Uttar Pradesh Rajarshi Tandon
Open University

UGPHS-102

Oscillation, Waves and Electrical Circuits

॥ सरस्वती नः सुभगा मयस्करत् ॥

Uttar Pradesh Rajarshi Tandon Open University

UGPHS-102

Oscillation, Waves and Electrical Circuits

BLOCK

1

ISBN :

Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2020.
Printed By : K.C.Printing & Allied Works, Panchwati, Mathura -281003.

UNIT-01 UNDAMPED OSCILLATOR

Structure

1.1 INTRODUCTION

In your school science courses you must have learnt about different types of motions. You are familiar with the motion of falling bodies, planets and satellites. A body released from rest and falling freely (under the action of gravity) moves along a straight line. But an object dropped from an aeroplane or a ball thrown up in the air follows a curved path (except when it is thrown exactly vertically). You must have also observed the motion of the pendulum of a wall clock and vibrating string of a violin or some other string instrument. These arc examples of oscillatory motion. The simplest kind of oscillatory motion which can be analysed mathematically is the Simple Harmonic Motion (SHM). We can analyse oscillatory motions of systems of entirely different physical nature in terms of SHM. For example, the equation of motion that we derive for a pendulum will be similar to the equation of motion of a charge in a circuit containing an inductor and a capacitor. The form of solutions of these equations and the time variation of energy in these systems show remarkable similarities. However, there are many important phenomena which arise due to superposition of two or more harmonic oscillations. For example, our ear drum vibrates under a complex combination of harmonic vibrations. But we shall discuss this aspect in the next unit. In this unit we will study oscillatory systems using simple mathematical techniques. Our emphasis would be on highlighting the similarities between different systems.

Any motion which repeats itself after regular interval is called *periodic or harmonic motion* and the time interval after which the motion is repeated is called its time period. Some examples of periodic motion include (see Fig. 1)

- motion of planets around the sun,
- motion of a piston inside a cylinder, used in automobile engines, or
- motion of a ball in a bowl.

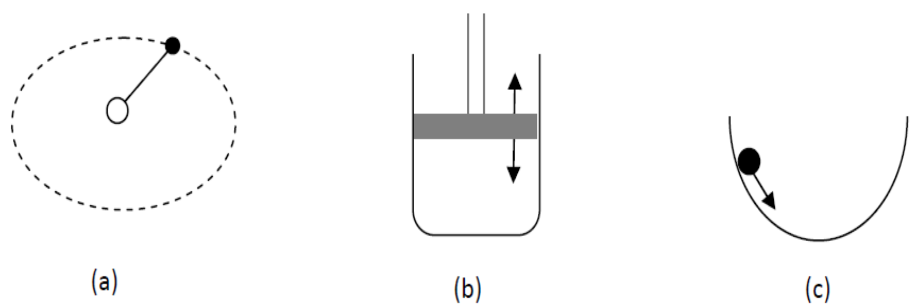


Fig. 1 : Some examples of periodic motion: (a) motion of the earth around the sun, or moon around the earth; (b) motion of a piston in a cylinder which is used in automobile engines; (c) motion of a ball in a bowl.

If in case of periodic motion, the body moves back and forth repeatedly about a fixed position (called equilibrium or mean position), the motion is said to be *oscillatory or vibratory*. For instance, the motion of the earth around the sun and the motion of the hands of the clock, are examples of periodic motion, but they are not oscillatory in nature. The motion of piston in an automobile engine, motion of a ball in a bowl, motion of needle of sewing machine or the bob of a pendulum clock are all examples of oscillatory motion.

An oscillating body is said to execute *simple harmonic motion* (SHM) if the magnitude of the forces acting on it is directly proportional to the magnitude of its displacement from the mean position and the force (called restoring force) is always directed towards the mean position. Thus, we can see that simple harmonic motion or SHM is actually a special case of oscillatory or

vibratory motion. We will study SHM in detail in this unit. Some examples of simple harmonic motion include (see Fig. 2)

- motion of a simple pendulum,
- a vibrating tuning fork, or
- a spring-mass system.

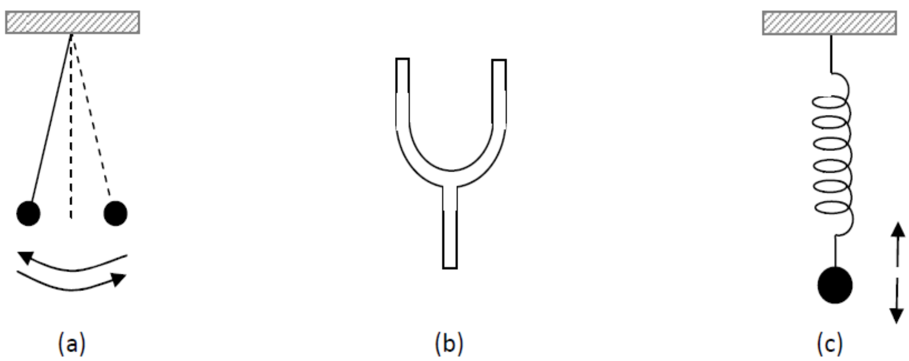


Fig. 2: Some examples of SHM: (a) A simple pendulum; (b) a vibrating tuning fork; (c) an oscillating spring-mass system.

OBJECTIVES :

After studying this unit, you should be able to

- Understand the concept simple harmonic motion
- Define Phase, amplitude, Time Period and Frequency
- Write down the general equation of simple harmonic motion
- Explain the concept of simple pendulum and compound pendulum

1.2 SIMPLE HARMONIC MOTION: BASIC CHARACTERISTICS

You know that each hand of a clock comes buck to a given position after the lapse of certain time. This is familiar example of *periodic. motion.* When a body in periodic motion moves to-and-fro (or hack and forth) about its position, the motion is vibratory or oscillatory motion is a common phenomenon. Well known examples of oscillatory motion are: oscillating bob of a pendulum clock, piston of' an engine. vibrating strings of a musical instrument, oscillating uranium nucleus before its fissions. Even large-scale buildings and bridges may at times undergo oscillatory motion. Many stars exhibit periodic variations in brightness, you must have observed that normally such oscillations, left to themselves, do not continue indefinitely i.e. they gradually die down to various damping factors like friction and air resistance etc. Thus, in actual practice, the oscillatory motion may be quite complex, as for instance vibrations of a violin string. We begin our study with the discussion of essential features of SHM. For this we consider an idealized model of a spring- mass system as an example of a *simple harmonic oscillator*.

1.2.1 OSCILLATIONS OF A SPRING-MASS SYSTEM

A spring-mass system consists of a spring of negligible mass whose one end is fixed to a rigid support *S* and other end carries a block of mass m which lies flat on a horizontal frictionless table: (Fig. I. 1). Let us take the x-axis to be along the length of the spring. When the mass is at rest, **we** mark a point on it and we define the origin of the axis by this point. That is at equilibrium the mark lies at x = 0.

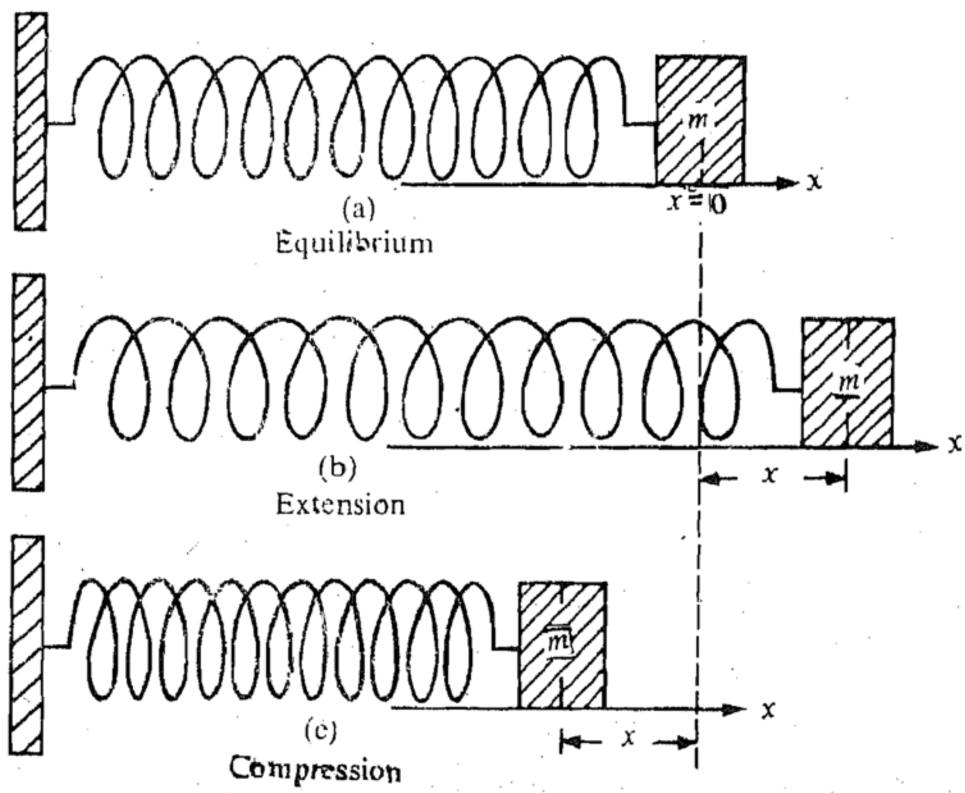


Fig. 1.1 A Spring-mass System as an ideal oscillator (a) A equilibrium configuration, (b) An extended configuration, (c) A compressed configuration.

If the spring is stretched by pulling the mass longitudinally, due to elasticity a restoring force comes into play which tends to bring the mass back towards the equilibrium position (Fig 1. lb). If the spring were compressed the restoring force would tend to extend the spring and restore the mass to its equilibrium position (Fig.l.1c). More you stretch/compress the spring, more will be the restoring force. ***So the direction of the restoring force is always opposite to the displacement.*** If total change in the length is small compared to the original length, then the magnitude of restoring force is linearly proportional to the displacement. Mathematically, we can write

$$F = -kx \qquad \textbf{1.1}$$

The negative sign signifies that the restoring force opposes the displacement. The quantity k is called the spring constant or the force constant of the spring. It is numerically equal to the magnitude of restoring force exerted by the spring for unit extention. Its SI unit is Nm^{-1}

Let us now study the effect of gravity on oscillations of spring-mass system. Consider a spring of negligible mass suspended from a rigid support with mass m attached to its lower end **(Fig.l.2)**

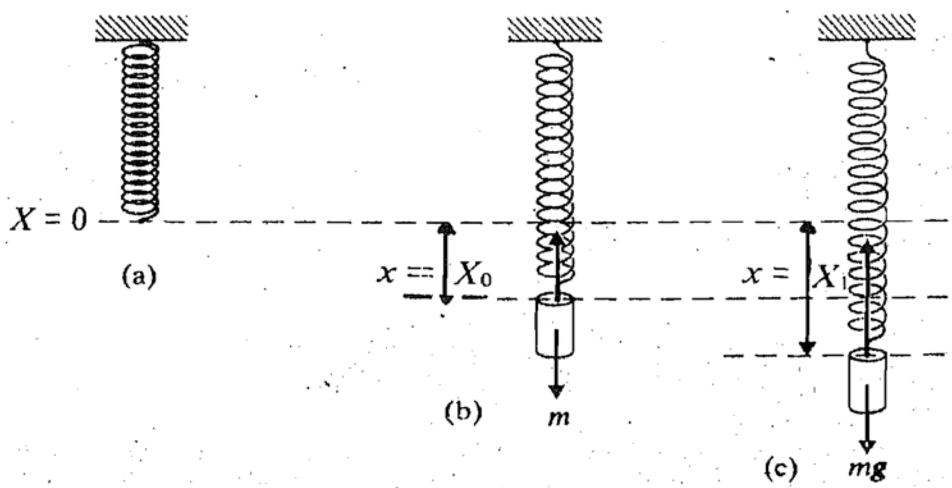


Fig. 1.2 A vertically hanging spring-mass system. (a) The spring with no object suspended from it (b) The spring in equilibrium with mass *m* suspended, (c) Spring-mass system displaced from equilibrium position.

Let us choose the X-axis along the length of the spring. We take the bottom of the spring as our reference point, X = 0, when no weight is attached to it (Fig. 1.2a). When a mass m is suspended from the spring, let the reference point move to x=Xo (Fig. 1.2b). At equilibrium, the weight, mg, balances the spring force, *kXo* Since the net force is zero, we have

$$mg - kX_0 = 0$$

or

$$mg = kX_0. \qquad \textbf{1.2}$$

Now if the mass is pulled downwards so that the reference mark shifts to X_I (Fig 1.2c), then the total restoring force will be kX_I and point in the upward direction. The net downward force will therefore be (using Eq. (1.2))

$$mg - X_I = k(Xo - X_I) = -kx$$

where $x = X_I - X_o$. Thus, the resulting restoring force on the mass is

$$F=-kx$$

where x is its displacement from the equilibrium position, X_O. This result is of the same form as Eq. (1.1) for the horizontal arrangement. It is thus clear that gravity has no effect on the frequency of oscillations of a mass hanging vertically from a spring, i.e., only displaces the equilibrium.

1.3 DIFFERENTIAL EQUATION OF SIMPLE HARMONIC MOTION

Let us now find the differential equation which describes the oscillatory motion of a spring-mass system. The equation of motion of such a system is given by equating the two forces acting on the mass:

mass X acceleration = restoring force

or

$$m\frac{d^2x}{dt^2} = -kx$$

where d^2x/dt^2 is the acceleration of the body,

It is important to note that in this equation, the equilibrium position of the body is taken as the origin, x = 0.

You will note that the quantity ***k/m*** has units of $\mathbf{Nm^{-1}kg^{-1} = (kg.ms^{-2})\ kg^{-1}m^{-1} = s^{-2}}$

Hence, we can replace ***k/m*** by $\omega_0^2$where ω_0 is called ***angular frequency,*** then the above equation takes the form,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad \mathbf{1.3}$$

It may be remarked here that Eq. (1.3) is the differential form of Eq. (1.1) and describes simple harmonic motion in one dimension.

A differential equation having terms involving only the first power of the variable and its derivatives is known as a linear differential equation. If such an equation contains no term independent of the variable, it is said to be homogeneous. We may, therefore, say that Eq. (1.3) is a second order linear homogeneous equation. Its solution will contain two arbitrary constants.

1.4 SOLUTION OF THE DIFFERENTIAL EQUATION FOR SHM

To find the displacement of the mass at any time t, we have to solve Eq. (1.3) subject to given initial conditions. A close inspection of Eq. (1.3) shows that **x** should be such a function that its second derivative with respect to time is the negative of the function itself, except for a multiplying factor ω_0^2 **.** From elementary calculus, we know that sine and cosine functions have this property. You can check that this property does not change even if sine and cosine functions have a constant multiplying factor.

A general solution for ***x*** (t) can thus be expressed as a linear combination of both sine and cosine terms, i.e.

$$x(t) = A_1 \cos \alpha t + A_2 \sin \alpha t \qquad (1.4)$$

Putting $A_1 = A \cos \phi$ and $A_2 = -A \sin \phi$, we get

$$x(t) = A\cos(\alpha t + \phi)$$

Differentiating this equation twice with respect to time and comparing, the resultant expression with Eq. (1.3), we obtain $\alpha = \pm \omega_0$. The negative sign is dropped as it gives negative frequency which is a physically absurd quantity.

Substituting $\alpha = \omega_0$ in the above equation, we get

$$x(t) = A\cos(\omega_0 t + \phi) \qquad (1.5)$$

Let us assume that the mass is held steady at some distance a from the equilibrium position and then released at t = 0. Thus the initial conditions are: at $t = 0$, x = a and $\frac{dx}{dt} = 0$. Then from Eq. (1.5) we would have, at t = 0.

$$A\cos\phi = a$$

And $-A\,\omega_0 \sin\phi = 0$

These conditions are sufficient to fix A and ϕ. The second condition tells us that ϕ is either zero or $n\pi$ (n = 1,2, ...). We reject the second option because the first condition requires $\cos\phi$ to be positive. Thus with the above initial conditions, Eq. (1.5) has the simple form

$$x = A \cos\omega_0 t \qquad (1.6)$$

1.4.1 PHASE AND AMPLITUDE

The quantity $(\omega_0 t + \phi)$ occurring in Eq. (15) is called the phase angle or the phase of vibration of the system at time t. At $t = 0$, the phase is ϕ and is called the initial phase or the *phase* constant. This gives us information about the initial position from where we start measuring the displacement. If at $t = 0$, the body is at $\mathbf{x} = \mathbf{x_o}$, then from Eq. (1.5) it follows that

$$\mathbf{x_o} = A\cos\phi$$

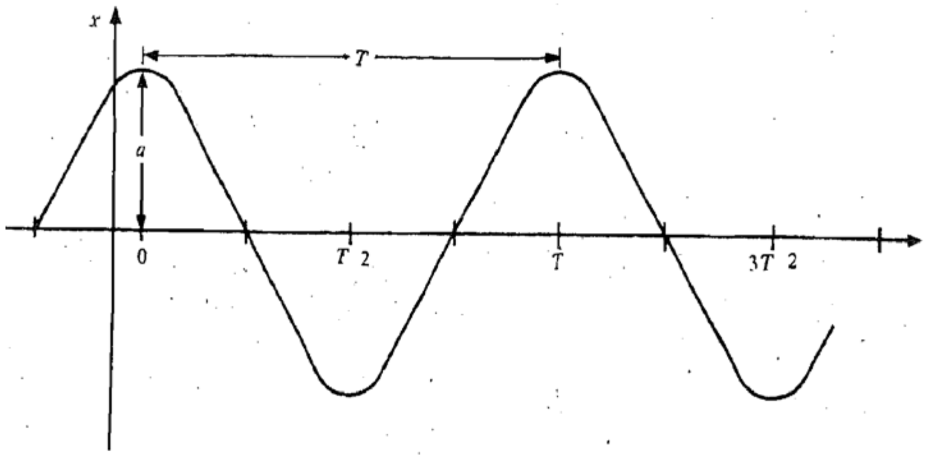


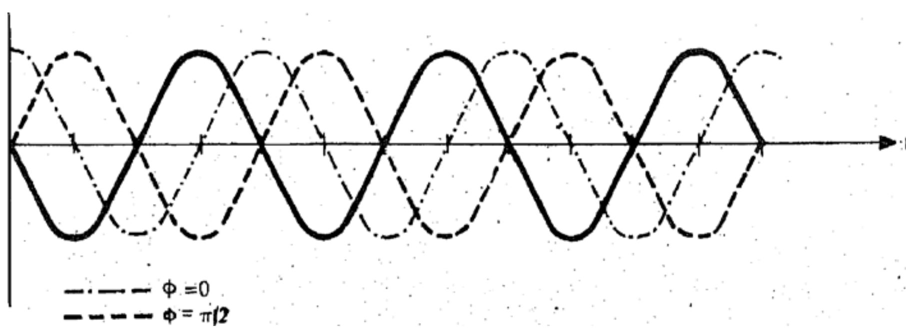
Fig. 1-.3 Displacement-time graph of simple harmonic motion with an initial phase ϕ.

We know that the value of the sine and cosine functions lie between 1 and -1. When $\cos(\omega_0 t + \phi) = 1$ or -1, the displacement has the maximum value. Let us denote it by a or -a. The quantity a is called the ***amplitude*** of oscillation.

We can, therefore, rewrite Eq. (1.5) as

$$\mathbf{x}(t) = a\cos(\omega_0 t + \phi) \qquad \textbf{(1.7)}$$

The displacement-time graphs for $\phi = 0$, π/2 and π are shown in Fig.l.4. In all the cases, the graphs have exactly the same shape if we shift the origin along the time axis. When the phase difference is π two oscillations are said to be in opposite phase or out of phase by π.



ϕ =π

Fig. 1.4 Plot of Eq. (1.7) for ϕ = 0, π /2 and π.

1.4.2 TIME PERIOD AND FREQUENCY

If we put $t = t + (2\pi/\omega_0)$ in Eq. 1.7, we obtain

$$x(t) = a\cos[\omega_0(t + 2\pi/\omega_0)\phi]$$

$$= a\cos[\omega_0 t + 2\pi + \phi]$$

$$= a\cos(\omega_0 t + \phi)$$

i.e., the displacement of the particle repeats itself after an interval of time $2\pi/\omega_0$.

In other words, the oscillating particle complete one vibration in time $2\pi/\omega_0$.

This time is called the period of vibration or the time period denoted by T.

$$T = 2\pi/\omega_0 \qquad \textbf{1.8}$$

For a spring-mass system $\omega_0^2 = \frac{K}{m}$, so that

$$T = 2\pi\sqrt{m/k} \qquad \textbf{1.9}$$

The number of vibrations executed by the oscillator per second is called frequency. The unit of frequency is Hertz (Hz) denoting by ν_0 therefore for a spring-mass system,

$$\nu_0 = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \textbf{1.10}$$

This means that stiffer the spring, higher will be the frequency of vibration .

1.4.3 VELOCITY AND ACCLERATION

We know that the displacement of a mass executing a simple harmonic motion is given as

$$x = a\cos(\omega_0 t + \phi)$$

Therefore, the instantaneous velocity, which is the first time derivative of the

displacement, is given by

$$v = \frac{dx}{dt} = -\omega_0 a\sin(\omega_0 t + \phi) \qquad \textbf{1.11}$$

We can rewrite it as

$$v = \omega_0 a \cos(\pi/2 + \omega_0 t + \phi) \qquad \textbf{1.12a}$$

You may also like to know the value of v at any point x . To this end, we rewrite

Eq. 1.11 as

$$\begin{aligned} v &= -\omega_0 [a^2 - a^2 \cos^2(\omega_0 t + \phi)]^{1/2} \\ &= -\omega_0 (a^2 - x^2)^{1/2} \qquad \text{for } -a \le x \le a \end{aligned} \qquad \textbf{1.12b}$$

We also know that acceleration is the first-time derivative of velocity. From Eq. 1.11 it readily follows that

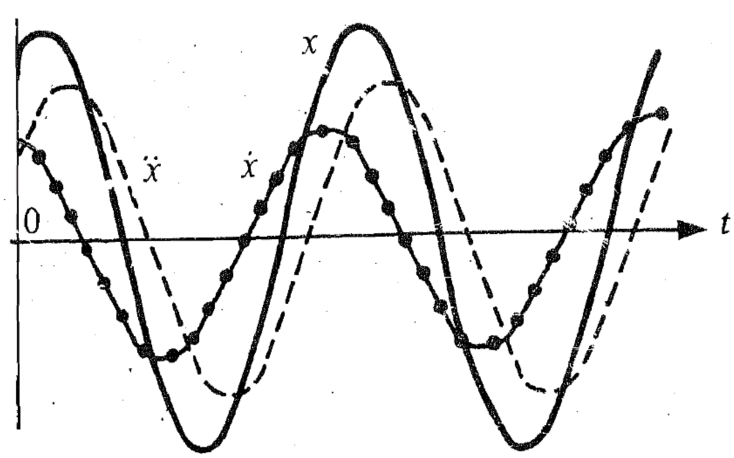
$$\begin{aligned} \frac{dv}{dt} &= -\omega_0^2 a \cos(\omega_0 t + \phi) \\ &= \omega_0^2 a \cos(\pi + \omega_0 t + \phi) \end{aligned} \qquad \textbf{1.13a}$$

Obviously, in terms of displacement

$$\frac{dv}{dt} = -\omega_0^2 x \qquad \textbf{1.13b}$$

If you compare Eqs. 1.7, 1.12 a and 1.13 a; you will note that (i) $\boldsymbol{\omega_0 a}$ is velocity amplitude and $\boldsymbol{\omega_0^2 a}$ is acceleration amplitude and (ii) velocity is ahead of, displacement by $\boldsymbol{\pi/2}$ and acceleration's ahead of velocity by $\boldsymbol{\pi/2}$.

If you plot displacement, velocity, and acceleration as functions of time you will get graphs as shown in Fig.l.5.



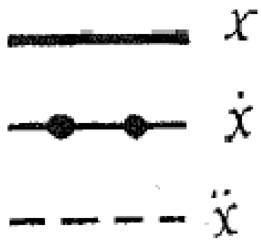


Fig. 1.5 Time variation of displacement, velocity and acceleration of a body executing SHM.

1.5 TRANSFORMATION OF ENERGY IN OSCILLATING SYSTEMS : POTENTIAL AND KINETIC ENERGIES

Consider the spring-mass system shown in Fig .1.1 when the mass is pulled, the

spring is elongated. The amount of energy required to elongate the spring through a distance ***dx*** is equal to the work done in bringing about this change. It is given by $dW = dU = -F_O\,dx$, where F_O is the applied force (such as by hand). This force is balanced by the restoring force. That is, its magnitude is same as that of ***F*** and we can write $F_0 = -kx$. Therefore, the energy required to elongate the spring through a distance **x** is

$$U = \int_0^x F_0\,dx = k\int_0^x x\,dx = \frac{1}{2}kx^2 \qquad \textbf{1.14}$$

This energy is stored in the spring in the form of potential energy and is responsible for oscillations of the spring-mass system.

On substituting for the displacement from Eq. (1.7) in Eq. (1.14), we get

$$U = \tfrac{1}{2}ka^2\cos^2(\omega_0 t + \phi) \qquad \textbf{1.15}$$

Note that at ***t*** = 0, the potential energy is

$$U_0 = \tfrac{1}{2}ka^2\cos^2\phi \qquad \textbf{1.16}$$

As the mass is released, it moves towards the equilibrium position and the potential energy starts changing into kinetic energy *(K.E)*. The kinetic energy at any time ***t*** is given by $K.E = 1/2\,mv^2$. Using Eq. (1. 1**),** we get

$$K.E = \tfrac{1}{2}m\omega_0^2 a^2\sin^2(\omega_0 t + \phi)$$

$$= \tfrac{1}{2}ka^2\sin^2(\omega_0 t + \phi) \qquad \textbf{1.17}$$

since $\omega_0^2 = k/m$.

One can also express ***K.E*** in terms of the displacement by writing

$$\sin^2\theta + \cos^2\theta = 1$$

$$K.E = \tfrac{1}{2}\, ka^2\,[1 - \cos^2(\omega_0 t + \phi)]$$
$$= \tfrac{1}{2}\, ka^2 - \tfrac{1}{2}\, ka^2 \cos^2(\omega_0 t + \phi)$$
$$= \tfrac{1}{2}\, ka^2 - \tfrac{1}{2}\, kx^2 = \tfrac{1}{2}\, k\,(a^2 - x^2) \qquad \textbf{1.18}$$

This shows that when an oscillating body passes through the equilibrium position (x = 0), its kinetic energy is maximum and equal to ***1 /2 ka²***.

1.6 CALCULATION OF AVERAGE VALUES OF QUANTITIES ASSOCIATED WITH SHM

In Fig. 1.5 we have plotted displacement, velocity and acceleration as a function of time. You will note that for any complete cycle in each case, the area under the curve for the first half is exactly equal to the area under the curve in the second half and the two are opposite in sign. Thus over one complete cycle the algebraic sum of these areas is zero. This means that average values of displacement, velocity and acceleration over one complete cycle are zero. If we plot x^2 **(or v^2)** versus **t**, the curves would lie in the upper half only so that the total area will be positive during one complete cycle. This suggests that we can talk about average values of kinetic and potential energies.

The time average of kinetic energy over one complete cycle is defined as

$$< K.E > = \frac{\int_0^T K.E\, dt}{T} \qquad \textbf{1.19a}$$

On substituting for *K.E* from Eq. (1.17), we get

$$< \boldsymbol{K.E.} > = \frac{k\alpha^2}{2T}\int_0^T \sin^2(\omega_\alpha t + \phi)\,dt \qquad \textbf{1.19b}$$

On solving the integral in Eq. (1.19b) you will find that its value is *T/2*. So, the

expression for average kinetic energy reduces to

$$< K.E > = \frac{ka^2}{4}$$ **1.20**

Similarly, one can show that the average value of potential energy over one cycle is

$$< U \geq \frac{Ka^2}{4}$$ **1.21**

That is, the average kinetic energy of a harmonic oscillator is equal to the average

potential energy over one complete period.

Thus the sum of average kinetic and average potential energies is equal to the total energy:

$$< K.E > + < U > = ¼\, ka^2 + ¼\, ka^2$$
$$= ½\, ka^2 = E$$

1.7 AN L-C CIRCUIT

We will now discuss harmonic oscillations of charge in an ideal (R = 0) *L-C* circuit depicted in Fig. 1.6. As we know that the electric and magnetic energies in such a circuit play roles analogous to potential and kinetic energies respectively for a spring-mass system. For simplicity, we assume that the inductor no resistance.

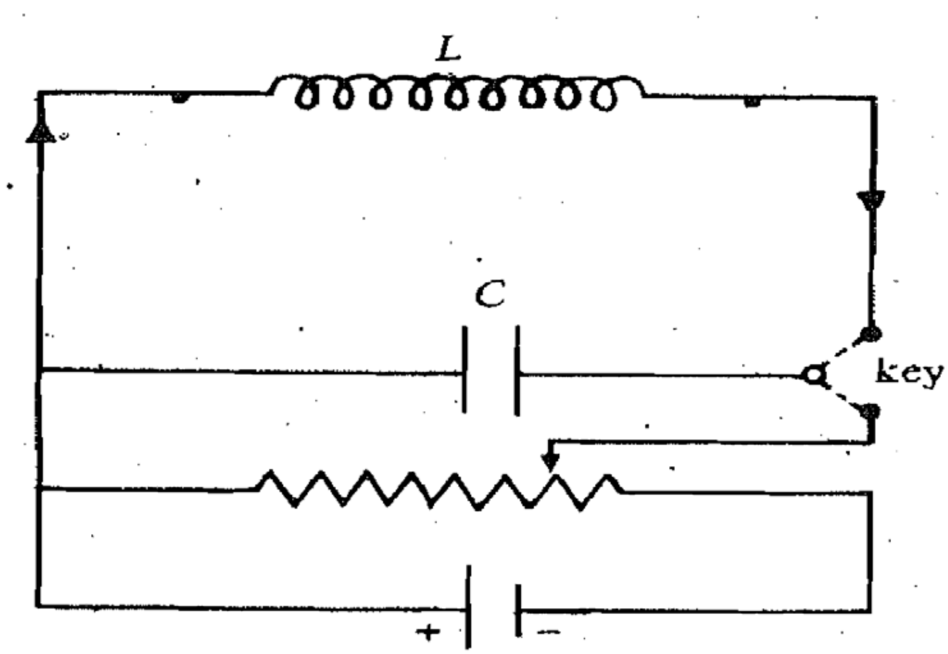


Fig. 1.6 . An ideal L-C circuit.

In a pendulum, the mean position is taken as the equilibrium state. What **is** the equilibrium state in an *L-C* circuit? It corresponds to the state when there is no current in the circuit. It may be disturbed by charging or discharging the capacitor.

Let the capacitor be given a charge Q_o coulomb. Then the voltage across the capacitor plates will be $\mathbf{Q_o / C}$. Now if the circuit is disconnected, the capacitor discharges through the inductor. As a result current starts building up in the circuit gradually and the charge on the plates of the capacitor decreases. At any time ***t,*** let the current in the circuit be ***I*** and the charge on capacttor plates be q. Then the voltage drop across the inductor will be

$$V_L = -L \frac{dI}{dt}$$

This must be equal to the voltage $V_C = q/C$ across the capacitor plates at that time. Thus, we can write

$$V_C = V_L$$

or $$\frac{q}{C} = -L \frac{dI}{dt} \qquad \textbf{1.22}$$

Since

$$I = dq/dt \text{ and } \frac{dI}{dt} = -\frac{d^2q}{dt^2}$$

Eq. (1.22) takes the form

$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0 \qquad \textbf{1.23}$$

$$\text{where } \omega_0^2 = \frac{1}{LC}$$

This means that one can have a wide range of frequencies by changing the values of L and C. That is how you tune different stations in your radio sets.

Eq. (1.23) represents SHM and has the solution

$$q = Q_0 \cos(\omega_0 t + \phi) \qquad \textbf{1.24}$$

This shows that charge oscillates harmonically with the period

$$T = 2\pi\sqrt{LC} \qquad \textbf{1.25}$$

Differentiating Eq.(1.24) with respect to time, we get the instantaneous current

$$I = -Q_0\,\omega_0 \sin(\omega_0 t + \phi)$$
$$= I_0 \cos(\omega_0 t + \phi + \pi/2)$$

where $I_0 = Q_0\,\omega_0$,

Thus the current leads the charge in phase by **π/2.**

Let us now calculate the energy stored in the inductor ***L*** and the capacitor **C** at any

instant **t**. As the current rises from zero to **I** in time ***t,*** the energy stored in the

inductor, E_L , is obtained by integrating the instantaneous power with respect to time,

$$E_L = -\int_0^t I\,V_L\,dt$$

The negative sign implies that work is done against, rather than by the emf. On

substituting for V_L, we get

$$E_L = L\int_0^I \frac{dI}{dt} I\,dt = \tfrac{1}{2} LI^2$$

The energy stored in the capacitor at time **t** is

$$E_C = q^2/2C$$

Thus the total energy

$$E = E_L + E_C = \tfrac{1}{2} LI^2 + \frac{1}{2}\frac{q^2}{C} \qquad \textbf{1.26}$$

This expression for total energy is similar to the one for mechanical oscillator ***(E= 1/2 m**v^2 **+1/2 k**x^2***).*** As ***q*** and **I** vary with time, the inductor and capacitor exchange energy periodically. This is similar to the energy exchange in the spring mass system. Further, the mass and inductor play analogous roles in mechanical and electrical systems, respectively.

1.8 EXAMPLES OF PHYSICAL SYSTEMS EXECUTING SHM

We have seen that for a system, to execute simple harmonic motion, it must have two parts: one which can store potential energy (like spring) and the other capable of', storing kinetic energy (such as mass). We will now study physical system executing SHM using techniques developed for our model spring-mass system.

1.8.1 SIMPLE PENDULUM

A simple pendulum is an idealized system consisting of a point mass (bob) suspended by an inextensible, weightless string. As the bob of mass ***m*** is displaced by an angle ***θ*** from its equilibrium position the restoring force is provided by the tangential component of the weight ***mg*** along the arc (Fig. 1.7). It is given by

$$F = -mg \sin\theta$$

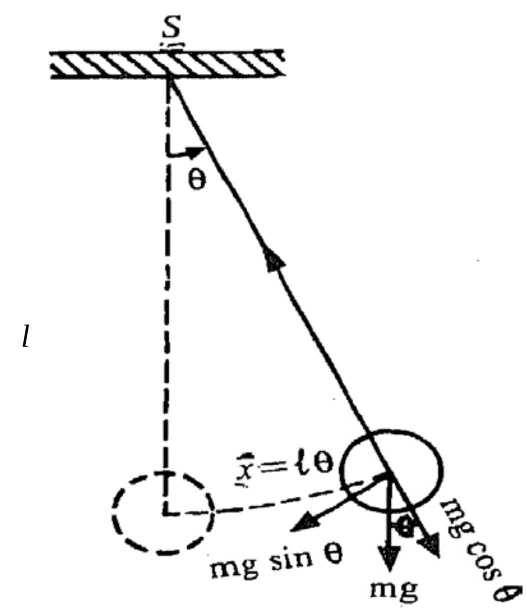


Fig. 1.7 A simple pendulum

The equation of motion of the bob is, therefore,

$$m\frac{d^2x}{dt^2} = -mg \sin\theta$$

The bob is moving along the arc whose length at any instant is given by **x.** 1f the corresponding angular displacement from the equilibrium position is θ, then the length of arc is

$$x = \ell\theta \tag{1.28}$$

where ***l*** is length of the string by which the bob is suspended.

Differentiating Eq. (1.28) twice with respect to ***t***, and substituting the result in Eq. (1.27), we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta \tag{1.29}$$

For small angular displacements, **sin*θ*** may be approximated to ***θ***. In this approximation, Eq. (1.29) takes the form

$$\frac{d^2\theta}{dt^2} + \omega_0^2\,\theta = 0. \tag{1.30}$$

where $\omega_0 = \sqrt{g/\ell}$.

Eq. (1.30) is exactly of the standard form (1.3.) showing that pendulum executes simple harmonic motion. The time period of oscillation is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\ell/g} \tag{1.31}$$

By analogy, we can write the general solution of the Eq. (1.30) as

$$\theta = \theta_m \cos(\omega_0 t + \phi) \tag{1.32}$$

where θ_m , is the maximum angular displacement.

From Eq. (1.31) we will note that for small angular displacements, the frequency of oscillation of a simple pendulum depends on ***g*** and ***l*** but not on the mass of the bob. The appearance of the factor ***g*** in Eq. (1.31) implies that a pendulum clock will move slower near the equator than at the poles. Do you know why? This is because the value of ***g*** varies with latitude. For the same reason, the period of a pendulum will be different on moons and planets.

