta) \qquad \text{............ (26)}$$

If this work is done in a small-time interval dt, then dividing both sides of equation (26) by dt, we get

$$\frac{dW}{dt} = t\left(\frac{d\theta}{dt}\right) \qquad \text{............ (27)}$$

Now, by definition, $\frac{dW}{dt} = P$, the average power associated with torque.

and $\frac{d\theta}{dt} = \omega$, average angular speed of the bod in this interval.

From (27), $\boldsymbol{P = t\,\omega}$

i.e. power associated with torque is given by the product of torque and angular speed of the body about the axis of rotation.

In linear motion, the corresponding relation for power is $\boldsymbol{P = F\,v}$

3.7.3 Relation between Torque and Force

We know that, Torque is defined as the time rate of change of angular momentum.

∴ Torque $\vec{\tau} = \frac{d\vec{L}}{dt}$ .... (28)

but $\vec{L} = \vec{r} \times \vec{p}$ .... (29)

Now $\frac{d\vec{L}}{dt} = \frac{d}{t}(r \times p) = \frac{d\vec{r}}{dt} \times \vec{p} \times \vec{r}\frac{d\vec{L}}{dt}$

But $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0$

$\therefore\ \vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}\ \left[\because \vec{F} = \frac{d\vec{p}}{dt}\right]$

$\vec{\tau} = \vec{r} \times \vec{F}$ .... (30)

∴ Torque is the vector product of position vector $\vec{r}$ and force $\vec{F}$.

3.7.4 Relation between Torque and Angular Momentum

As, we know that,

The vector relation between angular momentum $\vec{L}$ and linear momentum $\vec{p}$ is $\vec{L} = \vec{r} \times \vec{p}$

Differentiating both sides w.r.t. time we get $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$
,

Applying the product rule for differentiation on the right-hand side,

$\frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$ .... (31)

Now $\frac{d\vec{r}}{dt} = \vec{v} =$ velocity of the particle and $\vec{p} = m\vec{v}$ linear momentum of the particle

$\therefore \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} m \vec{v} = 0,$ as the cross product of two parallel vectors is zero. Further, as -- $\frac{d\vec{p}}{dt} = \vec{F},$ therefore, from (93),

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \vec{F} = \vec{\tau} \quad \text{or} \quad \frac{d}{dt}(\vec{L}) = \vec{\tau} \qquad \text{........... (32)}$$

Hence the time rate of change of angul r momentum of a particle
a
is equal to the torque acting on it.

Equation (32) is the rotational analogue of the equation

$$\frac{d}{dt}(\vec{p}) = \vec{F}, \qquad \text{.......... (33)}$$

Which represents Newton's second law for the translational motion of a single particle.

Example-9

A force $\vec{F} = (2\hat{\imath} + 3\hat{\jmath})N$ acts at a point having position vector $\vec{r} = (2\hat{\imath} - \hat{k})m.$

(i) Find the torque of the force about the origin.

(ii) Find the torque of the force about z-axis.

Solution:

(i) $\tau = \vec{r} \times \vec{F}$

(ii) z-component of τ is known as torque about z-axis.

(i) $\vec{\tau} = \vec{r} \times \vec{F} = (2\hat{\imath} - \hat{k}) \times (2\hat{\imath} + 3\hat{\jmath})$

$= (3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})N - m$

(ii) $\tau_z = 6\ N - m$

Example-10

A force $\vec{F} = 3\hat{\jmath}$ i.e., N acts at a point (2, 2)m. Find torque of the force about the origin.

Solution:

$\vec{\tau} = (2\hat{\imath} + 2\hat{\jmath})m;\ \vec{F} = 3\hat{\jmath}$

$\vec{\tau} = \vec{r} \times \vec{F} = 6\hat{k}\ N - m.$

Example-11

In the figure shown, find torque of force F about O. Given F = 20N, d = 1 m.

Solution:

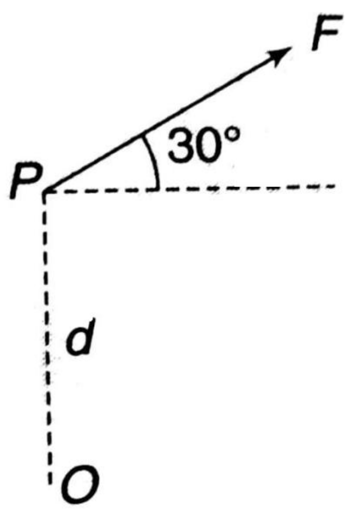


Figure - 23

Component of force perpendicular to OP is $F\cos 30^o = 10\sqrt{3}N$.

$\therefore \quad \tau = dF_{\perp} = 1 \times 10\sqrt{3} = 10\sqrt{3}N - m$

OR, $\quad \vec{r} = \overrightarrow{OP}$

Angle between $\vec{r}$ and $\vec{F}$ is 60°.

$\therefore \quad \tau = rFsin60^o = 1 \times 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}\ N - m$

Example-12

AB is a rod. Two forces of same magnitude F act at the two ends of the rod in opposite directions. Show that torque on the rod is same about A or B or any other point C. Such pair of equal and opposite forces are often known as a couple.

Solution:

Torque about C:

$\tau = Fx + Fy$

$= F(x + y) = FL$

Where L is the length of the rod. Thus, value of torque is independent of choice of point C.

Example-13 Choose the correct option:

During paddling of a bicycle, the force of friction exerted by the ground on the two wheels is such that it acts,

(A) in the backward direction on the front wheel and in the forward direction on the rear wheel.

(B) in the forward direction on the front wheel and in the backward direction on the rear wheel.

(C) in the backward direction on both the front and the rear wheels.

(D) in the forward direction on both the front and the rear wheels.

Solution:

This problem is explored in detail for conceptual clarity. Let a non-rotating $(\omega_0 = 0)$ disc of radius r having initial velocity u be gently placed on the rough surface.

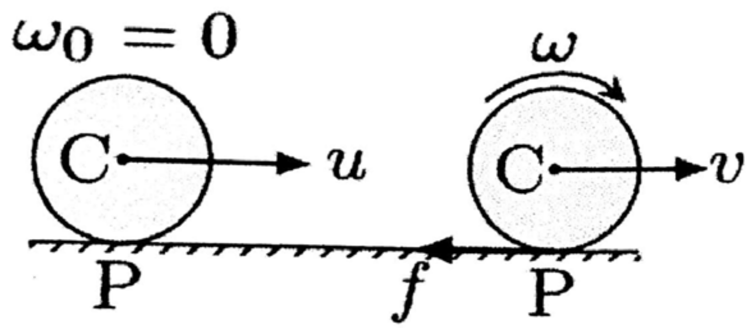


Figure- 24

Initial velocity of the contact point P, $\vec{u}_P = \vec{u}_C + \vec{\omega}_0 \times \vec{r}_{CP} = \vec{u}_C$ is same as that of the centre C. Thus, P moves towards right relative to the surface. To oppose this, frictional force at P acts towards left (sec figure). The friction force retards the velocity of C i.e., $v < u$. (clockwise). This torque gives clockwise angular acceleration $(\tau = I\alpha)$ and disc starts rotating clockwise. If the coefficient of friction is sufficiently large then retardation and angular acceleration continue till $u = \omega r$. At this instant, velocity of P relative to the surface becomes zero, making $f = 0$. After it, the disc continues to roll without slipping.

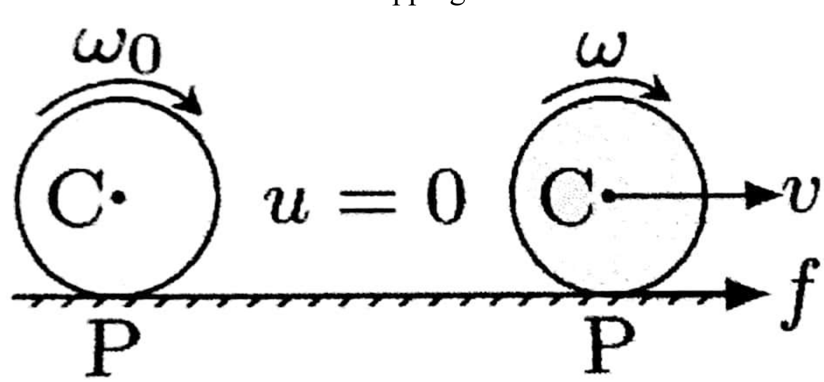


Figure- 25

Now, let a disc with non-zero angular velocity ω_0 and zero linear velocity *(u = 0)* be gently placed on a rough surface. Initial velocity of the contact point *P*, $\vec{u}_P = \vec{u}_C + \vec{\omega}_0 \times \vec{r}_{CP} = \omega_0 r$, is towards left relative to the surface. To oppose this, friction force at *P* acts towards the right (see figure). The friction force increases the velocity of *C*. The torque about *C* due to friction force is anti-clockwise. This torque gives anti-clockwise angular acceleration i.e., $\omega < \omega_0$. If the coefficient of friction is sufficiently large then acceleration and angular retardation continue till $v = \omega r$. At this instant, velocity of *P* relative to the surface becomes zero, making *f = 0*. After it, the disc continues to roll without slipping.

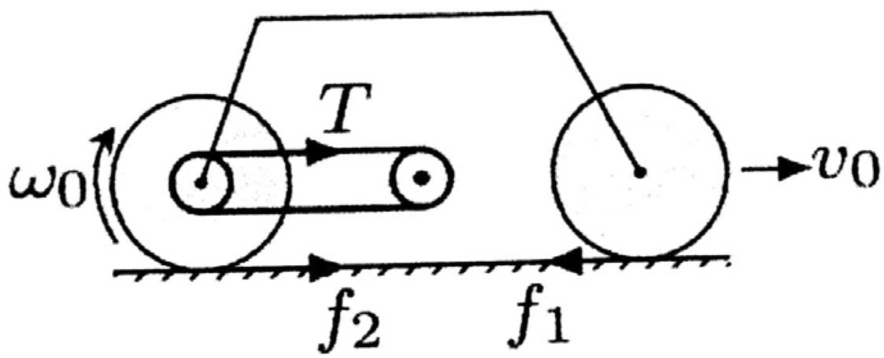


Figure- 26

Now, consider the bicycle. The front wheel is connected to rest of the bicycle by a rod passing through its centre (axle). The torque on the wheel about its centre by the force coming from the rest of the bicycle is zero. Thus, paddling can give linear velocity to the front wheel but cannot rotate it. The situation is similar to the first case discussed above and friction force acts in the backward direction. The situation of rear wheel is different. The rear wheel is connected to the rest of the bicycle by a rod passing through its centre and a chain connected to the paddles (see figure). Pressing the paddle increases the tension in the upper portion of the chain. This tension gives rise to a clockwise torque and wheel starts rotating in clockwise direction. Thus, situation of rear wheel is similar to the second case discussed above and friction force acts in the forward directio**Ans.A**

Self-Assessment Question (SAQ)

13. A constant torque of 500 N m turns a wheel of moment of inertia 100 kg m^2 about an axis through it centre. Find the gain in angular velocity in 2 s.

14. A flywheel of mass 25 kg has a radius of 0.2 m. It is making 240 r.p.m. What is the torque necessary to bring it to rest in 20 s? If the torque is due to a force applied tangentially on the rim of the flywheel, what is magnitude of the force? Assume that mass of the flywheel is concentrated at its rim.

3.8 ROTATIONAL KINETIC ENERGY

The rotating blade of a fan has some kinetic energy due to rotational motion which cannot be expressed directly as $K.E = \frac{1}{2}mv^2$ since all the points do not have same speed.

To find rotational K. E. we take the fan's blade as a collection of different very small particles called elements. One such elements has mass dm and is at distance r from the rotational axis as shown. Its kinetic energy can be given as

$$d(K.E.) = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(r\omega)^2$$

$$= \frac{1}{2}(dm)x^2\omega^2$$

.... (34)

The rotational kinetic energy of the body is given summing i.e. integrating the kinetic energy of all the elements of the body.

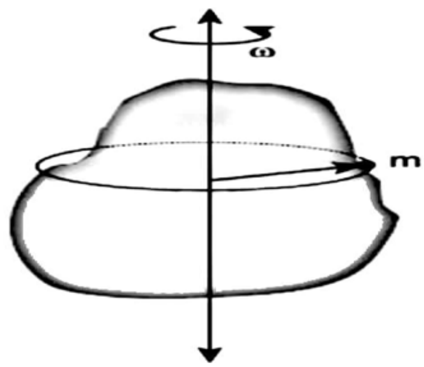


Figure -27

$$K.E. = \int d(K.E.) = \int \frac{1}{2}(dm)r^2\omega^2 \qquad ......(35)$$

Since ω is same for every element of a rigid body so we take ω^2 outside the integral.

$$\therefore\ K.E. = \frac{1}{2}\omega^2 \int r^2 dm \qquad ......(36)$$

We get,

$$\boldsymbol{K.E. = \frac{1}{2}I\omega^2} \qquad ......(37)$$

where $I = \int r^2 dm$

This is the expression for the rotational kinetic energy of a body.

The above equation (37) is analogous to the $K.E. = \frac{1}{2}mv^2$ i.e. Kinetic energy of boy having translational motion. Here ω is analogous to

v. Also, I is analogous to mass m i.e. I plays the same role in rotational motion as that of mass in translational motion. In other words, as the inertia to the translational motion is due to the mass, inertia to the rotational motion is due to the quantity Moment of Inertia.

3.8.1 Relation between Angular Momentum Rotational Kinetic Energy

Let a rigid body of moment of inertia I rotate with angular velocity ω. The angular momentum of a rigid body is, L = I ω. The rotational kinetic energy of the rigid body is, KE = 1/2 $I\omega^2$. By multiplying the numerator and denominator of the above equation with I, we get a relation between L and KE as,

$$KE = \frac{1}{2}\frac{I^2\omega^2}{I} = \frac{1}{2}\frac{(I\omega)^2}{I}$$

$$\therefore \quad KE = \frac{L^2}{2I} \qquad \text{...... (38)}$$

This is the relation between angular momentum and rotational kinetic energy.

Example-14

Find the kinetic energy of Earth related to its rotation about its axis. Given: Mass of Earth = 6 × 10^{24} kg, Radius of Earth = 6,400 km. Assume that the Earth is a uniform sphere.

Solution

We know that,

Angular speed of Earth is,

$$\omega = \frac{2\pi rad}{24h}$$

Rotational kinetic energy,

$$k = \frac{1}{2}I\omega^2$$

Now,

Angular speed $\omega = \dfrac{2\pi rad}{24\times 60\times 60s} = 7.3\times 10^{\;5} rad^{\;1}$

MI of Earth about its axis

$$I = \frac{2}{5}MR^2 = \frac{2}{5}\times 6\times 10^{24}\times (6400\times 10^3)^2$$

$$= 9.83\times 10^{37} kgm^2$$

$$\square k=\frac{1}{2}I\omega^2=\frac{1}{2}\times9.83\times10^{37}\times(7.3\times10^{\ 5})^2$$

$$=2.62\times10^{29}J$$

Self-Assessment Question (SAQ)

15. A particle performing uniform circular motion has angular momentum L. What will be its angular momentum, if its angular frequency is halved and kinetic energy is doubled?

16. Write an expression for the rotational kinetic energy of a body.

17. If angular momentum is conserved in a system, whose moment of inertia is decreased, will its rotational kinetic energy be also conserved? Explain.

3.9 ANGULAR IMPULSE – Momentum Theorem

In linear mechanics, we used Newton's second law to show that a net force acting on a system changed the momentum of the system.

$$\vec{F}_{net}=m\vec{a}$$

$$\vec{a}=\frac{\Delta\vec{v}}{\Delta t}$$

$$\vec{F}_{net}=m\frac{\Delta\vec{v}}{\Delta t}\Rightarrow\vec{F}_{net}\Delta t=m\Delta\vec{v}\Rightarrow\vec{F}_{net}\Delta t=\Delta\vec{p}$$

We can do the exact same thing in rotational mechanics by using Newton's second law for rotation. In this case a net torque acting on a system changes the angular momentum of the system. The product of net torque and time is angular impulse.

$$\vec{\tau}_{net}=I\vec{\alpha} \qquad \text{.........(39)}$$

$$\vec{\alpha}=\frac{\Delta\vec{\omega}}{\Delta t} \qquad \text{.........(40)}$$

$$\vec{\tau}_{net}=I\frac{\Delta\vec{\omega}}{\Delta t}\Rightarrow\vec{\tau}_{net}\Delta t=I\Delta\vec{\omega}\Rightarrow\vec{\tau}_{net}\Delta t=\Delta\vec{L}$$

$$\vec{\tau}_{net}\Delta t=\Delta\vec{L} \qquad \text{.........(41)}$$

The above equation (41) is analogous to impulse momentum theorem in translatory motion.

The L.H.S. term in the equation (41) i.e. $\vec{\tau}_{net}\Delta t$ is termed as angular impulse and R.H.S. term is the change in angular momentum.

Equation (41) is the relation between angular impulse of torque and change in angular momentum.

Example-15

A fan has moment of inertia 4 kg m^2 about its rotation axis. When switched on, its motor applies a constant torque of 10 Nm. Find the angular speed of the fan 4 s after it is switched on.

Solution

We know that,

$$\Delta L = \tau \Delta t$$

One can also find angular acceleration using $\tau = I\alpha$ and then use $\omega = \omega_0 + at$

Now,

$$\Delta L = \tau\ \Delta t\ \Rightarrow\ L_f - L_i = \tau \Delta t$$

$$\Rightarrow\ I\omega_F - 0 = 10 \times 4$$

$$\Rightarrow 4\omega_f = 40\ \ \Rightarrow\ \ \ \omega_f = 10\ rad\ s^{-1}$$

Self-Assessment Question (SAQ)

18. State the mathematical relationship between the Angular Impulse of a Torque and change in Angular Momentum produced by the impulse.
19. What is an Angular Impulse?

3.10 CONSERVATION OF ANGULAR MOMENTUM

Consider a system of n-particles. Suppose that the particles of the system are under the action of torques due to external forces acting on them. The internal forces between the particles do not contribute to the total torque on the system. If $\frac{d\vec{L}}{dt}$ is rate of change of angular momentum of the system, then the external torque acting on the system is given by

$$\vec{\tau} = (ext) = \frac{d\vec{L}}{dt}$$

In case external torque on the system is zero, then

$$\frac{d\vec{L}}{dt} = 0$$

Or $\vec{L}$ = a constant vector

If $\vec{L}_1, \vec{L}_2, \vec{L}_3, \ldots\ldots\ldots, \vec{L}_n$ are angular momenta of the different particles of the system about the axis of rotation then

$$\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_n = \text{ a constant vector}$$

It is the mathematical form of the law of conservation of angular momentum*, which states that if no external torque acts on a system, then the total angular momentum of the system always remains conserved.

Since angular momentum is a vector quantity; in absence of external torque, both its magnitude and direction must remain unchanged. For example, when a planet moves around the sun, the torque on the planet is due to the sun i.e. from within the system. As no external torque is acting on the planet, the constant magnitude of the angular momentum vector leads to the constant areal velocity of the planet (i.e. Kepler's second law of planetary motion) and the fixed direction of the angular momentum vector points out that the plane of the orbital motion of the planet around the sun must remain fixed.

It may be pointed out that if external forces do not act on the system, then both the linear momentum and the angular momentum of the system remain constant.

3.10.1 Application of Angular Momentum

The following are the Angular Momentum

1. **Deduction of Kepler's second law of planetary motion.** Consider a planet moving around the sun in elliptical orbit. Let $\vec{r}$ be the position vector of the planet w.r.t. the sun and $\vec{F}$ be gravitational force on the planet due to the sun. Then, torque on the planet due to the force exerted by the sun is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The force on the planet always acts along the line joining the centres of the planet and the sun and is directed towards the sun. As a result, the vectors $\vec{r}$ and $\vec{F}$ are parallel vectors and consequently $\vec{r} \times \vec{F}$ and hence$\vec{\tau} = 0$.

Therefore, for the planet moving under the gravitational force of the sun $(\vec{\tau} = 0)$,

$$\frac{d\vec{L}}{dt} = 0 \left(\because \vec{\tau} = \frac{d\vec{L}}{dt}\right)$$

or $\vec{L}$ = a constant vector

As proved in section 2.11, the angular momentum of a planet is also given by

$$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t},$$

where m is mass of the planet and $\frac{\Delta \vec{A}}{\Delta t}$ is its areal velocity. Since angular momentum of the planet is a constant vector.

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{\vec{L}}{2m} = \text{ a constant vector}$$

Therefore, areal velocity of the planet always remains constant. It is exactly what Kepler predicted about planetary motion in 1602, i.e. the line joining the planet to the sun sweeps out equal areas in equal intervals of time. It is known as the **Kepler's second law of planetary motion.**

2. **The orbit of the planet always lies in a fixed plane.** As discussed above, when a planet moves in its elliptical orbit, its angular momentum vector always remains constant i.e.

$$\vec{L} = \vec{r} \times m\vec{v} = \text{a constant vector}$$

Since the vector $\vec{L}$ is perpendicular to the plane containing vectors $\vec{r}$ and m $\vec{v}$, the direction of angular momentum vector will remain unchanged, only if the plane containing the vectors $\vec{r}$ and m $\vec{v}$ remains fixed i.e. the orbit of the planet remains in a fixed plane.

3. **Variation of linear speed of planet in its orbit.** Due to the effect of the gravitational force of the sun, a planet moves along elliptical orbit around the sun. Since torque on the planet due to the zero, the angular momentum of planet is a constant of motion i.e.

$$m\ v\ r = \text{constant}$$

Now, the distance of the planet from the sun continuously varies along its elliptical path. Therefore, it can hold only, if the speed of the planet also varies accordingly. Therefore, the linear speed of the planet is not constant in its orbit.

3.11 MOMENT OF INERTIA

❖ Moment of inertia plays the same role in rotational motion as mass plays in linear motion. It is the property of a body due to which it opposes any change in state of rest or of uniform rotation.

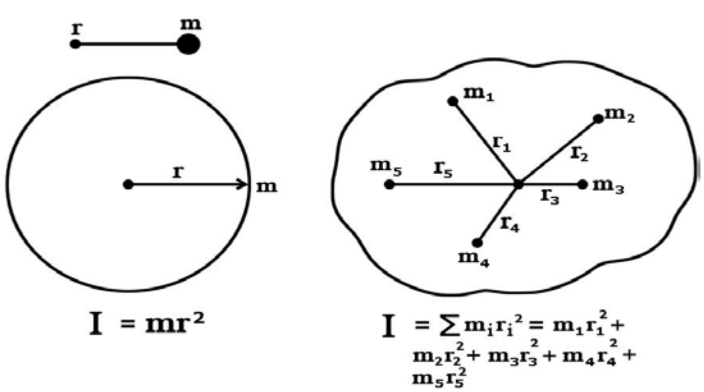


Figure - 28

- Moment of inertia of a particle I = mr^2; where r is the perpendicular distance of particle from rotational axis.
- Moment of inertia of a body made up of number of particle (discrete distribution)
  - $I = m_1 r_1^2 + m_2 r_3^2 + \cdots$
- Moment of inertia of a continuous distribution of mass, treating the element of mass dm at position r as particle
  - dI = dm r^2 i.e., $I = \int r^2 \, dm$
- Dimension: $[ML^2T^0]$
- S. I. unit: kgm^2.
- Moment of inertia depends on mass, distribution of mass and on the position of axis of rotation.
- Moment of inertia does not depend on angular velocity, angular acceleration, torque angular momentum and rotational kinetic energy.
- It is not a vector as direction (clockwise or anti-clockwise) is not to be specified and also not a scalar as it has different values in different directions. Actually, it is a tensor quantity.
- In case of a hollow and solid body of same mass, radius and shape for a given axis, moment of inertia of hollow body is greater than for the solid body because it depends upon the mass distribution.

3.11.1 Physical Significance of Moment of Inertia

Physical Significance. Moment of inertia plays the same role in rotatory motion as mass does in linear motion, i.e., moment of inertia is an analogue of mass in linear motion.

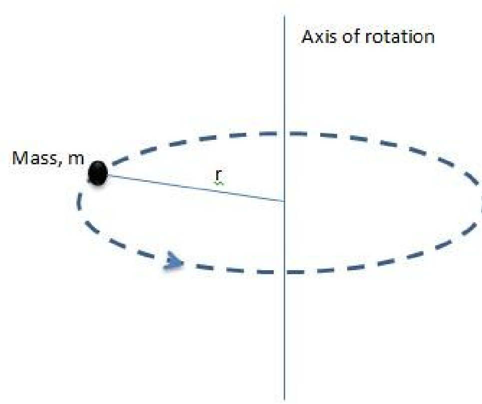


Figure - 29

According to Newton's first law of motion, a body continues in its state of rest or of uniform motion in a straight line unless some external force acts upon it. This property of matter is known as **inertia.** A body always resists an external force tending to change its state of rest or of linear motion. Greater the mass of the body greater is the force required to produce a given linear acceleration.

Similarly, bodies possess rotational inertia, i.e., a body free to rotate about an axis opposes any change in its state of rest or of rotation. Greater the moment of inertia of a body greater is the couple required to produce a given angular acceleration.

The moment of inertia depends not only on the mass of a body but also on the distribution of mass about the axis of rotation.

3.11.2 Moment of Inertia of a System of Particles

Let a system of a particles, having masses m_1, m_2, …., m_i, …. m_n which are situated at distances r_1, r_2, …., r_i, …… r_n respectively, from the axis of rotation, as shown in Figure 30. As the moment of inertia of a particle is a scalar quantity, the moment of inertia of the system of particles will be given by the sum of the moments of inertia of all the particles of the system. If I is the moment of inertia of the system of particles, then

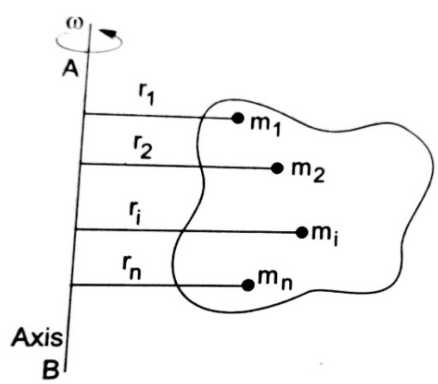


Figure - 30

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_i r_i^2 + \cdots + m_n r_n^2$$

$$\Rightarrow \qquad I = \sum_{i=1}^{n} m_i r_i^2.$$
…………….(42)

3.11.3 Expression for Moment of Inertia of Certain Regular Bodies

1. **Moment of Inertia of a Circular Ring**

   (a) Moment of inertia about an axis through its centre and perpendicular to its plane.

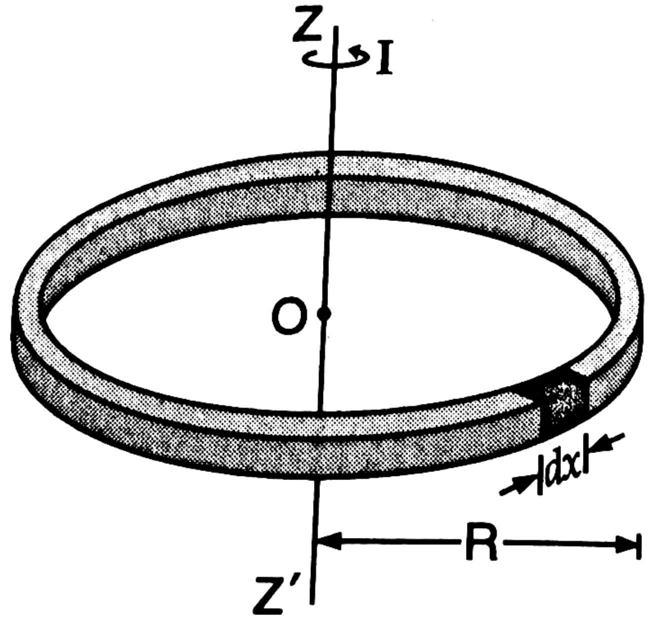


Figure - 31

$$I = MR^2 \qquad \text{................(43)}$$

(b) Moment of inertia about its diameter

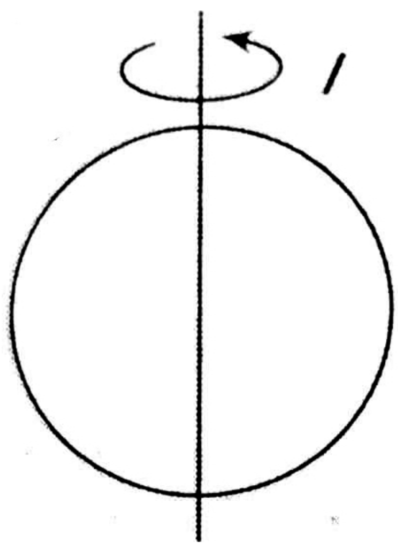


Figure - 32

$$I_d = \frac{1}{2}MR^2 \qquad \text{................(44)}$$

2. Moment of inertia of a Circular Disc

(a) Moment of inertia about an axis through its centre and perpendicular to its plane.

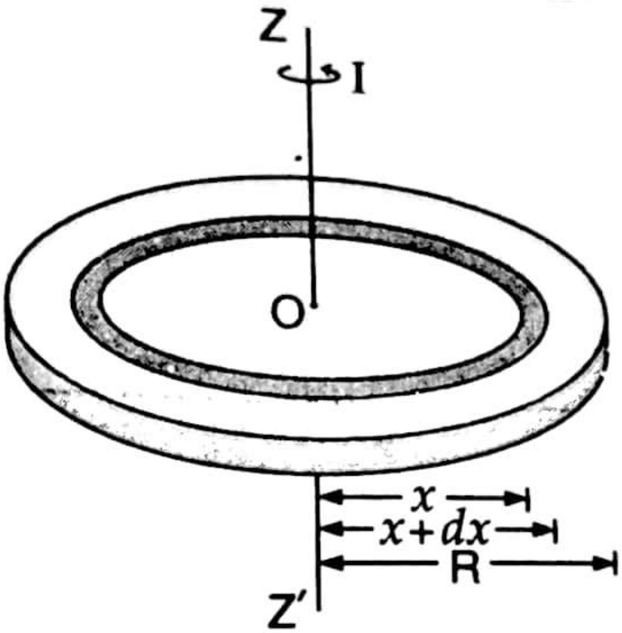


Figure - 33

$$I = \frac{1}{2}MR^2$$ …………….(45)

(b) Moment of inertia about its diameter.

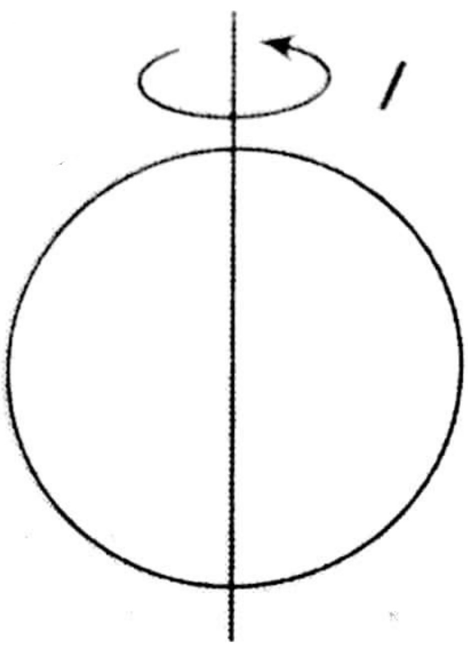


Figure - 34

$$I_d = \frac{1}{4}MR^2$$ …………….(46)

3. Moment of inertia of a Thin Rod

(a) When the axis is perpendicular to the length of the rod and passes through one of its ends.

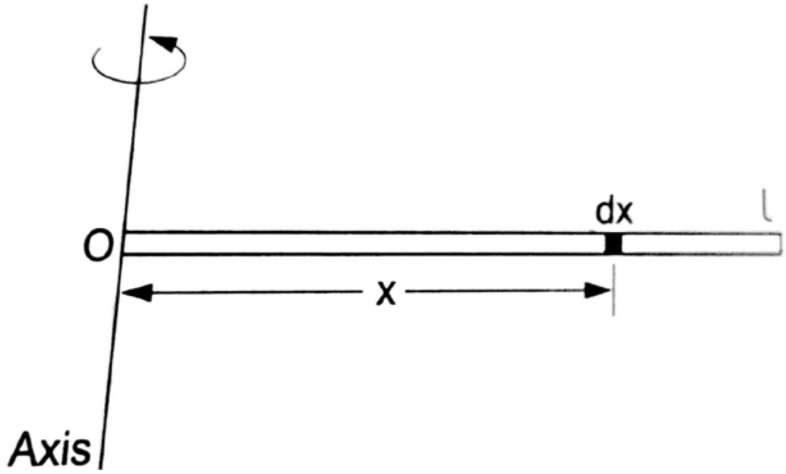


Figure - 35

$$I = \frac{ML^2}{3}$$

……………… (47)

(b) When the axis is perpendicular to the length of the uniform rod and passes through its centre.

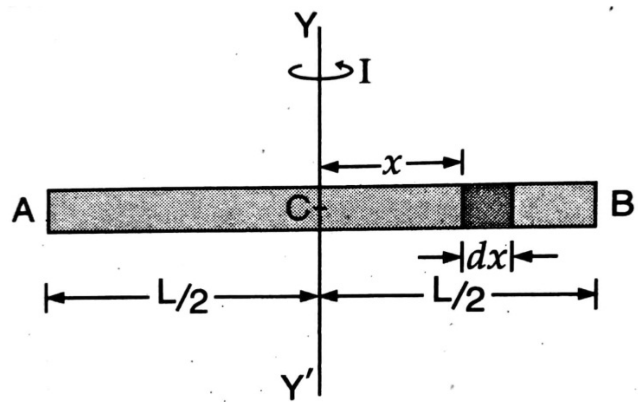


Figure - 36

$$I = \frac{ML^2}{12}$$

……………. (48)

4. **Moment of inertia of a Thin Rectangular Plates**

(a) When the axis lies in the plane of the plate, is perpendicular to its length and passes through its centre.

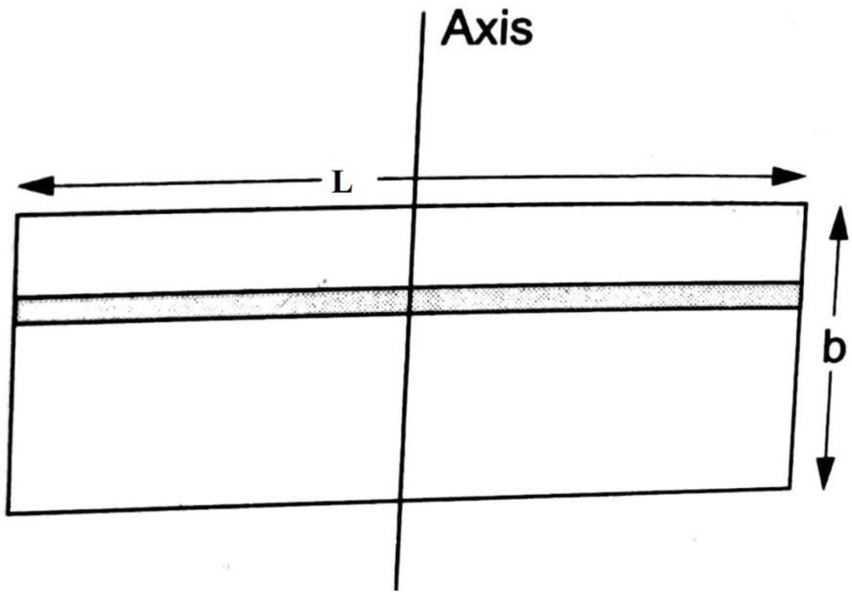


Figure - 37

$$I_y = \frac{ML^2}{12} \qquad \text{.......... (49)}$$

(b) When the axis lies in the plane of the plate, is perpendicular to its breath and passes through its centre.

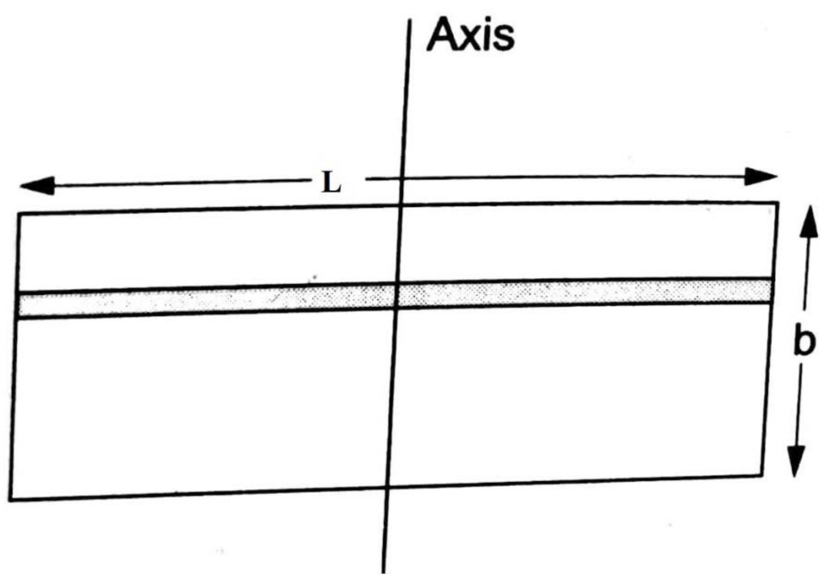


Figure - 38

$$I_x = \frac{ML^2}{12} \qquad \text{.......... (50)}$$

(c) When the axis is perpendicular to the plane of the plate and passes through its centre.

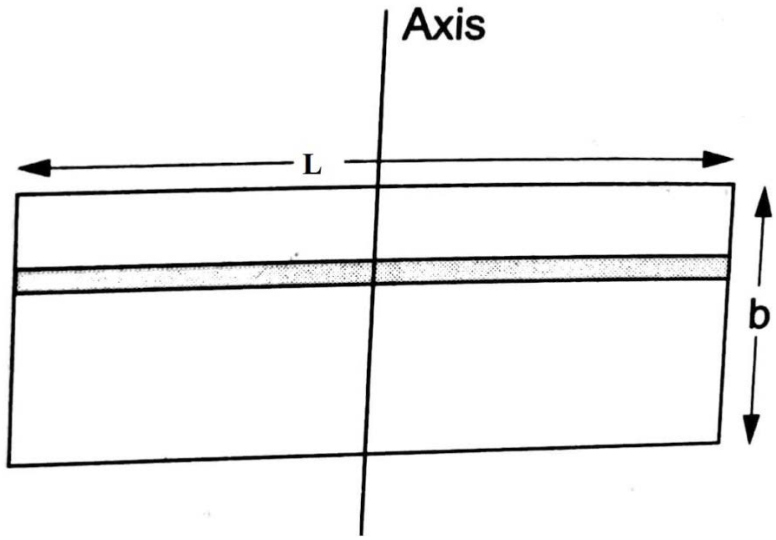


Figure - 39

$$I_z = \frac{M}{12}(L^2 + b^2)$$
………. (51)

5. (a) Moment of Inertia Hollow Cylinder

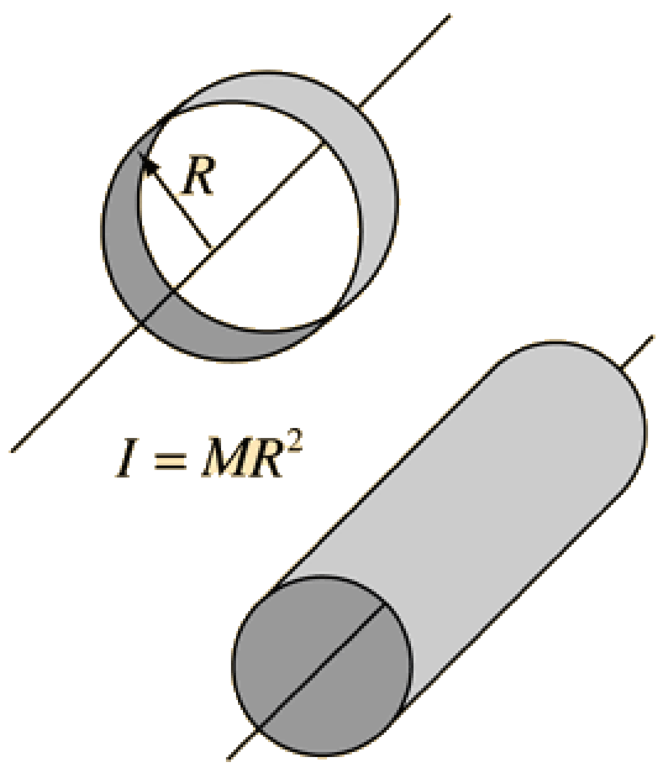


Figure - 40

$$I = Mr^2 \qquad \text{………. (52)}$$

(b) Moment of Inertia Solid Cylinder

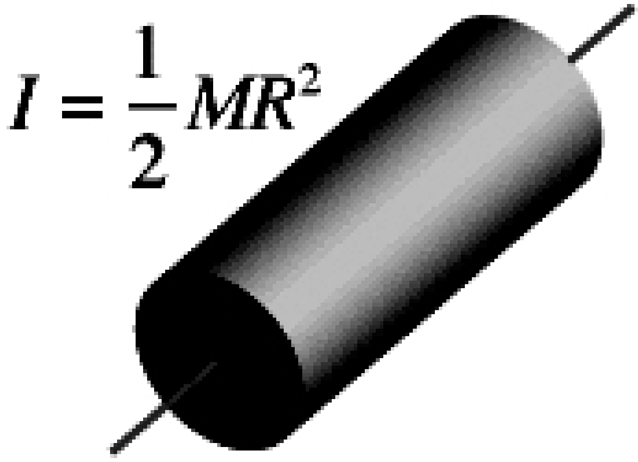


Figure - 41

$$I = \frac{Mr^2}{2} \qquad \text{.......... (53)}$$

6. (a) Moment of Inertia Hollow Sphere

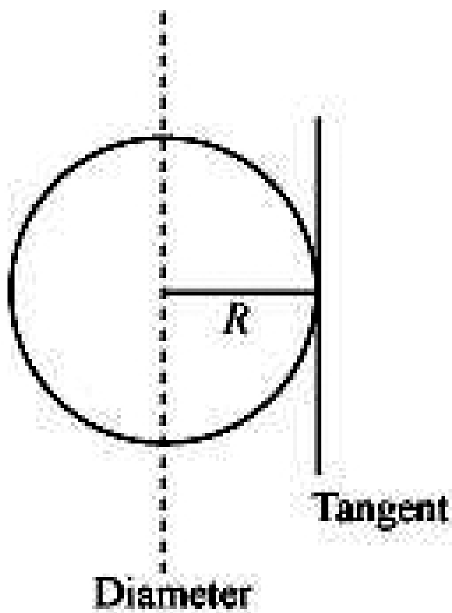


Figure - 42

$$I = \frac{2Mr^2}{3} \qquad \text{.......... (54)}$$

(b) Moment of Inertia Solid Sphere

Figure - 43

$$I = \frac{2Mr^2}{5} \qquad \text{………. (55)}$$

3.11.4 Radius of Gyration

Radius of gyration of a body about a given axis is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

When square of radius of gyration is multiplied with the mass of the body gives the moment of inertia of the body about the given axis.

$$I = Mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{M}}$$

………….. (56)

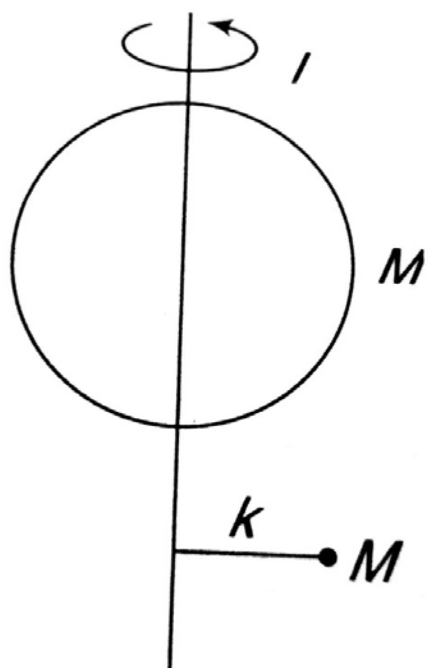


Figure-44

❖

Here, k is called radius of gyration.

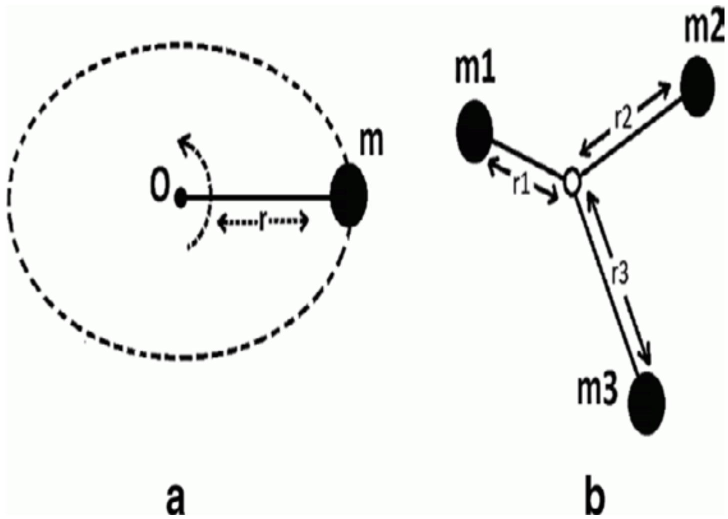


Figure-45

From the formula of discrete distribution

$I = mr_1^2 + mr_2^2 + mr_3^2 + \dots\dots\dots\dots + mr_n^2$

If $m_1 = m_2 = \dots\dots\dots\dots\dots\dots = m$ then

$I = m(r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots\dots\dots r_n^2)$
………….. (57)

From the definition of Radius of gyration,

$I = Mk^2$
………….. (58)

By equating (56) and (57)

$Mk^2 = m(r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots\dots\dots\dots + r_n^2)$

$nmk^2 = m(r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots\dots\dots\dots + r_n^2)$

$\therefore \quad k = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r^2 n}{n}}$

Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

❖ Radius of gyration (k) depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body w.r.t. the axis of rotation.

❖ Radius of gyration (k) does not depend on the mass of body.

❖ Dimension $[M^0L^1T^0]$.

❖ S.I. unit: Meter.

❖ Significance of radius of gyration: Through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion.

Example: In case of a disc rotating about an axis through its centre of mass and perpendicular to its plane

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{(1/2MR^2}{M}} = \frac{R}{\sqrt{2}}$$

So instead of disc we can assume a point mass M at a distance $(R/\sqrt{2})$ from the axis of rotation for dealing the rotational motion of the disc.

Example-16

Find radius of gyration of a uniform solid sphere about it diameter.

Solution:

We know that, radius of gyration of $MK^2 = I$

$$\therefore \quad MK^2 = \frac{2}{5}MK^2 \quad \Rightarrow \quad K = \sqrt{\frac{2}{5}}.R$$

A higher value of implies that the mass is effectively at a larger distance from the axis.

Example-17

Calculate moment of inertia of earth about its diameter, taking it to be a sphere of radius 6400 km and mass 6 × 10^{24} kg.

Solution:

Here, M = 6 × 10^{24} kg

R = 6400 km = 6.4 × 10^6 m

As $I = \frac{2}{5}MR^2$

$$\therefore \quad I = \frac{2}{5} \times 6 \times 10^{24}(6.4 \times 10^6)^2$$

= 9.83 × 10^{37} kg m^2

Example-18

If angular momentum is conserved in a system whose moment of inertia is decreased, will its rotational kinetic energy be also conserved? Explain.

Solution:

Here, $L = I\omega =$ constant

K.E. of rotation, $K = \frac{1}{2} I \omega^2$

$$K = \frac{1}{2I} I^2 \omega^2 = \frac{L^2}{2I}$$

As L is constant, $K \propto 1/I$

When moment of inertia (I) decreases, K.E. of rotation (K) increases. Thus K.E. of rotation is not conserved.

Example-19

If earth were to shrink suddenly, what would happen to the length of the day?

Solution:

If earth were to shrink suddenly, its radius R would decrease. The moment of inertia of earth $I = \frac{2}{5} MR^2$ would decrease. As no external torque is acting on earth, its angular momentum $L = I\omega = I\frac{2\pi}{T}$ remains constant. As I decreases, T must decrease. Hence the length of the day will decrease.

Self-Assessment Question (SAQ)

20. Find the moment of inertia of a rod of length 0.5 m and mass 0.2 kg about an axis (a) passing through its centre and (b) through one end of the rod, the axis being perpendicular to its length in both the cases.

21. Calculate the moment of inertia of a ring of mass 2 kg and radius 50 cm about an axis passing through it centre and perpendicular to its plane.

22. Fill in the blanks.

    (a) Moment of inertia is a ……………. Quantity.

    (b) The SI unit of moment of inertia is …………….

    (c) The dimensional formula for moment of inertia is …………….

    (d) The radius of gyration is a ……………… quantity.

    (e) The SI unit of radius of gyration is …………

    (f) The moment of inertia of a thin, uniform rod of mass M and length L about an axis passing through its centre and perpendicular to its length is ……….

3.12 THEOREM OF PARALLEL AXES

This theorem states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis passing through its centre of mass plus the product of the mass of the body and the square of the distance between the two parallel axes.

If I_c is the moment of inertia of the body about an axis passing through its centre of mass and I is moment of inertia about a parallel axis at a distance from it, then according to this theorem,

$$I = I_c + Ma^2 \qquad \text{.....................} (59)$$

where M is mass of the body.

Proof Let AB be an axis through the centre of mass c of the body and A' B' be an axis parallel to AB and at a distance a from it, as shown in Figure 46.

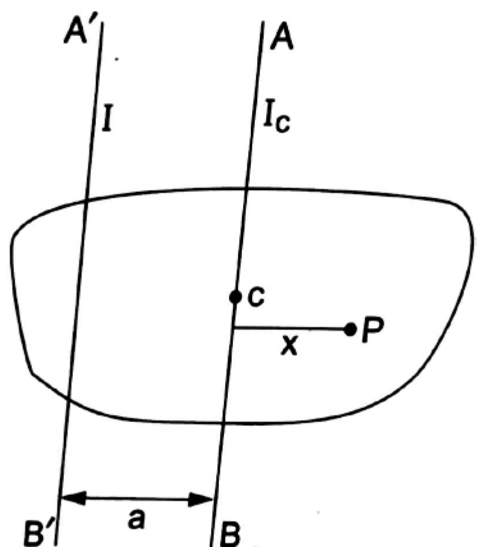


Figure - 46

Consider a particle of mass m at the point P, at a distance x from the axis AB. Then, the moment of inertia of this particle about the axis AB is mx^2 and its moment of inertia about the parallel axis A' B' is $m(x + a)^2$. The moment of inertia of the whole lamina about the axis A' B' is

$$I = \sum m(x+a)^2 \qquad \text{.....................} (60)$$

$$\Rightarrow \qquad I = \sum m(x^2 + a^2 + 2ax) = \sum mx^2 + \sum ma^2 + 2a\sum mx$$

$$\Rightarrow \qquad I = I_c + Ma^2 + 2a\sum mx. \qquad \text{.....................} (61)$$

Now, $\sum mx = 0$, as the body is balanced about its centre of mass, the algebraic sum of the moments of the weights (mg) of all the particles about an axis passing through its centre of mass must be zero. Equ. 61.

$$\boldsymbol{I = I_c + Ma^2} \qquad \text{.....................} (62)$$

3.13 THEOREM OF PERPENDICULAR AXES

This theorem states that the moment of inertia of a plane lamina about an axis OZ perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about two mutually perpendicular axes OX and OY in its plane, O being the point of intersection of the three axes.

If I_x, I_y and I_z are the moments of inertia of the lamina lying in the XY plane about OX, OY and OZ axes respectively, then according to this theorem,

$I_x, = I_y + I_z$ .... (63)

Proof Let OX, OY and OZ be three mutually perpendicular axes, OX and OY being in the plane of the lamina and OZ, perpendicular to it, as shown in Figure 47.

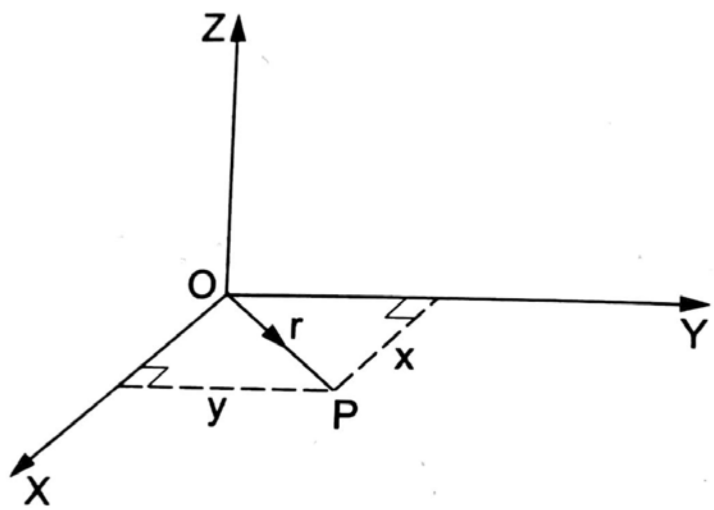


Figure - 47

Consider a particle of mass m at point P in the plane of the lamina, the distance OP being r. As is obvious from the figure, the moment of inertia of the particle about the x-axis and y-axis are my^2 and mx^2 respectively. The moments of inertia of the whole lamina about the x-axis and y-axis are, therefore,

$I_x = \sum my^2 \ and \ I_y = \sum mx^2$ .... (64)

The moment of inertia of the lamina about the z-axis is

$I_z = \sum mx^2.$ .... (65)

Now, $I_x + I_y = \sum mr^2 + \sum mx^2 = \sum m(x^2 + y^2) = \sum mr^2$

$\Rightarrow \quad \boldsymbol{I_x + I_y = I_z}$ .... (66)

Example-20

The moment of inertia of a uniform circular disc about a tangent of the disc in its own plane is given by $\frac{5}{4}MR^2$ where the symbols have their usual meanings. Using this relation, find its M.I. about an axis through its centre and perpendicular to the plane.

Solution:

If I_d is M.I. of the disc about its diameter, then from the theorem of parallel axes, we have

$$I_{tan} = I_d + M\,R^2$$

or $$I_d = I_{tan} - M\,R^2 = \frac{5}{4}M\,R^2 - M\,R^2 = \frac{1}{4}M\,R^2$$

From the theorem of perpendicular axes, M.I. of the disc about an axis through its centre and perpendicular to its plane is given by

$$I = I_d + I_d = 2I_d = 2 \times \frac{1}{4}M\,R^2 = \frac{1}{2}\boldsymbol{M\,R^2}$$

Example-21

A uniform thin rod has mass M and length L. Find its MI about an axis that is perpendicular to the rod and passes through it COM.