When the amplitude of oscillation is not small, we are required to solve the general Eq. (1.29). The time period, which can be expressed in the form of a series involving the maximum angular displacement $\boldsymbol{\theta_m}$, is given by

$$T = 2\pi\sqrt{l/g}\left[1 + \frac{1}{2^2}\sin^2\frac{\theta_m}{2} + \left(\frac{1\times 3}{2\times 4}\right)^2\sin^4\frac{\theta_m}{2} + \ldots\right] \qquad \mathbf{1.33}$$

we can check the accuracy of Eq. (1.31) by comparing the value of T obtained from Eq. (1.33). For example, you will find that when $\boldsymbol{\theta_m}$, is $\mathbf{15^O}$ (corresponding to a total to and fro-angular displacement of **30°),** the actual value of time period differs from that given by Eq. (1.31) by less than **0.5%.**

1.8.2 COMPOUND PENDULUM

A compound pendulum is a rigid body capable of oscillating freely about a horizontal axis passing through it (Fig 1.8). At equilibrium position, the centre of gravity **G** lies vertically below the point of suspension **S**. Let the distance **SG** be ***l*** **.** If the pendulum is given a small angular displacement θ at any instant, it oscillates over the same path. Is its motion simple harmonic? To answer this question we note that the restoring torque about *S* is $\boldsymbol{-mgl\sin\theta}$ and it tends to bring the pendulum towards the equilibrium position.

If **I** is the moment of inertia of the body about the horizontal axis passing through *S,* the restoring torque equals $\boldsymbol{I d^2\theta/dt^2}$. Hence the equation of motion can be written as

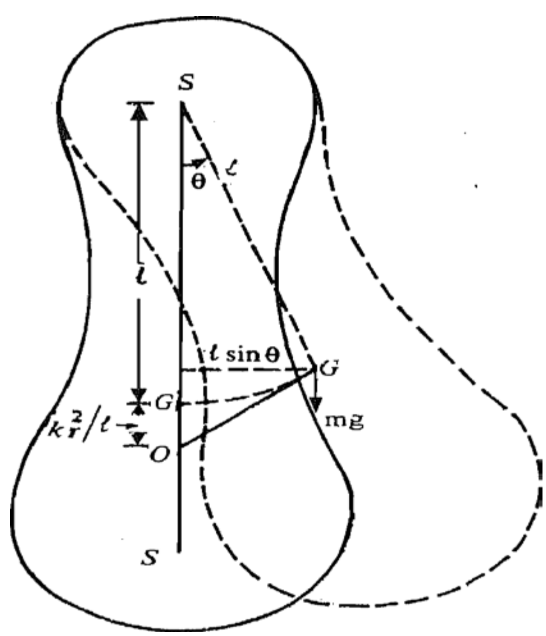


Fig. 1.8 A rigiri body oscillating about a horizontal axis: Compound pendulum

$$I\frac{d^2\theta}{dt^2} = -mg\ell\sin\theta \qquad 1.34$$

For small angular displacement, $\sin\theta \simeq \theta$ and Eq. (1.34) takes the form

$$\frac{d^2\theta}{dt^2} + \frac{mg\ell}{I}\theta = 0 \qquad 1.35$$

This equation shows that a compound pendulum executes SHM and the time period is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{I/mg\ell} \qquad 1.36$$

There is a very useful and important theorem of parallel axes in the study of moment of inertia. According to this theorem, the moment of inertia **I** of a body about any axis and its inertia **I**, about a parallel axis passing through its centre of gravity are connected by the relation

$$I = I_g + m\ell^2 \qquad 1.37$$

where ***l*** is the distance between the two axes and $I_g = mk_r^2$ The quantity k_r is theradius of gyration of the body about the axis passing through **G**. It is the radial distance at which the whole mass of the body could be placed without any change in the moment of inertia of the body about that axis.

On substituting the expression for **I** from **Eq.** (1.37) in Eq. (1.36), we obtain

$$T = 2\pi\sqrt{\frac{k_r^2 + \ell^2}{lg}} \qquad 1.38$$

On comparing this expression for **T** with that given by Eq. **(1.31)** for a simple pendulum we will note that two periods become equal if ***l*** in Eq. (1.31) is replaced by $L = \sqrt{(k_r^2/\ell) + \ell}$.This is called the length of an equivalent simple pendulum. If we produce the line ***SG*** and take a point **O** on it such that $SO = \frac{k_r^2}{\ell} + \ell$, then **O** is called the centre of oscillation.

1.8.3 VERTICAL OSCILLATIONS OF A FLOTING CYLINDER

When a partially submerged floating body is slightly depressed and released, the body executes vertical simple harmonic oscillation. Consider a solid cylinder of length ***l*** and area of cross section **A**, floating in a liquid of density **σ** with height **h** inside the liquid. Let **ρ** be the density of material of solid cylinder. At equilibrium, the up thrust of the liquid displaced balances. The weight of the cylinder. i.e.

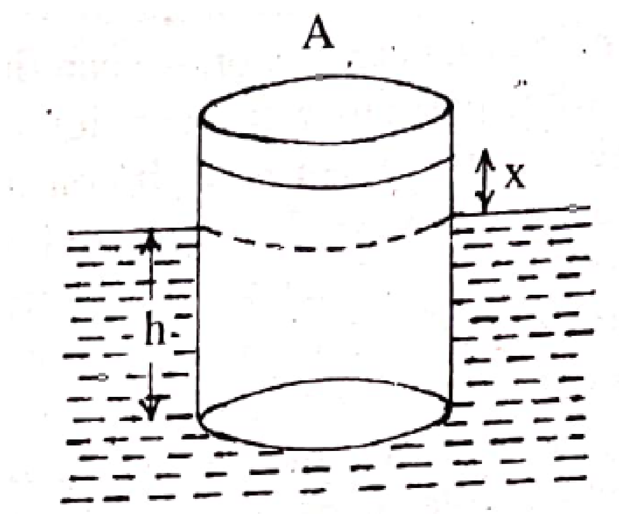


Fig. 1.9 Floting Cylinder

$Ah\,\sigma\,g = A\,l\,\rho\,g$

Now from this equilibrium position if it is depressed by **x** and then released, it oscillates up and down, the immersed length of cylinder will be **(h+x)** so that the upthrust is **A (h+x) σ g**. Therefore the resultant force acting upwards on the cylinder is **A (h+x) σ g - A *l* ρ g i.e. A (h+x) σ g- A h σ g or A x σ g.**

Therefore restoring force **= -A σ g x**

Since mass of solid cylinder is **A *l* ρ**, Therefore its acceleration is

$$f = \frac{Restoring\ force}{mass} = -\frac{A\,\sigma\,g\,x}{A\,l\,\rho}$$

$$\text{or } f = -\frac{\sigma\,g}{l\,\rho}x = -\omega_0^2 x$$

$$\text{where } \omega_0^2 = \frac{\sigma\,g}{l\,\rho}$$

So that time period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l\,\rho}{\sigma\,g}}$

If mass and radius of the solid cylinder have been given but length ***l*** and density **ρ**, not given then

$\mathbf{M = \pi r^2 \, \mathit{l} . \rho}$ or $\mathit{l}\,\rho = \frac{M}{\pi r^2}$

Therefore $T = 2\pi\sqrt{\frac{M}{\pi r^2 \sigma g}}$

From this expression it is clear that if density of liquid decreases time period will increase and vice versa.

And also $\mathbf{A\,h\,\sigma\,g = A\,\mathit{l}\,\rho\,g = Mg}$

or $\mathbf{M = A\,h\,\sigma = \pi\,r^2\,h\,\sigma}$

Therefore $T = 2\pi\sqrt{\frac{h}{g}}$ where h is the height of the solid cylinder inside the liquid.

1.8.4 UP AND DOWN OSCILLATION OF LIQUID CONTAINED IN A VERTICAL U- TUBE

If the liquid on one side in a verticle U-tube is depressed and then released, its motion up and down the two sides of the tube is simple harmonic.

Let us calculate its time period.

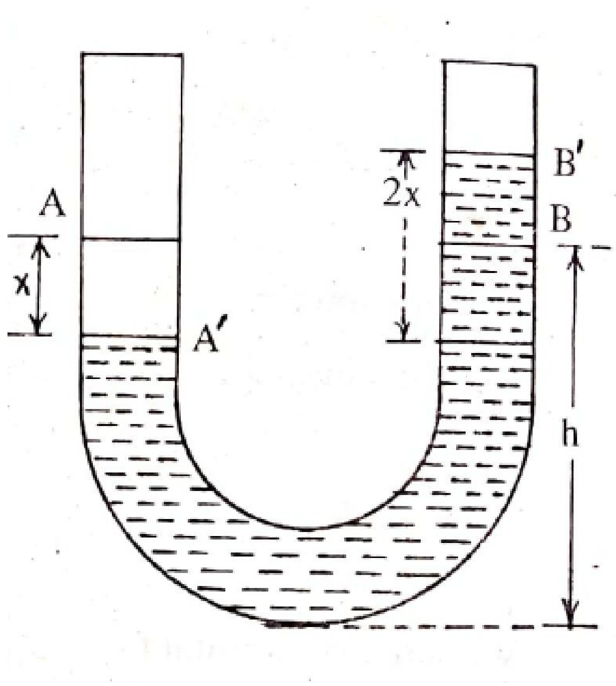


Fig. 2.0 Vertical U- tube

Let AB be the initial level of liquid in the U-tube and let the column on the left be depressed through distance **x** to **A'**, then the column on the right will rise up through the same distance **x** to the level **B'**, so that the difference in levels between the two columns in **A'B' = 2x.**

If **A** is internal cross sectional area of the tube, then restoring force

= **- Excess pressure × Area**

$= -2x\,\rho\,g\,A$

This is the force acting on total mass of liquid **m= 2 h A** ρ due to which the liquid acceleration is

$$\frac{d^2x}{dt^2} = \frac{Force}{mass} = \frac{-2\,x\,\rho\,g\,A}{2\,h\,A\,\rho} = -\frac{g}{h}.x$$

or $$\frac{d^2x}{dt^2} = -\omega_0^2 x \quad where\ \omega_0^2 = \frac{g}{h}$$

Thus the acceleration is propotional to the displacement **x** from initial level **AB** and is directed towards it. Hence the motion of the liquid is harmonic.

Therefore time period is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{h}{g}}$$

1.9 SUMMARY

In this unit, we have studied about simple harmonic motion, and what are the conditions and basic characteristics of SHM. Thereafter, we learned how to calculate the velocity and acceleration of a particle executing SHM. Now, after having understood the different characteristics of SHM, we are in a position to the forward and discuss some other physical systems executing SHM. In the present unit, we also studied about the concept of Simple Pendulum and Compound Pendulum.

1.10 TERMINAL QUESTIONS

1. What are the characteristics of simple harmonic motion?
2. Explain the concept of simple harmonic motion.
3. Discuss SHM as a oscillations of a Spring mass system.
4. Write down the differential equation of SHM.
5. Write short notes on :
   (a) Phase and amplitude
   (b) Time Period and Frequency
   (c) Simple Pendulum
   (d) Compound Pendulum
6. Explain the concept of Vertical Oscillations of a Floting cylinder.
7. Discuss up and down oscillation of Liquid contained in a vertical U-tube.

UNIT-02 DAMPED OSCILLATOR

Structure

2.1 INTRODUCTION

In Unit 1 you learnt that SHM is a universal phenomenon. Now you also know that in the ideal case the total energy of a harmonic oscillator remains constant in time and the displacement follows a sine curve. This implies that once such a system is set in motion it will continue to oscillate forever. Such oscillations are said to be free or ***undamped.*** Do you know of any physical system in the real world which experiences no damping? Probably there is none. You must have observed that oscillations of a swing, a simple or torsional pendulum and a spring-mass system when left to themselves, die down gradually. Similarly, the amplitude of oscillation of charge in an LCR circuit or of the coil in a suspended type galvanometer becomes smaller and smaller. This implies that every oscillating system loses some energy as time elapses. The question now arises: Where does this energy go? To answer this, we note that when a body oscillates in a medium it experiences resistance to its motion. This means that damping force comes into play. Damping force can arise within the body itself, as well as due to the surrounding medium (air or liquid). The work done by the oscillating system against the damping forces leads to dissipation of energy of the system. That is, the energy of an oscillating body is used up in overcoming damping. But in some engineering systems we knowingly introduce damping. A familiar example is that of brakes - we increase friction to reduce the speed of a vehicle in a short time. In general, damping causes wasteful loss of energy. Therefore, we invariably try to minimize it.

Many a time it is desirable to maintain the oscillations of a system. For this we have to feed energy from an outside agency to make up for the energy losses due to damping. Such oscillations are called ***forced oscillations.*** You will learn various aspects of such oscillations in the next unit.

In this unit you will learn to establish and solve the equation of motion of a damped harmonic oscillator. Damping may be quantified in terms of logarithmic decrement, relaxation time and quality factor. You will also learn to compute expressions for the logarithmic decrement, power dissipated in one cycle and the quality factor.

OBJECTIVES

After studying this unit, you should be able to

- Understand the concept of differential equation of clamped oscillator
- Understand the various types of damping effects
- Define average energy of a weakly damped oscillator
- Explain various parameters of weak damping
- Explain the concept of ‘damping in LCR circuit.

2.2 DIFFERENTIAL EQUATION OF A DAMPED OSCILLATOR

While considering the motion of a damped oscillator, some of the questions that come to our mind are : Will Eq.(1.2) still hold? If not, what modification is necessary? How to describe damped motion quantitatively? To answer these questions we again consider the spring-mass system of Unit 1. Let us imagine that the mass moves horizontally in a viscous medium, say inside a lubricated cylinder, as shown in Fig.2.1.

As the mass moves, it will experience a drag, which we denote by *Fd*. The question now arises : How to predict the magnitude of this damping force? Usually, it is difficult to quantify it exactly. However, we can make a reasonable estimate based on our experience. For oscillations of sufficiently small amplitude, it is fairly reasonable to model the damping force after Stokes' law. That is, we take F_d to be proportional to velocity and write

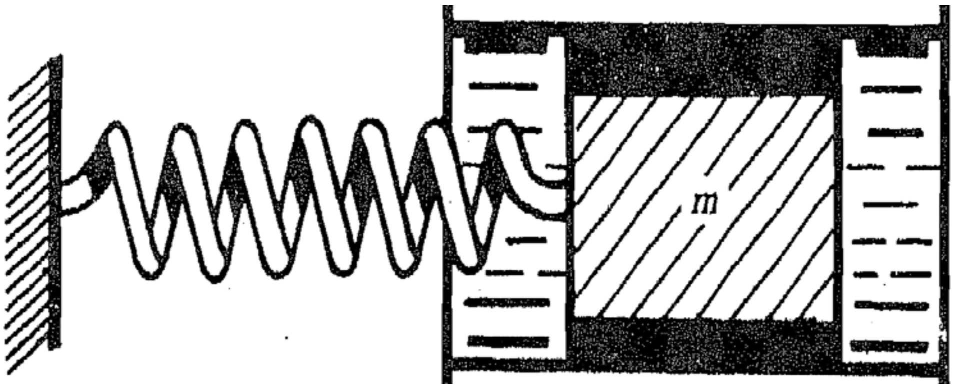


Fig. 2.1 A damped spring-mass system.

$$F_d = -\gamma v \qquad \textbf{2.1}$$

The negative sign signifies that the damping force opposes motion. The constant of

proportionality γ is called the *damping coefficient,* Numerically, it is equal to force

per unit velocity and is measured in

$$\frac{N}{ms^{-1}} = \frac{kgms^{-2}}{ms^{-1}} = kg\, s^{-1}$$

We will now, establish the differential equation which describes the oscillatory motion of a damped harmonic oscillator. Let us take the x-axis to be along the length of the spring. We define the origin of the axis ***(X*** = **0)** as the equilibrium position of the mass. Imagine that the mass (in the spring-mass system) is pulled longitudinally and then released. It gets displaced from its equilibrium position. At any instant, the forces acting on the spring-mass system are :

(i) *a restoring force* : is th$-kx$ where k and

(ii) a *damping force* : $-\gamma v$, where $v = \frac{dx}{dt}$ is the instantaneous velocity of the ***dt*** oscillator. This means that for a damped harmonic oscillator, the equation of motion must include the restoring force as well as the damping force. Hence, in this case Eq.(1 .2) is modified to

$$m \frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} \qquad 2.2$$

After rearranging terms and dividing throughout by ***m,*** the equation of motion of a damped oscillator takes the form

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega_0^2 x = 0 \qquad 2.3$$

where $\omega_0^2 = k/m$ and $2b = \gamma/m$.

(You will note that a factor of 2 has been introduced in the damping term as it helps us to obtain a neat expression for the solution of this equation.) The constant *b* has the dimension of

$$\frac{\text{force}}{\text{velocity X mass}} = \frac{MLT^{-2}}{LT^{-1}M} = T^{-1}$$

Hence, its unit is $\mathbf{s^{-1}}$ which is the same as that of $\boldsymbol{\omega_0}$.

We will note that like Eq. (1.3), Eq. (2.3) is a linear second order homogeneous

differential equation with constant coefficients. If there were no damping, the second term in Eq. (2.3) will be zero and the general solution of the resulting equation will be given by Eq. (1.5), i.e. $x = A\cos(\omega_0 t + \phi)$. On the other hand, if there is damping of no restoring force, the third term in Eq.(2.3) .will be zero. Then the general solution of the resulting equation is given by $x(t) = Ce^{-2bt} + D$ where C and D are constants. This means that the displacement will decrease exponentially in the absence of any restoring force. Thus, we expect that the general solution of Eq. (2.3), Will represent an oscillatory motion whose amplitude decreases with time.

2.3 SOLUTIONS OF THE DIFFERENTIAL EQUATION

How does damping influence the amplitude of oscillation? To discover this we have to solve Eq. (2.3) when both the restoring force and the damping force are present. The general solution, as discussed above, should involve both exponential and harmonic terms. Let us therefore take a solution of the form

x(t) = a exp (αt) **2.4**

when **a** and **α** are unknown constants.

Differentiating **Eq.(2.4)** twice with respect to time, we get

$\frac{dx}{dt}$ = **a α exp (αt)**

and

$$\frac{d^2x}{dt^2} = a\alpha^2 \exp(at).$$

Substituting these expressions in Eq. (2.3), we get

$$(a^2 + 2b\alpha + \omega_0^2)\, a \exp(cut) = 0 \quad \textbf{2.5}$$

For this equation to hold at all times, we should either have

a = 0

Which is trivial, or

$$\alpha^2 + 2b\alpha + \omega_0^2 = 0 \quad \textbf{2.6}$$

This equation is quadratic in ***α***. Let us call the two roots α_1 and α_2

$$\alpha_1 = -b + (b^2 - \omega_0^2)^{1/2} \quad \textbf{2.7a}$$

$$\alpha_2 = -b - (b^2 - \omega_0^2)^{1/2} \quad \textbf{2.7b}$$

These roots determine the motion of the oscillator. Obviously ***α*** has dimensions of inverse time. Did you not expect it from the form of exp (αt)?

Thus, the two possible solutions of Eq. (2.3) are

$$x_1(t) = \alpha_1 \exp\{-\{b + (b^2 - \omega_0^2)^{\frac{1}{2}}\}t]$$

$$x_1(t) = \alpha_1 \exp\{-\{b - (b^2 + \omega_0^2)^{\frac{1}{2}}\}t] \quad \textbf{2.8}$$

Since Eq.(2.3) is linear, the principle ,of superposition is applicable. Hence, the general solution is obtained by the superposition of x_1 and x_2

$$x(t) = \exp(-bt)\,[a_1 \exp\{(b^2 - \omega_0^2)^{1/2}\}t] + a_2 \exp\{-(b^2 - \omega_0^2)^{1/2}\}t] \quad \textbf{2.9}$$

We will note the quantity $(b^2-\omega^2_0)$ can be negative, zero or positive respectively depending on whether **b** is less than, equal. to or greater than ω_o respectively. We, therefore, have three possibilities :

(i) If $b > \omega_o$, we say that the system is over damped,

(ii) If $b = \omega_o$, we have a critically damped system,

(iii) If $b < \omega_o$, we have an under-damped system.

Each of these conditions gives a different solution, which describes a particular behaviour.

We will now discuss these solutions in order of their increasing importance.

2.3.1 HEAVY DAMPING

When resistance to motion is very strong, the system is said to be heavily damped.

Can you name a heavily damped system of practical interest? Springs joining wagons of a train constitute the most important heavily damped system. In your physics laboratory, vibrations of a pendulum in a viscous medium such as thick oil and motion of the coil of a dead beat galvanometer are heavily damped systems.

Mathematically, a system is said to be heavily damped if $b > \omega_o$. Then the quantity $(b^2-\omega^2_0)$ is positive definite. If we put

$\beta = \sqrt{b^2 - \omega_0^2}$ the general solution for damped oscillator given by Eq. (2.9) reduces to

$$x(t) = \exp(-bt)\,[a_1 \exp(\beta t) + a_2 \exp(-\beta t)]. \qquad 2.10$$

This represents non-oscillatory behaviour. Such a motion is called dead-beat. The actual displacement will, however, be determined by the initial conditions. Let us suppose that to begin with the oscillator is at its equilibrium position, i.e $x = 0$ at $t = 0$. Then we give it a sudden kick so that it acquires a velocity v_o, i.e $v = v_o$ at $t = 0$. Then from Eq. (2.10) we have

$$a_1 + a_2 = 0$$

$$-b(a_1 + a_2) + \beta(a_1 - a_2) = v_0$$

These equations may be solved to give

$$a_1 = -a_2 = \frac{v_0}{2\beta}$$

On substituting these results in Eq. (2.10), we can write the sblution in compact form:

$$x\,(t) = \frac{v_0}{2\beta}\ \exp\,(-\,bt)\,[\exp\,(\beta t) - \exp\,(-\,\beta t)]$$

$$= \frac{v_0}{\beta}\ \exp\,(-\,bt)\ \sinh\,\beta t \qquad \textbf{2.11}$$

Where $\sin h\, \beta t = [\exp(\beta t) - \exp(-\beta t)]/2$ is hyperbolic sine function. From Eq. (2.11) it is clear that ***x*** (t) will be determined by the interplay of an increasing hyperbolic function and a decaying exponential. These are plotted separately in Fig. 2.2(a). Fig. 2.2(b) shows the plot of Eq. (2.11) for a heavily damped system when it is suddenly disturbed from its equilibrium position. We will note that initially the displacement increases with time. But soon the exponential term becomes important and displacement begins to decrease gradually.

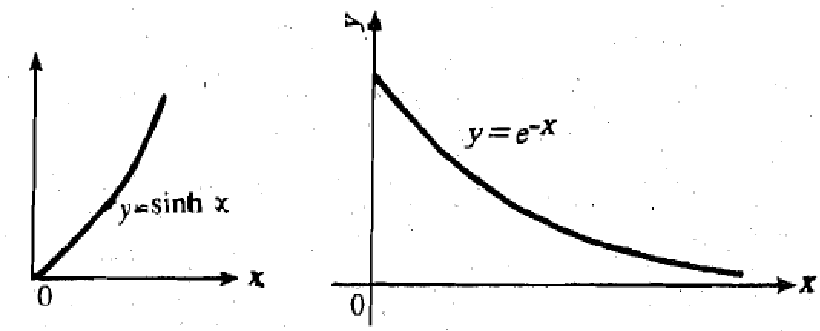


Fig. 2.2(a) Plot of sinh x and exp (- x)

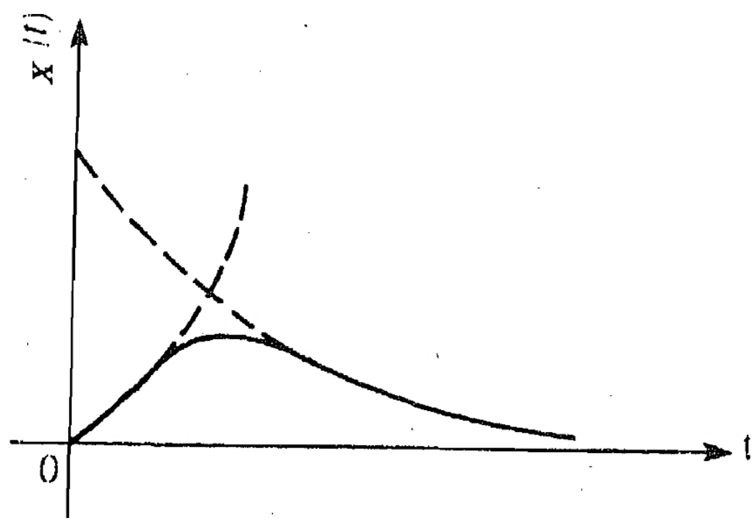


Fig. 2.2(b) Plot of Eq. (2.1I) for a heavily damped system

2.3.2 CRITICAL DAMPING

We may have observed that on hitting an isolated road bump, a car bounces up and down and the occupants feel uncomfortable. To minimize this discomfort, the bouncing caused by the road bumps must be damped very rapidly and the automobile is restored to equilibrium quickly. For this we use critically damped shock absorbers. Critical damping is also useful in recording instruments such as a galvanometer (pointer type as well as suspended coil type) which experience sudden impulses. We require the pointer to move to the correct position in minimum time and stay there without executing oscillations. Similarly, a ballistic galvanometer coil is required to return to zero displacement immediately.

Mathematically, we say that a system is critically damped if ***b*** is equal to the natural frequency. ω_0 **,** of the system. This means that $b^2 - \omega_0^2 = 0$ **,** so that Eq **(2.9)** reduces to

$$x(t) = (a_1 + a_2)\exp(-bt) = a\exp(-bt) \qquad \textbf{2.12}$$

where $a = a_1 + a_2$.

Let us pause for a minute and recall that the solution of the differential equation for SHM involves two arbitrary constants which are fixed by giving the initial conditions. But **Eq.** (2.12) has only one constant. Does this mean that it is not a complete solution? It is important to understand how this happens. The reason is simple : the quadratic equation for **α** **(Eq.** 2.6) has equal roots. So, the two terms in Eq. (2.9) give the same time dependence and reduce to one term. It can be easily verified that in this case the general solution Eq. (2.3) is

$$x(t) = (p+qt)\exp(-bt) \qquad \textbf{2.13a}$$

Where **p** and ***q*** are constants. **p** has the dimensions of length and ***q*** that of velocity. These can be determined easily from the initial conditions.

Let us assume that the system is disturbed from its mean equilibrium position by a sudden impulse. (The coil of a suspended type galvanometer receives some electric charge at $t = 0$). That is, at $t = 0$, $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = v_0$

This gives $p = 0$ and $q = v_0$ so that the complete solution is

$$x(t) = v_0 t \exp(-bt) \qquad \textbf{2.13b}$$

Fig. **2.3** illustrates the displacement time graph of a critically damped system described by Eq. (2.13b). At maximum displacement

$$\left.\frac{dx}{dt}\right|_{x=x_{max}} = 0 \text{ and } \left.\frac{d^2x}{dt^2}\right|_{x=x_{max}} < 0.$$

This occurs at time $t = 1/b$

$$x_{max} = v_0 t e^{-1} = 0.368 \frac{v_0}{b} = 0.736 \frac{m v_0}{\gamma}$$

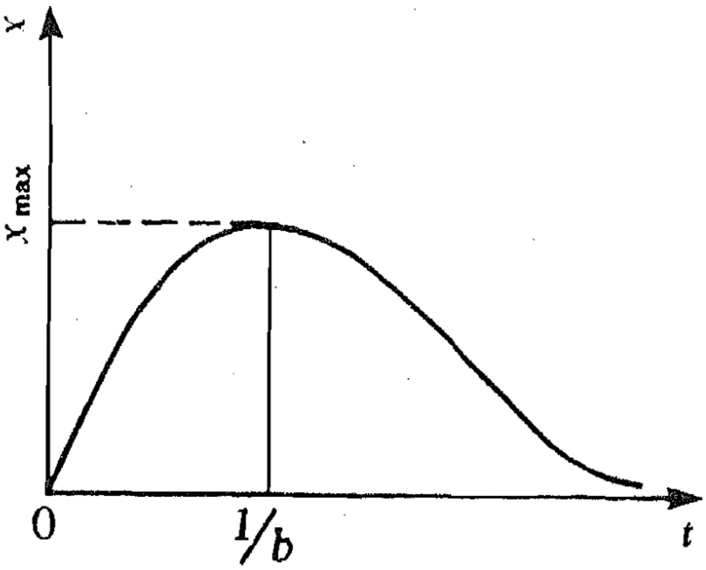


Fig. 2.3 Displacement-time graph for a critically damped system described by Eq. (2.13b).

2.3.3 WEAK OR LIGHT DAMPING

When $\mathbf{b < \omega_o}$ we refer to it as a case of weak damping. This implies that $\mathbf{(b^2-\omega^2_0)}$ is a negative quantity, i.e $\mathbf{(b^2-\omega^2_0)^{1/2}}$ is imaginary. Let us rewrite it as

$$(b^2 - \omega_0^2)^{1/2} = \sqrt{-1}\,(\omega_0^2 - b^2)^{1/2}$$
$$= \pm i\omega_d$$

where $i = \sqrt{-1}$ and

$$\omega_d = (\omega_0^2 - b^2)^{1/2} = \left[\frac{k}{m} - \frac{\gamma^2}{4m^2}\right]^{1/2} \qquad \textbf{2.14}$$

is a real positive quantity. We will note that for no damping **(b = 0)**, ω_d reduces to

ω_o **,** the natural frequency of the oscillator.

On combining Eqs. (2.9) and (2.14) we find that the displacement now has the form

$$x(t) = \exp(-bt)\,[a_1 \exp(i\omega_d t) + a_2 \exp(-i\omega_d t)] \qquad \textbf{2.15}$$

To compare the behaviour of a damped oscillator with that of a free oscillator, we

should recast Eq.(2.15) so that the displacement varies sinusoidally. To do this, we

write the complex exponential in terms of sine and cosine functions. This gives

$$x(t) = \exp(-bt)\,[a_1(\cos\omega_d t + i\sin\omega_d t)] - a_2(\cos\omega_d t - i\sin\omega_d t)]$$

On collecting coefficients of **cos ω_dt** and **sin ω_dt ,** we obtain

$$x(t) = \exp(-bt)\,[(a_1 + a_2)\cos\omega_d t + i(a_1 - a_2)\sin\omega_d t)] \qquad \textbf{2.16}$$

Let us now put

$$a_1 + a_2 = a_0 \cos\phi$$

and

$$-i(a_1 - a_2) = a_0 \sin\phi \qquad \textbf{2.17}$$

where **a_0 and** ϕ are arbitrary constants. These are given by

$$a_0 = 2\sqrt{a_1 a_2}$$

and

$$\tan\phi = -i\,\frac{a_1 - a_2}{a_1 + a_2} \qquad \textbf{2.18}$$

From the second of these results we note that tan ϕ is a complex quantity. Does this

mean that ϕ is also complex? How can we interpret a complex angle? To know this, we use the identity

$$\sec^2\phi = 1 + \tan^2\phi$$

and calculate **$\cos\phi$.** The result is

$$\cos\phi = \frac{a_1 + a_2}{2\sqrt{a_1 a_2}}$$

This means that $\cos\phi$, and hence ϕ, is real.

Substituting Eq. (2.17) into Eq. (2.16) we find that the expression within the parentheses is cosine of the sum of two angles. Hence, the general solution of Eq. ***(2.3)*** for a weakly damped oscillator $\mathbf{b < \omega_o}$ is

$$x(t) = a_0 \exp(-bt)\cos(\omega_d t + \phi) \qquad \textbf{2.19}$$

with ω_d given by Eq. (2.14) . We will note that the solution given by Eq. (2.19)

describes sinusoidal motion with frequency ω_d which remains the same throughout the motion. This property is crucial for the use of oscillators in accurate time -pieces. How is the amplitude modified vis-a-vis an ideal SHM? We will note that the amplitude decreases exponentially with time at a rate governed by **b**. So we can say that motion of a weakly damped system is not simple harmonic.

The damped oscillatory behaviour described by Eq. (2.19) is plotted in Fig.**2.4** for the particular case of ϕ = **0.** Since the cosine function varies between + 1 and - 1, we observe that the displacement-time curve lies between $a_0 \exp(-br)$ and $-a_0 \exp(-br)$. Thus, we may conclude that damping results in decrease of amplitude and angular frequency.

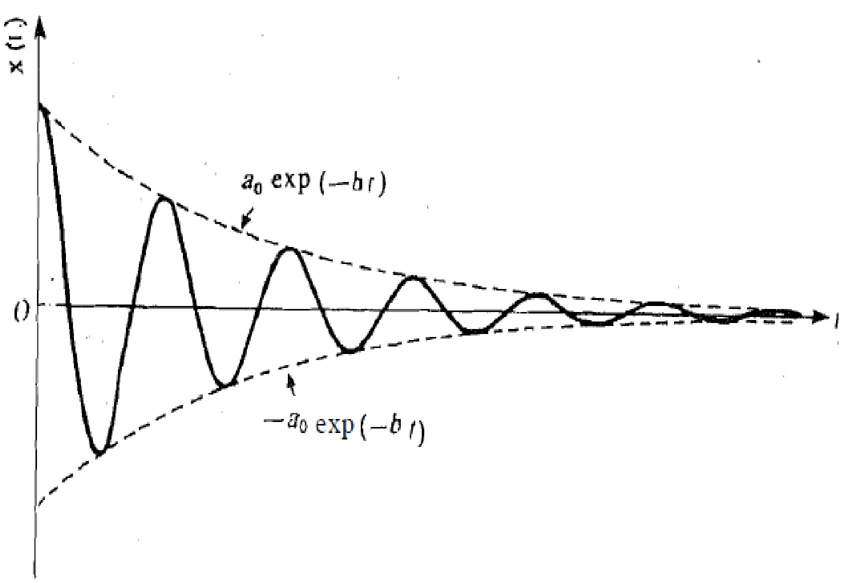


Fig. 2.4 Displacement-time graph for weakly damped harmonic oscillator

How does damping influence the period of oscillation? We can discover this effect by noting that the period of oscillation is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$

If **b > 0**, $\omega_d < \omega_O$. This means that the period of vibration of a damped oscillator is more than that of an ideal oscillator. Did you not expect it since damping forces resist motion?

2.4 AVERAGE ENERGY OF A WEAKLY DAMPED OSCILLATOR

In Unit 1 we calculated the average energy of an undamped oscillator. The question now arises: How does damping influence the average energy of a weakly damped oscillator? To answer this we note that in the presence of damping the amplitude of oscillation decreases with the passage of time. This means that energy is dissipated in overcoming resistance to motion. From Unit 1 we recall that at any time, the total energy of a harmonic oscillator is made up of kinetic and potential components. We can still use the same definition and write

$$E(t) = K.E(t) + U(t)$$

$$= \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2 \qquad \textbf{2.20}$$

where ***(dx/dt)*** denotes instantaneous velocity.

For a weakly damped harmonic oscillator, the instantaneous displacement is given by Eq. (2.19):

$$x(t) = a_0 \exp(-bt) \cos(\omega_d t + \phi)$$

By differentiating it with respect to time, we get instantaneous velocity:

$$\frac{dx(t)}{dt} = v = -a_0 \exp(-bt)\, [b \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)] \qquad \textbf{2.21}$$

Hence, kinetic energy of the oscillator is

$$K.E = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2} m a_0^2 \exp(-2bt)[b \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)]^2$$

$$= \tfrac{1}{2} m a_0^2 \exp(-2bt)[b^2 \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + 4)$$
$$+ b\,\omega_d \sin 2(\omega_d t + \phi)] \qquad \textbf{2.22a}$$

Similarly, the potential energy of the oscillator is

$$U = \tfrac{1}{2} kx^2 = \tfrac{1}{2} m\omega_0^2 x^2$$

since $k = m\omega_0^2$

On substituting for x, we get

$$U = \tfrac{1}{2} m a_0^2 \omega_0^2 \exp(-2bt) \cos^2(\omega_d t + \phi) \qquad \textbf{2.22b}$$

Hence, the total energy of the oscillator at any time ***t*** is given by

$$E(t) = \tfrac{1}{2} m a_0^2 \exp(-2bt)[(b^2 + \omega_0^2) \cos^2(\omega_d t + 4)$$
$$+ \omega_d^2 \sin^2(\omega_d t + \phi)$$
$$+ b\omega_d \sin 2(\omega_d t + \phi)] \qquad \textbf{2.23}$$

When damping is small, the amplitude of oscillation does not change much over one oscillation. So we may take the factor $\exp(-2bt)$ as essentially constant. Further, since

$\langle \sin^2(\omega_d t + \phi) \rangle = \langle \cos^2(\omega_d t + \phi) \rangle = \tfrac{1}{2}$ and $\langle \sin(\omega_d t ++) \rangle = 0$, the

energy of a weakly damped oscillator when averaged-over one cycle is given by

$$\langle E \rangle = \tfrac{1}{2} m a_0^2 \exp(-2bt) \langle [(b^2 + \omega_0^2) \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi)$$
$$+ b\omega_d \sin 2(\omega_d t + \phi)] \rangle$$

$$= \tfrac{1}{2} m a_0^2 \exp(-2bt) \left[\frac{b^2 + \omega_0^2}{2} + \frac{\omega_d^2}{2}\right]$$
$$= \tfrac{1}{2} m a_0^2 \omega_0^2 \exp(-2bt) \qquad \textbf{2.24a}$$

From Unit 1 we recall that $E_0 = \tfrac{1}{2} m a_0^2 \omega_0^2$ is the total energy of an undamped oscillator. Hence, we can write

$$\langle E \rangle = E_0 \exp(-2bt) \qquad \textbf{2.24b}$$

This shows that the ***average energy*** of ***a weakly damped oscillator decreases***

exponentially with time. This is illustrated in Fig. ***2.5.*** From ***Eq, (2.24 b)*** we will also observe that the rate of decay of energy depends on the value of **b**; larger the value of **b**, faster will be the decay.