Solution:

We will use parallel Axis theorem, for solving:

We know that, Moment of Inertia about an axis through one end and perpendicular to the rod is

$$I = \frac{ML^2}{3}$$

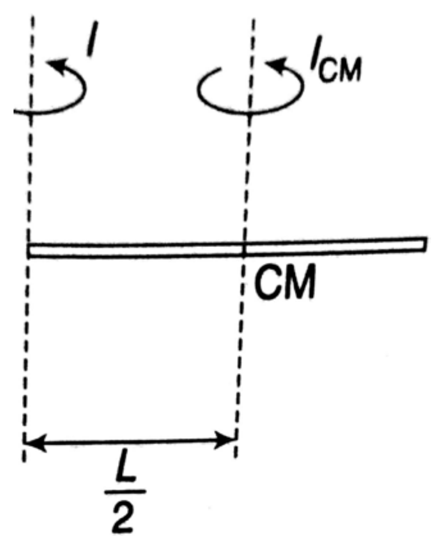


Figure - 48

Other axis is parallel to this axis and passes through COM.

∴ $I = I_{CM} + Md^2$ gives

$$\frac{ML^2}{3} = I_{CM} + M\left(\frac{L}{2}\right)^2$$

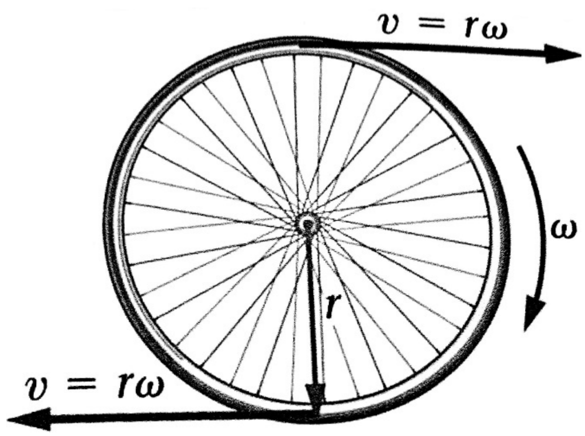
⇒ $$I_{CM} = \frac{ML^2}{3} - \frac{ML^2}{4} = \frac{ML^2}{12}$$

Self Assessment Question (SAQ)

23. State and prove the theorem of parallel axes.
24. State and prove the theorem of perpendicular axes.

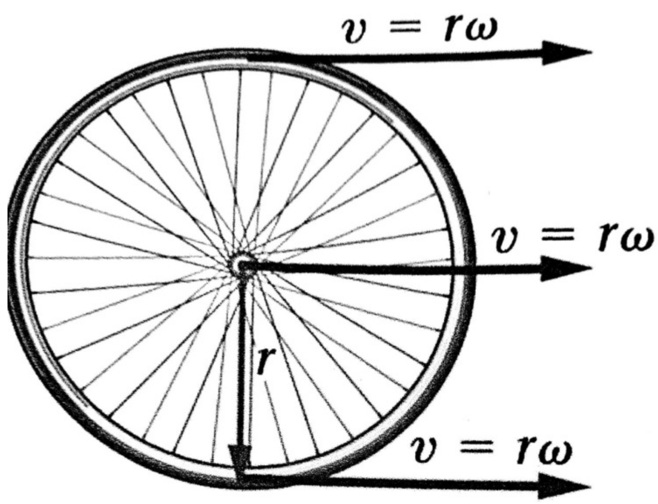
3.14 WHAT IS ROLLING MOTION?

A motion that is a combination of rotational and translational motion is called Rolling Motion e.g. a wheel rolling down that road.



(Rotational Motion)

Figure – 49



(Translational Motion)
Figure – 50
(Rotational Motion + Translational Motion = Rolling Motion)

3.14.1 Types of Rolling Motion

Rolling Motion is classified mainly two types:

(a) Pure Rolling or Rolling without slipping/sliding or perfect rolling motion.

(b) Impure rolling or rolling with slipping sliding or imperfect rolling motion.

Rolling motion of a body can be analyzed by considering it as a superposition of translational motion of center of mass of body plus rotational motion of body about an axis passing through center of mass of body.

(a) Pure Rolling Motion : If the relative velocity of the contact (between body & platform) is zero then the rolling motion is said to be pure rolling motion.

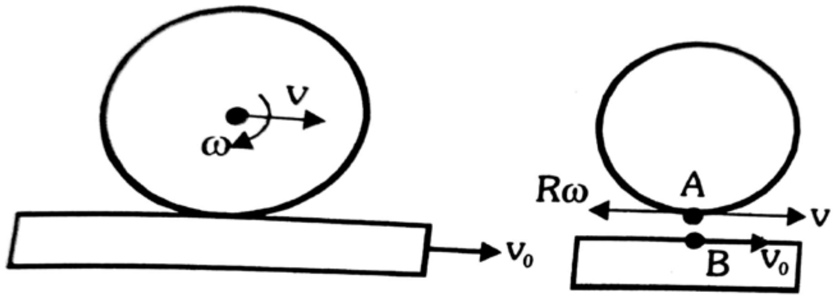


Figure - 51

(b) Impure Rolling Motion : In impure rolling motion, the point of contact of the body with the platform is not relatively at rest with respect to platform on which it is performing rolling motion, as a result sliding. For impure rolling motion, $v_{AB} \neq 0$ $i.e. v - R\omega \neq v_0$

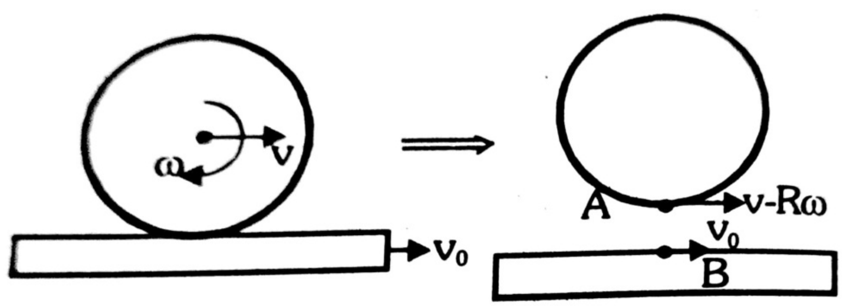


Figure - 52

If platform is stationary i.e. $v_0 = 0$ then condition for impure rolling motion is $v \neq R\omega$.

For impure rolling, as $v \neq R\omega, so\ a \neq R\alpha$.

3.14.2 Kinetic Energy of a Rolling Body

When a body rolls on a surface, it has two types of motion – translational motion and rotational motion. Accordingly, it has two types of kinetic energy – translational kinetic energy and rotational kinetic energy. If I is the moment of inertia of the body about the axis through its centre of mass and parallel to the surface and, ω, its angular velocity, then

rotational kinetic energy $= \frac{1}{2} I_{CM}\omega^2$. ………….. (67)

If m is the mass and v_{CM} be the linear velocity of the centre of mass of body, then

translational kinetic energy $= \frac{1}{2} mv_{CM}^2$. ………….. (68)

Hence, the total kinetic energy E of the rolling body is equal to the sum of its translational kinetic energy and rotational kinetic energy. That is,

$$E = \frac{1}{2} mv_{CM}^2 + \frac{1}{2} I_{CM}\omega^2 = \frac{1}{2} mv_{CM}^2 + \frac{1}{2} . mK^2 . \frac{v_{CM}^2}{r^2}$$

$$\Rightarrow \boldsymbol{E = \frac{1}{2} mv_{CM}^2 \left(1 + \frac{K^2}{r^2}\right)} \qquad \text{………….. (69)}$$

This is the expression for the kinetic energy of rolling body.

Therefore,

As $\left(1 + \frac{K^2}{r^2}\right) > 1$, it is clear from Equation (69) that the kinetic energy of a body when it is rolling is greater than when it simply slides with the velocity v_{CM}.

3.14.3 Condition for Rolling Without Slipping

Consider a circular/spherical body of radius R rolling on a horizontal surface, as shown in Figure 10.31. Let the velocity of its centre of mass be v_{CM}. If ω is its angular velocity of rotation, then the linear velocity of any point on its circumference due to rotation will be $v = \omega R$, its direction being different at different points on its surface. At the topmost point A, its direction will be parallel to $\vec{v}_{CM}$, whereas at the lowest point, its direction will be opposite to $\vec{v}_{CM}$.

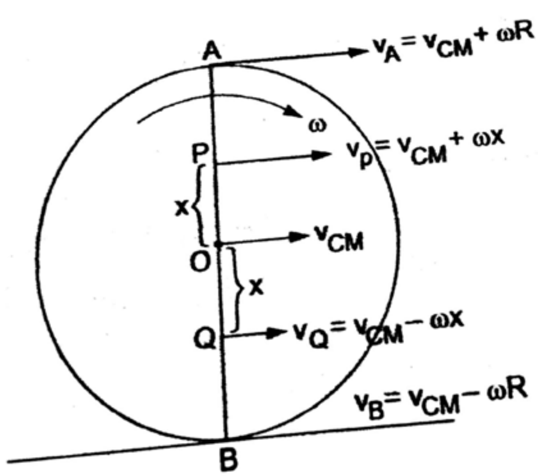


Figure - 53

Therefore, the linear velocity of the topmost point A is

$$v_A = v_{CM} + \omega R,$$

and that of the lowest point B is

$$v_B = v_{CM} - \omega R.$$

If the body rolls without slipping, then its velocity at the lowest point, which is in contact with the horizontal surface, must be zero. That is, for rolling without slipping,

$$v_B = 0$$

$$\Rightarrow \quad v_{CM} = \omega R. \qquad \text{………….. (70)}$$

The relation $v_{CM} = \omega R$ expressed by Equation 10.86 is the condition for pure rolling motion.

Join the diameter AB and on this vertical diameter, take two points P and Q at distances $\pm x$ from the centre of mass O. Then, at point P, v_{CM} and ωx are in the same direction. Hence, the instantaneous velocity of the point P is

$$v_P = v_{CM} + \omega x = v_{CM} + \frac{v_{CM}}{R}.x$$

$$\Rightarrow \quad v_P = v_{CM}\left(1 + \frac{x}{R}\right) \qquad \text{………….. (71)}$$

Again, at the point Q, v_{CM} and ωx are in the opposite directions. Hence, the instantaneous velocity at the point Q is

$$v_Q = v_{CM} - \omega x = v_{CM} + \frac{v_{CM}}{R}.x$$

$$\Rightarrow \quad v_Q = v_{CM}\left(1 - \frac{x}{R}\right) \qquad \text{………….. (72)}$$

Obviously, velocity at the highest point A is

$$v_A = v_{CM}\left(1 + \frac{R}{R}\right) = 2v_{CM} \qquad \text{………….. (73)}$$

and that at the lowest point B is

$$v_B = v_{CM}\left(1 - \frac{R}{R}\right) = 0 \qquad \text{………….. (74)}$$

The velocities at different point of the vertical diameter are shown in Figure 10.31 in magnitude and direction. If the motion of the body on a rough horizontal surface is not pure rolling, then the velocity at the point B of the body with respect to the horizontal surface will not be zero and work will be done by the frictional forces.

We now consider the following two cases.

1. $\boldsymbol{v_{CM} > \omega R}$: In this case, point B in contact with the horizontal surface will have a tendency to move forward and hence, a frictional force will act in the backward direction. This frictional

force will decrease the value of v_{CM} and will produce a torque in the clockwise direction, thus increasing the value of ω. At a certain instant, when v_{CM} becomes equal to ωR, the relative velocity of the point B in contact with the horizontal surface becomes zero and pure rolling results.

2. $\boldsymbol{v_{CM} < \omega R}$: In this case, point B in contact with the horizontal surface will have a tendency to move backward (towards left) and the frictional force will act towards right in Figure 53. The frictional force will increase the value of v_{CM} and will produce a torque in the anticlockwise direction, thus decreasing the value of ω. At a certain instant, when v_{CM} becomes equal to ωR, the relative velocity of the point B in contact with the horizontal surface becomes zero and pure rolling results.

In both the cases, therefore, pure rolling results after some time, whatever be the initial conditions.

As discussed above, in pure rolling motion there is no sliding of the rigid body over the rough surface, hence no work is done by the frictional force. This is strictly true only when the rolling bodies and the surface over which they roll are perfectly rigid and no deformation occurs.

3.14.4 Motion of a Rolling Body on an Inclined Plane

Consider a body (hollow sphere, solid sphere, ring, disc, hollow cylinder or solid cylinder) of mass M, radius R, and moment of inertia I about the axis of rotation, rolling down an inclined plane of inclination θ with the horizontal, as shown in Figure 54. The forces acting on the rolling body are

1. its weight Mg, vertically downward,
2. normal reaction N, by the inclined plane in the direction shown, and
3. frictional force f, opposite to the direction of motion of the body (that is, up the plane).

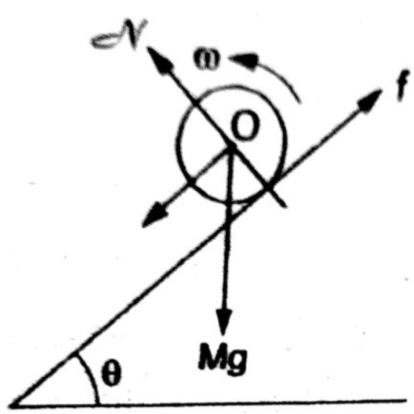


Figure - 54

The line of action of the weight Mg of the body and the normal reaction N, both pass through the centre *O* of the rolling body. Hence, the moments of the forces Mg and N about the centre *O* of the body is zero. These to not play any role in increasing the angular velocity ω of the rolling body. The only force that may increase the angular velocity of the rolling body is the frictional force f whose line of action does not pass through the centre *O*. Its moment about the point *O* is f. R in anticlockwise direction and this increases the angular velocity of the rolling body.

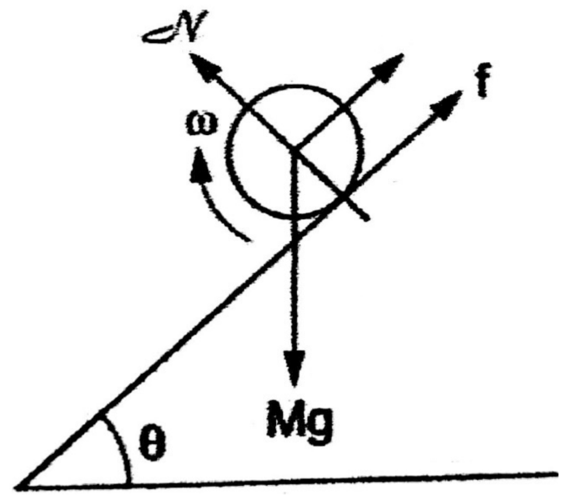


Figure - 55

If the body rolls up the plane, as shown in Figure 55, then to decrease its angular velocity, the frictional force f on it at the point of contact with the inclined plane will act along the plane and in the upward direction. The moment of this force f, or torque, on the body in this case will be f. R in the anticlockwise direction which will slow down the angular velocity of the rolling body. Note that in either case, the frictional force acts up the plane.

3.15 ACCELERATION OF A BODY ROLLING DOWN AN INCLINED PLANE

Consider a body of mass *M* and radius *R* rolling down a rough inclined plane, having an inclination θ with the horizontal. Let the body start from point *A* as shown in Figure 56. As the body starts rolling down the plane, it gains kinetic energy, but loses potential energy. Consider a point *B* at a distance *s* away from the point *A*. If *v* is the linear velocity of the centre of mass of the body and, *K*, its radius of gyration, then

gain in kinetic energy of the body $= \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$,

and loss in its potential energy $= Mgs\ sin\theta$, where $s\ sin\theta$ is the height through which it falls in moving from point A to point B. Applying the principle of conservation of energy, we get

gain in kinetic energy = loss in potential energy

$$\Rightarrow \quad \frac{1}{2}M\,v^2\left(1+\frac{K^2}{R^2}\right) = Mgs\,sin\theta$$

$$\Rightarrow \quad v^2 = \frac{2gs\,sin\theta}{\left(1+\frac{K^2}{R^2}\right)} \qquad \text{……….. (75)}$$

As the body starts from rest point A, its velocity is given by

$$v^2 = u^2 + 2as = 2as \qquad \text{……….. (76)}$$

where a is the linear acceleration of the body down the plane.

Using equations 75 and 76, we get

$$2as = \frac{2gs\,sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

$$\Rightarrow \quad \boldsymbol{a} = \frac{\boldsymbol{g\,sin\theta}}{\left(\boldsymbol{1+\frac{K^2}{R^2}}\right)} \qquad \text{……….. (77)}$$

This is an expression for acceleration of a body rolling down an inclined plane.

Example-22

A solid sphere starting from rest rolls 5.3 as without slipping along a smooth inclined plane 1 m in length, the angle of inclination being $\boldsymbol{\theta = 0.573^o}$. What is the acceleration due to gravity at that place?

Solution:

If a is the acceleration down the plane, then

$$a = \frac{g\,sin\theta}{1+\frac{K^2}{R^2}} = \frac{g\,sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\,sin\theta. \qquad \text{(i)}$$

With this acceleration the body covers a distance of $s = 1\ m$ in time $t = 5.3\ s$.

Hence, using

$$s = ut + \frac{1}{2}at^2$$

we get $$1 = 0 + \frac{1}{2}\times a \times (5.3\ s)^2$$

$$\Rightarrow \quad a = 0.07\ m\,s^{-2}. \qquad \text{(ii)}$$

Using equations (i) and (ii), we get

$$\frac{5}{7} g\ sin\theta = 0.07\ m\ s^{-2}.$$

Solving this, we get $g = 9.8 m\ s^{-2}$.

Self-Assessment Questions (SAQs)

25. Calculate the kinetic energy of rolling ring of mass 0.2 kg about an axis passing through its center of mass and perpendicular to it, if is center of mass is moving with a velocity of 3 m/s.

26. A solid sphere is rolling down an inclined plane without slipping of height 20 m. Calculate the maximum velocity with which it will reach the bottom of the plane. ($g = 10\ m/s^2$)

3.16 SUMMARY

In the present unit, we have studied about kinematics of rotational motion, angular velocity, angular displacement, angular acceleration, torque, moment of inertia, angular impulse, rotational kinetic energy and radius of gyration etc.

Angular Acceleration: The rate of change of angular velocity.

Angular Displacement: Rotational analog of change of position.

Angular Velocity: A measure of the rotation rate of a rotating object.

Radian: The natural measure of angle and, also, the official SI unit of angle; it is the ratio of the arc length to the radius on a circle or circular arc.

Rotational Motion: Motion about fixed axis.

Torque: The rotational analog of force; torque depends on force and where that force is applied.

Translational Motion: Moving from place to place. (only linear direction)

General Motion: Consists of both linear and rotational motion.

In the unit, we have studied about moment of inertia and its Physical Significance with some practical afflictions. We have also proved theorem of Parallel and Perpendicular axes with some examples.

These theorems make easy to find out the moments of inertia of certain regular bodies. We have contained several examples conceptual questions and self-assessment questions (SAQs) to check your progress.

3.17 TERMINAL QUESTIONS

1. Write answers to the following questions.

(a) What is rotatory motion? Give two examples.

Ans. Rotatory motion - A body is said to be in a rotatory motion or a circular motion if it moves about a fixed axis without changing the radius of its motion.

Example- The blades of a fan, a spinning wheel.

(b) What is meant by circular motion? Give one example.

Ans. The motion of a body along a circular path is called circular motion.

Example- A girl is whirling a stone tied at the end of a string in a circular path.

(c) How does a rotatory motion differ from the circular motion?

Ans. (i) In rotatory motion, the axis of rotation passes from a point in the body itself whereas in circular motion, the axis of revolution passes through a point outside the body. Thus, the motion of earth around the sun is the circular motion whereas the motion of earth about its own axis is the rotational motion.

(ii) In the circular and rotatory motions, the distance of a point of the body from a fixed point always remains same, whereas it is not same in curvilinear motion.

(d) What do you mean by translatory motion? Give one example.

Ans. If an object like a vehicle, moves in a line in such a way that every point of the object moves through the same distance in the same time, then the motion of the object is called translatory motion.

Example- The motion of an apple falling from a tree the motion of a man walking on a road, the motion of a box when pushed from one corner of a room to the other, are all the translatory motion.

(e) State the two theorems of moment of inertia.

(f) Write down the physical significance of moment of inertia.

2. Define moment of inertia and radius of gyration. What is the physical significance of moment of inertia?

3. State and explain the following:

(i) Theorem of parallel axes

(ii) Theorem of perpendicular axes.

4. Define the terms torque and moment of inertia. Establish the relation between these quantities.

5. Define torque. Derive the relation between torque and moment of inertia.

6. Establish the relation between kinetic energy and moment of inertia for a rigid body.

7. Derive an expression for the rotational kinetic energy of a rigid body rotating with an angular velocity ω and hence define moment of inertia.

8. What must be the relation between *l* and *R* if the moment of inertia of the cylinder about its axis is to be the same as the moment of inertia about the equatorial axis?

M.I. of a cylinder about its own axis $I = \frac{1}{2}MR^2$

M.I. about equatorial axis is given by $I' = M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$

When both are equal i.e. $I = I'$ we get $\frac{1}{2}MR^2 = M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$

or $l^2 = 3R^2 \quad \therefore l = \sqrt{3}R$

9. A flat thin uniform disc of radius a has a hole of radius b in it at a distance c from the centre of the disc {c < (a – b)}. If the disc were free to rotate about a smooth circular rod of radius b passing through the hole, calculate the moment of inertia about the axis of rotation.

Let M be the mass of the disc of radius a and having a hole of radius b at a distance c from the centre of the disc.

If the hole were supposed to be at the centre , then the moment of inertia of the disc about an axis through O ad perpendicular to the plane of the disc

$$I_0 = \frac{M(a^2 + b^2)}{2}$$

Applying the principle of parallel axis, moment of inertia about an axis, passing through the centre of a circle of radius b at a distance c from the centre of the disc

$$I = I_0 + Mc^2$$

$$= \frac{M(a^2 + b^2)}{2} + Mc^2$$

10. Explain the concept of angular momentum and discuss the physical meaning of angular momentum.

11. Write down the relationship between the Angular Impulse of a Torque and change in Angular Momentum Produced by the impulse.

12. Define the terms:

(a) Angular Velocity

(b) Uniform Angular Velocity

(c) Angular Displacement

(d) Angular Acceleration

(e) Angular Impulse

(f) Moment of Force

13. What is rolling motion. Write down the acceleration of a body rolling down an inclined plane.

14. Explain different type of rolling motion.

15. Obtain expression for Kinetic Energy of rolling motion.

16. When a body is under pure rolling, the fraction of its total kinetic energy which is the purely rotational is $\frac{2}{5}$. Identify the body.

Solution:

In this type of questions calculate the expression for moment of inertia I and that helps in identifying the body.

Rotational $KE = \frac{1}{2}I\omega^2$

Translational $KE = \frac{1}{2}Mv^2$

$\therefore$ Total $KE = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$

Given, $\frac{2}{5} = \frac{Rotational\ KE}{Total\ KE}$

or $\frac{2}{5} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2}$

For pure rolling put $v = \omega R$

or $\frac{2}{5} = \frac{I\omega^2}{I\omega^2 + M\omega^2R^2}$

$\therefore$ $I = \frac{2MR^2}{3}$

So, the rolling body is hollow sphere.

17. Establish the relation between Torque and Force of a rigid body.

18. Derive the relation between Angular Acceleration and Linear Acceleration.

19. Derive an expression for Angular Momentum of a Rigid Body and Torque.

20. Draw analogy/comparison between Rotational Motion and Translational Motion.

21. Use the theorem of parallel axis to calculate moment of inertia of a disc of mass 400 g and radius 7 cm about an axis passing through its edge and perpendicular to the plane of the disc.

Ans. Given, M = 400 g ; R = 7 cm

M.I. of the disc about an axis through the edge and perpendicular to its plane

$$I' = I + M\,R^2 = \frac{1}{2}M\,R^2 + M\,R^2 = \frac{3}{2}M\,R^2$$

$$= \frac{3}{2} \times 400 \times (7)^2 = \mathbf{29{,}900\ g\ cm^2}$$

22. A disc of mass 5 kg and radius 0.5 m rolls on the ground at the rate of 10 ms^{-1}. Calculate the kinetic energy of the disc.

Ans. Given, M = 5 kg ; R = 0.5 m ; v = 10 m s^{-1}

K.E. of translational motion $= \frac{1}{2}M\,v^2$

and K.E. of rotation $= \frac{1}{2}I\omega^2 = \frac{1}{2} \times \left(\frac{1}{2}M\,R^2\right) \times \left(\frac{v}{R}\right)^2$

$$= \frac{1}{4}M\,v^2$$

Therefore, total kinetic energy of the disc

$$= \frac{1}{2}M\,v^2 + \frac{1}{2}M\,v^2 = \frac{3}{4}M\,v^2$$

$$= \frac{3}{4} \times 5 \times 10^2 = \mathbf{375\,J}$$

23. Is radius of gyration of a body constant quantity?

24. What is moment of inertia of a (a) ring (b) disc about its diameter?

25. Match the following:

Column A		**Column B**	
(a)	Circular Motion	(i)	a running fan
(b)	Periodic Motion	(ii)	a car moving in a market
(c)	Vibratory Motion	(iii)	revolution of earth around the sun
(d)	Rotatory Motion	(iv)	motion of wire of a guitar
(e)	Non-Uniform Motion	(v)	motion of pendulum of a clock

26. Match the following:

Column A		Column B	
(a)	Rotational Analogue of mass	(i)	Moment of Inertia
(b)	Moment of Inertia hollow sphere	(ii)	MR^2
(c)	Moment of Inertia of the Rod	(iii)	$\frac{1}{12}ML^2$
(d)	Rotational Analogue of Force	(iv)	Torque

27. Write down the types of motion being performed by each of the following:

(a) Vehicle on a straight road

(b) Blades of an electric fan in motion

(c) Pendulum of a wall clock

(d) Smoke particles from chimney

(e) Hands of a clock

(f) Earth around the sun

(g) A spinning top

Ans. (a) Rectilinear motion

(b) Rotatory motion

(c) Oscillatory motion, periodic motion

(d) Non-periodic motion

(e) Uniform circular and periodic motion

(f) Rotatory motion, circular motion and periodic motion

(g) Rotatory motion

3.18 SOLUTIONS AND ANSWERS

Self-Assessment Question (SAQ)

1. Hint (Section – 3.3)

2. Hint (Section – 3.3.1, 3.3.2)

3. Hint (Section – 3.3.3)

4. Hint (Section – 3.3.5)

5. Given, $\omega_0 = 100\ rpm = \frac{100}{60} \times 2\pi = \frac{10\pi}{3} rad/s$

$\omega = 0, t = 15\ s$

From $\omega = \omega_0 + \alpha\ t$

$$0 = \frac{10\pi}{3} + \alpha \times 15 \qquad or \qquad \alpha = \frac{-10\pi}{45}$$

Angle traced by the motor in the process is

$$\theta = \omega_0\, t + \frac{1}{2}\alpha t^2 = \frac{10\pi}{3} \times 15 + \frac{1}{2}\left(\frac{-2\pi}{9}\right)(15)^2$$

$$\theta = 50\pi - 25\pi = 25\pi$$

$\therefore$ No of revolution $= \frac{\theta}{2\pi} = \frac{25\pi}{2\pi} = \mathbf{12.5}$

6. Given, $\omega_0 = 120\ r.p.m. = 2\ r.p.s. = 4\pi\ rad\ s^{-1}$

$$\omega = 0 \qquad and \qquad t = 10\ s$$

Now, $\omega = \omega_0 + \alpha t$

$\therefore$ $0 = 4\pi + \alpha \times 10 \qquad or \qquad \alpha = -0.4\pi\ rad\ s^{-1}$

Also, the angle covered by the motor,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = 4\pi \times 10 + \frac{1}{2} \times (-0.4\pi) \times 10^2 = 40\pi - 20\pi$$

$$= 20\pi\ rad$$

Hence, the number of revolutions completed,

$$n = \frac{\theta}{2\pi} = \frac{20\pi}{2\pi} = \mathbf{10}$$

7. HINT

Hands of a watch move uniformly. For uniform motion $\boldsymbol{\omega = \frac{d\theta}{dt} = \frac{\Delta\theta}{\Delta t}}$

For the seconds hand:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\ rad}{60\ s} = \frac{\pi}{30} rad\ s^{-1}$$

For the minute hand:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\ rad}{60\ min}$$

$$= \frac{2\pi}{60 \times 60} rad\ s^{-1} = \frac{\pi}{1800} rad\ s^{-1}$$

Note that angular speed does not depend on the length of the hands of the watch.

8. **HINT**

$\boldsymbol{v = \omega r}$

For the hour hand, angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\ rad}{12\ h} = \frac{2\pi}{12\times60\times60} rad\ s^{-1}$$

The tip of the hand rotates in a circle of radius 1 cm.

$$\therefore \quad v = \omega r = \frac{2\pi}{12\times60\times60} \times 1\ cm\ s^{-1}$$

$$= 1.45 \times 10^{-4}\ cm\ s^{-1}$$

3.19 SUGGESTED READINGS

1. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons.
2. Elementary Mechanics, IGNOU, New Delhi.
3. College Physics, Hugh D. Young.
4. An Introduction to Mechanics Daniel Kleppner and Robert J. Kolenkow.

UNIT 4

DYNAMICS OF MANY PARTICLES

Structure:

4.1 INTRODUCTION

In the earlier units, we Primarily considered the motion of a single particle. In this unit, we shall study objects consist of many Particles. In Principles, studying many – particle system is complicated by the need to consider all the forces acting among the Particles, as well as any forces applied from outside the system. This unit, we introduce the concept of centre of mass of a body, a concept that is essential for describing the motion of rigid bodies, and also deal with the centre of gravity, COM and laboratory frame of reference. In this unit, we have also covered concept of linear momentum, angular momentum, Kinetic energy, Potential energy for a system of Particles. In this unit, we also study different aspects of a of rotational motion of a rigid body.

4.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand Concept of Center of Mass and Center of Gravity.
- Compute Numerical based on COM and GOG.
- Define Equilibrium of Rigid Body.
- Explain the Concept of Motion of Center of Mass of a System.
- Relate Comparisons Between Conservation Laws.

4.3 CENTRE OF MASS

When we consider the motion of a system of particles, there is one point in it which behaves as though the entire mass of the system (i.e., the sum of the masses of all the individual particles) is concentrated there and its motion is the same as would ensue if the resultant of all the forces acting on all the particles were applied directly to it. This point is called the centre of mass (COM) of the system.

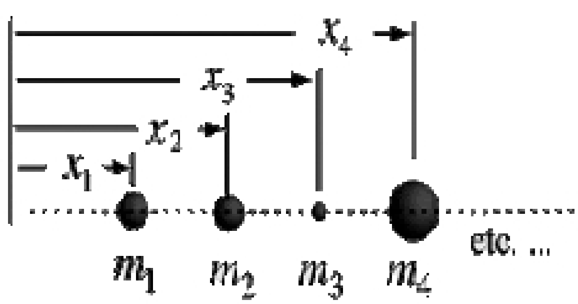
The Point in the system, where, the whole mass of the system can be supposed to be concentrated, is called center of mass of the system.

It is point in a system which moves as if whole mass of the system is concentrated at the point and all external forces are acting on it. Its positon is given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{M} \quad \text{............. (1)}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i y_i}{M} \quad \text{............. (2)}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i z_i}{M} \quad \text{............. (3)}$$



$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

Figure - 1

Where

CM – centre of mass

M_i – mass of the i^{th} particle

X_i – x coordinate of the i^{th} particle

Y_i – y co-ordinate of the particle

Note: The concept of COM is very useful in solving many problems, in particular, those concerned with collision of particles.

4.3.1 Definition of Centre of Mass (COM) of a System

If we have a system consisting of n particles, of mass $m_1, m_2, \ldots\ m_n$ with $\vec{r}_1, \vec{r}_2, \ldots.\vec{r}_n$ as their position vectors at a given instant of time. The position vector …. of the COM of the system at that instant is given by :

$$\vec{r}_{COM} = \frac{m_1\ \vec{r}_1 + m_2\ \vec{r}_2 + \cdots + m_n \vec{r}_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_1\ \vec{r}_1}{\sum_{i=1}^{n} m_1}$$

………….. (4)

or $$\vec{r}_{COM} = \frac{\sum_{i=1}^{n} m_1\ \vec{r}_1}{M}$$

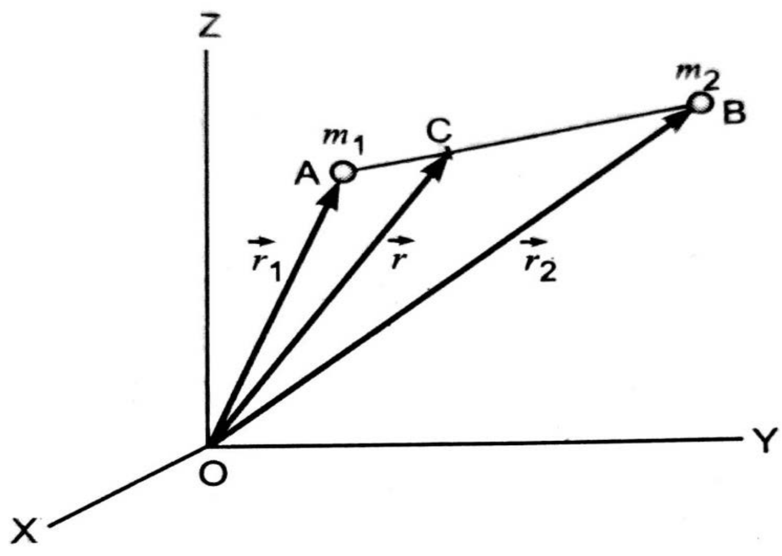


Figure - 2

Here, M = $m_1 + m_2 + \cdots + m_n$ and $\sum m_1 \vec{r}_1$ is called the first moment of the mass.

Further, $\vec{r}_i = x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}$

and $\vec{r}_{COM} = x_{COM} \hat{\imath} + y_{COM} \hat{\jmath} + z_{COM} \hat{k}$

So, the cartesian co-ordinates of the COM will be

$$x_{COM} = \frac{m_1x_1+m_2x_2+\cdots+m_nx_n}{m_1+m_2+\cdots+m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum m_i}$$

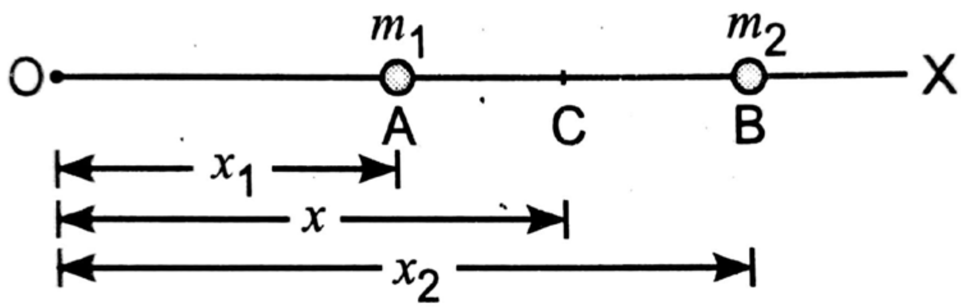


Figure - 3

or $$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}$$

Similarly, $$y_{COM} = \frac{\sum_{i=1}^{n} m_iy_i}{M}$$

and $$z_{COM} = \frac{\sum_{i=1}^{n} m_iz_i}{M}$$

4.3.2 Importance of Centre of Mass

The centre of mass (COM) of a system of particles moves as if the entire mass were concentrated there and all external forces were applied there. When a stick is tossed in air, each particle experiences its weight (m_1g, m_2g, m_3g …..) as an external force. But, acceleration of particle of mass m_1 is not g, because it experiences forces applied by neighboring particles also.

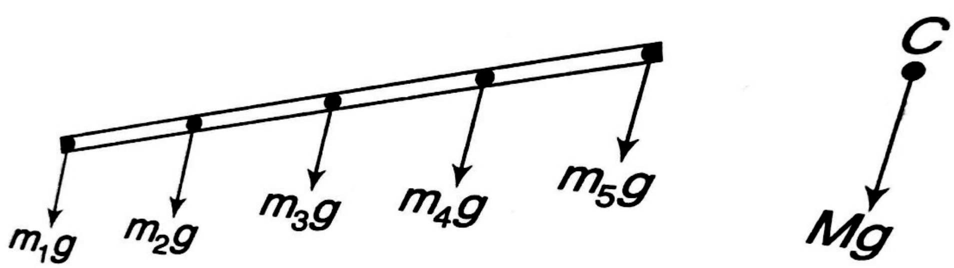


Figure - 4

The special point in the stick whose acceleration is decided solely by the external forces is called its COM.

If mass of the stick is M = $m_1 + m_2 + m_3 + \ldots$, then net external force on it is

$$F_{ext} = m_1g + m_2g + m_3g + \cdots$$

$$\boxed{F_{ext} = Mg} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

Acceleration of COM is

$$\boxed{a_{cm} = \frac{F_{ext}}{M} = g} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

For describing translational motion of the stick, we consider its entire mass at its COM and the sum of all external forces (m_1g, m_2g, m_3g …..) acting at that point.

4.3.3 Centre of Mass of a Rigid Body

Mathematically, position coordinates of the centre of mass of rigid body are given by

$$x_{cm} = \frac{\int x dm}{\int dm}; y_{cm} = \frac{\int y dm}{\int dm}; z_{cm} = \frac{\int z dm}{\int dm} \qquad \text{……………} (7)$$

4.3.4 Velocity and Acceleration of Centre of Mass

Velocity of Centre of Mass

The instantaneous velocity of centre of mass is given by

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots m_n\vec{v}_n}{M}; \text{ or } \vec{v}_{cm} = \frac{\vec{P}_{system}}{M} \qquad \text{……………} (8)$$

Where $\vec{P}_{system}$ is the total linear momentum of centre of mass

Acceleration of Centre of Mass

We know that,

$$\Rightarrow \qquad \vec{v}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{v}_i \qquad \text{……………} (9)$$

Differentiating equation (9) $\vec{v}_{cm}$ w.r.t. time we get $\vec{a}_{cm}$ as

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots m_n\vec{a}_n}{M}; \text{ or } \vec{a}_{cm} = \frac{\sum \vec{F}_{ext}}{M} \qquad \text{……………} (10)$$

If a force is applied along a line passing through the centre of mass of the body. All the particles of the body move with same linear velocity and acceleration.

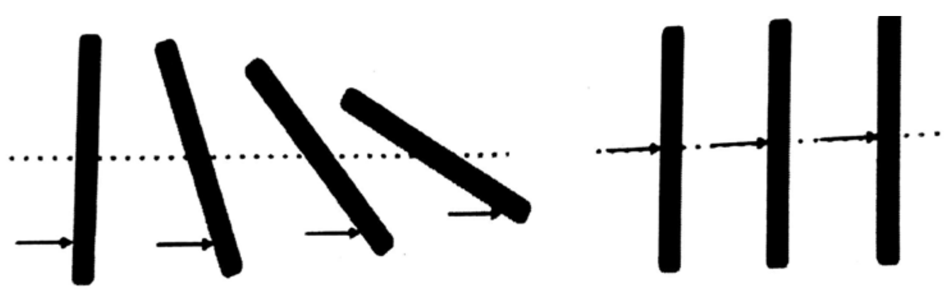


Figure – 5

Where $\sum \vec{F}_{ext}$ is the vector sum of forces acting on the particles of system.

Example: 1

What is the unit of Center of Mass?

Solution:

The unit of Centre of Mass is meter(m).

Example: 2

Write down expression for the position vector of the centre of mass of a system consisting of two objects in terms of their masses and position vectors.

Solution:

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Self-Assessment Questions (SAQs)

1. What is centre of mass of a system of particles?
2. What is the position vector of centre of mass of two particles of equal masses?
3. Write down Importance of Centre of Mass.

4.4 LOCATION OF CENTRE OF MASS

First of all, we find position of COM of a system of particles. Just to make the subject easy we classify a system of particles in three groups:

1. System of two particles.
2. System of a Many Particles on a Straight Line.
3. Center of Mass of three-Dimensional Distribution of Particles

Now let us take them separately.

4.4.1 System of Two Particles

The given figure shows two particles of masses m_1 and m_2, placed on the x-axis, at co-ordinates x_1 and x_2 respectively.

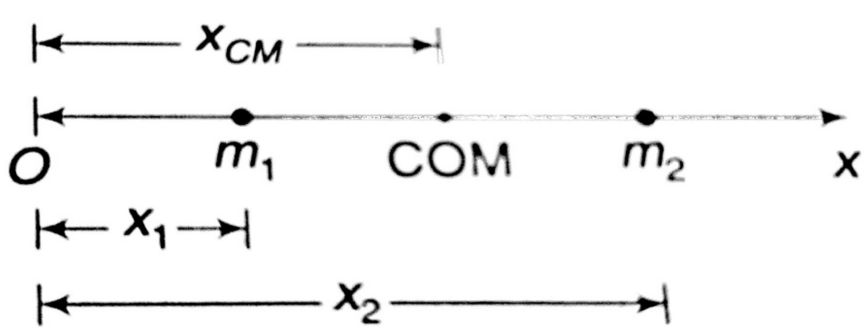


Figure - 6

The position of COM for the system of two particles is given by

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M} \qquad \text{.............. (11)}$$

Where, $M = m_1 + m_2$ is the total mass of the system. The COM of a two-particle system lies somewhere between them on the line joining the two.

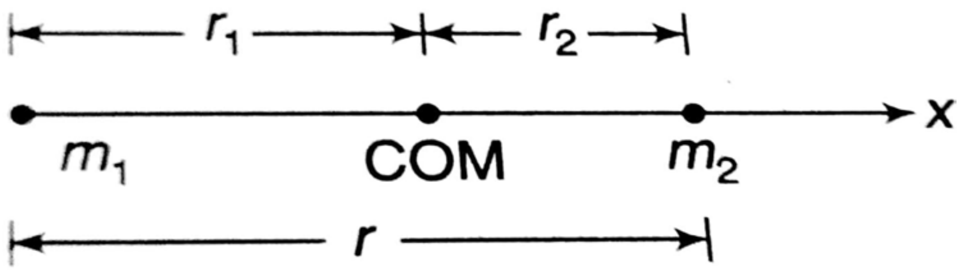


Figure - 7

If the origin is chosen at the position of mass m_1, then x co-ordinate of COM is he distance of the COM from the particle of mass m_1. Let us assume this distance as r_1 and let the distance between the two particles be r.

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or $$r_1 = \frac{0 + m_2 r}{m_1 + m_2} = \frac{m_2 r}{m_1 + m_2}$$

Distance of the COM from m_2 is

$$r_2 = r - r_1 \frac{m_1 r}{m_1 + m_2}$$

It is easy to see that $\frac{r_1}{r_2} = \frac{m_2}{m_1}$

$\Rightarrow \qquad m_1 r_1 = m_2 r_2$

.... (12)

The last equation (12) leads us to conclude that the COM of the system will be closer to the heavier mass.

4.4.2 System of a Many Particles on a Straight Line

Particles of masses $m_1, m_2, m_3, \ldots\ m_n$ are located on the x-axis at co-ordinates $x_1, x_2, x_3, \ldots\ x_n$ respectively. The COM of this collection of particles lies on x-axis with x co-ordinate of COM given by

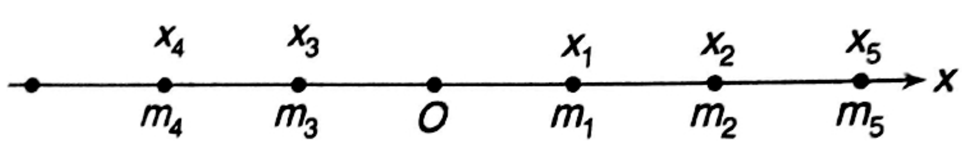


Figure - 8

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$\boldsymbol{x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i} \qquad \dots\dots\dots\dots (13)$$

where, M is total mass of the system.

If the position of COM is chosen as the origin.

then $\boldsymbol{\sum_{i=1}^{n} m_i x_i = 0}$ .... (14)

4.4.3 COM of Three–Dimensional Distribution of Particles

When particles in a system are scattered in a three-dimensional space, then the position of COM is defined by three co-ordinates, given as:

$$x_{CM} = \sum_{i=1}^{n} m_i x_i \,;\; y_{CM} = \sum_{i=1}^{n} m_i y_i \,;\; z_{CM} = \sum_{i=1}^{n} m_i z_i \dots\dots (15)$$

We can also write the above relations in terms of a single vector relation. The position vector of the COM of a system of particles is given by

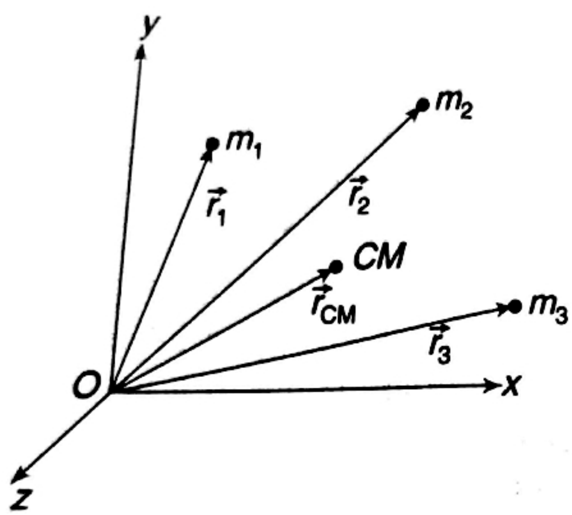


Figure - 9

$$\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_n \vec{r}_n}{M}$$

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_1 \vec{r}_1 \qquad \dots\dots (16)$$

Where, $\vec{r}_1, \vec{r}_2 \dots \vec{r}_n$ are position vectors of particles of masses $m_1,\ m_2 \dots m_n$ respectively.

$$\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{\boldsymbol{k}}$$

and $\vec{r}_{CM} = x_{CM}\hat{\imath} + y_{CM}\hat{\jmath} + z_{CM}\hat{\boldsymbol{k}}$

If we choose the COM of the system as the origin, then

$$x_{CM} = y_{CM} = z_{CM}$$

$$\Rightarrow \quad \boldsymbol{\sum m_i x_i = \sum m_i y_i = \sum m_i z_i = 0} \qquad \dots\dots\dots\dots\dots (17)$$

4.4.4 Position of Centre of Mass of Bodies of Regular Shape

Table – 1

The table given below gives us the position of the centre of mass of some of the bodies of regular shape.

S. No.	Body	Position of Centre of Mass
1.	Uniform hollow sphere	Centre of the sphere
2.	Uniform solid sphere	Centre of the sphere
3.	Uniform circular ring	Centre of the ring
4.	Uniform circular disc	Centre of the disc
5.	Uniform rod	Centre of the rod
6.	A plane square lamina	Point of intersection of diagonals
7.	Triangular lamina	Point of intersections of the medians
8.	Rectangular or cubical block	Point of intersection of diagonal
9.	Hollow cylinder	Middle point of the axis of the cylinder
10.	Solid cylinder	Middle point of the axis of the cylinder
11.	Cone or pyramid	On the axis of the cone at a point distant $3h/4$ from the vertex, where h *is* the height of the cone.

Example: 3

A particle of mass m is projected vertically up from a point with initial velocity 20 ms^{-1}. Another particle of mass 2m is projected simultaneously from the same point, with velocity 20 ms-1 at an angle of 30o to the horizontal.

(i) Find the velocity of the COM of the system of two particles, one second after their projection.

(ii) **Final acceleration of the COM of the system, one second after their projection.**

Solution:

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2} \quad \textbf{and} \quad \vec{a}_{CM} = \frac{m_1\vec{a}_1+m_2\vec{a}_2}{m_1+m_2}$$

(i) Velocity of the first particle after 1 s is

$$v = 20 - gt = 20 - 10 \times 1 = 10ms^{-1}(\uparrow)$$

Horizontal and vertical components of initial velocity of the second particle are (Horizontal-x and vertical-y-direction)

$$u_x = 20cos30^o = 10\sqrt{3}ms^{-1} \quad \text{and}$$

$$u_y = 20sin30^o = 10ms^{-1}$$

Velocity components after 1 s

$$v_x = 10\sqrt{3}ms^{-1}; = 10 - gt = 10 - 10 \times 1 = 0$$

∴ $$\vec{v}_{CM} = \frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2} = \frac{m(10\hat{j})+2m(10\sqrt{3}\hat{i})}{m+2m} = \left(\frac{20}{\sqrt{3}}\hat{i} + \frac{10}{3}\hat{j}\right)ms^{-1}$$

(ii) $$\vec{a}_{CM} = \frac{m_1 g(\downarrow)+m_2(\downarrow)}{m_1+m_2} = g(\downarrow)$$

Example: 4

Two blocks of masses m_1 and m_2 are connected by an ideal spring, having force constant k. The system is placed on a smooth floor, and pushed against a wall with compression in the spring equal to x. System is released from this position.

(i) **Find the acceleration of the COM of the system, immediately after its release.**

(ii) **Find the acceleration of the COM of the system, after m_1 breaks-off the wall.**

Solution:

(i) **The compressed spring pushed the block of mass m1 against the wall. The wall applies a normal force on m_1. This is the external force that causes the COM to accelerate.**

(ii) **Once m1 leaves contact with the wall, there is no external force on the system. Now, $a_{CM} = 0$.**

(i) Spring force on the block of mass m_1, at the instant the system is released, is kx.

Normal force exerted by the wall on the block of mas m_1 is N = kx

∴ For the system $F_{ext} = N = kx$

∴ $a_{CM} = \frac{F_{ext}}{m_1+m_2} = \frac{kx}{m_1+m_2}$

(ii) As long as the spring is compressed, it will keep the block pressed against the wall. When the block of mass m_2 moves to the right by a distance x, the spring attains its natural length. At this instant, the spring does not exert any force on the blocks. The normal force by wall on m_1 becomes zero. The system leaves the wall. After leaving the wall, $F_{ext} = 0$

⇒ $a_{CM} = 0$

Example: 5

If one of the particles is heavier than the other, to which side will their centre of mass shift?

Solution:

The centre of mass will shift closer to the heavier particle.

Example: 6

Does centre of mass of a sytem of two particles lie on the line joining the particles?

Solution:

Yes, always.

Self-Assessment Questions (SAQs)

4. Derive an expression for the position vector of the centre of mass of a system consisting of two particles.
5. Two bodies of masses 1 kg and 2 kg are lying in xy plane at (-1, 2) and (2, 4) respectively. What are the coordinates of the centre of mass?

4.5 CENTRE OF GRAVITY

We have discussed, the centre of mass, which is the point where the whole mass of the body is supposed to be concentrated. Now in this section (4.5) we will discuss the centre of gravity.

The centre of gravity is that point of the body, where the whole weight of the body is supposed to be concentrated.

Consider a rigid body as shown in the figure given below. If $\vec{r}_i$ is the position vector of the i^{th} particle of an extended body with respect to the centre of gravity of the body, then the torque about the centre of gravity, due to the force of gravity on the particle is given by

$$\vec{t}_1 = \vec{r}_i \times m\overline{g} = 0 \qquad \text{............ (18)}$$

Now,

The total torque about the centre of gravity is zero.

$$\sum_i \vec{t}_i = \sum_R r_i \times m_1\overline{g} = 0 \qquad \text{............ (19)}$$

Therefore, we may define the centre of gravity of a body as that point where the total gravitational torque acting on the body id zero.

$\overline{g}$ is same for all particles. If the body is small them, we can take g out of the summition sign. Now since $\overline{g}$ is non-zero therefore $\sum_i m_i\vec{r}_i = 0,$ in the above discussion we have taken $\vec{r}_i$ as the position vector of i^{th} particle with respect to C.G. In the case of centre of mass we have discussed that if $\sum_i m_i\vec{r}_i = 0$ this implies that centre of mass must lie at the origin, and now C.G. also be at origin.

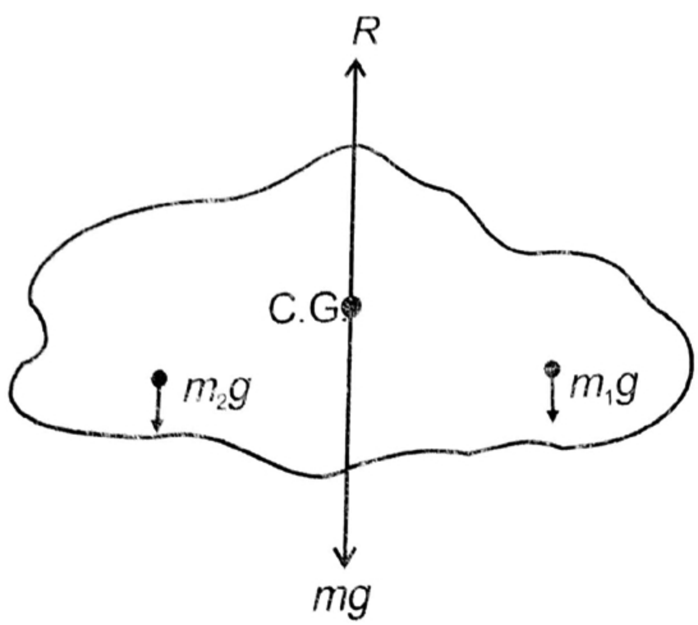


Figure - 10

Thus, we conclude that the centre of gravity coincides with the centre of mass because the body being small, $\overline{g}$ does not vary from one point of the body to the other. If the body is so extended that g varies from part to part of the body, then the centre of gravity and centre of mass will not coincides.

Remember: *Centre of mass has nothing to do with gravity, it depends only on the distribution of mass of the body.*

Concept of Centre of Gravity (with example)

Let us take an irregular shaped cardboard and a narrow-tipped object like a pencil. You can locate by trial and error a point G on the cardboard where it can be balanced on the tip of the pencil. (The cardboard remains horizontal in this position). This point of balance is the centre of gravity (CG) of the cardboard. The tip of the pencil provides a vertically upward force due to which the cardboard is in mechanical equilibrium.