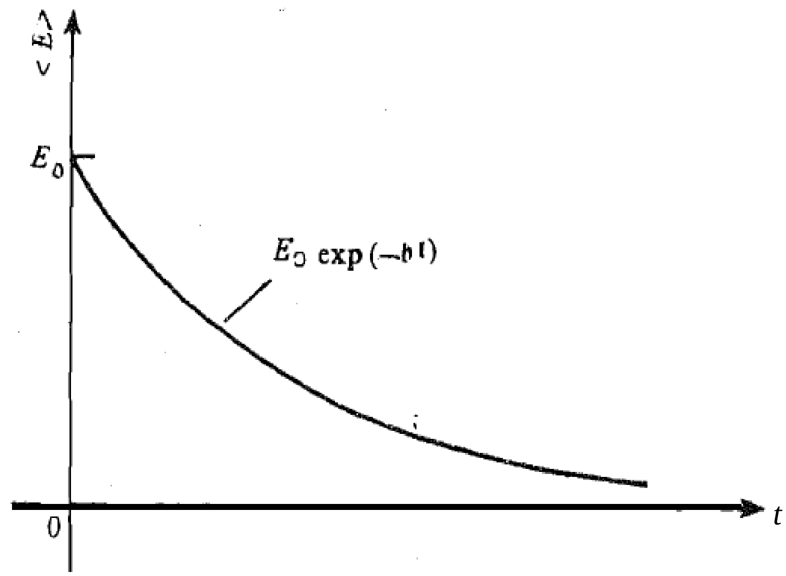


Fig. 2.5 Time variation of average energy for a weakly damped system.

2.4.1 AVERAGE POWER DISSIPATED OVER ONE CYCLE

Since energy of a damped oscillator does not remain constant in time $\left(\frac{dE}{dt}\right)$ not zero. In fact, it is negative. The rate of loss of energy at any time gives instantaneous power dissipated. From Eq. **(2.20)** we can write

$$\frac{dE}{dt} = P(t) = \left[m \frac{d^2x}{dt^2} + kx \right] \frac{dx}{dt}$$

On combining this result with Eq. ***(2.2)*** we find that power dissipated by a damped

oscillator is given by

$$P(t) = -\gamma\left(\frac{dx}{dt}\right)^2$$

This relation shows that the rate of doing work against the frictional force is directly proportional to the square of instantaneous velocity. On substituting for

$\left(\frac{dx}{dt}\right)$ from Eq. **(2.21),** we obtain

$$P(t) = -\gamma a_0^2 \exp(-2bt)\,[b^2\cos^2(\omega_d t+\phi) + \omega_d^2\sin^2(\omega_d t+\phi) + b\omega_d\sin(2\omega_d t+\phi)]$$

Hence, the average power dissipated over one cycle is given by

$$\begin{aligned} \langle P \rangle &= -\tfrac{1}{2}\gamma a_0^2\,\omega_0^2 \exp(-2bt) \\ &= -\frac{\gamma}{m}\langle E \rangle \\ &= -2b\langle E \rangle \end{aligned} \qquad \textbf{2.25}$$

The negative sign here signifies that power is dissipated.

2.5 METHODS OF CHARACTERISING DAMPED SYSTEMS

We now know that in the viscous damping model, a damped oscillator is characterized by γ and ω_o. We also know that this model applies to vastly different physical systems. Therefore, you may ask: Are there other ways of characterizing damped oscillations? Experience tells us that in certain cases it is more convenient to use other parameters to characterize damped motion. In all cases we can relate these to γ and ω_o . We will now discuss these briefly.

2.5.1. LOGARITHMIC DECREMENT

The most convenient way to determine the amount of damping present in a system is to measure the rate at which *amplitude* of oscillation dies away. Let us consider the damped vibration shown graphically in Fig. 2.6. Let a_0 and a_1 be the first two successive amplitudes of oscillation separated by one period.

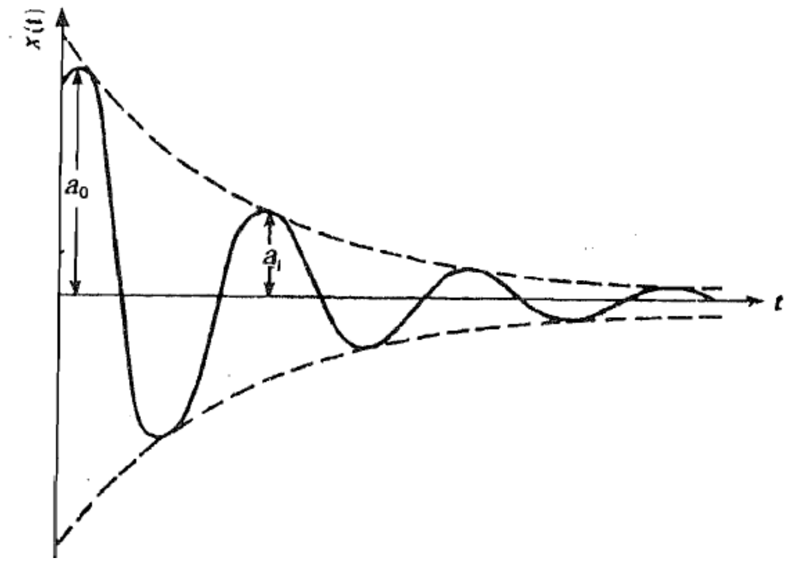


Fig. 2.6 A damped oscillation the first two amplitudes arc a_o and a_1.

We will note that these amplitudes lie in the same direction/quadrant. If T is the

period of oscillation, then using Eq. (2.19) for a weakly damped oscillator, we can

write

$$a_1 = a_0 \exp(-bT)$$

so that $$\frac{a_0}{a_1} = \exp(bT) = \exp(\gamma T/2m) \qquad \textbf{2.26}$$

We will note that in the ratio a_o / a_1 the larger amplitude is in the numerator. That is why this ratio is called the ***decrement.*** It is denoted by the symbol d. You may now ask: Is the decrement same for *any* two consecutive amplitudes? The answer is: yes, it is. To show this let us consider the ratio of the second and the third amplitudes. These are observed for ***t*** **= T** and ***t*** **= 2T**, respectively in Eq. (2.19). Then, we can write

$$\frac{a_1}{a_2} = \frac{a_0 \exp(-bT)}{a_0 \exp(-2bT)} = \exp(bT)$$

So, we may conclude that for any two consecutive amplitudes separated by one period, we have

$$\frac{a_{n-1}}{a_n} \doteq \exp(bT) = d \qquad \textbf{2.27}$$

That is, ***decrement is the same for two successive amplitudes*** and we can write

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_{n-1}}{a_n} = d \qquad \mathbf{2.28}$$

The logarithm of the ratio of successive amplitudes of oscillation separated by one period is called the logarithmic decrement. It is usually dcnoted by the symbol λ:

$$\lambda = \ln\left(\frac{a_{n-1}}{a_n}\right) = \frac{\gamma T}{2m} \qquad \mathbf{2.29a}$$

This equation shows that we can measure λ by knowing two successive amplitudes. But from an experimental point of view it is more convenient and accurate to compare amplitudes of oscillations separated by n periods. That is, we measure $\mathbf{a_o/a_n}$. To compute this ratio, we first invert Eq. (2.29 a) to write

$$\frac{a_{n-1}}{a_n} = \exp(\lambda) \qquad \mathbf{2.29b}$$

The ratio $\mathbf{a_o/a_n}$ can now be written as

$$\frac{a_0}{a_n} = \left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\left(\frac{a_2}{a_3}\right)\ldots\left(\frac{a_{n-1}}{a_n}\right) = [\exp(A)]''$$

$$= \exp(n\lambda) \qquad \mathbf{2.30}$$

Since the ratio of any two consecutive amplitudes is the same,

taking log of both sides, we get the required result:.

$$\lambda = \frac{1}{n}\ln\left(\frac{a_0}{a_n}\right) \qquad \mathbf{2.31}$$

This shows that if we plot **In** $\mathbf{(a_o/a_n)}$ versus n for different values of **n**, we will obtain a straight line. The slope of the line gives us λ.

2.5.2 RELAXATION TIME

In Physics we often measure decay of a quantity in terms of the fraction e^{-1} of the initial value. This gives us another way of expressing the damping effect by means of the time taken by the amplitude to decay to e^{-1}

= 0.368 of its original value. This time is called the relaxation time. To understand this, we recall that the amplitude of a damped oscillation is given by

$$a(t) = a_0 \exp(-bt)$$

If we denote the amplitude of oscillation after an interval of time τ by a (t + τ)

We can write

$$a(t+T) = a_0 \exp[-b(t+\tau)]$$

By taking the ratio **a (t + τ)/a(t)** we obtain

$$\frac{a(t+\tau)}{a(t)} = \exp(-b\tau)$$

$$= 0.368 \text{ for } b\tau = 1 \qquad 2.32$$

This shows that for $\mathbf{b = \tau^{-1}}$ the amplitude drops to **1/e = 0.368** of its initial value.

Using this result in Eq. **(2.25),** we get

$$\langle P \rangle = \frac{2\langle E \rangle}{\tau}$$

The relaxation time τ, is therefore a measure of the rapidity with which motion is

damped.

2.5.3 THE QUALITY FACTOR

Yet another way of expressing the damping effect is by means of the rate of decay of energy. From Eq. (2.24**b)** we note that the average energy of a weakly damped oscillator decays to $E_0 e^{-1}$ in time $t = \frac{1}{2b} = \frac{m}{\gamma}$ seconds. If ω_d is its angular frequency, then in this time the oscillator will vibrate through $\omega_d\, m/\gamma$ radians. *The number of radians through which a weakly damped system ostillates as its average energy decays to $E_0 e^{-1}$ is a measure of the quality factor, Q* :

$$Q = \frac{\omega_d m}{\gamma} = \frac{\omega_d}{2b} = \frac{\omega_d \tau}{2} \qquad 2.33$$

We will note that Q is only a number and has no dimensions. In general, **γ** is small so that Q is very large. A tuning fork has Q of a thousand or so, where as a rubber band exhibits a much lower (-10) Q. This is due to the internal friction generated by the coiling of the long chain of molecules in a rubber band. An undamped oscillator (**γ = 0**) has an infinite quality factor.

For a weakly damped mechanical oscillator, the quality factor can be expressed

in terms of the spring factor and damping constant. For weak damping,

$$\omega_d \cong \sqrt{\frac{K}{m}}$$

Hence $Q = \sqrt{\frac{km}{\gamma}}$

That is, the quality factor of a weakly damped oscillator is directly proportional to the square root of **k** and inversely proportional to **γ**.

We can rewrite Eq. (2.33) in a more physically meaningful form using Eq. (2.25)

$$Q = \frac{\omega_d}{2b} = \frac{2\pi}{T_d} \times \frac{\langle E \rangle}{\langle P \rangle} \qquad \textbf{2.34}$$

$$= 2\pi \; \frac{\text{average energy stored in the system in one cycle}}{\text{average energy lost in one cycle}}$$

The quality factor is related to the fractional change in the frequency of an undamped oscillator. To establish this relation, we note that

$$\omega_d = \sqrt{\omega_0^2 - b^2}$$

or

$$\frac{\omega_d^2}{\omega_0^2} = 1 - \frac{b^2}{\omega_0^2}$$

$$\cong \mathbf{1} - \frac{\mathbf{1}}{\mathbf{4Q^2}}$$

where we have used Eq. (2.33). This result can be rewritten as

$$\frac{\omega_d}{\omega_0} = \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

$$= 1 - \frac{1}{8Q^2}$$

where in the binomial expansion we have retained terms upto first order in $\boldsymbol{Q^2}$

Hence, the fractional change in $\boldsymbol{\omega_o}$ is $\mathbf{1/(8\,Q^2)}$.

2.6 EXAMPLES OF DAMPED SYSTEMS

We know that all harmonic oscillators in nature have some damping, which in general, is quite small. To enable you to appreciate the effect of damping, we will consider two specific cases: (i) Oscillations of charge in an ***LCR*** circuit, and (ii) motion of the coil in a suspension type galvanometer. These are of particular interest to us as the former has wide applications in radio engineering and the latter is used in the Physics laboratory.

2.6.1 AN LCR CIRCUIT

In Unit 1 we observed that in an ideal **LC** circuit, charge executes SHM. Do you expect any change in this behaviour when a resistor is added? To answer this question, we consider Fig. 2.7. If a current **I** flows through the circuit due to discharging/charging of the capacitor, the voltage drop across the resistor is ***RI***. Thus Eq. (1.36)

now modifies to

$$\frac{q}{C} = -L\frac{dI}{dt} \qquad \frac{dq}{dt} \qquad \mathbf{2.35}$$

Eq. (2.35) may be rewritten as

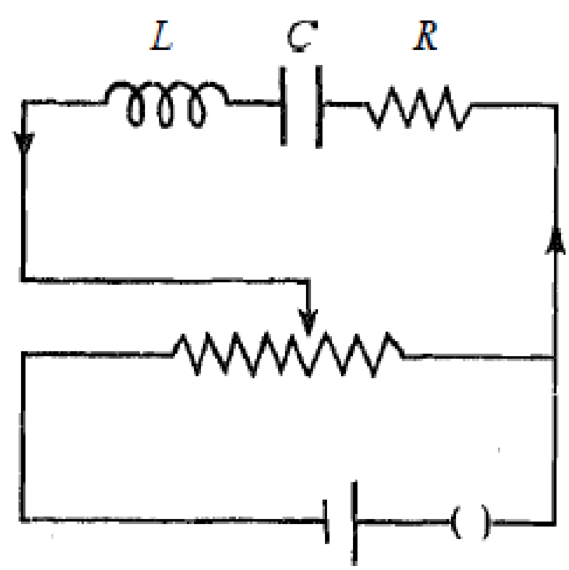


Fig. 2.7 An LCR circuit

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \qquad \textbf{2.36}$$

Comparing it with Eq. (2.2) we find that **L, R** and **1/ C** are respectively analogous to **m,** $\boldsymbol{\gamma}$ and **k**. This means that a resistor in an electric circuit has an exactly analogous effect as that of the viscous force in a mechanical system.

To proceed further, we divide Eq. (2.36) throughout by **L** obtaining

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}\, q = 0 \qquad \textbf{2.37}$$

In this form, Eq. (2.37) is analogous to **Eq.** (2.3) and the two may be compared

directly. This gives

$$\omega_0^2 = \frac{1}{LC}$$

$$\text{and} \qquad b = \frac{R}{2L} \qquad \textbf{2.38}$$

We know that b has dimensions of time inverse. This means that ***R / L*** has the unit of $\mathbf{s^{-1}}$, same as that of $\boldsymbol{\omega_o}$. That is why $\boldsymbol{\omega_o L}$ is measured in ohm.

With these analogles all the results of Section 2.3 apply to Eq. (2.37). For a weakly damped circuit, the charge on the capacitor plates at time t is

$$q(t) = q_0 \exp\left(-\frac{R}{2L}\,t\right)\cos(\omega_d t + \phi) \qquad \textbf{2.39a}$$

with angular frequency

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad \textbf{2.39b}$$

Eq. 2.39a shows that the charge amplitude $q_0 \exp\left(-\frac{R}{2L}\,t\right)$ will decay at a rate

which depends on the resistance. Thus in an ***LCR*** circuit, resistance is the only dissipative element; an increase in **R** increases the rate of decay of the charge and

decreases the frequency of oscillations.

When $1/LC >> R^2/4L$,

$$\omega_d^2 \cong \omega_0^2 = \frac{1}{LC}$$

or $$\omega_0 L = \frac{1}{\omega_0 C}$$

Since $\omega_0 L$, is measured in ohms, $1/\omega_0 C$ is also measured in ohms. These are respectively referred to as ***inductive reactance and capacitive reactance.***

For $R = 0$, **Eq. (2.39**a) reduces to **Eq. (1.38)** and $\omega_d = \omega_o$. The Q value of a

weakly damped *LCR* circuit is

$$Q = \frac{\omega_d}{2b} \simeq \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \textbf{2.40}$$

This equation shows that for a purely inductive circuit **(R = 0)** , quality factor will be infinite.

2.6.2 A SUSPENSION TYPE GALVANOMETER

A suspension type galvanometer consists of a current carrying coil suspended in a magnetic field. The field is produced by a horse-shoe magnet. The magnet is shaped so that the coil is aligned always along the magnetic lines of force. To ensure uniform strength, an iron cylinder is suspended between the poles of the magnet, as shown in Fig. **(2.8).** When we pass charge through the galvanometer coil, it rotates through some angle **θ.** Since the coil is mechanically a torsional pendulum, it experiences a restoring couple $-k_t\theta$ and a damping couple $-\gamma\frac{d\theta}{dt}$. Do you know how damping creeps in, in this case? It has origin in air friction and electromagnetic induction.

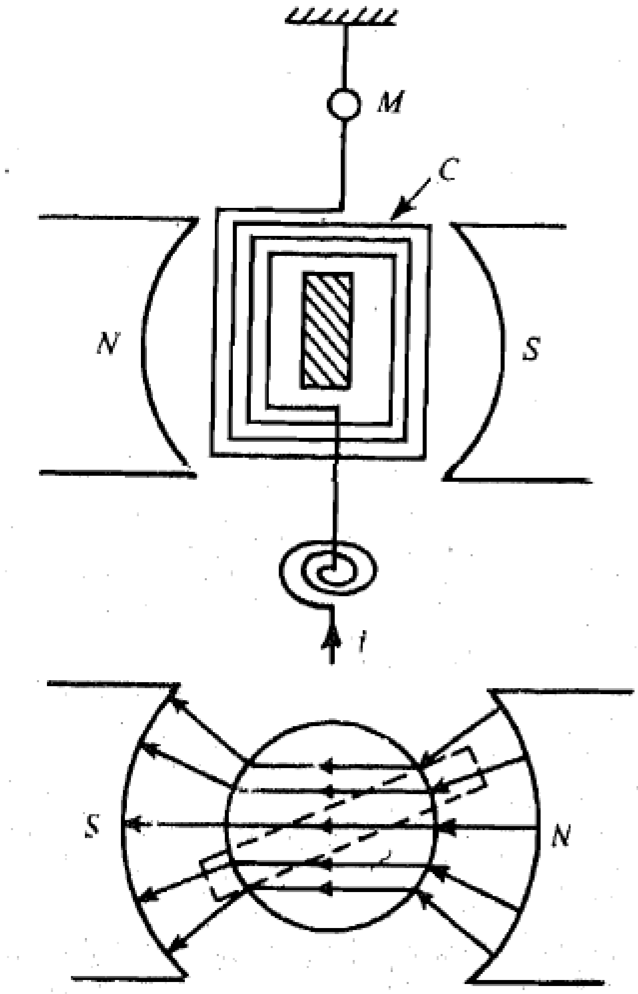


Fig. 2.8 A schematic representation of a suspension type galvanometer

Hence, for the motion of the coil, Eq**. (1.35)** modifies to

$$I \frac{d^2\theta}{dt^2} = - k_t\theta - \gamma \frac{d\theta}{dt}$$

2.41

where ***I*** is moment of inertia of the coil about the axis of suspension. Comparing it

with Eq. (3.2) we find that ***I*** and $\boldsymbol{k_t}$ are analogous to **m** and **k** respectively.

Dividing throughout by **I** and defining

$$2b = \gamma / I$$

and $$\omega_0^2 = k_t / I$$

2.42

we get $$\frac{d^2\theta}{dt^2} + 2b \frac{d\theta}{dt} + \omega_0^2 \theta = 0$$

2.43

This equation is of the same form as Eq. ***(2.3).*** Hence, all results deduced earlier will apply to the motion of the coil described by Eq. ***(2.43).***

For low damping, the solution of Eq. (2.43) is

$$\theta = \theta_0 \exp(-bt)\cos(\omega_d t + \phi) \qquad \textbf{2.44}$$

where θ_0 **exp (-** bt**)** is the amplitude of oscillation. Eq. ***(2.44)*** describes oscillatory

motion with the period of oscillation **T** given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left[\frac{k_t}{I} - \frac{\gamma^2}{4I^2}\right]^{1/2}} \qquad \textbf{2.45}$$

This explains why a weakly damped suspension type galvanometer is called a ***ballistic*** galvanometer. We will note that for damping to be small, we must decrease **γ** and increase **I**. The question now arises: How can we reduce **γ?** As mentioned earlier, air damping is usually small. Nevertheless, it will always be present. To reduce electromagnetic damping, we must minimise induced emf. To ensure this, we wind the coil over a nonconducting bamboo or ivory frame. If the frame is metallic, it is cut at one place, so that no current can flow through it.

The quality factor of a ballistic galvanometer is

$$Q = \frac{\omega_d}{2b} = \frac{I}{\gamma}\sqrt{\frac{k_t}{I} - \frac{\gamma^2}{4I^2}} \qquad \textbf{2.46a}$$

If $\frac{k_t}{I} >> \frac{\gamma^2}{4I^2}$, this expression reduces to

$$Q = \sqrt{\frac{k_t I}{\gamma^2}} \qquad \textbf{2.46b}$$

This relation shows that a lightly damped suspension type galvanometer will have

high quality factor.

2.7 SUMMARY

In the present unit, we have studied about different effect of damping, and also discuss the differential equation of a damped harmonic oscillator. We have also discuss relaxation time Logarithmic decrement and quality factor.

2.8 TERMINAL QUESTIONS

1. Obtain differtial equation of a damped oscillator.
2. Explain the concept of Heavy damping and critical damping
3. Discuss average energy of a weakly damped oscillator.
4. Define Light damping.
5. What is Relaxation time?

UNIT-03 FORCED OSCILLATOR

Structure

3.1 INTRODUCTION

In the previous unit we studied how the presence of damping affects the amplitude and the frequency of oscillation of a system. However, in systems, such as a wall clock or an ideal LC circuit, oscillations do not seem to die out. To maintain oscillations we have to feed energy to the system from an external agent called a ***driver.*** In general, the frequencies of the driver and the driven system may not match. But in Steady State, irrespective of its natural frequency, the system oscillates with the frequency of the applied periodic force. Such oscillations are called ***forced oscillations***. When the frequency of the driving force exactly matches the natural frequency of the vibrating system a spectacular effect is observed, the amplitude of forced oscillations becomes very large and we say that ***resonance*** occurs. Do you know that Galileo was the first physicist who understood how and why resonance occurs?

Resonances are desirable in many mechanical and molecular phenomena. Butresonance can be disastrous also; it can literally break an oscillating system apart. For instance, fast blowing wind may set a suspension bridge in oscillation. If the frequency of the fluctuating force produced by the wind matches the natural frequency of the bridge, it gains in amplitude and may ultimately collapse. In 1940, the Tacoma Narrows bridge in Washington State collapsed within 4 months of its being opened. Similarly, when the army marches on a suspension bridge, soldiers are instructed to / break step to avoid resonant vibrations. In practice, isolated systems are rare. In solid state and molecular physics, two or more systems are coupled through interatomic forces. In an electric circuit we have inductive and capacitive couplings. The oscillations of such systems will be studied in the next unit.

In this unit we shall study, in detail, the response of a system when it is driven by an external harmonic force.

OBJECTIVES :

After studying this unit, you should be able to –

- Define Forced oscillator
- Write down differential equation for a weekly damped forced oscillator
- Understand the concept of Transient solution steady state solution
- Explain resonance frequency, low driving frequency
- Define Quality factor

3.2 DIFFERENTIAL EQUATION FOR A WEAKLY DAMPED FORCED OSCILLATOR

To establish the differential equation of a forced weakly damped harmonic oscillator, let us again consider the spring-mass system**.** It is now also subjected to an external driving force, ***F(t).*** That is, instead of allowing the model oscillator to oscillate at its natural frequency, we push it back and forth periodically at a frequency **ω** (Fig. 3.1). We can write the driving force as

$$F(t) = F_0 \cos\omega t \qquad 3.1$$

where $\mathbf{F_O}$ is a constant

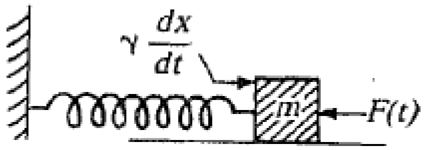


Fig. 3.1 A weakly damped forced spring-mass system.

Let the mass be displaced from its equilibrium position and then released. At any instant, it is subject to (i) a restoring force, $-kx$, (ii) a damping force, $-\gamma \frac{dx}{dt}$ and (iii) a driving force, $F_0 \cos \omega t$.

So for a forced oscillator Eq. **(2.2)** is modified to

$$m \frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} + F_0 \cos \omega t \qquad \mathbf{3.2}$$

Dividing by *m* and rearranging terms, the equation of motion of a forced oscillator

takes the form

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t \qquad \mathbf{3.3}$$

where $2b = \gamma/m$, $\omega_0^2 = k/m$ and $f_0 = F_0/m$ is a measure of the driving force.

This equation can be apply to any oscillator whose natural frequency is ω_o and is subject to a harmonic driving force. You will note that Eq. **(3.3)** is an inhomogeneous second order linear differential equation with constant coefficients. We will now solve this equation to learn about the motion of a forced oscillator.

3.3 SOLUTIONS OF THE DIFFERENTIAL EQUATION

Before we solve Eq. **(3.3),** let us analyses the situation physically. From the previous unit, i.e. under damped oscillation, when there is no applied force, a weakly damed system **(** $b < \omega_0$ **)** oscillates harmonically with angular frequency $\omega_d = \sqrt{\omega_0^2 - b^2}$. But when a driving force of angular frequency ω is applied, it imposes its own frequency on the oscillator. Thus, we expect that the actual motion will be the result of superposition of two oscillations; one of frequency ω_d (of damped oscillations) and the other of frequency ω (of the driving force). Thus, when $\omega \neq \omega_0$ the general solution of Eq. **(3.3)** can be written as,

$$x(t) = x_1(t) + x_z(t)$$

where $\mathbf{x_1}$ ***(t)*** is *a* solution of the equation obtained by replacing the RHS of Eq. (3.3) by zero.

On substituting this result in Eq. (3.3), we will find that $\mathbf{x_2}$ ***(t)*** satisfies the equation

$$\frac{d^2x_2}{dt^2} + 2b\frac{dx_2}{dt} + \omega_0^2 x_2(t) = f_0 \cos \omega t$$

It is thus clear that $(\mathbf{x_1} + \mathbf{x_2})$ is the complete solution of Eq. (3.3). In your course on differential equations you must have learnt that $\mathbf{x_1}$ is called the *complementary function* and $\mathbf{x_2}$ is called the *particular integral.*

You may recall that when there is no driving force, the displacement of a weakly damped ***(*** $b<\omega_o$ ***)*** system at any instant is given by Eq. **(2.19)**

$$x_1(t) = a_0 e^{-bt} \cos(\omega_d t + \phi) \qquad \text{(A)}$$

3.3.1 TRANSIENT SOLUTION

Equation (A) this complementary function decays exponentially and after some time it will disappear. That is why it is also referred to as the ***transient solution***. In the transient state, the system oscillates with some frequency which is other than its natural frequency or the frequency of the driving force.

3.3.2 STEADY STATE

After a sufficiently long time ($t >> \tau$), natural oscillations of the spring mass system will disappear due to damping. However, we know that the general solution of Eq. **(3.3)** will not decay with time. That is, the system wiil oscillate with the frequency of the driving force. The system is then said to be in the ***steady-state.*** We will now obtain the steady-state solution of Eq. **(3.3).**

3.3.3 STEADY-STATE SOLUTION

To obtain the steady state solution of Eq. **(3.3)**, let us suppose that the displacement of the forced oscillator is given by

$$x_2(t) = a \cos(\omega t - \theta) \qquad \textbf{3.4}$$

where a and θ are unknown constants. By comparing Eqs. **(3.1)** and **(3.4)** you will

note that the driving force leads the displacement in phase by an angle θ.

To determine ***a*** and **θ** we differentiate Eq. **(3.4)** twice with respect to time. This gives,

$$dx_2 = a\omega \sin(\omega t - \theta)$$

and $$\frac{d^2x_2}{dr^2} = a\omega^2 \cos(\omega t - \theta)$$

Substituting these results back in Eq. **(3.3),** we get

$$\left[(\omega_0^2 - \omega^2)a\cos\theta + 2ab\omega\sin\theta - f_0\right]\cos\omega t$$

$$+\left[\omega_0^2 - \omega^2\, a\sin\theta - 2\,ab\omega\cos\theta\right]\sin\omega t = 0 \qquad \textbf{3.5}$$

We know, that both **cosωt** and **sinωt** never simultaneously become zero. When one vanishes, the other takes a maximum value. Therefore, Eq. **(3.5)** can be satisfied only when both terms within the square brackets become zero separately, **i.e.**

$$\left(\omega_0^2 - \omega^2\right)a\cos\theta + 2ab\omega\sin\theta = f_0 \qquad \textbf{3.6a}$$

and $$\left(\omega_0^2 - \omega^2\, a\sin\theta - 2\,ab\omega\cos\theta = 0\right) \qquad \textbf{3.6b}$$

Eq**. (3.6 b)** readily gives the phase by which the driving force leads the displacement:

$$\theta = \tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2} \qquad \textbf{3.7a}$$

The amplitude of steady-state displacement can be determined from Eq. **(3.6a)** once we know the values of **sinθ** and **cos θ**. To get these values we construct the so-called caustic impedance triangle, as shown in Fig. 3.2. We can write

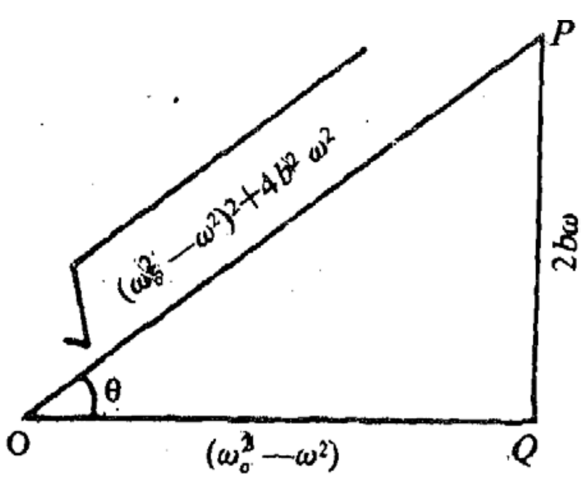


Fig. 3.2 An acoustic impedance triangle

$$\sin\theta = \frac{2b\omega}{\left[\left(\omega_0^2-\omega^2\right)^2+4b^2\omega^2\right]^{1/2}}$$

and $$\cos\theta = \frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}$$

Using these values of **sin θ** and **cos θ** in Eq. **(3.6 a)** and rearranging terms, we get

$$a = \frac{f_0}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}} = \frac{F_0}{m\,[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}} \qquad \textbf{3.7b}$$

Thus, we find that the steady-state amplitude of forced oscillations depends on (i) amplitude and angular frequency of the driving force, (ii) mass and the natural angular frequency of the oscillating system and (iii) the damping constant. Putting this value of **a** in Eq. **(3.4)** we can write the steady-state solution of Eq**.(3.3)** as

$$x_2(t) = \frac{F_0}{m\,[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}\cos(\omega t-\theta) \qquad \textbf{3.8}$$

The important point to note here is that the steady-state solution has the frequency of the driving force and its amplitude is constant. Moreover, its phase is also defined completely with respect to the driving force. Therefore, it does not depend on the initial conditions. In other words, the motion of a driven system in steady-state is independent of the way we start the oscillation.

The transient solution, steady-state solution and their sum,

$$x(t) = a_0e^{-bt}\cos(\omega_d t+\phi) + \frac{F_0\cos(\omega t-\theta)}{m\,[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}$$

i.e. the complete general solution of Eq**. (3.3)** are shown in Fig. 3.3. The contribution, of the transient part diminishes with time and ultimately disappears completely. The time for which transients persist is determined by ***b*** and hence by the damping factor **γ.** The greater the value of ***b,*** more quickly do the transients die out.

For an undamped system, the steady-state solution is obtained by putting **b = 0**

in Eqs **(3.7a)** and **(3.8).** This gives

$$\theta = 0$$

and $$x_2(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \qquad \textbf{3.9}$$

That is, the driving force and the displacement are in the same phase (θ = **0).** From this we may conclude that phase lag is essentially a consequence of damping. We further note that if the frequency of the driving force equals the frequency of the undamped oscillator, its amplitude will become infinitely large. Then **resonance** ***is*** said to occur. You may now ask: Do we observe infinitely large amplitude in practice? No, the amplitude is finite since some damping is always present in every system.

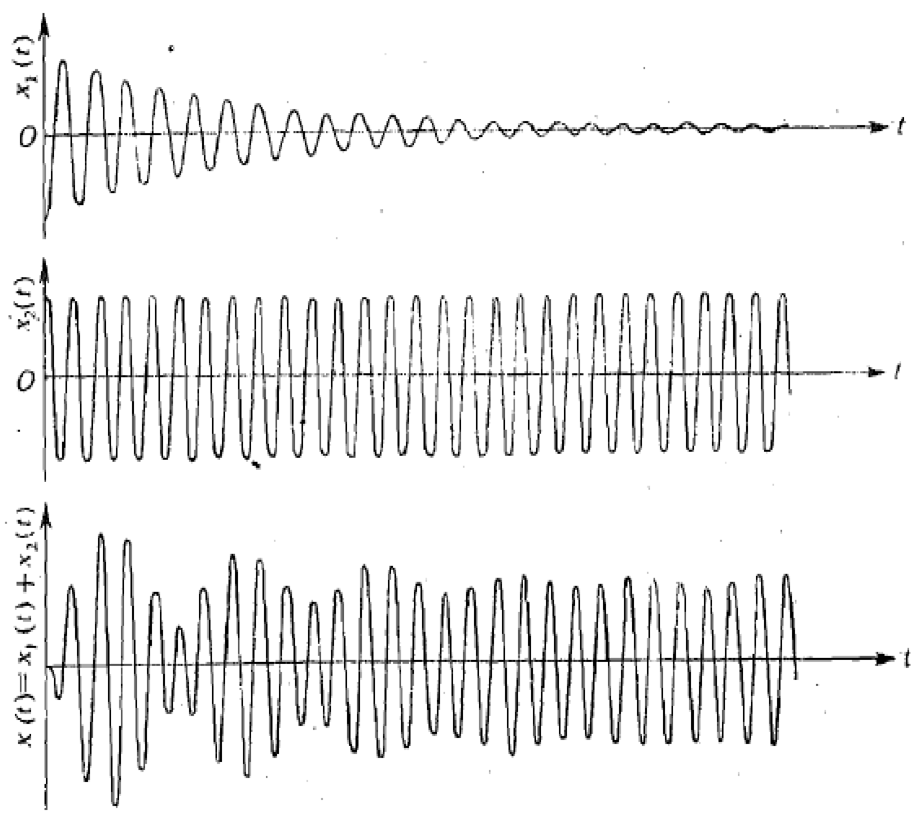


Fig. 3.3 Time variation of the transient solution, steady-state solution and the general solution of Eq. (3.3) for a weakly damped system.

3.4 EFFECT OF THE FREQUENCY OF THE DRIVING FORCE ON THE AMPLITUDE AND PHASE OF STEADY-STATE FORCED OSCILLATIONS

We know that the variation with the frequency of the driving force of the steady-state amplitude ***a (ω)*** of a forced system is given by Eq**.**

(3.7b). Depending on the relative magnitudes of the natural and the driving frequencies, three cases arise. We will now discuss these separately in detail.

3.4.1 LOW DRIVING FREQUENCY *(Ω << Ω$_O$)*

To know the behaviour of ***a (ω)*** at low driving frequencies, we first rewrite

Eq. **(3.7b)** as

$$a(\omega) = \frac{f_0}{\omega_0^2\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{4b^2\omega^2}{\omega_0^4}\right]^{1/2}}$$

For $\omega << \omega_0$, we note that the ratio ω^2/ω_0^2 will be much less than one.

So, we neglect terms containing ω^2/ω_0^2. This gives

$$a(\omega) = \frac{f_0}{\omega_0^2} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \quad \textbf{3.10a}$$

Thus, at very lo; driving frequencies, the steady-state amplitude of the oscillation is controlled by the stiffness constant and the magnitude of the driving force.