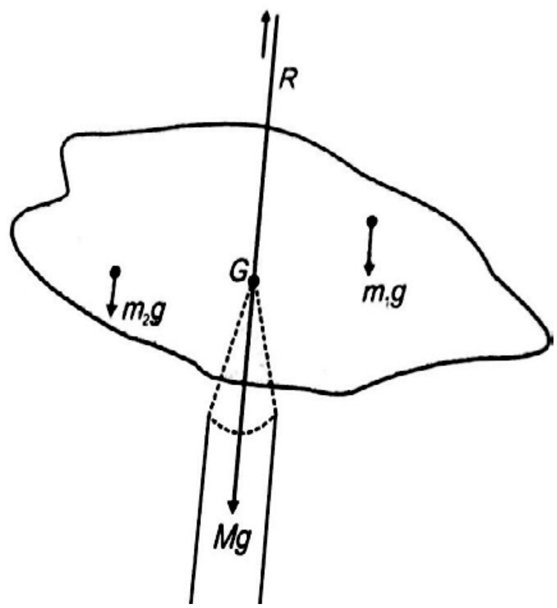


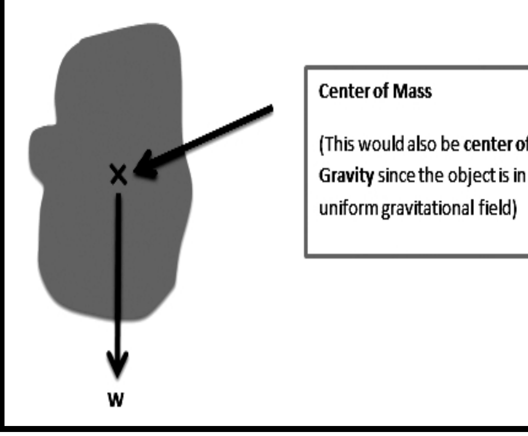
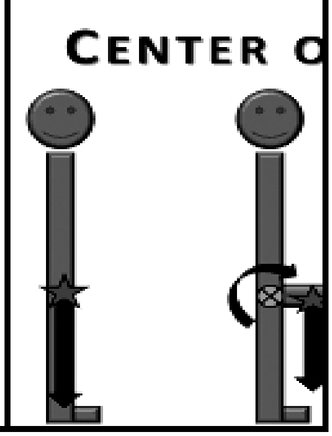
Figure - 11

As shown in the figure given below the reaction of the tip is equal and opposite to *Mg*, the total weight of (i.e., the force of gravity on) the cardboard and hence the cardboard is in translational equilibrium. It is also in rotational equilibrium. If it were not so, due to the unbalanced torque it would tilt and fall. There are torques on the cardboard due to the forces of gravity like m_1g, m_2g ….. etc, acting on the individual particles that make up the cardboard. The CG of the cardboard is so located that the total torque on it due to the forces m_1g, m_2g … etc. is zero.

4.5.1 Difference Between Centre of Mass and Center of Gravity

Table-2

The table given below gives us the Difference Between Centre of Mass and Center of Gravity

Centre of Mass		Centre of Gravity	
1	The centre of mass of a body is a point, where the mass of the body can be supposed to be concentrated. In fact, nothing exists at the location of the centre of mass. It is only a mathematical concepts.	1	The centre of gravity of a body is a point, where the whole weight of the body may be supposed to act.
2	It refers to mass of the body.	2	It refers to weight of the body.
3	The concept of centre of mass is useful to study the complicated motion of the body.	3	The concept of centre of gravity is useful to study the stability of the body.
4	Centre of mass depends on mass distribution.	4	Centre of gravity depends on acceleration due to gravity ‘g’.
5	In case of small and symmetrical bodies where gravitational field is uniform, centre of mass and centre of gravity of the body coincide with each other.	5	In case of extended and non-symmetrical bodies, where gravitational field is non-uniform, centre of mass and centre of gravity of the body does not coincide with each other.
6	Centre of mass may or may not be inside the body.	6	Centre of gravity always be inside the body.
7		7	

Example: 7

A metal rod of length 50 cm having mass 2 kg is supported on two edges placed 10 cm from each end. A 3 kg load is suspended at 20 cm from one end. Find the reactions at the edges. (take g = 10 m/s^2)

Solution:

AB is the rod, C is the centre of gravity and W is the weight of the rod acting downward and W_1 is the weight of the load suspended at point D. Rod is supported at two edges E and F as shown in the figure. R_1 and R_2 are the reaction force at E and respectively.

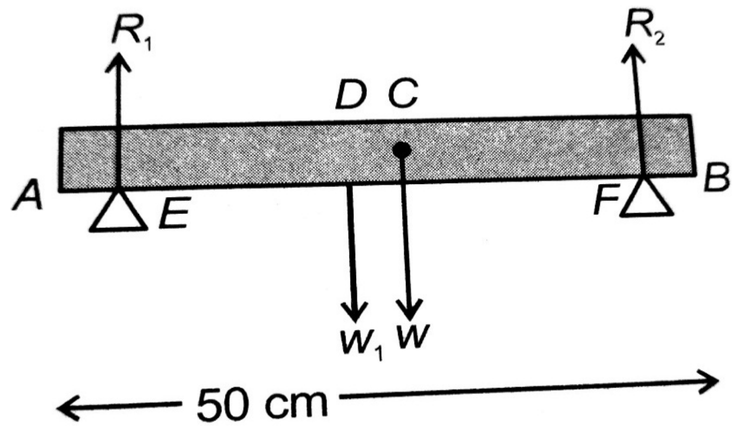


Figure - 12

As the rod is uniform and homogeneous,

therefore, G is at the centre

AB = 50 cm, AC = 25 cm, AD = 20 cm,

CD = 5 cm, AE = BF = 10 cm, ED = 10 cm,

EC = FC = 15 cm

$W = mg = 2 \times 10 = 20$ N

$W_1 = 3 \times 10 = 30$ N

For translational equilibrium, $\sum_i \vec{F}_i = 0$

$R_1 + R_2 - W - W1 = 0$

{W_1 and W act in the downward direction and R_1 and R_2 act in the vertically upward direction}

$R_1 + R_2 - 20 - 30 = 0$

$\Rightarrow \quad R_1 + R_2 = 50$ ………….. (i)

For rotational equilibrium, $\sum_i \vec{\tau}_i = 0$

$\Rightarrow \quad -R_1(EC) + W_1(CD) + R_2(FC) = 0$

$\Rightarrow \quad -R_1(15) + 30(5) + R_2(15) = 0$

$\Rightarrow \quad R_1 - R_2 = 10$ ………….. (ii)

Adding (i) & (ii), we get

$2R_1 = 60$, $R_1 = 30$ N

and R2 = 50 – 30 = 20 N

$\boxed{R_1 = 30\ N}$, $\boxed{R_2 = 20\ N}$

Self-Assessment Questions (SAQs)

6. Explain what is meant by centre of gravity.
7. Can the centre of gravity of a body be situated outside its material of the body? Give an example.

4.6 LABORATORY AND COM FRAMES OF REFERENCE

A reference frame is the space determined by a rigid body regarded as the base. The rigid body is supposed to extend in all directions as far as necessary. A point in space is located by the three co-ordinates taken with respect to the origin of the reference system.

If the origin of the reference system is a point rigidly fixed to the laboratory it is known as the laboratory frame.

The laboratory frame is inertial so long as earth is taken to be an inertial frame.

Centre of mass system (Frame of reference). If the origin of the reference system is a point rigidly fixed to the centre of mass of a system of particles on which no external force is acting it is known as the **centre of mass frame of reference.**

In the centre of mass reference frame the position vector of the centre of mass $\vec{R} = 0$ as the centre of mass is itself the origin of the reference system.

∴ **The velocity of centre mass $\vec{V} = \frac{d\vec{R}}{dt} = 0$**

……………. (19)

and the **linear momentum $\vec{P} = M\vec{V}$ of the system is also = 0. Hence it is known as a zero-momentum frame.**

Advantages of studying collision process in centre of mass system.

(a) In the absence of any external force the velocity of the centre of mass is a constant. In other words, the centre of mass reference frame moves with a constant velocity with respect to the laboratory frame. Hence the centre of mass frame is also an inertial frame.

Various physical quantities measured in the two systems are related to each other by Galelian transformations provided the velocity of centre of mass is small as compared to the velocity of light.

(b) A system of two particles requires six co-ordinates to describe the motion in the laboratory system. Three co-ordinates are required to describe the motion of centre of mass and three more co-ordinates are required to describe the relative motion. But in the centre of mass frame we require only three co-ordinates as the centre of mass is itself at rest in this frame.

The discussion of a collision process, therefore, becomes much simpler in the centre of mass frame of reference than in the laboratory frame.

4.6.1 Perfectly Elastic Collision in One Dimension

In This section (4.6.1), we have discussed laboratory frame and center of mass frame.

4.6.2 Laboratory Frame:

Let m_1 and m_2 be the masses of the two particles $\vec{u}_1$ and $\vec{u}_2$ and $\vec{v}_1$, $\vec{v}_2$ their respective velocities before and after an elastic one-dimensional collision i.e., a head on collision along the line joining their centres, then

According to the principle of conservation of linear momentum

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \quad \text{.............. (20)}$$

and according to the law of conservation of energy

$$m_1 u_1^2 + m_2 u_2^2 = m_1 v_1^2 + m_2 v_2^2 \quad \text{.............. (21)}$$

Rewriting equations (i) and (ii) and taking magnitudes only we have

$$m_1(u_1 - v_1) = m_2(u_2 - v_2) \quad \text{............... (22)}$$

and $$m_1(u_1^2 - v_1^2) = m_2(u_2^2 - v_2^2) \quad \text{............... (23)}$$

Dividing (23) by (24) we have $u_1 + v_1 = v_2 + u_2$ or $u_1 - u_2 = -(v_1 - v_2)$ ....(24)

This shows that in an elastic one-dimensional collision the relative velocity with which the two particles approach each other before collision is equal to the relative velocity with which they recede away from each other after collision.

Velocity after collision. From equation (24) we have $v_1 = v_2 + u_2 - u_1$

and $v_2 = v_1 + u_1 - u_2$

Substituting the value of v_2 in (22), we have

$m_1(u_1 - v_1) = m_2(v_1 + u_1 - u_2 - u_2)$

or $v_1(m_1 - m_2) = (m_1 - m_2)u_1 + 2m_2u_2$ or

$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2$

……. (25)

Similarly, $v_2 = \frac{m_2 - m_1}{m_1 + m_2}u_1 + \frac{2m_1}{m_1 + m_2}u_1$ ……. (26)

Special cases. Case (i): One of the colliding particles is initially at rest. Let m_2 be initially at rest, then $u_2 = 0$.

Hence $v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1$ and $v_2 = \frac{2m_1}{m_1 + m_2}u_1$ ……. (27)

Case (ii): The particles have the same mass. In such a case $m_1 = m_2$

Putting the value in (25), we have $v_1 = u_2$ and putting the value in (26), we have $v_2 = u_1$

Hence in one dimensional elastic collision of two particles of equal mass, the particles simply interchange their velocities after collision.

If m_2 is also initially at rest, then $u_2 = 0$ $\therefore$ $v_1 = 0$ and as before $v_2 =$

……..… (28)

Hence the first particle of mass m_1 comes to rest after collision and the second particle of mass m_2 acquires the initial velocity of the first.

Case (iii): The particle at rest is very massive. If m_2 is very heavy as compared to m_1 and $u_2 = 0$, then $m_1 = 0, m_1 - m_2 = -m_2$ and $m_1 + m_2 = m_2$ $\therefore$ $v_1 = -u_1$ and $v_2 = 0$

This shows that when a very light particle collides against a very massive particle at rest, the heavy particle continues to remain at rest and the velocity of the light particle is reversed.

A familiar example of this is the dropping of a steel ball on an equally hard horizontal surface on the ground. This is in fact a collision between the light ball and the massive ground at rest. The velocity of the ball is reversed on impact. This is judged from the fact that the ball rises to the same height from which it was dropped.

Case (iv): Particle at rest is very light. If the particle at rest is very light

$m_2 = 0;\ m_1 - m_2 = m_1$ and $m_1 + m_2 = m_1$

Putting the value in relation (27) we have $v_1 = u_1$ and $v_2 = 2u_1$

This shows that the velocity of the heavy particle remains almost the same after collision and the light particle acquires nearly twice the velocity of the heavy particle.

4.6.3 Centre of Mass Frame:

When no external force is acting, the velocity of centre of mass is given by

$$\vec{V}_{cm} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

As the collision is the dimensional, therefore taking magnitudes only $V_{cm} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$ Velocity of the particle of mass m_1 before collision relative to centre of mass frame according to Galelian transformations is given by

$$\vec{u}'_1 = \vec{u}_1 - \vec{V}_{cm} = \vec{u}_1 - \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

$$= \frac{m_1\vec{u}_1 + m_2\vec{u}_2 - m_1\vec{u}_1 - m_2\vec{u}_2}{m_1 + m_2} = \frac{m_2(\vec{u}_2 - \vec{u}_1)}{m_1 + m_2}$$

Taking magnitudes only $\vec{u}'_1 = \frac{m_2(u_1 - u_2)}{m_1 + m_2}$ …….. (29)

Velocity of the particle of mass m_2 before collision relative to centre of mass frame according to Galelian transformations is given by

$$\vec{u}'_1 = \vec{u}_2 - \vec{V}_{cm} = \vec{u}_2 - \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

$$= \frac{m_1\vec{u}_2 + m_2\vec{u}_2 - m_1\vec{u}_1 - m_2\vec{u}_2}{m_1 + m_2} = \frac{m_1(\vec{u}_2 - \vec{u}_1)}{m_1 + m_2}$$

Taking magnitudes only $\vec{u}'_2 = \frac{m_1(u_2 - u_1)}{m_1 + m_2}$ ………..…. (30)

Velocity after collision. The velocity of the particle of mass m_1 after collision relative to centre of mass frame according to Galelian transformations is given by $\vec{v}'_2 = \vec{v}_1 - \vec{V}_{cm}$

Taking magnitudes only

$$\vec{v}\,'_2 = v_1 - V_{cm} = \frac{(m_1 - m_2)u_1 + 2m_2 u_2}{m_1 + m_2} - \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

$$= \frac{m_1 u_1 + m_2 u_1 + 2m_2 u_2 - m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{-m_2 (u_1 - u_2)}{m_1 + m_2} \qquad \text{........ (31)}$$

The velocity of the particle of mass m_2 after collision in the centre of mass frame according to Galelian transformations is given by

$$\vec{v}\,'_2 = \vec{v}_1 - \vec{V}_{cm}$$

Taking magnitudes only

$$v\,'_2 = v_2 - V_{cm} = \frac{(m_2 - m_1)u_2 + 2m_1 u_1}{m_1 + m_2} - \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

$$= \frac{m_2 u_2 + m_1 u_2 + 2m_1 u_1 - m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{-m_1 (u_2 - u_1)}{m_1 + m_2} \qquad \text{........ (32)}$$

The centre of mass is at rest before and after collision relative to the centre of mass reference frame.

4.6.4 Velocities of Particles in COM frame and Laboratory Frame

(In Elastic Collision)

Consider a particle mass m_1 moving with a velocity $\vec{u}_1$ in the laboratory frame and let it suffer a perfectly elastic collision with a particle of mass m_2 at rest.

The velocity of centre of mass of a system of two particles relative to the laboratory frame is given by $\vec{V}_{cm} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$

Since we have assumed the mass m2 to be initially at rest

$$\vec{u}_2 = 0 \qquad \text{and} \qquad \vec{V}_{cm} = \frac{m_1}{m_1 + m_2} \vec{u}_1$$

This shows that $\vec{V}_{cm}$ and $\vec{u}_1$ have the same direction. Let $\vec{u}\,'_1$ and $\vec{u}\,'_2$ be the initial velocity of the particles of mass m_1 and m_2 before collision in the centre of mass frame, then according to Galilean transformation equations.

$$\vec{u}\,'_1 = \vec{u}_1 - \vec{V}_{cm} = \vec{u}_1 \left(1 - \frac{m_1}{m_1 + m_2}\right) = \frac{m_2}{m_1 + m_2} \vec{u}_1$$

and $\quad \vec{u}\,'_2 = \vec{u}_2 - \vec{V}_{cm} = -\vec{V}_{cm} = \frac{m_1}{m_1 + m_2} \vec{u}_1$

$[\because \vec{u}_2 = 0]$

We also $\vec{u}\,'_1 = -$also $\vec{u}\,'_2 = \vec{u}_1 - \vec{u}_2$ i.e., the relative velocity between the two particles in the laboratory frame and centre of mass frame is the same.

Taking magnitude only $\qquad \vec{u}\,'_1 = \frac{m_2}{m_1 + m_2} \vec{u}_1 \qquad$ .... (33)

$$\vec{u}'_2 = \frac{m_1}{m_1+m_2}\vec{u}_1 \qquad \text{........ (34)}$$

4.6.5 Relation Between Final Velocities and Initial Velocities in COM Frame

Let $\vec{v}'_1$ and $\vec{v}'_2$ be the final velocities of the particles of mass m_1 and m_2 after collision in the centre of mass frame, then $\vec{v}'_1 = \vec{v}_1 - \vec{V}_{cm}$ and $\vec{v}'_2 = \vec{v}_2 - \vec{V}_{cm}$ where $\vec{v}_1$ and $\vec{v}_2$ are the final velocities of m_1 and m_2 in the laboratory frame.

In the centre of mass frame, the centre of mass is always at rest, therefore, the total linear momentum before and after collision is not only conserved but is also equal to zero.

$$\therefore \quad m_1\vec{u}'_1 + m_2\vec{u}'_2 = 0 \quad \text{and} \quad m_1\vec{v}'_1 + m_2\vec{v}'_2 = 0$$

From the above relation we get $\vec{u}'_1 = -\frac{m_2}{m_1}\vec{u}'_2$ and $\vec{v}'_1 = -\frac{m_2}{m_1}\vec{v}'_2$

Taking magnitudes only $\vec{u}'_1 = -\frac{m_2}{m_1}\vec{u}'_2$ .... (35)

$$\text{and} \quad \vec{v}'_1 = -\frac{m_2}{m_1}\vec{v}'_2 \qquad \text{......... (36)}$$

The negative signs indicate that $\vec{u}'_1$ and $\vec{u}'_2$ act along the same straight line in opposite directions. Similarly, $\vec{v}'_1$ and $\vec{v}'_2$ also act along the same straight line in opposite direction.

We can say, in other words, in the C.M. system the two particles move towards each other before collision and away from each other after collision.

As collision is elastic, the kinetic energy is also conserved.

$$\therefore \quad \frac{1}{2}m_1u'^2_1 + \frac{1}{2}m_2u'^2_2 + \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2 \qquad \text{....... (37)}$$

Substituting the values of u'_1 and u'_2 from (35) and (36) in equation (37), we have

$$\frac{1}{2}m_1\frac{m_2^2}{m_1^2}\,u'^2_2 + \frac{1}{2}m_2\,u'^2_2 = \frac{1}{2}m_1\frac{m_2^2}{m_1^2}\,v'^2_2 + \frac{1}{2}m_2\,v'^2_2$$

$$\text{or} \quad u'^2_2\left[\frac{m_2^2}{2m_1} + \frac{1}{2}m_2\right] = v'^2_2\left[\frac{m_2^2}{2m_1} + \frac{1}{2}m_2\right]$$

$$\therefore \quad u'_2 = v'_2 \qquad \text{........ (38)}$$

Similarly $u'_1 = v'_1$ .... (39)

In order words, in an elastic collision in the centre of mass frame, the magnitudes of the velocities of the particles do not change i.e., there is only a change in direction.

4.6.5 Perfectly Inelastic Collision in One Dimension

A collision is said to be perfectly inelastic if the two particles stick together after collision. We shall discuss the problem in the laboratory frame as well as in the centre of mass frame.

(a) **Laboratory frame.** Let m_1 and m_2 be the masses and $\vec{u}_1$ and $\vec{u}_2$ the velocity of two particles before collision. Since the two particles stick together on impact let their velocity after collision be V.

Initial linear momentum of mass m_1 moving with a velocity $\vec{u}_1 = m_1\vec{u}_1$. Initial linear momentum of mass m_2 at rest = 0

Let the combined mass $(m_1 + m_2)$ move with a velocity $\vec{v}$ in the initial direction then

Final linear momentum of the system $= (m_1 + m_2)\vec{v}$

According to the principle of conservation of linear momentum

$$m_1\vec{u}_1 = (m_1 + m_2)\vec{v} \qquad \therefore \qquad \vec{v} = \frac{m_1}{m_1 + m_2}\vec{u}_1$$

Kinetic energy of m_1 before collision $= \frac{1}{2}m_1u_1^2$

Kinetic energy of m_2 before collision = 0

$[\because m_2 \; is \; at \; rest]$

$\therefore$ Total kinetic energy before collision $T_1 = \frac{1}{2}m_1u_1^2$

Kinetic energy of combined mass $(m_1 + m_2)$ after collision

$$T_1 = \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2}(m_1 + m_2)\frac{m_1^2u_1^2}{(m_1 + m_2)^2} = \frac{1}{2}\frac{m_1^2}{m_1 + m_2}u_1^2$$

$$\therefore \qquad \frac{T_2}{T_1} = \frac{1}{2}\frac{m_1^2u_1^2}{m_1 + m_2} \times \frac{1}{\frac{1}{2}m_1u_1^2} = \frac{m_1}{m_1 + m_2}$$

As $m_1 < m_1 + m_2$, $T_2 < T_1$ i.e. the final kinetic energy after collision is less than the kinetic energy before collision in the lab system.

Decrease in energy. The decrease in energy $E = T_1 - T_2 = \frac{1}{2}\frac{m_1^2 u_1^2}{m_1+m_2}$

$$= \frac{1}{2}\left[\frac{m_1^2 u_1^2 + m_1 m_2 u_1^2}{m_1+m_2}\right] = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} u_1^2 = \frac{1}{2}\mu u_1^2 \qquad \text{......... (40)}$$

, where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

(b) Centre of Mass Frame. Velocity of centre of mass $\vec{V}_{c.m.} = \frac{m\vec{u}_1}{m_1+m_2}$ $[\because \vec{u}_2 = 0]$

Velocity of m1 before collision in C.M. System $= \vec{u}\,'_1 = \vec{u}_1 - \vec{V}_{c.m.}$

$$= u_1 - \frac{m_1}{m_1+m_2}\vec{u}_1 = \frac{m_2}{m_1+m_2}\vec{u}_1$$

Velocity of m_2 before collision in C.M. System

$$\vec{u}\,'_2 = \vec{u}_2 - \vec{V}_{c.m.} = -\vec{V}_{c.m.} = -\frac{m_1}{m_1+m_2}\vec{u}_1$$

After collision the two particles stick together and the combined $(m_1 + m_2)$ moves with a velocity equal to the velocity of centre of mass with respect to the lab, system and is at rest with respect to the centre of mass system itself. Hence according to the principle of conservation of liner momentum $m_1\vec{u}\,'_1 + m_2\vec{u}\,'_2 = 0$

[$\because$ The final linear momentum = 0]

or $\qquad m_1\vec{u}\,'_1 + m_2\vec{u}\,'_2$

Initial kinetic energy of m_1 in C.M. system $= \frac{1}{2}m_1\vec{u}\,'^2_1$

Initial kinetic energy of m_2 in C.M. system $= \frac{1}{2}m_2\vec{u}\,'^2_2$

$\therefore$ Total 41initial kinetic energy in C.M. system $(m_1 + m_2) = T\,'_2 = 0$

[$\because$ $(m_1 + m_2)$ is at rest w.r.t. C.M. system]

Decrease in kinetic energy. The decrease in kinetic energy

$$E = T\,'_1 - T\,'_2 \qquad [\because T\,'_2 = 0]$$

$$= \frac{1}{2}m_1 u'^2_1 + \frac{1}{2}m_1 u'^2_2$$

$$= \frac{1}{2}m_1\left(\frac{m_2 u_1}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1 u_1}{m_1+m_2}\right)^2$$

$$= \frac{1}{2} \frac{m_1 m_2 u_1^2}{(m_1+m_2)^2} [m_1 + m_2] = \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} u_1^2$$

$$= \frac{1}{2} \mu\, u_1^2 \qquad \text{....... (41)}$$

, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

Comparing relations (40) and (41), we find that the decrease in kinetic energy after collision is the same in the lab, system as well as in the C.M. system.

This decrease in K.E. may appear as excitation energy of the scattered particle.

It may also be noted that decrease in energy of the combined mass in C.M. system is equal to the initial kinetic energy of the particles in the same system.

Note: Reduced mass is a useful concept in dealing with bonded system – a system in which particles are bound to each other. For example, two blocks connected with a spring, two stars moving under mutual gravitational pull, H_2O molecule, etc.

In general, the reduced mass of a n-particle system is defined as

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \cdots + \frac{1}{m_n}$$

Example: 8

Two identical billiard balls collide. Just before collision, they were moving with speeds *u* and 2*u* in opposite directions. Each ball has mass m.

(i) Find the momentum of each ball in COM frame before collision.

(ii) Find the momentum of the system of two balls in COM frame after collision.

Solution:

(i) $|\vec{P}_{1CM}| = |\vec{P}_{2CM}| = \mu|\vec{v}_{12}|$

(ii) COM frame is zero momentum frame.

(i) $\mu = \frac{m.m}{m+m} = \frac{m}{2}$

$v_{12} = 3u (= v_{12})$

$\therefore \quad |\vec{P}_{1CM}| = \mu v_{12} = \frac{3}{2}\, mu$

Also $\quad |\vec{P}_{2CM}| = \mu v_{21} = \frac{3}{2}\, mu$

(ii) Momentum of a system is always zero in COM frame.

Self-Assessment Questions (SAQs)

7. Obtain an expression Relation Between Final Velocities and Initial Velocities in COM Frame.

8. Write shorts on Laboratory and COM Frame of Reference.

4.7 MOTION OF CENTRE OF MASS OF A SYSTEM

We know that, Positin vector $\vec{F}$ of a system of a particles is

$$M\vec{R} = \sum_{i=1}^{n} m_i\vec{r}_1 = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots m_n\vec{r}_n \qquad \ldots\ldots\ldots\ldots (42)$$

Differentiating the two sides of the equation with respect to time we get.

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt} \qquad \ldots\ldots\ldots\ldots (43)$$

The rate of change of position $\frac{d\vec{r}_1}{dt}$, $\frac{d\vec{r}_2}{dt}$, $\frac{d\vec{r}_n}{dt}$ can be replace $\frac{d\vec{R}}{dt}$ with $\vec{v}_{cm}$ where $\vec{v}_{cm}$ is the velocity of the centre of mass. Similarly $\frac{d\vec{r}_1}{dt}, \frac{d\vec{r}_2}{dt}, \ldots, \frac{d\vec{r}_n}{dt}$ can be replaced by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We have written the above equations by assuming that masses m_1, m_2, ……. m_n do not change with time, hence can be taken out of the differentiation sign and can be treated as constants.

$$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots m_n\vec{v}_n \qquad \ldots\ldots\ldots.(44)$$

Again, differentiating the above equation (44), we get

$$M\frac{d\vec{v}_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + \frac{m_2 d\vec{v}_2}{dt} + \cdots m_n\frac{d\vec{v}_n}{dt} \qquad \ldots\ldots\ldots.(45)$$

Change in velocity is acceleration, so we get

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots m_n\vec{a}_n \qquad \ldots\ldots\ldots.(46)$$

where a_1, a_2, …….. a_n are the acceleration of first, second, and …….n^{th} particle respectively and $\vec{a}_{cm}$ is the acceleration of the centre of mass of the system of particles.

From Newton's second law, we know that $\vec{F} = m\bar{a}$

Therefore, the force acting on the first particle given by $\vec{F}_1 = m\vec{a}_1$, is the force acting on the second particle and $\vec{F}_n$ is the force acting on the n^{th} particle therefore equation (46) becomes

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n \qquad \ldots\ldots\ldots.(47)$$

Thus, we conclude from equation (47) that the product of the mass of the system and the acceleration of the centre of mass is the vector sum of all the forces acting on the system of particles.

Here you should remember that, when we talk about the forces $\vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n$ on particles of masses m_1, m_2 …. m_n respectively then it is not a single force. Let us consider the force $\vec{F}_1$ acting on particle of mass m_1 then, it is not a single force, rather it is the sum of all the force acting on particle m_1. Similarly $\vec{F}_1 + \vec{F}_2 + \cdots \vec{F}_n$ are the sum of all forces acting on particles of masses m_2, m_3, ….. m_n respectively and this "sum" include all the external and internal forces of the particles.

Now from Newton's third law, we know that internal forces occur in equal and opposite pairs and therefore, in the sum of forces their contribution is zero, as they cancel each other. Hence, only external forces contribute to equation (47) and we can rewrite the equation (47) as

$$\boxed{\boldsymbol{M\vec{a}_{cm} = \vec{F}_{ext}}} \qquad \dots\dots\dots.(48)$$

, where $\vec{F}_{ext}$ is the sum of all external forces acting on the particles of the system.

Equation (48) states that the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at the point.

***Note:* 1.** *To determine the motion of the centre of mass no knowledge of internal forces of the system of particles is required. For this purpose, we need to know only the external forces.*

2. *To obtain equation (vii) we did not need to specify the nature of the system of particles. The system may be a collection of particles or it may be a rigid body.*

4.7.1 Velocity of COM

$$\Rightarrow \qquad \vec{v}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i v_i \qquad \dots\dots\dots\dots\dots\dots\dots (49)$$

$$M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n \dots\dots\dots\dots\dots\dots\dots (50)$$

$$P_{system} = M\vec{v}_{CM} \qquad \dots\dots\dots\dots\dots\dots\dots (51)$$

4.7.2 Acceleration of COM

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n \qquad \dots\dots\dots\dots\dots\dots\dots (52)$$

$$\therefore \qquad M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n \qquad \dots\dots\dots\dots\dots\dots\dots (53)$$

$$\therefore \qquad M\vec{a}_{CM} = \vec{F}_{ext} \qquad \dots\dots\dots\dots\dots\dots\dots (54)$$

Example: 9

A particle of mass m is projected vertically up from a point with initial velocity 20 ms^{-1}. Another particle of mass 2m is projected simultaneously from the same point, with velocity 20 ms-1 at an angle of 30o to the horizontal.

(i) **Find the velocity of the COM of the system of two particles, one second after their projection.**

(ii) **Final acceleration of the COM of the system, one second after their projection.**

Solution:

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \quad \textbf{and} \quad \vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

(i) Velocity of the first particle after 1 s is

$$v = 20 - gt = 20 - 10 \times 1 = 10ms^{-1}(\uparrow)$$

Horizontal and vertical components of initial velocity of the second particle are (Horizontal-x and vertical-y-direction)

$$u_x = 20cos30^o = 10\sqrt{3}ms^{-1} \quad \text{and}$$

$$u_y = 20sin30^o = 10ms^{-1}$$

Velocity components after 1 s

$$v_x = 10\sqrt{3}ms^{-1}; = 10 - gt = 10 - 10 \times 1 = 0$$

∴

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m(10\hat{j}) + 2m(10\sqrt{3}\hat{i})}{m + 2m} = \left(\frac{20}{\sqrt{3}}\hat{\imath} + \frac{10}{3}\hat{j}\right) ms^{-1}$$

(ii) $$\vec{a}_{CM} = \frac{m_1 g(\downarrow) + m_2(\downarrow)}{m_1 + m_2} = g(\downarrow)$$

Example: 10

In the arrangement shown in figure, the horizontal surface is smooth but there is friction between the two blocks. Coefficient of friction is μ. Mass of B is twice that of A. When a horizontal force F is applied to B, the two blocks slip on one another. Find the acceleration of the COM of the system of two blocks, A and B. Mass of A is m.

Solution:

Acceleration of the COM depends on external force only.

Friction force between the blocks is an internal interaction when we are considering (A+B) as our system. F is the only unbalanced external force on the system.

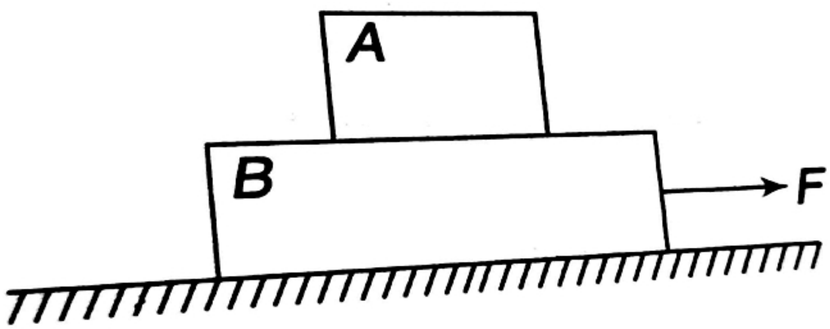


Figure - 13

$\therefore \qquad a_{CM} = \frac{F}{m+2m} = \frac{F}{3m}$

Self-Assessment Questions (SAQs)

9. Explain the concept of Motion of Centre of Mass of a System.

4.8 LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

- The **linear momentum** of a particle with mass m and velocity **v** is defined as a **vector p,**
- **p** = m**v**
- With **p** and **v** in the **same direction**.
- **SI unit:** kilogram-meter per second or kg.m/s.
- **Newton's Second Law of motion in terms of momentum:**
- The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.
- In equation form:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

- The linear momentum of an object can only be changed by a net external force!

Substitute $\vec{p} = m\vec{v}$ into $\vec{F}_{net} = \frac{d\vec{p}}{dt}$

Then $\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$

$$\vec{F}_{net} = m\vec{a} \qquad \text{............(55)}$$

We Studied about in our previous section, we conclude that the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point. Now, what will be the motion of the centre of mass if no external forces are applied? To answer this question let us discuss the linear momentum of the system of particles. Let us recall that the linear momentum of a particle is defined as

$$\vec{P} = m\vec{v}$$

and according to Newton's second law.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

i.e. the rate of change of linear momentum of a particle is equal to the net force acting on the object.

We know that,

$$M\vec{v}_{cm} = \sum_{i=1}^{n} m_i\vec{v}_i$$

and

$$\sum_{i=1}^{n} \vec{P}_i = \sum_{i=1}^{n} m_i\vec{v}_i$$

L.H.S. is the summation of linear momentum of n particles of the system, which is equal to product of the total mass of the system and velocity of the center of mass of the system.

So, $\vec{P} = M\vec{v}_{cm}$ ………… (56)

Differentiating the above equation (56) w.r.t. time, we get

$$\frac{d\vec{p}}{dt} = M\frac{d\vec{v}_{cm}}{dt} = M\vec{a}_{cm} = \vec{F}_{ext}$$ ………… (57)

Now, if the next external force on the system is zero, the linear momentum of the system, is conserved and he centre of mass will move with constant velocity.

Thus, we conclude that if the total external force acting on a system of particles is zero, the velocity of the centre of mass remains constant.

4.8.1 Conservation of Linear Momentum

- Assume that the net external force on a system of particles is zero (an isolated system) and that no particles leave or enter the system (the system is closed).

- If $F_{net} = 0$ is substituted into $\vec{F}_{net} = \frac{d\vec{P}}{dt}$ Eq. 57, then dp/dt = 0
- P = constant (closed, isolated system)
- In words: If no external force act on a system of particles, the total linear momentum of the system does not change.
- Known as: **Law of Conservation of Linear Momentum**:
  - $P_i = P_f$ (closed, isolated system)
  - [Tot. Lin. Momentum at a initial time t_i] = [Tot. Lin. Momentum at a later time t_f]

For a system of **n** particles, each with its own mass, velocity, and linear momentum interacting with each other, the system as a whole has a **total linear momentum P,** which is defined to be **the vector sum of the individual particles' linear momentum.**

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.$$
.... (58)

We get, $\vec{P} = \vec{M}\vec{V}_{com}$
.... (58a)

(linear momentum of system of Particles)

- This gives us another way to define the linear momentum of a system of particles:
- The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.
- Differentiating eq. 58a with respect to time gives:

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com} \quad \text{...................... (59)}$$

- Comparing $F_{net} = Ma_{net}$, eq.55, with eq. then:
- $\vec{F}_{net} = \frac{d\vec{P}}{dt}$ **system of particles)** .... (60)

Example: 11

Two particles of mass 1 kg and 2 kg are moving along the same line with speeds 2 m/s and 4m/s respectively. Calculate the speed of the centre of mass of the system if both the particles are moving in the same direction.

Solution:

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Substituting the value, we get

$$V_{CM} = \frac{1\times2+2\times4}{3} = \frac{2+8}{3} = \frac{10}{3} m/s$$

Example: 12

No external force acts on a system at rest. What is the velocity of the centre of mass?

Solution:

The centre of mass of such a system remains at rest.

Self-Assessment Questions (SAQs)

10. Two particles of equal mass are moving along the same line with the same speed in the same direction. What is the speed of the centre of mass of the system?

HINT

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Solution:

Let the speed of the particles be v and mass of the particle be m.

$$V_{CM} = \frac{mv + mv}{2m} = \frac{2mv}{2m} = v$$

11. Write short notes on the following:

(a) Linear Momentum of a System of Particles

(b) Conservation of Linear Momentum

4.9 ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES

We know that, angular moment of individual particles.

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i$$

The angular momentum of the ith particle is given by $\vec{L}_i = \vec{r}_i \times \vec{p}_i$, where $\vec{r}$ is the position vector of ith particle w.r.t. a given origin and $\vec{p}_i = (m_i \vec{v}_i)$ is the linear momentum of the ith particle. So, the total angular momentum of a system of particles is $\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{P}_i$.
.... (61)

As, we know the relation between the angular momentum ($\vec{L}$) of a Particle and Torque ($\vec{\tau}$).

That is $\frac{d\vec{L}}{dt} = \vec{\tau}_i$, where $\frac{d\vec{L}_i}{dt}$ is the rate of change of angular momentum of the ith particle and $\vec{\tau}_i$ is the torque acting on it. We can generalize the above equation for a system of particles and get $\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum \vec{L}_i\right) = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i$ and $\vec{\tau}_i$ is given by the equation $\vec{\tau}_i = \vec{r}_i \times \vec{F}_i$. …………….. (62)

The force $\vec{F}_i$ is the force acting on the ith particle, which is the vector sum of the external forces $\vec{F}_i^{ext}$ acting on the particle and the internal forces $\vec{F}_i^{ext}$ exerted on it by the other particles of the system. So, the torque acting on the particle can be internal or external. Hence, we may therefore separate the contribution of the external and the internal forces and can write

$$\vec{\tau}_{ext} = \sum_i \vec{r}_i \times \vec{F}_i^{ext} \quad \text{…………….. (63)}$$

$$\vec{\tau}_{int.} = \sum_i \vec{r}_i \times \vec{F}_i^{int.} \quad \text{…………….. (64)}$$

From Newton's third law we know that the forces between any two particles of the system are equal and opposite. But we know that equal forces which are parallel to each other can form couple (Torque) but these forces (internal forces) act along the line joining the two particles. So, in this case the contribution of the internal forces to the total torque on the system is zero, since the torque resulting from each action-reaction pair of forces is zero. We thus have, $\vec{\tau}_{int} = 0$ and therefore $\vec{\tau} = \vec{\tau}_{ext}$.

Since, $\frac{d\vec{L}}{dt} = \vec{\tau} \Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau}_{ext.}$

Thus, the time rate of change of the total angular momentum of a system of particles about a point is equal to the sum of the external torques (i.e. the torque due to external forces) acting on the system taken about the same point. The above equation holds good for any system of particles, whether it is a rigid body or its individual particles have all kinds of rotational motion.

4.9.1 Conservation of Angular Momentum

If the external forces acting on the system of particles is zero

$\Rightarrow \quad \vec{\tau}_{ext.} = 0$, hence

$$\frac{d\vec{L}}{dt} = 0 \quad \text{…………….. (65)}$$

or $\quad \vec{L} = constant$

Thus, if the total external torque on a system of particles is zero then, the total angular momentum of the system is conserved i.e., remains constant. According to the above equation $\vec{L}_x, \vec{L}_y, \vec{L}_z$ all are conserved, $\vec{L}_x, \vec{L}_y$ and $\vec{L}_z$ are the components of angular momentum vector $\vec{L}$ along x, y and z axes respectively. The statement that the total angular momentum is conserved means that each of these three components is conserved. The above equation is the rotational analogue of the conservation law of the total momentum, for a system of particles.

Example: 13

A body of mass m is moving with a constant velocity along a line parallel to the x-axis, away from the origin. Its angular momentum with respect to the origin.

(A) **is zero** **(B)** **remains constant**

(C) **goes on increasing** **(D)** **goes on decreasing**

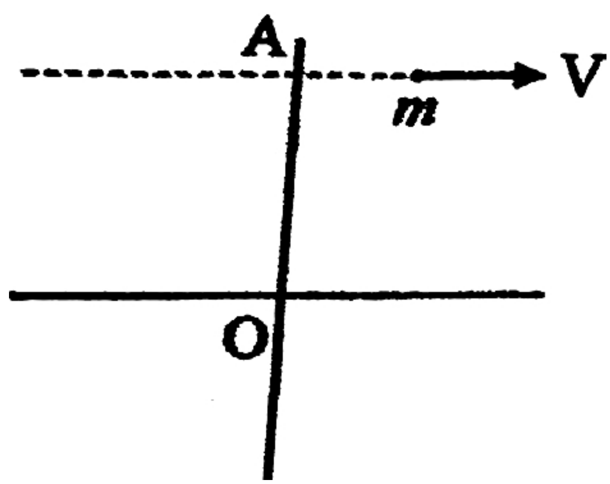


Figure - 14

Solution:

We know that,

$$L = mvr_{\perp}$$

$$L = mv(OA)$$

As OA and v are both constant, the angular momentum remains constant.

option (B)

Example: 14

A particle of mass 1 kg is moving along the line y = x + 2 (here, x and y are in meters) with speed 2m/s. The magnitude of angular momentum of particle about origin is:

(A) $4kg - m^2/s$ (B) $\sqrt[2]{2}\, kg - m^2/s$

(C) $\sqrt[4]{2}\, kg - m^2/s$ (D) $2kg - m^2/s$

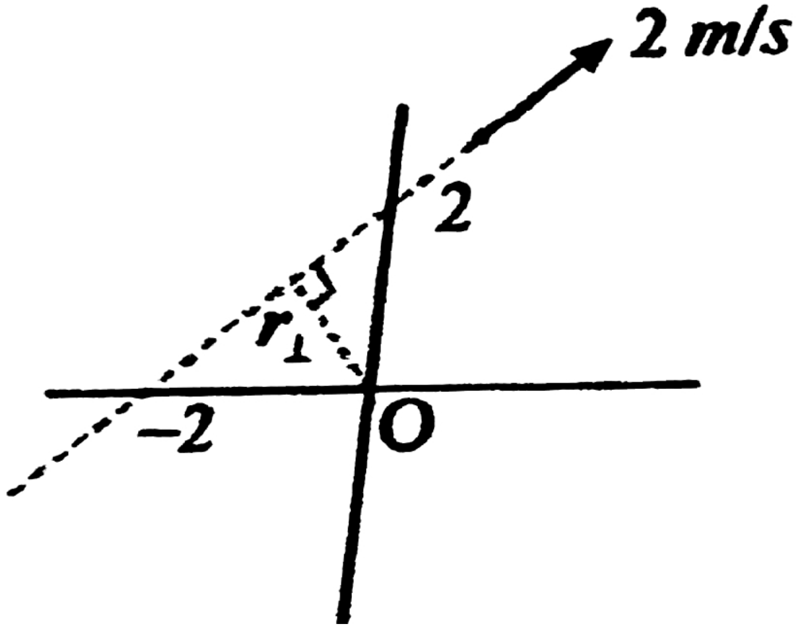


Figure - 15

Solution:

We know that,

$L = mvr_\perp$ $L = (1kg)(2m/s)\ (\sqrt{3}m)$

where $r_\perp = \frac{|0-0+2|}{\sqrt{1^2+1^2}} = \sqrt{2}$ $L = \sqrt[2]{2}\, kg\ m^2/s$

option (B)

Example: 15

A particle of mass m is projected with a velocity v_0 making angle of 45° with the horizontal. The magnitude of the angular momentum of the projectile about the point of projection when the particle is at its maximum height h is:

(A) $zero$ (B) $mv^3/(\sqrt[4]{2}g)$

(C) $mv^3/(\sqrt{2}g)$ (D) $\sqrt[m]{2gh^3}$

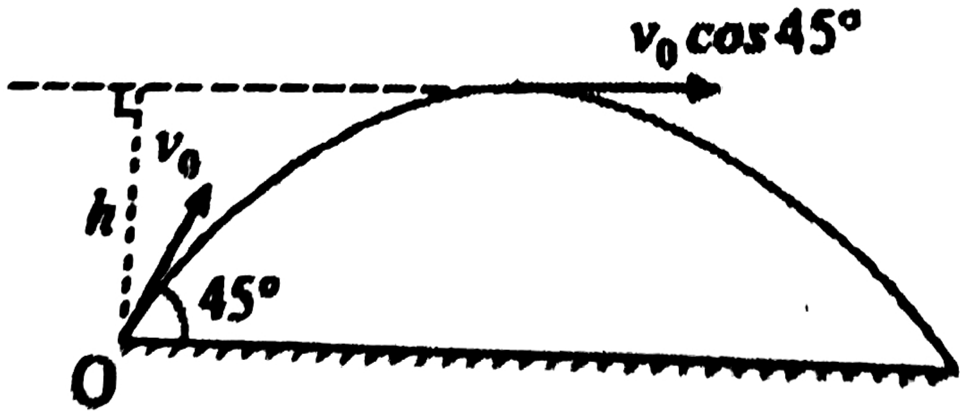


Figure - 16

Solution:

We know that,

$L_0 = mvr_\perp$

At the highest point, V = speed = $V_0 \cos 45^o$

$$r_\perp = h = \frac{V_0^2 \sin^2 45^0}{2g}$$

$\Rightarrow \quad L_0 = m\left(\frac{V_0}{\sqrt{2}}\right)\left(\frac{V_0^2}{4g}\right) = \frac{mV_0^3}{4\sqrt{2}g}$

option (B)

Self-Assessment Questions (SAQs)

12. Obtain an expression for the Angular Momentum for a System of Particle.
13. Write short notes on Conservation of Angular Momentum.

4.10 EQUILIBRIUM OF A RIGID BODY

A rigid body is said to be in mechanical equilibrium, if its linear momentum and angular momentum are not changing with time i.e. the body has neither linear acceleration nor angular acceleration. This means

(a) The total forces, i.e. the vector sum of the forces acting on the rigid body is zero,

$$\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. So, the above equation gives the condition for the translational equilibrium of the body.

(b) If the total torque acting on the rigid body vanishes i.e. the vector sum of the torques on the rigid body is zero,

$$\vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_n = \sum_{i=1}^{n} \vec{\tau} = 0. \qquad \text{................ (67)}$$

The total angular momentum of the body does not change with time. So, the above equation gives the condition for the rotational equilibrium of the body.

Example: 16

In the figure (17) given below, what is the equilibrium of the rod i.e. is it translational or rotational?

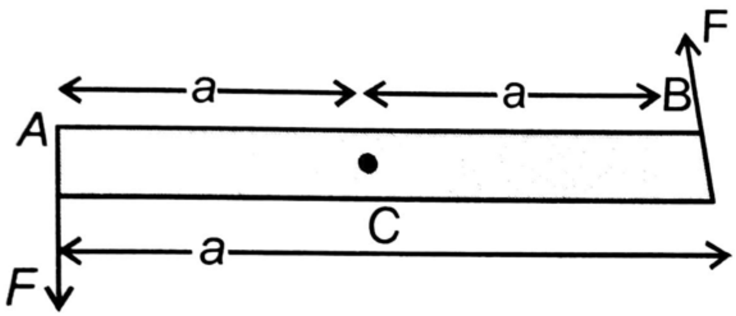


Figure - 17

Solution:

As the forces are acting in opposite directions therefore $\sum \vec{F} = 0$ hence the rod is in translational equilibrium.

Self-Assessment Questions (SAQs)

14. Write down condition for Equilibrium of a Rigid Body.

4.11 TORQUE OF A SYSTEM OF PARTICLES

Torque is a quantity which measures the capability of a force to rotate a body. Torque due to a force is also known as the moment of a force. It is defined as the product of the force and the perpendicular distance between the line of action of the force and the axis of rotation. This perpendicular distance is known as the force arm.

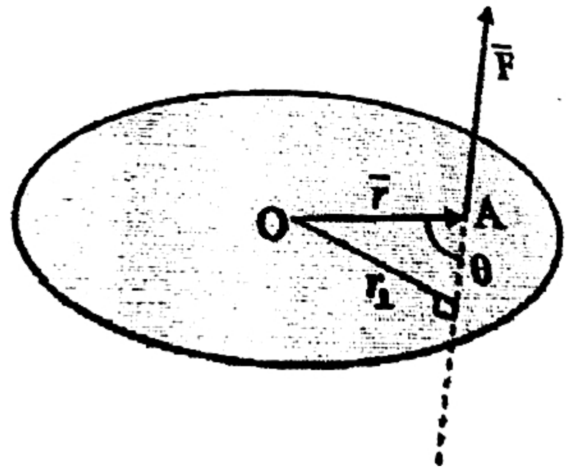


Figure - 18

Torque = τ = (force) × (force arm)

$$\tau = Fr_{\perp} \qquad \text{................ (68)}$$

Units of torque are N-m.

Torque is a vector quantity. Torque is clock-wise if the tendency of force is to produce clock-wise rotation and vice-versa. In vector torque due to a force F acting at a point A (whose position vector is r) is:

$$\bar{\tau} = \bar{r} \times \bar{F} \qquad \text{................ (69)}$$

Taking magnitude, we can see that: $\tau = r\,F \sin\theta$

$$\Rightarrow \qquad \tau = F(r \sin\theta) = Fr_{\perp}$$

$$\Rightarrow \qquad \boldsymbol{\tau = Fr_{\perp}} \qquad \text{................ (70)}$$

Couple

A pair of two equal and opposite forces acting along parallel lines but having different lines of action constitutes a couple.

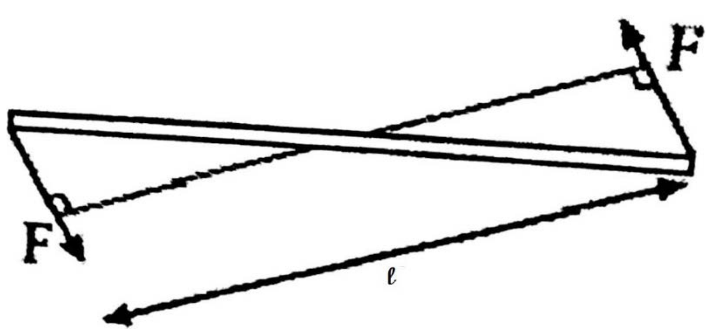


Figure - 19

Moment of couple or torque $= F\ell$

= (force) × (perpendicular distance between forces)

4.11.1 Work done by a Torque

Consider a rigid body acted upon by a force F at perpendicular distance r from the axis of rotation. Suppose that under this force, the body rotates through an angle $\Delta\theta$.

Work done = force × displacement

$$W = Fr.\ \Delta\theta$$

$$W = \tau\ \Delta\theta \qquad \text{……………..} (71)$$

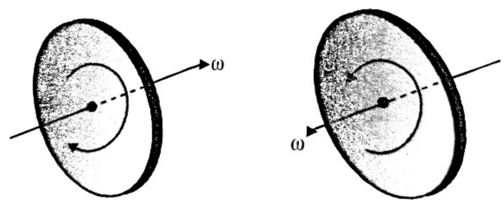


Figure - 20

Work done = (torque) × (angular displacement)

$$Power = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

$$\therefore \quad Power = \tau\omega \qquad \text{……………..} (72)$$

4.11.2 Torque and Newton's IInd Law for Rotation

According to Newton's IInd Law of rotation:

The rate of change angular momentum of a body is equal to the net torque acting on it.

$$\frac{\Delta L}{\Delta t} = \tau \qquad \text{……………..} (73)$$

$$I\frac{\Delta\omega}{\Delta t} = \tau \qquad \text{……………..} (74)$$

where,

$$\left(\alpha = \frac{\Delta\omega}{\Delta t}\right)$$

Remember: *This important equation $\tau = I\alpha$ can be compared with Newton's IInd Law of motion (F = ma). Hence in rotation, torque (τ) plays the role corresponding to force in linear motion.*

Example: 17

A grind-stone is in the form of a solid cylinder has a radius of 0.5 m and a mass 50 kg.

(a) What torque will bring it from rest to an angular velocity of 300 rev/min in 10 s?

(b) What is the kinetic energy when it is rotating at 300 rev/min?

Solution:

Let $\omega_i = 0 rad/s,$

$\omega_f = 2\pi\left(\frac{300}{60}\right) = 10\pi\ rad/s^2$

(a) $\alpha = \frac{\omega_f - \omega_i}{t} = \frac{10\pi - 0}{10} = \pi\ rad/s^2$

Torque required $= \tau = I\alpha = \left(\frac{1}{2}MR^2\right)\alpha$

$\Rightarrow \quad \tau = 1/2(50)(0.5)^2\pi = 19.6\ Nm$

(b) Kinetic energy of a rotting body (RKE):

$R\ K\ E = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$

$= \frac{1}{2}\left(\frac{1}{2}(50)(0.5)^2\right)(10\pi)^2$

R K E = 3084 *J*

Example: 18

Calculate the torque by an airplane engine whose output is 2000 hp at an angular velocity of 2400 rev/min.

Solution:

$\omega = 2\pi\left(\frac{2400}{60}\right) = 80\pi\ rad/s.$

Work done by torque

= (torque) × (angular displacement)

Power = work done per sec $= \tau\frac{\Delta\theta}{\Delta t}$

Power $= \tau\omega$

$\tau = \frac{2000 \times 746}{80\pi} = 5937\ Nm$

Self-Assessment Questions (SAQs)

15. Calculate the instantaneous power of a wheel rotating with an angular velocity of 20 rad/s, when a torque of 10 Nm is applied to it.

16. Derive an expression of work done by a Torque.

4.12 KINETIC ENERGY OF A SYSTEM OF PARTICLE

Consider a system consisting of particles of masses $m_1, m_2, m_3 \dots m_n$ and velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_n$ respectively. Let the velocity of the COM of the system be $\vec{v}_{CM}$.

In reference frame attached to the COM, the velocity of i^{th} particle will be

$$\vec{v}_{iCM} = \vec{v}_i - \vec{v}_{CM} \qquad \Rightarrow \qquad \vec{v}_i = \vec{v}_{iCM} + \vec{v}_{CM}$$

kinetic energy of the system of particles in ground frame is

$$k = \frac{1}{2}\sum m_i(\vec{v}_i.\vec{v}_i)$$

$$= \frac{1}{2}\sum m_i(\vec{v}_{iCM} + \vec{v}_{iCM}).(\vec{v}_{iCM} + \vec{v}_{iCM})$$

$$K = \frac{1}{2}m_i v_{iCM}^2 + \frac{1}{2}m_i v_{iCM}^2 + (\sum m_i \vec{v}_{iCM})\vec{v}_{CM} \qquad \dots\dots\dots\dots\dots\dots (75)$$

[Since $\vec{a}.\vec{a} = a^2$]

The third term is zero, since $\sum m_i \vec{v}_{iCM}$ is the momentum of the system in COM frame and it must be zero.

$$\therefore \qquad k = \frac{1}{2}\sum m_i v_{iCM}^2 + \frac{1}{2}(\sum m_i) v_{CM}^2$$

$$= \frac{1}{2}\sum m_i v_{iCM}^2 + \frac{1}{2}M v_{CM}^2$$

$$k = k_{wrtCM} + \frac{1}{2}\, M v_{CM}^2 \qquad \dots\dots\dots\dots\dots\dots (76)$$

k_{wrtCM} is the kinetic energy of the system in COM frame and $\frac{1}{2}\, M v_{CM}^2$ is the kinetic energy associated with the translational motion of the system as a whole.

If no external force is acting on a system, v_{CM} will not change. Only k_{wrtCM} can change due to internal interactions. This has a very important implication. When no external force is acting on a system, change in KE in ground frame will be same as the change in COM frame.

We can easily prove that for a two-particle system

$$\boxed{k_{wrtCM} = \frac{1}{2}\mu v_{rel}^2} \qquad \dots\dots\dots\dots\dots\dots (77)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and $|v_{rel}| = |\vec{v}_1 - \vec{v}_2|$

REMEMBER:

(a) COM frame is 'zero momentum frame'. Linear momentum of a system in its COM frame is always zero.

(b) In a system having two particles, the particles move with equal and opposite momentum in COM frame. Magnitude of momentum of each particle in COM frame is

$$P = \mu v_{rel}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, know as reduced mass and v_{rel} is the relative speed of the two particles.

(c) KE of a system in COM frame, for a two-particle system, is $k_{wrtCM} = \frac{1}{2}\mu v_{rel}^2$

(d) KE of a system in ground frame is

$$k = k_{wrtCM} = \frac{1}{2}M v_{CM}^2$$

where M is the total mass of the system.

(e) If $F_{ext} = 0$, then change in KE in ground frame is same as the change in KE in COM frame.

Example: 19

Two identical particles, each of mass m are moving in perpendicular directions with speed v. Find the kinetic energy of the system in a reference frame attached to the COM of the system.

Solution:

$$k_{wrtCM} = \frac{1}{2}\mu v_{rel}^2$$

$$\mu = \frac{mm}{m+m}\frac{m}{2}$$

$$\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2 \qquad \Rightarrow \qquad |\vec{v}_{rel}| = \sqrt{v_1^2 + v_2^2}$$

$$\therefore \qquad k_{wrtCM} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\frac{m}{2}(v_1^2 + v_2^2) = \frac{1}{4}m(v_1^2 + v_2^2)$$

Self-Assessment Questions (SAQs)

17. Write short notes on Kinetic Energy of a System of Particle.

4.13 GRAVITATIONAL POTENTIAL ENERGY OF AN EXTENDED BODY

The potential energy of a body is simply the sum of potential energies of its constituent particles. In the figure shown,

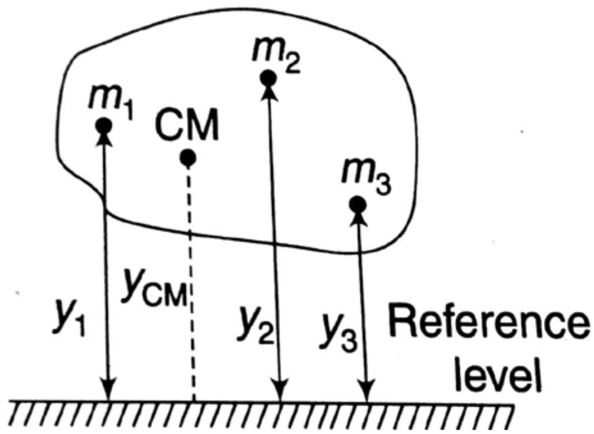


Figure - 21

the potential energy of the body can be written as

$$u = m_1 g y_1 + m_2 g y_2 + \cdots$$

$$= \frac{(m_1 y_1 + m_2 y_2 + \cdots)g}{M} . M \qquad [M = m_1 + m_2 + \cdots]$$

$$\therefore \quad u = M g y_{CM}$$

It means, we can assume the entire body to be a point mass placed at its COM, for purpose of writing potential energy.

Self-Assessment Questions (SAQs)

18. Write down an Expression of Gravitational Potential Energy of an Extended Body.

4.14 DIFFERENCE BETWEEN CONSERVATION LAWS

According to law of Conservation of momentum, angular & Linear Moments both should be conserved.

4.14.1 Conservation of Linear Momentum for a Particle and System of Particle

We know that,

According to Newton's Third Law of Motion

$$F_1 = -F_2$$

$$\frac{m_2(v_2 - u_2)}{t} = \frac{m_1(v_1 - u_1)}{t}$$

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

Sum of linear momentum after collision = Sum of linear momentum before collision.

The Principle of Conservation of Linear Momentum and Newton's Third Law of Motion are consistent with each other.

- Suppose, that the net external force on a system of particles is zero (an isolated system) and that no particles leave or enter the system (the system is closed).
- If $F_{net} = 0$ is substituted into $\vec{F}_{net} = \frac{d\vec{P}}{dt}$ ………….(78)

  then dp/dt = 0
- P = constant (closed, isolated system)
- In words: If no external force act on a system of particles, the total linear momentum of the system does not change.
- Known as: **Law of Conservation of Linear Momentum**:
  - $P_i = P_f$ (closed, isolated system)
  - [Tot. Lin. Momentum at a initial time t_i] = [Tot. Lin. Momentum at a later time t_f]

For a system of **n** particles, each with its own mass, velocity, and linear momentum interacting with each other, the system as a whole has a **total linear momentum P,** which is defined to be **the vector sum of the individual particles' linear momentum.**

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n. \quad \text{………… (79)}$$

We get,

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{M}}\vec{\boldsymbol{V}}_{com} \quad \text{………… (80)}$$

(linear momentum of system of Particles)

- This gives us another way to define the linear momentum of a system of particles:
- The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.
- Differentiating eq. 80 with respect to time gives: $\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com}$ …… ……………. (81)

  We get,
- $\vec{\boldsymbol{F}}_{net} = \frac{d\vec{\boldsymbol{P}}}{dt}$ **system of particles)** …………………. (82)

4.14.2 Conservation of Angular Momentum for a Particle and System of Particle

(a) For a particle

Angular momentum about origin (O) is given as:

$\vec{L} = \vec{r} \times (m\vec{v})$

Therefore, $\vec{L} = \vec{r} \times \vec{P}$

where $\bar{r}$ = position vector of the particle; $\vec{v}$ = velocity,

$\vec{P} = linear\ momentum$

$\Rightarrow \quad L = mv\, r \sin\theta = mv(OA)\sin\theta = mvr_{\perp}$

where $r_{\perp}$ = perpendicular distance of velocity vector from O.

(b) For a particle moving in a circle of radius r with a speed v, its linear momentum is mv, its angular momentum (L) is given as:

$L = mvr_{\perp} = mvr$

(c) For a rigid body (about a fixed axis)

Angular momentum of a rigid body (or any system of Particle) about an axis is the sum of angular momentum of individual particles.

L = sum of angular momenta of all particles

$= m_1v_1r_1 + m_2v_2r_2 + m_3v_3r_3 + \cdots$

$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + \cdots \qquad (v = r\omega)$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^3 \ldots\ldots)\omega$

$$L = I\omega$$

(compare with linear momentum p = mv in linear motion)

L is also a vector and its direction is same as that of ω (i.e., clockwise or anticlockwise)

In a nutshell, we can say if the external torque is zero, angular momentum of the system is conserved i.e. as $\vec{\tau} = 0$ and

$\vec{\tau} = \frac{d\vec{L}}{dt}$

$\Rightarrow \quad \frac{d\vec{L}}{dt} = 0$

$\Rightarrow \quad \vec{L} = constant$

We know, $\vec{L} = I\omega = constant$

Hence, $I_1\omega_1 = I_2\omega_2$

……………………(83)

The above equation (83) applies to many situations that we come across in daily life.

(a) A circus acrobat performs feats involving spin by bringing his arms and legs closer to his body or vice-versa. This is because, on bringing the arms and legs closer to the body, his moment of inertia/decreases.

Hence his angular velocity ω increases.

$\vec{L} = I\vec{\omega} = constant$

as I decreases, ω (increases) so as to keep the angular momentum constant.

(b) On the basis of conservation of angular momentum an ice skater performs the feats.

Suppose an ice skater is rotating with her arms and legs stretched outwards. When she folds her arms and brings the stretched leg close to the other leg, her moment of inertia decreases and hence, her angular velocity increases, as shown in the given figure (22a) & (22b). As $\vec{L} = I\vec{\omega} = constant$ and as I (decreases), ω (increases) to keep the angular momentum constant.

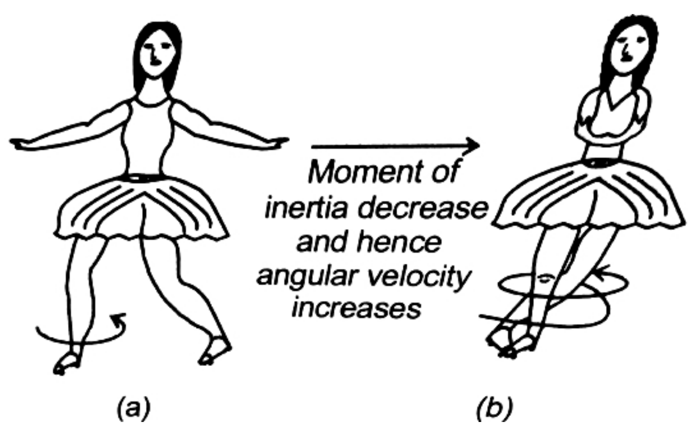


Figure – 22a & 22b

4.14.3 Applications of Conservation of Angular Momentum

(a) A rotating disc picks up two particles that stick to its periphery

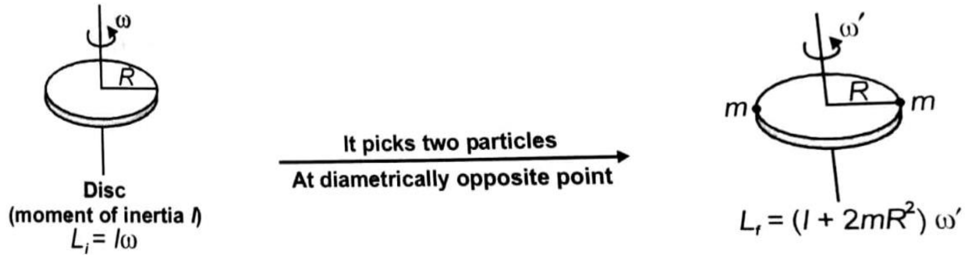


Figure - 23

$$L_i = L_f \Rightarrow \omega' = \frac{I\omega}{I+2mR^2}$$

(b) An standing on a rotating platform starts walking on it.

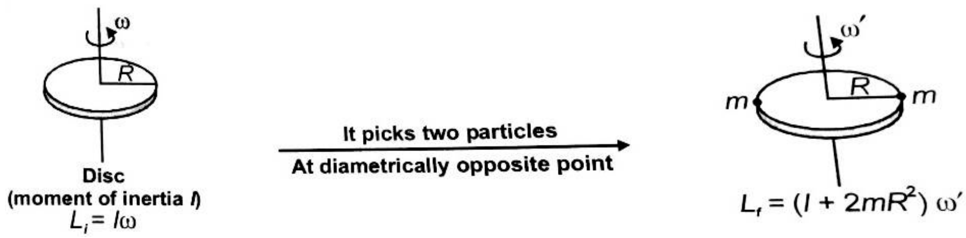


Figure - 24

Let moment of inertia of platform be I and mass of the man standing at an edge be m $L_i = (I + mR^2)\omega$

The man starts walking with speed v in same sense w.r.t. the platform. The angular velocity of the platform becomes

$$\omega' = \omega - \frac{mvR}{I + mR^2}$$

4.15 SUMMARY

In the present unit, we have studied about centre of mass of a system and centre of gravity. We have studied about location of COM, zero momentum frame. In this unit, we have also established the relation between the angular momentum and torque. We have included examples and self-assessment questions (SAQs) to check your progress.

4.16 TERMINAL QUESTIONS

1. Fill in the blanks.

(a) The centre of mass of a body …………………. With the redistribution of mass.

(b) The centre of mass of a body ……………….. on the distribution of mass.

(c) The centre of mass of a thin uniform circular wire lies at its ………………

(d) The centre of mass of an equilateral triangle lies at its …………….. .

(e) The acceleration of the centre of mass is given by ………………….. .

(f) In an elastic collision, the kinetic energy of the system remains …………… .

(g) In an inelastic collision, the kinetic energy of the system ……………… conserved.

2. Write answer to the following questions:

   (a) What is centre of mass?

   (b) Define the term ' centre of gravity of a body'.

   (c) State factor on which the position of centre of gravity of a body depend?

   (d) What is the position of centre of gravity of a:

      (i) rectangular lamina

      (ii) cylinder

3. Differentiate between the centre of mass and centre of gravity of a rigid body.

4. Explain the equilibrium of a rigid body.

5. Obtain an expression for the position vector of the center of mass of a two-particle system in one dimension.

6. Discuss the motion of centre of mass with the help of example.

7. Where does the centre of mass of a Uniform circular ring lie?

8. Give the location of the centre of mass of a

   (i) Sphere

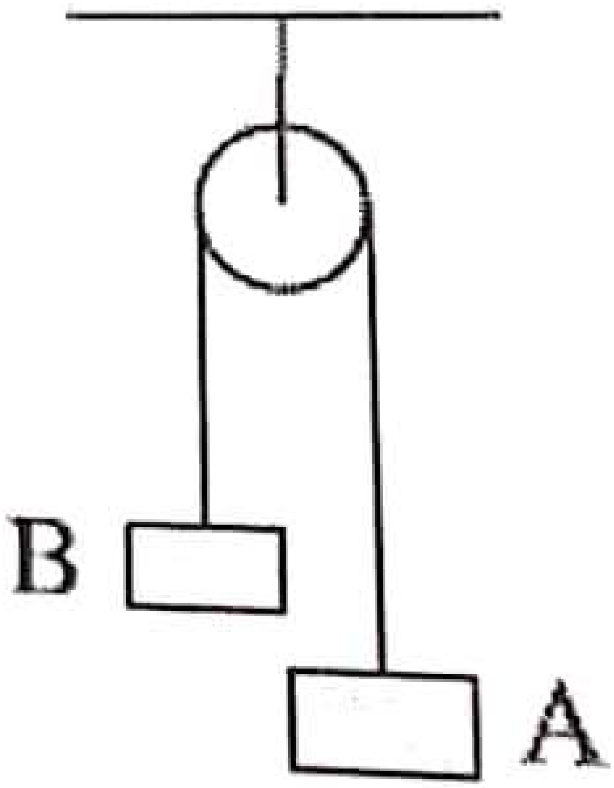
   (ii) Cylinder

   (iii) Ring

   (iv) Cube

9. Does the centre of mass of a body necessarily lie on the body.

10. In the given figure, calculate the linear acceleration of the blocks.



Mass of block A = 10 kg

Mass of block B = 8 kg

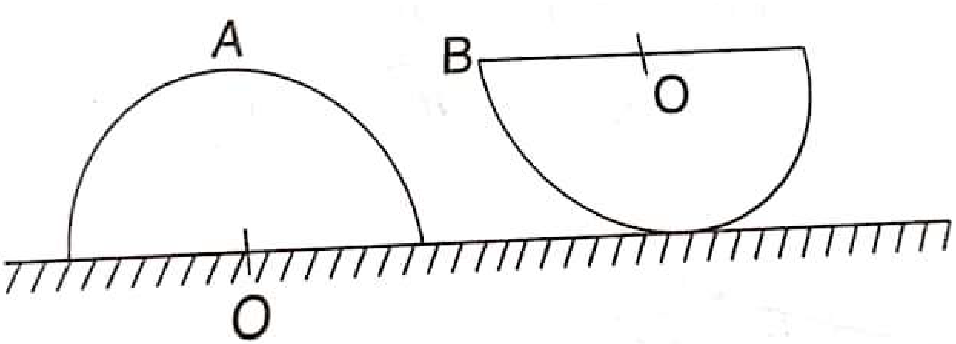
Mass of disc shaped pulley = 2 kg

(take g = 10 m/s^2)

11. Two balls of masses, m and 2m, approach each other with a relative velocity u and collide. After collision, they separate from one another at a relative speed $\frac{u}{2}$.

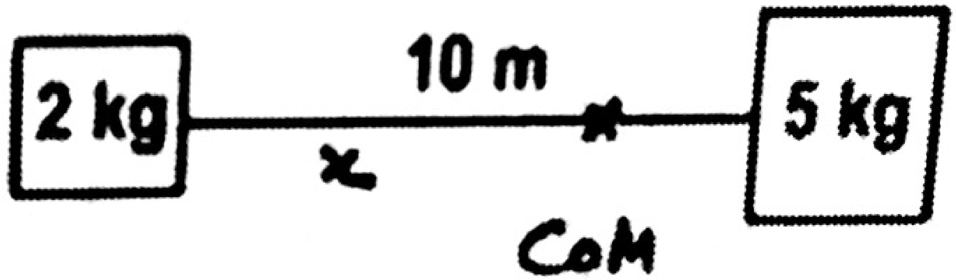
    Find loos in kinetic energy of the system of the two balls due to collision.

12. A uniform solid hemisphere is kept on a horizontal table in two ways – shown as A and B in figure.

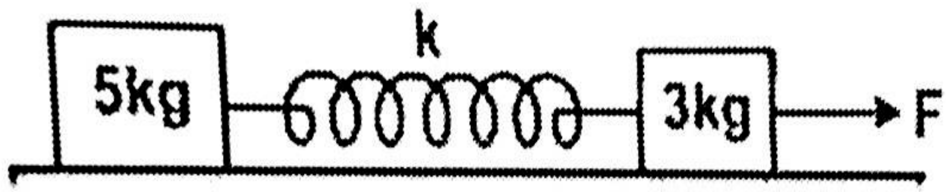


In which case is the potential energy of the hemisphere higher?

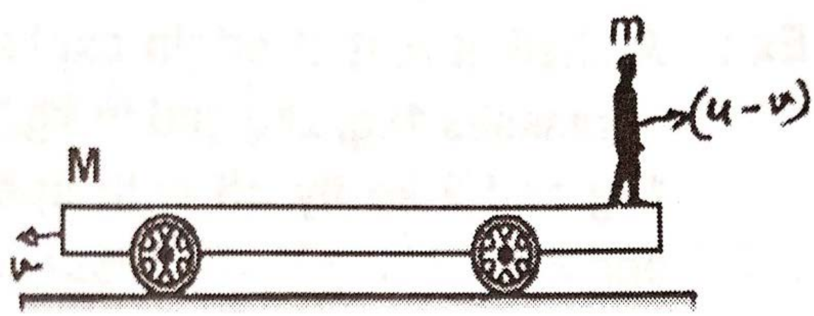
13. (a) Find location of center of mass of this system from 2 kg mass.



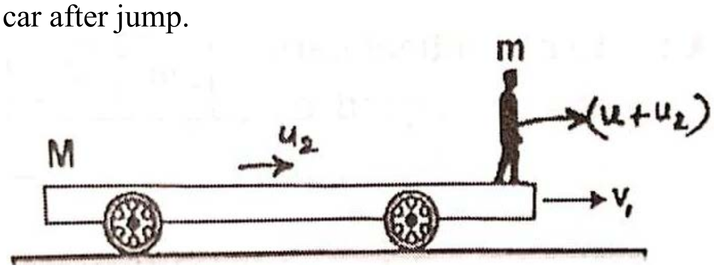
(b) Two blocks of masses 3 kg and 5 kg are connected by an ideal spring and placed on a smooth surface as shown. A force of 40 N is applied on 3 kg block and it is observed that at an instant acceleration of 3 kg mass is 2 m/s^2. Find acceleration of 5 kg mass at this instant.



14. A flat car of mass M is at rest on a frictionless floor with a child of mass m standing at the edge. If child jumps off from the car toward right with initial velocity u, with respect to car, find the velocity of car after the jump.



15. A flat car of mass M with a child of mass m moving at speed v_1. The child jumps in the direction of motion of car with a velocity u with respect to car. Find the final velocities of the child and that of car after jump.



16. (a) Obtain an expression of Kinetic Energy of a System of Particle.

(b) Write down the expression of Gravitational Potential Energy of an Extended Body.

17. Obtain the Relation between final velocities and Initial velocities in COM frame.

18. Explain the concept of Angular momentum for a System of Particles.

19. Deduce relation between Torque and Newton's IInd law of Rotation.

20. Discuss applications of Conservation of Angular Momentum of Particle.

21. Write short notes on

(a) Velocity and Acceleration of the Center of Mass

(b) Conservation of Linear Momentum

(c) Motion of Center of Mass

(d) Center of Gravity

22. Match the following: (for a Body and Center of Mass).

Column A (Body)		**Column B (Position of Center of Mass)**	
(a)	Uniform Rod	(i)	Center of Rod
(b)	A Plane Square Plate (Lamina)	(ii)	Point of Intersection of diagonals
(c)	Hollow Cylinder	(iii)	Middle Point of the axis of the Cylinder
(d)	Cone or Pyramid	(iv)	On the axis of the cone at a point distant 3h/4 from the vertex, where h is the height of the cone.

23. Match the following:

Column A (Body)		Column B (Position of Center of Mass)	
(a)	Uniform hollow sphere	(i)	Center of the sphere
(b)	Uniform circular ring	(ii)	Center of the ring
(c)	Uniform circular disc	(iii)	Center of the disc
(d)	Solid cylinder	(iv)	Middle point of the axis of the cylinder

4.17 SOLUTIONS AND ANSWERS

1. Hint (Section 4.3)
2. It is the average of the Position vectors of two Particles.
3. Hint (Section 4.3.2)
4. Hint (Section 4.4.1)
5. Let the coordinates of the centre of mass be (x, y)

$$x = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

$$= \frac{1\times(-1)+2\times 2}{3}$$

$$= \frac{-1+4}{3} = 1$$

$$y = \frac{m_1y_1+m_2y_2}{m_1+m_2}$$

$$= \frac{1\times 2+2\times 4}{3}$$

$$= \frac{2+8}{3} = \frac{10}{3}$$

Therefore, the coordinates of centre of mass be $\left(1, \frac{10}{3}\right)$.

6. Yes, the centre of gravity of a body can be situated outside its material of the body.

Example: Centre of Gravity of a rign.

7. Hint (Section 4.6.5)

8. Hint (Section 4.6)

9. Hint (Section 4.7)

10. Let the speed of the particles be v and mass of the particle be m.

$$V_{CM} = \frac{mv+mv}{2m} = \frac{2mv}{2m} = v$$

11. (a) Hint (Section 4.8)

(b) Hint (Section 4.8.1)

12. Hint (Section 4.9)

13. Hint (Section 4.9.1)

14. Hint (Section 4.10)

15. $P = \tau\omega$

$\tau = 10\ Nm, \omega = 20\ rad/s$

∴ $P = 10 \times 20 = 200\ W$

16. Hint (Section 4.11)

17. Hint (Section 4.12)

18. Hint (Section 4.13)

ANSWERS TERMINAL QUESTONS:

1. (a) changes

(b) depends

(c) geometrical center

(d) centroid

(e) $\frac{F}{m}$

(f) conserved

(g) does not remain

2. (a) Hint (Section 4.3)

(b) Hint (Section 4.5)

(c) Centre of gravity of a body of given mass position depends on its shape i.e. on the distribution of mass. For example: Uniform wire's centre of gravity is at its mid-point. But if this wire is bent to make a circle, its centre of gravity will then be at the centre of circle.

(d) (i) The position of centre of gravity of a rectangular lamina is at the point of intersection of its diagonals.

(ii) The position of centre of gravity of a cylinder is at the midpoint on the axis of cylinder.

3. Hint (Section 4.5.1)

4. Hint (Section 4.10)

5. Hint (Section 4.4.1)

6. Hint (Section 4.7)

7. Hint (Section 4.4.4)

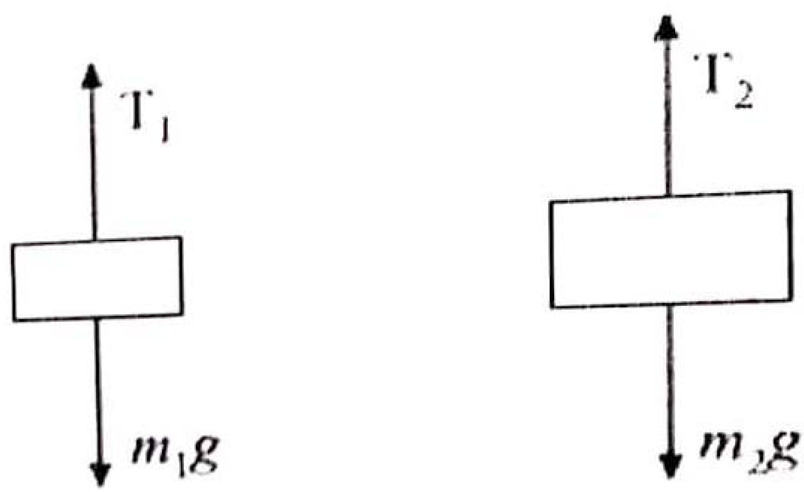
8. Hint (Section 4.4.4)

9. No, it is not necessary that center of mass of a body lies insides the body.

10. Let R be the radius of the pulley and T_1 and T_2 be the tensions in the left and right portions of the string.

Let $m_1 = 10$ kg ; $m_2 = 8$ kg ; M = 2 kg.

Let a be the acceleration of blocks.



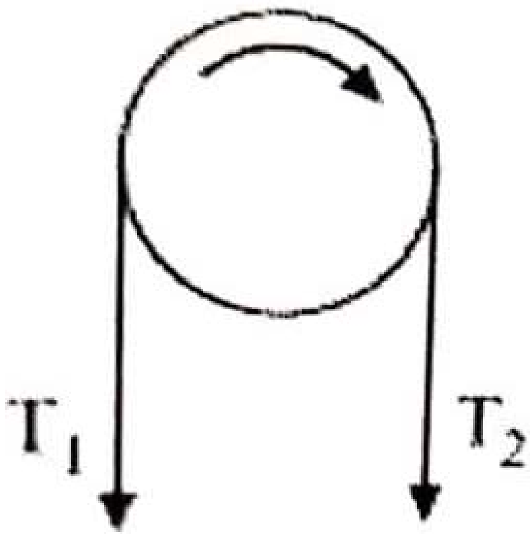
For the blocks (linear motion)

(i) $T_1 - m_1g = m_1a$

(ii) $m_2g - T_2 = m_2a$

For the pulley (rotation)

Net torque $= I\alpha$



(iii) $T_2R - T_1R = \frac{1}{2}MR^2\alpha$

The linear acceleration of blocks is same as the tangential acceleration of any point on the circumference of the pulley which is $R\alpha$.

(iv) $a = R\alpha$

Diving (iii) by R and adding to (i) and (ii),

$$m_2g - m_1g = m_2a + m_1a + \frac{M}{2}R\alpha$$

$$\Rightarrow \quad m_2g - m_1g = \left(m_2 + m_1 + \frac{M}{2}\right)a$$

$$a = \frac{m_2 - m_1}{m_2 + m_1 + \frac{M}{2}}g$$

$$= \frac{(10-8)g}{10+8+\frac{2}{2}} = \frac{20}{19}m/s^2$$

11. When no external force acts, change in KE of a system in COM frame is same as the change in KE in ground frame.

KE of system in COM frame before collision is

$$k_{wrtCM} = \frac{1}{2}\mu u^2 = \frac{1}{2}\left(\frac{m.2m}{3m}\right)u^2 = \frac{1}{2}mu^2$$

KE in COM frame after collision is

$$k_{wrtCM} = \frac{1}{2}\mu\left(\frac{u}{2}\right)^2 = \frac{1}{2}\left(\frac{2}{3}m\right)\frac{u^2}{4} = \frac{1}{12}mu^2$$

Loss in KE in COM frame = Loss in KE in ground frame

$$= \frac{1}{3}mu^2 - \frac{1}{12}mu^2 = \frac{1}{4}mu^2$$

12. For writing PE, we can assume that the entire mass is concentrated at its COM.

Distance of COM from centre O is $\frac{3R}{8}$

$\therefore$ PE in position A is $u_A = Mg\left(\frac{3R}{8}\right)$

PE in position B is $u_B = Mg\left(R - \frac{3R}{8}\right) = Mg\left(\frac{5R}{8}\right)$

$\therefore \qquad u_B > u_A$

13. (a) $x = \frac{5\times10}{2+5} = \frac{50}{7} m.$

(b) As F is the only external force present

Acc^n of COM $a_c = \frac{F}{8} = \frac{40}{8} = 5m/s^2$

As we know $a_c = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} \Rightarrow 5 = \frac{3\times2+50}{8}$

$5a = 40 - 6 = 34$

$a = \frac{34}{5} m/s^2$

14. By using Conservation of Momentum, we use23.

Column A (Body)		Column B (Position of Center of Mass)	
(a)	Uniform hollow sphere	(i)	Center of the sphere
(b)	Uniform circular ring	(ii)	Center of the ring
(c)	Uniform circular disc	(iii)	Center of the disc
(d)	Solid cylinder	(iv)	Middle point of the axis of the cylinder

4.18 SUGGESTED READINGS

1. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd.

2. Introductory Mechanics, David Morin.

3. Berkeley Physics Course Vol. I, Mechanics, C. Kittel et al, McGraw – Hill Company.

4. Concepts of Physics, Part I, HC Verma, Bharati Bhawan, Patna.

5. Elementary Mechanics, IGNOU, New Delhi.

UNIT 5

DYNAMICS OF RIGID BODY

Structure:

5.1 INTRODUCTION

In the previous units, we have studied about centre of mass and centre of gravity of a system. To simplify our study of particles, we consider extended bodies as rigid bodies. Ideally a rigid body is a body with a definite and unchanging shape. Since real bodies deform under the influence of forces therefore, no real body is truly rigid. Next, we introduced the concept of inertia tensor. Afterwards, some important properties of inertia tensor. In this unit, we shall also establish the relationship among angular momentum, moment of inertia and angular velocity in tensor form.

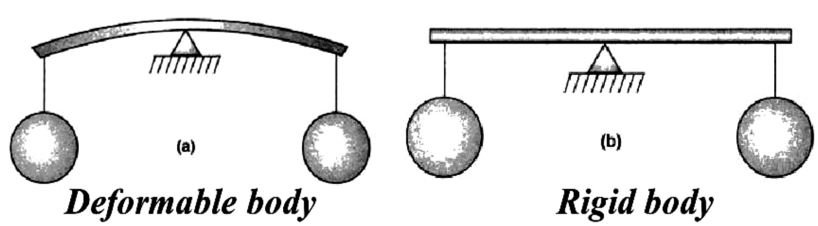
5.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand the concept of rigid body
- Relate equation among angular momentum, moment of inertia angular velocity in tensor form.
- Define Moment of Inertia
- Explain the concept inertia tensor
- Understand what is meant by Precessional Motion

5.3 CONCEPT OF RIGID BODY

Anybody which doesn't undergo deformation (change in length or change in area or change in shape) under the action of forces is said to be rigid body



5.3.1 RIGID BODY

A body is said to be rigid if the relative position of parts of the body remains unchanged during motion or under the action of external forces. During motion the body as a whole move. It can also be considered a system consisting of a large number of particles such that the distances between pairs of particles remain constant. That is, $r_{ij} = c_{ij}$, where r_{ij} is the distance between the *i*th and *j*th particles, and c_{ij}'s are constants.

Next, we try find the number of independent co-ordinates needed to describe the position of a rigid body in space. A rigid body in space is defined by three points which do not lie on the same straight line. Each point is specified by three co-ordinates and therefore 9 co-ordinates are needed to specify a rigid body. But these 9 co-ordinates are connected by the 3 equations of constraints.

$$r_{12} = c_{12} r_{13} = c_{13} \quad \text{and} \quad r_{223} = c_{23}$$

Hence, we require 6 co-ordinates to specify the position of a rigid body. Apart from the constraints of rigidity, there may be additional constraints on the rigid body; for example, the body may be constrained to move on a surface, or it may be allowed to move with one point fixed. These additional constraints will further reduce the number of independent co-ordinates. There are several ways of selecting the independent co-ordinates.

The two important types of motion of a rigid body are translational motion and rotational motion. The translation of a rigid body will be given by the single particle in motion. The remaining three co-ordinates are used to specify the rotational motion.

5.3.2 LOCATION

Location of a rigid body tells us where it is placed and can be measured by position coordinates of any particle of the body or its mass center. It is also known as position.

5.3.3 ORIENTATION

Orientation of a body tells us how it is placed with respect to the coordinate axes. Angles made with the coordinate axes by any linear dimension of the body or a straight line drawn on it. provide suitable measure of orientation.

5.3.4 CHARACTERISTICS OF RIGID BODY:

The Characteristics of Rigid Body are Classified in the form of Translatory and Rotatory Motions

The motion of a rigid body may be either translational or Rotational or the combination of the two. Translatory motion is a progressive motion while the rotatory motion is a kind of stationary motion. In translatory motion, every particle of the moving body moves with the same velocity at any time. On the other hand, in rotatory motion different particles of the moving body move with different linear velocities, however, the angular velocity of each particle is the same. Translatory motion is caused and altered by a single force or a set of forces equivalent to a single force, rotatory motion, on the other hand, is caused and altered by a single couple or a set of couples equivalent to a single couple. In translatory motion the consequent linear acceleration at any instant is the same for every particle while the consequent linear accelerations of different particles in rotatory motion are different, yet the angular acceleration of each particle is the same.

TRANSLATORY MOTION

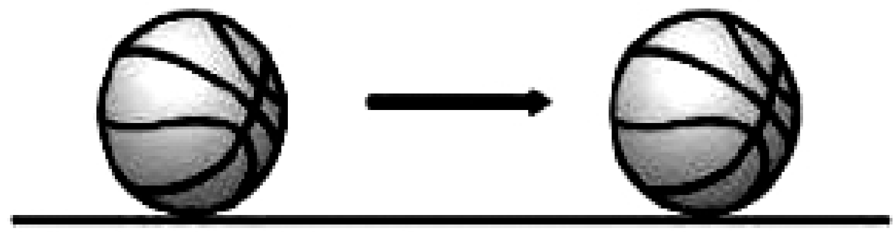


Figure -1

ROTATORY MOTION

Figure - 2

COMBINATION OF TRANSLATORY AND ROTATORY MOTION

Figure - 3

Force and mass have opposite effects on translatory motion, the force causing and the mass opposing acceleration. In rotatory motion, the place of mass is taken by moment of inertia of the body about the axis of rotation. Hence the rotatory motion is caused or altered by a couple and is opposed by the moment of inertia of the body about the axis of rotation.

5.4 RELATION AMONG ANGULAR MOMENTUM, MOMENT OF INERTIA AND ANGULAR VELOCITY IN TENSOR FORM

Consider a rigid body rotating about a fixed point with angular velocity $\vec{\omega}$. Take the origin O at this fixed point and the three co-ordinate axes *X, Y* and *Z* as shown figure 5.

The linear velocity of a particle *i,* having position vector $\vec{r}_i$.

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

If m_i is the mass of this particle, then the angular momentum of the particle *i* about the fixed-point O.

$$\vec{l}_i = \vec{r}_i \times m_i \vec{v}_i$$

$$= \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i) = m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Then, total angular momentum of the rigid body .... (1)

where $\sum_i$ represents summation over all the particles of the rigid body

Using the vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}),$$ we get

$$\vec{L} = \sum_i m_i [\vec{\omega}(\vec{r}_i.\vec{r}_i) - \vec{r}_i(\vec{r}_i.\vec{\omega}_i)]$$

$$= \sum_i m_i [\vec{\omega} r_i^2 - \vec{r}_i(\vec{r}_i.\vec{\omega})] \quad .... (2)$$

If (x_i, y_i, z_i) are the Cartesian co-ordinates of the particle *i* and $(\omega_x, \omega_y, \omega_z)$. The component of angular velocity $\vec{\omega}$ along the three co-ordinate axes, then

$$\vec{r}_i = x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}$$ and therefore $r_1^2 = x_i^2 + y_i^2 + z_i^2$

and $\vec{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$

$\therefore$ $\vec{r}_i.\vec{\omega} = (x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}).(\omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k})$

$$= x_i \omega_x + y_i \omega_y + z_i \omega_z$$

Putting the values of ω, r_i^2 and $\vec{r}_i.\omega$ in component form in Eq. (2) we get

$$\vec{L} = \sum_i m_i\left[\left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right)(x_i^2 + y_i^2 + z_i^2) - \left(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\right)\left(x_i\omega_x + y_i\omega_y + z_i\omega_z\right)\right]$$

$$= \sum_i m_i\left[\hat{\imath}\left(\omega_x x_i^2 + \omega_x y_i^2 + \omega_x z_i^2 - \omega_x x_i^2 - \omega_y x_i y_i - \omega_z x_i z_i\right) + \hat{\jmath}\left(\omega_y x_i^2 + \omega_y y_i^2 + \omega_y z_i^2 - \omega_x x_i y_i - \omega_y y_i^2 - \omega_z y_i z_i\right) + \hat{k}(\omega_z x_i^2 + \omega_z y_i^2 + \omega_z z_i^2 - \omega_x x_i z_i - \omega_z y_i z_i - \omega_z z_i^2)\right]$$

If L_x, L_y and L_z are components of $\vec{L}$ along the three co-ordinate axes, then

$$L_x = \sum_i m_i(y_i^2 + z_i^2)\omega_x - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z \quad \text{....... (3)}$$

$$L_y = \sum_i m_i y_i x_i \omega_x + \sum_i m_i(z_i^2 + x_i^2)\omega_y - \sum_i m_i y_i z_i \omega_z \quad \text{........ (4)}$$

and $$L_z = \sum_i m_i z_i x_i \omega_x - \sum_i m_i z_i y_i \omega_y + \sum_i m_i(x_i^2 + y_i^2)\omega_z \quad \text{........ (5)}$$

We now substitute

$$\sum_i m_i(y_i^2 z_i^2) = I_{xx}; -\sum_i m_i x_i y_i = I_{xy}; -\sum_i m_i x_i z_i = I_{xz}$$

$$-\sum_i m_i y_i x_i = I_{yx}; \sum_i m_i(z_i^2 + x_i^2) = I_{yy}; -\sum_i m_i y_i z_i = I_{yz}$$

and $$-\sum_i m_i z_i x_i = I_{zx}; \sum m_i z_i y_i = I_{zy}; \sum_i m_i(x_i^2 + y_i^2) = I_{zz}$$

Equations (3), (4) and (5) now become

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \quad \text{....... (6)}$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \quad \text{....... (7)}$$

and $$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \quad \text{....... (8)}$$

or $$\boxed{\boldsymbol{L} = \boldsymbol{I\omega}} \quad \text{....... (9)}$$

$$\boxed{\therefore \vec{L} = \hat{\imath}\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right) + \hat{\jmath}\left(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right) + \hat{k}\left(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\right)}$$

.... (10)

This equation shows that the angular momentum vector $\vec{L}$ is, in general, not in the same direction as the angular velocity vector $\vec{\omega}$ nor it is the direction of axis of rotation.

Also, Equations (9) and (10) shows that the relation among the angular momentum vector, the inertia tensor and the angular velocity vector.

Equation (6), (7) and (8) may be written in the matrix notation as under

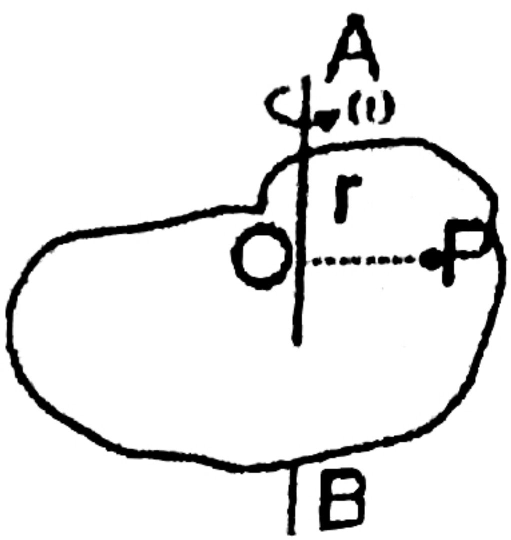
$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{......... (11)}$$

Self-Assessment Question (SAQ) 1: Show that the angular momentum $\vec{L}$ of a rotating rigid body is :

$L = I\omega$	where, ω is the angular velocity and I is the inertia tensor.

5.5 EQUATIONS OF ROTATIONAL MOTION WHEN THE DIRECTIONS OF ANGULAR MOMENTUM COINCIDE

Let a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on axis. We choose it at the centre of the circle. In this case $\vec{r}$ and $\vec{P}$ are perpendicular to each other and $\vec{r} \times \vec{P}$ is along the axis. Thus, component of $\vec{r} \times \vec{P}$ along the axis is mvr itself. The angular momentum of the whole rigid body about AB is sum of components of all particles, i.e



$$L = \sum_i m_i r_i v_i$$

Here, $v_i = r_i \omega$

$$\therefore \quad L = \sum_i m_i r_i^2 \omega \text{ or } L = \omega \sum_i m_i r_i^2 \text{ or } L = I_\omega \left(as \sum_i m_i r_i^2 = I \right) \text{ ....(12)}$$

5.6 PRINCIPAL MOMENTS OF INERTIA

In vector notation the angular momentum $\vec{L}$ of a rigid body may be expressed as

$$\vec{L} = \overleftrightarrow{I}\vec{\omega} \text{ where } \overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad \text{.......... (13)}$$

is called the moment of inertia tensor.

The nine quantities $I_{xx}, I_{xy}, I_{xz}; I_{yx}, I_{yy}, I_{yz};$ and I_{zx}, I_{zy}, I_{zz} are the components of the moments of inertia of the body about the fixed X, Y and Z axes.

The diagonal elements I_{xx}, I_{yy} and I_{zz} are the moments of inertia of the rigid body about X-axis, Y-axis and Z-axis respectively and are called principal moments of inertia (or principal moments).

5.6.1 PROPERTIES OF MOMENTS OF INERTIA TENSOR

The moment of inertia tensor is a symmetric tensor i.e. its off diagonal elements are equal

$$\therefore \quad I_{xy} = I_{yx}; I_{xz} = I_{zx}; \; I_{yz} = I_{zy} \quad \text{............ (14)}$$

As a result of this, there are only six independent components

$$I_{xx}, I_{yy} \; I_{xz} = I_{zx}; \; I_{yz} = I_{zy} \quad \text{............ (15)}$$

As the products of inertia about the three principal axes zero, i.e.

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0 \quad \text{............ (16)}$$

Only three components are left I_{xx}, I_{yy} and I_{zz} which are sometimes written as $I_x, I_y, I_z,$

(i) **Spherical top.** A rigid body for which

$$I_{xx} = I_{yy} = I_{zz} \quad \text{............ (17)}$$

is called a spherical top. In a spherical top all the axes are symmetric. A sphere is an example of a spherical top.

(ii) **Symmetric top.** A rigid body for which

$$I_{xx} = I_{yy} \neq I_{zz} \qquad .... (18)$$

is called a symmetric top. A cylinder satisfies this condition. If the axis of the cylinder is taken as principal Z-axis, then X and Y-axes are symmetric axes. But a cylinder is not called a symmetric top. On the other hand all rigid bodies which do not have cylindrical shape but satisfy the condition given above are considered as a symmetric top. The earth flattened at the poles and bulging at the equator satisfies the above condition and is taken to be a symmetrical top.

(iii) Asymmetric top. A rigid body for which

$$I_{xx} \neq I_{yy} \neq I_{zz} \qquad .... (19)$$

is called an asymmetric top. A rigid body, in general is an asymmetric top.

(iv) Rotor. A rigid body for which

$$I_{xx} = I_{yy} \quad \text{and} \quad I_{zz} = 0 \qquad .... (20)$$

is called a rotor. Example, a diatomic molecule.

5.7 PRODUCTS OF INERTIA

The off diagonal elements $I_{xy}, I_{xz}, I_{yx}, I_{yz}$; and I_{zx}, I_{zy} are called products of inertia. These occur I n symmetric pairs i.e.,

$$I_{xy} = I_{yx};\ I_{yz} = I_{zy};\ \text{and} \quad I_{xz} = I_{zx} \qquad .... (21)$$

5.7.1 Importance of Product of Inertia

The rotational behavior of a rigid body about a given point is determined by a set of six quantities, the three principal moments of inertia and the three products of inertia.

5.8 PRINCIPAL AXES OF INERTIA

A set of three mutually perpendicular axes drawn through a point in the rigid body taken as origin, such that the products of inertia (I_{xy}, I_{yx}; I_{yz}, I_{zy}; I_{xz}, I_{zx}) about then vanish i.e. each is equal to zero whereas (I_{xx}, I_{yy}, I_{zz}) the principal moments of inertia are non-zero are called principal axes of inertia or simply principal axes.

5.8.1 Importance of Principle Axis of Inertia

In terms of principal axes, the angular momentum of a rigid body is given by

$$\vec{L} = I_{xx}\omega_x\hat{\imath} + I_{yy}\omega_y\hat{\jmath} + I_{zz}\omega_z\hat{k} \qquad .... (22)$$

5.9 INERTIA TENSOR

We know that,

$$L = \sum_i m_i[r_i^2\omega - (r_i.\omega)r_i]$$

Then, $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \text{............ (23)}$$

or $L = I\omega$

The nine elements $I_{xx}, I_{xy}, \dots, I_{xz}$ of the (3 × 3) matrix may be regarded as components of a single entity ***I***. This entity ***I*** is called **inertia tensor**. Since $I_{xx} = I_{xy}$ etc., I is a **symmetric tensor**. If we denote x, y, z by x1, x2, x3 respectively, then in general any element of the inertia tensor is given by

$$I_{\alpha\beta} = I_{\alpha\beta} = \sum_{i=1}^{N} m_i\left[\delta_{\alpha\beta} r_i^2 - x_{i\alpha}x_{i\beta}\right]$$

where $\alpha, \beta = 1,2,3.$

5.9.1 Properties of Inertia Tensor

Inertia Tensor is symmetric

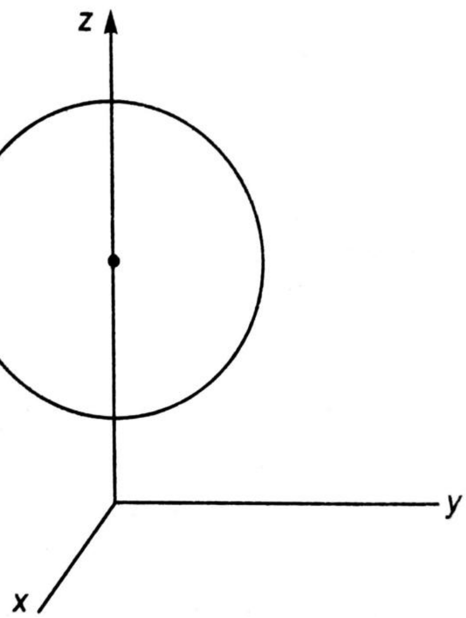
The moment of inertia tensor is given by

$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad \text{............ (24)}$$

It is called symmetric because its off diagonal elements known as products of inertia are equal i.e.

$$I_{xy} = I_{yx};\ I_{xz} = I_{zx};\ I_{yz} = I_{zy} \qquad \text{............ (25)}$$

Example 1: Consider a solid sphere of mass *M,* radius *R w*ith centre on the *Z* axis at a point distant *I* from the origin. Find the inertia tensor with respect to the origin.



We are required to find $I_{xx}, I_{xy}, I_{xz}, I_{yy}$, etc.

First, we find the inertia tensor with respect to its centre of gravity, $(I_0)_{ij}$

$$(I_0)_{zz} = \sum m_i (x_i^2 + y_i^2)$$

This has the same form as the moment of inertia of the sphere about a line passing through its centre of mass (here the z-axis)

So, $(I_0)_{zz} = \frac{2}{5} MR^2$

similarly, $(I_0)_{xx} = (I_0)_{yy} = (I_0)_{zz} = \frac{2}{5} MR^2$

and $(I_0)_{xy} = -\sum m_i x_i y_i = 0$

Because of symmetry, and $\vec{R}$ has the coordinates (0, 0, l)

So, $I_{xx} = (I_0)_{xx} + M(Y^2 + Z^2)$

$= \frac{2}{5} MR^2 + Ml^2$

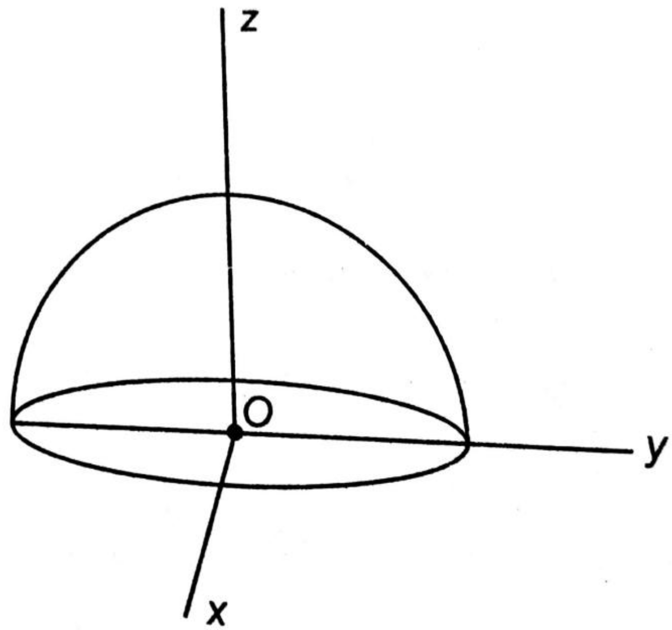
$I_{yy} = \frac{2}{5} MR^2 + Ml^2$

$I_{zz} = \frac{2}{5} MR^2$

$I_{xy} = (I_0)_{xy} - MXY = 0$

Infact, $I_{ij} = 0 \; if \; i \neq j$

Example 2: Find the tensor of inertia of a uniform hemisphere whose mass is M and radius of base is R about the centre of its base.



Tensor of inertia about the centre of base

Set up the coordinates such that the origin is at the centre of its base

$I_{zz} = \sum m_i(x_i^2 + y_i^2)$

$= \frac{2}{5}MR^2$ (using an earlier result)

$I_{yy} = \sum m_i(x_i^2 + z_i^2)$

This is the moment of inertia of the hemisphere about the y-axis. If we complete the sphere, the moment of inertia about the y-axis would be

$\frac{2}{5}M'R^2$, where M' is the mass of the whole sphere.

And since moments of inertia are additive, the moment of inertia of the hemisphere is half the moment of inertia of the sphere.

Thus, $I_{yy} = \frac{2}{5}MR^2$, where M is the mass of the hemisphere.

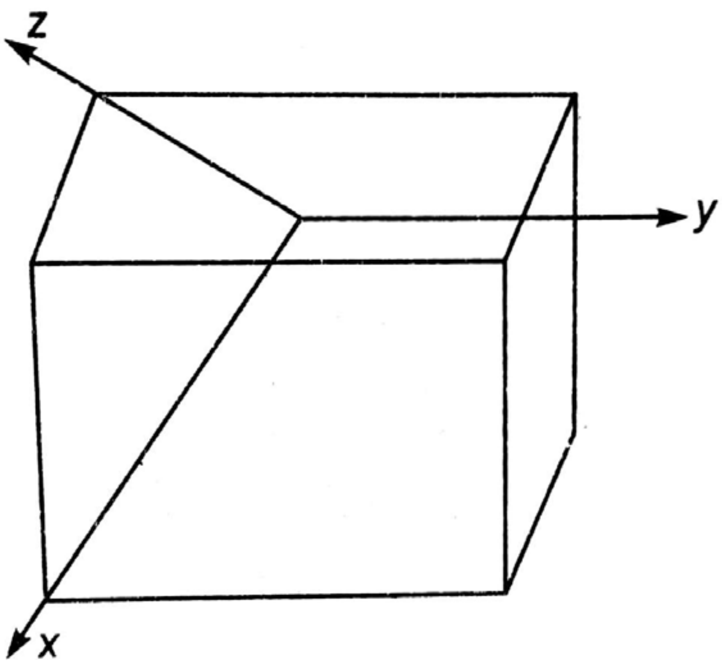
$I_{xy} = 0$, by symmetry about the z-axis.

$I_{xx} = 0$ {for every $(x_i y_i)$ there is a point $(-x_i, z_i)$ so, $\sum_i m_i x_i z_i = 0$}

So, $I_{ij} = 0,$ where $i \neq j$

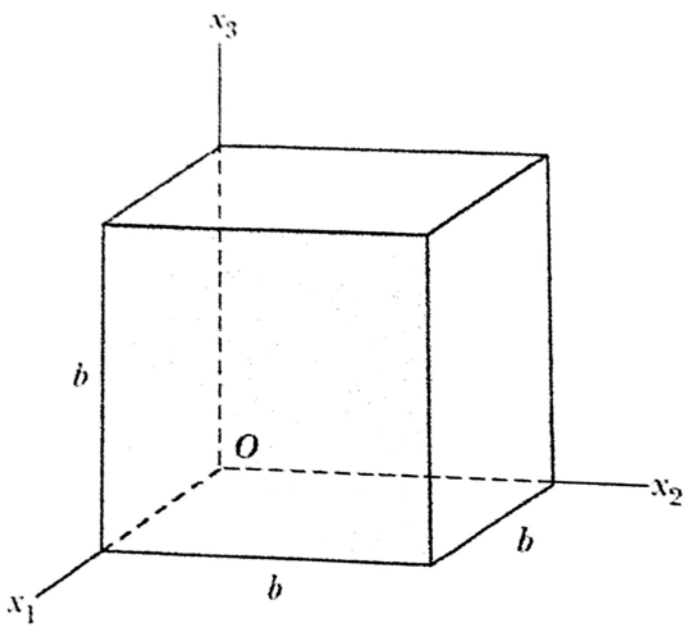
Self-Assessment Question (SAQ) 2

Find the inertia tensor for a uniform solid cube having edge a and mass M about a vertex.



Self-Assessment Question (SAQ) 3:

Calculate the inertia tensor for a homogeneous cube of density ρ, mass M, and side length b. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes.



First, we calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia I_{11}, I_{22}, and I_{33} are equal and that same holds for all of the products of inertia. So,

$$I_{11} = \int_0^b \int_0^b \int_0^b \rho\,(x_2^2 + x_3^2) dx_1 dx_2 dx_3$$

$$= \rho \int_0^b dx_3 \int_0^b dx_2\,(x_2^2 + x_3^2) \int_0^b dx_1$$

$$= \rho b \int_0^b dx_3 \left(\frac{b^2}{3} + b x_3^2\right) = \rho b \left(\frac{b^4}{3} + \frac{b^4}{3}\right)$$

$$= \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2.$$

5.10 PRECESSIONAL MOTION

One of the most interesting predictions of the vector torque-angular momentum relation $\vec{\tau} = d\vec{L}/dt$, is the phenomenon of precessional. This is defined as the cone-shaped motion of a rotational axis due to the action of a constant torque on a spinning object. Such motion is commonly seen in gyroscopes and tops.