Under this condition Eq. **(3.7**a) yields

$$\tan\theta = \frac{2bw}{\omega_0^2 - w^2} \longrightarrow 0 \text{ for } \frac{\omega}{\omega_0} << 1. \quad \textbf{3.10b}$$

That is, the driving force and the steady-state displacement are in the same phase**.**

3.4.2 RESONANCE FREQUENCY (Ω =Ω$_O$)

To calculate the value of ***a (ω)*** at resonance, we set **ω =ω₀** in Eq. **(3.7b).** The first

term in the denominator vanishes and the amplitude is given by

$$a(\omega_0) = \frac{f_0}{2b\omega_0} \quad \textbf{3.11a}$$

From this we note that at resonance the amplitude depends upon the damping; it is inversely proportional to *b.* That is why in actual practice the amplitude never become infinite.

Similarly by setting $\omega = \omega_0$ in Eq. (3.7a) we find that

$$\tan\theta \rightarrow \infty$$

so that

$$\theta = \pi/2 \qquad \textbf{3.11b}$$

This means that the driving force and the displacement are out of phase by ***π/2.*** You may be thinking that the value of **a** (ω_0) given by Eq. **(3.11*a)*** is maximum. This however is not true. Why? To answer this, let us maximize **a** (ω)**.** That is, differentiate Eq. **(3.7 b)** with respect to ω and set the resulting expression equal to zero. The frequency at which the first derivative becomes zero and the second derivative is negative gives the correct answer :

$$\frac{da(\omega)}{d\omega} = \frac{d}{d\omega}\left[\frac{f_0}{[(\omega_0^2-\omega^2)^2+4\,b^2\omega^2]^{1/2}}\right]$$

$$= -\frac{f_0[-4\omega(\omega_0^2-\omega^2)+8b^2\omega]}{2[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{3/2}} = 0 \quad \text{for } \omega = \omega_r$$

This equality will hold only when the numerator vanishes identically, i.e.

$$-4\omega(\omega_0^2-\omega^2)+8b^2\omega = 0 \quad \text{for } \omega = \omega_r$$

We ignore the root $\omega_r = 0$, which is trivial. Then we must have

$$\omega_r^2 - \omega_0^2 + 2b^2 = 0$$

This equation is quadratic in ω_r, and the acceptable root is

$$\omega_r = (\omega_0^2 - 2b^2)^{1/2} \qquad \textbf{3.12}$$

The root corresponding to the negative sign is physically meaningless and is ignored. For **a** (ω) to be maximum, its second derivative with respect to ω should be negative.

You can easily verify that at

$$\omega_r = (\omega_0^2 - 2b^2)^{1/2}, \text{ and } \left.\frac{d^2a}{d\omega^2}\right|_{\omega=\omega_r} \text{ is negative.}$$

Thus, we can conclude that the peak value of amplitude is attained at a frequency slightly below ω_0. The shift is caused due to damping. We can visualize it as follows: When the driver imparts maximum push, the driven system does not accept it instantly due to a finite phase difference between ***x(t)*** and ***F(t).***

On substituting for ω_0 from Eq. (3.12) in Eq. (3.7b) and simplifying the resulting expression, we get the peak value of steady-state amplitude:

$$a_{max} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}}$$ **3.13**

When at a particular frequency, the amplitude of the driven system becomes

maximum, we say that ***amplitude resonance*** occurs. The frequency ω_r is referred to as the ***resonance frequency.*** It is instructive to note that ω_r is less than ω_0 as well as $\omega_d = \sqrt{\omega_0^2 - b^2}$.

3.4.3 HIGH DRIVING FREQUENCY ($\Omega \gg \Omega_O$)

For $\omega \gg \omega_0$ we rewrite Eq. **(3.7b)** as

$$a(\omega) = \frac{f_0}{\omega^2\left[\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 + \frac{4b^2}{\omega^2}\right]^{1/2}}$$

and neglect terms containing ω_0^2/ω^2 as well as $(2b/\omega)^2$, as they are both much s

than unity. Then the amplitude of resulting vibration is given by

$$a(\omega) = \frac{f_0}{\omega^2}$$ **3.14a**

That is, at high frequencies the amplitude decreases as $1/\omega^2$ and ultimately becomes zero.

Similarly from Eq. (3.7a), the phase is given by

$$\tan\theta = \frac{2b\omega}{(\omega_0^2 - \omega^2)} \sim -\frac{2h}{\omega} \xrightarrow[\omega \to \infty]{} 0$$

or $\theta = \pi.$ **3.14b**

This means that at high frequencies the driving force and displacement are out of phase by π.

We may thus conclude that

(i) The amplitude of oscillation in steady-state varies with frequency. It becomes maximum at $\omega_r = \sqrt{\omega_0^2 - 2b^2}$ and has value $f_0/2b\sqrt{(\omega_0^2 - b^2)}$.

For $\omega > \omega_r$, $a(\omega)$ decreases as ω^{-2}.

(ii) 'The displacement lags behind the driving force by an angle θ, which increases from zero at $\omega = 0$ to π at extremely high frequencies. At $\omega = wo$, $\theta = \pi/2$.

The frequency dependence of ***a(ω)*** and ***θ***(ω) is shown in Fig. 3.4.

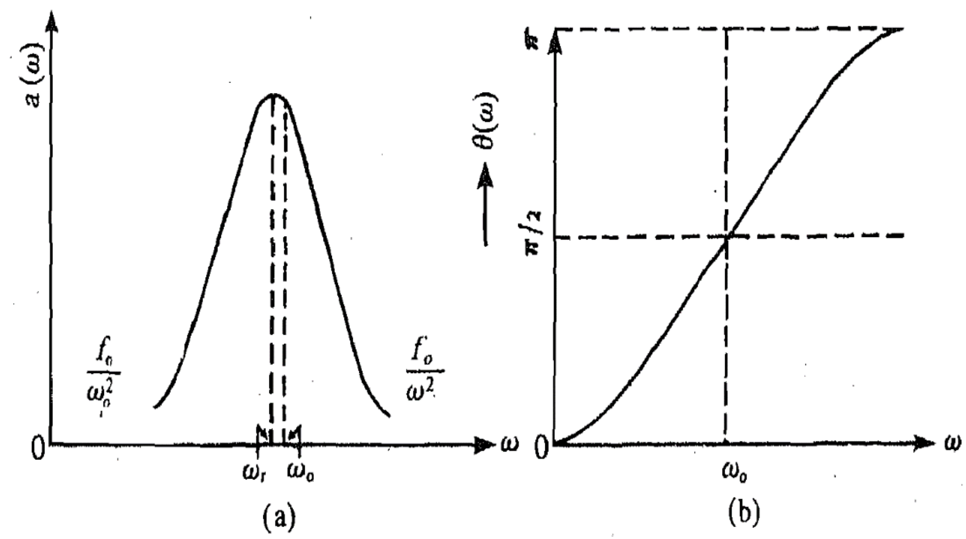


Fig. 3.4 Frequency variation of (a) Steady state Amplitude, (b) Phase of a Forced oscillator.

3.5 POWER ABSORBED BY A FORCED OSCILLATOR

You now know that every oscillating system loses energy in doing work against damping. But oscillations of a forced oscillator are maintained by the energy supplied by the driving force. It is, therefore, important to know the average rate at which energy must be supplied to the system to sustain steady-state oscillations. So, we now calculate the average power absorbed by the oscillating system.

By definition, the instantaneous power is given by

$$P(t) = \text{force} \times \text{velocity}$$
$$= F(t) \times v$$

Differentiating Eq. (3.8) with respect to time, we get

$$V = -v_0 \sin(wt - 0) = v_0 \cos(\omega t - \phi) \qquad \textbf{3.15}$$

where

$$v_0 = \frac{f_0 \omega}{m\,[\omega_0^2 - \omega^2)^2 + 4b^2\,\omega^2]^{1/2}} \qquad \textbf{3.15a}$$

is the velocity amplitude and

$$\phi = \theta - \pi/2 \qquad \textbf{3.15b}$$

is the phase difference between velocity and the applied force. On substituting for

F(t) and **v** from Eqs. (3.1) and (3.15), respectively and find that the instantaneous

power absorbed by the oscillator is given by

$$P(t) = F_0 v_0 \cos \omega t \cos (wt - \phi)$$

Since $\cos (wt - \phi) = \cos \omega t \cos 4 + \sin \omega t \sin \phi$, we can rewrite the expression for instantaneous power as

$$P(t) = F_0 v_0 [\cos^2 wt \cos 9 + \cos wt \sin \omega t \sin \phi]$$

From this we can easily calculate the average power absorbed over one cycle:

$$\langle P \rangle = \frac{1}{2} F_0 v_0 \sin \phi \langle \sin 2 \omega t \rangle + F_0 v_0 \cos \phi \langle \cos^2 \omega t \rangle \qquad \textbf{(3.16)}$$

From Unit 1 you may recall that **<sin 2ωt> = 0** so that the first term on the **RHS** of

Eq. **(3.16)** drops out. Also **<cos² ωt> = 1/ 2**. Then Eq. **(3.16)** reduces to

$$\langle P \rangle = \frac{1}{2} F_0 v_0 \cos '4 = \frac{1}{2} F_0 v_0 \sin \theta \qquad \textbf{3.17}$$

On substituting for **sin θ** from Fig. 3.2 and v_o from Eq. **(3.15a)** in Eq. **(3.17),** we get

$$\langle P \rangle = \left(\frac{bF_0^2}{m} \right) \frac{\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4b^2 \omega^2]} \qquad \textbf{3.18}$$

From Eq. (3.17) we note that the average power absorbed by a forced oscillator will ,be maximum when **sin θ = 1 = cos θ i.e. ϕ = π/2 (ϕ = 0)**.This happens for **ω = ω_o** .Using this result in Eq. **(3.18)**, we get

$$\langle P \rangle_{max} = \frac{1}{4bm} F_0^2 \qquad \textbf{3.19}$$

That is, the peak value of average power absorbed by a maintained system is determined by damping, and the amplitude of the driving force. The frequency variation of **<P>** is shown. in Fig. 3.5

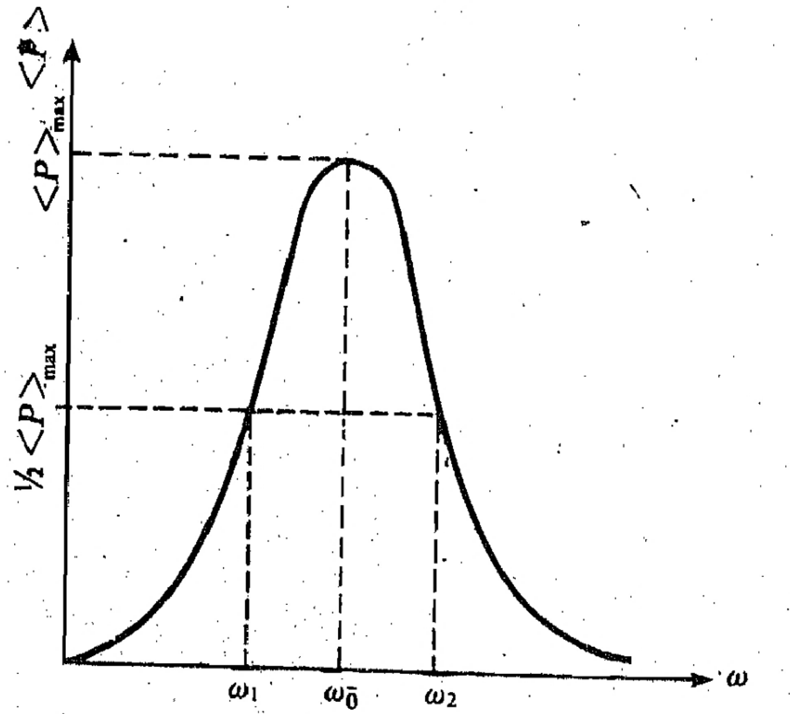


Fig. 4.5 Frequency variation of average Power. ω_1 **and** ω_2 **correspond to half-power points.**

It is important to note that unlike the case of amplitude resonance, maximum average power is transferred at the natural frequency of ,the system. This arises because velocity and driving force are in phase.

3.6 QUALITY FACTOR

We defined the quality factor of a damped oscillator as

$$Q = 2\pi \frac{\text{Average Energy Stored in one cycle}}{\text{Average Energy Dissipated in one cycle}}$$

You can use the same definition to calculate Q of a forced oscillator once you know **<*E*>** and **<*P*>.**

Now we show that the average energy of a forced oscillator is

$$<E> = \frac{1}{4} m (\omega^2 + \omega_0^2) a^2$$

and the quality factor is given by

$$Q = \frac{\omega^2 + \omega_0^2}{4b\omega}$$

$$\langle K.E \rangle = (1/2)\, m \langle v^2 \rangle$$
$$= (1/2)\, m\, w^2 a^2 \langle \sin^2 (\omega t - \theta) \rangle = (1/4)\, m\omega^2 a^2$$
$$\langle U \rangle = (1/2)\, k \langle x^2 \rangle$$
$$= (1/2)\, m\omega_0^2\, a^2 \langle \cos^2 (\omega t - \theta) \rangle = (1/4)\, m\omega_0^2 a^2$$

Therefore, time-averaged energy is

$$\langle E \rangle = (1/4)\, m\, (\omega^2 + \omega_0^2)\, a^2$$

Average energy dissipated per second is $\langle \gamma\, v^2 \rangle = mb\omega^2 a^2$

By definition,

$$Q = 2\pi \frac{\text{Average energy stored in one cycle}}{\text{Average energy dissipated in one cycle}}$$

$$= 2\pi \frac{\text{Average energy stored in one cycle}}{\text{Time period X Average energy dissipated in one second}}$$

$$= 2\pi \frac{m(\omega_0^2 + \omega^2)a^2}{4\, Tmb\omega^2 a^2}$$

$$= \frac{\omega_0^2 + \omega^2}{4\omega b} \qquad \left(\because\ \omega = \frac{2\pi}{T} \right)$$

Another equivalent and more useful interpretation of the quality factor is in terms of amplitudes. The Q factor is defined as the ratio of the amplitude at resonance to the amplitude at low frequencies $(\omega \to 0)$. Using this definition, the value of the quality factor can be calculated rather easily on dividing Eq. (3.13) by Eq. (3.10a).

$$Q = \frac{a_{max}}{a(\omega \to 0)} = \frac{f_0}{2b(\omega_0^2 - b^2)^{1/2}} \times \frac{\omega_0^2}{f_0} = \frac{\omega_0^2}{2b(\omega_0^2 - b^2)^{1/2}}$$ **3.20a**

If damping is small, $b^2 \ll \omega_0^2$ and the expression for the quality factor reduces to

$$Q = \frac{\omega_0}{2b} = \frac{\omega_0 \tau}{2}$$ **3.20b**

winch is the same as Eq. (2.33) with b = 0.

Using Eq. **(3.20b)**, we show that the amplitude and phase of a weakly damped forced oscillator can be expressed as

$$a(\omega) = a_0 \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

and $$\tan\theta = \frac{1/Q}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)}$$

where $a_0 = F_0/m\omega_0^2$.

From Eq. (3.7b) we recall that amplitude of a weakly damped forced oscillator is

given by

$$a(\omega) = \frac{F_0}{m[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$

$$= \frac{F_0/m}{\omega\omega_0\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{4b^2}{\omega_0^2}\right]^{1/2}}$$

If we put $a_0 = \frac{F_0}{m\omega_0^2}$ and use Eq. **(3.20)**, we get the required result

$$a(\omega) = a_0 \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Similarly, from Eq. **(3.7 a)** we recall that

$$\tan\theta = \frac{2b\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{2b\omega}{\omega\omega_0\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)}$$

or $$\tan\theta = \frac{1/Q}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)}$$

For different values of **Q,** frequency variation of **a (ω)** and **θ(ω)** based on these equations is shown in Fig.3.6. We observe that as **Q** increases (i.e., damping decreases), the value of **a (ω)** increases.

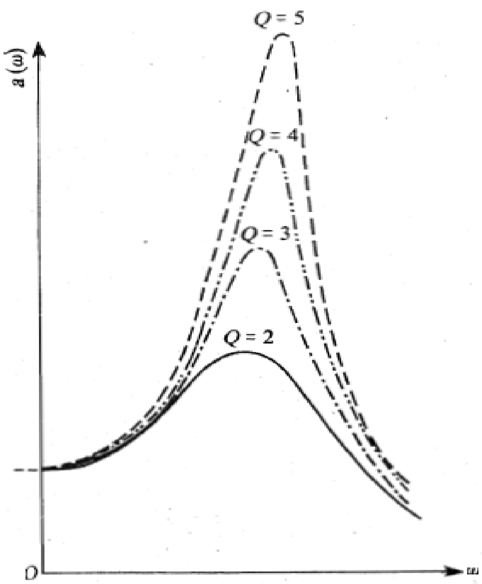


Fig. 3.6 (a) Amplitude as a function of driving frequency for different values of Q, (b) Phase difference *θ* as a function of driving frequency for different values of Q.

Now we show that

$$\langle P\rangle = \frac{1}{2}\frac{F_0^2\,\omega_0}{k}Q$$

From Eq. (3.18)

$$\langle P\rangle = \frac{bF_0^2}{m}\frac{\omega^2}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]}$$

Putting $Q = \frac{\omega_0}{2b}$, **we get**

$$\langle P\rangle = \frac{\omega_0 F_0^2}{2mQ}\left[\frac{\omega^2}{(\omega_0^2-\omega^2)^2+(\omega^2\omega_0^2/Q^2)}\right]$$

At $\omega = \omega_o$ the denominator in the parentheses will become minimum and the I average power absorbed by the oscillator becomes maximum

$$\langle P \rangle_{max} = \frac{F_0^2 Q}{2m\omega_0}$$

$$= \frac{1}{2} \frac{F_0^2 \omega_0 Q}{m \omega_0^2}$$

$$= \frac{1}{2} \frac{F_0^2 \omega_0 Q}{k}$$

3.6.1 Q IN TERMS OF BAND WIDTH: SHARPNESS OF A RESONANCE

The ***Q*** of a system can also be defined as

$$Q = \frac{\text{Frequency at which power resonance occurs}}{\text{Full width at half-power points}} \quad \textbf{3.21}$$

To calculate the frequency at which average power drops to half its maximum value we can write

$$\frac{1}{2} \frac{F_0^2 \omega_0}{kQ} \frac{\omega_0^2 \omega^2}{\left[(\omega_0^2 - \omega^2)^2 + \frac{\omega_0^2 \omega^2}{Q^2} \right]} = \frac{1}{4} \frac{F_0^2 \omega_0 Q}{k}$$

On simplification we can write

$$(\omega_0^2 - \omega^2)^2 = \frac{\omega^2 \omega_0^2}{Q^2}$$

so that

$$(\omega_0^2 - \omega^2) = \pm \frac{\omega \omega_0}{Q}$$

This equation has 4 roots. Of these two roots correspond to negative frequencies and are physically unacceptable. The other two acceptable roots are

$$\omega_1 = -\frac{\omega_0}{2Q} + \omega_0 \left(1 + \frac{1}{4Q^2} \right)^{1/2}$$

and $$\omega_2 = \frac{\omega_0}{2Q} + \omega_0 \left(1 + \frac{1}{4Q^2} \right)^{1/2} \quad \textbf{3.22}$$

Obviously, the second of these roots is greater than ω_0 and the other root is smaller than ω_0 This is illustrated in Fig. **3.5.**

The frequency interval between two half-power points is

$$\omega_2 - \omega_1 = 2\Delta\omega = \frac{\omega_0}{Q} \quad \textbf{3.23}$$

From Eq. **(3.23)** it is clear that a high **Q** system has small band width and the resonance is said to be sharp. On the other hand, a low **Q** system has a large band width and the resonance is said to be flat. This is illustrated in Fig.3.6. Thus, the sharpness of resonance refers to the rapid rate of the fali of power with frequency on either side of resonance. We measure it in terms of the Q-value of the system. The Q factor has its greatest importance in reference to electrical circuits which we will discuss now.

3.7 AN LCR CIRCUIT

We have so far discussed the resonant behaviour of a simple mechanical system subject to a periodic force. Another physical system which also exhibits resonant behaviour is a series **LCR** circuit containing a source of alternating **e.m.f.** We will discuss the behaviour of this system by drawing similarities with a mechanical system.

From Unit 2 we know that in an **LCR** circuit charge oscillations die out because of power losses in the resistance. What changes do you expect in this behaviour when a source of alternating e.m.f. of frequency ω is introduced? To answer this question, let us consider Fig. 3.7. Let **I** be the current in the circuit at a given time. Then, the applied EMF is equal to the sum of the potential differences across the capacitor, resistor and the inductor, Then Eq. **(2.35)** modifies to

$$\frac{q}{C} + RI + L\frac{dI}{dt} = E_0 \cos \omega t \quad \textbf{3.24}$$

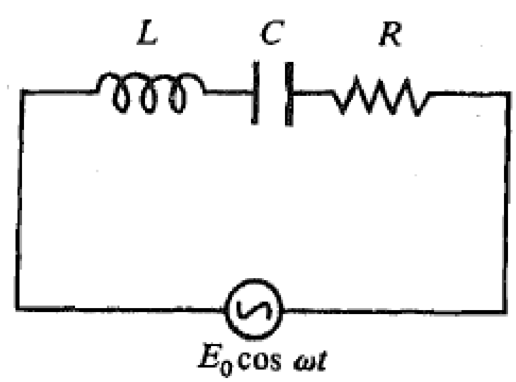


Fig. 3.7 A harmonically driven LCR circuit

Since $I = \frac{dq}{dt}$, this equation can be rewritten as

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E_0 \cos \omega t$$

3.25

Dividing throughout by **L**, we get

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L} \cos \omega t$$

3.26

In this form Eq. (3.26) is similar to Eq. (3.3). Hence its steady-state solution can be

written by analogy. For a weakly damped system, the charge on capacitor plates at

any instant of time is given by

$$q = \frac{E_0/L}{\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{\omega R}{L}\right)^2\right]^{1/2}} \cos(\omega t - \theta)$$

3.27

where $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ is the angular frequency of oscillation and

$$\tan\theta = \frac{(\omega R/L)}{\frac{1}{LC} - \omega^2}$$

3.28

defines the phase with respect to the applied EMF.

The current in the circuit is obtained by differentiating Eq. **(3.27)** with respect to ***t.***

The result is

$$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi)$$

3.29

where $\phi = \theta - \pi/2$ is the phase difference between E_0 and I. Since

$$\tan\phi = -\cot\theta = -\frac{\frac{1}{LC} - \omega^2}{\omega R/L}$$

we find that

$$\phi = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

3.30

From **Eq.(3.29)** we note that current in a **LCR** circuit is a function of the frequency.

When ***ωL*** **=1/ωC** , the Circuit is capacitive in nature and we can write

$$\left(\omega L - \frac{1}{\omega C}\right)^2 \simeq \frac{1}{\omega^2 C^2}$$

Thus, if we are working at low frequencies and *R* is also small, the current amplitude will be small. What will be its magnitude for ***ω - 0****?* In this limit ***I - 0*** and leads the applied EMF by ***π/ 2.***

As the driving frequency increases, the reactance $\left(\omega L - \frac{1}{\omega C}\right)$ decreases and currentamplitude increases. When

$$\omega L = \frac{1}{\omega C}$$

3.31

the term under the radical sign in Eq. **(3.29)** becomes minimum; equal to R. Then the current attains its peak value $\mathbf{I_0 = E_0 / R}$ and the circuit is said to resonate with

frequency

$$\nu_r = \frac{1}{2\pi\sqrt{LC}}$$

3.32

At resonance, the current and applied **e.m.f.** are in phase. When the driving frequency is high, the circuit will be inductive and the current lags behind EMF by **π/2.**

For different values of R, the frequency variation of peak current and phase is shown in Fig. 3.8. You will observe that lower the resistance, higher is the peak value of the current and sharper is the resonance.

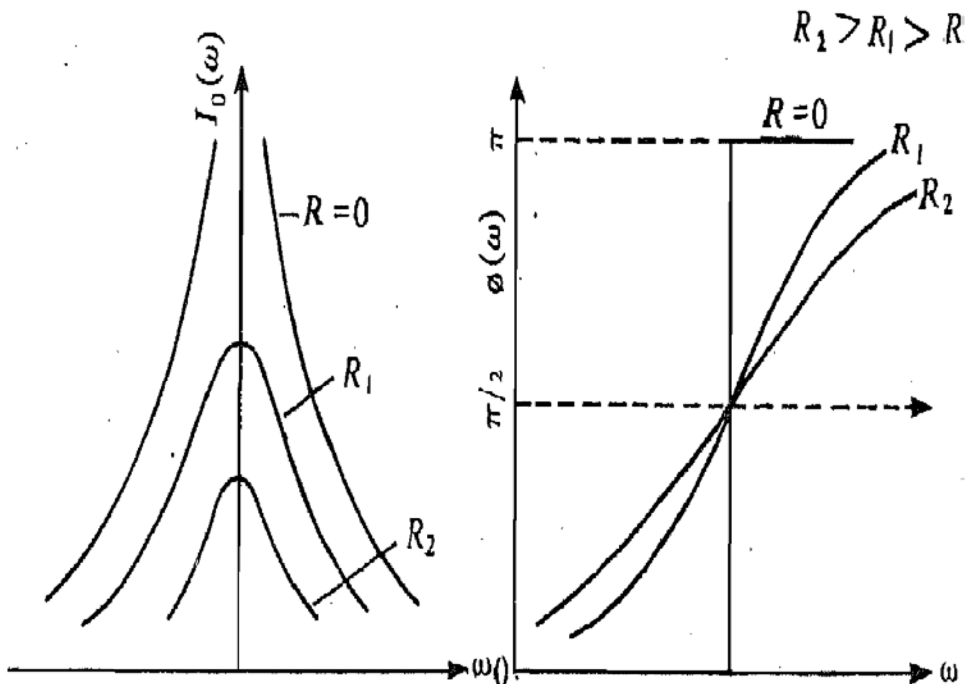


Fig. 3.8 Frequency variation of peak current and phase for different values of R in a driven *LCR* circuit.

The power in an electric circuit is defined as the product of current and EMF. For an **LCR** circuit, we can write

$$P = EI = E_0 I_0 \cos \omega t \cos (wt - \phi)$$

where $I_0 = E_0 \Big/ \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}$

Using the formula $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$, we can rewrite the above expression for power as

$$P = \frac{E_0 I_0}{2} [\cos 4 + \cos (2\omega t - \phi)]$$

Power averaged over one complete cycle is obtained by noting that

$< \cos (2wt - \phi) > = 0$, Hence

$$\langle P \rangle = \frac{E_0 I_0}{2} \cos\phi$$

$$= E_{rms} I_{rms} \cos\phi \qquad \textbf{3.33}$$

where $E_{rms} = E_0/\sqrt{2}$ and $I_{rms} = I_0/\sqrt{2}$ are, respectively the root mean square values of e.m.f. and current. Since $\langle P \rangle$ varies with $\cos\phi$, it is customary to call $\cos\phi$ as the power factor.

The quality factor of an LCR circuit is given by

$$Q = \frac{\omega_0 L}{R} \qquad \textbf{3.34}$$

where $\omega_0 = 1/\sqrt{LC}$

You can verify that the band width of power resonance curve for an ***LCR*** circuit is

given by

$$\omega_2 - \omega_1 = \frac{2}{\tau} = \frac{R}{L} = \frac{\omega_0}{Q} \qquad \textbf{3.35}$$

so that

$$Q = \frac{\text{Frequency at resonance}}{\text{Full width at half-power points}}$$

The ***Q*** of a circuit determines its ability to select a narrow band of frequencies from a wide range of input frequencies. This, therefore, acquires particular importance in relation to radio receivers. Signals of various frequencies from all stations are present around the antenna. But the receiver selects just one particular station to which we wish to tune and discard others. Normally radio receivers operating in **MHz** region have Q values of the order of $\mathbf{10^2}$ to $\mathbf{10^3}$**.** Microwave cavities have Q values of the order of $\mathbf{10^5}$.

3.8 SUMMARY

Free Oscillation

The free oscillation possesses constant amplitude and period without any external force to set the oscillation. Ideally, free oscillation does not undergo damping. But in all-natural systems damping is observed unless and until any constant external force is supplied to overcome damping. In such a system, the amplitude, frequency, and energy all remain constant.

Damped Oscillation

The damping is a resistance offered to the oscillation. The oscillation that fades with time is called damped oscillation. Due to damping, the amplitude of oscillation reduces with time. Reduction in amplitude is a result of energy loss from the system in overcoming external forces like friction or air resistance and other resistive forces. Thus, with the decrease in amplitude, the energy of the system also keeps decreasing. There are two types of damping

- Natural Damping
- Artificial Damping

Forced Oscillation

When a body oscillates by being influenced by an external periodic force, it is called forced oscillation. Here, the amplitude of oscillation, experiences damping but remains constant due to the external energy supplied to the system.

The Q factor (quality factor) of a resonator is a measure of the strength of the damping of its oscillations, or for the relative linewidth. The term was originally developed for electronic circuits, e.g. LC circuits, and for microwave **cavities**, but later also became common in the context of **optical resonators**.

The Q factor is 2π times the ratio of the stored energy to the energy dissipated per oscillation cycle, or equivalently the ratio of the stored energy to the energy dissipated per radian of the oscillation. For a microwave or **optical resonator**, one oscillation cycle is understood as corresponding to the field oscillation period, not the round-trip period

Q Factor of an Oscillator

The term Q factor is sometimes also applied to continuously operating oscillators, such as active **optical frequency standards**. In that case, only the definition via the bandwidth can be used; the bandwidth is then the **linewidth** of the output signal.

If the oscillator is based on some resonator (which is virtually always the case), the effective *Q* factor of the oscillator may deviate substantially from the intrinsic *Q* value of the resonator. Particularly measurements on atomic transitions (such as in a cesium atomic clock) have a limited measurement time, so that the effective linewidth of the reference transition is increased. (This problem can be severe for cesium clocks; cesium fountain clocks represent a significant advance towards longer measurement times.) On the other hand, a carefully stabilized oscillator can have a linewidth which is a tiny fraction of the linewidth of the underlying frequency standard; for cesium atom clocks, the quartz oscillator is often stabilized e.g. to a millionth of the linewidth of the signal from the cesium beam apparatus. Effectively, the good short-term stability of the quartz oscillator is combined with the high accuracy and low long-term drift of the cesium apparatus.

3.9 TERMINAL QUESTIONS

1. Define Forced oscillator

2. Discuss Deferential equation for a weekly damped forced oscillator.

3. Write short notes on:

   (a) Steady – State Solution

   (b) Resonance frequency

4. What is quality factor?

5. If x = a conwt + b sinwt, show that it represents SHM. Also find the amplitude of SHM.

UNIT-04 COUPLED OSCILLATOR

Structure

4.1 INTRODUCTION

In this unit you have studied isolated (single) oscillating systems such as a spring-mass system, a pendulum or a torsional oscillator. In nature we aiso come across many examples of coupled oscillators. We know that atoms in a solid are coupled by interatomic forces. In molecules, say the water molecule, two hydrogen? toms are coupled to an oxygen atom while in a carbon dioxide molecule oxygen atoms are coupled to one carbon atom. In all these cases, oscillators of one atom are affected by the presence of other atom(s). In radio and TV transmission, we use electrical circuits with inductive capacitive couplings. Therefore, it is important to extend our study of preceding units to cases where such simple systems are coupled.

OBJECTIVES

After studying this unit, you should be able to –

- Understand the concept of oscillation in two waves
- Explain about Lissajous figures
- Define energy of two coupled masses
- Write down the differential equation of coupled massses

4.2 DEFINE OSCILLATIONS IN TWO DIMENSIONS

We have confined our discussion to harmonic oscillations in one dimension. But oscillatory motion in two dimensions is also possible. Most familiar example is the motion of a simple pendulum whose bob is free to swing in any direction in the x - y plane. (We call this arrangement a spherical pendulum.) We displace the pendulum in the x-direction and as we release it, we give it an impulse in the y-direction. What happens when such a pendulum oscillates? The result is a composite motion **whose** maximum x-displacement occurs when y-displacement is zero and y-velocity is maximum and vice versa. Remember that since the time period of the pendulum depends only on acceleration due to gravity and the length of the cord, the frequency of the superposed **SHM's** will be the same. The result is a curved path, in general, an ellipse. We now apply the principle of superposition to the case where two harmonic oscillations are perpendicular to each other.

4.2.1 SUPERPOSITION OF TWO MUTUALLY PERPENDICULAR HARMONIC OSCILLATIONS OF THE SAME FREQUENCY

Consider two mutually perpendicular oscillations having amplitudes $\mathbf{a_1}$ and $\mathbf{a_2}$ such

that $\mathbf{a_1} > \mathbf{a_2}$ and angular frequency ω_o. These are described by equations

$$x = a_1 \cos \omega_0 t \qquad \textbf{4.1}$$

and $$y = a_2 \cos (\omega_0 t + \phi) \qquad \textbf{4.2}$$

Here we have taken the initial phase of the vibrations along the x and the y-axes to

be zero and ϕ respectively. That is, ϕ is the phase difference between the two vibrations.

We shall first find out the resultant oscillation for a few particular values of phase difference ϕ .

Case I. $\phi = 0$ or π.

For $\phi = 0$

and $x = a_1 \cos \omega_0 t$

$y = a_2 \cos \omega_0 t$

Therefore

$y/x = a_2/a_1$

or $y = (a_2/a_1)\, x$ **4.3**

Similarly, for $\phi = \pi$

$x = a_1 \cos \omega_0 t$

and $y = -a_2 \cos \omega_0 t$

so that

$$y = -(a_2/a_1)\, x \quad \textbf{4.4}$$

Eqs. 4.3 and 4.4 describe straight lines passing through the origin. This means .

that the resultant motion of the particle is along a straight line. However, for ϕ = **0**

the motion is along one diagonal (PR in Fig. 4.1a) but when ϕ = π the motion is

along the other diagonal (QS in Fig. 4.1b).

Case II. $\phi = \pi/2$

In this case the two vibrations are given by

$x = a_1 \cos \omega_0 t$

and $y = a_2 \cos(\omega_0 t + \pi/2) = -a_2 \sin \omega_0 t.$

On squaring these expressions and adding the resultant expressions, we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = \cos^2\phi + \sin^2\phi = 1 \quad \textbf{4.5}$$

This is the equation of an ellipse. Thus the resultant motion of the, particle is along an ellipse whose principal axes lie along the ***x***- and the y-axes. The semi-major and semi-minor axes of the ellipse are a_1 and a_2. Note that as time increases x decreases from its maximum positive value but **y** becomes more and more negative. Thus the ellipse is described in the clockwise direction as shown in Fig. 4.1c. We analyse the case when ϕ = **3π/ 2 or** ϕ = **- π /2**, we will obtain the same ellipse. But the motion will be in anticlockwise direction (Fig. **4.1d).**

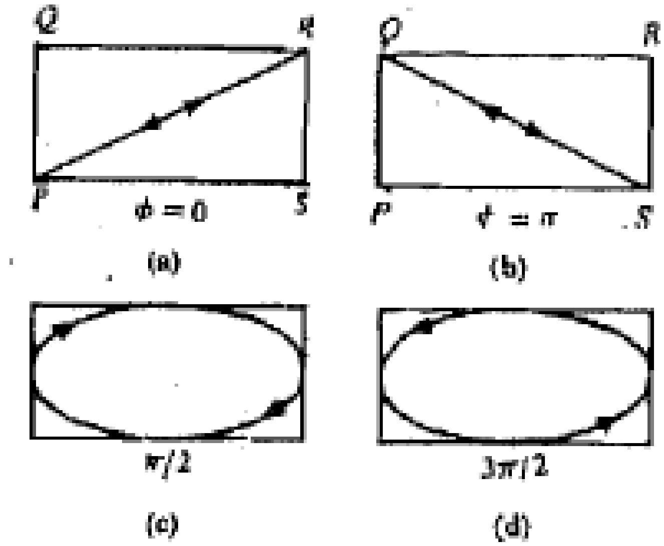


Fig. 4.1 Superposition of two mutually perpendicular harmonic oscillations having the same frequency different phases.

When amplitudes a_1 and a_2 are equal, i.e, $a_1 = a_2 = a$, Eq. (4.5) reduces to

$$x^2 + y^2 = a^2$$

This equation represents a circle of radius **a**. This means that the ellipse reduces to a circle.

4.2.2 SUPERPOSITION OF TWO RECTANGULAR HARMONIC OSCILLATIONS OF NEARLY EQUAL FREQUENCIES : LISSAJOUS FIGURES

We now know that when two orthonormal vibrations have exactly the same frequency, the shape of the curve traced out by the resultant oscillation depends on the phase difference between component vibrations. For a few values of the phase difference ϕ in the range 0 to 2π radian, these curves are shown in Fig. 4.1. When the two individual rectangular vibrations are of slightly different frequencies, the resulting motion is more complex. This is because the relative phase $[\phi = \omega_2 t + \phi_0 - \omega_1 t = (\omega_2 - \omega_1) t + \phi_0]$ of the two vibrations gradually changes with time. This makes the shape of the figure to undergo a slow change. If the amplitudes of vibrations are $\mathbf{a_1}$ and $\mathbf{a_2}$, respectively, then the resulting figure always lies in a rectangle of sides $\mathbf{2a_1}$ and $\mathbf{2a_2}$. The patterns which are traced out are called Lissajous figures. When the two vibrations are in the same phase, i.e. $\phi = 0$, the Lissajous figure reduces to a straight line and coincides with the diagonal $\mathbf{y = (a_2/a_1) x}$ of the rectangle. As ϕ changes from 0 to $\pi/2$ the Lissajous figure is an ellipse and passes through oblique positions in the rectangle. When ϕ

increases from π/2 to π, the ellipse closes into a straight line which coincides with the (other) diagonal **$y = (a_2/a_1)\,x$** of the rectangle. Further, as ϕ changes from π to 2 π, the series of changes mentioned above take place in the reverse order. In general, the shape of curve depends on the amplitudes, frequencies and the phase difference. All these changes are shown in Fig 4.2.

The phase ϕ changes by 2π in the time intervial **$2\pi/(\omega_2 - \omega_1)$.** Therefore, the period of the complete cycle of changes is **$2\pi/(\omega_2 - \omega_1)$** and its

frequency is $\frac{\omega_2 - \omega_1}{2\pi} = \nu_1 - \nu_2$ i.e., equal to the difference of the

frequencies of individual vibrations.

Lissajous figures can be illustrated easily by means of a cathode ray oscilloscope (CRO). Different alternating sinusoidal voltages are applied at XX and YY deflection plates of the CRO. The electron beam traces the resultant effect on the flourescent screen. When the applied voltages have the same frequency, we can obtain various curves of Fig. 4.2 by adjusting the phases and amplitudes.

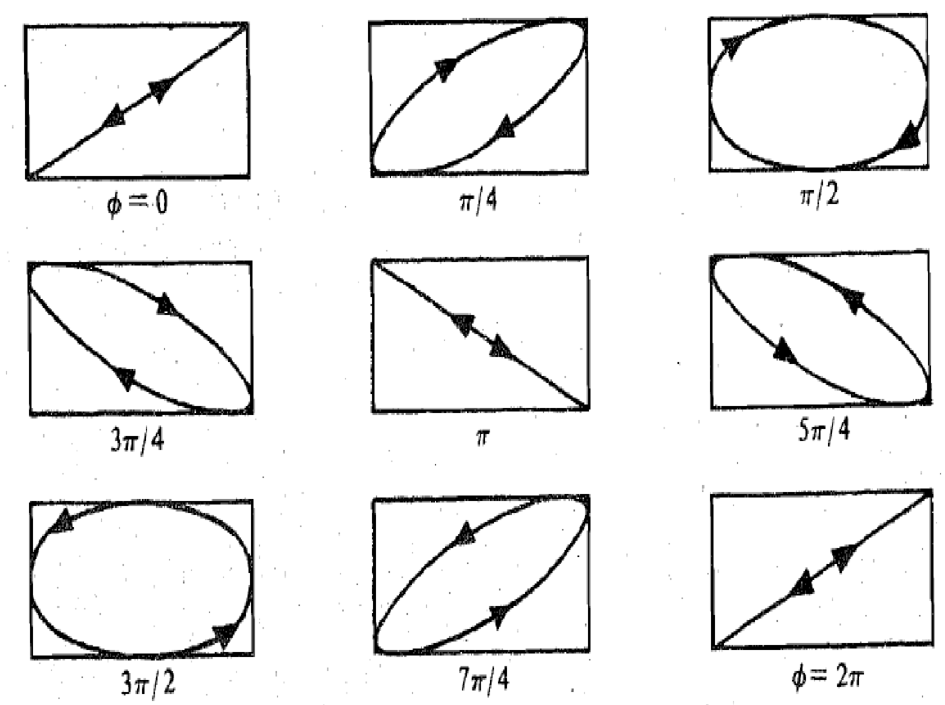


Fig. 4.2 Superposition of two mutually perpendicular harmonic oscillations or same frequency and having values of ϕ lying between 0 and 2π.

If the frequenqies of individual perpendicular vibrations are in the ratio 2: 1, the

Lissajous figures are relatively complex. It has the shape of parabola for ϕ = $\mathbf{0}$ or $\boldsymbol{\pi}$

and for ϕ = $\boldsymbol{\pi/2}$ its shape is that of figure '8'. To clarify this let us study the following example .

Two rectangular harmonic vibrations having frequencies in the ratio 2: 1 are

represented as follows :

$$\boldsymbol{x = a_1 \cos(2\omega_0 t + 4)} \qquad \mathbf{4.6}$$

$$\boldsymbol{y = a_2 \cos \omega_0 t} \qquad \mathbf{4.7}$$

We will calculate the resultant motion for ϕ = **0, π/2 and π**

(i) When $\phi = 0,\ x = a_1 \cos 2\omega_0 t = a_1 (2 \cos^2 \omega_0 t - 1)$

and $y = a_2 \cos \omega_0 t$

Since $\frac{y}{a_2} = \cos \omega_0 t$, we can rewrite the above equation as

$$\frac{x}{a_1} = \frac{2y^2}{a_2^2} - 1$$

On rearranging term, we get

$$y^2 = \frac{a_2^2}{2a_1}(x + a_1) \qquad \mathbf{4.8}$$

This equation represents a parabola (Fig. 4.3 a)

(ii) When $\phi = \frac{\pi}{2}$

$$x = -a_1 \sin 2\omega_0 t$$

or $-\frac{x}{a_1} = 2 \sin \omega_0 t \cos \omega_0 t$

and $y = a_2 \cos \omega_0 t$

Since we can write

$$\cos \omega_0 t = \frac{y}{a^2}$$

$$\sin \omega_0 t = \sqrt{1 - \frac{y^2}{a_2^2}}$$

the first of these equations reduces to

$$-\frac{x}{a_1} = \frac{2y}{a_2}\sqrt{1-\frac{y^2}{a_2^2}}$$

On squaring and rearranging teems, we get

$$\frac{4y^2}{a_2^2}\left(\frac{y^2}{a_2^2}-1\right)+\frac{x^2}{a_1^2}=0$$

which represents figure '8' in shape (Fig. 4.3b),

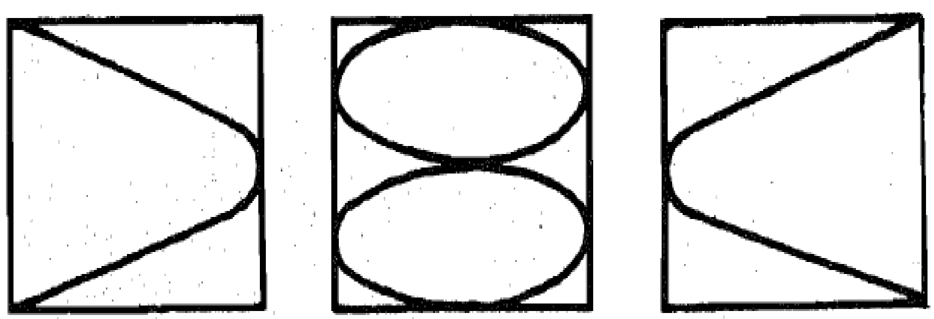
(iii) When $\phi = \pi$

$$x = -a_1 \cos 2\omega_0 t$$

$$-\frac{x}{a_1} = 2\cos^1 \omega_0 t - 1$$

$$y = a_2 \cos \omega_0 t$$

On combining these equations, we get



$\phi = 0$ (a) $\phi = \pi/2$ (b) $\phi = \pi$ (c)

Fig. 4.3 Superposition of two harmonic oscillations having frequencies in the ratio 2: 1 and phase difference (a) $\phi = 0$ (b) $\phi = \pi/2$ (c) $\phi = \pi$

$$\frac{2y^2}{a_2^2} = -\frac{x}{a_1} + 1$$

$$y^2 = -\frac{a_2^2}{2a_1}(x - a_1)$$

This represents a parabola which is oppositely directed to the case when ϕ **= 0.**

(Fig. 4.3c)

4.3 OSCILLATIONS OF TWO COUPLED MASSES

To analyse the effect of coupling we start again with the model spring-mass system. We consider two such identical systems connected (coupled) by a spring, as shown in Fig. (4.4a). In this system we have two equal masses attached to springs of stiffness constant ***k'*** and coupled to each other by a spring of stiffness constant ***k.*** In the equilibrium position, springs do not exert any force on either mass. The motion of this system will depend on the initial conditions. That is, the motion may be transverse or longitudinal depending on how the masses are disturbed. For simplicity, we first consider longitudinal motion of these two coupled masses.

We pull one of the masses longitudinally and then release it. The restoring force will tend to bring it back to its equilibrium position. As it overshoots the equilibrium mark, the coupling spring will pull the other mass. As a result both masses start oscillating longitudinally. This means that motion imparted to one of the two coupled masses is not confined to it only; it is transmitted to the other mass as well. We now establish the equation of motion of these masses.

4.3.1 THE DIFFERENTIAL EQUATION OF COUPLED MASSES

We choose x-axis along the length of the spring with **O** as the origin (Fig. 4.4a).

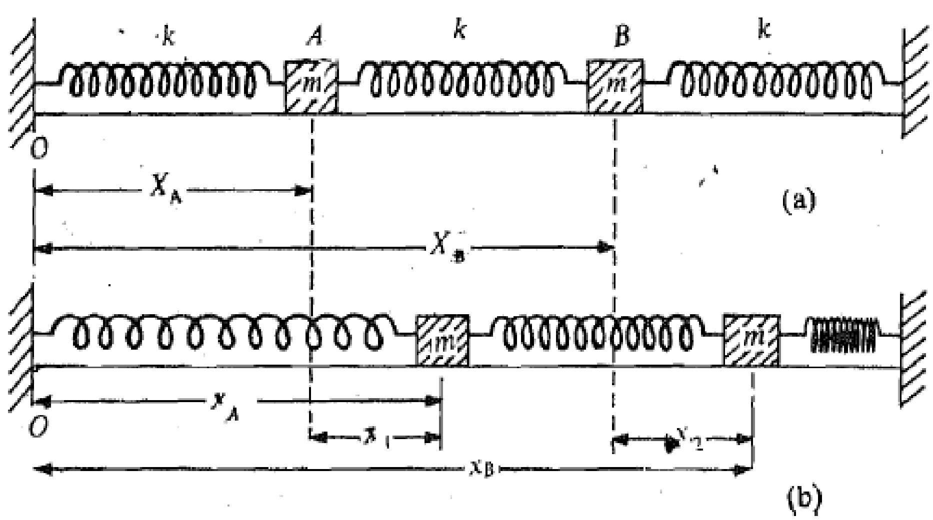


Fig. 4.4 Longitudinal oscillations of two coupled masses. (a) Equilibrium configuration (b) Configuration at time t

Let X_A and X_B be the coordinates of the centre of the masses *A* and B respectively.

When mass B is displaced towards the right and then released, mass *A* will also get

pulled towards the right due to the coupling spring. The coupled system would then start oscillating. Suppose x_A and x_B are the instantaneous positions of masses *A* and *B* respectively. Then their displacements from their respective equilibrium positions are given by

$$x_2 = x_B - X_B \text{ and } x_1 = x_A - X_A$$

Now at any instant of time during oscillation, the forces acting on mass **A** are

(i) restoring force : $-k'(x_A - X_A) = -k'x_1$; and
(ii) a coupling force : $k(x_B - x_A) - (X_B - X_A) = k(x_2 - x_1)$

We are here assuming that the masses are moving on a frictionless surface. By Newton's second law, the equation of motion of mass *A* is thus given by

$$m\frac{d^2x_A}{dt^2} = -k'(x_A - X_A) + k[x_B - X_B) - x_A - X_A]$$

$$m\frac{d^2(x_A - X_A)}{dt^2} = m\frac{d^2x_1}{dt^2} = -k'x_1 + k(x_2 - x_1) \quad \textbf{4.9}$$

Dividing throughout by m and rearranging terms, we get

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 - \omega_s^2(x_2 - x_1) = 0 \quad \textbf{4.10}$$

where $\omega_0^2 = \frac{k'}{m}$ and $\omega_s^2 = \frac{k}{m}$

Similarly the equation of motion of the mass ***B*** is

$$m\frac{d^2x_2}{dt^2} = -k'x_2 - k(x_2 - x_1) \quad \textbf{4.11}$$

This can also be rewritten as

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 + \omega_s^2(x_2 - x_1) = 0 \quad \textbf{4.12}$$

Let us pause for a minute and ask, Do Eqs. ***(4.10)*** and ***(4.12)*** represent simple harmonic motion? No, we cannot, in general, identify the motion described by these equations as simple harmonic because of the presence of the coupling term $\omega_t^2 (x_2 - x_1)$. This means that the analysis of previous units will not work since these equations are coupled in $\mathbf{x_1}$ and $\mathbf{x_2}$. The question now arises: How to solve these equations? These equations will have to be solved simultaneously. For this purpose we first add Eqs. **(4.10)** and **(4.12)** to obtain

$$\frac{d^2}{dt^2}(x_1 + x_2) + \omega_0^2 (x_1 + x_2) = 0 \quad \textbf{4.13a}$$

Next we subtract Eq. ***(4.12)*** from Eq. ***(4.10)*** and rearrange terms: This gives

$$\frac{d^2}{dt^2}(x_1 - x_2) + (\omega_0^2 + 2\omega_s^2)(x_1 - x_2) = 0 \quad \textbf{4.13b}$$

By looking at Eqs. (4.13a) and (4.13b) you will recognise that these are standard

equations for SHM. This suggests that if we.introduce two new variables defined as

$$\xi_1 = x_1 + x_2 \quad \textbf{4.14a}$$

$$\xi_2 = x_1 - x_2 \quad \textbf{4.14b}$$

the motion of a coupled system can be described in terms of two uncoupled and

independent equations:

$$\frac{d^2\xi_1}{dt^2} + \omega_1^2\xi_1 = 0 \quad \textbf{4.15}$$

$$\frac{d^2\xi_2}{dt^2} + \omega_2^2\xi_2 = 0 \quad \textbf{4.16}$$

where we have put

$$\omega_1^2 = \omega_0^2 = k'/m \quad \textbf{4.17}$$

$$\omega_2^2 = \omega_0^2 + 2\omega_s^2 = \frac{k' + 2k}{m} \quad \textbf{4.18}$$

We therefore find that new co-ordinates ξ_1 and ξ_2 have decoupled Eqs, (4.10) and (4.12) into two independent equations which describe simple harmonic motions of frequencies ω_1 and ω_2 and $\omega_2 > \omega_1$. The new coordinates are referred to as normal coordinates and simple harmonic motion associated with each coordinate is called a ***normal mode*** . Each normal mode has its own characteristic frequency called the normal ***mode frequency.***

4.3.2 NORMAL COORDINATES AND NORMAL MODES

The normal coordinates ξ_1 and ξ_2 are not a measure of displacement like ordinary co-ordinates x_1 and x_2 .e t they specify the configuration of a coupled system at any instant of time. Using the analysis of Unit **1,** you can readily write the general solution of Eqs. (4.15) and (4.16) as

$$\xi_1(t) = a_1 \cos(\omega_1 t + \phi_1) \qquad \textbf{4.19}$$

$$\xi_2(t) = a_2 \cos(\omega_2 t + \phi_2) \qquad \textbf{4.20}$$

where a_1 and a_2 are the amplitudes of normal modes and, ϕ_1 and ϕ_2 are their initial

phases.

Since $x_1(t) = (\xi_1 + \xi_2)/2$, we can write the displacement of mass A as

$$x_1(t) = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)] \qquad \textbf{4.21}$$

Similarly, we can write the displacement of the mass ***B*** as

$$x_2(t) = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) - a_2 \cos(\omega_2 t + \phi_2)] \qquad \textbf{4.22}$$

The constants a_1 , a_2, ϕ_1 , ϕ_2 are fixed by the initial conditions. Once we know these, we can completely determine the motion of the coupled masses. Now we

solve Eqs. (4.21) and (4.22) subject to the following initial conditions:

(A) $\quad x_1(0) = a,\ x_2(0) = a,\quad \left.\frac{dx_1}{dt}\right|_{t=0} = 0, \text{ and } \left.\frac{dx_2}{dt}\right|_{t=0} = 0$

(B) $\quad x_1(0) = a,\ x_2(0) = -a,\quad \left.\frac{dx_1}{dt}\right|_{t=0} = 0, \text{ and } \left.\frac{dx_2}{dt}\right|_{t=0} = 0$

$$x_1 = \frac{\xi_1 + \xi_2}{2} = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)] \qquad \textbf{4.23}$$

and $x_2 = \frac{\xi_1 - \xi_2}{2} = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) - a_2 \cos(\omega_2 t + \phi_2)]$ **4.24**

$$\frac{dx_1}{dt} = -\frac{1}{2}[a_1 \omega_1 \sin(\omega_1 t + \phi_1) + a_2 \omega_2 \sin(\omega_2 t + \phi_2)] \quad \textbf{4.25}$$

$$\frac{dx_2}{dt} = \frac{1}{2}[-a_1 \omega_1 \sin(\omega_1 t + \phi_1) + a_2 \omega_2 \sin(\omega_2 t + \phi_2)] \quad \textbf{4.26}$$

(A) Using the initial conditions, we get

$$2a = a_1 \cos\phi_1 + a_2 \cos\phi_2,\ 2a = a_1 \cos\phi_1 - a_2 \cos\phi_2$$

and

$$0 = a_1\omega_1 \sin\phi_1 + a_2\omega_2 \sin\phi_2,\ 0 = a_1\omega_1 \sin\phi_1 - a_2\omega_2 \sin\phi_2$$

Hence

$$a_1 \cos\phi_1 = 2a,\ a_2 \cos\phi_2 = 0,\ a_1\omega_1 \sin\phi_1 = 0,\ a_2\omega_2 \sin\phi_2 = 0$$

As $a_1, a_2, \omega_1, \omega_2$ are not equal to zero,

$$\phi_1 = \phi_2 = 0,\ a_1 = 2a,\ a_2 = 0$$

$$\therefore\ x_1 = a \cos\omega_1 t,\ x_2 = a \cos\omega_1 t$$

That is, $\xi_1 = a \cos\omega_1 t$ and $\xi_2 = 0$.

(B) Substitution of the initial conditions in Eqs. (4.23) to (4.26) gives

$$\phi_1 = 0,\ \phi_2 = 0,\ a_1 = 0,\ a_2 = 2a$$

$$\therefore\ x_1 = a \cos\omega_2 t,\ x_2 = -a \cos\omega_2 t$$

and $\xi_1 = 0,\ \xi_2 = 2a \cos\omega_2 t$

4.3.3 ENERGY OF TWO COUPLED MASSES

If the coupling between two masses is weak, $\boldsymbol{\omega_2}$ will be only slightly different from $\boldsymbol{\omega_1}$,so that $\boldsymbol{\omega_{mod}}$ will be very small. Consequently $\mathbf{a_{mod}}$ and $\mathbf{b_{mod}}$ will take quite some time to show an observable change. That is, $\mathbf{a_{mod}}$ and $\mathbf{b_{mod}}$ will be practically constant over a cycle of angular frequency $\boldsymbol{\omega_{av}}$ .Then Eqs. (4.27) and **(4.28)** can be regarded as characterjsing almost simple harmonic motion. Let us now calculate the energies of masses ***A*** and *B* using these equations.

$$x_1(t) = a_{mod}(t) \cos\omega_{av} t \quad \textbf{4.27}$$

and

$$x_2(t) = b_{mod}(t) \sin \omega_{av} t$$ **4.28**

where

$$a_{mod}(t) = 2a \cos \omega_{mod} t$$ **4.29**

and

$$b_{mod}(t) = 2a \sin \omega_{mod} t$$ **4.30**

are modulated amplitude.

We know that the energy of an oscillator executing SHM is given by

$$E_1 = \frac{1}{2} m\omega_{av}^2 a_{mod}^2(t) = 2 ma^2 \omega_{av}^2 \cos^2 \omega_{mod} t$$ **4.31a**

$$E_2 = \frac{1}{2} m\omega_{av}^2 b_{mod}^2(t) = 2 ma^2 \omega_{av}^2 \cos^2 \omega_{mod} t$$ **4.31b**

The total energy of two masses coupled through a spring which stores almost no

energy is given by

$$E = E_1 + E_2 = 2ma^2\omega_{av}^2$$ **4.32**

which remains constant with time.

Using Eq. (4.32), we can.rewrite Eqs. (4.31a) and (4.31b) as

$$E_1 = \frac{E}{2} [1 + \cos (wz - \omega_1) t]$$ **4.33a**

$$E_2 = \frac{E}{2} [1 - \cos (\omega_2 - \omega_1) t]$$ **4.33b**

These equations show that at $t = 0$, $E_1 = E$ and $E_2 = 0$. That is, to begin with mass at **A** possesses all energy. As time passes, energy of mass at A starts decreasing. But mass at B begins to gain energy such that the total energy of the system remains constant.

When $(\omega_2 - \omega_1)$ **r** $= \pi/2$, two masses share energy equally. When $(\omega_2 - \omega_1)$ **r** $= \pi$,

$E_1 = 0$ and $E_2 = E$, i.e: mass ***B*** possesses all the energy. As time passes, the energy

exchange process continues. That is, the total energy flows back and forth twice

between two masses in time T, given by

$T = 2\pi/(\omega_2 - \omega_1)$

4.3.4 GENERAL PROCEDURE FOR CALCULATEDLY NORMAL MODE FREQUENCIES

In most physical situations of interest, coupled masses may not be equal. Then the above analysis is not of much use; it has to be modified. To calculate normal mode frequencies in such cases, we follow the procedure outlined below (i) Write down the equation of motion of coupled masses (ii) Assume a normal made solution (iii) Substitute it in the equation of motion and compare the ratios of' normal mode amplitudes (iv) Solve the resultant equation. We now illustrate this procedure for two unequal masses m1 and m2 coupled through a spring of force constant k. The equation of motion of two coupled masses are

$$m_1 \ddot{x}_1 = -k' x_1 + k(x_2 - x_1) \quad \textbf{4.34a}$$

and

$$m_2 \ddot{x}_2 = k' x_2 - k(x_2 - x_1) \quad \textbf{4.34b}$$

Let us assume solutions of the form

$$x_1 = a_1 \cos(\omega t + \phi)$$

and

$$x_2 = a_2 \cos(\omega t + \phi)$$

where ω is angular frequency and ϕ is initial phase.

Then

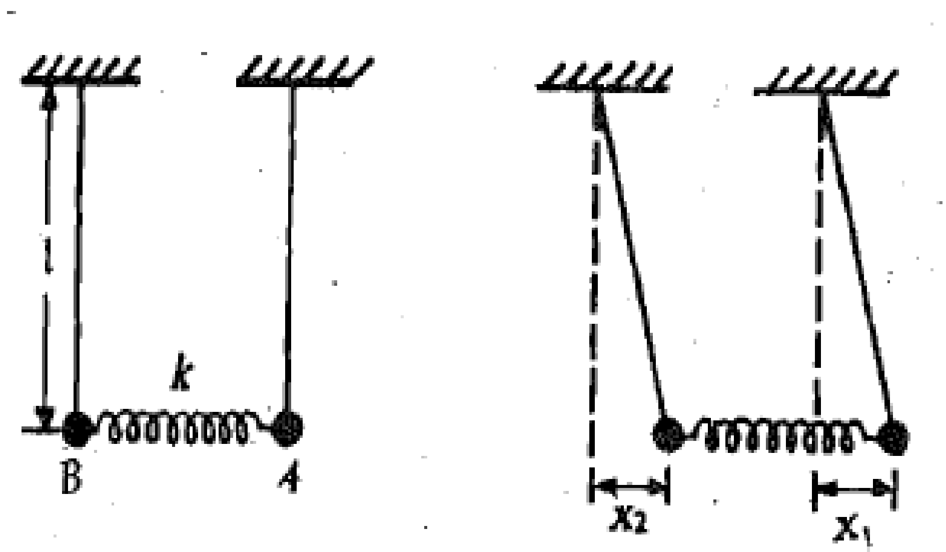
$$\ddot{x}_1 = -\omega^2 x_1$$

and

$$\ddot{x}_2 = -\omega^2 x_2$$

equation of motion of a simple pendulum is

$$m \frac{d^2x}{dt^2} = -\frac{mg}{l} x$$



(a) (b)

Fig. 4.5 Two identical pendulums simple together (a) Equilibrium configuration (b) Instantaneous configuration

In the present case, the equations of motion of bobs,A and B are

$$m \frac{d^2x_1}{dt^2} = -\left(\frac{mg}{l}\right) x_1 - k(x_1 - x_2)$$

and

$$m \frac{d^2x_2}{dt^2} = -\left(\frac{mg}{l}\right) x_2 + k(x_1 - x_2)$$

The term $\pm$ **k (x_l - x_2)** arises due to the presence of coupling. Dividing throughout by **m** and rearranging terms, we get

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_2 - \omega_s^2 (XI - x_2) = 0$$

4.35a

and

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 - \omega_s^2 (x_1 - x_2) = 0$$

4.35b

where we have substituted

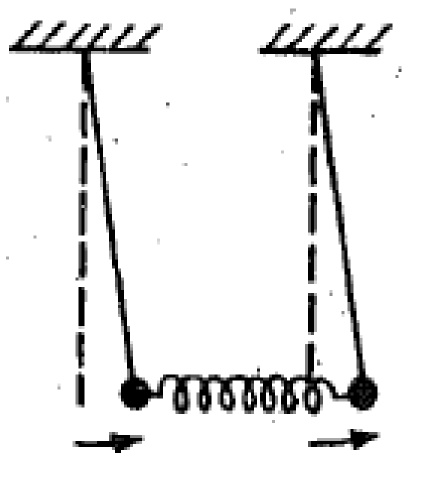
$\omega_0^2 = g/l$ and $\omega_1^2 = k/m$.

You will recognize that these equations are respectively identical to Eqs. (4.10) and (4.12). Thus the entire analysis of preceding sections applies and we can describe the motion of coupled pendulums by drawing analogies. The normal modes of this system are shown in Fig. 4.6. In mode **1($x_1 = x_2$)**, the bobs are in phase and oscillate with frequency

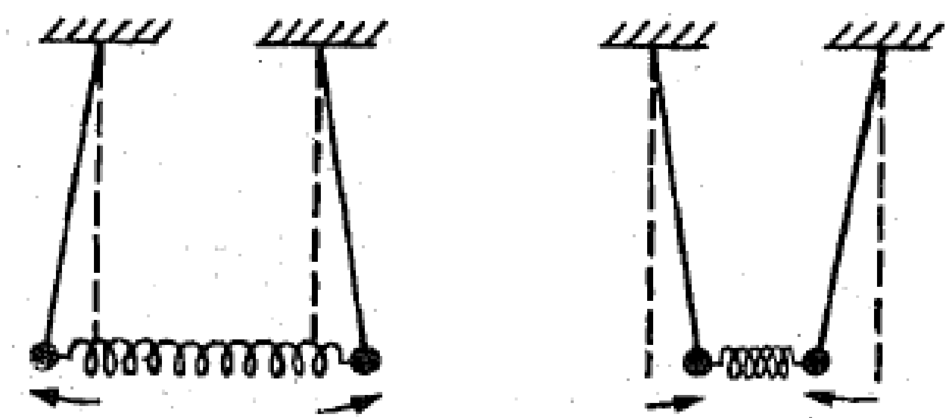
$$\omega_1 = \omega_0 = \sqrt{\frac{g}{l}}$$

But in mode **2($x_1 = -x_2$ or $x2 = -x_1$) ,** the bobs are in opposite phase and oscillate with frequency

$$\omega_2 = [\omega_0^2 + 2\omega_s^2]^{1/2} = [(g/l) + 2\,(k/m)]^{1/2}$$



(a)



(b)

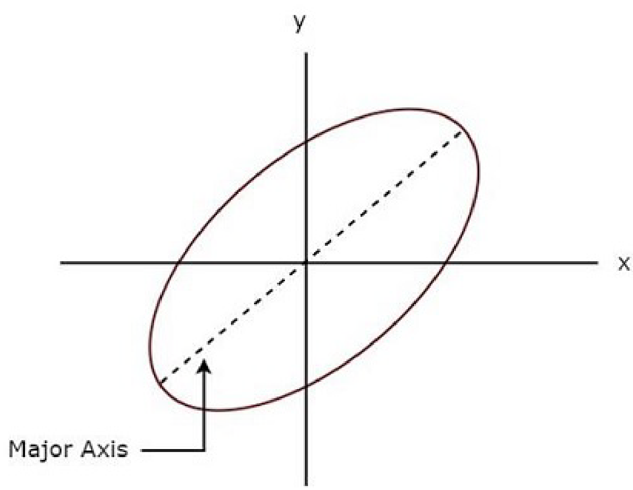
Fig. 4.6 Normal modes of a coupled pendulum (a) In-phase normal mode (b) Out-of-phase normal mode

4.4. SUMMARY

An **oscillator** is a **type** of circuit that controls the repetitive discharge of a signal, and there are two main **types** of **oscillator**; a relaxation, or an harmonic **oscillator**. This signal is often used in devices that require a measured, continual motion that can be used for some other purpose.

Lissajous figure is the pattern which is displayed on the screen, when sinusoidal signals are applied to both horizontal & vertical deflection plates of CRO. These patterns will vary based on the amplitudes, frequencies and phase differences of the sinusoidal signals, which are applied to both horizontal & vertical deflection plates of CRO.

The following figure shows an **example** of Lissajous figure.



The above Lissajous figure is in **elliptical shape** and its major axis has some inclination angle with positive x-axis.

Measurements using Lissajous Figures

We can do the following **two measurements** from a Lissajous figure.

- Frequency of the sinusoidal signal
- Phase difference between two sinusoidal signals

Now, let us discuss about these two measurements one by one.

Measurement of Frequency

Lissajous figure will be displayed on the screen, when the sinusoidal signals are applied to both horizontal & vertical deflection plates of CRO. Hence, apply the sinusoidal signal, which has standard **known frequency** to the horizontal deflection plates of CRO. Similarly, apply the sinusoidal signal, whose **frequency** is **unknown** to the vertical deflection plates of CRO

4.5 TERMINAL QUESTIONS

1. What are Lissajous figure used for?
2. How do you get Lissajous figures?
3. Explain the concept of Lissajous figures.
4. Find the differential equation of coupled masses.
5. Write short notes on:
   (a) Normal Co-ordinates and Normal Modes
   (b) Energy of two coupled masses
   (c) Normal modes frequencies

UGPHS-102

Oscillation, Waves and Electrical Circuits

Uttar Pradesh Rajarshi Tandon
Open University

BLOCK

2

ISBN :

Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2020.
Printed By : K.C.Printing & Allied Works, Panchwati, Mathura -281003.

UNIT-05 WAVE MOTION

Structure

5.1 INTRODUCTION

The term wave means propagation of some kind of **disturbance** in a medium and hence, wave motion is nothing but the transmission of disturbance from one point to another without the actual physical transfer or flow of matter as a whole. In physics, we come across different types of waves, viz., heat waves, sound waves, light waves, electromagnetic waves, matter waves (de Broglie waves), etc. The waves which require a material medium either for their production or for their propagation or both, are known as **mechanical waves**. In this unit, we shall study expression and properties of plane progressive wave and also study general equation of wave motion.

5.2 OBJECTIVES

After studying this unit, you should be able to –

- Define Wave Motion
- Compare Transverse wave and Longitudinal wave
- Apply General equation of wave motion
- Compute Numerical based on equation of a Plane Progressive Wave
- Explain the Concept of Phase Velocity

5.3 WHAT IS WAVE MOTION?

A wave is a disturbance which propagates with a definite speed from the point it is created. In other words, wave motion is the process of energy transfer, in which energy is transmitted in the form of disturbance from one place to another without migration of particles of the medium. If the medium is infinite in extent, the travelling disturbance is called **progressive** or **traveling wave**. If the medium is limited in extent, the progressive wave suffers reflection at the boundary of the medium, the incident and the reflected waves superpose giving rise to a **standing** or **stationary wave**.

Wave which require medium for propagation are called **mechanical waves**, e.g., Sound Waves, Water Waves, Waves in stretched strings. There is another kind of wave, the electromagnetic wave which requires no medium for propagation. e.g., radio wave, light wave, x-rays, r-rays etc.

5.3.1 TYPES OF MECHANICAL WAVES

Mechanical waves are of two types:-

(a) **Transverse Waves : -** If on propagation of mechanical wave in a medium, the particles of medium vibrate perpendicularly to the direction of propagation of the wave, then the wave is said to be **transverse wave**. Examples of such waves are waves produced in the stretched string, waves on the surface of water.

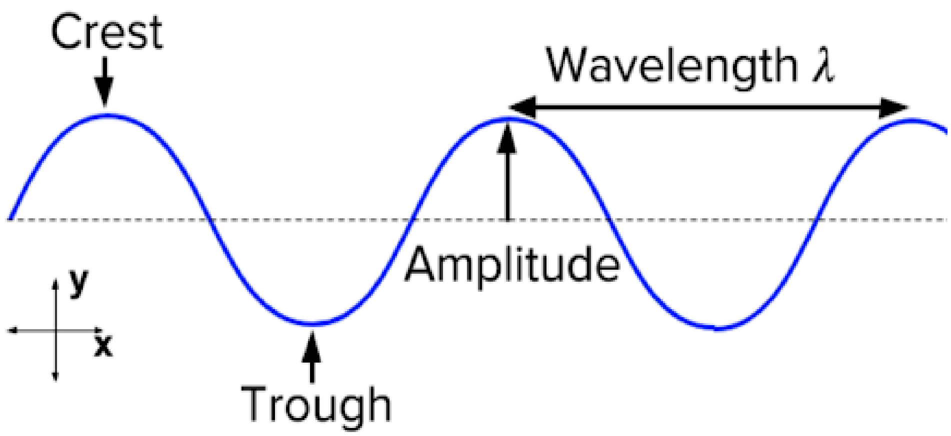


Figure – 1(a)

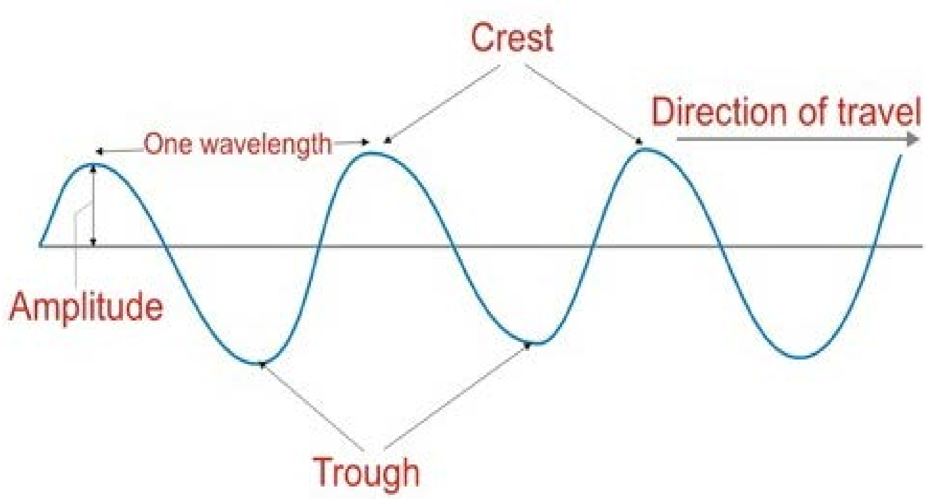


Figure – 1(b)

(b) **Longitudinal Waves : -** If on propagation of mechanical wave in a medium, the particles of the medium vibrate along the direction of propagation of wave, then the wave is said to be **longitudinal wave**. Examples of the waves are sound waves in air, waves produced in spiral spring etc.