5.10.1 What is Gyroscope and Elementary Gyroscope

Any symmetrical body rotating on an axis such that the axis can freely change its direction is called a gyroscope.

An elementary gyroscope is a circular disc mounted on three gimbals or rings so that it can turn about three mutually perpendicular axes one gimble is mounted in the next gimble with the help of bearings.

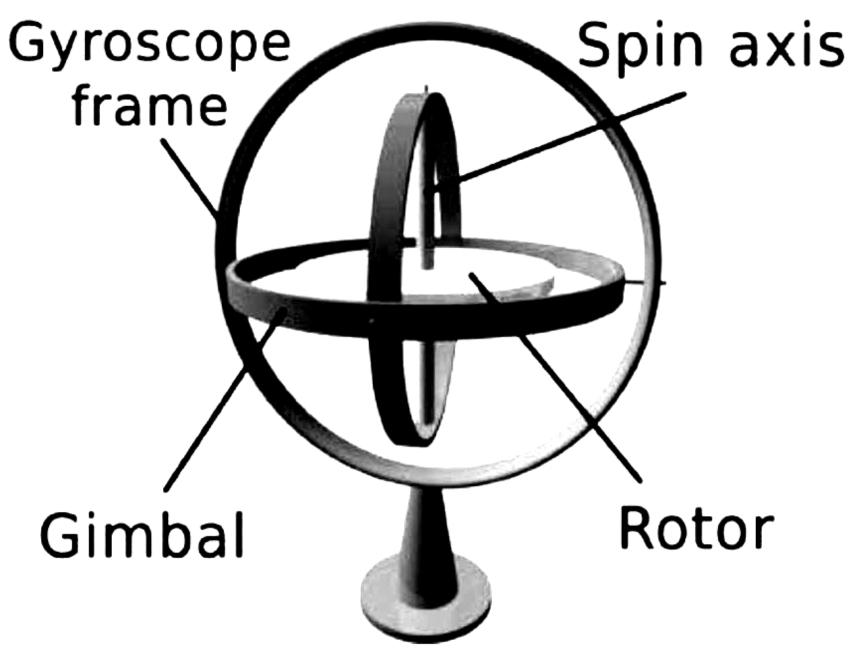


Figure - 6

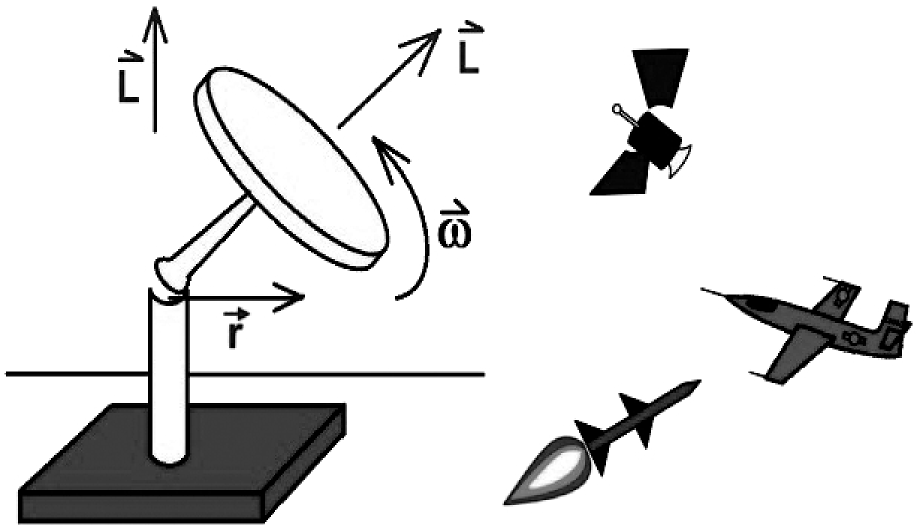


Figure - 7

5.10.2 Concept of Precessional Motion

Consider a circular disc spinning (or rotating) about its axis. Let O the centre of mass of the disc be the origin of the co-ordinate system. The axle of the disc lies along the YOY axis perpendicular to the disc and the other two axes. XOX and ZOZ lie in the plane of the disc. YOY is, therefore, the rotation axis. The angular velocity of rotation of the disc is $\vec{\omega}$ in a direction Z to X axis (anticlock wise). If I is the moment of inertia of the disc about the axis of rotation, then angular momentum of the disc (about the axis of rotation) $\vec{L} = I\vec{\omega}$. The angular momentum vector $\vec{L}$ acts along the direction OY.

In now two equal and opposite forces $\vec{F}$ and $\vec{F}$ are applied to the axle of the disc along the direction OX and OX, as shown, then these forces will produce a torque $\vec{\tau}$ acting along OZ in the direction of Z-axis as shown. The Z-axis is, therefore, known as the torque axis.

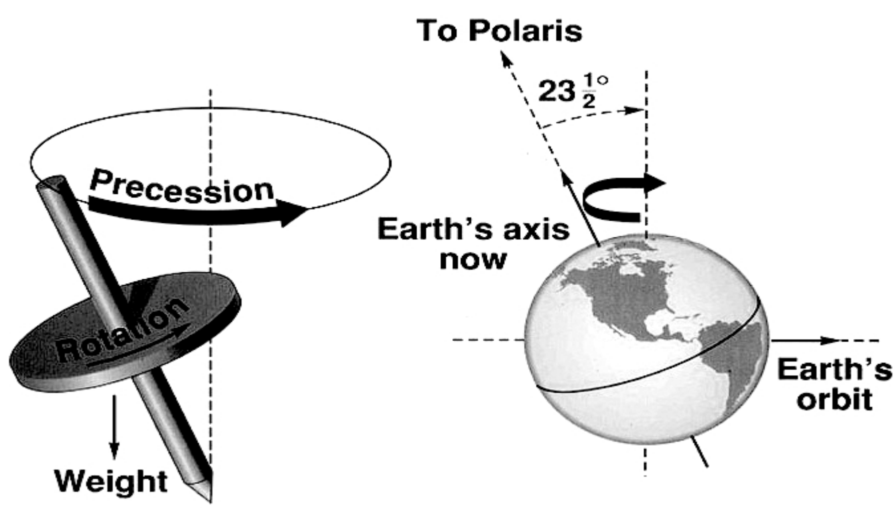


Figure - 8

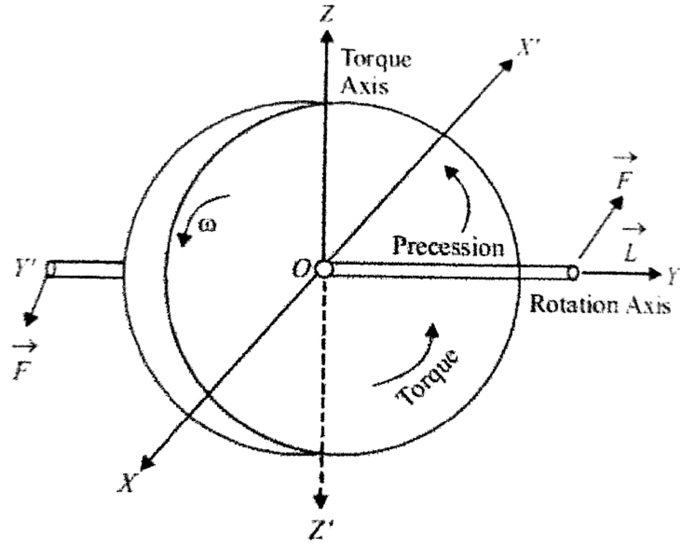


Figure - 9

If the disc were stationary, this torque will bring about rotation of the disc about the axis ZOZ. But if the disc is rotating in the direction as shown, the disc as a whole rotates about the axis XOX in the anticlockwise direction i.e., Y end of the axis turns towards Z and Y and towards Z. This motion of the axis of rotation of the disc is called **precession.** Precession is, therefore, defined as under:

When a torque is applied to a rotating body in a direction perpendicular to its axis of rotation the rotation produced in the direction of its axis of rotation, is called precession.

5.10.2 Precessional torque. The torque which brings about the rotation of the axis of rotation (precession) of the rotating is called rate of precession or precessional angular velocity.

5.10.3 Precessional angular velocity. The time rate of rotation of the axis of rotation of the rotating body due to precession is called rate of precession or precessional angular velocity.

Units. The units of precessional angular velocity denoted as Ω are radian per second (rad s^{-1}).

Expression between angular velocity and Torque. Consider a disc rotating in the XOZ plane with constant angular velocity $\vec{\omega}$ in the anticlockwise direction from Z to X about the axis YOY' perpendicular to the plane of the discs and passing through its centre of mass O (Figure - 9).

If I is the moment of inertia of the disc about the axis of rotation, then angular momentum of the disc (about the axis of rotation) $\vec{L} = I\vec{\omega}$. The angular momentum vector $\vec{L}$ acts along the direction OY.

If now, two equal and opposite forces $\vec{F}$ and $\vec{F}$ are applied to the axle of the disc along the directions OX and OX' as shown in Figure - 9, then these forces will produce a torque $\vec{\tau}$ acting along OZ in the direction of Z-axis. As a result of this the axis of rotation of the disc will turn about OZ in the XOY plane.

The torque vector $\vec{\tau}$ acts along Z-axis and the angular momentum vector $\vec{L}$ acts along the Y-axis. Thus, the direction of the applied torque is at right angles to the direction of angular momentum $\vec{L}$. This torque, therefore, only produces a change in direction of $\vec{L}$ without making any change in the magnitude of $\vec{L}$.

Suppose the torque $\vec{\tau}$ acts on the disc for a small time δt and changes the direction of angular momentum vector $\vec{L}$ through a small angle $\delta\theta$. Let the vector OA taken along Y-axis represent the initial magnitude and direction of angular momentum vector $\vec{L}$, then the magnitude and direction of the vector $\vec{L}$ after a small time δt is represented by the vector OB, where $OB = OA - \vec{L}$ and angle $\angle BOA = \delta\theta$. According to the triangle law of vector addition, $AB = \delta\vec{L}$ represents the change in angular momentum vector. As $\delta\vec{L}$ is a small length, it can be taken to be the arc of a circle so that

$$\frac{|\delta\vec{L}|}{|\vec{L}|} = \delta\theta \qquad \text{........ (26)}$$

Also, torque $\vec{\tau}$ is the time rate of change of angular momentum

$$\therefore \qquad \vec{\tau} = \frac{dL}{dt} = \frac{\delta L}{\delta t} \qquad \text{...... (27)}$$

Comparing relations (i) and (ii), we have

$$\vec{L}\delta\theta = \vec{\tau}\delta t$$

or

$$\frac{\delta\theta}{\delta t} = \frac{\frac{\vec{\tau}}{\vec{L}}}{} = \frac{\tau}{L} = \frac{\tau}{I\omega}$$

$[\because \vec{L} = I\omega]$

But $\frac{\delta\theta}{\delta t}$ represents the angular velocity of precession (or rate of precession) Ω.

$\therefore$ **Angular velocity of precession:** $\boxed{\Omega = \frac{\tau}{I\omega}}$ **............**(28)

Thus, the rate of precession is

(a) directly proportional to the torque applied

(b) inversely proportional to the moment of inertia of the disc and

(c) inversely proportional to the period of precession is given by

$$T = \frac{2\pi}{\Omega} = \frac{2\pi\ I\omega}{\tau} \qquad \ldots\ldots (29)$$

5.11 SUMMARY

In the Present Unit, we have studied about rigid body, Location, Orientations, Inertia Tensor, Moment of Inertia and Product of Inertia.

In this Unit, we have also studied about and derived the relation among angular momentum, moment of inertia and angular velocity in tensor form $L = I\omega$.

We have contained examples and self-assessment questions (SAQs) to check your progress.

5.12 TERMINAL QUESTIONS

1. Write answers to the following questions.

(a) What do you mean by a rigid body?

(b) Define Moments of Inertia.

(c) Explain the term Inertia Tensor.

(d) What is Symmetric top for a rigid body?

2. Briefly explain the concept of a rigid body. Also, define translatory and rotatory motion.

3. Establish a relation among angular momentum, momentum of inertia and angular velocity in tensor form.

4. (a) A rigid body is in such a way that all particles in the body have the same instantaneous velocity at all times. What do you conclude about the motion of the body?

(b) A rigid body is moving in such a way that two particles in the body have the same instantaneous velocity at all times. What do you conclude about motion of rigid body?

5. (a) Why spin angular velocity of a star is greatly enhanced when it collapses under gravitational pull and becomes a neutron star?

(b) Can a body in translatory motion have angular momentum?

6. What is precession angular velocity? Establish a relation between angular velocity of precession and torque.

7. (a) Write down equation of rotational motion when direction of angular momentum. Coincide

(b) A solid sphered of diameter 2 cm and mass 50 gm has a pivot pin 5 mm long fixed normally to its surface. When it spins like a top, it makes 20 revolutions per second. Find its precessional angular velocity.

8. Find the moments and products of inertia of a homogeneous cube of side a for an origin at one corner, with axes directed along the edges.

9. Calculate the inertia tensor for the system of four-point masses 1 gm, 2 gm, 2 gm and 4 gm, located at the points (1, 0, 0), (1, 1, 0), (1, 1, 1) and (1, 1, -1) cm.

10. Find the tensor of inertia of a uniform hemisphere whose mass is M and radius of base is R about the centre of its base.

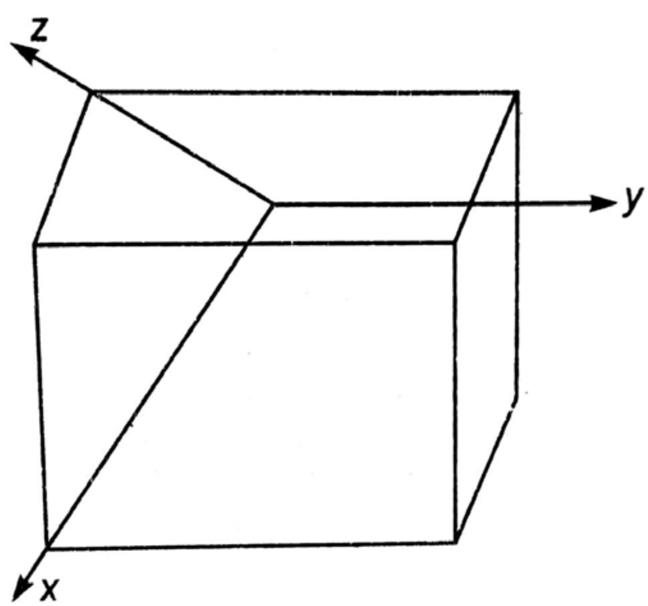
11. What do you mean by inertia tensor? Explain what do you understand by Principal axes and the Principal moments of inertia. Write down the expression of the Principal moments of inertia of a rigid body.

5.13 SOLUTION AND ANSWERS:

Self-Assessment Questions (SAQs):

1. **Hint (Section 5.4)**

2.



We may directly calculate I_{ij}

$I_{xx} = \int \rho(y^2 + z^2)dx\, dy\, dz$

$= \int \rho y^2\, dx\, dy\, dz + \int \rho z^2\, dx\, dy\, dz$

$= \int a^2 \frac{a^3}{3} + \rho a^2 . \frac{a^3}{3} = 2\frac{\rho a^5}{3}$

$= \frac{2}{3} M a^2$

Similarly, $\quad I_{yy} = I_{zz} = \frac{2}{3} M a^2$

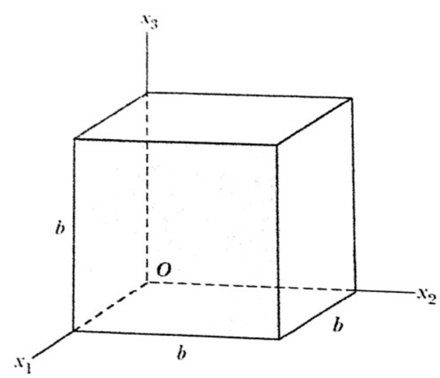
$I_{xy} = -\int \rho\, xy\, dx\, dy\, dz$

$= -\int \rho\, x\, dx\, \int y\, dy\, \int dz$

$= -\rho\, \frac{a^2}{2} . \frac{a^2}{2} . a = -\frac{Ma^2}{4} 2$

Thus, $\quad I = \begin{pmatrix} \frac{2}{3} M a^2 & -\frac{Ma^2}{4} & -\frac{Ma}{4} \\ -\frac{Ma}{4} & \frac{2}{3} M a^2 & -\frac{Ma}{4} \\ -\frac{Ma}{4} & -\frac{Ma}{4} & \frac{2}{3} M a^2 \end{pmatrix}$

3.



First we calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia I_{11}, I_{22}, and I_{33} are equal and that same holds for all of the products of inertia. So,

$$I_{11} = \int_0^b \int_0^b \int_0^b \rho\, (x_2^2 + x_3^2) dx_1 dx_2 dx_3$$

$$= \rho \int_0^b dx_3 \int_0^b dx_2\, (x_2^2 + x_3^2) \int_0^b dx_1$$

$$= \rho b \int_0^b dx_3 \left(\frac{b^3}{3} + b x_3^2\right) = \rho b \left(\frac{b^4}{3} + \frac{b^4}{3}\right)$$

$$= \frac{2}{3} \rho b^5 = \frac{2}{3} M b^2.$$

ANSWERS: TERMINAL QUESTION

2. HINT (Section 5.3)

3. HINT (Section 5.4)

4.

5. (a) On collapsing under gravitational pull, size of star decreases. Therefore, its moment of inertia decreases. As angular momentum $(L = I\omega)$ is conserved, and I decreases, therefore, spin angular velocity ω increases.

(b) Yes, a particle in translatory motion always has an angular momentum, unless the point (about which angular momentum is calculated) lies on the line of motion.

6. HINT (Section 5.10.3)

7. (a) HINT (Section 5.5)

(b) We know that,

Precessional angular velocity $\Omega = \frac{mgr}{I\omega}$

Here mass of the sphere = 50 gm.

Diameter of the sphere = 2 cm. ∴ Radius of the sphere R = 1 cm.

Moment of Inertia of the sphere $I = \frac{2}{5} m R^2 = \frac{2}{5} \times 50 \times 1^2 = 20\ gm\ cm^2$

Length of the pivot pin = 5 mm = 0.5 cm

∴ Distance of centre of mas from the tip of the pivot r = 1 + 0.5 = 1.5 cm

Number of revolutions per second n = 20

∴ Angular velocity $\omega = 2\pi n = 2\pi \times 20 = 40\pi\ rad\ s^{-1}$

Here precessional angular velocity $\Omega = \frac{mgr}{I\omega} = \frac{50 \times 980 \times 1.5}{20 \times 40\pi} = 29.2\ rad\ s^{-1}$

8. $I_{xx} = \int_v \rho(y^2 + z^2) dV = \rho \int_0^a dx \int_0^a \left[\int_0^a (y^2 + z^2) dy\right] dz$

$$= \rho a \int_0^a \left(\frac{a^3}{3} + az^2\right) dz = \rho a \left(\frac{a^4}{3} + \frac{a^4}{3}\right)$$

$$= \frac{2}{3}\rho a^5 = \rho a^3 \frac{2}{3} a^2 = \frac{2}{3} M a^2$$

Where M is the mass of the cube. By similar arguments

$$I_{yy} = I_{zz} = \frac{2}{3} M a^2$$

$$I_{xy} = -\int_v \rho xy\, dV = -\rho \int_0^a dz \int_0^a \left[\int_0^a xy\, dx\right] dy$$

$$= -\rho a \int_0^a \frac{a^2}{2} y\, dy = -\rho \frac{a^5}{4} = -\frac{1}{4} Ma^2$$

$$I = \begin{pmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma2^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{pmatrix}$$

9. $I_{xx} = \sum_{i=1}^4 m_i(y_i^2 + z_i^2) = 1 \times 0 + 2 \times 1 + 3 \times 2 + 4 \times 2 = 16\ gm - cm^2$

Similarly, $I_{yy} = \sum_i^4 m_i(x_i^2 + z_i^2) = 17\ gm - cm^2$ and

$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 19\ gm - cm^2$

Also

$$I_{xy} = I_{yx} = -\sum_{i=1}^4 m_i x_i y_i$$
$$= -[0 + 2 \times 1 \times 1 + 3 \times 1 \times 1 + 4 \times 1 \times 1]$$
$$= -9\ gm - cm^2$$

Similarly, $I_{xz} = I_{zx} = -\sum_i m_i x_i y_i = 1\ gm - cm^2$

10. Tensor of inertia about the centre of base Set up the coordinates such that the origin is at the centre of its base

$$I_{zz} = \sum m_i(x_i^2 + y_i^2)$$

$= \frac{2}{5} MR^2$ (using an earlier result)

$$I_{yy} = \sum m_i(x_i^2 + z_i^2)$$

This is the moment of inertia of the hemisphere about the y-axis. If we complete the sphere, the moment of inertia about the y-axis would be

$\frac{2}{5} M' R^2$, where M' is the mass of the whole sphere.

And since moments of inertia are additive, the moment of inertia of the hemisphere is half the moment of inertia of the sphere.

Thus, $I_{yy} = \frac{2}{5} MR^2$, where M is the mass of the hemisphere.

$I_{xy} = 0$, by symmetry about the z-axis.

$I_{xx} = 0$ {for every $(x_i y_i)$ there is a point $(-x_i, z_i)$ so, $\sum_i m_i x_i z_i = 0$}

So, $I_{ij} = 0,$ where $i \neq j$

and $I_{yz} = I_{zy} = -\sum_i m_i x_i y_i = 1\, gm - cm^2$

Thus, the inertia tensor *I* is

$$I = \begin{pmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{pmatrix}$$

11. HINT (Section 5.9)

5.14 SUGGESTED READINGS:

1. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons.

2. Elementary Mechanics, IGNOU, New Delhi.

3. College Physics, Hugh D. Young.

4. An Introduction to Mechanics Daniel Kleppner and Robert J. Kolenkow.

Uttar Pradesh Rajarshi Tandon
Open University

UGPHS-101

Vector, Mechanics and General Physics

BLOCK

2

GENERAL PHYSICS

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UNIT 6

GRAVITATION

Structure:

6.1 INTRODUCTION

In Block-I, we discussed about vectors and dynamics of a particles. As we all know about Aryabhatt, the great Indian astronomer and mathematician, who first studied the motion of celestial bodies such as the moon, the earth, Mercury, Venus, etc., in 5^{th} century AD and made it known that the different planets move around the sun not in circular orbits, but in elliptical orbits. About a thousand years later, Tycho Brahe and Johannes, Kepler studied by motion of planets and formulated laws which came to be known as Kepler's laws of planetary motion. In the seventeenth century, Newton, after a number of observations on the motion of planets, came up with his famous laws of gravitation.

6.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand the Concept of Gravitation and Gravity.
- Compute the Gravitational Constant G.
- Define Acceleration due to Gravity (g).
- Understand the concept of Escape velocity of body.

6.3 HISTORY OF THE LAW OF GRAVITATION

The way the law of universal gravitation was discovered is often considered the paradigm of modern scientific technique. The major steps involved were.

- The hypothesis about planetary motion given by **Nicolaus Copernicus (1473–1543)**.
- The careful experimental measurements of the positions of the planets and the Sun by **Tycho Brahe (1546 –1601)**.
- Analysis of the data and the formulation of empirical

laws by **Johannes Kepler (1571–1630)**.

- The development of a general theory by **Isaac Newton (1642 –1727)**.

6.4 NEWTON'S LAW OF GRAVITATION

Newton's law of universal gravitation is usually stated as that every particle attracts every other particle in the universe with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. The publication of the theory has become known as the "first great unification", as it marked the unification of the previously described phenomena of gravity on Earth with known astronomical behaviors.

This is a general physical law derived from empirical observations by what Isaac Newton called inductive reasoning. It is a part of classical mechanics and was formulated in Newton's work *Philosophiæ Naturalis Principia Mathematica* ("the *Principia*"), first published on 5 July 1687. When Newton presented Book 1 of the unpublished text in April 1686 to the Royal Society, Robert Hooke made a claim that Newton had obtained the inverse square law from him.

In today's language, the law states that every point mass attracts every other point mass by a force acting along the line intersecting the two points. The force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them.

The equation for universal gravitation thus takes the form:

$$F = G\frac{m_1 m_2}{r^2} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where F is the gravitational force acting between two objects, m_1 and m_2 are the masses of the objects, r is the distance between the centers of their masses, and G is the gravitational constant.

6.5 GRAVITATIONAL CONSTANT 'G'

The **gravitational constant** (also known as the **universal gravitational constant**, the **Newtonian constant of gravitation**, or the **Cavendish gravitational constant**), denoted by the letter G, is an empirical physical constant involved in the calculation of gravitational effects in Sir Isaac Newton's law of universal gravitation and in Albert Einstein's general theory of relativity.

In Newton's law, it is the proportionality constant connecting the gravitational force between two bodies with the product of their masses and the inverse square of their distance. In the Einstein field equations, it quantifies the relation between the geometry of spacetime and the energy–momentum tensor (also referred to as the stress–energy tensor). The measured value of the constant is known with some certainty to four significant digits. In SI units its value is approximately 6.674×10^{-11} $m^3\cdot kg^{-1}\cdot s^{-2}$.

6.5.1 Measurement of Gravitational Constant 'G'

Value of G was first measured in 1798 by a gifted English scientist, Henry Cavendish (1731-1810). Cavendish made many contributions to science but his measurement of G was the most prolific of all. The experiment required a very delicate set up.

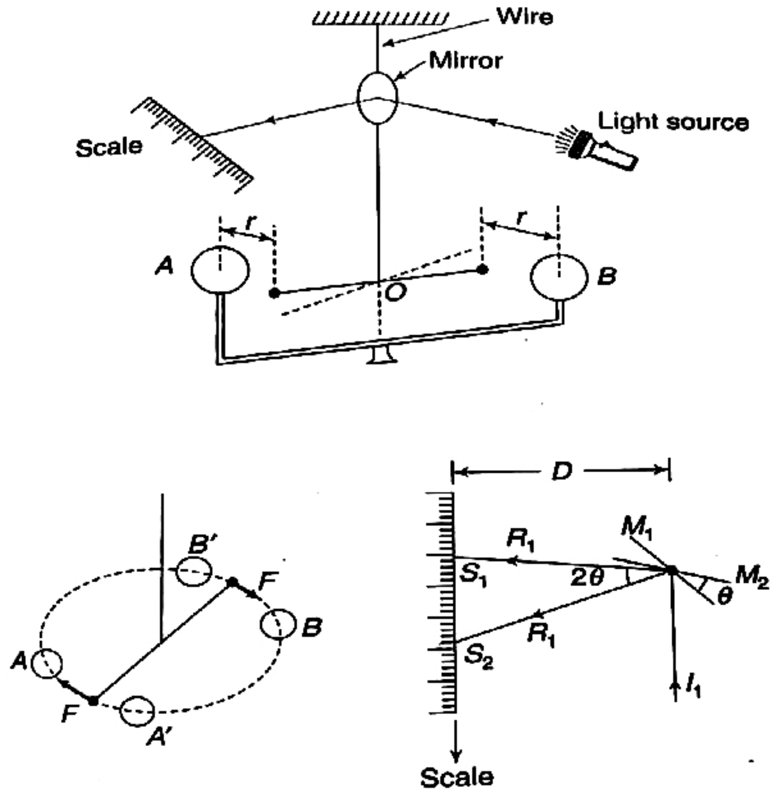


Figure - 1

Two small balls, each of mass m, are attached to the ends of a light rod to form a dumb bell. The rod is suspended by a fine fiber or a thin metal wire. There is a small mirror attached to the wire, which reflects a sharp beam of light incident on it to the scale fixed at some distance D from the mirror. Initially, there is no twist in the wire.

Two heavy spheres (A and B), of mass M each, are brought near the smaller spheres such that the centres of the four spheres fall on a horizontal circle. Let distance between the centres of a heavy ball and the smaller ball near it be r.

Gravitational pull of larger ball on the smaller ball is

$$F = \frac{G\,Mm}{r^2} \qquad \text{............... (2)}$$

Torque on the dumb bell due to gravitational force is

$$\tau = 2Fl \qquad [2l = \text{length of the dumb bell}]$$

$$= \frac{2GM\,ml}{r^2}$$

This torque causes the dumb bell to rotate and the wire gets twisted. The twisted wire produces a counter-torque. After some time, an equilibrium is established with torque produced by wire balancing the gravitational torque.

If a wire is twisted by θ, the torque it develops is given by $\tau = k\theta$, where k is a constant for the given wire, known as its torsional constant.

If equilibrium is attained with wire twisted by θ,

$$\frac{2GM\,ml}{r^2} = k\theta$$

$$\Rightarrow \qquad G = \frac{kr^2}{2Mml} \qquad \text{…………… (3)}$$

6.6 ACCELERATION DUE TO GRAVITY (g)

The gravitational pull of earth on a body is often referred to as gravity. Acceleration produced by this force is called acceleration due to gravity. When a body of mass m is near the surface of Earth, force of gravity acting on it is

$G = \frac{GMm}{R^2}$ towards the centre of earth

Thus, acceleration due to gravity (denoted by g) near the surface of Earth is

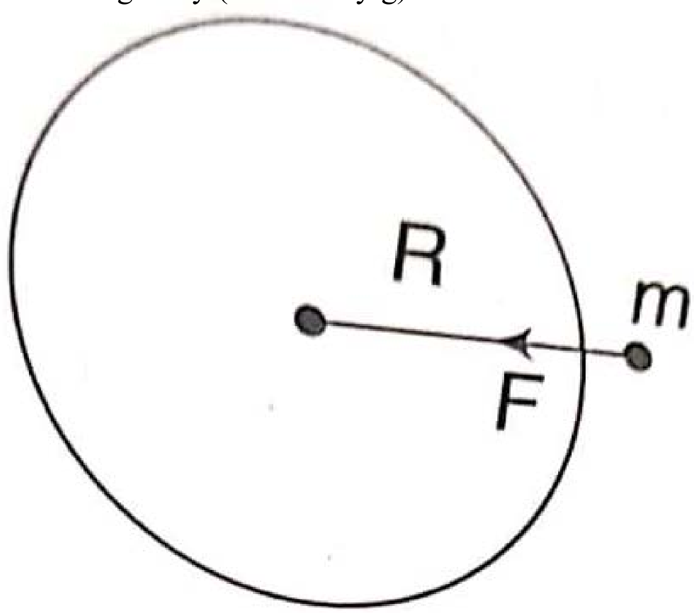


Figure - 2

$$g = \frac{F}{m} = \frac{GM}{R^2} \quad \text{............... (4)}$$

The above expression has been written assuming Earth to be a uniform sphere.

Measured value of g near the surface of Earth is nearly 9.8 ms^{-2}.

Acceleration due to gravity (g) has the same value as gravitational field intensity due to Earth.

6.6.1 "Weighing" the Earth

Value of acceleration due to gravity (g) can be measured easily with the help of simple experiments. Radius of Earth (R) is also known. The day Cavendish measured the value of universal constant G, the mass of the Earth became known. It is often said that Cavendish was the first person to weigh the Earth.

Putting g = 9. Ms^{-2}, G = 6.67×10^{-11} N m^2 kg^{-2},

R = 6.4×10^6 m in equation (4) gives, the mass of Earth as

$$M = 6 \times 10^{24} \text{ kg} \quad \text{............... (5)}$$

6.6.2 Variation in 'g' on the Surface of Earth

Acceleration due to gravity at a point on the surface of the Earth differs from the value predicted by equation (4), due to various reasons. Prominent reasons are:

(a) Non-Uniform Earth

(b) Non-Spherical Earth

(c) Rotation of the Earth

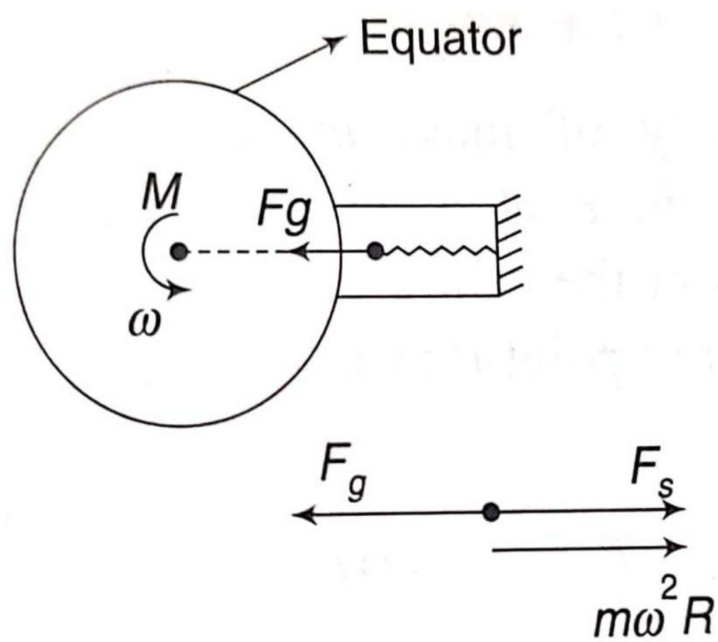


Figure – 3

Gravitational pull on the ball is

$$F_g = \frac{GMm}{R^2} = mg \qquad \text{............... (6)}$$

$$F_S = F_g - m\omega^2 R$$

F_S = Weight recorded by the balance = mg'

Where g' is the effective acceleration due to gravity at the place,

$$\therefore \qquad mg' = mg - m\omega^2 R$$

$$\Rightarrow \qquad g' = g - \omega^2 R \qquad \text{............... (7)}$$

The above equation gives the value of free fall acceleration at the equator. It is less than the gravitational acceleration (g).

6.6.3 What is Gravity

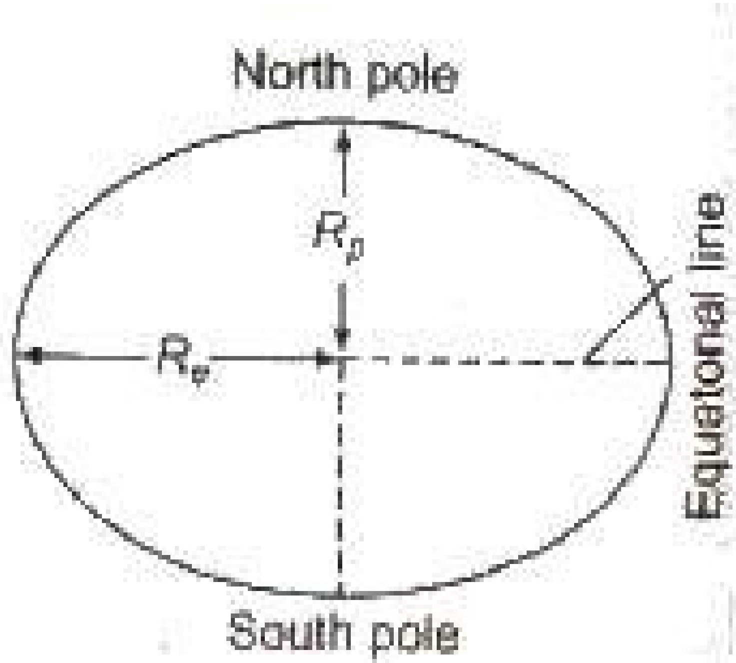
Gravity is the weakest of the four universal forces which also include nuclear force, weak radiation force, and electromagnetism. Gravity is the force exerted by any object with mass on any other object with mass. Gravity is ubiquitous, omnipresent and causes objects to accelerate towards the centers of other objects exerting gravitational attraction (like the center of the Earth). When shuttle astronauts are in space, they experience gravity at approximately 80% of Earth's surface gravity. The missing 20% allows astronauts to float, "seeming weightless. Objects outside of the Earth's gravitational field are held in the Sun's gravitational field. Outside of the solar system, objects are held by the gravity of other stars and the galaxy.

Weight is mass being pulled by gravity towards the center of the closest object exerting gravitational pull. Therefore, weight varies from place to place. On Earth, the difference is negligible. But in space, objects are continuously into another object's gravity well (such as the Earth, Sun or Moon) and experience free fall. In this situation, the objects are weightless. On other planets, the objects experience different intensities of gravity, and therefore have different weights.

6.7 FACTORS AFFECTING DUE TO ACCELERATION DUE TO GRAVITY

6.7.1 Shape of Earth

Acceleration due to gravity g & infi; 1 / R^2 Earth is elliptical in shape. Its diameter at poles is approximately 42 km less than its diameter at equator.



Therefore, g is minimum at equator and maximum at poles.

Figure - 4

6.7.2 Rotation of Earth about Its Own Axis

If ω is the angular velocity of rotation of earth about its own axis, then acceleration due to gravity at a place having latitude λ is given by

$g' = g - R\omega^2 \cos^2 \lambda$

At poles $\lambda = 90°$ and $g' = g$

Therefore, there is no effect of rotation of earth about its own axis at poles. At equator $\lambda = 0°$ and $g' = g - R\omega^2$

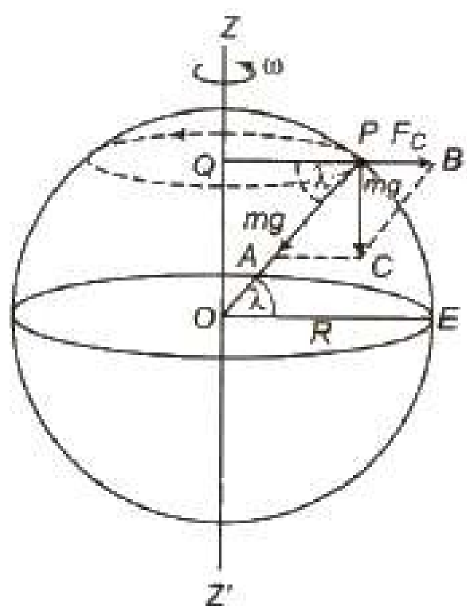


Figure - 5

The value of g is minimum at equator

If earth stapes its rotation about its own axis, then g will remain unchanged at poles but increases by $R\omega^2$at equator.

6.7.3 Effect of Altitude

The value of g at height h from earth's surface $g' = g / (1 + h / R)^2$

Therefore, g decreases with altitude.

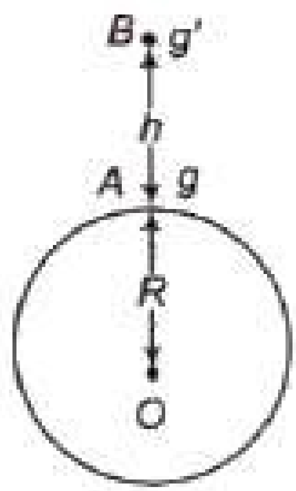


Figure - 6

6.7.4 Effect of Depth

The value of gat depth h A from earth's surface $g' = g * (1 - h / R)$

Therefore, g decreases with depth from earth's surface. The value of g becomes zero at earth's centre.

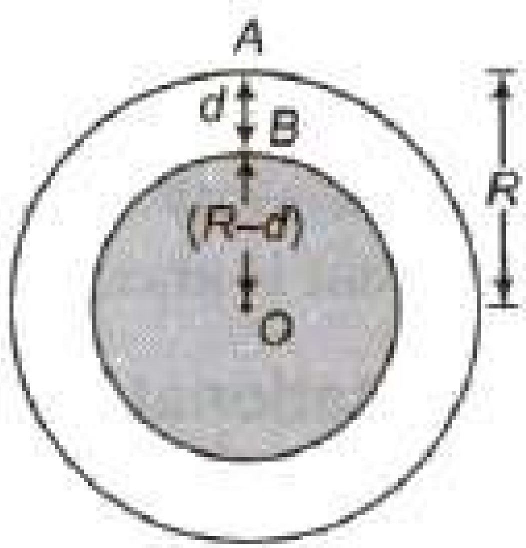


Figure - 7

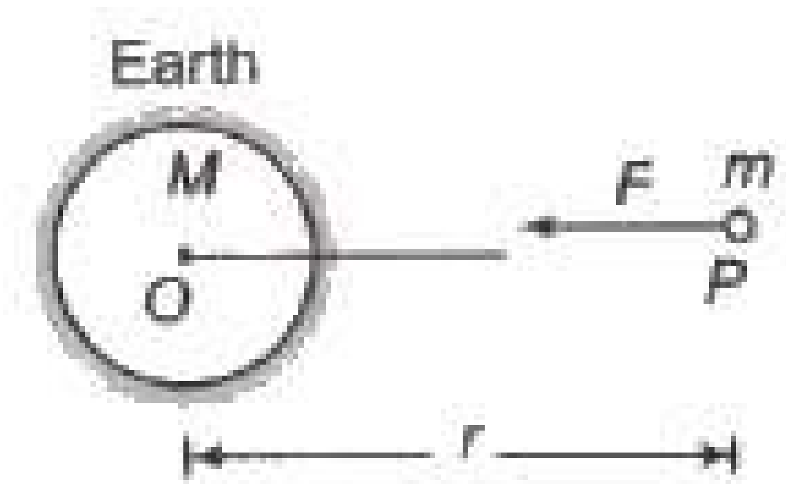
6.8 GRAVITATIONAL FIELD

The space in the surrounding of a body in which its gravitational pull can be experienced by other bodies is called **gravitational field**.

6.8.1 Intensity of Gravitational Field

The gravitational force acting per unit mass at Earth any point in gravitational field is called intensity of gravitational field at that point.

Itis denoted by E_g or I. E_g or I = F / m



Intensity of gravitational field at a distance r from a body of mass M is given by E_g or $I = GM / r^2$

It is a vector quantity and its direction is towards the centre of gravity of the body. Its S1 unit is N/m and its dimensional formula is $[LT^{-2}]$.

Gravitational mass M_g is defined by Newton's law of gravitation. $M_g = F_g / g = W / g$ = Weight of body / Acceleration due to gravity

$\therefore (M_1)g / (M_2)g = F_{g1g2} / F_{g2g1}$

6.9 GRAVITATIONAL POTENTIAL

Gravitational potential at any point in gravitational field is equal the work done per unit mass in bringing a very light body from infinity to that point.

It is denoted by V_g.

Gravitational potential, $V_g = W / m = – GM / r$

Its SI unit is J / kg and it is a scalar quantity. Its dimensional formula is $[L^3r^{-2}]$.

Since work W is obtained, that is, it is negative, the gravitational potential is always negative.

6.9.1 Gravitational Potential Energy

Gravitational potential energy of any object at any point in gravitational field is equal to the work done in bringing it from infinity to that point. It is denoted by U.

Gravitational potential energy U = – GMm / r

The negative sign shows that the gravitational potential energy decreases with increase in distance.

Gravitational potential energy at height h from surface of earth U_h = – GMm / R + h = mgR / 1 + h/R

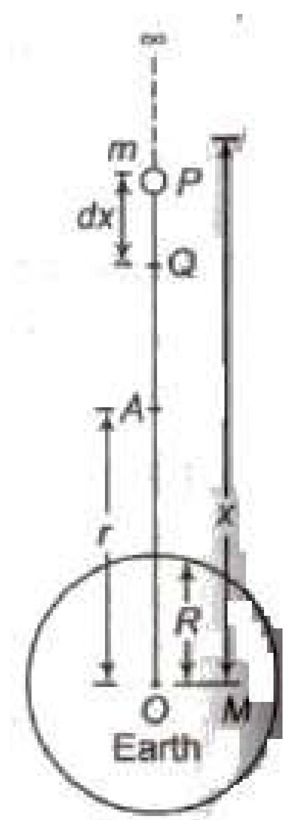


Figure - 9

6.10 WHAT IS SATELLITE?

A heavenly object which revolves around a planet is called a satellite. Natural satellites are those heavenly objects which are not man made and revolve around the earth. Artificial satellites are those heaven objects which are manmade and launched for some purposes revolve around the earth.

Time period of satellite $T = 2\pi \sqrt{r^3} / GM$

$= 2\pi \sqrt{(R + h)^3} / g$ [$g = GM / R^2$

Near the earth surface, time period of the satellite $T = 2\pi \sqrt{R^3 / GM} = \sqrt{3\pi / G\rho}$

$T = 2\pi \sqrt{R / g} = 5.08 * 10^3$ s = 84 min. where p is the average density of earth. Artificial satellites are of two types:

6.10.1 Geostationary or Parking Satellites

A satellite which appears to be at a fixed position at a definite height to an observer on earth is called geostationary or parking satellite.

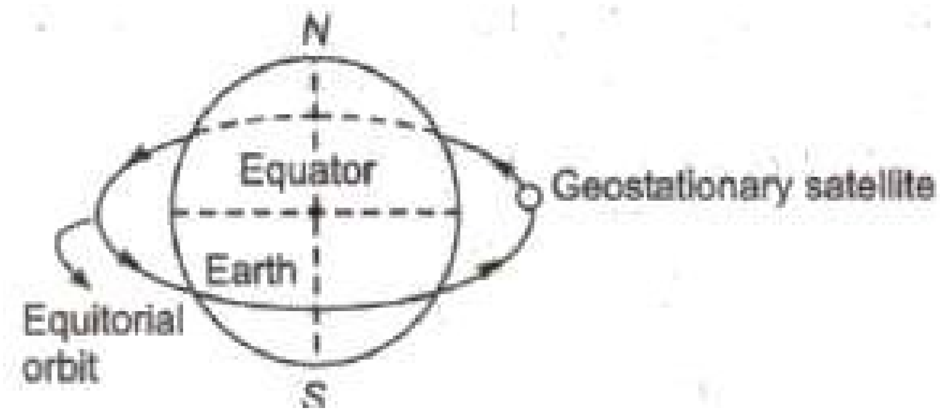


Figure - 10

6.10.2 Polar Satellite

These are those satellites which revolve in polar orbits around earth. A polar orbit is that orbit whose angle of inclination with equatorial plane of earth is 90°.

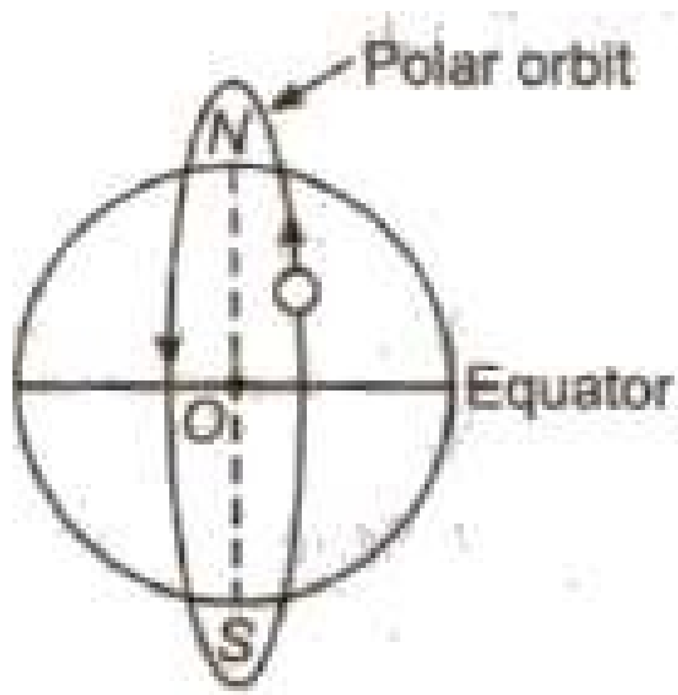


Figure - 11

Height from earth's surface = 880 km Time period = 84 min

Orbital velocity = 8 km / s

Angular velocity = 2π / 84 = π / 42 rad / min.

There satellites revolve around the earth in polar orbits.

These satellites are used in forecasting weather, studying the upper region of the atmosphere, in mapping, etc.

PSLV series satellites are polar satellites of India.

6.11 GRAVITATIONAL SELF ENERGY

The gravitational self-energy Us of a uniform solid sphere is equal to the amount of work done in assembling together its infinitesimal particles initially lying infinite distance apart.

Let us consider the sphere to be formed by continuous deposition of mass particles in the form of successive spherical shells around an inner spherical core of radius r until it becomes a full-fledged solid sphere of radius R as shown in figure.

If ρ be the density of the material of the sphere and hence of the spherical core,

we have mass of the inner core = mass * density $=\frac{4}{3}\pi r^3$

And, if the thickness of the spherical shell deposited on it be dr , we have

Mass of the shell = surface area*thickness * density

$4\pi r^2 \, dr \, \rho$

Therefore, change in self-energy of the core due to additional mass deposited on it

dUs=Potential of the core (i.e. P.E. per unit mass × additional mass

$$= -\frac{mass\ of\ the\ core}{radius\ of\ the\ core} G\, 4\pi r^2\, dr\, \rho$$

$$= -\frac{\left(\frac{4}{3}\pi r 3\rho\right)(4\pi r^2 \,dr\, \rho}{r}$$

$$= -\frac{16}{3}\pi^2 \rho^2 G r^4 \, dr$$

The integral of this expression for dUs, between limit r=0 and r=R, then give the self-energy of the whole sphere on being built from the very start i.e.,

$$Us = -\int_0^R \frac{16}{3}\pi^2\rho^2 G r^4 dr = -\frac{16}{3}\pi^2\rho^2\ G[\frac{r^5}{5}]_0^R\ R = -\frac{16\,\pi^2\rho^2 R^5 G}{15} =$$

$$-\frac{3}{5}\frac{(\frac{4}{3}\pi r^3\rho)^2}{R}G = -\frac{3}{5}\frac{M^2}{R}G$$

Therefore, gravitational self-energy of the solid sphere,

$$Us = -\frac{3}{5}\frac{M^2}{R}G$$

Negative sign indicates that this much energy is actually evolved or released during the process of assembling the solid sphere.

6.12 ORBITAL VELOCITY

Orbital velocity of a satellite is the minimum velocity required to the satellite into a given orbit around earth.

Orbital velocity of a satellite is given by $v_o = \sqrt{GM / r} = R \sqrt{g / R + h}$

where, M = mass of the planet, R = radius of the planet and h = height of the satellite from planet's surface.

If satellite is revolving near the earth's surface, then r = (R + h) =- R Now orbital velocity,

$v_o = \sqrt{gR}$

= 7.92km / h

if v is the speed of a satellite in its orbit and v_o is the required orbital velocity to move in the orbit, then.

(a) If $v < v_o$, then satellite will move on a parabolic path and satellite falls back to earth.

(b) If $V = v_o$ then satellite revolves in circular path/orbit around earth.

(c) If $v_o < V < v_e$ then satellite shall revolve around earth in elliptical orbit.

6.12.1 Energy of a Satellite in Orbit

Total energy of a satellite E = KE + PE

$= GMm / 2r + (- GMm / r)$

$= – GMm / 2r$

6.12.2 Binding Energy

The energy required to remove a satellite from its orbit around the earth (planet) to infinity is called binding energy of the satellite.

Binding energy of the satellite of mass m is given by $BE = + GMm / 2r$

6.13 ESCAPE VELOCITY

Escape velocity on earth is the minimum velocity with which a body has to be projected vertically upwards from the earth's surface so that it just crosses the earth's gravitational field and never returns.

Escape velocity of any object $v_e = \sqrt{2GM / R}$

$= \sqrt{2gR} = \sqrt{8\pi p GR^2 / 3}$

Escape velocity does not depend upon the mass or shape or size of the body as well as the direction of projection of the body.

Escape velocity at earth is 11.2 km / s.

Some Important Escape Velocities

Heavenly body	Escape velocity
Moon	2.3 km/s
Mercury	4.28 km/s
Earth	11.2 km/s
Jupiter	60 km/s
Sun	618 km/s
Neutron star	2×10^5 km/s

Relation between escape velocity and orbital velocity of the satellite $v_e = \sqrt{2}\, v_o$

If velocity of projection U is equal the escape velocity ($v = v_e$), then the satellite will escape away following a parabolic path.

If velocity of projection u of satellite is greater than the escape velocity ($v > v_e$), then the satellite will escape away following a hyperbolic path.

6.14 WEIGHTLESSNESS CONDITION

It is a situation in which the effective weight of the body becomes zero, Weightlessness is achieved

(a) during freely falling under gravity

(b) inside a space craft or satellite

(c) at the centre of the earth

(d) when a body is lying in a freely falling lift.

6.15 COMMUNICATION SATELLITE

A **communications satellite** is an artificial satellite that relays and amplifies radio telecommunications signals via a transponder; it creates a communication channel between a source transmitter and a receiver at different locations on Earth. Communications satellites are used for television, telephone, radio, internet, and military applications. There are about 2,000 communications satellites in Earth's orbit, used by both private and government organizations. Many are in geostationary orbit 22,236 miles (35,785 km) above the equator, so that the satellite appears stationary at the same point in the sky, so the satellite dish antennas of ground stations can be aimed permanently at that spot and do not have to move to track it.

The high frequency radio waves used for telecommunications links travel by line of sight and so are obstructed by the curve of the Earth. The purpose of communications satellites is to relay the signal around the curve of the Earth allowing communication between widely separated geographical points. Communications satellites use a wide range of radio and microwave frequencies. To avoid signal interference, international organizations have regulations for which frequency ranges or "bands" certain organizations are allowed to use. This allocation of bands minimizes the risk of signal interference.

6.16 SUMMARY

❖ **Gravitational Force:** It is a force of attraction the two bodies by the virtue of their masses.

❖ **Acceleration due to Gravity:** The acceleration produced in the motion of a body freely falling towards earth under the force of gravity is known as acceleration due to gravity.

❖ **Orbital Speed:** The minimum speed required to put the satellite into the given orbit around earth is called orbital speed.

❖ **Satellite:** It is body which revolves continuously in an orbit around a comparatively much larger body.

- **Polar Satellite:** It is satellite which revolves in polar orbit around the earth.
- **Geostationary Satellite:** It is the satellite which appears at a fixed position and at a definite height to an observer on earth.

6.17 TERMINAL QUESTIONS

1. Fill in the blanks:
   (a) The gravitational force is an ……………………. force.
   (b) According to Newton's law of gravitation, F = ……………… .
   (c) In the Newton's formula for gravitation, G is a …………… .
   (d) The dimensional formula for gravitational constant is ………………….. .
   (e) The gravitational constant and the acceleration due to gravity are related by ………………. .
   (f) The value of acceleration due to gravity at the poles is ……………….. .
   (g) The gravitational field at a point is a …………………. quantity.
   (h) The gravitational potential at a point is a ……………….. quantity.
   (i) The escape velocity is given by ……………………… .
   (j) Moon is the natural ………………… of the earth.
2. Write Newton's law of gravitation.
3. Define gravitational constant G. Why is it called universal constant?
4. Distinguish between gravity.
5. Define gravitational field at a point.
6. Explain Escape velocity.
7. Explain Newton's law of gravitation. Hence, define gravitational constant and give its SI unit. On what factors does the gravitational constant depend?
8. Define Weightlessness in an artificial satellite.

6.18 ANSWERS TERMINAL QUESTIONS

1. (a) attractive / conservative

   (b) $\frac{Gm_1 m_2}{r^2}$

   (c) Universal Constant

   (d) $M^{-1}L^3T^{-2}$

   (e) $g = \frac{GM}{R^2}$

   (f) maximum

   (g) vector

   (h) scalar

   (i) $v_e = \sqrt{2gR}$

   (j) satellite

2. Hint (Section 6.4)
3. Hint (Section 6.5, 6.5.1)
4. Hint (Section 6.6, 6.6.3)
5. Hint (Section 6.8)
6. Hint (Section 6.13)
7. Hint (Section 6.4, 6.5, 6.5.1)
8. Hint (Section 6.14)

6.19 SUGGESTED READINGS

1. Concepts of Physics, Part I, H. C. Verma.
2. Modern Physics, Beiser, Tata McGraw Hill.
3. Fundamental of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons.
4. College Physics, Hugh D. Young.

UNIT : 7

ELASTICITY

Structure:

1. INTRODUCTION

In block I, the basic concepts of mechanics have been discussed in detail in which one of the important aspects was gravitation. Gravity is a conservative force. and there are many others. Elastic (Hooke's Law) forces. electric forces, etc. are conservative forces (which only depends on their initial and final positions and not on the paths followed to reach from initial to final position). It has also been mentioned that these forces are central forces. Now, the basic question arises *(i)* What are central force and why is it so special? *(ii)* What are the properties of the central forces? and *(iii)* Is three any physical world application of this forces? Now in this unit the main forces will be on central forces. All these questions have been answered at appropriate place.

In the early 1600s. Johannes Kepler summarizes the carefully collected data of his mentor Tycho Brahe which describes the motion of planets in a sun-centered solar system in the form of three laws of planetary motion which are called as Kepler's Law. Kepler's efforts to explain the underlying reasons for such motions are no longer accepted: the actual laws themselves are still considered for an accurate description of the motion of any plant and any satellite. Kepler's laws and their applications are discussed in detail in this unit.

In this unit. we shall first understand conservative and non-conservative forces and show that the conservative force can be represented as gradient of potential energy. We then discuss about central forces and their properties. Then we move on the reduction of two body central problem to one body problem. We also show that the angular momentum is conserved which results in the restriction of the motion of the particle in a plane perpendicular to the angular momentum. In SEc. IX we deduce the form of velocity and acceleration in central force field and show that the equation of motion can be written in component form. In Sec XL. we study the inverse square law in detail and derive the equation of the orbits which are analyzed in detail. Keplar's law are studied in Sec. XIII. Finally, the law of gravitation has been derived with the help of Kepler's law.

Objectives: The complete study of this unit will help you in.

(i) identifying and understanding the central forces.

(ii) solve problems by applying the properties of motion under central conservative force.

(iii) determine the possible orbits under a given inverse square central conservative forces.

(iv) understanding the nature of the orbits physically.

II. CENTRAL FORCES

A central force is by definition a force that points radially and whose magnitude depends only on the distance from the source (that is, not on the angle around the source). If interaction between any two objects is represented by a central force. then the force is directed along the line joining the centers of the two objects. Equivalently. we may say that a central force is one whose potential depends only on the distance from the source. That is if the source is located at the origin, then the potential energy is of the form $v(\bar{r}) = v(r)$. Mathematically it is represented as $F=f(r)\ \bar{r}$,from a fixed point. Unit vector is a vector radius vector $\bar{r}$ of the particle with respect to fixed point.

III. EXAMPLES

Some of the well-known examples are:

1. The gravitational force acting on a particle by another particle which is stationary in an inertial frame of reference of reference is a central force. which is always directed towards the Sum.

2. The electrostatic force acting on a charged on a charged particle by another. The electron in Hydrogen atom moves under a central force which is always directed towards the nucleus.

3. Certain two-body nuclear interaction such as the scattering of α -particles nuclei.

4. A particle attached to one end of a spring whose other end is stationary in an inertial frame of reference. The spring always pulls towards the fixed end.

Many more such examples can be found in physics that are governed by this force which suggest that these forces are ubiquitous. that is they are found everywhere in the physical world. Hence, it is better to learn how to deal with them, however dealing with them is much easier than one might think, because crucial simplifications occur in the equations of motion when V is a function of r only. these simplifications will become evident in the following Sections.

IV. CHARACTERISTICS OF CENTRAL FORCES

If a particle moves in a central force field then the following properties hold:

1. Central forces are long range forces.

2. It acts along the line joining the centers of the two objects.

3. It is conservative force, that is $\overline{\nabla} \times \overline{F}$.

4. It can be written as $\overline{F} = -\overline{\nabla} V\, (\,\overline{r}\,)$=0.

5. The path of the particle must be a plane curve, i.e., it must lie in a plane.

6. The angular momentum of the particle is conserved.

We will describe these properties in detail. and prove it, in appropriate sections.

V. CENTRAL CONSERVATIVE FORCES:

A conservative force is a force that acts on a particle. such that the work done by this force in moving this particle from one point to another is independent of the path taken. In other words, it can be said that, the work done depends only on the initial and final position of the particle. Now we will show that the central force is conservative in nature.

For this purpose, let us consider a central force that is either directed towards or away from a fixed point which is called of force. This can be expressed in mathematical form as.

$$\overline{F} = -F\ \overline{r} \quad (1)$$

where is a unit vector pointing from the center of force of the particle as shown in the FIG. 1. The center of force of the particle *P* is represented by *O*. As noted earlier in the

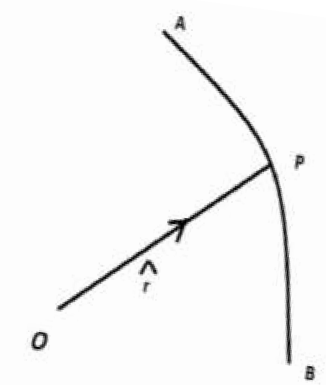


FIG. 1: A particle moving under a central force

case of electrostatic or gravitational forces, *F* depends only on the separation between the center of force particle. Hence. Eq. (1) can be rewritten as.

$$\overline{F} = -f\,(\mathrm{r})\,\overline{r}\ . \quad (2)$$

The aim here is to prove that Eq. (2) is conservative in nature.

For this purpose. Let ns consider points A and B which are connected by two random path say path and path 2. These paths are intersected by two hypothetical parts of radii r and r + dr as shown in Fig.2.

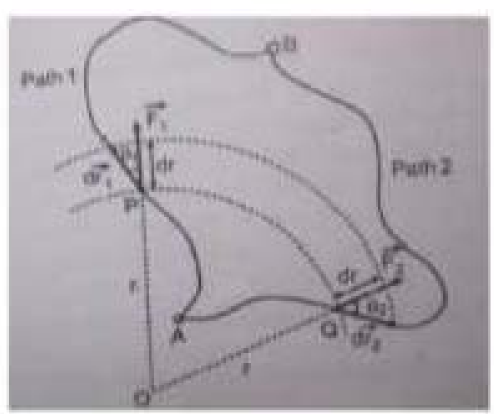


FIG. 2. Work done on a particle moving from A to B

Consider two forces $\overline{F}_1$ and $\overline{F}_2$ are acting on particles on particles P and Q due to which there is a displacement $d\overline{r}_1$. and $d\overline{r}2$ along path 1 and 2 if θ_1 and $\theta 2$ are angles between F_1 and $d\overline{r}_1$ and F_2 and $d\overline{r}2.$ respectively, then we can wrife.

$$F_1\, d\overline{r}_1 = F_1\, d\overline{r}_1\, \cos\theta_1, \tag{3}$$

$$F_2\, d\overline{r}_2 = F_2\, d\overline{r}_2\, \cos\theta 2, \tag{4}$$

It is evident from Fig. 2 that the magnitude of central forces is equal. that is $F_1=F_2$ as P and Q are at the equal distances from O. Therefore, the projection of $d\overline{r}_1$ and $d\overline{r}_2$ on F_1 and F_1 will be equal. It means.

$$F_1\, d\overline{r}_1 = F_2\, d\overline{r}_2 \tag{5}$$

Integrating Eq. (5) throughout the path that is from A to B we get

$$\int_A^B \mathrm{Fld}\overline{\mathrm{r}}_1 = \int_A^B \mathrm{F}_2.\mathrm{d}\overline{\mathrm{r}}_2 \tag{6}$$

Equation (6) implies that the work done during both the paths are smae. that is

$$\mathrm{dw} = \int_A^B \overline{\mathrm{F}}.\mathrm{d}\overline{\mathrm{r}} \tag{7}$$

This it can be concluded that the central force given by Eq. (2) is conservative.

SAQ1 : Show that the curl of conservative force is zero.

SAQ2 : Show that the work done around the closed path is zero.

VI. CONSEVATIVE FORCE AS NEGATIVE GRADIENT OF POTENTIAL ENERGY.

Consider a particle acted upon by a conservative force with corresponding potential energy V(f). The work done by in a small displacement from to is:

$$W(\overline{r} \rightarrow \overline{r} + d\overline{r}) = \overline{F}.d\overline{r} = F_x dx + F_y dy + F_2 dz.$$

(8)

where F_x, F_y, F_x are components of force along its axes.

The potential energy of a body or the system of bodies is infact a form of stored energy which can be recovered force such into kinetic energy, which is represented by V. When a body under conservative force such as gravitational force of elastic force is taken from one position to another. then the work done in this process is stored as potential energy in the body. The difference in the potential energy of the body at two different positions is defined as the work done in moving the body form one position to the other in the absence of frictional forces. Hence, we can write.

$$W(\overline{r} \rightarrow \overline{r} + d\overline{r}) = -dV = -[V(\overline{r} + d\overline{r}) - V(\overline{r})].$$

$$= -[V(x+dx, y+dy, z+dz) - V(x,y,z)] \tag{9}$$

We know from the definition of a derivative that

$$df = f(x+dx) - f(x) = \frac{df}{dx}dx. \tag{10}$$

This suggests that the potential energy can be written as

$$dV = V(x+dx, y+dy, z+dz) - V(x,y,z)$$

$$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{11}$$

where the derivatives of V are now partial derivatives with respect to x,y,z, Hence from Eqs. (9) and (11)

$$W(\bar{r} \to \bar{r} + d\bar{r}) = -dV = -\frac{\partial V}{\partial x}dy - \frac{\partial V}{\partial y}dy - \frac{\partial V}{\partial z}dz. \tag{12}$$

Comparing (8) with (12) we get

$$F = -\dot{x}\frac{\partial V}{\partial V} - \dot{y}\frac{\partial V}{\partial y} - \dot{z}\frac{\partial V}{\partial z} = \dot{\nabla}V. \tag{13}$$

where the operator $\dot{\nabla} = -\dot{x}\frac{\partial}{\partial x} - \dot{y}\frac{\partial}{\partial y} - \dot{z}\frac{\partial}{\partial z}$ is pronounced "grad". That is conservative force is the negative gradient of potential.

SAQ3: Show that the force $\overline{F} = (2ry + z^2)\dot{x} + x^2\dot{y} + 2rz\bar{z}$ is conservative.

VII. REDUCTION TO ONE-BODY PROBLEM

Consider in isolated system consisting of two particles of masses m_1 and m_2 with their corresponding position vectors as $\bar{r}_1$ and $\bar{r}_2$ as shown in Fig. 3 Let $\vec{r}_1$ and $\vec{r}_2$ be their position vectors with respect to the centre of mass (CM) and $\overline{R}$ be the position vector of the centre of mass. Here the only forces are due to an interaction potential V. From the Fig. 3 it is clear that.

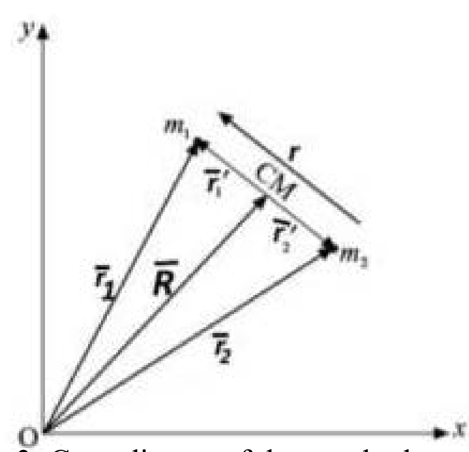


FIG. 3: Co-ordinates of the two-body system

$$\bar{r} = \bar{r}_1 - \bar{r}_2 = \vec{r}_1 - \vec{r}_2, \tag{14}$$

$$\bar{r}_1 = \overline{R} + \vec{r}_1 \tag{15}$$

$$\vec{r}2 = \overline{R} + \vec{r}2. \tag{16}$$

Now the centre of mass can be defined as

$$R = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{17}$$

with $m_1\vec{r}_1 + m_2\vec{r}_2 = 0$. which on solving on solving provides

$$\vec{r}2 = -\frac{m_1\vec{r}_1}{m_2} \tag{18}$$

Equations (14) and (15) upon simplification gives

$$\vec{r}_1 = -\frac{m_2\vec{r}}{m_1 + m_2} \tag{19}$$

In a similar way one can write

$$\vec{r}_2 = -\frac{m_1\vec{r}}{m_1 + m_2} \tag{20}$$

Utilizing Eqs. (15), (16) and (20) one can wrie

$$\vec{r}_1 = \overline{R} + \frac{m_2\vec{r}}{m_1 + m_2} \text{ and } \vec{r}_2 = \overline{R} - \frac{m_1\vec{r}}{m_1 + m_2} \tag{21}$$

As mentioned earlier the forces acting are directed along the line joining the masses. The kinetic energy of the system can be written as.

$$\mathrm{T} = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_1\dot{r}_2^2,$$

$$\Rightarrow T = \frac{1}{2}\left(\dot{R} + \frac{m_2\dot{r}}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\dot{R} - \frac{m_1\dot{r}}{m_1+m_2}\right)^2$$

$$\Rightarrow T = \frac{1}{2}(m_1 + m_2)\dot{R}2 + \frac{1}{2}\frac{m_1 m_2^2\dot{r}^2}{(m_1+m_2)^2} + \frac{1}{2}\frac{m_1^2 m_2\dot{r}2}{(m_1+m_2)^2},$$

$$\Rightarrow T = \frac{1}{2}(m_1 + m_2)\dot{R}2 + \frac{1}{2}\frac{m_1 m_2}{m_1 m_2}\dot{r}2,$$

$$\Rightarrow T = \frac{1}{2}M\dot{R}2 + \frac{1}{2}\mu\dot{R}2. \tag{22}$$

Here the quantities M and are defined as

$$\mathrm{M{=}m_1{+}m_2} \text{ and } \mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{23}$$

where μ is called as reduced mass of system. Thus, the kinetic energy is transformed to the form two effective particles of mass M and μ. Here, μ is the mass of the orbiter and M is the stationary central mass.

Thus, the central force motion of two bodies about their center of mass can always be reduced to an equivalent on body problem.

VIII. CONSERVATION OF ANGULAR MOMENTUM

Angular momentum an important aspect of central forces because it is constant over time For a point mass, the angular momentum $\overline{L}$ is defined as.

$$\overline{L} = \overline{r} \times \overline{p}.$$

$\overline{L}$ depends on $\overline{r}$, so it therefore depends on where the origin of the coordinate system has been picked. Note that $\overline{L}$ is a vector, and that it is orthogonal to both $\overline{r}$ and $\overline{p}$ by nature of the cross product.

Theorem 1: If a particle is subject to a central force only. then its angular momentum is conserved. That is, If V($\overline{r}$) = V(r), then $\frac{dL}{dt} = 0.$

Proof: From Eq. (24) we have.

$$\frac{dL}{dt} = \frac{d}{dt}(\overline{r} \times \overline{p}),$$

$$= \frac{d\overline{r}}{dt} \times \overline{p} + \overline{r} \times \frac{d\overline{p}}{dt},$$

$$= \overline{u} \times (\mu \overline{u}) + \overline{r} \times \overline{F},$$

$$= 0.$$

The second term is zero because $\overline{F} \alpha \overline{r}$ and the cross product of two parallel vectors is zero. Hence it can be written that

$$\overline{L} = \text{constant}.$$

That is, the angular momentum $\overline{L}$ of a body under the action of a central force is conserved. Next consider the dot product of $\overline{L}$ with $\overline{r}$

$$\overline{L}.\overline{r} = (\overline{r} \times \mu \dot{\overline{r}}).\overline{r} = (\overline{r} \times \overline{r}).\mu \overline{r} = 0.$$

It means that the angular momentum $\overline{L}$ is normal to the vector $\overline{r}$. In other words, throughout the motion, the radius vector of the particle lies in a plane perpendicular to the angular momentum. That is, the motion is confined to a plane which is perpendicular to $\overline{L}$ as shown in Fig.4 Thus, the problem has been simplified to a motion in two dimensions instead of three dimensions.

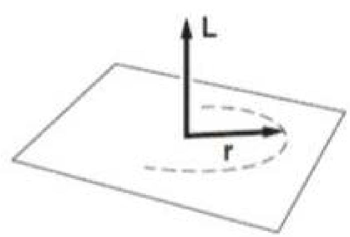


FIG. 4: Conservation of angular momentum implies that the relative motion occurs in a plane

IX. EXPRESSION FOR TRANSVERS AND RADIAL ACCELERATION OF A BODY MOVING UNDER CENTRAL FORCE.

In the previous section. we have seen that the motion under central forces lies in a plane. Hence it will be much easier to work with polar coordinates rather than Cartesian. For this we briefly discuss some relevant aspects of polar coordinates.

To start with let us consider the motion in (x,y) plane such that the angular momentum points in the direction of z. Then we can write.

$$\bar{r} = x\hat{x} + y\hat{y}, \tag{28}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the direction of Cartesian axes and x and y are the components of the vector. Here we will use polar coordinates as it is convenient in this case. Let the polar coordinates are specified by r = $|r|$ and the angle θ between r and $\hat{x}$, see Fig. 5. The relations between the polar and Cartesian coordinates are given as.

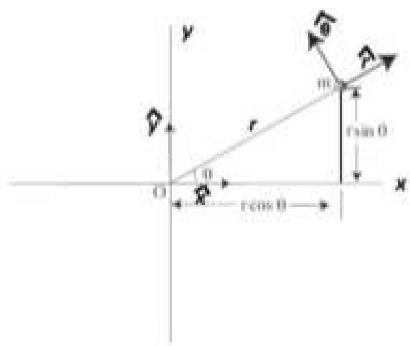


FIG. 5: Polar coordinate system associated with a particle moving in the xy plane.

$$x = r\cos\theta \qquad y = r\sin\theta \tag{29}$$

A. Velocity

To find the velocity in central force motion We need the derivatives of vector expressed in polar coordinates. Thus differentiating Eq. (38) with respect to 't' we get

$$\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \dot{r}\dot{r} + r\,\dot{\bar{r}}, \tag{29}$$

Here in Eq. (39) the form of is not known. To find this we proceed as follows

$$\dot{\bar{r}} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\sin\theta\,\dot{x} + \dot{\theta}\cos\theta\,\dot{y} = \dot{\theta}(-Sin\theta\,\dot{x} + \cos\theta\,\dot{y}) = \dot{\theta}\dot{\theta} \tag{40}$$

From Eqs. (39) and (40) we arrive at the form of velocity as

$$\bar{r} = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\bar{\theta}. \quad (41)$$

Her, velocity is sum of two components given as

(i) Radial velocity: It is defined as the component of velocity along the radius vector and is given

(ii) Transverse velocity: The component of velocity perpendicular to the radius vector is defined as transverse velocity and is given as

B. Acceleration

To find the acceleration of the particle we differentiate Eq. (41) once

$$\bar{a} = \breve{r} = \ddot{r}\hat{r} + \dot{r}\dot{\bar{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\theta + r\dot{\theta}\dot{\bar{\theta}}. \quad (42)$$

It is to ben noted that the only new derivative appearing in (42) is which can be found by differentiating (37) once. That is

$$\dot{\bar{\theta}} = \dot{\theta}(\cos\theta\,\bar{x} - \sin\theta\,\hat{y}) = -\dot{\theta}\,\hat{r}. \quad (43)$$

Using Eqs. (40) and (43) in (42) and simplifying we finally get the form of acceleration as.

$$\bar{a} = \hat{r} = (\ddot{r} - \dot{\theta}^2)\,\dot{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\dot{\theta}. \quad (44)$$

Here again we see that the acceleration is sum of two components

(i) Radial acceleration: It is defined as the component of acceleration along the radius vector and is given as.

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad (45)$$

(ii) Transverse acceleration: The component of acceleration of acceleration perpendicular to the radius vector is called as transverse acceleration and is of form.

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad (46)$$

EQUATION OF MOTION

Let r and θ be the polar coordinates of a point where a system of particles of reduced mass μ is situated. The radial and transverse accelerations are given by Eqs. (45) and (46) Now. from Newton's law the equation of motion along r can be written with the help of (45) as.

$$\mu\alpha,= F(r).$$

$$\Rightarrow \mu(\ddot{r} - r\dot{\theta}^2) = F(r). \qquad (47)$$

Similarly, the equation of motion corresponding to transverse component of acceleration becomes (using (46))

$$\mu a_\theta = 0.$$

$$\Rightarrow \mu(\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0. \qquad (48)$$

SAQ4: From equation of motion for transverse component show that angular momentum is conserved.

Solution: Let us consider a particle of reduced mass at a distance r. the separation distance between tow particles. If the origin be the centre of force then the motion of this particle is equivalent is equivalent to a two-body motion. For such motion we know from Eq. (48)

$$\mu(\overline{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\theta}) = \theta.$$

$$\Rightarrow \mu\frac{d}{dt}(r^2\dot{\theta}) = 0.$$

$$\Rightarrow \frac{d}{dt}(\mu r^2\dot{\theta}) = 0.$$

$$\Rightarrow \mu r^2\dot{\theta} = constant,$$

$$\Rightarrow L = Iw = \mu r^2\dot{\theta} = constant. \qquad (49)$$

where we have used I= and Hence it is clear that the angular momentum is conserved in a central force. It is also another form of first equation of motion corresponding to θ.

X. EXPRESSION FOR TOTAL ENERGY

To find the total energy of the system we need the kinetic and potential energies of the particle of reduced mass in central force field. The kinetic energy can be written with the help of Eq. (41) as.

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu(\overline{v}.\overline{v}).$$

$$\Rightarrow T = \frac{1}{2}\mu v_r^2 + \frac{1}{2}Iw^2.$$

$$\Rightarrow T = \frac{1}{2}\mu v_r^2 + \frac{1}{2}\frac{\mu^2 r^4\dot{\theta}^2}{\mu r^2}.$$

$$\Rightarrow T = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2}.$$

Now if the potential energy is the total energy of the system can be written with the help of Eqs. (50) as.

$$E = T + V = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2} + V(r). \tag{51}$$

Equation (51) is the expression for total energy under central forces. It is known as second equation of motion corresponding to polar coordinate.

A. Effe ctive Potential

Consider a particle of mass subject to a central force describe by the potential V(r), Equation (51) looks like the equation for a particle moving in one dimension under the influence of the potential.

$$V_{eff}(r) \sim \frac{L^2}{2\mu r^2} + V(r). \tag{52}$$

Here, is called as effective potential. Since the effective force can be written with the help of Eq. (52) as

$$F_{eff}(r) = \frac{L2}{\mu r3} - V'(r) \text{ where } V' = -\frac{dv}{dr'} \tag{53}$$

Which agrees with $F_{eff} = V_{eff}^{'}(r)$. This effective potential concept says that if we want to solve a two-dimension problem involving a central force, then we can recast the problem into a simple one-dimensional problem with a slightly modified potential.

B. Con servation of total energy

We know that for conservative force

$$F(r) = \frac{dV}{dr} \tag{54}$$

Substituting Eq. (54) in (47) we get

$$\mu r - \mu r\dot{\theta}2 = -\frac{dV}{dr}.$$

$$\Rightarrow \mu r = -\frac{dV}{dr} + \frac{L^2}{\mu r^3}.$$

$$\Rightarrow \mu r = \frac{dV}{dr} - \frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) \text{ where } L = \mu r^2\dot{\theta},$$

$$\Rightarrow \mu\ddot{r} = -\frac{d}{dr}\left(V + \frac{L^2}{2\mu r^2}\right)$$

Multiplying both sides of above equation by $\dot{r}$ we get

$$\mu \ddot{r} = \dot{r}\frac{d}{dr}\left(V + \frac{L^2}{2\mu r^2}\right).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{L^2}{2\mu r^2}\right).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)\right) = 0$$

$$\Rightarrow \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = constant.$$

$$\Rightarrow E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = constant. \quad (56)$$

XI. INVERSE SQUARE LAW FORCE

The most import ant type of central force is the one in which the force varies inversely as the square of the radial distance. that is.

$$F(r) = -\frac{k}{r^2} \Rightarrow v(r) = -\frac{k}{r}, \quad (57)$$

where k is a positive constant for an attractive force and negative for a repulsive force. The two most important cases under this category are gravitational force and coulomb force. For the gravitational force $k=Gm_1m_2$ where G is the gravitational constant.

To understand the central force motion quantitatively, we need to solve the equations of motion given by Eqs. Eq. (49) and (51). which is nothing but the conservation of L and E statement, that is

$$\mu r^2\dot{\theta} = L. \quad (58)$$

$$\frac{1}{2}\mu \dot{r}^2 + \frac{L}{2\mu r^2} + V'(r) = E. \quad (59)$$

There are following two possibilities of solving above equations of motion.

(i) We can solve for r and θ in terms of t. This method has an advantage of immediately yielding velocities and in turn provide the information of the particle at any time t.

(ii) We can solve for r in terms of θ However. this method shows explicitly the form of the trajectory in space. even though we do not about its rate of evolution in space.

Here, we will focus on second approach as it will provide the form of the trajectories under various circumstances. For this purpose. Eq. (58) can be rewritten as

$$\frac{d\theta}{dt} = \frac{L}{\mu r^{2'}}$$

$$\Rightarrow \theta = \int \frac{L}{\mu r^2} dt + \theta_{0,} \quad (60)$$

Where, θ_0 is an integration constant. Now, Eq. (59) can also be written as

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{L}{2\mu r^2} - V(r) \right)}} \quad (61)$$

Substituting this value of *dt* in (60). we get

$$\theta = \int \frac{(L/r^2)dr}{\sqrt{2\mu(E - V - L2/r2)}} + \theta_{o.} \quad (62)$$

Let us consider

$$u = \frac{1}{r} \Rightarrow dr = -u^2 du. \quad (63)$$

Substitution Eq. (63) in Eq. (62) and simplifying we get

$$\theta = \theta o - L\int \frac{du}{\sqrt{2\mu(E - V) - L^2 u^2}} \quad (64)$$

If the form of the potential V is known further integration can be done to find the explicit form of θ.

Substituting Eq. (57) in Eq. (64) and simplifying we get

$$\theta = \theta o - \int \frac{du}{\left[(2\mu E/L^2) + (2\mu k/L^2)u - u^2\right]^{1/2}} \quad (65)$$

The integral on the right side of equation (65) is a standard one of the type

$$\int \frac{dr}{(\alpha + \beta x + rx^2)^{(1/2)}} = \frac{1}{\sqrt{-'1}} cos^{-1}\left(\frac{\beta + 2'1x}{(\beta 2 - 4\alpha'1)^{1/2}} \right) \quad (66)$$

Comparing Eqs. (65) and (66) we get

$$\alpha = \frac{2\mu E}{L^2}.\ \beta = \frac{2\mu k}{L^2} \qquad \text{and} \qquad `1 = -1. \quad (67)$$

Keeping the values of α, β and `1 in mind and integrating Eq. (65) with the help of formula given by (66) yields

$$\theta - \theta_o = cos^{-1}\left[\frac{uL2 / \mu k - 1}{\sqrt{1 + (2EL^2 / \mu k^{.2})}}\right]$$

$$\Rightarrow cos(\theta - \theta) = \frac{uL2 / \mu k - 1}{\sqrt{1 + (2EL^2 / \mu k^{.2})}}$$

$$\Rightarrow u = \frac{1}{r} = \frac{\mu k}{L_2}\left[\sqrt{1 + \frac{2EL^2}{\mu k^{.2}} cos(\theta - \theta_0)}\right] \tag{68}$$

which is the equation of the orbit. it may be noted that only three (θ_0, E and L) of the four constants of integration appear int he orbits equation. The fourth constant can be obtained by finding the solution of the other equation of motion. that is, Eq. (58) Equation (68) can be rewritten after simplification as.

$$\frac{1}{r} = \frac{\mu k}{L^2}(1 + c \cos\theta). \tag{69}$$

Where

$$\in = \sqrt{1 + \frac{2EL^2}{\mu k^{.2}}} \tag{70}$$

is the eccentricity of the particle's motion, Equation (69) is the general equation of a conic with one focus at the origin. Hence, the basic motion of objects under the influence of gravity, which takes care of virtually all of the gazillion tons of stuff in the universe is given by Eq. (69)

Limits on r in Eq. (69) : To understand Eq. (69) completely we need to know the maximum and minimum values of r. It will give an idea of the region in which the motion of the particle is restricted and in turn will provide the nature of the orbit.

(i) Value of r_{min}: The minimum value of r is obtained when the right-hand side reaches its maximum value. This is possible when $cos\,\theta = 1.$ In this case the value of r_{min} is

$$r_{min} = \frac{L2}{\mu k(+ \in)} \tag{71}$$

(ii) Value of rmi : The answer depends on whether $\in$ is greater than or less than 1

(a): If $\in$<1 (which corresponds to circular or elliptical orbits, as we will see in the Subsec. (A) then the minimum value of the right-hand side of Eq. (69) is $\overset{\mu\Omega}{L^2}(1-\in)$ Therefore.

$$r_{min} = \frac{L2}{\mu k(-\in)} \tag{72}$$

(b): If $\in \geq 1$ (which corresponds to parabolic or hyperbolic orbits. as we will see in the Subsec. A). then the right-hand side of Eq. (69) can become zero (when $\cos\theta = -1/\in)$ = −1/) Therefore.

$$r_{min} = \infty(if\ \in\geq 1). \tag{73}$$

A. The Orbits

From Eq. (69) it is clear that the form of the orbit will depend on which is given by (70) Hence in this subsection we will examine the various cases for

Case (i) : Circle $(\in= 0)$

If $\in= 0$,then Eq. (70) says that $E = -\frac{\mu\kappa^2}{2r^2}$. The negative E means that potential energy is more negative than the kinetic energy is positive. It implies that the particle is trapped in the potential well. Equations (71) and (72) give $r_{min} = r_{min} = \frac{L^2}{\mu k}$, which suggests that the particle is moving in a circular orbit with radius $\frac{L^2}{\mu k}$, It is to be noted that Eq. (69) is independent of r and θ.

For a given L, the energy $-\frac{\mu k^2}{2L^2}$, is the minimum value that E can have due to Eqs. (58) and (59). This is true because to achieve the minimum. we certainly want r = 0. This can be shown that by minimizing the effective potential $\frac{L^2}{2\mu r^2} - \frac{k}{r}$ we get the same value for E If we plot $V_{eff}(r)$, we have the situation shown in Fig. 6. The particle is trapped at the bottom of the potential well, so it has no motion in the r direction.

Case (ii): Ellipse (0 < $\in$ < 1)

If 0 < $\in$ < 1, then Eq. (70) implies the condition $\frac{\mu k^2}{\mu r^2} < E < 0.$

Moreover. form Eqs. (71)

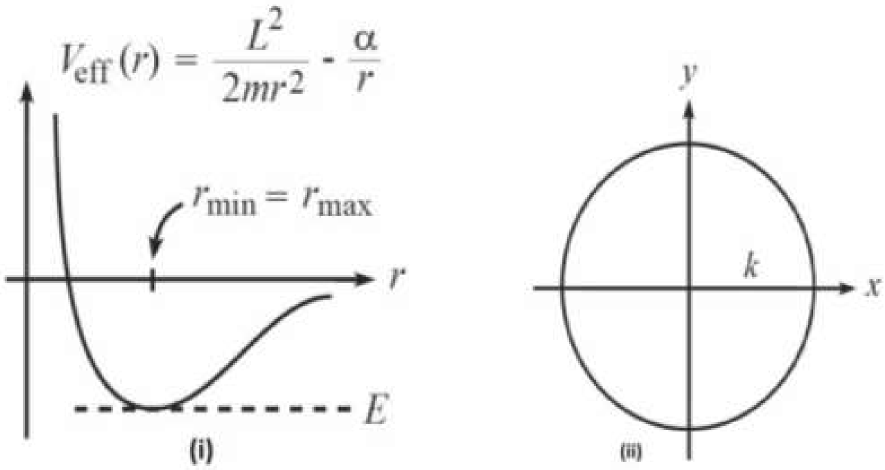


FIG. 6: (i): Position of r_{min} and $r_{min.}$ **(ii):** Orbit of the particle

and (72) we get different values of r_{min} and $r_{min.}$ However, is not sufficient to conclude that the resulting motion is an ellipse. For this purpose, we proceed as follows:

Equation (70) can be rewritten as

$$\in 2 = 1 + \frac{2EL^2}{\mu k^{.2}}.$$

$$\Rightarrow E = \frac{k^2 \mu}{2L^2}(\in^2 - 1). \tag{74}$$

We know that for an attractive force, energy is negative. Hence

$$E < 0 \Rightarrow \frac{k^2 \mu}{2L^2}(\in^2 - 1) < 0.$$

$$\Rightarrow \in^2 - 1 \Rightarrow \in < 1. \tag{75}$$

From this it can be conclude that the path is an ellipse,

If we plot V, we have the situation shown in Fig. 7. The particle oscillates between and r_{min} and r_{min} The energy is negative, so the particle is trapped in the potential well.

Case (iii): Parabola ($\in = 1$)

If $\in = 1$, then Eq. (70) says that $E = 0$. This value of E implies that the particle barely makes it out to infinity (its speed approaches zero as $r \to \infty$). Equation (71) gives $r_{min} = L^2 / 2m\alpha.$ and Eq. (73) gives Again, it isn't obvious that the resulting motion is a parabola. We'll demonstrate this below.

If we plot $V_{eff}(r)$, we have the situation show in Fig. 8. The particle does not oscillate back and forth in the r direction. It moves inward (or possibly not, if it was initially moving

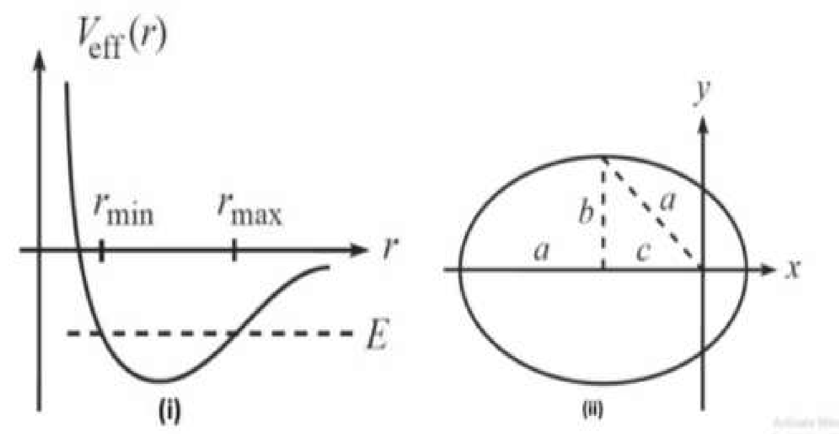


FIG. 7: (i): Position of r_{min} and r_{min} **(ii):** Orbit of the particle

out ward), turns around at r_{min} and $r_{min} = L2/2\mu\kappa'$. then heads out to infinity forever.

Case (iv): Hyperbola ($\in > 1$)

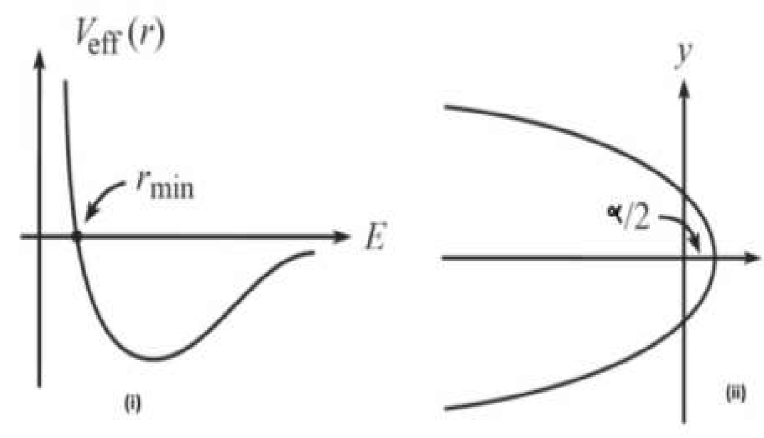


FIG. 8: (i): Position of r_{min} and r_{min} **(ii):** Orbit of the particle

If $\in > 1$,. then Eq. (70) says that E > 0. This value of E implies that the particle makes it out to infinity with energy to spare. The potential goes to zero as $r \to \infty$. so the particle's speed approaches the nonzero value $\sqrt{2E/\mu}$ as $r \to \infty$. Equation (71) gives and Eq. (73) gives $r_{max} = \infty$. Again, it isn't obvious that the resulting is a hyperbola. We'll demonstrate this below.

If we plot $V_{eff}(r)$, we have the situation shown in Fig. 9. As in the parabola case. the particle does not oscillate back and forth in the r direction. It moves inward (or possibly not. If it was initially moving outward). turns around at and then heads out to infinity forever.

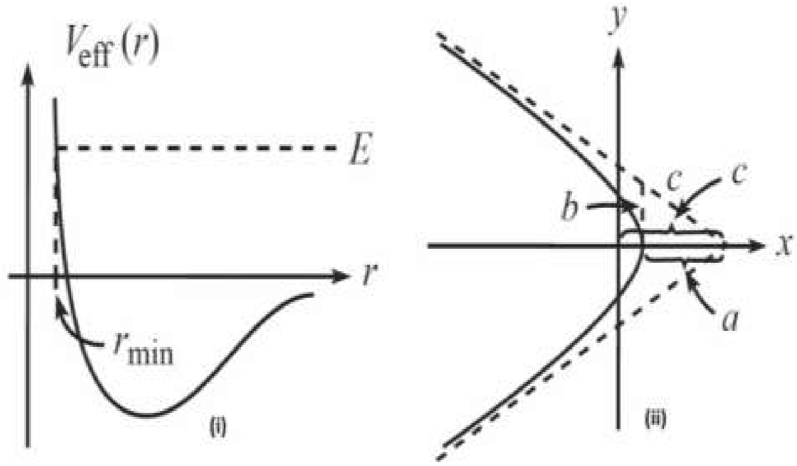


FIG. 9. (i): Position of r_{min} and r_{min} **(ii):** Orbit of the particle

XII. PROOF OF CONIC ORBITS

To prove the Eq. (69) describe the conic sections as discussed in **Subsection, A**. let us consider.

$$\delta \equiv \frac{l2}{\mu k}. \tag{76}$$

Substituting Eq. (76) in Eq. (69) and using cos $\theta = x/r$ gives

$$\frac{1}{r} = \frac{1}{k}\left(1 + \frac{x}{r}\right). \tag{77}$$

Multiplying Eq. (77) through by kr, we get

$$\alpha = r + \in x \ \ \Rightarrow r = k - \in x. \tag{78}$$

Squaring above equation yields

$$x^2 + y^2 = \delta^2 - 2\delta \in x + \in^2 x^2. \tag{79}$$

Now we will discuss various cases for.

(i) **Circle** $(\in = 0)$ **:** In this case, Eq. (70) becomes $x^2 + y^2 = \delta^2$. we have a circle with radius $\delta = L^2 / mk.$ with its center at the origin (see Fig. 6).

(ii) **Ellipse** (0 < $\in$ < 1) In this case. Eq. (79) can be rewritten after rearranging the terms as

$$\frac{\left(x+\frac{\delta r}{1-\in^2}\right)}{a^2}-\frac{y^2}{b^2}=1,\ \text{where}\ a=\frac{\delta}{1-\in^2},\ \text{and}\ b=\frac{\delta}{\sqrt{1-\in^2}} \quad (80)$$

This is the equation for an ellipse with its center located at the semi-major and semi-minor axes are a and b, respectively.

(iii) **Parabola** : $(\in=1)$ In this case Eq. (79) becomes $x^2+y^2=\delta^2$ which can be written $as\ y^2=\delta^2(x-\delta/2)$. This is the equation for a parabola with vertex at $(\delta/2,0)$. So we have a parabola with its focus located at the origin (see Fig. 7).

(iv) **Hyperbola:** $(\in=1)$ In this case, Eq. (79) after completing the square for the x terms and simplifying we get.

$$\frac{\left(x+\frac{\delta r}{x^2-1}\right)}{a^2}-\frac{y^2}{b^2}=1,\ \text{where}\ a=\frac{\delta}{\in^2-1}\ \text{and}\ b=\frac{\delta}{\sqrt{\in^2-1}} \quad (81)$$

This is the equation for a hyperbola with its center located at

XIII. KEPLER'S LAWS

Based on the detailed astronomical data of Tycho Brahe, kepler enunciated three general laws regarding planetary motion. They can be stated as follows:

Law of orbits: Planets move in elliptical orbits with the sun at one focus.

Proof: We have already shown this in Sec. XII.

Law of areas: The radius vector to a planet sweeps out area at a rate is independent of its position in the orbit.

Proof: This law is nothing but another way of writing conservation of angular momentum. The area swept out by the radius vector during a short period of time is $dA=r(rd\theta)/2$ because is the base of the thin triangle in Fig. 10. Therefore. we have (using $L=mr2\,\theta.$)

$$\frac{dA}{d}=\frac{r^2\dot{\theta}}{2}=\frac{L}{2m} \quad (82)$$

which is constant, because L is constant for a central force.

Law of periods: The square of the period of revolution about the sun is proportional to

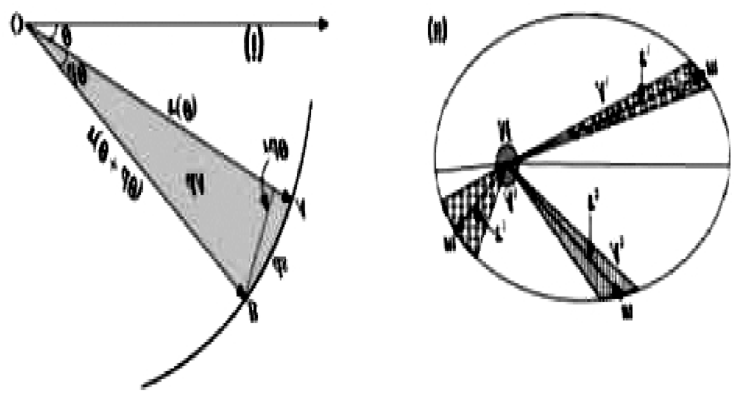


FIG. 10: (i) Area swept by a radius vector r in time dt (ii) Area swept by a radius vector r in time period *T*.

the cube of the semi-major axis of its orbit. Mathematically, it can be written as

$$T2 = \frac{4\pi^2 \alpha^3}{Gm_1 m_2}. \tag{83}$$

where a is the semi-major axis.

Proof: In the case of ellipse, the perihelion $(r_1 = r_{min})$ and $(r_2 = r_{min})$ aphelion distances are the values of r when and respectively: Now from Eq. (69)

$$r_{min} = r_1 = \frac{1}{C(1+\in)} \qquad r_{min} = r_2 = \frac{1}{C(1+\in)}. \tag{84}$$

Then one can write the semi-major axis as

$$a = \frac{r_1 + r_2}{2} = \frac{1}{C(1-\in^2)} \tag{85}$$

Let *T* be the time period of an elliptical orbit. Then integrating Eq. (82) over the time of a whole orbit gives.

$$A = \int_O^T \frac{dA}{dt} = \int_O^T \frac{L}{2m} = \frac{LT}{2m}. \tag{86}$$

The area of the ellipse is also equal to $A = \pi ab,$ where b is the length of the semi-minor axis.

Then comparing Eq. (86) with $A = \pi ab,$ we get

$$\frac{LT}{2\mu} = \pi ab \Rightarrow T = \frac{2\pi ab\mu}{L}. \tag{87}$$

From the properties of ellipse, we know that

$$b = a\sqrt{1-\in^2} \tag{88}$$

From Eqs. (85) and (88) we get

$$a^2(1-\in^2) = \frac{1}{C^2(1-\in^2)}(1-t^2) = \frac{1}{C^2(1-t^2)} \tag{89}$$

$$\Rightarrow 1-\in^2 = \frac{1}{aC}.$$

Substituting (89) in (88) yields

$$b2 = \frac{a}{C} = \frac{aL^2}{\mu k}.$$

Substituting (90) in (87) provides

$$T2 = \frac{4\pi 2a3}{G(m_1 m_2)}.$$

which is the statement of Kepler's third law.

The third law can be written in an alternative μ by form by $(m_1m_2)/(m_1+m_2)$ and k by its value Gm_1m_2 as.

$$T2 = \frac{4\pi 2a3}{G(m_1 m_2)}.$$

These three laws describe the motion of all the planets (and asteroids, comets, and such) in the solar system.

XIV. LAW OF GRAVITATION FROM KEPLER'S LAWS

To obtain the law of gravitation from Kepler's law let us rewrite Eq. (59) as

$$\mu\bar{r} - \mu r\dot{\theta}2 = F(r). \tag{93}$$

Let us consider then

$$\theta = \frac{L}{\mu r2} = \frac{L}{\mu}u^2 \tag{94}$$

and

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{1}{u^2}\frac{du}{dt} = \frac{1}{u^2 d\theta}\dot{\theta} = -\frac{L}{\mu}\frac{du}{\mu d\theta}. \tag{95}$$

Similarly, we can get the second derivative of r as

$$\bar{r} = -\left(\frac{Lu}{\mu}\right)^2 \frac{d^2\mu}{d\theta^2}. \tag{96}$$

Substituting Eqs. (94) and (96) in (93) and simplifying we get

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2}F\left(\frac{1}{u}\right). \tag{97}$$

Now, re-inverting the above relation in the form of yields

$$\frac{d^2u}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu}{L^2}r^2F(r). \tag{98}$$

Equation Eq. (69) is

$$\frac{1}{r}=C(1+\cos\theta) \qquad C=\frac{\mu k}{L^2}. \tag{99}$$

Replacing 1/r on the left-hand side of (99), we get

$$C\frac{d^2}{d\theta^2}(1+\in\cos\theta)+C(1+\in\cos\theta)=-\frac{\mu}{L^2}r^2F(r). \tag{100}$$

$$\Rightarrow C=-\frac{\mu}{L2}r2F(r) \tag{101}$$

$$\Rightarrow F(r0=\frac{CL^2}{\mu}\frac{1}{r^2}=-\frac{K'}{r^{2\prime}} \quad K'=\frac{CL^2}{\mu}=cons\tan t \tag{102}$$

The negative sign indicates that the force is of attraction. Equation (86) can be rewritten as.

$$T2=\frac{4\pi^2\mu^2}{L^2}a^2b^2 \tag{103}$$

Since, Substituting this value of b2

$$T^2=4\pi^2\mu\frac{\mu}{CL2}a^3=\frac{4\pi^2\mu a^3}{K'} \tag{104}$$

Comparing this with (92) we get

$$\frac{\mu}{K'}=\frac{1}{G(m_1+m_2)}\Rightarrow\frac{m_1m_2}{m_1m_2}=\frac{1}{G(m_1+m_2)}.$$

$$\Rightarrow K'=Gm_1m_2. \tag{105}$$

With this value of Eq. (102) reduces to

$$F(r)=-\frac{Gm_1m_2}{r^2}\dot{r} \tag{106}$$

which is gravitational force of sun of planet.

XV. SUMMARY

- A central force is by definition a force that points radially and whose magnitude depends only on the distance from the distance from the source. It is represented mathematically as

  $$\overline{F}=F\overline{r}$$

- Central force is conservative in nature and hence it can be represented as negative gradient of potential. that is.

❖ In central force motion of two bodies about their center of mass can always be reduced to an equivalent on body problem.

❖ The angular momentum L is conserve in central forces due to which the motion is confined to a plane perpendicular to.

❖ Conservation of and E provides two equations of motion and they are also known as integrals of motion.

❖ The equation of orbit for an inverse square law, that is is represented by a conic equation as

$$\frac{1}{r} = \frac{\mu k}{L2}(1 + \cos\theta). \tag{109}$$

where $\in = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$.

❖ For repulsive central conservative force, the orbit will be a hyperbola whereas for attractive force its shape depends on the form of $\in$.

Terminal Question

1: Find the components of the force on the body when it is in position (–2,0,5). if the potential energy is given by where V is in jule and in meter.

UNIT: 8

ELASTICITY

Structure:

8.1 INTRODUCTION

In Block-I, we studied about rotation of the bodies and then realized that the motion of a body depends on how mass is distributed within the body. We restricted ourselves to simpler situation of rigid bodies. Rigid bodies don't bend, stretch, or squash when forces act on them. But the rigid body is an idealization. All real materials are elastic and do deform to some extent. In the present unit we will introduce the concepts of stress, strain, Simple Principle called Hooke's Law that help us predict what deformation will occur when forces are applied to a real kind of (not perfect rigid) body.

8.2 OBJECTIVES

After studying this unit, you should be able to –

- ❖ Define Elasticity
- ❖ Understand the Concept of Stress and Strain
- ❖ Explain Poisson's Ratio
- ❖ Related Poisson ratio and elastic constants
- ❖ Understand Elastic Potential Energy

8.3 WHAT IS ELASTICITY

If the distance between any two particles in a body remains unattired whatever the external forces applied to it, the body is said to be rigid body. In practice it is not possible to have a perfect rigid body. Everybody gets deformed under the action of forces to a smaller or larger extent. The property of the body by virtue of which it tends to regain its original shape and size on the removal of external (deforming) forces is elasticity.

If a body recovers completely is original shape, size or volume as soon as the deforming forces have been removed, it is said to be perfectly elastic body while if it completely retains its altered size and shape, it is said to be perfectly plastic body. In general, there is no perfectly elastic or plastic body, actual bodies the between the two extremes. The nearest approach to a perfectly plastic body is putty and perfectly elastic body is a quartz fiber.

8.4 STRESS AND STRAIN

When an external (deforming) force acts upon a body, relative displacements of its various particles take place and consequently the body gets deformed. Elastic body offers appreciable resistance to the deforming force inside the body. These internal forces are equal in magnitude and opposite to the deforming force so long as there is no permanent change produced in the body. Finally, these internal forces, restoring forces restore the body to its original form when deforming forces are removed. The restoring per unit area comes into play inside the body called stress. If the force F is acting on the area cross section a, then

$$Stress = \frac{F}{a}$$

If stress is normal to the surface, it is called normal stress e.g., stress is normal in case of a change in length of a wire or in change in the volume of a body.

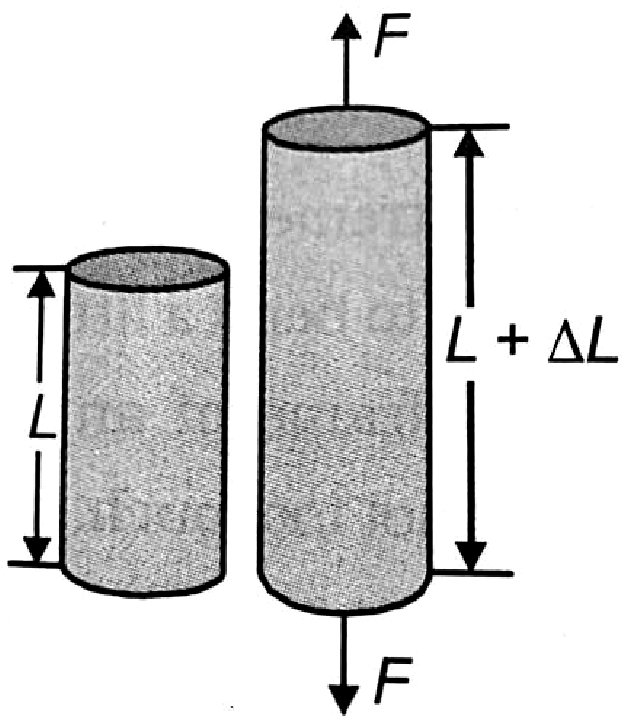


Figure – 1a

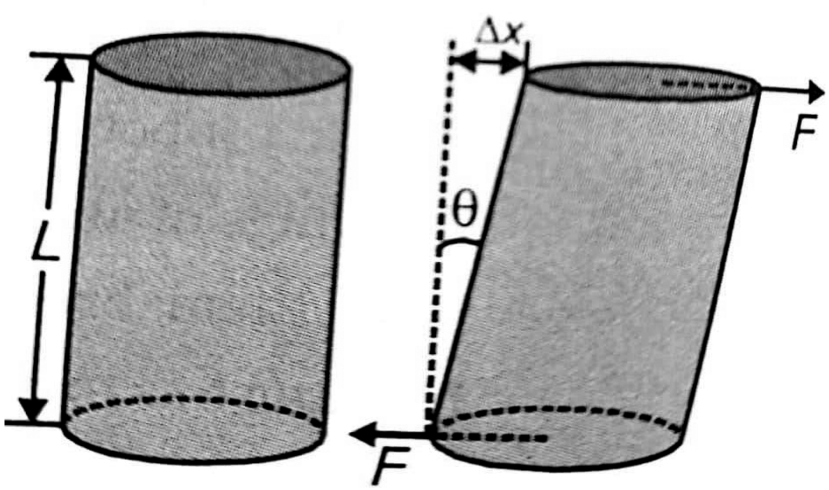


Figure – 1b

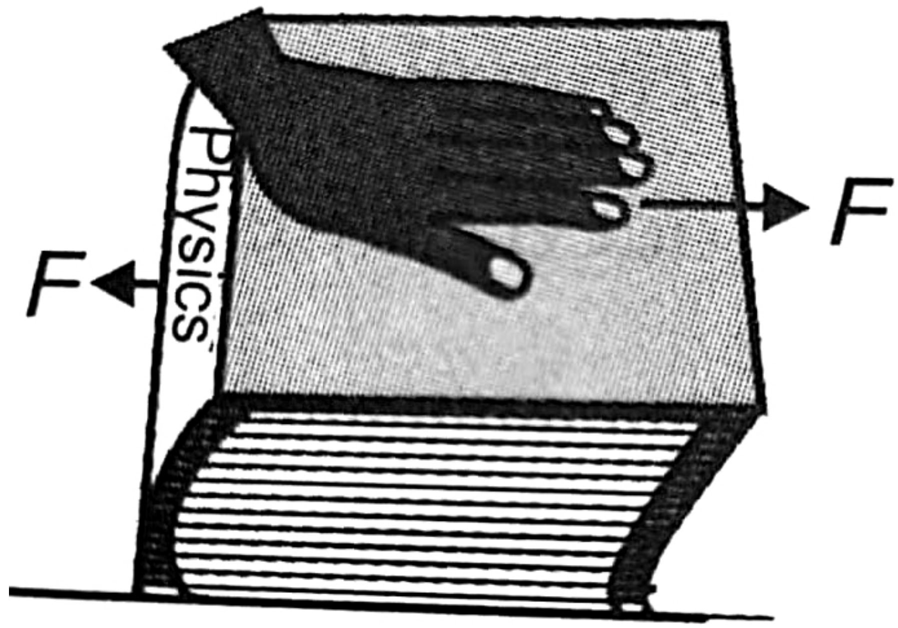


Figure – 1c

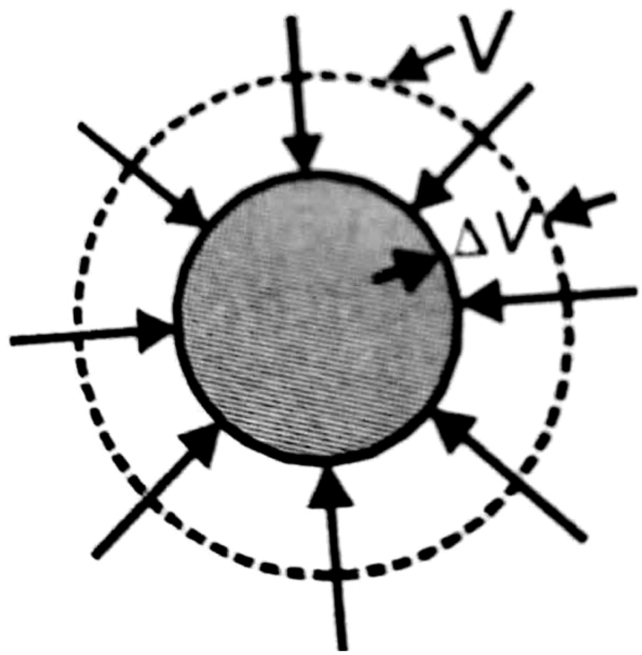


Figure – 1d

Figures: (a) Cylinder subjected to tensile stress stretches it by an amount ΔL. (b) A cylinder subjected to shearing (tangential) stress deforms by an angle θ. (c) A book subjected to a shearing stress (d) a solid sphere subjected to a uniform hydraulic stress shrinks in volume by an amount ΔV.