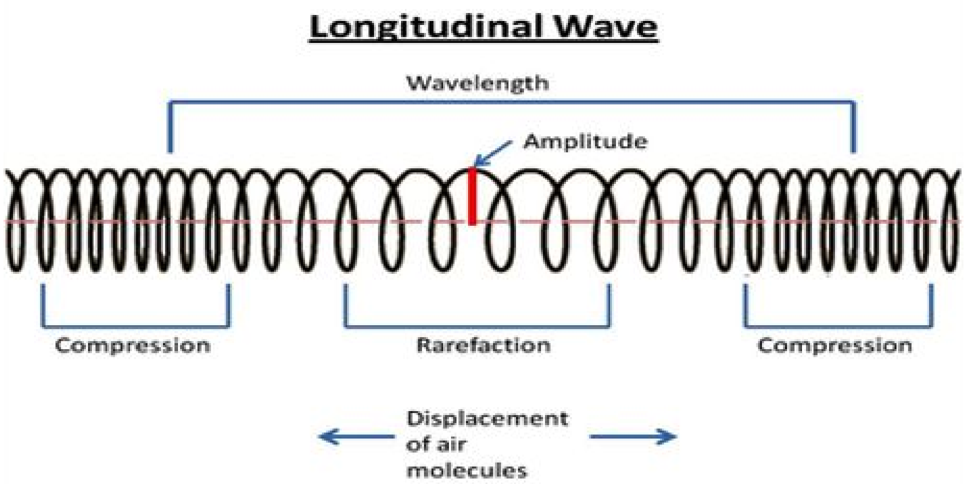


Figure – 2(a)

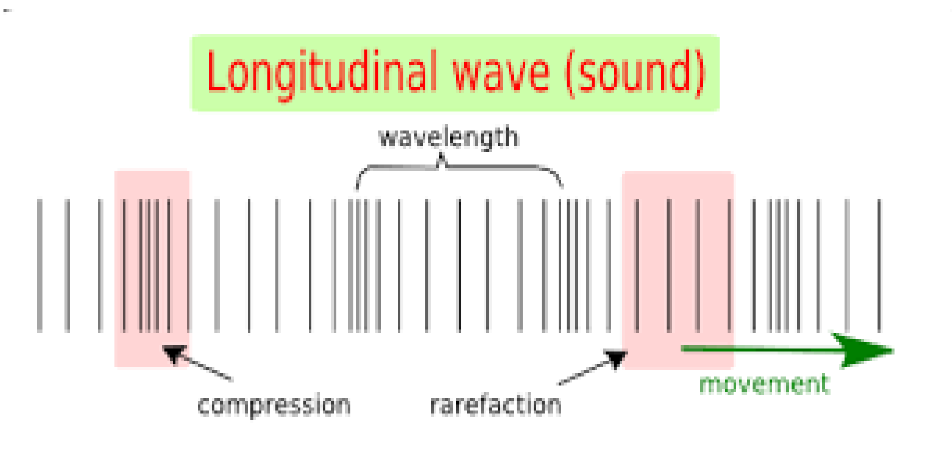


Figure – 2(b)

5.3.2 Difference between transverse and longitudinal wave :-

S. No.	***Transverse Wave***	***Longitudinal Wave***
1.	In these waves the particles of the medium vibrate perpendicularly to the direction of propagation of wave.	In these waves the particles of the medium vibrate along the direction of propagation of wave.
2.	These waves propagates in the form of crests and troughs.	These waves propagates in the form of compression and rarefaction.

3.	These waves can be produced in the interior of solids and on the surface of liquids, also in light.	These wave can be produced in all types of media, solid, liquid, gas.
4.	In these waves there are no variation of pressure and density along the direction of propagation of wave.	In these waves the pressure and density vary along the direction of propagation of wave.

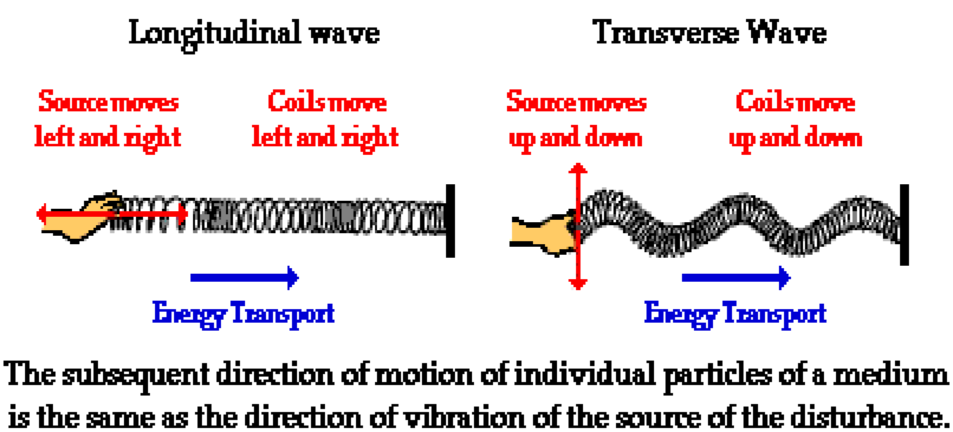


Figure – 2(c)

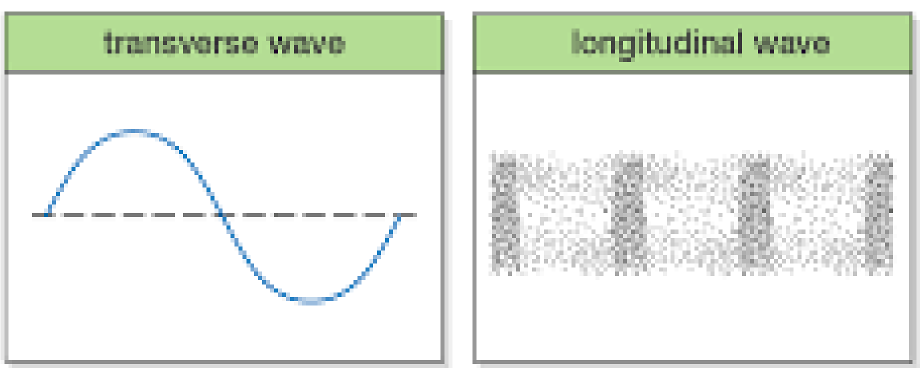


Figure – 2(d)

5.4 GENERAL EQUATION OF WAVE MOTION

Consider a wave is moving with a velocity v along x-axis. The displacement of the particle at any instant ‘t’ can be represented as

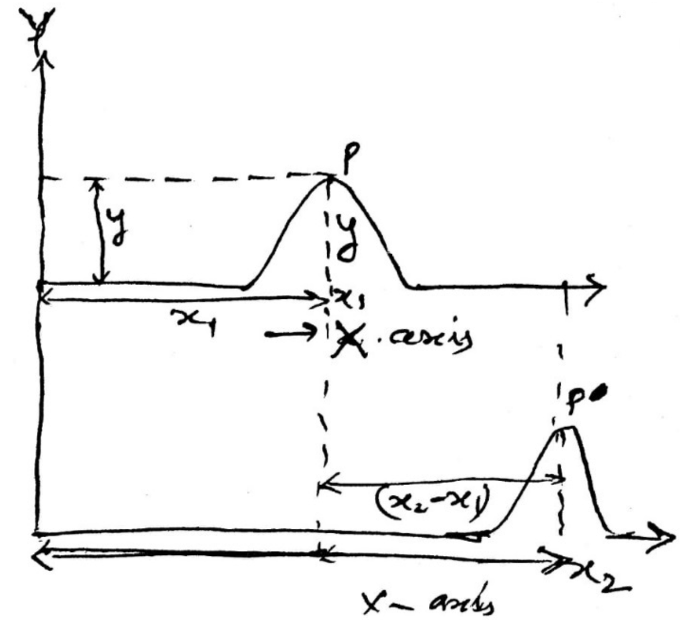


Figure - 3

$y = f(x, t)$ ………………… (1)

from figure, the displacement of wave at point x_1 & at time t_1 is

$y = f(x_1, t_1)$ ………………… (2)

The wave moves without change of shape i.e. its displacement, to point x_2 at time t_2 then,

$y = f(x_2, t_2)$ ………………… (3)

$\because \quad x_2 - x_1 = v(x_2 - t_1)$ (where v is the velocity)

or $x_1 - vt_1 = x_2 - vt_2$ ………………… (4)

from equation (2) & (3)

$f(x_1, t_1) = f(x_2, t_2)$ ………………… (5)

Hence the function which satisfies both equation (4) & (5) is,

$f(x - vt)$ or $f(vt - x)$ ………………… (6)

So, the equation of a wave travelling along +ve x direction is

$\boxed{y = f(x - vt) \;\; or\; y = f(vt - x)}$ ………………… (7)

Similarly the equation of a wave travelling in –x direction is,

$\boxed{y = f(x + vt) \quad or \quad y = f(vt + x)}$ ………………… (8)

Equation (7) and (8) are general equation of one dimensional wave of all types.

5.5 WAVE TERMINOLOGY

Some terms that is used in connection with wave motion.

(a) **Amplitude:-** The maximum displacement of any particle vibrating on either side of its equilibrium position is called the amplitude. Generally it is dented by ‘a’.

(b) **Time Period:-** The time taken by a particle of medium in completing one vibration is called the time period. It is dented by ‘T’.

(c) **Frequency:-** The number of vibrations completed by a vibrating particle of the medium is called frequency. It is denoted by ‘n’ or ‘f’.

$n = f = \frac{I}{T}$.

(d) **Speed of Wave:-** The distance transverse by the wave per second is called speed of wave or wave speed. It is denoted by ‘v’.

(e) **Wave Length:-** The distance traversed by a wave in one time-period is called the wave length. In other words, the distance between two nearest particles of medium vibrating in the same phase is called the **‘wave length’**. It is denoted by ‘λ’.

(f) **Phase:-** The phase of a vibrating particle at any instant represents the position and direction of motion of the vibrating particle at that instant. It is denoted by ‘ϕ’.

5.6 EQUATION OF A PLANE PROGRESSIVE WAVE

Consider a plane progressive wave is propagating in a medium along +ve (positive) x-axis. When wave propagates, the particles of medium starts to vibrate about their mean position. Let the particle begin to vibrate from origin at time t = 0. If y is the displacement of the particle at time t, then equation of the particle executes simple Harmonic motion about mean position is

$$y = A \sin \omega t \qquad \text{………………… (1)}$$

Where A is amplitude and ω is angular velocity. ωt is called the phase of the oscillation. This disturbance or wave reaches the point x after a time $\frac{x}{v}$, where v is velocity of the wave. The phase of the point x will be less than that at the point x = 0 and which is equal to $\omega\left(t - \frac{x}{v}\right)$. Hence the displacement of the particle at point x at time t is,

$$y = A \sin \omega \left(t - \frac{x}{v}\right) \qquad \text{………………… (2)}$$

If T is time period and λ is the wave length then $\omega = \frac{2\pi}{T}$

$$\therefore \qquad y = A\sin\frac{2\pi}{T}\left(t - \frac{x}{v}\right)$$

$$= A\sin 2\pi\left(\frac{t}{T} - \frac{x}{vT}\right)$$

$$vT = \lambda \qquad \left(\because v = n\lambda;\ n = \frac{1}{T}\right)$$

$$\therefore \qquad y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

It can also be expressed as

$$y = A\sin\frac{2\pi}{\lambda}(vt - x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (4) may also be expressed as,

$$y = A\sin(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

When, $\omega = 2\pi n$ and $K = \frac{\omega}{v} = \frac{2\pi}{\lambda} = propagation\ constant.$

We can also write the equation of plane progressive wave travelling in +ve x-direction. In exponential from as

$$y = A\, e^{i(\omega t - kx)};\ i = \sqrt{-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The equation of plane progressive wave travelling in –x-direction can be written as,

$$y = A\sin(\omega t + kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and in exponential form,

$$y = A\, e^{i(\omega t + kx)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

5.6.1 PHASE DIFFERENCE

The phases of different particles at the same instant are different or the phase of the same particle at different instants is different. The difference of the two phases is called the phase difference.

When a wave is propagating along the direction of +ve (positive) x-axis then the displacement of a particle at a distance x from the origin is given by,

$$y = a\sin(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$\omega t - kx$ represents the phase of the wave,

$$\text{i.e. } \phi = \omega t - kx \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let ϕ_1 & ϕ_2 be the phases of the two particles at distances x_1 and x_2 from origin, respectively then,

$\phi_1 = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$, $\phi_2 = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$ $\left(\because \omega = \frac{2\pi}{T} \& K = \frac{2\pi}{\lambda}\right)$

$$\phi_1 - \phi_2 = \frac{2\pi}{\lambda}(x_2 - x_1)$$

$$\Delta\phi = \frac{2\pi}{\lambda} \times \Delta x$$

$$Phase\ difference = \frac{2\pi}{\lambda} \times path\ difference$$

If $\Delta x = \lambda$ then $\Delta\phi = 2\pi$, i.e., the two particles of medium at sparation λ are always in the same phase and then λ is called wavelength.

Example: 1

The equation for displacement of a wave is given by,

$$\boldsymbol{y = 8 \sin 2\pi\left(\frac{t}{0.02} - \frac{x}{100}\right)}$$

where y and x are in cm and t in seconds.

(a) Is the wave is progressive or stationary?

(b) What is the amplitude of the wave?

(c) What is the wavelength?

(d) What is the velocity of propagation?

(e) What is the frequency?

Solution:

We know that, equation of plane progressive is,

$\boldsymbol{y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)}$**; on comparison we get,**

(a) The wave is progressive

(b) Amplitude = 8 cm

(c) Propagation constant, $k = \frac{2\pi}{\lambda}$

$\frac{2\pi}{\lambda} = \frac{2\pi}{100} \Rightarrow \lambda = 100\ cm$

(d) Velocity of propagation.

$v = \frac{\omega}{k}$, $\omega = \frac{2\pi}{0.02} = 100\pi$

$\omega = 100\pi,$ $k = \frac{2\pi}{100}$

$\therefore$ $v = \frac{100\pi}{2\pi} \times 100 = 5 \times 10^3 cm/s$

(e) Frequency,

$\omega = 100\pi,$

$2\pi n = 100\pi$

$n = 50\ Hz.$

Example; 2

The propagation constant of the wave is 1.5 × 10⁴/m and velocity is 380 m/sec. calculate

- **(a) Wave length,**
- **(b) Wave number and**
- **(c) Frequency of wave**

Solution :

(a) $k = 1.5 \times 10^4/m$

$k = \frac{2\pi}{\lambda} \quad \therefore \quad \lambda = \frac{2\pi}{k}$

$\lambda = \frac{2\pi}{1.5\times10^4} m = 4.19 \times 10^{-4} m$

(b) Wave number $= \frac{1}{\lambda} = \frac{1}{4.19\times10^{-4}} = 2387\ /m$

(c) Frequency, $\nu = \frac{v}{\lambda} = \frac{380}{4.19\times10^{-4}} = 9.07 \times 10^5\ cycle/second$

Example: 3

A wave has a frequency 400 Hz and velocity 320 m/s. Find the distance between the points which are 45° out of phase.

Solution :

Given, $n = 400\ Hz$; $v = 320\ m/s$

$$\Delta\phi = 45^o = \frac{\pi}{4}$$

$$\Delta x = ?$$

$$v = n\lambda$$

$$\lambda = \frac{v}{n} = \frac{3200}{400} = \frac{4}{5} m$$

$$\Delta\phi = \frac{2\pi}{\lambda} \times \Delta x$$

$$\Delta x = \Delta\phi \times \frac{\lambda}{2\pi}$$

$$\Delta x = \frac{\pi}{4} \times \frac{4}{5 \times 2\pi} = \frac{1}{10} m = 0.1\ m$$

5.7 PHASE VELOCITY OR WAVE VELOCITY OF A PLANE PROGRESSIVE WAVE

The equation of progressive wave is,

$$y = A\sin(\omega t - kx)$$

The phase at a point x at the time t is given by,

$$\phi = \omega t - kx$$

At a fixed point, ϕ increases with time & at a fixed time, ϕ decreases with distance x. Now we have chosen a point on the wave whose phase ϕ has a fixed value, say ϕ_o. Then the position of this point is given by,

$$\omega t - kx = \phi_o$$

$$x = \frac{\omega}{k}t - \frac{\phi_o}{k}$$

The velocity with which a point of constant phase moves on the wave is called the **phase velocity** and is given by,

$$\boxed{\frac{dx}{dt} = \frac{\omega}{k} = v_p} = v$$

Which is same as the velocity of the wave.

5.8 PARTICLE VELOCITY OF WAVE

Let us consider a progressive wave travelling in tx-direction. Its equation is of the form, be,

$y = A\sin(\omega t - kx)$ ………………. (1)

The velocity of a particle located at point x is given by $\frac{\partial y}{\partial t}$. This velocity is called the particle velocity

$u = \frac{\partial y}{\partial t} = A\,\omega\, cos\,(\omega t - kx)$ ………………. (2)

$u_{max} = A\omega$ ………………. (3)

u_{max} is the maximum velocity.

5.9 DIFFERENTIAL EQUATION OF WAVE MOTIONS

Let us consider a plane progressive wave travelling through a medium along x-axis. The equation of wave is,

$y = a\sin(\omega t - kx)$ ………………. (1)

The instantaneous velocity u at point x is $\frac{\partial y}{\partial t}$. Then,

Particle velocity, $u = \frac{\partial y}{\partial t} = a\,\omega\, cos(\omega t - kx)$ ………. (2)

Now, differentiating equation (1) with r.t. 'x', we get,

$\frac{\partial y}{\partial x} = -ak\, cos(\omega t - kx)$ ………………. (3)

Divide equation (2) by (3), we get,

$$\frac{u}{\left(\frac{\partial y}{\partial x}\right)} = \frac{-\omega}{k}$$

$$\frac{\omega}{k} = v = wave\ velocity$$

$$\boxed{u = -v\frac{\partial y}{\partial x}} \quad \text{………………. (4)}$$

Thus, the particle velocity at point x is equal to the wave velocity and multiplied by the slope of the medium at point x. This is true for transverse as well as longitudinal waves.

Differentiating equation (2) w.r.t. 't', we get,

$\frac{\partial^2 y}{\partial t^2} = -a\,\omega^2 sin(\omega t - kx)$ ………………. (5)

Again differentiating equation (3) w.r.t. 'x' we get,

$\frac{\partial^2 y}{\partial x^2} = -ak^2\, sin(\omega t - kx)$ ………………. (6)

Dividing (5) by (6)

$$\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{-a\omega^2}{-ak^2} = \frac{\omega^2}{k^2} = v^2$$

$$\boxed{\boldsymbol{\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}}} \quad \text{………………. (7)}$$

This is differential equation of wave motion and this equation is also known as classical wave equation.

Example: 4

Which of the following are the solution of one dimensional wave equation?

(a) $\boldsymbol{y = x^2 + v^2 t^2}$ **(b)** $\boldsymbol{y = 5\sin x \cos vt}$

(c) $\boldsymbol{y = 3\sin 2x \cos vt}$ **(d)** $\boldsymbol{y = 2x - 5t}$

Solution:

The general differential equation for a progressive wave is

$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ………………. (1)

The equation of the problem, which satisfies the condition (1) will be the solution of one dimensional wave equation.

(a) $y = x^2 + v^2t^2$

$\frac{\partial y}{\partial t} = 2v^2t; \quad \frac{\partial^2 y}{\partial t^2} = 2v^2$ ………………. (a)

$\frac{\partial y}{\partial x} = 2x; \quad \frac{\partial^2 y}{\partial x^2} = 2$ ………………. (b)

from (a) and (b) we get,

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$

Hence the equation (1) is the solution of the one dimensional wave equation.

(b) $y = 5 \sin x \cos vt$

$\frac{\partial y}{\partial t} = -5v \sin x \sin vt$

$\frac{\partial^2 y}{\partial t^2} = -5v^2 \sin x \cos vt$ ………………. (a)

$\frac{\partial y}{\partial x} = 5 \cos x \cos vt$

$\frac{\partial^2 y}{\partial x^2} = -5 \sin x \cos vt$ ………………. (b)

from (a) and (b) we get,

Hence the equation (b) is the solution of one dimensional wave equation.

(c) $y = 3 \sin 2x \cos vt$

$\frac{\partial y}{\partial t} = -3v \sin 2x \sin vt$

$\frac{\partial^2 y}{\partial t^2} = -3v^2 \sin 2x \cos vt$ ………………. (a)

$\frac{\partial y}{\partial x} = 6 \cos 2x \cos vt$

$\frac{\partial^2 y}{\partial x^2} = -12 \sin 2x \cos vt$ ………………. (b)

from (a) and (b) we get,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

Hence the equation (c) is not the solution of one dimensional wave equation.

(d) $y = 2x - 5t$

$\frac{\partial y}{\partial t} = -5$

$\frac{\partial^2 y}{\partial t^2} = 0$ ………………. (a)

$$\frac{\partial y}{\partial x} = 2$$

$$\frac{\partial^2 y}{\partial x^2} 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(b)}$$

from (a) and (b) we get,

$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ does not exist

So, $y = 2x - 5t$ is not the solution of classical wave equation.

5.10 ENERGY AND INTENSITY IN A PLANE PROGRESSIVE WAVE

Let us consider a wave is propagating along the (+ve x)-direction and the displacement of the particle from mean position is y. The equation of wave is

$$y = a \sin(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial y}{\partial t} = a\omega \cos(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial^2 y}{\partial t^2} = -a\omega^2 \sin(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Consider a medium whose density is ρ and cross sectional area is unity. Then the mass of thickness (layer)dx = ρdx.

The Kinetic energy of layer = $\frac{1}{2}\rho dx.\left(\frac{\partial y}{\partial t}\right)^2$

Using equation (1),

$$K.E. = T = \frac{1}{2}\rho dx\big(a\omega\ (\omega t - kx)\big)^2$$

$$= \frac{1}{2}\rho dx\ a^2\omega^2\ cos^2(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The force acting on this layer,

$$F = (\rho dx)\frac{\partial^2 y}{\partial t^2} = \rho dx\big(-a\omega^2\ sin(\omega t - kx)\big)$$

$= -\rho dx\ \omega^2 y$ (using (4))

$\therefore$ Potential energy of layer, $U = -\int_o^y F dy$

$$= -\int_o^y (-\rho\ dx\omega^2 y) dy$$

$$= \omega^2 \rho\ dx\ \int_o^y y\ dy$$

$$= \omega^2 \rho\ dx. \frac{y^2}{2}$$

$$= \frac{1}{2}\omega^2 \rho\ dx. a^2\ sin^2(\omega t - kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

∴ Total energy of layer = K.E. + P.E. = T + U

$$\boxed{E = \frac{1}{2}\rho\, dx.\ \omega^2 a^2} \qquad \text{………………} (7)$$

Which is the energy of wave.

Volume of the layer = area × thickness

= *unity* × *dx* = *1* × *dx* = *dx*

∴ Energy of wave per unit volume or energy density is,

$$u = \frac{1}{2}\omega^2\ \rho a^2$$

Thus, the energy density of the wave does not depend on x and t.

5.10.1 INTENSITY OF WAVE

The intensity of wave is defined as the total energy of the wave passing per unit cross-sectional area per second.

If the distance traversed by wave in one second = ρ

∴ Intensity of wave = u. ρ

$$\boxed{\boldsymbol{I = \frac{1}{2}\omega^2\ \rho\ a^2.\rho}}$$

5.11 PLANE PROGRESSIVE WAVE IN FLUID

Suppose a long cylindrical tube of uniform cross section containing a fluid.

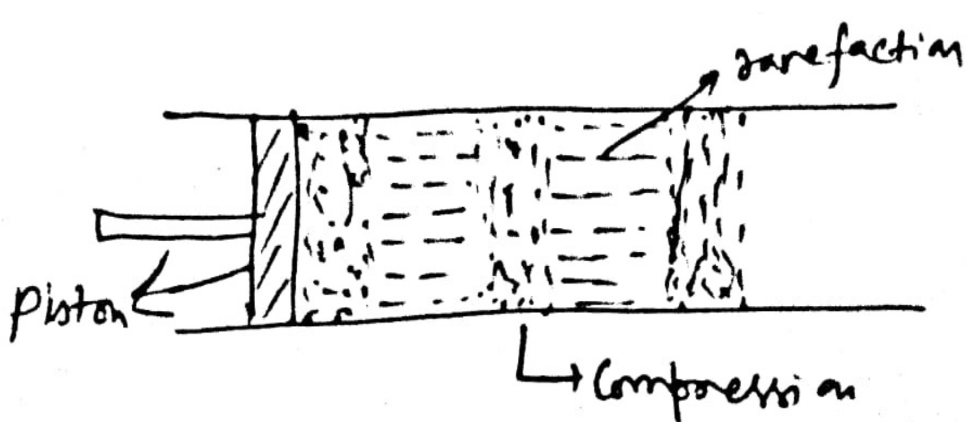


Figure - 4

After supplying sinusoidal wave through position in the fluid of rod, the compressions and rarefaction occurs in the fluid & they travel along the axis of the tube. These compressions and rarefactions travelling wave are called the longitudinal wave. At places where there is a compression, the pressure and density of the fluid are above the equilibrium value and at those places where rarefaction occurs, the pressure and density of the fluid below the equilibrium value.

The displacement of the fluid particle in +ve (positive) x direction is given by,

$$y = a\cos(\omega t - kx) \quad or \quad a\sin(\omega t - kx)$$
$$= ae^{i(\omega t - kx)})$$

Condensation

It is defined as the ratio of the change in density to the original density.

$$S = \frac{\rho - \rho_o}{\rho_o}$$

Volume Strain or Dilation

It is the ratio of change in volume to original volume.

$$S = \frac{v - v_o}{v_o}$$

5.11.1 VELOCITY OF WAVE IN A FLUID

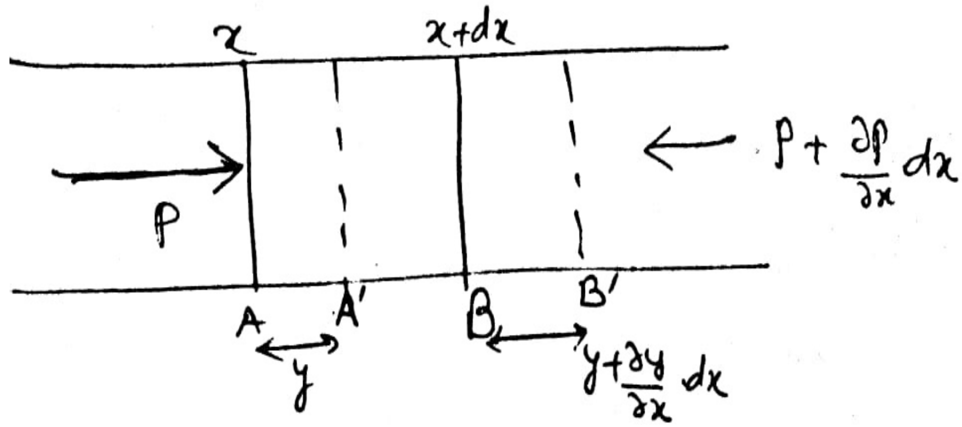


Figure - 5

Consider a small volume element of fluid in cylindrical tube layer A and B which are located at x & x + dx respectively. After applying wave, this volume element is subjected to pressure and volume change. If p is excess pressure and v is volume change in original volume v_o of the element, then Bulk modulus is,

$$B = \frac{excess\ pressve}{-\frac{u}{v_o}} = \frac{p}{\delta} = \frac{p}{S}$$

$\therefore \qquad p = B.s \qquad \qquad$ ……………… (1)

Let initial positions of layer A and B be

$$x_A = x$$

$$x_B = x + dx$$

After propagation of wave, layer A shifted to new position A' and layer B shifted to B', consider $AA' = x$ & $BB' = y + \frac{\partial y}{\partial x}.dx,$ then new positions of the layer A' and B' are,

$$x_{A'} = x + y$$

$$x_{B'} = x + dx + y + \frac{\partial y}{\partial x}.dx$$

If area of cross-section of tube be α then volume between two layers A and B is α.dx, and between new layers A' & B' is $\alpha.\left(dx + \frac{\partial y}{\partial x}.dx\right)$. Therefore change in volume of the fluid element be

$$= \alpha.\left(dx + \frac{\partial y}{\partial x}.dx\right) - \alpha dx$$

$$= +\alpha\frac{\partial y}{\partial x}.dx$$

∴ Volume strain or Dilation is given by,

$$\delta = \frac{\alpha\frac{\partial y}{\partial x}.dx}{\alpha.dx} = \frac{\partial y}{\partial x}.$$

Hence from (1) excess pressure, $(\because \delta = -S)$

$$p = -B.\frac{\partial y}{\partial x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let the pressure acting on the layer A be P. then the pressure acting on the layer B would be $P + \frac{\partial P}{\partial x}.dx$.

Thus, the net pressure acting on the element in +ve x direction is given by,

$$P - \left(P + \frac{\partial P}{\partial x}.dx\right) = -\frac{\partial P}{\partial x}.dx$$

If P_o is the equilibrium value of pressure, then $P = P_o + p$.

Hence net pressure $= -\frac{\partial}{\partial x}(P_o + p)dx$

$$= -\frac{\partial p}{\partial x}.dx$$

The force acting on the element is,

$$F_x = \left(-\frac{\partial p}{\partial x}.dx\right).\alpha$$

$\because \qquad p = -B\frac{\partial y}{\partial x}$ (Using equation (2))

$\therefore \qquad F_x = \alpha B\frac{\partial^2 y}{\partial x^2}.dx$ ……………… (3)

The mass of the element = $\alpha\ dx\ \rho_o$; $\rho_o \rightarrow density\ of\ fluid$

Hence,

$$(\alpha\ dx\ \rho_o)\frac{\partial^2 y}{\partial t^2} = \alpha B\frac{\partial^2 y}{\partial x^2}dx$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho_o}\ \frac{\partial^2 y}{\partial x^2}dx$$ ……………… (4)

Which is similar to wave equation,

$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ……………… (5)

Comparing (4) & (5) we get,

$$v = \sqrt{\frac{B}{\rho_o}}$$ ……………… (6)

Which gives the velocity of longitudinal waves produce in fluid.

5.12 LONGITUDINAL WAVE AS A PRESSURE WAVE

Let the equation of wave travelling in +ve x-direction is

$y = a\sin\ (\omega t - kx)$ ……………… (1)

The excess pressure at a point x is given by,

$$p = +B\frac{\partial y}{\partial x}$$

$$= +Bk\ a\cos\ (\omega t - kx)$$

$\because \qquad v = \sqrt{\frac{B}{\rho_o}} \Rightarrow B = v^2\rho_o$

$\therefore \qquad p = v^2\rho_o k\ a\cos(\omega t - kx)$

$v = \frac{\omega}{k} \Rightarrow \omega = vk$

$p = a\ v\omega\rho_o\cos\ (\omega t - kx)$

$\boxed{p = p_o\cos(\omega t - kx)}$ ……………… (2)

$p_o = a\ v\omega\ \rho_o \rightarrow pressure\ amplitude$

This is the pressure equation of a longitudinal wave. It is also convenient to discuss the behaviour of wave in terms of pressure variations instead of displacement of fluid particles. The equation (1) and (2) shows that the displacement wave is 90^o out of phase with pressure wave.

5.12.1 INTENSITY AND PRESSURE AMPLITUDE

The Intensity of a wave is given by,

$$I = \frac{1}{2}\omega^2\rho_0 a^2 . l \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The equation (1) can be written as,

$$I = \frac{1}{2}\frac{\omega^2\rho_0^2 l^2 a^2}{\rho_0 l}$$

Since pressure amplitude $p_0 = \omega v a \rho_0$

$$\therefore \qquad \boxed{\boldsymbol{I = \frac{p_0^2}{2\rho_0 v}}}$$

Which is relation between Intensity and pressure amplitude.

5.12.2 EFFECT OF VARIOUS FACTORS ON VELOCITY OF FLUID (GAS)

(a) Effect of Pressure :-

The velocity of longitudinal wave in fluid is

$$v = \sqrt{\frac{B}{\rho_0}} = \sqrt{\frac{rp}{\rho_0}}$$

Due to change in pressure, the density of the medium also changes in such a way that -$\frac{P}{\rho_0}$ remain constant for a given temperature. Thus there is no effect of pressure change on the speed of wave if the temperature remains constant.

(b) Effect of Temperature:-

$V = volume = \frac{M}{\rho}$

$\therefore$ for Gas equation

PV = RT

$\frac{PM}{\rho} = RT \qquad \Rightarrow \qquad \frac{P}{\rho} = \frac{RT}{M}$

$\therefore \qquad V = \sqrt{\frac{rRT}{M}}$

$$v \, \alpha \sqrt{T}$$

(c) Effect of Frequency and Amplitude:-

Then is no effect of frequency and amplitude and velocity of the wave in fluid.

5.13 SUMMARY

- A wave is a kind of disturbance created in an elastic medium.
- A wave motion is a process of transmission of energy and momentum from one point of a medium to another without any actual transfer of the particles of the medium.
- The essential requirements for producing a wave motion are:
  (a) a vibrating body, called the source of wave, that may create a disturbance;
  (b) an elastic medium through which the wave will propagate, and
  (c) the particles of the medium for participating in the process of transmission of disturbance.
- Two distinct types of mechanical waves are:
  (a) Transverse waves in which the particles of the medium execute simple harmonic motion about their respective mean positions at right angles to the direction of wave propagation. The electromagnetic waves are transverse in nature. Waves spreading over the surface of water are transverse.
  (b) Longitudinal waves in which the particles of the medium execute simple harmonic motion about their respective mean positions along the direction of wave propagation. Sound waves are longitudinal.

5.14 TERMINAL QUESTIONS

1. Fill in the blanks.

(a) A wave is a ………………

Ans.- disturbance

(b) A mechanical wave requires a …………….. for its propagation.

Ans.- material medium

(c) Waves formed on the water surface in a lake are …………

Ans.- transverse

(d) An electromagnetic wave …………….. a material medium for its propagation.

Ans.- does not require

(e) Sound waves are ……………..

Ans.- longitudinal

(f) Light waves are ………………

Ans.- transverse

(g) In a longitudinal wave, particles of the medium vibrate ………… the direction of wave propagation.

Ans.- along

(h) In a liquid, longitudinal waves propagate only …………….. of the liquid.

Ans.- in the interior

(i) In a liquid, transverse waves propagate only …………….. of the liquid.

Ans.-on the surface

(j) Transverse waves cannot be produced in a ……………… medium.

Ans.- gaseous

(k) In a transverse wave, particles of the medium vibrate …………… the direction of wave propagation.

Ans.- in a direction perpendicular to

(l) In a progressive wave, …………. Are transferred from one point of the medium to the other.

Ans.- energy and momentum

(m) For transverse waves to propagate in a medium, the medium must possess ………….

Ans.- rigidity

(n) Sound waves propagate in a gaseous medium by ……………… and ……….

Ans.- compressions, rarefactions

(o) Phase difference and path difference are related by ……………

Ans.- $\phi = \frac{2\pi}{\lambda}.x$

2. Write down the equation of Phase velocity of a plane progressive wave.

3. Define a wave.

4. What do you mean by wave motion?

5. Distinguish between longitudinal and transverse waves.

6. Given below are some examples of wave-motion. State in each case, if the wave-motion is transverse, longitudinal or a combination of both:

   (a) Motion of a kink in a long coil-spring produced by displacing one end of the spring sideways.

   (b) Waves produced in a cylinder containing a liquid by moving its piston back and forth.

   (c) Waves produced by a motor boat sailing in water.

   (d) Light waves travelling from the sun to the earth.

   (e) Ultrasonic waves in air produces by a vibrating quartz crystal.

   (f) Radio waves broadcasted from a radio station.

   Ans.-

   (a) When the spring is pulled sideways, motion of the kink will be a transverse wave. (If the spring is pulled parallel to its length, the motion of the kink will be a longitudinal wave).

   (b) The liquid molecules oscillate along the direction of motion of the piston, that is, along the direction of propagation of the wave. Hence it is example of longitudinal wave.

   (c) The propeller of a motor boat cuts the water surface laterally and pushes it backward. Hence the wave-motion is a combination of both longitudinal ana transverse waves.

   (d) Light waves are electromagnetic waves which are transverse in nature.

   (e) Ultrasonic waves are longitudinal sound waves of high frequency.

   (f) Radio waves are electromagnetic waves of large wavelength which are transverse in nature.

7. Write down the Equation of a Plane Progressive Wave.

8. Explain differential Equation of Wave Motions.

9. Write short notes on: -

(a) Plane Progressive Wave in Fluid

(b) Velocity of Wave in a Fluid

(c) Longitudinal Wave as a Pressure Wave.

UNIT-06 WAVES AT BOUNDARIES OF TWO MEDIA

Structure

6.1 INTRODUCTION

In the Previous unit 5, was concerned with waves that could be imagined as travelling uninterrupted in a specified medium. You have also studied about the expression and properties of Plane Progressive Wave. This unit is chiefly about some of the effects that take place when a travelling wave encounters a barrier/boundaries or a different medium or small obstacles . In this unit we will study about bounded medium, acoustic impedance and characteristic impedance. In this unit, we shall also study some important examples and expression for reflection and transmission of Energy Waves at joint of two media.

6.2 OBJECTIVES

After studying this unit, you should be able to –

- Concept of Bounded Medium
- Define acoustic impedance, characteristic impedance

- ❖ Solve problems based on acoustic impedance and characteristic impedance
- ❖ Understand concept of reflection and transmission of Energy Waves at joint of two media.

6.3 FREE AND BOUNDED MEDIUM

A "bounded medium" is one which has a definite boundary and whose boundaries are separated from other media by distinct surfaces. Such a medium can vibrate with only certain definite frequencies and these frequencies are the characteristic frequencies of that medium.

The boundary of a bounded medium may be of two types: rigid (or closed) and free (or open). For example, both the ends of a sitar string are rigid, the closed end of the tube of an air column is rigid while its open end is free. When a wave travels in a medium having no boundary, then the wave continues to travel as such. If the medium has a boundary (rigid or free), then the wave is reflected from the boundary.

The physical channels (the media) that carry data are of two types: bounded and unbounded. In a bounded medium, the signals are confined to the medium and do not leave it (except for smaller leakage amounts). A pair of wires, coaxial cable, waveguide, and optical-fiber cable are examples of bounded media.

6.4 ACOUSTIC IMPEDANCE

Impedance

One of the important physical characteristics relating to the propagation of sound is the acoustic impedance of the medium in which the sound wave travels.. Acoustic impedance (Z) is given by the ratio of the wave's acoustic pressure (p) to its volume velocity (U):

$$\boxed{Z = \frac{p}{U}}$$

Like its analogue electrical impedance (or electrical resistance), acoustic impedance is a measure of the ease with which a sound wave propagates through a particular medium. Also, like electrical impedance, acoustic impedance involves several different effects applying to different situations. For example, specific acoustic impedance (Z), the ratio of acoustic pressure to particle speed, is an inherent property of the medium and of the nature of the wave. Acoustic impedance, the ratio of pressure to volume velocity, is equal to the specific acoustic impedance per unit area. Specific acoustic impedance is useful in discussing waves in confined mediums, such as tubes and horns. For the simplest case of a plane wave, specific acoustic impedance is the product of the equilibrium density (ρ) of the medium and the Velocity of Wave (v)

$$\boxed{Z = \rho v}$$

The unit of specific acoustic impedance is the pascal second per metre, often called the rayl, after Lord Rayleigh. The unit of acoustic impedance is the pascal second per cubic metre, called an acoustic ohm, by analogy to electrical impedance.

Impedance Mismatch

Mediums in which the speed of sound is different generally have differing acoustic impedances, so that, when a sound wave strikes an interface between the two, it encounters and impedance mismatch. As a result, some of the wave reflects while some is transmitted into the second medium. In the case of the well-known bell-in-vacuum experiment, the impedance mismatches between the bell and the air and between the air and the jar result in very little transmission of sound when the air is at low pressure.

Acoustic Filtration

Filtration of sound plays an important part in the design of air-handling systems. In order to attenuate the level of sound from blower motors and other sources of vibration, regions of larger or smaller cross-sectional area are inserted into air ducts. The impedance mismatch introduced into a duct by a change in the area of the duct or by the addition of a side branch reflects undesirable frequencies, as determined by the size and shape of the variation. A region of either larger or smaller area will function as a low-pass filter, reflecting high frequencies; an opening or series of openings will function as a high-pass filter, removing low frequencies. Some automobile mufflers make use of this type of filter.

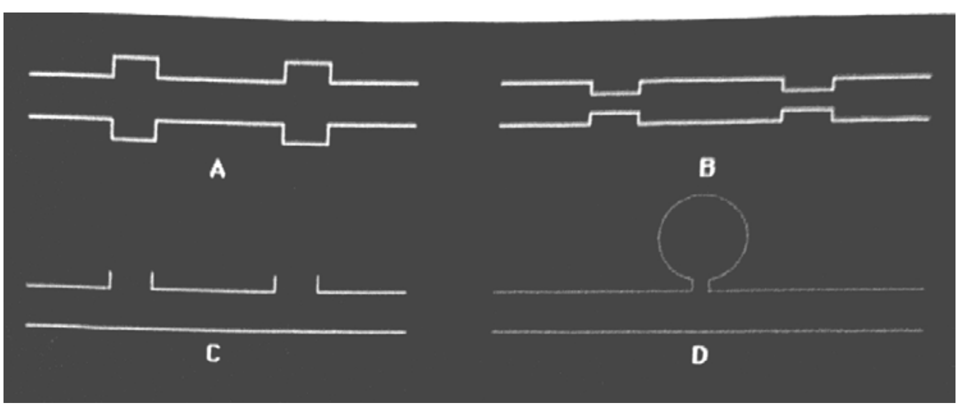


Figure - 1

Acoustic Filters Typically used in Air-handing System

Acoustic filters typically used in air-handing system. (A) and (B) Low-pass filters; (C) a high-pass filter; (D) a band-pass filter, which actually filters out vibrations within a narrow frequency range (see text).

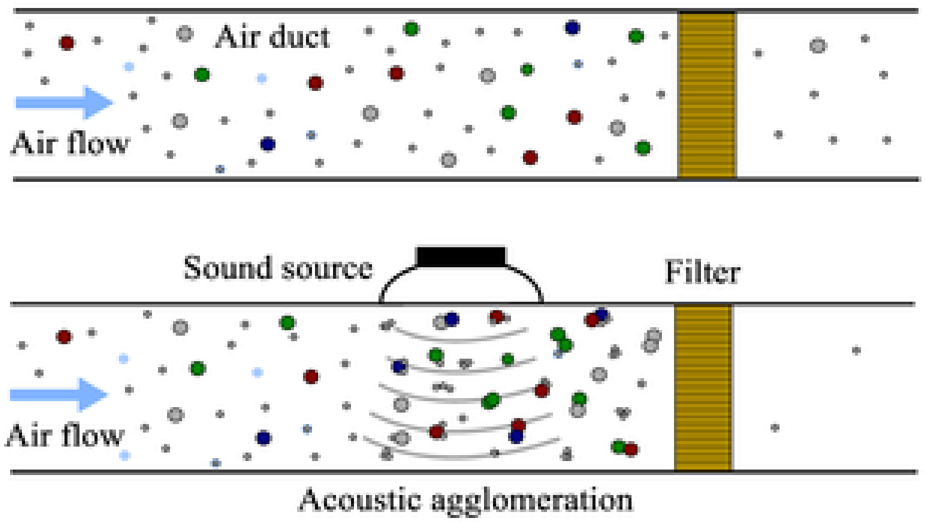


Figure - 2

A connected spherical cavity, forming what is called a band-pass filter, actually functions as a type of band absorber or notch filter, removing a band of frequencies around the resonant frequency of the cavity (see below, standing waves: The Helmholtz resonator).

Sound travels through materials under the influence of sound pressure. Because molecules or atoms of a solid are bound elastically to one another, the excess pressure results in a wave propagating through the solid.

The **acoustic impedance** (**Z**) of a material is defined as the product of its density (ρ) and acoustic velocity (v).

$$Z = \rho v$$

Acoustic impedance is important in

1. the determination of acoustic transmission and reflection at the boundary of two materials having different acoustic impedances.
2. the design of ultrasonic transducers.
3. assessing absorption of sound in a medium.

The following applet can be used to calculate the acoustic impedance for any material, so long as its density (ρ) and acoustic velocity (v) are known. The applet also shows how a change in the impedance affects the amount of acoustic energy that is reflected and transmitted. The values of the reflected and transmitted energy are the fractional amounts of the total energy incident on the interface.

Now, the acoustic impedance on a medium is defined as the ratio of excess pressure to the particle velocity of the medium,

$$Z = \frac{excess\ pressure}{particle\ velocity} = \frac{p}{\frac{\partial y}{\partial t}}$$

$$p = -B\frac{\partial y}{\partial x}$$

If $y = a\sin(\omega t - kx)$

$$\frac{\partial y}{\partial t} = a\ \omega\ cos(\omega t - kx)$$

$$\frac{\partial y}{\partial x} = a\ k\ cos(\omega t - kx)$$

∴ $Z = \frac{-B\ a.k\cos(\omega t - kx)}{a\ \omega\cos(\omega t - kx)} = \frac{Bk}{\omega} = \rho\ v^2.\frac{k}{\omega}$

$= \frac{\rho\ v^2}{v}$

$= \rho v$

$$\boxed{Z = \rho v}$$

Which is impedance of medium and known as ohm's law.

6.5 CHARACTERISTIC IMPEDANCE

- ❖ Any medium through which waves propagate will present an impedance to those waves.
- ❖ If the medium is lossless, and possesses no resistive or dissipation mechanism, for a string the impedance is determined by inertia and elasticity.
- ❖ The presence of a loss mechanism will introduce a complex term into the impedance.

Suppose a string lying along x-axis. The equilibrium tension of the string is T_o. At any instant, the displacement of the string at x = 0 is y.

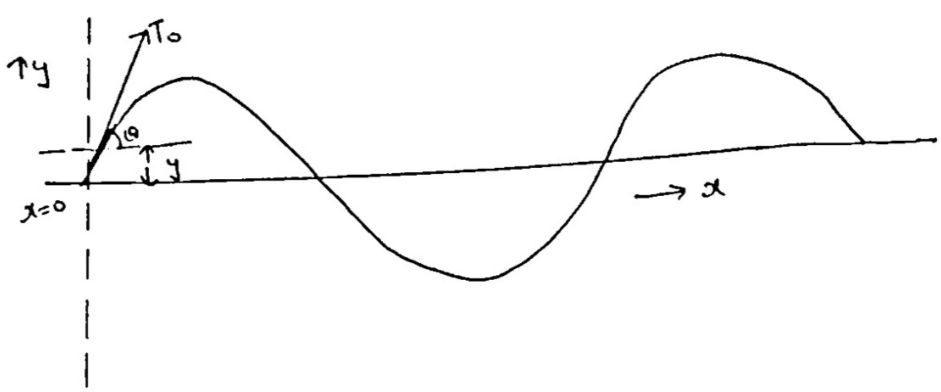


Figure - 3

The string exerts a transverse force (i.e. along y-axis) $T_0 \, sin\theta$ on the output terminal.

i.e. $F_y(on \; oscillator) = T_0 \, sin\theta = T_0 tan\theta$ (θ is small $sin\theta = tan\theta$)

$= T_0 \frac{\partial y}{\partial x}$

$\because \quad \frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial x}$

$\therefore \quad F_y = -\frac{T_0}{v} \frac{\partial y}{\partial x}$

when the string emits a travelling wave, it experiences a drag force which is negatively proportional to the velocity it imposes the medium. Therefore, force on the string equal to F_y but in opposite direction.

$$F_{String} = -F_y = \frac{T_0}{v} \frac{\partial y}{\partial x}$$

the characteristic impedance Z is defined as,

$$Z = \frac{Transverse \; applied \; force}{Transverse \; velocity}$$

$$= \frac{T_0}{v}$$

But $\quad v = \sqrt{\frac{T_0}{\rho_0}} \;\Rightarrow\; T_0 = v^2 \rho_0$

Hence $Z = \frac{v^2 \rho_0}{v} = \rho_0 v$

$$\boxed{Z = \rho_0 v}$$

Example : 1

A one-meter long string weighing one gram is stretched with a force of 10N. Calculate the speed of transverse wave.

Solution :

Linear density F_y, $\quad \rho_0 = \frac{mass}{length}$

mass = one gram = 0.001 kg

length = 1 m

$\therefore \quad \rho_0 = 0.001 \; kg/m$ given T = 10 N

$\therefore \quad v = \sqrt{\frac{T}{\rho_0}} = \sqrt{\frac{10}{0.001}} = \sqrt{10^4} = 10^2 m/sec.$

6.6 STRING

A string is a solid body which is extremely thin, perfectly elastic over its entire length and has no stiffness at all. A string can vibrate only when it is subjected to a certain amount of tension. Both longitudinal and transverse waves can be set up in a string. Longitudinal waves can be set up in a stretched string when it is rubbed by a small piece of chamois-leather along the length. On the other hand, transverse waves can be se up in a stretched string by bowing or plucking. The quality of sound produced in a bowed or plucked string depends on the point of bowing or plucking. Transverse waves are more common than the longitudinal waves in a stretched string.

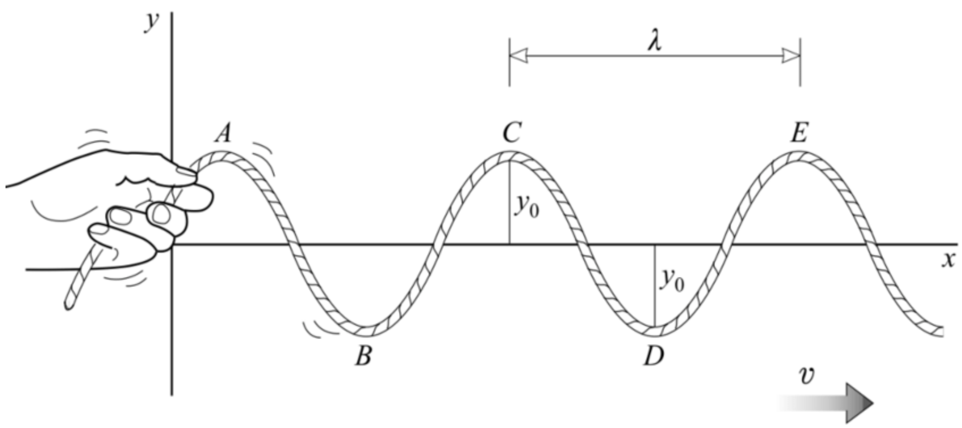


Figure - 4

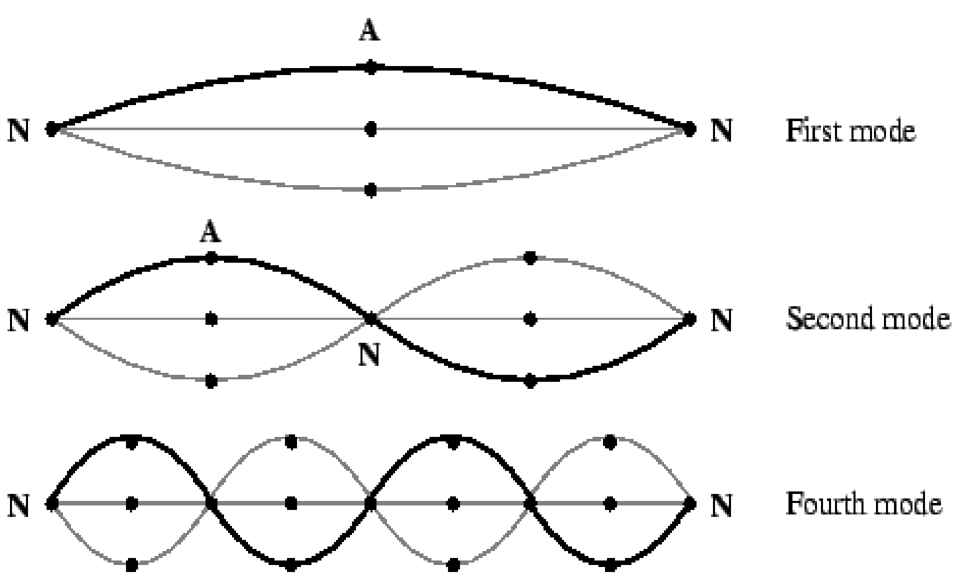


Figure - 5

6.7 PLANE PROGRESSIVE WAVE IN STRETCHED STRING

Consider a uniform stretched string along x-axis having mass per unit length ρ_0. Under equilibrium conditions, it can be considered to be straight. Let us choose x-axis drop the length of the string in its equilibrium state. Suppose that we displace the string normal to its length by a small amount so that a small section of length dx is displaced through a distance y from its mean position. It results in wave motion. we shall assume that the string satisfies the following condition –

(a) Its length is large compared with its diameter.

(b) String is flexible, perfectly.

(c) Tension remain constant at all time.

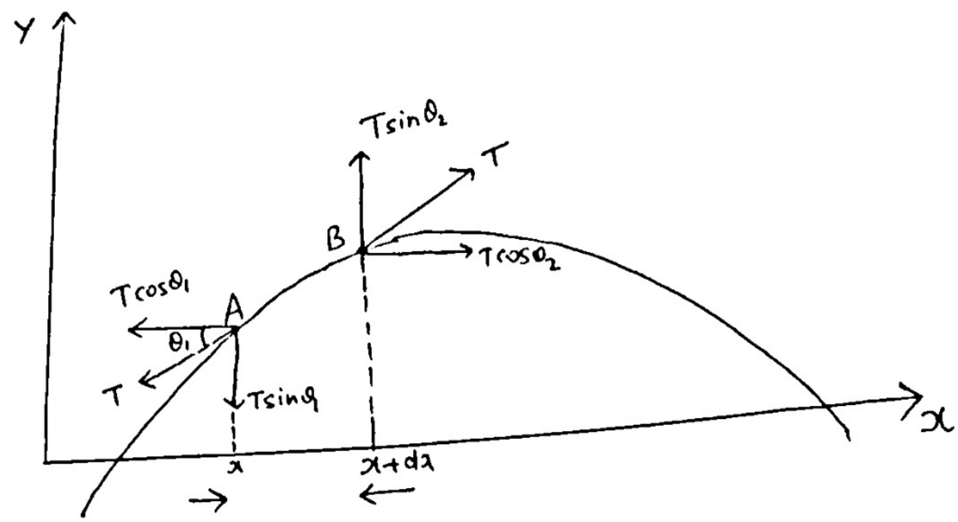


Figure - 6

Suppose that tension at each end of element AB is T. consider a small part of the string dx lying between x and x+dx. The displacement at pint x is y. Let θ_1 and θ_2 be the slope of the tangents at the ends of the segment AB. The net transverse force on the segment is,

$$F = T\ sin\theta_2 - T\ sin\theta_1$$

$$= T\ tan\theta_2 - T\ tan\theta_2$$

($sin\theta_1 = tan\theta_1$ & $sin\theta_2 = tan\theta_2$ because if the curvature of string is not very large, θ_1 & θ_2 will be small)

$$F = T\left(\frac{\partial y}{\partial x}\right)_{x+dx} - T\left(\frac{\partial y}{\partial x}\right)_x$$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)_x . dx + - - - - - - - - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$= T\frac{\partial^2 y}{\partial x^2}dx.$ (use Taylor series expansion)

The equation of motion of the segment is,

$$(\rho_0 . dx).\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}.dx$$

or $\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho_0}\frac{\partial^2 y}{\partial x^2}$ ……………… (1)

Comparing this equation with general wave equation,

$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ……………… (2)

We get, $v^2 = \frac{T}{\rho_0}$

$\Rightarrow$ $\boxed{v = \sqrt{\frac{T}{\rho_0}}}$

which is the velocity of transverse wave in stretched string.

6.8 REFLECTION AND TRANSMISSION COEFFICIENT OF AMPLITUDE OF WAVES ON STRING AT JOINTS OF TWO MEDIA/BOUNDARY

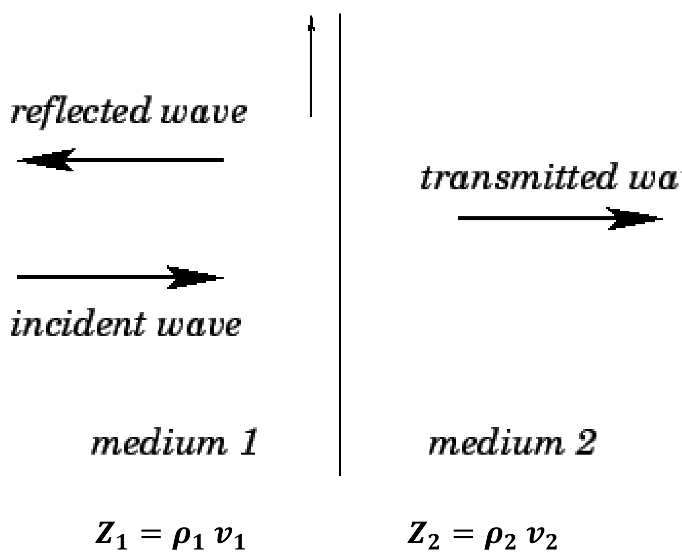


Figure - 7

Suppose string is under a tension T_o.

Let a plane progressive wave is travelling along +x-direction. If ρ_1 and v_1 be the linear mass density and velocity of the wave in **medium 1** respectively then the characteristic impedance of the string is given by,

$$Z_1 = \rho_1 v_1 \qquad \text{……………} (1)$$

Let the two **medium 1** and **2** are joined at x = 0. If Z_2 is the characteristic importance of **medium 2** then

$$Z_2 = \rho_2 v_2 \qquad \text{……………} (2)$$

where, ρ_2 is the linear mass density and v_2 is the velocity of wave in **medium 2** respectively.

When a travelling wave meets at boundary, a part of its amplitude reflected and rest is transmitted. Let the incident, reflected and transmitted wave be,

$$y_i = a_i \sin\left(\omega t - \frac{\omega x}{v_1}\right) = a_i \sin\omega\left(t - \frac{x}{v_1}\right) \qquad \text{……………} (3)$$

$$y_r = a_r \sin\left(\omega t + \frac{\omega x}{v_1}\right) = a_r \sin\omega\left(t + \frac{x}{v_1}\right) \qquad \text{……………} (4)$$

$$y_t = a_t \sin\left(\omega t - \frac{\omega x}{v_2}\right) = a_t \sin\omega\left(t - \frac{x}{v_2}\right) \qquad \text{……………} (5)$$

where, a_i, a_r and a_t are the incident, reflected and transmitted amplitude respectively.

The following boundary condition must hold:-

(a) The displacement is continuous at x = 0

(b) The transverse force $T\frac{\partial y}{\partial x}$ is continuous across the boundary.

Applying first condition (1), we get,

$$y_i + y_r = y_t \qquad at \quad x = 0$$

Using equation (3), (4) and (5)

$$a_i\, sin\omega t + a_r\, sin\omega t = a_t\, sin\omega t$$

Hence, $$a_i + a_r = a_t \qquad \text{……………} (6)$$

Differentiating equation (3), (4), (5) w.r.t. ‘x’ we get,

$$\frac{\partial y_i}{\partial x} = -\frac{a_i}{v_1}\cos\omega\left(t - \frac{x}{v_1}\right)$$

$$\frac{\partial y_r}{\partial x} = +\frac{a_r}{v_1}\cos\omega t\left(t+\frac{x}{v_1}\right)$$

$$\frac{\partial y_t}{\partial x} = -\frac{a_t}{v_2}\cos\omega\left(t-\frac{x}{v_2}\right)$$

Applying boundary condition (2) we get,

$$T\frac{\partial y_i}{\partial x}\bigg|_{x=0} + T\frac{\partial y_r}{\partial x}\bigg|_{x=0} = T\frac{\partial y_t}{\partial x}\bigg|_{x=0}$$

$$T\left(-\frac{a_i}{v_1}\right)cos\omega t + \left(+T\frac{a_r}{v_1}\ cos\omega t\right) = T\left(-\frac{a_t}{v_2}\ cos\omega t\right)$$

$$T\left(\frac{a_i}{v_1} - \frac{a_r}{v_1}\right) = -T\frac{a_t}{v_2}$$

$$T\frac{a_i - a_r}{v_1} = +T\frac{a_t}{v_2}$$

Since $v_1 = \sqrt{\frac{T}{\rho_1}}$ $\therefore$ $\frac{T}{v_1} = \rho_1\, v_1 = Z_1$

Similarly $\frac{T}{v_2} = \rho_2\, v_2 = Z_2$

Hence $(a_i - a_r)Z_1 = Z_2.a_t$ *or* $\frac{a_t}{a_r}$

$$\boxed{a_i - a_r = \frac{Z_2}{Z_1}.a_t} \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

Add equation (6) and (7) we get,

$$2a_i = \left(1+\frac{Z_2}{Z_1}\right)a_t$$

$$\Rightarrow \boxed{\frac{a_t}{a_i} = \frac{2}{1+\frac{Z_2}{Z_1}} = \frac{2Z_1}{Z_2+Z_1}} \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

$\frac{a_t}{a_i}$ is known as transmission amplitude coefficient and is represented by T, as

$$\boxed{T = \frac{2Z_1}{Z_2+Z_1}} \qquad \ldots\ldots\ldots\ldots\ldots (9)$$

Multiplying equation (7) by $\frac{Z_1}{Z_2}$ we get,

$$\frac{Z_1}{Z_2}(a_i - a_r) = a_t \qquad \ldots\ldots\ldots\ldots\ldots (10)$$

Subtracting equation (10) from (4) we get,

$$(a_i - a_r) - \frac{Z_1}{Z_2}(a_i - a_r) = 0$$

$$a_i\left(1-\frac{Z_1}{Z_2}\right)+a_r\left(1+\frac{Z_1}{Z_2}\right)=0$$

$$\boxed{\frac{a_r}{a_i}=\frac{-\left(1-\frac{Z_1}{Z_2}\right)}{\left(1+\frac{Z_1}{Z_2}\right)}=\frac{-(Z_2-Z_1)}{Z_1+Z_1}=\frac{Z_1-Z_2}{Z_1+Z_2}} \qquad \text{…………… (11)}$$

$\frac{a_r}{a_i}$ is known as reflection amplitude coefficient and is represented as.

$$\boxed{R=\frac{Z_1-Z_2}{Z_1+Z_2}} \qquad \text{…………… (12)}$$

⇒ T = 1+R ⇒ R lies between -1 and +1 and T between 0 & 2.

Conclusions:-

(a) If $Z_2 > Z_1$, then from equation (12)

$$R=\frac{Z_1-Z_2}{Z_1+Z_2}\rightarrow -ve$$

It means that reflected amplitude is opposite in sign; which means that reflected wave suffers phase change of Π. Example is, sound waves are incident from air to water.

(b) If $Z_2 < Z_1$, then R is positive, the reflected wave has no change of phase.

(c) If $Z_2 = 0$, then,

R = 1

T = 2

Reflected wave has the same amplitude as the incident wave and there is no phase change.

(d) If $Z_1 = Z_2$ (In absence of discontinuity) i.e. two strings are identical

R = 0

T = 1

Which means that incident wave is transmitted as such without any reflection. This is called impedance matching. This phenomenon is important in transfer of energy.

(e) $Z_2 = \infty$ then R = -1 and T = 0. The incident wave is totally reflected with a phase change of Π.

6.9 REFLECTION AND TRANSMISSION OF ENERGY WAVES AT JOINT OF TWO MEDIA

Let us consider two strings of linear mass density ρ_1 & ρ_2 joined at x = 0. Suppose a plane harmonic wave of amplitude a_i and angular frequency ω is travelling along the first string at a velocity v_1. The energy flow in the string is,

$$\frac{1}{2}\,\omega^2 a_i^2 \rho_1 v_1 = \frac{1}{2}\omega^2 a_i^2 Z_1 \qquad \text{……………} (1)$$

Where $Z_1 = \rho_1 v_1$ is the impedance of first string. This is the energy arrives at x = 0.

The partial reflection and transmission of energy of the wave which incident on the boundary at x = 0. Let a_r and a_t be the reflected and transmitted amplitudes and Z_2 be the impedance of second string, then energy at boundary is,

$$\frac{1}{2}\,\omega^2 a_r^2 Z_1 + \frac{1}{2}\omega^2 a_r^2 Z_2 \qquad \text{……………} (2)$$

we can write the equation (2) as,

$$= \frac{1}{2}\omega^2 a_i^2 \left(\frac{a_r}{a_i}\right)^2 Z_1 + \frac{1}{2}\omega^2 a_i^2 \left(\frac{a_t}{a_i}\right)^2 Z_2$$

$$= \frac{1}{2}\omega^2 a_i^2 \left[\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 Z_1 + \left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 Z_2\right]$$

$$= \frac{1}{2}\omega^2\, a_i^2\, Z_1 \left[\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 + \frac{4Z_1^2}{(Z_1 + Z_2)^2}\frac{Z_2}{Z_1}\right]$$

$$= \frac{1}{2}\omega^2\, a_i^2\, Z_1 \left[\frac{(Z_1 + Z_2)^2}{(Z_1 + Z_2)^2}\right] = \frac{1}{2}\omega^2\, a_i^2\, Z_1$$

Which is same as the energy incident at boundary x = 0. Hence the total energy is conserved.

The Reflection energy coefficient is given by,

$$\frac{Reflected\ energy}{Incident\ energy} = \frac{\frac{1}{2}\omega^2\, a_r^2\, Z_1}{\frac{1}{2}\omega^2\, a_i^2\, Z_1} = \left(\frac{a_r}{a_i}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

Similarly transmission energy coefficient is given by,

$$\frac{Transmited\ energy}{Incident\ energy} = \frac{\frac{1}{2}\omega^2\, a_t^2\, Z_1}{\frac{1}{2}\omega^2\, a_i^2\, Z_1} = \left(\frac{a_t}{a_i}\right)^2 . \frac{Z_2}{Z_1}$$

$$= \frac{4Z_1^2}{(Z_1 + Z_2)^2} \cdot \frac{Z_2}{Z_1}$$

$$= \frac{4Z_1\, Z_2}{(Z_1 + Z_2)^2}$$

If the wave is incident from second string or medium, then reflection coefficient would be $\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$. Thus the boundary reflects the same fraction of incident energy in either direction.

Example: 2

A travelling wave in SI unit is given by,

$$\boldsymbol{y = 0.04\, e^{-\left(\frac{t}{2} - \frac{x}{0.3}\right)^2}}$$

Find (a) the speed of wave (b) initial shape of string.

Solution:

The equation is,

$$y = 0.04\, e^{-\left(\frac{t}{2} - \frac{x}{0.3}\right)^2} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

(a) equation (1) can be written as,

$$y = 0.04\, e^{-\left(\frac{x}{0.3} - \frac{t}{2}\right)^2}$$

$$y = 0.04\, e^{-\left(\frac{1}{(0.3)^2}\ \left(x - \frac{0.3t}{2}\right)^2\right)}$$

Comparing it with the general equation, $y = f(x - vt)$

$$v = \frac{0.3}{2} = 0.15\ m/s$$

(b) at t = 0.

$$y = 0.04\, e^{-\frac{x^2}{0.09}}$$

Example: 3

A simple harmonic wave of frequency 75 Hz and amplitude 2 cm is travelling along the axis with a velocity of 45 m/s. Find the velocity of particle at a distance of 15 m from the origin after and interval of $\frac{1}{3}$ seconds.

Solution:

The equation of wave travelling along the +ve x axis is,

$$y = a \sin(\omega t - kx)$$

$a = 2cm = 2 \times 10^{-2} m, \qquad \omega = 2\Pi v = 2\Pi \times 75 = 150\Pi$

$v = 45\ m/s,$

$$v = \frac{\omega}{k} \ \Rightarrow k = \frac{\omega}{v} = \frac{150\Pi}{45} = \frac{10}{3}\Pi\ m^{-1}$$

velocity of particle is given by,

$$v = \frac{\partial y}{\partial t} = a\,\omega \cos(\omega t - kx)$$

at $\qquad x = 15m, \qquad t = \frac{1}{3}\ second.$

$$v = 2 \times 10^{-2} \times 150\Pi \cos\left(150\Pi \times \frac{1}{3} - \frac{10\Pi}{3} \times 15\right)$$

$$= 2 \times 10^{-2} \times 150\Pi = 3\Pi$$

6.10 SUMMARY

- The **acoustic impedance** (**Z**) of a material is defined as the product of its density (ρ) and acoustic velocity (v).

$$Z = \rho v$$

- A string is a solid body, which is infinitely thin, perfectly elastic over its entire length and has no stiffness at all. It can vibrate only if it is subjected to a tension.
- Longitudinal waves can be se up in a stretched string when it is rubbed by a small piece of chamois leather along it length.
- Transverse waves can be set up in a stretched string by bowing or plucking.
- The velocity of transverse vibration in a string is given by $v = n\lambda\sqrt{\frac{F}{\mu}}$.
- When a string is fixed between two rigid supports and transverse waves are set up in it, the waves travel along the string and get reflected from the fixed end. Hence, there is superposition of two identical progressive waves travelling in opposite directions, the result of which is the formation of stationary waves with definite nodes and antinodes. Tat the two fixed ends, there are nodes and in between these two nodes, there may be any number of antinodes and nodes.

6.11 TERMINAL QUESTIONS

1. A transverse wave on a string with a mass per unit length of 0.04 kg/m is given by

$y(x,t) = 0.02 \sin(30\Pi\ t - 5\Pi\ x)$.

Find the frequency, wavelength and amplitude of the wave?

2. A plane progressives wave travelling along -ve x-axis is characterized by frequency 500 Hz, amplitude 0.02 m and phase velocity 400 ms^{-1}. Write down its equation.

3. Calculate the characteristic impedance offered by a sonometer wire stretched by a force of 20N. It weighs 2g per meter.

4. Calculate acoustic impedance of air and water at NTP. Use $\rho_{air} = 1.29\ kgm^{-3}$, $v_{air} = 332\ ms^{-1}, \rho_{water} = 10^3 kg\ m^{-3}$ and $v_{water} = 1500\ m5^{-1}$.

5. Two strings are joined together and stretched under the same tension. For transverse wave, calculate the reflection and transmission amplitude coefficients when the ratio of their linear densities 1:9.

6. Define Impedance.

7. What is Acoustic Impedance?

8. Write down the expression of Acoustic Impedance?

9. Obtain expression for Plane Progressive Wave in Stretched String.

10. Explain the concept of Characteristic Impedance.

11. Obtain expression of reflection and transmission of energy waves at joint of two media.

UNIT-07 SUPERPOSITION OF WAVES

Structure

7.1 INTRODUCTION

When two or more progressive waves travelling in a medium meet at a point, they are said to have superposed on one another, and the Phenomenon itself is known as Superposition. This leads to the phenomena of interference, beats and the formation of stationary waves in the medium. In this unit, we shall study about the Superposition of Waves and its Consequences. Moreover, we shall also study stationary waves, harmonics and overtones.

7.2 OBJECTIVES

After studying this unit, you should be able to –

- Understand the concept of Principle of Superposition
- Define Standing Waves
- Solve Problems based on standing or stationary Waves
- Comparison between Interference and Beats
- Understand Concept of Energy of Stationary Waves.

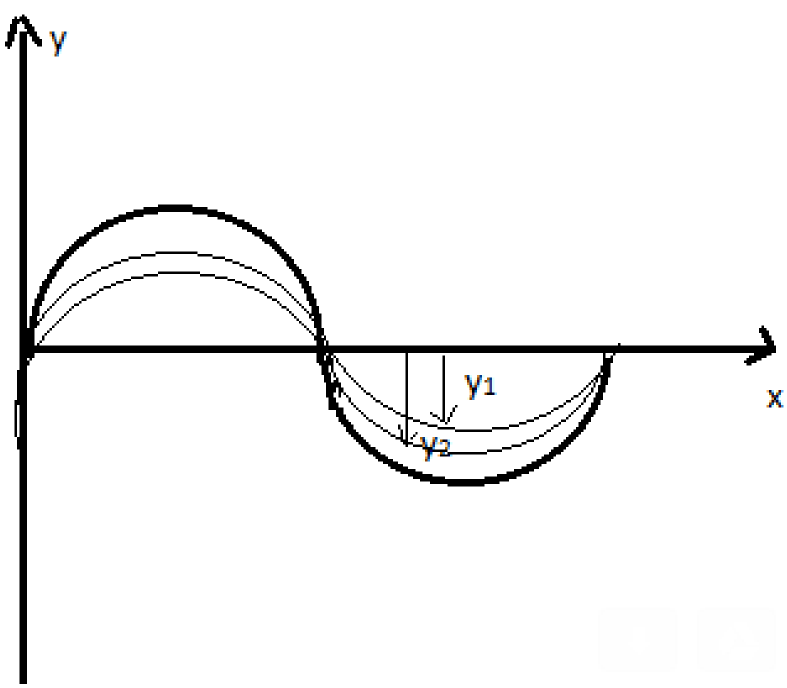
7.3 SUPERPOSITION OF WAVES

When two or more waves travel the same path & independent of one another, the resultant displacement of a particle at a given time is found to be equal to the algebraic sum of its displacements due to individual waves. In other words, the resultant displacement of a particle can be determined by algebraically sum of the displacements of individual waves. This is known as **principle of superposition of waves**.

A wave is characterized by its amplitude, angular frequency, wave vector and phase. Therefore, depending on which of these components is same or different, superposition of waves give rise to significantly different but very interesting phenomena. We now discuss some of these:-

What is Superposition of Waves?

According to the principle of superposition. The resultant displacement of a number of waves in a medium at a particular point is the vector sum of the individual displacements produced by each of the waves at that point.



7.3.1 PRINCIPLE OF SUPERPOSITION

Considering two waves, travelling simultaneously along the same stretched string in opposite directions as shown in the figure above. We can see images of waveforms in the string at each instant of time. It is observed that the net displacement of any element of the string at a given time is the algebraic sum of the displacements due to each wave.

Let us say two waves are travelling alone and the displacements of any element of these two waves can be represented by $y_1(x, t)$ and $y_2(x, t)$. When these two waves overlap, the resultant displacement can be given as $y(x,t)$.

Mathematically, $y(x, t) = y_1(x, t) + y_2(x, t)$

What is Interference of Light?

The phenomena of formation of maximum intensity at some points and minimum intensity at some other point when two (or) more waves of equal frequency having constant phase difference arrive at a point simultaneously, superimpose with each other is known as interference.

Types of Superposition of Waves

According to the phase difference in superimposing waves, interference is divided into two categories as follows.

Constructive Interference

If two waves superimpose with each other in the same phase, the amplitude of the resultant is equal to the sum of the amplitudes of individual waves resulting in the maximum intensity of light, this is known as constructive interference.

Destructive Interference

If two waves superimpose with each other in opposite phase, the amplitude of the resultant is equal to the difference in amplitude of individual waves, resulting in the minimum intensity of light, this is known as destructive interference.