The normal stress may be either compressive or expansive (tensile) according as decrease or increase in volume takes place. When the stress is tangential to the surface, it is called shearing or tangential stress.

8.5 HOOKE'S LAW

There is a simple relationship between stress and strain discovered by Robert Hooke in 1676. According to this law, stress is directly proportional to strain for small deformations i.e., within elastic limit. Mathematically,

Stress × Strain

or Stress = E. Strain: E is a constant

$$\Rightarrow \quad E = \frac{Stress}{Strain}$$

The constant E is called elasticity coefficient or modules of elasticity. The value of coefficient of elasticity depends upon the nature of material and also the condition to which it has been subjected after manufacturing. The S.I. unit of coefficient of elasticity E is N/m^2.

8.6 KINETIC MODEL FOR SOLIDS (F – r and U – r graphs)

Solid is one of the fundamental states of matter. A Solid is characterized by structural rigidity and resistance to a free applied to the surface. In solid, the intermolecular forces are very strong and the constituent particles are closed packed. Hence, solids are incompressible and have high density. These interatomic forces give an explanation of some of elastic and thermal properties of a solid.

If U is the potential energy of two molecules separated by a distance r, the force between them, $F = -\frac{dU}{dr}$. The variations of potential energy U of two molecules and force F between them with their separation r have been shown in Figure-1.

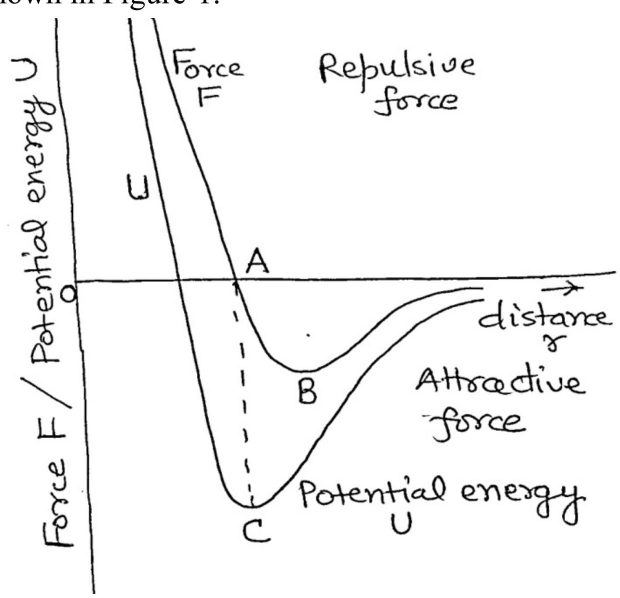


Figure – 1e

There are two forces act between the molecules – repulsive force and attractive force. We can see from the graph that when the molecules are close to each other the repulsive force predominates while at larger distance, the attractive force is large. The resultant force between the molecules is repulsive from O to A, attractive from A to B but increasing with distance and, attractive from B to infinity but decreasing with distance. There is a position A on the graph, where the two forces balance other. This is the equilibrium position for molecules in solid. The potential energy is minimum at this point. Any disturbance from this position produces a force tending to return the molecules to A.

8.7 BEHAVIOUR OF A WIRE UNDER LOAD

Let a wire be clamped at one end and loaded at the other end. If the stress or load is increased continuously until the wire breaks down, we observe the following behaviour of the wire as shown in Figure-2. The figure shows the variation between stress and strain of a loaded wire and is known as stress – strain diagram.

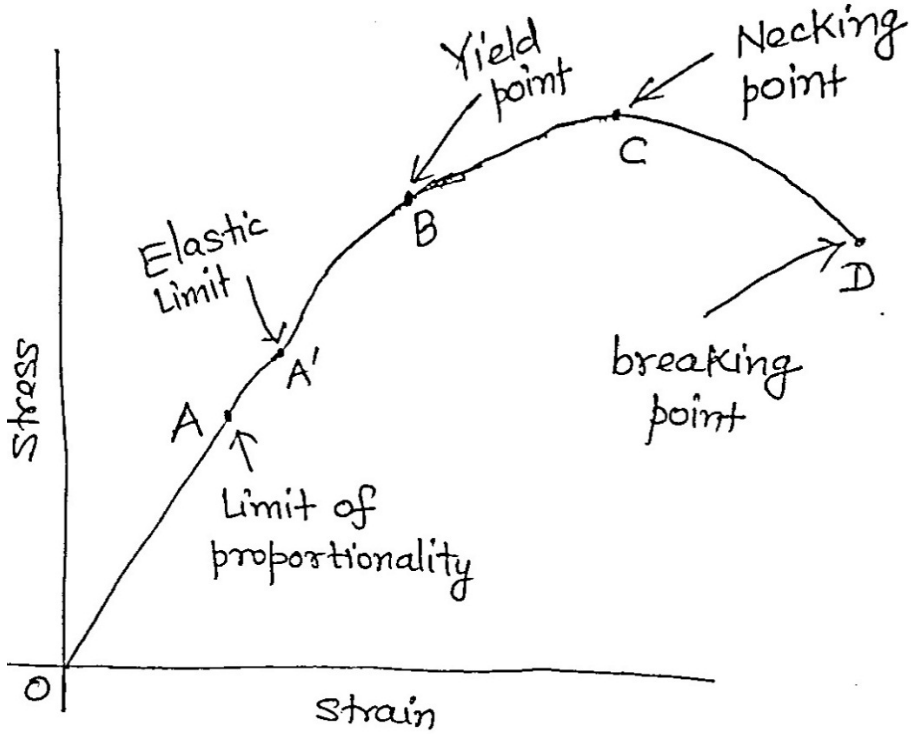


Figure - 2

The part OA of the curve is a straight line which shows that stress is proportional to strain upto the point A. Hooke's law holds good only for the straight-line portion OA of the curve. The point A is called the limit of proportionality. If stress is further increased, a point A' is reached, which is known as elastic limit. Below the elastic limit the body regains its original position or shape or size when deforming force is removed.

8.8 POISSON'S RATIO

Poisson's ratio is **"the ratio of transverse contraction strain to longitudinal extension strain in the direction of the stretching force."** Here,

- Compressive deformation is considered negative
- Tensile deformation is considered positive.

Symbol	Greek letter 'nu',ν
Formula	Poisson's ratio = – Lateral strain / Longitudinal strain
Range	-1.0 to +0.5
Units	Unitless quantity
Scalar / Vector	Scalar quantity

8.8.1 Poisson's Ratio Formula

Imagine a piece of rubber, in the usual shape of a cuboid. Then imagine pulling it along the sides. What happens now?

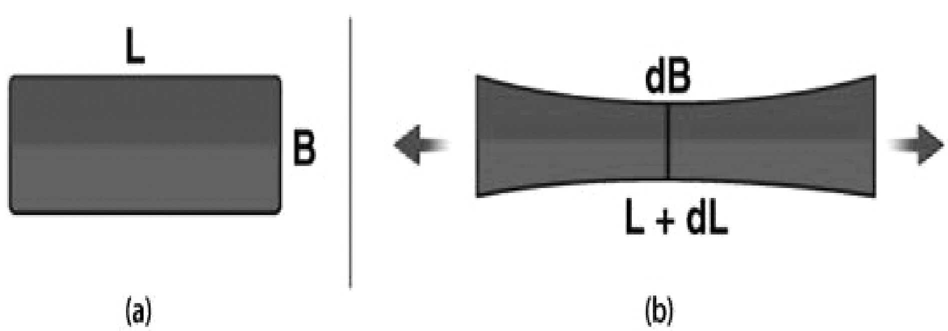


Figure - 3

It will compress in the middle. If the original length and breadth of the rubber are taken as L and B respectively, then when pulled longitudinally, it tends to get compressed laterally. In simple words, length has increased by an amount dL and the breadth has increased by an amount dB.

In this case,

$$\in_t = -\frac{dB}{B} \; \in_l = -\frac{dL}{L}$$

The formula for Poisson's ratio is,

$$Poisson's\,ratio = \frac{Transverse\ strain}{Longitudinal\ strain} \Rightarrow v = -\frac{\in_t}{\in_l}$$

where,

εt is the Lateral or Transverse Strain

εl is the Longitudinal or Axial Strain

ν is the Poisson's Ratio

The strain on its own is defined as the change in dimension (length, breadth, area…) divided by the original dimension.

8.8.2 Poisson Effect

When a material is stretched in one direction, it tends to compress in the direction perpendicular to that of force application and vice versa. The measure of this phenomenon is given in terms of Poisson's ratio. For example, a rubber band tends to become thinner when stretched.

8.8.3 Poisson's Ratio Values for Different Material

It is the ratio of transverse contraction strain to longitudinal extension strain, in the direction of the stretching force. There can be a stress and strain relation which is generated with the application of force on a body.

- ❖ For tensile deformation, Poisson's ratio is positive.
- ❖ For compressive deformation, it is negative.

Here, the negative Poisson ratio suggests that the material will exhibit a positive strain in the transverse direction, even though the longitudinal strain is positive as well.

For most materials, the value of Poisson's ratio lies in the range, **0 to 0.5.**

A few examples of Poisson ratio is given below for different materials.

Material	Values
Concrete	0.1 – 0.2
Cast iron	0.21 – 0.26
Steel	0.27 – 0.30
Rubber	0.4999
Gold	0.42 – 0.44
Glass	0.18 – 0.3
Cork	0.0
Copper	0.33
Clay	0.30 – 0.45
Stainless steel	0.30 – 0.31
Foam	0.10 – 0.50

8.9 DERIVE THE RELATIONSHIP BETWEEN THE ELASTIC CONSTANTS, I.E. E, K AND G

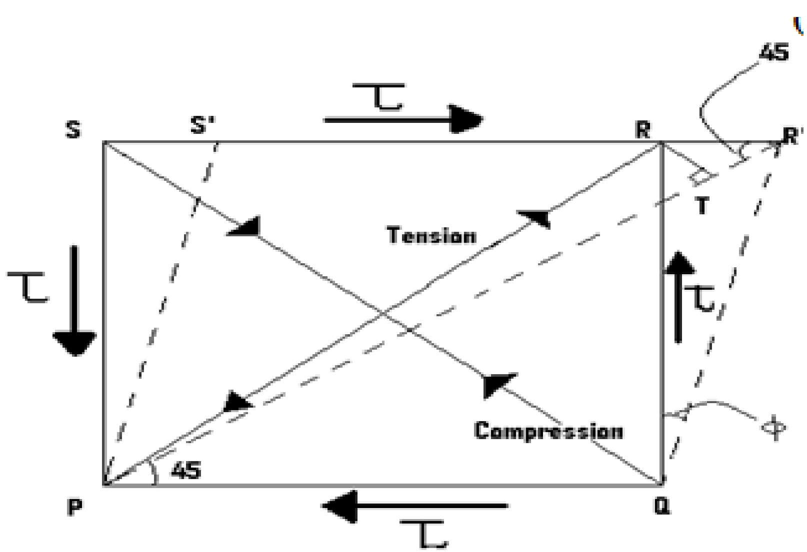


Figure - 4

Consider a solid cube, subjected to a Shear Stress on the faces PQ and RS and complimentary Shear Stress on faces QR and PS. The distortion of the cube, is represented by the dotted lines. The diagonal PR distorts to PR'.

(a) Relationship between E and G

Modulus of Rigidity, $G = \frac{Shear\ Stress}{Shear\ Strain}$

$$Shear\ Strain = \frac{Shear\ Stress}{G}$$

From the diagram, Shear Strain $\phi = \frac{PR'}{QR}$

Since Shear Stress = τ,

$$\frac{RR'}{QR} = \frac{\tau}{G} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From R, drop a perpendicular onto distorted diagonal PR'

The strain experienced by the diagonal $= \frac{TR'}{PR}$ (Considering that PT ≈ PR)

$$= \frac{RR' \cos 45}{\frac{QR}{\cos 45}} = \frac{RR'}{2QR} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Strain of the Diagonal $PR = \frac{RR'}{2QR} = \frac{\tau}{2G} (Form\ I)$

Let f be the Direct Stress induced in the diagonal PR due to the Shear Stress τ

Strain of the diagonal $= \frac{\tau}{2G} = \frac{f}{2G}$ $\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

The diagonal PR is subjected to Direct Tensile Stress while the diagonal RS is subjected to Direct Compressive Stress.

The total strain on Diagonal PR would be $= \frac{f}{G} + \frac{1}{m}\left(\frac{f}{E}\right)$

$$= \frac{f}{E}\left(1 + \frac{1}{m}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Comparing Equations (III) and (IV), we have

$$\frac{f}{2G} = \frac{f}{E}\left(1 + \frac{1}{m}\right)$$

Re – arranging the terms, we have,

$$E = 2G\left(1 + \frac{1}{m}\right) \quad \text{……………………. (5)}$$

(b) Relationship between E and K

Instead of Shear Stress, let the cube be subjected to direct stress f on all faces of the cube.

We know,

$$e_v = \frac{f_x + f_y + f_z}{E}\left[1 - \frac{2}{m}\right]$$

Since $f = f_x = f_y = f_z$

$$e_v = \frac{3f}{E}\left[1 - \frac{2}{m}\right] \quad \text{……………………. (6)}$$

Also, by the definition of Bulk Modulus,

$$e_v = \frac{f}{K}$$

Equating (6) and (7), we have:

$$\frac{f}{K} = \frac{3f}{E}\left[1 - \frac{2}{m}\right]$$

$$E = 3K\left[1 - \frac{2}{m}\right] \quad \text{……………………. (8)}$$

(c) Relationship between E, G and K

From the equation **(5)**,

$$\frac{1}{m} = \frac{E - 2G}{2G}$$

From the equation **(8)**

$$\frac{1}{m} = \frac{3E - E}{6K}$$

Equating both, we get,

$$\frac{E - 2G}{2G} = \frac{3K - E}{6K}$$

Simplifying the equation, we get,

$$E = \frac{9KG}{3K + G}$$

This is the relationship between E, G and K.

8.10 ANGLE OF TWIST

- Deformation of a circular shaft subjected to pure torsion
  - (a) Fix left end of shaft
  - (b) A moves to A'
  - (c) ϕ = angle of twist (in radians)
- What are the boundary conditions on ϕ?
  - (a) $\phi(x) = 0$ at $x = 0$
  - (b) $\phi(x) = \phi$ at $x = F$
- For pure torsion, ϕ is linear.

$$\phi(x) = \frac{\phi x}{L}$$

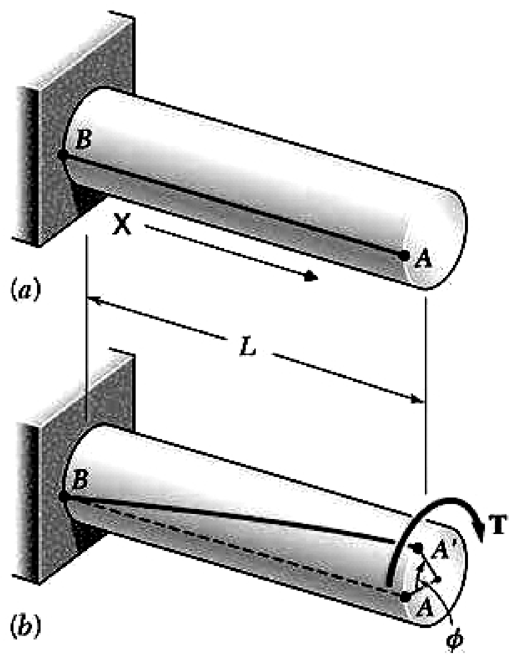


Figure - 5

8.11 ANGLE OF SHEAR

A form of stress resulting from equal and opposite forces that do not act along the same line. If a thick hard-bound book is lying flat, and one pushes the front cover from the side so that the covers and pages no longer constitute parallel planes, this is an example of shear.

The angle of deformation on the sides of an object exposed to shearing stress. Its symbol is φ (the Greek letter phi), and its value will usually be well below 90°.

8.12 CONTILEVER

When a beam of uniform cross section is fixed horizontally at one end and can be bent by a load applied at or near the free end, the system is called a cantilever. When the free end of the cantilever is loaded by a weight Mg, the beam bends with curvature changing along its length. The curvature is zero at the fixed end and increases with distance from this end becoming maximum at the free end. Taking X-axis horizontally in the direction of the length of the unbent beam, Y-axis vertically downwards, let us consider the equilibrium of a transverse section of the beam at a point P whose co-ordinates are (x, y), if the beam is of length l the distance of this section at P from the applied mass at the free end is (l × x) and hence the moment of the external couple is evidently equal to Mg (l × x). For equilibrium of the section, this moment of the external couple must be equal to the bending moment at the section.

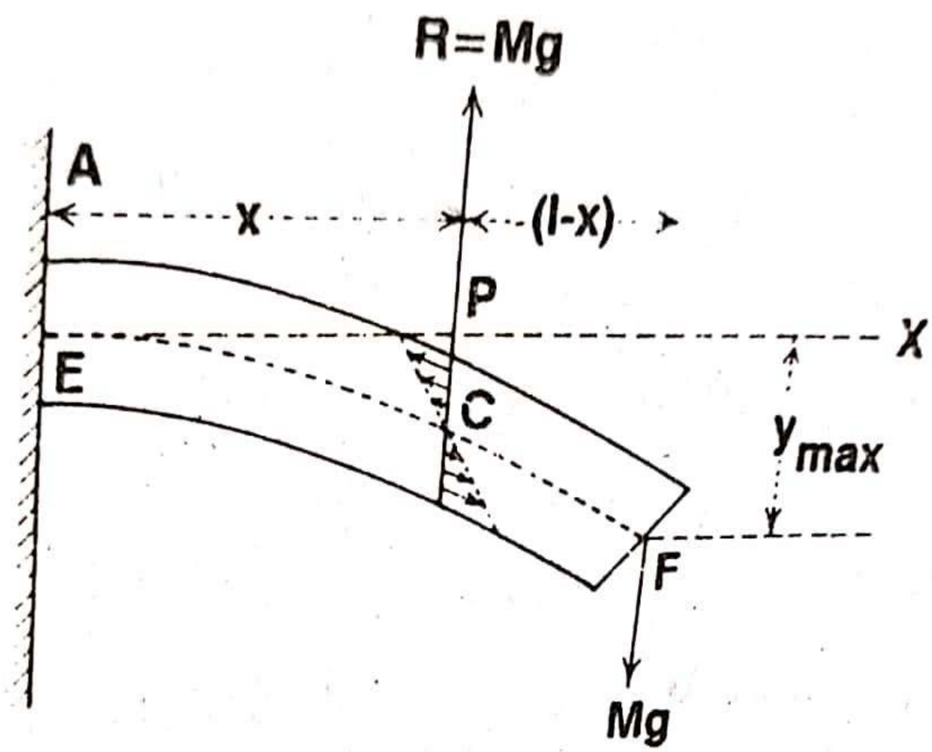


Figure - 6

Hence $$\frac{YI}{R} = Mg\,(l \times x)$$

or $$\frac{1}{R} = \frac{Mg}{YI}(l \times x)$$

Now the radius of curvature R of the neutral axis at P distant x from fixed end and having depression y is given by

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

Thus $\frac{d^2y}{dx^2} = \frac{Mg}{YI}(l \times x)$

Integrating once, we get

$$\frac{dtdy}{dx} = \frac{Mg}{YI}\left(lx - \frac{x^2}{2}\right) + c_1$$

when $x = 0 \; \frac{dy}{dx} = 0 \; \square \; c_1 = 0$

Hence $\frac{dy}{dx} = \frac{Mg}{YI}\left(lx - \frac{x^2}{2}\right)$

Integrating again, we get

$$y = \frac{Mg}{YI}\left(\frac{l\,x^2}{2} - \frac{x^2}{6}\right) + c_2$$

when $x = 0 \; y = 0 \;\; \square \; c_2 = 0$

Hence $y = \frac{Mg}{YI}\left(\frac{l\,x^2}{2} - \frac{x^3}{6}\right)$

This gives the depression of the beam at distance x from the fixed end.

At the loaded end when $x = l$, the depression is maximum and is given by

$$\delta = \frac{Mg\,l^3}{3\,YI}$$

If the beam is of a rectangular cross section (breadth b and thickness d), the geometrical moment of inertia

$$I = \frac{bd^3}{12}$$

Therefore, $\delta = \frac{4\,Mg\,l^3}{Y\,bd^3}$

For beam of circular cross section of radius r, $I = \frac{\pi\,r^4}{4}$

Therefore, $\delta = \frac{4\,Mg\,l^3}{3\,Y\,\pi\,r^4}$

8.13 ELASTIC POTENTIAL ENERGY

In natural state of a body, the molecules settle in their equilibrium position and the potential energy corresponding to this position is minimum (Refer to the graph given in section 2 of this chapter). When deformed, the molecular separation changes and potential energy increases. This happens due to appearance of an internal forces, against whom work has to be performed to deform a body. The increase in potential energy when body is deformed is known as elastic potential energy.

Let us calculate the elastic potential energy of a wire, which is stretched by applying a force.

Consider a wire of length L and cross-section A. we pull it so as to stretch it slowly. When extension is x, the tension force is such that

$$Y = \frac{\frac{F}{A}}{\frac{x}{L}}$$ [Assuming a to remain constant]

$$\Rightarrow \quad F = \left(\frac{YA}{L}\right)x = kx$$

Where $k = \frac{YA}{L}$ is a constant

Tension (F) changes with stretch (x) just like spring force. [Fact is that we assumed that an ideal spring is one which obeys Hooke's law; implying that $F \propto x$].

Therefore, the elastic potential energy of the wire, when its length is increased by ΔL is given as

$$U = \frac{1}{2}k(\Delta L)^2 = \frac{1}{2}\left(\frac{YA}{L}\right)(\Delta L)^2$$

This may be written as

$$U = \frac{1}{2}\left(Y\frac{\Delta L}{L}\right)\left(\frac{\Delta L}{L}\right)(LA)$$

$$\Rightarrow \quad U = \frac{1}{2}(\text{stress})\,(\text{strain})\,(\text{volume of wire})$$

Energy stored in unit volume of the wire is known as elastic energy density (u)

$$\therefore \quad u = \frac{1}{2} \times Stress \times Strain$$

Unit of energy density (u) is Jm^{-3}.

It can also be written as,

$$U = \frac{1}{2}\left(AY\frac{\Delta L}{L}\right)\Delta L$$

$= \frac{1}{2}$ (maximum value of stretching force) (extension)

8.14 BENDING OF BEAMS

A rod of uniform cross section whose length is much larger compared to its other dimensions is called beam. When a beam gets bent by two equal and opposite couples applied at its ends, its longitudinal filaments are lengthened on the convex side and subjected to tension while those on the concave side are shortened and compressed. Following are some definitions connected with bending of beams.

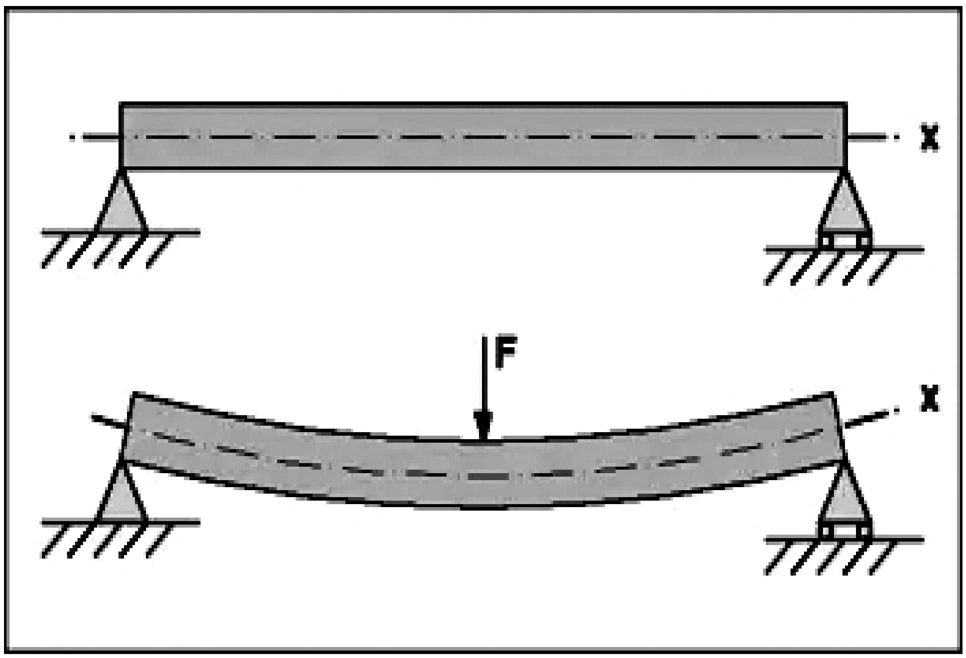


Figure - 7

8.14.1 Neutral Surface

When a beam is bent, its filaments on one side are elongated while the filaments on the inner concave side are compressed and get shortened. In the middle of the beam there will be layer of filaments in which they are neither shortened nor elongated but remain constant in length. This layer of filaments is known as the neutral surface and the filaments lying on this surface are called neutral filaments.

8.15 BENDING MOMENT

When a beam is clamped at one end and a load is applied to the other end, it is bent due to the moment of the load. Now, when bending occurs, the filaments of the beam above the neutral axis are lengthened. Consequently, they must be in tension. The filaments of the beam below the neutral axis are shortened. Consequently, they must be under pressure. These forces of tension and compression constitute a restoring couple. In the position of equilibrium, the internal restoring couple is equal and opposite to the external couple producing bending of the beam, the moment of this balancing couple formed by forces of tension and compression is called the bending moment.

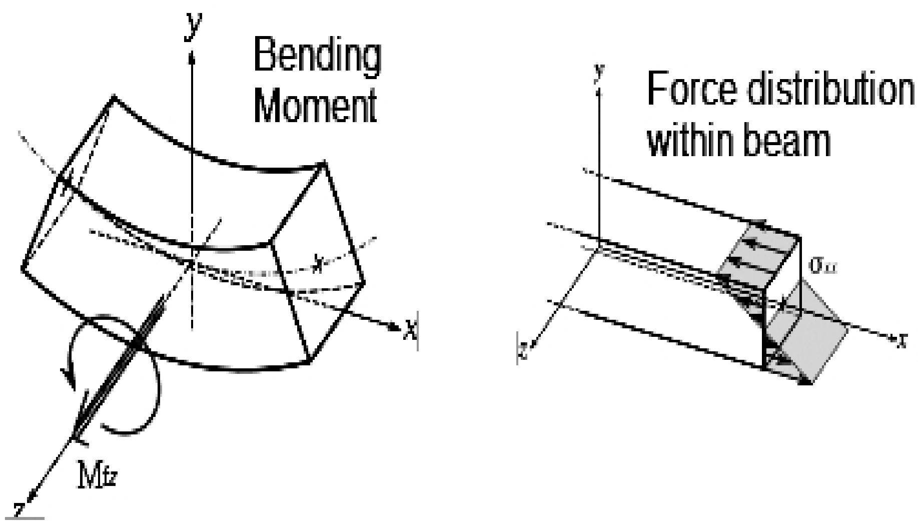


Figure - 8

8.16 GEOMETRICAL INERTIA

Bending moment is

$$G = \sum \frac{Y\, y^2 \alpha}{R} = \frac{Y}{R} \sum \alpha y^2$$

The expression $\sum \alpha\, y^2$ is called the geometrical moment of inertia of the strained transverse section of the beam.

8.17 FLEXURAL RIGIDITY

Flexural rigidity of a beam is defined as the external bending moment required to produce unit radius of curvature. Thus, quantity Y Ak^2 is called flexural rigidity.

8.18 SUMMMARY

- ❖ Elasticity: Elasticity is that property of the material of a body due to which the body opposes any change in its shape and size when deforming force are applied on it and recovers its original configuration partially or wholly as soon as the deforming forces are removed.
- ❖ Stress: It is defined as the internal restoring force per unit area of cross-section of object.
- ❖ Strain: The change in dimensions of an object per unit original dimensions is called strain.
- ❖ Hooke's Law: For small deformation, the stress is proportional to strain.
- ❖ Shear Modulus: It is ratio of shear stress to shearing strain.
- ❖ Poisson's Ratio: The lateral strain is proportional to longitudinal strain within the elastic limit and the ratio of two strains is called Poisson's ratio.

8.19 TERMINAL QUESTIONS

1. Define Elasticity.
2. Define the terms:
   (a) Stress
   (b) Strain
3. Derive an expression for the elastic potential energy of a wire under stress.
4. State Hooke's law in elasticity.
5. Prove that elastic energy density $= \frac{1}{2} stress \times strain$
6. Write short notes on:
   (a) Angle of Twist
   (b) Angle of Shear
   (c) Cantilever
   (d) Poisson's Ratio

ANSWERS TERMINAL QUESTIONS

1. Hint (Section 8.3)
2. (a) Hint (Section 8.4)
   (b) Hint (Section 8.4)

3. Hint (Section 8.13)
4. Hint (Section 8.5)
5. Hint (Section 8.5)
6. (a) Hint (Section 8.10)
   (b) Hint (Section 8.11)
   (c) Hint (Section 8.12)
   (d) Hint (Section 8.8)

8.20 SUGGESTED READINGS

1. Introduction to Solid Mechanics, H. Shames, Prentice Hall India.
2. Strength of Materials, G. H. Ryder McMillan, India Ltd.
3. College Physics, Hugh D. Young
4. Concept of Physics, H. C. Verma.

UNIT 9

FLUID MECHANICS AND VISCOSITY

Structure:

9.1 INTRODUCTION

In Block-I, we studied about Mechanics and dynamics of a Particle in depth. As, we know Mechanics is the oldest physical science that deals with both stationery and moving boundaries under the influence of forces. In this unit, we shall study about Fluid Mechanics, which deals with behavior of fluids at rest or in motion and the interaction of fluids with solids or other fluids at the boundaries. This unit also cover classification of fluids and equation of continuity.

9.2 OBJECTIVES

After studying this unit, you should be able to –

- Define Fluid Mechanics
- Understand classification of fluid
- Derive Poiseuille's law and Equation
- Explain Application of Stoke's law
- State and derive Equation of Continuity

9.3 WHAT IS FLUID MECHANICS?

Fluid Mechanics is the science that deals with behavior of fluids at rest (fluid statics) or in motion (fluid dynamics) and the interaction of fluids with solids or other fluids at the boundaries.

9.4 CLASSIFICATION OF FLUIDS

Classification of Fluids are as follows:

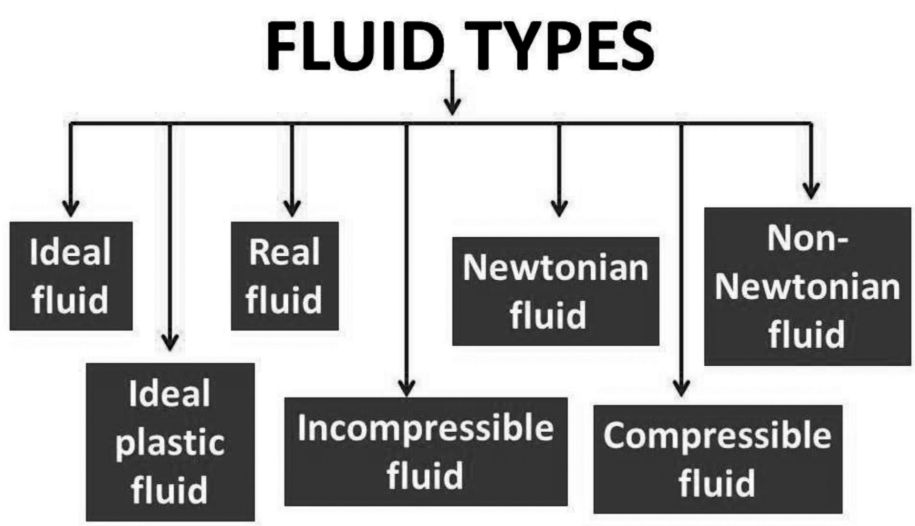


Figure – 1

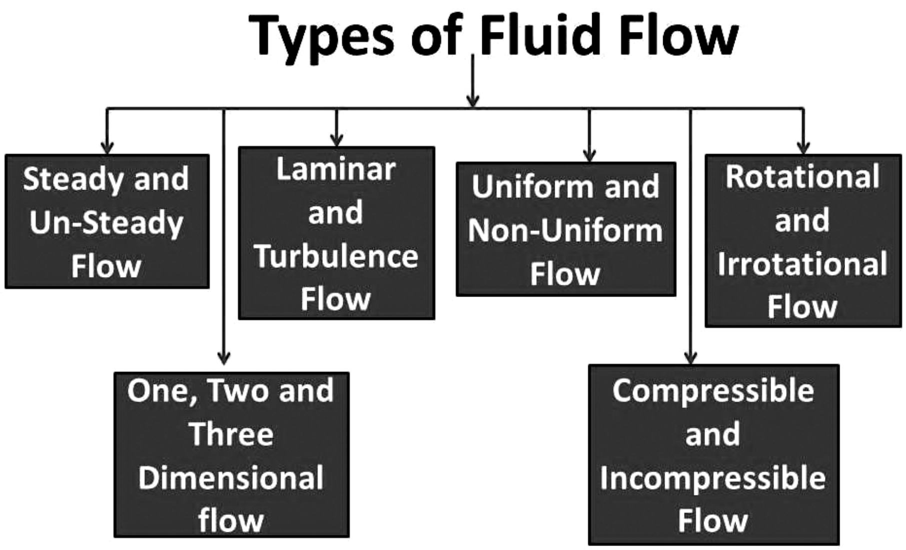


Figure - 2

(a) Ideal fluids and Real or Practical fluids.

(b) Newtonian fluids and Non-Newtonian fluids.

9.4.1 Ideal Fluids:

Ideal fluids are having following properties.

(a) It is incompressible.

(b) It has zero viscosity.

(c) Shear force is zero when the fluid is in motion i.e. No resistance is offered to the motion of any fluid particles.

9.4.2 Real or Practical Fluids:

(a) It is compressible.

(b) They are viscous in nature.

(c) Some resistance is always offered by the fluid when it is in motion.

(d) Shear stress always exists in such fluids.

9.4.3 Newtonian Fluids:

In Newtonian fluids a linear relationship exists between the magnitudes of shear stress τ and the resulting rate of deformation (du/dy). i.e. the constant of proportionality μ does not change with the rate of deformation.

$$\tau = \mu \frac{du}{dy}$$

Example: Water, Kerosene,

The viscosity at any given temperature and pressure is constant for a Newtonian fluid and is independent of the rate of deformation.

9.4.4 Non-Newtonian Fluid:

In Non-Newtonian fluids, there is a non-linear relation between the magnitude of the applied shear stress and the rate of deformation. The viscosity will vary with variation in rate of deformation. They do not obey Newton's law of viscosity.

The Non-Newtonian fluids can be further classified into five groups. They are simple Non-Newtonian, ideal plastic, shear thinning, and shear and real plastic fluids.

Simple Non-Newtonian has already explained.

In plastics, up to a certain value of shear stress there is no flow. After the limit it has a constant viscosity at any given temperature.

In shear thinning materials, the viscosity will increase with rate of Deformation (du/dy).

In shear thinning materials, viscosity will decrease with rate of Deformation (du/dy).

Example: For Non-Newtonian fluids are paint, toothpaste, and ink.

9.5 CRITICAL VELOCITY

Critical velocity is defined as the speed at which a falling object reaches when both gravity and air resistance are equalized on the object.

The other way of defining critical velocity is the speed and direction at which the fluid can flow through a conduit without becoming turbulent.

9.5.1 Critical Velocity Formula

Following is the mathematical representation of critical velocity with the dimensional formula:

VC=Reηρr

Where,

Vc: critical velocity

Re: Reynolds number (ratio of inertial forces to viscous forces)

η: coefficient of viscosity

r: radius of the tube

ρ: density of the fluid

Dimensional formula of:

- Reynolds number (Re): $M^0L^0T^0$
- Coefficient of viscosity (η): $M^1L^{-1}T^{-1}$
- Radius (r) : $M^0L^1T^0$
- Density of fluid (ρ): $M^1L^{-3}T^0$
- Critical velocity: Vc=[M0L0T0][M1L−1T−1][M1L−3T0][M0L1T0]

∴ Vc=M0L1T−1

SI unit of critical velocity is ms^{-1}

9.5.2 Reynolds Number

Reynolds number is defined as the ratio of inertial forces to viscous forces. Mathematical representation is as follows:

Re=ρuLμ=uLν

Where,

ρ: density of the fluid in $kg.m^{-3}$

μ: dynamic viscosity of the fluid in m^2s

u: velocity of the fluid in ms^{-1}

L: characteristic linear dimension in m

ν: kinematic viscosity of the fluid in m^2s^{-1}

Depending upon the value of Reynolds number, flow type can be decided as follows:

- If Re is between 0 to 2000, the flow is streamlined or laminar
- If Re is between 2000 to 3000, the flow is unstable or turbulent
- If Re is above 3000, the flow is highly turbulent

Reynolds number with respect to laminar and turbulent flow regimes are as follows:

- When the Reynolds number is low that is the viscous forces are dominant, laminar flow occurs and are characterized as a smooth, constant fluid motion
- When the Reynolds number is high that is the inertial forces are dominant, turbulent flow occurs and tends to produce vortices, flow instabilities and chaotic eddies.

Following is the derivation of Reynolds number:

Re=maτA=ρV.dudtμdudy.A∝ρL3dudtμdudyL2=ρLdydtμ=ρu0Lμ= u0Lν

Where,

t: time

y: cross-sectional position

u=dxdt : flow speed

τ: shear stress in Pa

A: cross-sectional area of the flow

V: volume of the fluid element

u_0: maximum speed of the object relative to the fluid in ms^{-1}

L: a characteristic linear dimension

μ: dynamic viscosity of the fluid in Pa.s

ν: kinematic viscosity in m^2s

ρ: density of the fluid in $kg.m^{-3}$

9.6 STREAMLINE FLOW

A streamline flow or laminar flow is defined as one in which there are no turbulent velocity fluctuations. In consequence, the only agitation of the fluid particles occurs at a molecular level. In this case the fluid flow can be represented by a streamline pattern defined within an *Eulerian description* of the flow field. These *streamlines* are drawn such that, at any instant in time, the tangent to the streamline at any one point in space is aligned with the instantaneous velocity vector at that point. In a steady flow, this streamline pattern is identical to the flow-lines or path-lines which describe the trajectory of the fluid particles within a *Lagrangian description* of the flow field, whereas in an unsteady flow this equivalence does not arise.

The definition of a streamline is such that at one instant in time streamlines cannot cross; if one streamline forms a closed curve, this represents a boundary across which fluid particles cannot pass. Although a streamline has no associated cross-sectional area, adjacent streamlines may be used to define a so-called *stream tube*. This concept is widely used in fluid mechanics since the flow within a given stream tube may be treated as if it is isolated from the surrounding flow. As a result, the conservation equations may be applied to the flow within a given stream tube, and consequently the streamline pattern provides considerable insight into the velocity and pressure changes. For example, if the streamlines describing an incompressible fluid flow converge (i.e. the cross-sectional area of the stream tube contracts), this implies that the velocity increases and the associated pressure reduces.

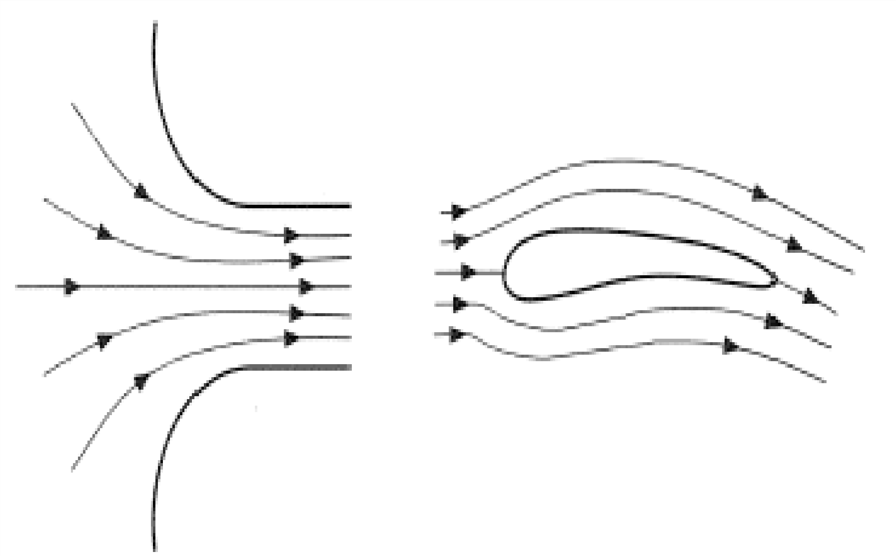


Figure – 3

9.7 TURBULENT MOTION

Turbulent Motion is a flow regime characterized by chaotic property changes. This includes rapid variation of pressure and flow velocity in space and time. In contrast to laminar flow the fluid no longer travels in layers and mixing across the tube is highly efficient. Flows at Reynolds numbers larger than 4000 are typically (but not necessarily) turbulent, while those at low Reynolds numbers below 2300 usually remain laminar. Flow in the range of Reynolds numbers 2300 to 4000 and known as transition.

9.8 COMPRESSIBLE FLUID

All real fluids are compressible, and almost all fluids expand when heated. Compression waves can propagate in most fluids: these are the familiar sound waves in the audible frequency range, and ultrasound at higher frequencies. Thermal expansion gives rise to heat convection, especially in the presence of a gravitational field: hot air rises and cold air sinks.

In general, heat transfers and fluid motions are coupled and should be treated together by using the equations of fluid dynamics along with those of thermodynamics and heat diffusion. However, the coupled equations are complicated, and we will start with the simplified assumption that fluid motions occur either *isothermally* (at constant temperature) or *adiabatically* (with negligible heat transfer), as a first approximation.

In order to use thermodynamics, it must be possible to define a temperature T(r, t) that varies with position r and time *t*, in the same way as one defines other hydrodynamic variables such as the mass density p(r,t), the pressure p(r,t) and the fluid velocity v(r,t). One must be able to consider a volume *V* that is large enough to be macroscopic (it contains many particles) and small enough to be infinitesimal with respect to variations of *T*; in addition, the particle velocities within *V* must be given by the thermal equilibrium distribution, when viewed in a frame moving along with the fluid with the local velocity v(r,t).

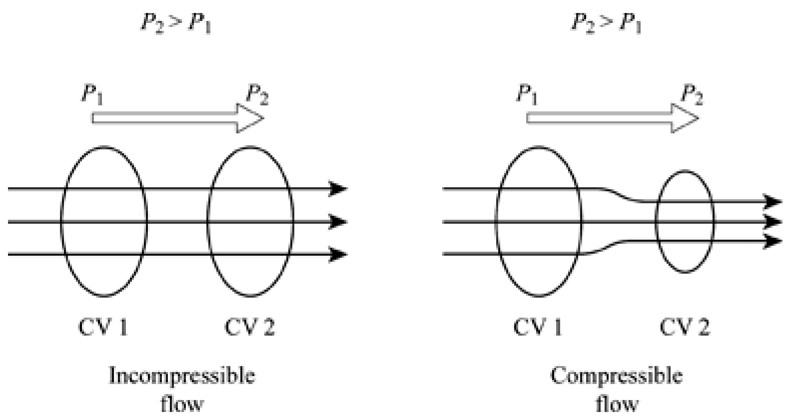


Figure - 4

We will not treat variations in the chemical composition of the fluid, so that in effect we can suppose that the volume *V* contains *N* particles of average mass *m*, and we also assume that the fluid is uncharged and non-magnetic. Then the first law of thermodynamics (just energy balance) can be stated as follows: the heat transfer *dQ* to the fluid element containing *N* particles causes a change of the internal energy *E* and of the volume *V* according to

$$dE = dQ - pdV$$

An infinitesimal heat transfer *dQ* corresponds to a change of the entropy *S*, according to *dQ*=*TdS*: thus, an adiabatic volume change (*dQ*=0, no heat transfer) is also isentropic (*dS*=0). In thermodynamics, the advantage of introducing *S* is that a system contains a well-defined amount of entropy, but not a definite amount of heat (since work can be turned into heat and, to some extent, heat can be used to do work). The entropy *S* also has the important property that it never decreases for a closed system (second law of thermodynamics). For simplicity, we do not use the second law in this chapter, although we will often refer to the entropy content of the fluid. When we need to carry out a derivation involving *S*, we assume that the fluid is an ideal gas, work up to the final formula, and then simply quote its general form for any fluid.

9.9 INCOMPRESIBLE FLUID

Incompressible Fluid: The fluid whose density doesn't vary in any sort of flow is considered as incompressible fluid.

Incompressible flow does not imply that the fluid itself is incompressible.

Example of incompressible fluid flow:

The stream of water flowing at high speed from a garden hose pipe. Which tends to spread like a fountain when held vertically up, but tends to narrow down when held vertically down. The reason being volume flow rate of fluid remains constant.

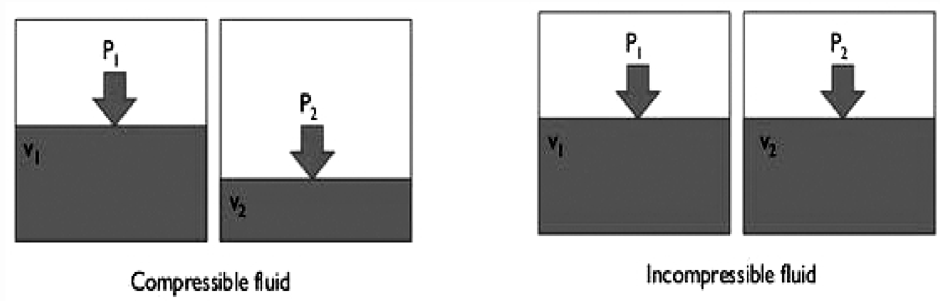


Figure – 5

9.10 EQUATION OF CONTINUITY

Continuity equation represents that the product of cross-sectional area of the pipe and the fluid speed at any point along the pipe is always constant. This product is equal to the volume flow per second or simply the flow rate. The continuity equation is given as:

R = A v = constant

Where,

- ❖ R is the volume flow rate
- ❖ A is the flow area
- ❖ v is the flow velocity

9.10.1 Assumption of Continuity Equation

These are the assumptions of continuity equation:

- ❖ The tube is having a single entry and single exit
- ❖ The fluid flowing in the tube is non-viscous
- ❖ The flow is incompressible
- ❖ The fluid flow is steady

9.10.2 Derivation of Equation of Continuity

Consider the following diagram:

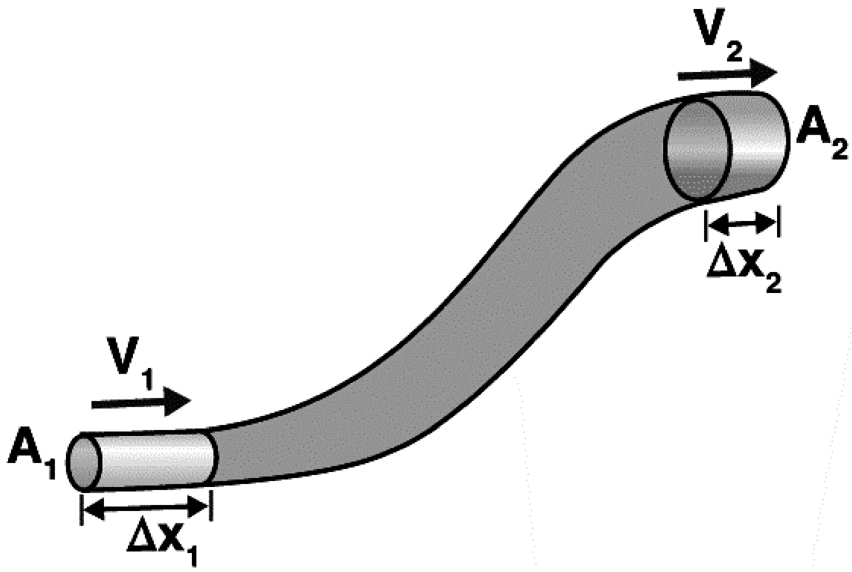


Figure - 6

Now, consider the fluid flows for a short interval of time in the tube. So, assume that short interval of time as Δt. In this time, the fluid will cover a distance of Δx_1 with a velocity v_1 at the lower end of the pipe.

At this time, the distance covered by the fluid will be:

$$\Delta x_1 = v_1 \Delta t$$

Now, at the lower end of the pipe, the volume of the fluid that will flow into the pipe will be:

$$V = A_1 \Delta x_1 = A_1 v_1 \Delta t$$

It is known that ***mass (m) = Density (ρ)*** × ***Volume (V).*** So, the mass of the fluid in Δx_1 region will be:

$$\Delta m_1 = Density \times Volume$$

$$\Rightarrow \Delta m_1 = \rho_1 A_1 v_1 \Delta t \qquad \text{…………..(1)}$$

Now, the mass flux has to be calculated at the lower end. Mass flux is simply defined as the mass of the fluid per unit time passing through any cross-sectional area. For the lower end with cross-sectional area A_1, mass flux will be:

$$\Delta m_{1/\Delta t} = \rho_1 A_1 v_1 \qquad \text{…………..(2)}$$

Similarly, the mass flux at the upper end will be:

$$\Delta m_{2/\Delta t} = \rho_2 A_2 v_2 \qquad \text{…………..(3)}$$

Here, v_2 is the velocity of the fluid through the upper end of the pipe i.e. through Δx_2, in Δt time and A_2, is the cross-sectional area of the upper end.

In this, the density of the fluid between the lower end of the pipe and the upper end of the pipe remains the same with time as the flow is steady. So, the mass flux at the lower end of the pipe is equal to the mass flux at the upper end of the pipe i.e. ***Equation 2 = Equation 3.***

Thus,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \qquad \text{.............(4)}$$

This can be written in a more general form as:

$$\rho\, A\, v = constant \qquad \text{.............(5)}$$

The equation proves the law of conservation of mass in fluid dynamics. Also, if the fluid is incompressible, the density will remain constant for steady flow. So, $\rho_1 = \rho_2$.

Thus, ***Equation 4*** can be now written as:

$$A_1 v_1 = A_2 v_2 \qquad \text{.............(6)}$$

This equation can be written in general form as:

$$A\, v = constant \qquad \text{.............(7)}$$

Now, if ***R*** is the volume flow rate, the above equation can be expressed as:

$$\mathbf{R = A\, v = constant} \qquad \text{.............(8)}$$

This is the derivation of continuity equation.

9.11 POISEUELLI'S LAW AND EQUATION

Consider a solid cylinder of fluid, of radius *r* inside a hollow cylindrical pipe of radius *R*.

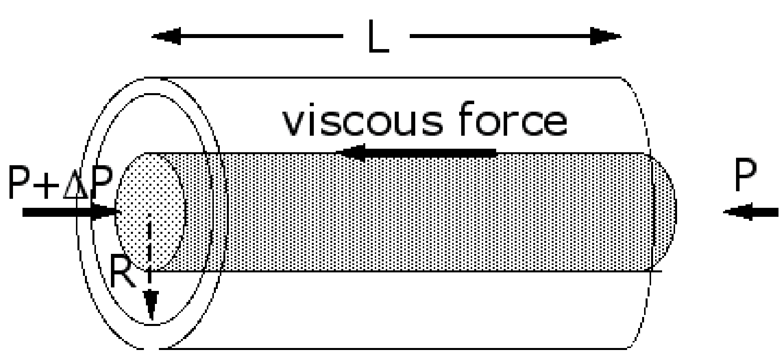


Figure – 7

The driving force on the cylinder due to the pressure difference is:

$$F_{pressure} = \Delta P\left(\pi r^2\right) \qquad \text{.............(9)}$$

The viscous drag force opposing motion depends on the *surface area of the cylinder* (length L and radius r):

$$F_{viscosity} = -\eta\left(2\pi r L\right)\frac{dv}{dr} \qquad \text{.............(10)}$$

In an equilibrium condition of constant speed, where the net force goes to zero.

$$F_{pressure} + F_{viscosity} = 0$$

$$\Delta P\left(\pi r^2\right) = \eta\left(2\pi r L\right)\frac{dv}{dr}$$

so

$$\frac{dv}{dr} = \frac{\Delta P\left(\pi r^2\right)}{\eta\left(2\pi r L\right)} = \left(\frac{\Delta P}{2\eta L}\right)\cdot r$$

We know empirically that the velocity gradient should look like this:

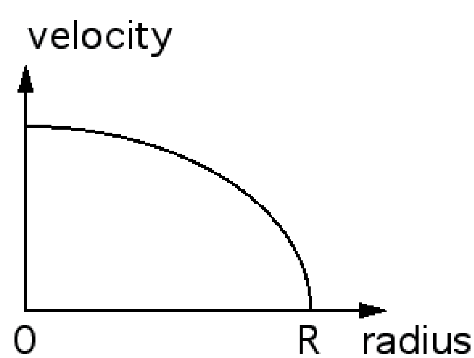


Figure – 8

At the centre

r=0

$$\frac{dv}{dr} = 0$$

v is at its maximum.

At the edge

r=R

v=0

From the velocity gradient equation above, and using the empirical velocity gradient limits, an integration can be made to get an expression for the velocity.

$$\frac{dv}{dr} = \left(\frac{\Delta P}{2nL}\right).r$$

rewriting

$$\int_v^0 dv = \left(\frac{\Delta P}{2nL}\right).\int_r^R r dr$$

$$v(r) = \left(\frac{\Delta P}{4nL}\right)[R^2 - r^2] \qquad \text{.............(11)}$$

Which has a parabolic form as expected.

Now the equation of continuity giving the volume flux for a variable speed is:

$$\frac{dV}{dt} = \int v \bullet dA \qquad \text{.............(12)}$$

Substituting the velocity profile equation and the surface area of the moving cylinder:

$$\frac{dV}{dt} = \int v \bullet dA = \int_0^R \left(\frac{\Delta P}{4\eta L}\right)\left[R^2 - r^2\right] \bullet (2\pi r dr)$$

$$= \left(\frac{\pi \cdot \Delta P}{2\eta L}\right)\int_0^R \left(R^2 r - r^3\right)dr$$

$$= \left(\frac{\pi \cdot \Delta P}{2\eta L}\right)\left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{\pi \cdot \Delta P \cdot R^4}{8\eta L}$$

Poiseuille's equation

$$\frac{dV}{dt} = \frac{\pi \cdot \Delta P \cdot R^4}{8\eta L} \qquad \text{.............(13)}$$

9.12 STOKE'S LAW FOR VISCOUS FORCE

Stoke 's Law is a mathematical equation that expresses the settling velocities of the small spherical particles in a fluid medium. The law is derived considering the forces acting on a particular particle as it sinks through the liquid column under the influence of gravity. The force that retards a sphere moving through a viscous fluid is directly proportional to the velocity and the radius of the sphere, and the viscosity of the fluid. Sir George G. Stokes, an English scientist expressed clearly the viscous drag force F as:

Stokes's law finds application in several areas such as:

- Settling of sediment in freshwater
- Measurement of the viscosity of fluids

In the next section, let us understand the derivation of Stoke's Law.

Stoke's Law Derivation

The viscous force acting on a sphere is directly proportional to the following parameters:

- the radius of the sphere
- coefficient of viscosity
- the velocity of the object

Mathematically, this is represented as

F∝ηarbvc

Now let us evaluate the values of a, b and c.

Substituting the proportionality sign with an equality sign, we get

F=kηarbvc …………..(14)

Here, k is the constant of proportionality which is a numerical value and has no dimensions.

Writing the dimensions of parameters on either side of equation (14), we get

$[MLT^{-2}] = [ML^{-1}T^{-1}]^a [L]^b [LT^{-1}]^c$

Simplifying the above equation, we get

$[MLT^{-2}] = M^a \cdot L^{-a+b+c} \cdot T^{-a-c}$ …………..(15)

According to classical mechanics, mass, length and time are independent entities.

Equating the superscripts of mass, length and time respectively from equation (15), we get

$a = 1$ …………..(16)

$-a + b + c = 1$ …………..(17)

$-a - c = 2$ or $a + c = 2$ …………..(18)

Substituting (16) in (18), we get

$1 + c = 2$

$c = 1$ …………..(19)

Substituting the value of (16) & (19) in (17), we get

$-1 + b + 1 = 1$

$b = 1$ …………..(20)

Substituting the value of (16), (19) and (20) in (14), we get

$F = k\eta rv$ …………..(21)

The value of k for a spherical body was experimentally obtained as 6π

Therefore, the viscous force on a spherical body falling through a liquid is given by the equation

$F = 6\pi\eta rv$ …………..(22)

9.12.1 Application of Stoke's Law

Applications of Stoke's law are as follows:

(a) Rain drop do not acquire alarmingly high velocity during their free fall. If this does not happen a person moving in rain would get hurt.

(b) While jumping from an airplane, parachute helps us to land safely on the earth.

(c) It is used to determine the value of charge on an electron. (Millikan's oil drop method)

9.13 TERMINAL VELOCITY

A falling object in the air, which is not influenced by wind or other sideways forces, has a maximum of **two** forces acting on it: **weight** and **air resistance** (also known as drag). The weight **does not change**. The air resistance is **zero** when the object is stationary but **increases as the object speeds up**.

The resultant (net) forces the falling object experiences is equal to the force of gravity minus the air resistance. [$F = mg - \text{Drag}$] [F=mg−Drag]

Newton's Second Law states that the acceleration of the object is **proportional to** the resultant force on the object when its mass is **constant**.

- The longer the object falls for, **the faster it falls (due to its acceleration)**,
- So, the greater the **air resistance**, which increases with speed.
- Eventually, the air resistance upwards will equal the **force of gravity** downwards.

- At this point, the resultant force is **zero**,
- hence, the velocity remains constant. This is called the **terminal velocity** of the object.

An object with a constant force acting on it in the direction it is travelling and a frictional force (related to the object's velocity) acting in the opposite direction, will reach terminal velocity given enough time. This is true for cars driving along a road, or for an anchor falling through water towards the seafloor.

9.14 BERNOULLI'S THEOREM

Bernoulli's principle states that

The total mechanical energy of the moving fluid comprising the gravitational potential energy of elevation, the energy associated with the fluid pressure and the kinetic energy of the fluid motion, remains constant.

Bernoulli's principle can be derived from the principle of conservation of energy.

Bernoulli's Theorem Formula

Bernoulli's equation formula is a relation between pressure, kinetic energy, and gravitational potential energy of a fluid in a container.

The formula for Bernoulli's principle is given as:

$$p + \frac{1}{2}\rho v^2 + \rho g h = constant$$

Where,

- p is the pressure exerted by the fluid
- v is the velocity of the fluid
- ρ is the density of the fluid
- h is the height of the container

Bernoulli's equation gives great insight into the balance between pressure, velocity, and elevation.

Bernoulli's Equation Derivation

Consider a pipe with varying diameter and height through which an incompressible fluid is flowing. The relationship between the areas of cross sections A, the flow speed v, height from the ground y, and pressure p at two different points 1 and 2 is given in the figure below.

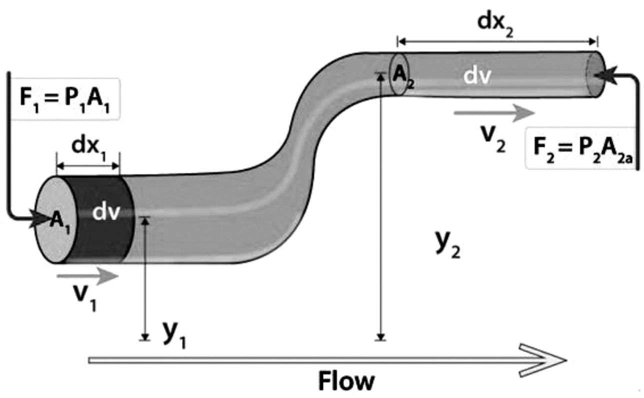


Figure - 9

Assumptions:

- ❖ The density of the incompressible fluid remains constant at both the points.
- ❖ Energy of the fluid is conserved as there are no viscous forces in the fluid.

Therefore, the work done on the fluid is given as:

$dW = F_1dx_1 - F_2dx_2$

$dW = p_1A_1dx_1 - p_2A_2dx_2$

$dW = p_1dV - p_2dV = (p_1 - p_2)dV$

We know that the work done on the fluid was due to conservation of gravitational force and change in kinetic energy. The change in kinetic energy of the fluid is given as:

$$dK = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}\rho dV(v_2^2 - v_1^2)$$

The change in potential energy is given as:

$$dU = mgy_2 - mgy_1 = \rho dVg(y_2 - y_1)$$

Therefore, the energy equation is given as:

$$dW = dK + dU$$

$$(p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho dVg(y_2 - y_1)$$

$$(p_1 - p_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)$$

Rearranging the above equation, we get

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

This is Bernoulli's equation.

9.15 SUMMARY

In this unit, we introduced and discussed some basic concepts of fluid mechanics.

- A substance in the liquid or gas phase is referred to as a fluid. Fluid mechanics is the science that deals with the behavior of fluids at rest or in motion and the interaction of fluids with solids or other fluids at the boundaries.
- We also studied in this unit equation of continuity.

9.16 TERMINAL QUESTIONS

1. What an Ideal fluid?
2. Explain Newtonian fluids and Non-Newtonian fluids.
3. State and Prove Bernoulli's Theorem for a liquid having stream line flow. Give one practical application.
4. Derive equation of continuity for steady and irrotational flow of a perfectly mobile and incompressible fluid. What conclusion is drawn from it?
5. Explain the terms:
   (a) Stream lire flow
   (b) Turbulent Motion
6. Derive an expression for Poiseuille's law.

ANSWERS TERMINAL QUESTIONS

1. Hint (Section 9.4.1)
2. Hint (Section 9.4.3, 9.4.4)
3. Hint (Section 9.14)
4. Hint (Section 9.10, 9.10.1, 9.10.2)

5. (a) Hint (Section 9.6)

   (b) Hint (Section 9.7)

6. Hint (Section 9.11)

9.17 SUGGESTED READINGS

1. Mechanics of Fluids, Bernard S, Massey and John Ward – Smith.
2. An introduction to Fluid Dynamics, G. K. Batchelor.
3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd.
4. Concepts of Physics, HC Verma, Bharati Bhawan, Patna.

UNIT 10

SURFACE TENSION

Structure:

10.1 INTRODUCTION

In the previous unit, we studied Bernoulli's equation, which is applicable for steady flow of ideal fluids. In the present unit, we introduce the concept of Surface Tension. A Liquid Surface has properties very different from its bulk. Surface of a Liquid is like a stretched membrance it is under tension. Surface tension of water – the most abundant liquid is higher than most other liquids. Many small insects live in a world dominated by surface tension of water. In this unit, we also study about capillary action.

10.2 OBJECTIVES

After studying this unit, you should be able to –

- ❖ Understand the concept of Surface Tension.
- ❖ Define Adhesive and Cohesive Force.
- ❖ Solve Problems based on Surface Tension.
- ❖ Distinguish between Drop and Bubbles.
- ❖ Explain the Concept of Capillarity

10.3 WHAT IS SURFACE TENSION?

The cohesive forces between liquid molecules are responsible for the phenomenon known as surface tension. The molecules at the surface do not have other like molecules on all sides of them and consequently they cohere more strongly to those directly associated with them on the surface. This forms a surface "film" which makes it more difficult to move an object through the surface than to move it when it is completely submersed.

Surface tension is typically measured in dynes/cm, the force in dynes required to break a film of length 1 cm. Equivalently, it can be stated as surface energy in ergs per square centimeter. Water at 20°C has a surface tension of 72.8 dynes/cm compared to 22.3 for ethyl alcohol and 465 for mercury.

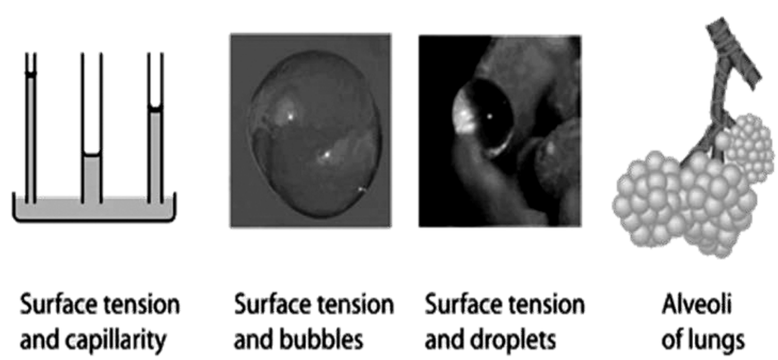


Figure – 1a

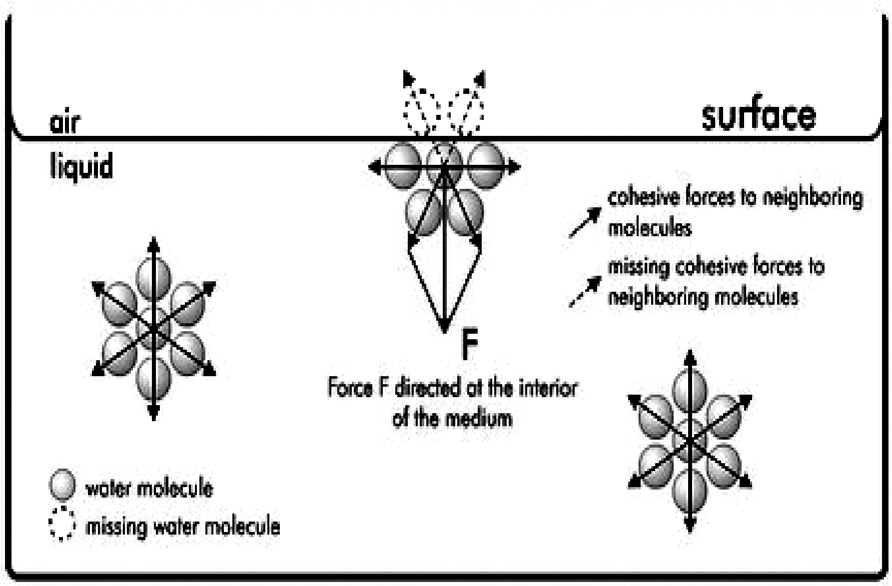


Figure – 1b

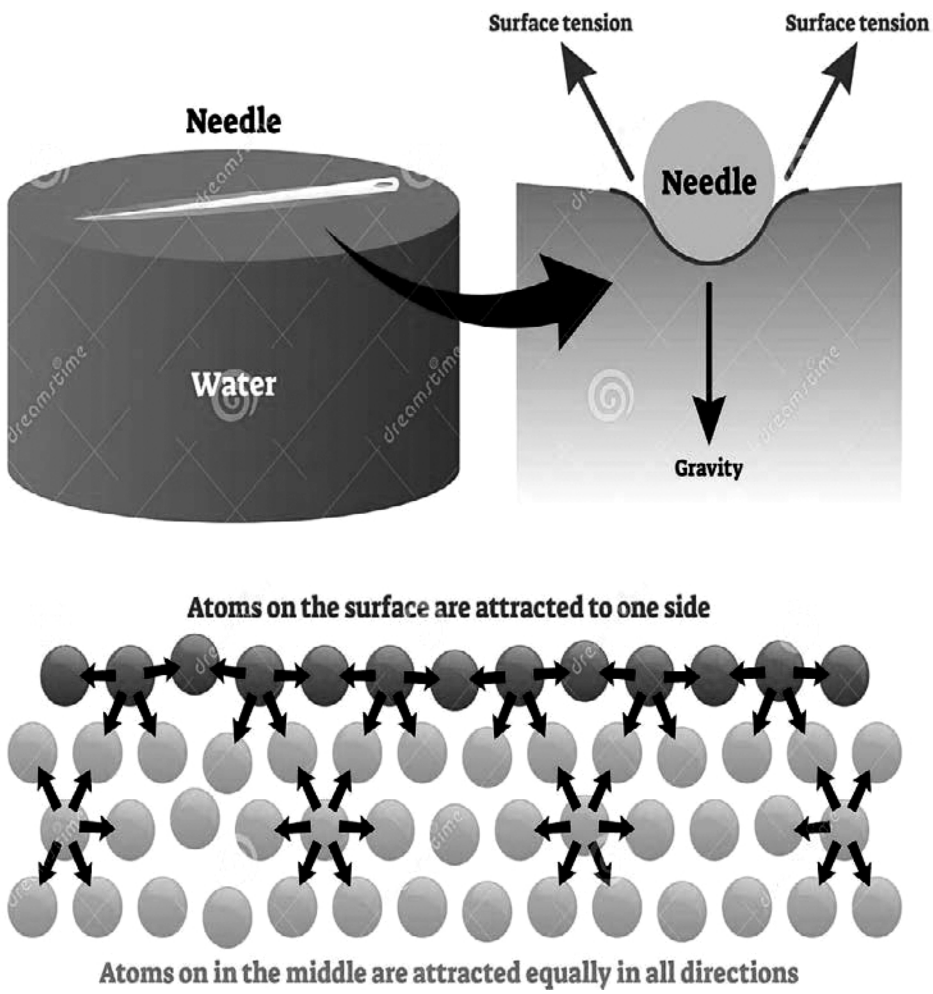


Figure – 1c

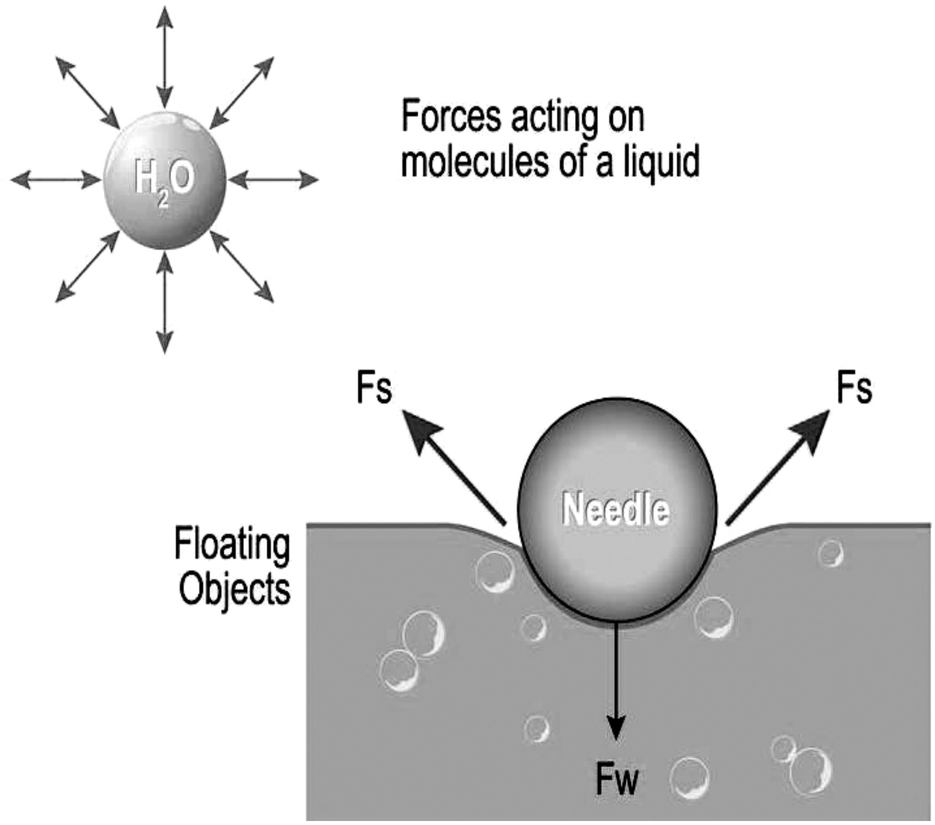


Figure – 1d

Surface tension is the tendency of fluid surfaces to shrink into the minimum surface area possible. Have you noticed when you fill a glass up to the brim with water, you can still add a few more drops till it spills out? Or have you ever broken a thermometer and observed how the fallen mercury behaves? By understanding the concept of surface tension, all these questions can be answered.

Surface Tension:

Surface tension is the phenomenon that occurs when the surface of a liquid is in contact with another phase (it can be a liquid as well). Liquids tend to acquire the least surface area possible. The surface of the liquid behaves like an elastic sheet. In physics,

Surface tension is the tension of the surface film of a liquid caused by the attraction of the particles in the surface layer by the bulk of the liquid, which tends to minimize surface area.

Given below in a table is the surface tension of various liquids:

Liquid	*Surface Tension (N/m)*
Hydrogen	2.4
Helium	0.16
Water	72.7
Ethanol	22.0
Sodium Chloride	114

10.3.1 Causes of Surface Tension

Intermolecular forces such as Van der Waals force, draw the liquid particles together. Along the surface, the particles are pulled toward the rest of the liquid. Surface tension is defined as,

The ratio of the surface force F to the length L along which the force acts.

Mathematically, surface tension can be expressed as follows:

T=F/L

Where,

- ***F*** is the force per unit length
- ***L*** is the length in which force act
- ***T*** is the surface tension of the liquid

What is the Unit of Surface Tension?

The SI unit of Surface Tension is **Newton per Meter** or **N/m**. Check other units in the table provided below.

SI Unit	N/m
CGS Unit	dyn/cm

Dimension of Surface Tension

As we know, surface tension is given by the formula,

Surface tension = F/L

We know that F = ma, substituting the value in the equation, we get

=ma/L

Equating the fundamental quantities into the equation, we get

$=MLT^{-2}L^{-1}$

Solving further, we get

$=\mathbf{MT^{-2}}$

Hence the dimensional formula of surface tension is $\mathbf{MT^{-2}}$.

Examples of Surface Tension

Water strider, which are small insects, can walk on water as their weight is considerably less to penetrate the water surface.

10.3.2 Application of Surface tension

- Insects walking on water
- Floating a needle on the surface of the water.
- Rainproof tent materials where the surface tension of water will bridge the pores in the tent material
- Clinical test for jaundice
- Surface tension disinfectants (disinfectants are solutions of low surface tension).
- Cleaning of clothes by soaps and detergents which lowers the surface tension of the water
- Washing with cold water
- Round bubbles where the surface tension of water provides the wall tension for the formation of water bubbles.
- This phenomenon is also responsible for the shape of liquid droplets.

Example: 1

Compute the surface tension of a given liquid whose dragging force is 7N and length in which the force acts is 2m?

Solution: Given,

- **F** = 7N
- **L** = 2m

According to the formula,

T = F/L

⇒ T = 7/2

⇒ T = 3.5 N/m

10.4 ADHESIVE FORCE

Adhesive forces come into play when two different substances are brought in contact. When we pour water on a glass plate, the plate becomes wet because the molecules of water stick to the molecules of glass under adhesive forces. In order to try the wet plate it should be wiped by a substance whose adhesion for water molecules is greater than of glass, for example rough dry cloth. Silken and nylon cloths cannot be used to dry wet glass plate because their adhesion for water is less.

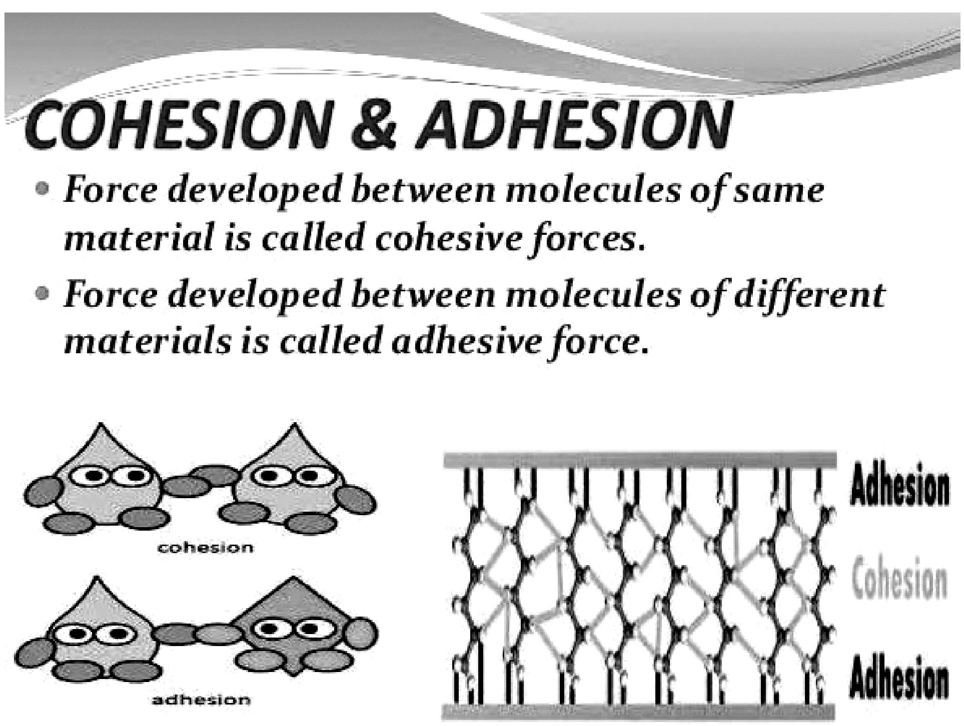


Figure – 2a

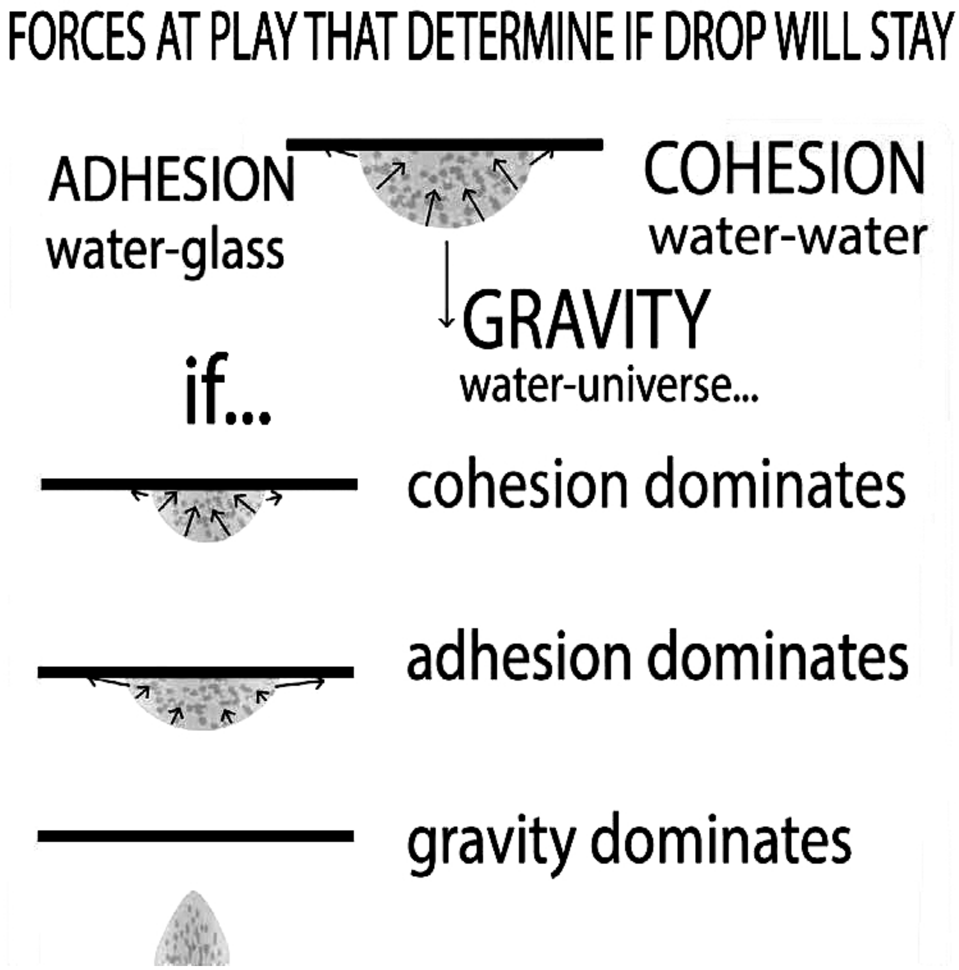


Figure – 2b

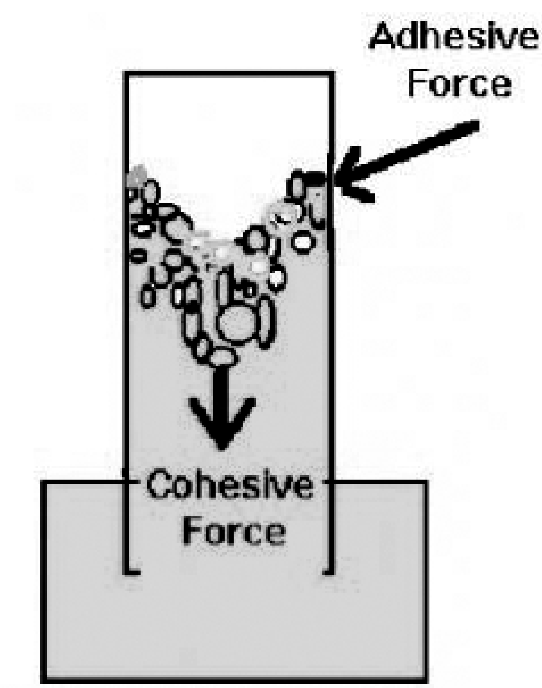


Figure – 2c

10.5 COHESIVE FORCE

The effects of cohesive and adhesive forces are observed in daily life. It is due to the cohesive force that two drops of a liquid when brought in mutual contact coalesce into one. It is difficult to separate two sticky plates of glass wetted with water because quite a large force has to be applied against the cohesive force between the molecules of water. The definite shape of solid substances is also due to the cohesive force present between its molecules. In general, we cannot adhere to pieces of solid simply by pressing them together. The reason is that ordinary pressure cannot bring the molecules of the two pieces so close (10^{-9}m) that cohesive force may become effective between them. But if their surfaces in contact are melted by heating, the molecules in the liquid state fill up the space between the solid surfaces. Then, on cooling, the surface adhere together. This is the process to join metals by welding. However, by special machines two pieces of metals can be pressed to an extent that their molecules come within the molecular range and stick together. This is called 'cold welding'.

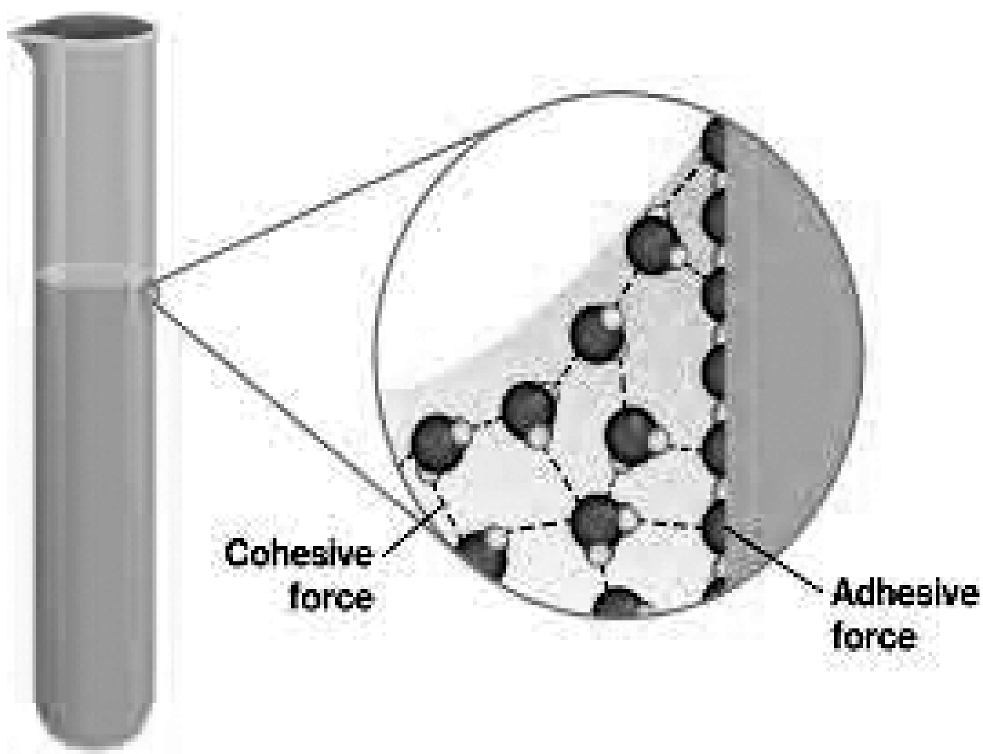


Figure – 3a

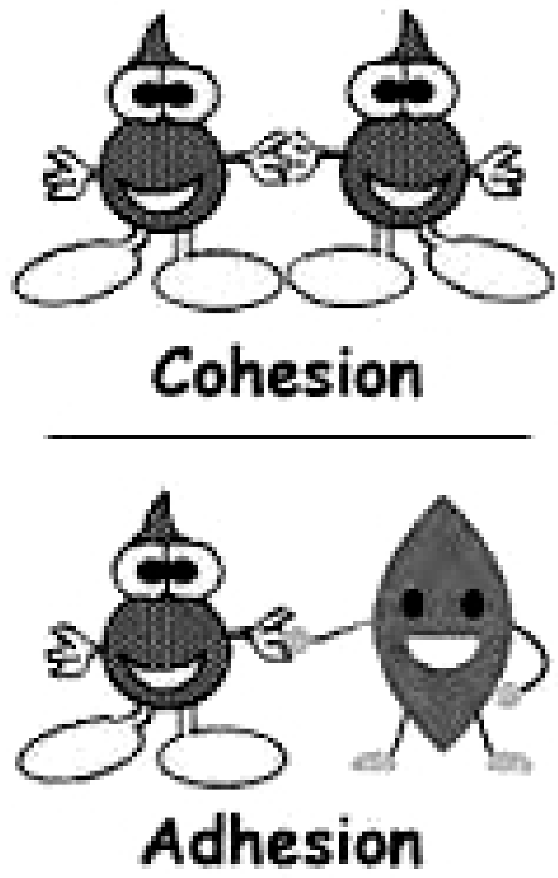


Figure – 3b

10.6 ANGLE OF CONTACT

When a liquid is in contact with a surface, various surface tensions come into play along various interface.

These are:

S_{la}-along liquid -air interface

S_{sl} -along solid -liquid interface

S_{sa} -along solid -air interface

Here angle of contact is important in the context that it decides the shape of the liquid surface near its plane of contact with another medium.

The angle of contact is defined as the angle that the tangent to the liquid surface at the point of contact makes with the solid surface inside the liquid.

The angle of contact depends on the nature of the solid and the liquid in contact.at the point of contact, the surface forces between the three media must be equilibrium.

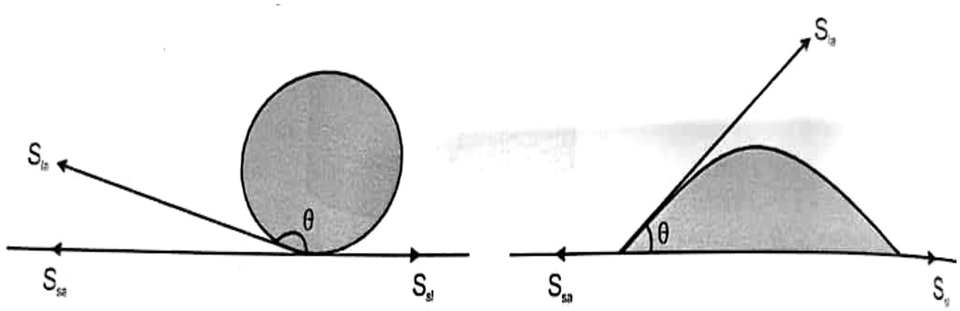


Figure - 4

10.7 EFFECT OF TEMPERATURE AND IMPURITY ON SURFACE TENSION

In general, surface tension decreases when temperature increases because cohesive forces decrease with an increase of molecular thermal activity. The influence of the surrounding environment is due to the adhesive action of liquid molecules that they have at the interface.

Example: 2

The lower end of a capillary tube of diameter 2 mm is dipped 8 cm below the surface of water in a beaker. What is the pressure required in the tube to blow an air bubble at its end in water? Given that surface tension of water = 73 × 10-3 N/ m^2 and P_{atm} = 105 Pa, g = 10 m/s^2•

Solution:

we know that,

$$P_A - P_B = 2T/R$$

$$P_A = \frac{2T}{r} + P_{atm} + h\rho g$$

$$= \frac{2\times73\times10^{-3}}{10^{-3}} + 10^5 + 0.008 \times 1000 \times 10$$

$$= 1.00946 \times 10^5 \text{Pa}$$

10.8 DROPS AND BUBBLES

10.8.1 Why Water and Bubbles are Drops?

- ❖ Whenever liquid is left to itself it tends to acquire the least possible surface area so that it has least surface energy so it has most stability.
- ❖ Therefore, for more stability they acquire the shape of sphere, as sphere has least possible area.

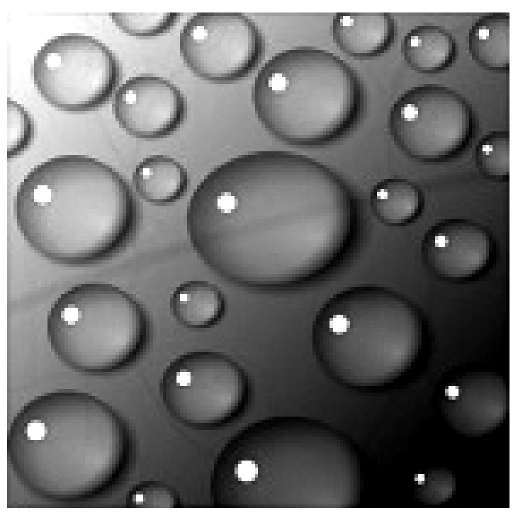
Spherical Shape

Figure - 5

10.8.2 Difference Between Drop, Cavity and Bubble

- ❖ Drop: - Drop is a spherical structure filled with water.
  - (a) There is only one interface in the drop.
  - (b) The interface separates water and air.

Example: Water droplet.



Water droplets

Figure - 6

- Cavity: -Cavity is a spherical shape filled with air.
- In the surroundings there is water and in middle there is cavity filled with air.
- There is only one interface which separates air and water.
- Example: - bubble inside the aquarium.

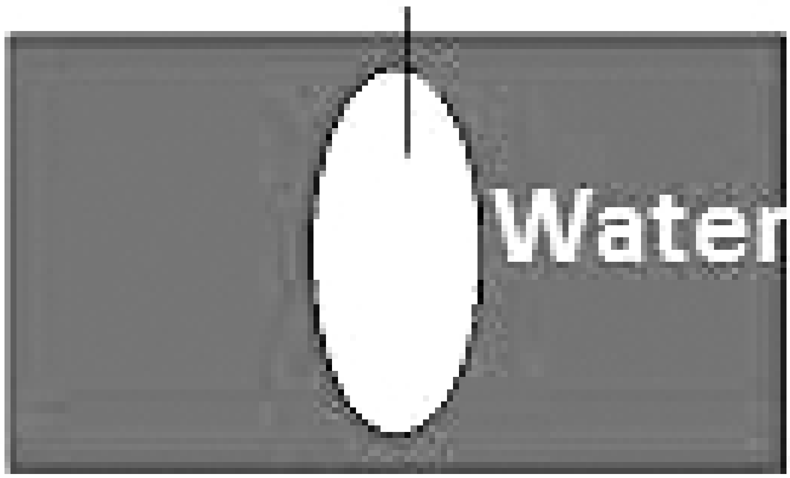


Figure - 7

- Bubble: - In a bubble there are two interfaces. One is air water and another is water and air.
- Inside a bubble there is air and there is air outside.
- But it consists of thin film of water.

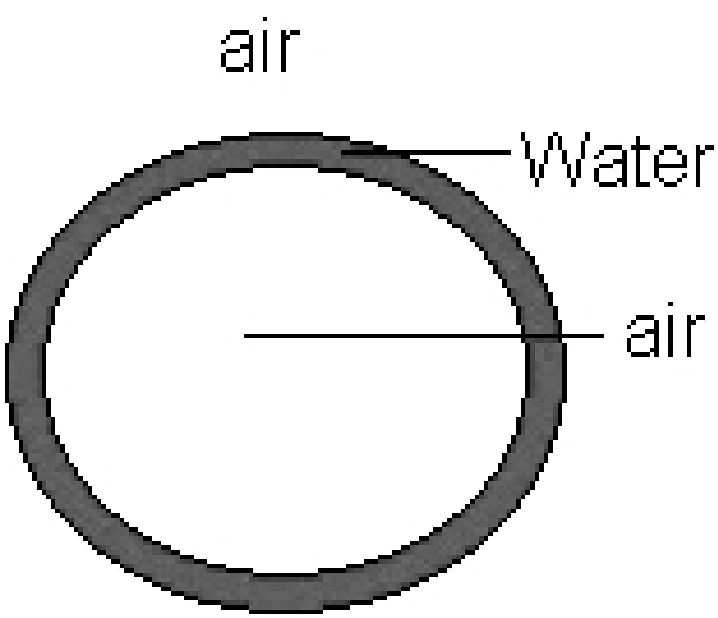
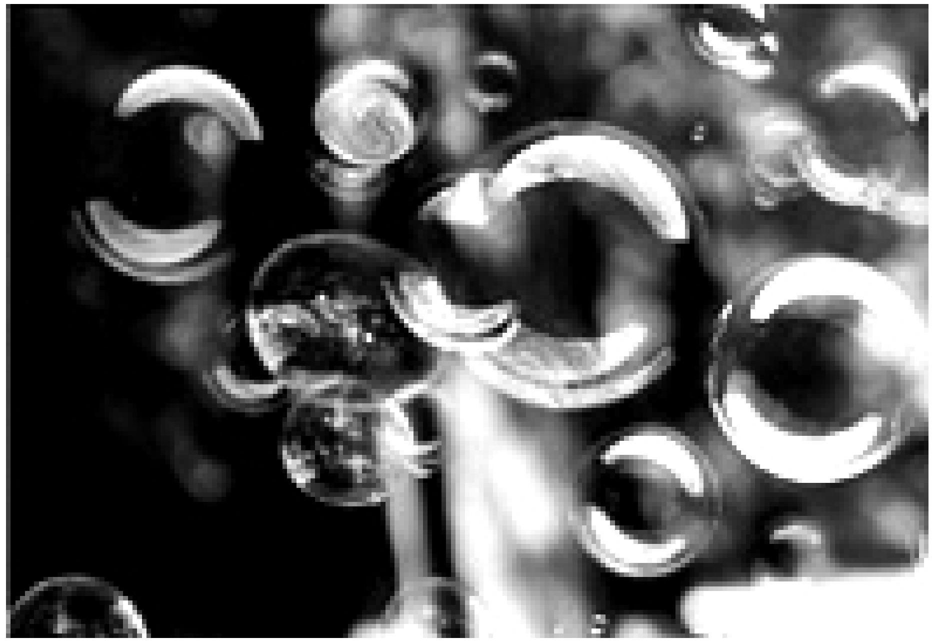


Figure – 8



Soap bubbles

Figure - 9

10.8.3 Pressure Inside a Drop and a Cavity

- Pressure inside a drop is greater than the pressure outside.
- Suppose there is a spherical drop of water of radius 'r' which is in equilibrium.
- Consider there is increase in radius which is Δr.
- Therefore, Extra Surface energy = Surface Tension(S) x area
- $= S_{la} \times 4\pi(r+\Delta r)^2 - S_{la} \times 4\pi r^2$
- After calculating
- **Extra Surface energy=$8\pi r\ \Delta r\ S_{la}$**
- At Equilibrium, Extra Surface energy = Energy gain due to the pressure difference
- $8\pi r\ \Delta r\ S_{la} = (P_i - P_o)\ 4\pi r^2 \times \Delta r$

where P_i= Pressure inside the drop and P_o = Pressure outside the drop.

After calculation $\mathbf{P_i - P_o = 2\ S_{la}/r}$

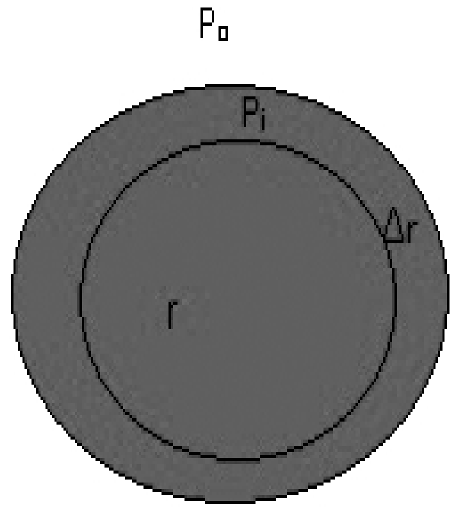


Figure - 10

10.8.4 Pressure Inside a Bubble

- Pressure inside a bubble is greater than the pressure outside.
- As bubble has 2 interfaces, P_i-P_o=2S_{la}/r x 2
- Therefore, **P_i-P_o=4S_{la}/r**

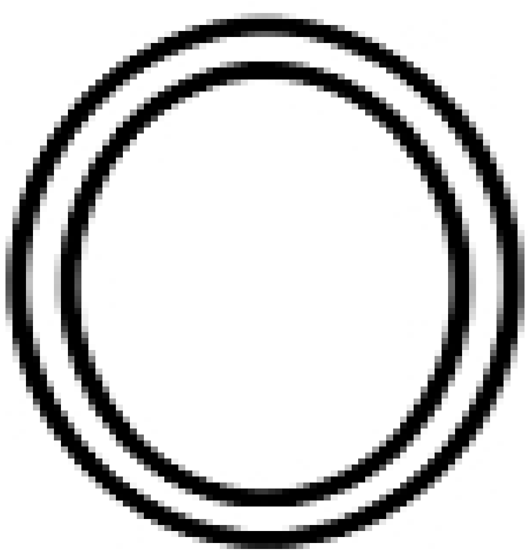


Figure - 11

10.9 SHAPE OF MENISCUS

The free surface of a liquid in a large container is always horizontal but when liquid meets a solid surface, the surface of liquid near the place of contact becomes, in general, curved. The nature of curvature depends upon the nature of liquid and the solid surface which are brought in contact. Referring to figure ( ), let AB be the liquid surface and CD the plane surface of the solid in contact with the liquid. Considering the equilibrium of a molecule A of the liquid on its surface and in contact with solid surface, there are three forces acting on it; viz.

(i) The weight mg acting vertically downward.

(ii) Adhesive force P due to attraction of molecules of the solid acting normal to CD.

(iii) A resultant cohesive force Q of the liquid molecules acting at an angle of 45^o to tangent plane to liquid surface and towards its interior.

Resolving the force Q horizontally and vertically these three forces are equivalent to two forces -

(a) A horizontal force $\left(P \sim Q\sqrt{2}\right)$ directed towards the substance of the solid or the liquid according as $Q < or > P\sqrt{2}$ and

(b) A vertical force $\left(mg + \frac{Q}{\sqrt{2}}\right)$ acting downwards.

10.10 IMPORTANCE AND APPLICATION OF CAPILLARITY

10.10.1 What is Capillarity?

- ❖ Ability of a liquid to flow in narrow spaces without the assistance of, and in opposition to, external forces like gravity
- ❖ Capillary action is sometimes called capillarity, capillary motion, or wicking

EXAMPLES

- ❖ Drawing up of liquids between the hairs of a paint-brush
- ❖ Drainage of constantly produced tear fluid from the eye
- ❖ Observed in thin layer chromatography
- ❖ draws ink to the tips of fountain pen nibs
- ❖ moving groundwater from wet areas of the soil to dry areas
- ❖ Capillarity is a physical phenomenon in which liquids flow without the help of gravity. Liquids even rise to a height against gravity, through narrow tubes.
- ❖ Capillary action is due to the phenomenon of Surface tension of liquid as well as adhesive forces between liquid molecules and

molecules of the narrow tube. Surface tension is due to cohesive attraction among liquid molecules.

- **Derivation:**
- When a thin (open or closed at the top) tube is inserted into a liquid in a container, the liquid inside the tube rises to a height h above the liquid surface outside. Let the diameter of the tube be D. The density of liquid be ρ. The surface tension of the liquid be S.

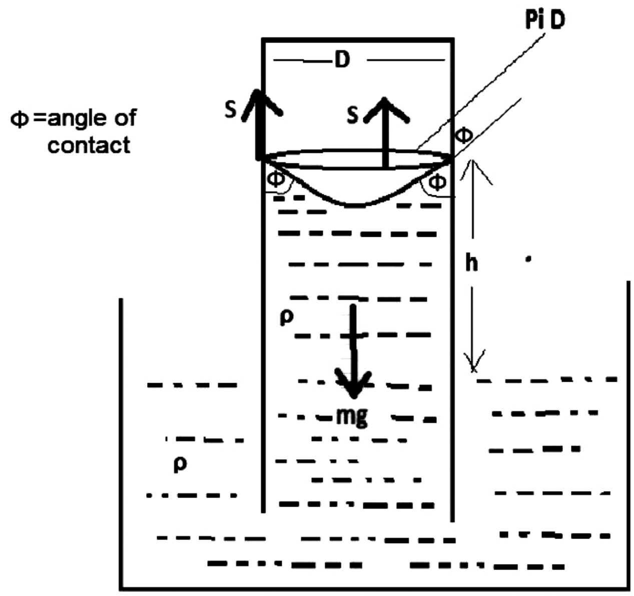


Figure – 12

- Weight of liquid column acting downwards = m g
- $W = \rho (\pi D^2/4) h g$ --(1)
- The surface on the top liquid inside the capillary tube has a trough (cup) like shape. Assume the angle of contact with the walls be Φ. Surface tension is the contact force per unit length along the circumference of top surface. This force pulls the liquid vertically upwards.
- Force upwards = $S * \pi D * \cos\Phi$ ----- (2)
- => $h = 4 S \cos\Phi / (\rho D g)$

10.10.2 Types of Capillarity

- Capillarity are of two types

(a) Capillarity rise

(b) Capillarity fall

10.10.3 Capillarity Rise

- ❖ Tendency of liquids to rise in tubes of small diameter in opposition to, external forces like gravity

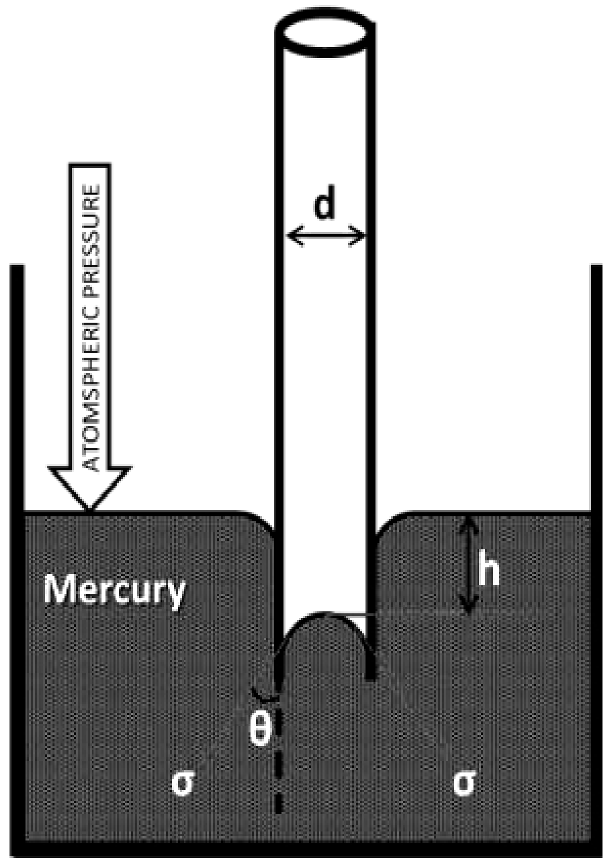


Figure - 13

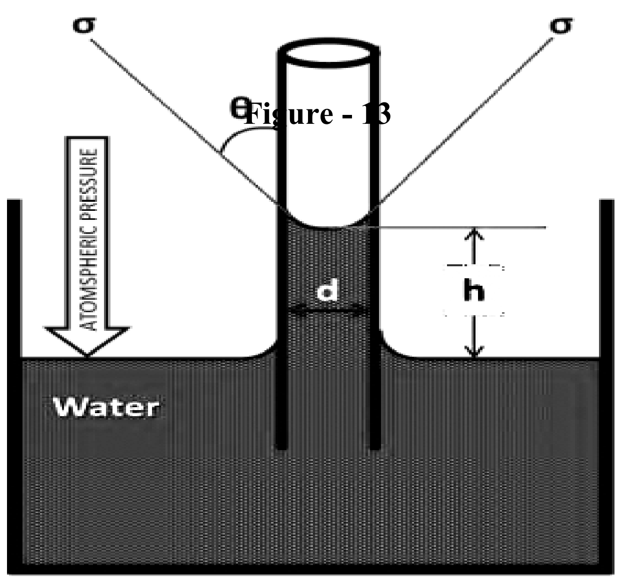


Figure - 14

10.10.4 Capillarity Fall

- ❖ Tendency of liquids to be depressed in tubes of small diameter in opposition to, external forces like gravity

10.10.5 Application of Capillarity

- ❖ Lubricating oil spread easily on all parts because of their low surface tension.
- ❖ Cotton dresses are preferred in summer because cotton dresses have fine pores which act as capillaries for sweat.
- ❖ Dirt get removed when detergents are added while washing clothes because surface tension of water is reduced.
- ❖ The absorption of ink by a blotting paper is due to capillary action, as the blotting paper is porous. When it is placed over the ink, the ink raises into the pores.

  Also rise of oil in the wick of a lamp is due to capillary action.
- ❖ If one end of a towel is dipped into a bucket of water and the other end hangs over the bucket, the entire towel soon becomes wet due to capillary action.
- ❖ Supply of water to the leaves at the top of even a tall tree is through capillary rise.
- ❖ A fabric can be waterproof, by adding suitable waterproofing materials to the fabric. This addition increases the angle of contact, thereby making the fabric waterproof.

10.11 SUMMARY

In the present unit, we have studied about Surface Tension, causes of Surface Tension, Angle of Contact, Shape of Meniscus and Capillarity.

- ❖ **Surface Tension:** The Property of liquids due to which their free surface behaves like an elastic stretched membranes is called Surface Tension.
- ❖ **Capillary Action:** The Phenomenon of a Liquid rising or falling through a tube of very fine bore (capillary), is called Capillary action.

10.12 TERMINAL QUESTIONS

1. Define Surface Tension.

2. Explain the Cause of Surface Tension. What are the units of Surface Tension?

3. What is Capillarity? Establish a relation among the height 'h' of Water column in a glass capillary tube, the diameter 'D' of the tube and the Surface Tension 'S' of water.

4. What are Cohesive and adhesive Forces.

5. Explain the difference between Cohesive and Adhesive Force.

ANSWERS TERMINAL QUESTIONS

1. Hint (Section 10.3)
2. Hint (Section 10.3.1)
3. Hint (Section 10.10)
4. Hint (Section 10.4, 10.5)
5. Hint (Section 10.4, 10.5)

10.14 SUGGESTED READINGS

1. Mechanics and Thermodynamics, G Basavaraju Dipan Ghosh, Tata McGraw Hill Publishing Company Limited.
2. Concept of Physics, H. C. Verma.
3. Surface Tension and the Spreading of liquids, R. S. Burdon.
4. Fluid Mechanics and Surface Tension, S. R. Stanley.