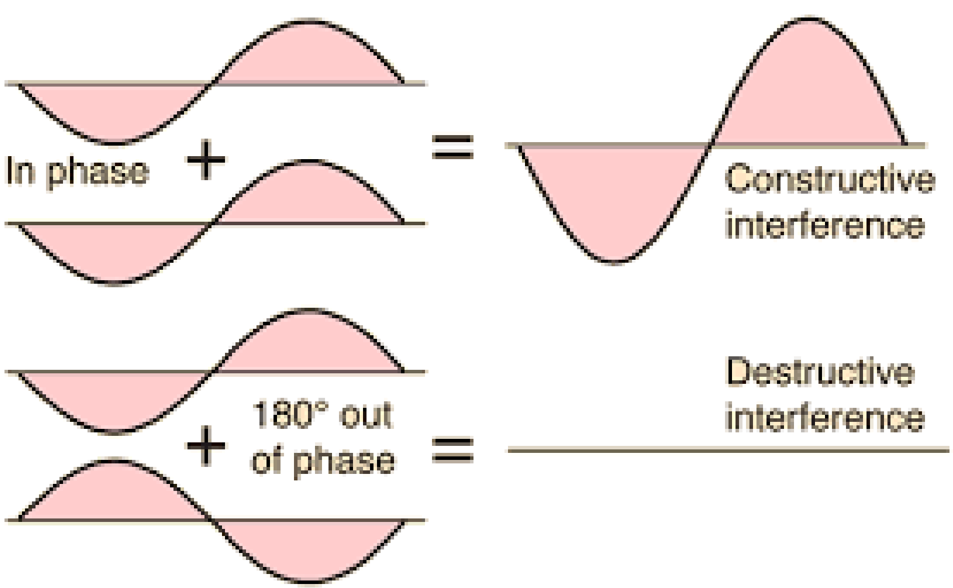


Figure – 1

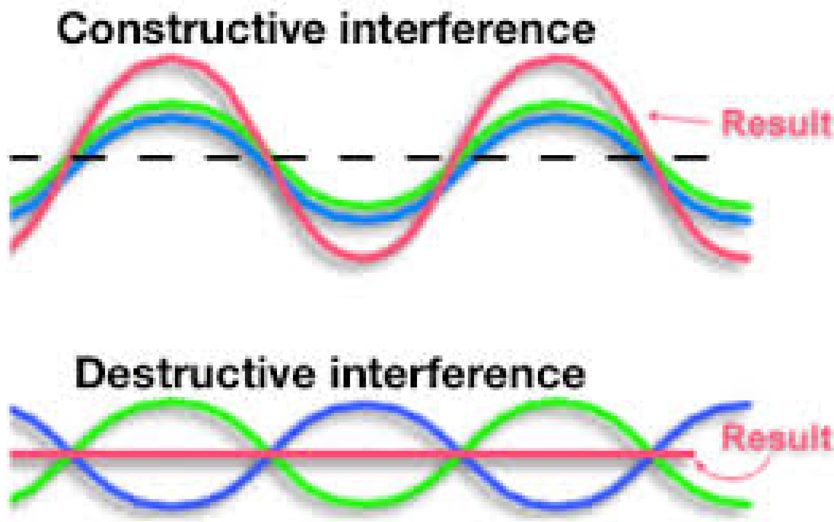


Figure – 2

7.3.2 LIMITATIONS OF PRINCIPLE OF SUPERPOSITION

- Superposition theorem doesn't work for power calculation.
- Power calculations involve either the product of voltage and current, the square of current or the square of the voltage.
- They are not linear operations.
- This statement can be explained with a simple example.

(a) Superposition of in Phase Wave of Different Amplitudes:-

Consider two waves which are in phase having same frequency, wave vector and phase but have different amplitudes propagating along the x-axis.

$$y_1(x,t) = a_1 \sin(\omega t - kx)$$

$$y_2(x,t) = a_2 \sin(\omega t - kx),$$

then Resultant wave is given by,

$$y = y_1 + y_2$$

$$= (a_1 + a_2) \sin(\omega t - kx)$$

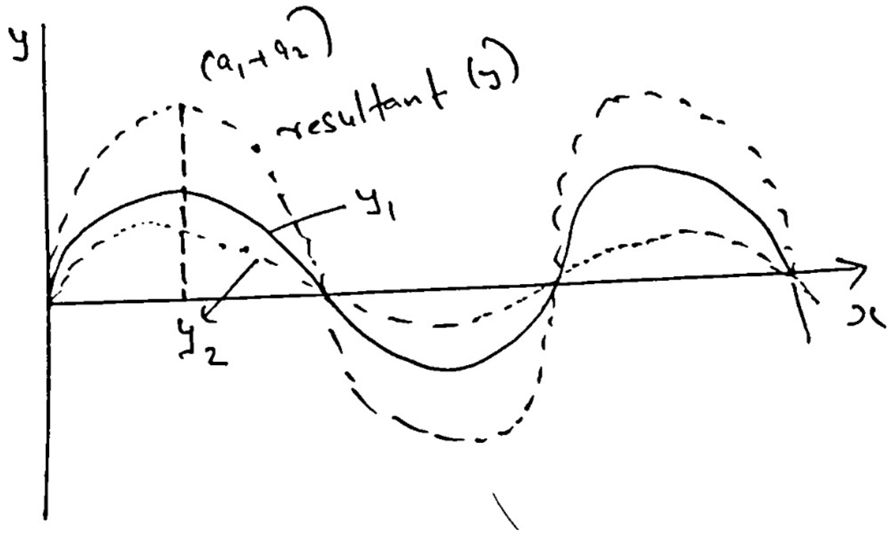


Figure - 3

(b) Superposition of Identical Out of Phase Waves:-

Let us consider that two identical but out of phase waves moving in the same direction the x-axis.

$$y_1(x,t) = a\sin(\omega t - kx)$$

$$y_2(x,t) = a\sin(\omega t - kx + \phi)$$

when such waves superpose, the amplitude of resultant wave can very from 0 to 2a. This leads to phenomenon of **interference**.

(c) Superposition of Identical Waves of Slightly Different Frequencies:-

When two waves having slightly different frequencies but equal amplitudes more in the same direction, we can write,

$$y_1(x,t) = asin(\omega_1 t - kx)$$

$$y_2(x,t) = a\sin(\omega_2 t - kx)$$

when such waves superpose, their superposition leads to beats.

(d) Superposition of identical Waves Moving in Opposite Directions:-

Let us consider superposition of two identical waves propagating in opposite directions in a medium.

We can write,

$$y_1(x,t) = a\sin(\omega t - kx)$$

$$y_2(x,t) = a\sin(\omega t + kx)$$

These waves are identical and superpose in a medium and they give rise to **stationary** or **standing waves**.

7.4 STANDING OR STATIONARY WAVES

When two identical waves (either longitudinal or transverse) travelling in opposite directions along the same line, superpose to each other they give rise to a new wave called **stationary** or **standing waves**.

For example, when a wave is produced in string or air column in organ pipes, it is reflected at the other end and super impose upon the incident wave to produce stationary wave.

The stationary waves can be formed only in a linear bounded medium. A medium, in which the wave is propagating linearly, must be finite in length, i.e. medium should have a boundary. This is an important condition for the formation of standing waves.

Mathematical Analysis:-

Case (a) Closed end pipe or string fixed at other end (one boundary is rigid and the other is free):-

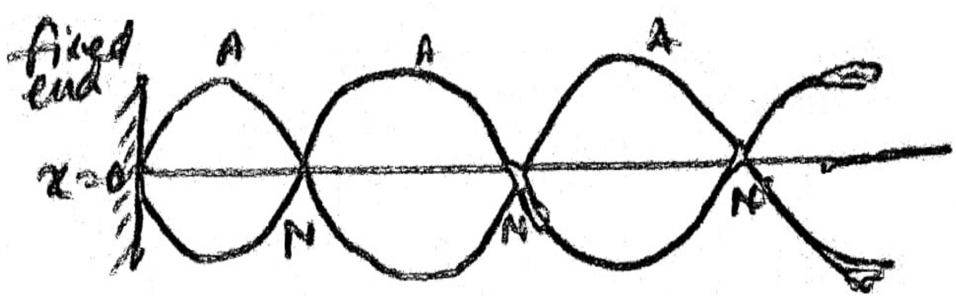


Figure - 2

Let the displacement of wave at any point of the medium having amplitude 'a' and wavelength 'λ' moving in +ve x-direction is,

$y_1 = a\sin\frac{2\Pi}{\lambda}(vt - x) = a\sin(\omega t - kx)$ ……………… (1)

this wave is incident normally on the fixed end at x = 0 and then reflected. The reflected wave moving along –x-direction will be given by,

$y_2 = a'\sin\frac{2\Pi}{\lambda}(vt + x) = a\sin(\omega t + kx)$ ……………… (2)

where a' is the amplitude of reflected wave.

The resultant displacement at point x is given by the principle of superposition, as,

$$y = y_1 + y_2$$

$y = a\sin(\omega t - kx) + a'\sin(\omega t + kx)$ ……………… (3)

at the fixed boundary x = 0, y = 0

$\therefore \qquad 0 = a \sin \omega t + a' \sin \omega t$

$\Rightarrow \qquad a = -a'$

It simply means that phase change of Π occurs at fixed end (rigid boundary). Hence equation (3) becomes,

$$y = a\sin(\omega t - kx) - a\sin(\omega t + kx)$$

$$= a[sin(\omega t - kx) - sin(\omega t + kx)]$$

$$\boxed{y = -2a \sin kx \cos \omega t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

$$\boxed{y = A \cos \omega t}$$

where, $A = -2a \sin kx$ is the amplitude of resultant wave. The equation (4) is the equation of **stationary** or **standing waves**.

The particle velocity of the standing wave is,

$$\frac{dy}{dt} = 2a\omega \; sinkx \; sin\omega t \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

The Acceleration of the particle is,

$$\frac{d^2y}{dt^2} = 2a\omega^2 \; sinkx \; cos\omega t \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

The pressure variation is,

$$p = -k\frac{dy}{dt} = 2k.ka \cos kx \; cos \; \omega t$$

$$p = 2k^2 \; a \cos kx \; cos \; \omega t \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

The displacement, amplitude, velocity, acceleration and pressure variation all depends position and time.

CHANGES W.R.T. POSITION

(a) At the position where, *sin kx = 0* *(at x = 0, y = 0)*

$$kx = n\Pi$$

$$x = \frac{n\Pi}{k} = \frac{n\Pi\lambda}{2\Pi} = \frac{n\lambda}{2}$$

$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, ------$; n = 0, 1, 2, ------

At this point, we find displacement y = 0

Velocity, $\frac{dy}{dt} = 0$

Acceleration, $\frac{d^2y}{dt^2} = 0$

Pressure variation, $-k\frac{dy}{dx}$ = maximum.

These points are called 'notes'. Thus the modal points are separated by a distance $\frac{\lambda}{2}$.

(b) At the position, where,

$$\sin kx = \pm 1$$

$$kx = n\frac{\Pi}{2}$$

$$x = n\frac{\Pi}{2} \times \frac{1}{k} = \frac{n\Pi}{2} \times \frac{\lambda}{2\Pi} = \frac{n\lambda}{4}.$$

$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, - - - - -$ n = 1, 3, 5, ------

At this point, we find displacement, y = maximum

Velocity, $\frac{dy}{dt}$ = maximum.

Acceleration, $\frac{d^2y}{dt^2}$ = maximum

Presume variation, $-k\frac{dy}{dx} = 0$

These points are called 'Antinodes'. Thus antinodal points are separated by $\frac{\lambda}{2}$.

7.4.1 MODES OF VIBRATION

Suppose one end fixed at x = 0 and other end be free at $x = l$.

Boundary conditions are

(a) at x = 0, y = 0

(b) and at $x = l$, $\frac{dy}{dx} = 0$

(at free boundary, which has no inertia, the pressure change $p = \frac{-dy}{dx}$. K is always zero, Hence $\frac{dy}{dx} = 0$)

$\because$ $y = -2a \sin kx \cos \omega t$ ………….. (1)

Applying condition (a), we get,

At x = 0, y = 0 using equation (1)

Applying condition (b) in equation (1) we get,

$$\frac{dy}{dx} = -2ak \cos kl \cos \omega t$$

$$0 = -2ak \cos kl \cos \omega t$$

This condition hold for all valves of t hence.

$$\cos kl = 0$$

$kl = (2p+1)\frac{\Pi}{2}$, p = 0, 1, 2, -----

$$\frac{2\Pi}{\lambda} l = (2p+1)\frac{\Pi}{2}$$

or, $\lambda = \frac{4l}{2p+1}$, p = 0, 1, 2, -----

Hence allowed frequencies are

$$n = \frac{v}{\lambda} = \frac{v}{4l}, \frac{3v}{4l}, \frac{5v}{4l}, \frac{7v}{4l}, - - - - - -$$

Hence fundamental frequency is $\frac{v}{4l}$ and higher harmonics are odd multiples of it. Thus only odd harmonies or overtones are present when one boundary is rigid.

Case (b)   Open end pipe or string free at the other end:-

(both the boundaries are free)

e.g.,   Example of such a medium is a pipe open at both ends and a string clamped at the middle.

Let the equation of incident simple harmonic wave moving along +x-direction be

$$y_1 = a\sin(\omega t - kx)$$

The equation of reflected wave be,

$y_2 = a' \sin(\omega t - kx)$.

The resultant displacement at any point x as time t is

$$y = y_1 + y_2$$

$y = a\sin(\omega t - kx) + a' \sin(\omega t - kx)$ .... (1)

When reflection takes place at free boundary, the phase of displacement does not get reversed. Hence the boundary condition are,

At $x = 0$, $\frac{dy}{dx} = 0$

& also at $x = l$, $\frac{dy}{dx} = 0$ for all values of t

Applying boundary condition, $\frac{dy}{dx} = 0$ at x = 0, we get,

From equation (1) as,

$$0 = -ka\cos\omega t + a'k\cos\omega t$$

$\Rightarrow$ $a = a'$

It means that there is no phase reversal. Hence the resultant displacement is,

$$y = a[sin(\omega t - kx) + sin(\omega t - kx)]$$

$$y = 2a \cos kx \sin \omega t$$

$\boxed{y = A \sin \omega t}$, where, $A = 2a \cos kx$

This represents the resultant vibration of a particle whose amplitude is $A = 2a \cos kx$.

Then the particle velocity,

$$\frac{dy}{dt} = 2a\omega \cos kx \cos \omega t$$

the Acceleration,

$$\frac{d^2y}{dt} = -2a\omega^2 \cos kx \sin \omega t$$

the pressure variation,

$p = -k\frac{dy}{dx} = +2ak^2 \sin kx \sin \omega t.$

all these physical parameter change with respect to position and time.

CHANGES W.R.T. POSITION

(a) At the position where,

$\sin kx = 0$

$(\because \frac{dy}{dx} = 0 \; at \; x = 0)$

$kx = n\Pi$; n = 0, 1, 2,-------

$x = \frac{n\Pi}{k} = \frac{n\Pi}{2\Pi}\lambda = \frac{n\lambda}{2}$

$x = \frac{n\lambda}{2}$, n = 0, 1, 2, ---------

or $x = 0, \frac{\lambda}{2}, \lambda \; \frac{3\lambda}{2}, - - - - - - -$

The displacement y = maximum

Velocity $\frac{dy}{dt}$ = maximum

Acceleration $\frac{d^2y}{dt^2}$ = maximum

Pressure variation, $p = 0$

These points are called Nodes. The distance between two nodal points is $\frac{\lambda}{2}$.

(b) Positions where,

$$\sin kx = \pm 1$$

$$kx = \frac{\Pi}{2}, \frac{3\Pi}{2}, \frac{5\Pi}{2}, --------$$

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ------$$

the displacement, y = 0

velocity, $\frac{dy}{dt} = 0$

Acceleration $\frac{d^2y}{dt^2} = 0$

Pressure variation p = maximum.

These points are called Antinodes. Antinodes are separated by $\frac{\lambda}{2}$ distance from each other.

MODES OF VIBRATION

When both the boundaries are free, we have,

$$y = 2a \cos kx \sin \omega t$$

Differentiating w.r.t. ‘x’ we get,

$$\frac{dy}{dx} = -2ka\ sinkx\ sin\omega t$$

when both ends are free i.e. at x = 0, $\frac{dy}{dx} = 0$

& at x = l, $\frac{dy}{dx} = 0$

$$-2ka \sin kl\ sin\omega t = 0$$

it holds for values of ‘t’ then,

$$\sin kl = 0$$

$kl = p\Pi,$ þ = 0, 1, 2, -----

$$k = \frac{p\Pi}{l}$$

$$\frac{2\Pi}{\lambda} = \frac{p\Pi}{l}$$

$$\lambda = \frac{2l}{p}$$

Thus allowed frequencies an,

$$v = n\lambda$$

$\Rightarrow$ $n = \frac{v}{\lambda} = \frac{þv}{2l} = \frac{v}{l}, \frac{2v}{2l}, \frac{3v}{2l}, \frac{5v}{2l} ----$

The fundamental frequency is $\frac{v}{l}$- and higher frequencies are multiples of this fundamental frequency $\frac{v}{l}$. Thus all harmonics or overtones are present in this case.

7.4.2 ENERGY OF STATIONARY WAVES

The stationary waves are formed by superposition of two waves travelling in opposite directions. The energy carried by one wave is equal to that by the other wave but in opposite direction. Thus the resultant energy transfer to any direction is zero. We can see it mathematically as follows:-

Let the equation for displacement be

$y = 2a\cos kx \sin \omega t$ (when both the boundaries is free).

The pressure variation is $p = -k\frac{dy}{dx}$

$$\frac{dy}{dx} = -2ka\ sinkx\ sin\omega t$$

$$p = -k\frac{dy}{dx}$$

$$p = 2k^2a\ sinkx\ sin\omega t$$

where, $p_0 = 2k^2a\ sin\omega t$

particle velocity, $u = \frac{\partial y}{\partial t} = 2a\omega\ coskx\ cos\omega t$

$$\boxed{u = u_0\ cos\omega t}$$

where, $u_0 = 2a\omega\ coskx$

Distance travelled by a particle in time dt = u.dt. Therefore energy transfer (work done per unit area) in small interval of time dt is,

$$p = u.dt$$

Hence energy transmitted in one complete time period is,

$$\int_0^T p.udt$$

∴ Rate of flow of energy per unit area or power is,

$$\frac{1}{T}\int_0^T p.udt$$

$$= \frac{1}{T}\int_0^T p_0\ sin\omega t\ u_0\ cos\omega t\ dt$$

$$= \frac{1}{T}\ (p_0\, u_0) \int_0^T sin\omega t\ cos\omega t\ dt$$

$$= \frac{p_0\, u_0}{2T} \int_0^T sin2\omega t\ dt = 0$$

Thus, no energy is transferred from one section or loop to another section or loop in a stationary waves.

7.4.3 CHARACTERISTICS OF STATIONARY WAVES

1. Nodes and antinodes are formed alternately.
2. Nodes are points where the particle are vibrate with zero amplitude and velocity but strain i.e. change in pressure is maximum.
3. Antinodes are points where the particles are vibrate with maximum amplitude and velocity and having minimum strain.
4. The distance between two adjacent nodes or two adjacent antinodes is $\frac{\lambda}{2}$.
5. All the particles have same frequency, vibrate simple harmonically.
6. There is no net flow of energy in any direction.

7.5 STANDING WAVE RATIO (SWR)

When the incident wave and partially reflected wave having unequal amplitudes superposed they give rise to a standing wave in which there will not be zero displacement at nodes. The ratio of the maximum amplitude to the minimum amplitude is called the standing wave ratio (SWR).

Let us consider the equation of the incident and reflected wave is,

$$y_i = A\cos(\omega t - kx)$$

$$y_r = RA\cos(\omega t + kx)$$

where 'R' is reflection coefficient.

The equation of standing wave is

$$y = y_i + y_r$$

$$= A\cos(\omega t - kx) + RA\cos(\omega t + kx)$$

$$= A\ (1 + R)\cos\omega t\cos kx + A\ (1 - R) sin\omega t\ sinkx.$$

$$= A\cos\omega t + A\ c\sin\omega t\ sin\phi$$

Where,

$c\ cos\phi = (1 + R)\cos kx$ & $c\ sin\phi = (1 - R)\sin kx.$

$$\boxed{y = A\,c\cos(\omega t - \phi)}$$

where, $c = (1 + R^2 + 2R\cos 2kx)^{\frac{1}{2}}$

standing wave Amplitude is

$$Ac = A\,(1 + R^2 + 2R\cos 2kx)^{\frac{1}{2}}$$

The maximum amplitude accures at cos 2kx = 1

$\therefore \quad (Ac)_{max} = A(1 + R^2 + 2R)^{\frac{1}{2}} = A(1 + R)$

The minimum amplitude occurs at cos2kx = - 1

$\therefore \quad (Ac)_{min} = A(1 + R^2 - 2R)^{\frac{1}{2}} = A(1 - R)$

Hence,

$$\boxed{SWR = \frac{Maximum\ amplitude}{Minimum\ amplitude} = \frac{1+R}{1-R}}$$

7.6 DISTINCTION BETWEEN INTERFERENCE AND BEATS

S. No.	Interference	S. No.	Beats
1	It occurs when two waves travel in same or opposite directions.	1	It occurs only when two waves travel in same direction.
2	Frequency of two waves are exactly equal.	2	Frequency of two waves are nearly equal.
3	The position of maxima and minima are fixed in the interference pattern.	3	Position of maxima and minima are not fixed in the beat pattern.
4	The amplitude of resultant wave varies from point to point but remains fixed for a particular point.	4	The amplitude of the resultant waves varies with time.
5	It is case of interference in space.	5	It is a case of interference in time.

Example : 8

A note produces 4 beats/second with a tuning fork of frequency 512 Hz and 6 beats/second with a tuning fork of frequency 514 Hz. Find the frequency of the note.

Solution :

Let n be the possible frequency of the note. Then in first case,

$$n = 512 \pm 4 = 516, 508\ Hz$$

and in second case, $n = 514 \pm 6 = 520, 508\ Hz$

Hence the frequency of the note is 508 Hz.

Example: 9

Show that in a stationary wave, all particles between any two consecutive nodes are in phase but they are in opposite phase with the particles between the next pair of nodes.

Solution:

The displacement of a standing wave is given by,

$$y = 2a \cos kx \sin \omega t$$

for any two points say, $x = x_1$ and $x = x_2$

$$y = 2a \cos kx_1 \sin \omega t$$

$$y = 2a \cos kx_2 \sin \omega t$$

the phase $\boldsymbol{\omega t}$ is same.

But if $x_2 = x_1 + \frac{\lambda}{2}$, i.e. the points belong to adjacent loops (they lie between consecutive pairs of nodes. The corresponding displacement are,

$$y_1 = 2a \cos kx_1 \sin \omega t$$

and

$$y_2 = 2a \cos kx_2 \sin \omega t = 2a\ cosk\left(x_1 + \frac{\lambda}{2}\right) \sin \omega t$$

$$y_2 = 2a\ cos\left(kx_1 + k\frac{\lambda}{2}\right) \sin \omega t$$

$$= 2a\ cos\left(kx_1 + \frac{2\Pi}{\lambda}.\frac{\lambda}{2}\right) \sin \omega t$$

$$= 2a\ cos(kx_1 + \Pi) \sin \omega t$$

$$= -2a \cos kx_1 \sin \omega t = 2a \cos kx_1\ sin(\omega t + \Pi)$$

Which shows that the phases of y_1 and y_2 are differ by Π.

Example: 10

The fundamental frequency of an organ pipe is 110 Hz. Other frequencies of tones produced by it are 220, 440, 550, 660 Hz. Is this pipe open at both ends or closed at one end and open at the other end? Find the length of pipe (speed of sound = 330 m/s).

Solution:

The given frequencies are 220, 440, 550, 660,

= 110 × (2, 4, 5, 6)

= fundament frequency × (2, 4,, 5, 6)

these are even and odd multiples of fundamental frequency 110 Hz. Thus pipe is open at both ends.

Fundament frequency = 110 Hz

$$n = \frac{v}{2l}$$

$$l = \frac{v}{2n} = \frac{330}{2 \times 100} = 1.5m$$

7.7 SUMMARY

- The principle of superposition of waves sates that if a number of progressive waves travelling through a medium meet simultaneously at a point, the resultant displacement of any particle at that point is equal to the vector sum of the displacements produced by the component waves at that point.
- Interference effect can be observed only if the two interfering waves are coherent. Two waves are said to be coherent, if their phase difference at any point is independent of time.
- There is one node between two successive antinodes and one antinode between two successive nodes.

7.8 TERMINAL QUESTIONS

1. What do you understand by stationary waves? State their two main characteristics. In what respects do they differ from progressive waves?
2. State the main characteristics of stationary waves and compare them with progressive waves.

3. Explain the principle of superposition of sound waves.

4. How does a stationary wave differ from a progressive wave?

5. Write the characteristics of stationary waves.

6. What do you mean by the term beat?

Uttar Pradesh Rajarshi Tandon Open University

UGPHS-102

Oscillation, Waves and Electrical Circuits

BLOCK

3

Course Design Committee

Course Preparation Committee

Dr. Vivesk Agrahari — **Author**
Asst. Professor , Dept. of Physics — **Block 1(Unit 1, 2, 3,4)**
ISDC, Prayagraj
Dr. Rakesh Kumar, Yasmin Begawadi — **Author**
Asst. Professor, Dept. of Physics — **Block 2(Unit 5, 6, 7)**
CMPPG College, Prayagraj
Mr. Ashish Shnakwar — **Author**
Dept. of electrical Engg, Asst. Professor — **Block 3(Unit 8, 9, 10)**
MJPRU, Bareilly
Dr. Pankaj Kumar — **Editor**
Associate Professor, Dept. of Physics, — **Block 1, 2 and 3**
BMM, Bharwari, Kaushambi
Mr. Arvind Kumar Mishra — **SLM Coordinator**
Academic Consultant (Physics), UPRTOU, Prayagraj
Prof. Ashutosh Gupta
Director, School of Computer and Information Science,
UPRTOU, Prayagraj

Faculty Members, School of Sciences

ISBN :

Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2020.
Printed By : K.C.Printing & Allied Works, Panchwati, Mathura -281003.

UNIT-8 ELECTRICAL CIRCUITS

Structure

8.1 INTRODUCTION

The difference of analysis of circuits with energy storage elements (inductors or capacitors) & time-varying signals with resistive circuits is that the equations resulting from KVL (Kirchhoff Voltage Law) and KCL (Kirchhoff Voltage Law) are now differential equations rather than algebraic linear equations resulting from the resistive circuits.

Transient region: the region where the signals are highly dependent on time. Steady-state region: the region where the signals are not time dependent (time rate of change of signals is equal to zero) or periodic.

A galvanometer is an electromechanical instrument used for detecting and indicating an electric current in the circuit. A galvanometer works as an actuator, by producing a rotary deflection (of a "pointer"), in response to electric current flowing through a coil in a constant magnetic field. Early galvanometers were not calibrated, but their later developments were used as measuring instruments, called ammeters, to measure the current flowing through an electric circuit. Resistance is one of the most basic elements encountered in electrical and electronics engineering. The value of resistance in engineering varies from very small value like, resistance of a transformer winding, to very high values like, insulation resistance of that same transformer winding. Although a multimeter works quite well if we need a rough value of resistance, but for accurate values and that too at very low and very high values we need specific methods. In this article we will discuss various methods **of** resistance measurement

8.2 OBJECTIVES

- Study and identify Transient phenomenon and galvanometer
- Explain and identify Transient state and steady state, Time constant.
- Study and identify Transient response LR, CR, LC and LCR circuits.
- Explain and identify Theory of moving coil galvanometer (dead beat and ballistic), critical resistance and damping.
- Study and identify Sensitivity (current, charge and voltage) of moving coil galvanometer.
- Explain and identify Applications to measurement of high resistance by leakage method.

8.3 TRANSIENT PHENOMENON AND GALVNOMETER

Transient phenomena is Rapidly changing actions occurring in a circuit during the interval between closing of a switch and settling to a steady-state condition, or any other temporary actions occurring after some change in a circuit.

Galvanometer: -

Definition: The galvanometer is the device used for detecting the presence of small current in the circuit. The galvanometer is mainly used in the bridges and potentiometer where they indicate the null deflection or zero current.

Principle of Galvanometer: -

The potentiometer is based on the premise that the current sustaining coil is kept between the magnetic field experiences a torque.

Construction of the Galvanometer: -

The construction of the potentiometer is shown in the figure below.

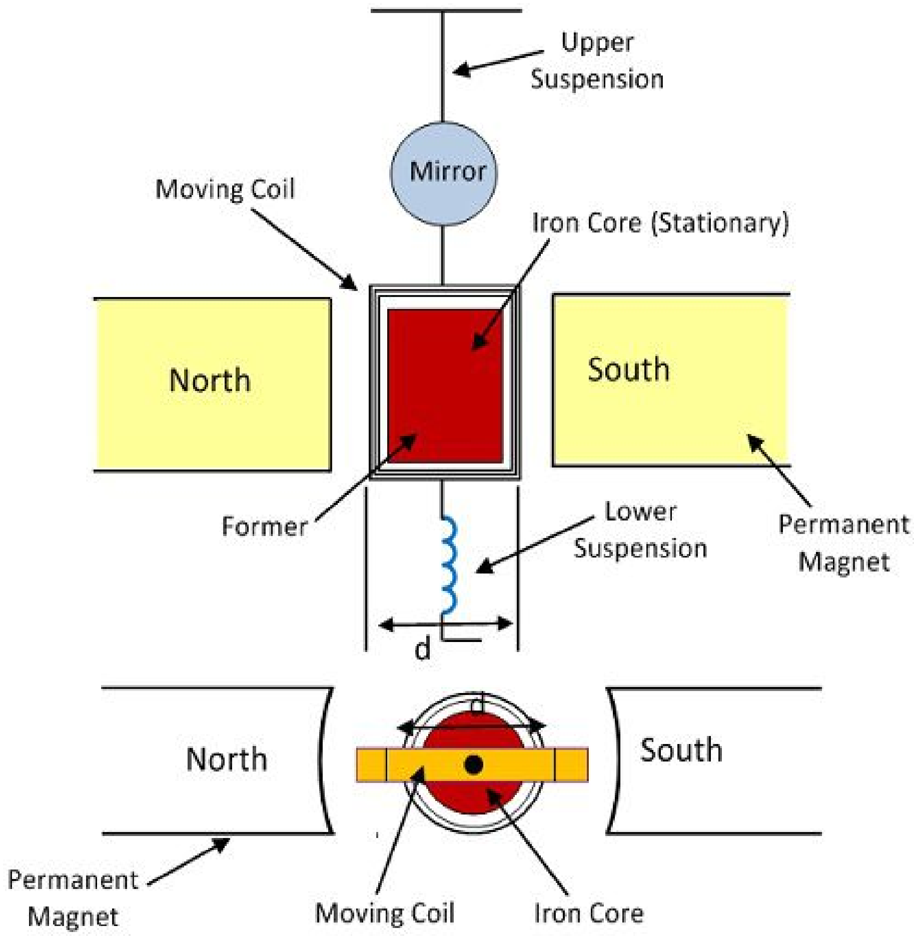


Fig 8.1 Moving Coil Galvanometer

The moving coil, suspension, and permanent magnet are the main parts of the galvanometer.

Moving Coil – The moving coil is the current carrying part of the galvanometer. It is rectangular or circular and has the number of turns of fine copper wire. The coil is freely moved about its vertical axis of symmetry between the poles of a permanent magnet. The iron core provides the low reluctance flux path and hence provides the strong magnetic field for the coil to move in.

Suspension – The coil is suspended by a flat ribbon which carries the current to the coil. The other current carrying coil is the lower suspension whose torque effect is negligible. The upper suspension coil is made up of gold or copper or phosphor wire which is made in the form of a ribbon. The mechanical strength of the wire is not very strong, and hence the galvanometers handle carefully without any jerks.

Mirror – The suspension carries a small mirror which casts the beam of light. The beam of light placed on the scale on which the deflection is measured.

Applications of Galvanometer

The galvanometer has following applications. They are

- It is used for detecting the direction of current flows in the circuit. It also determines the null point of the circuit. The null point means the situation in which no current flows through the circuit.
- It is used for measuring the current.
- The voltage between any two points of the circuit is also determined through galvanometer.

Working of Galvanometer

Let, l, d – the length of respective vertical and horizontal side of the coil in the meter.
N - number of turns in the coil,
B – Flux density in the air gap, wb/m^2
i – current through moving coil in Ampere
K – spring constant of suspension, Nm/rad
θ_f – final steady-state deflection of moving coil in radiance
When the current flows through the coil, it experiences a torque which is expressed as

$$\tau_d = Force \times Distance$$

The force on each side of the coil is given as,

$$F = NBil$$

Hence deflecting torque becomes,

$$\tau_d = NBild$$

$$\tau_d = NBAi$$

Where,

$$A = l \times d$$

N, B, A are the constant of the galvanometer.

$$\tau_d = Gi$$

The G is called the displacement constant of the galvanometer, and their value is equal NBA = NBld.

The controlling torque exerted by the suspension at deflection θ_F is

$$\tau_d = K\theta_F$$

For final steady deflection,

$$\tau_d = \tau_c$$

$$K\theta_F = Gi$$

Hence final steady deflection,

$$\theta_F = \frac{Gi}{K}$$

For the small deflection angle, the deflection is expressed as the product of the radius and angle of the turned. By the reflected beam, it is expressed as $1000 \times 2\theta_F = 2000\ Gi / K$ in millimetre.

The above equation shows that when the mirror turns through an angle θ_F the reflected beam turns through an angle $2\theta_F$ shown in the figure below.

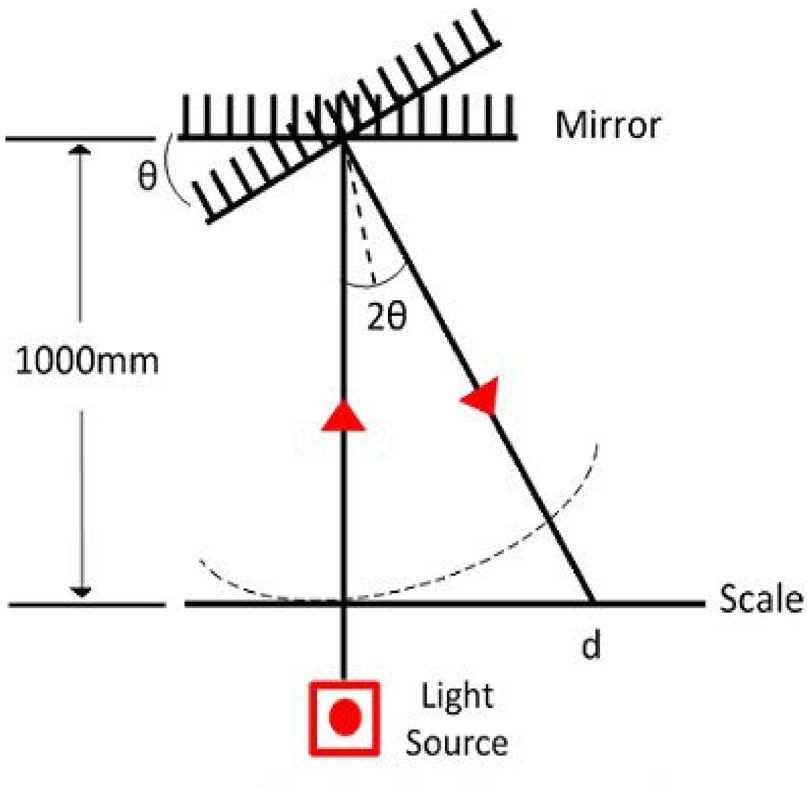


Fig 8.2 Measurement of Deflection with lamp and scale Arrangement

Conversion of Galvanometer into an Ammeter

The galvanometer can be converted into an ammeter by connecting the low resistance wire in parallel with the galvanometer. The potential difference between the voltage and the shunt resistance are equal.

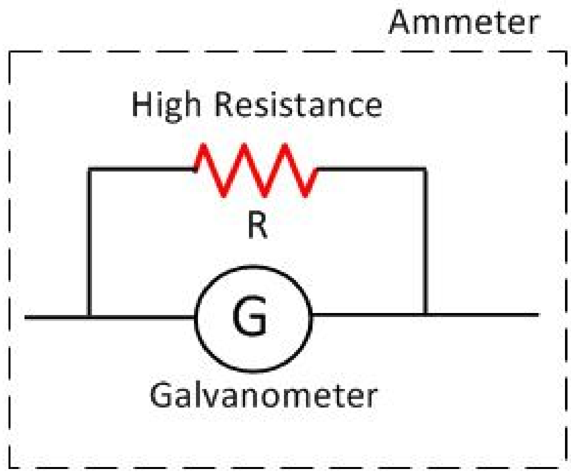


Fig 8.3 Conversion of Galvanometer into an Ammter

$$Voltage\ across\ galvanometer\ V_g = Galvanometer\ Resistance\ G \times Galvanometer\ Current\ I_g$$

$$Votlage\ across\ Shunt\ Resistance\ V_s = SI_s$$

Where, S = shunt resistance and I_s = current across the shunt.

Then current through shunt will be

$$\mathbf{I_s = I - I_g}$$

As the galvanometer and the shunt resistance are connected in potential with the circuit, their potentials are equal.

$$V_g = V_s$$

$$GI_g = SI_s$$

$$\boldsymbol{GI_g = S(I - I_g)}$$

Thus, the shunt resistance is given as,

$$\boldsymbol{S = \frac{GI_g}{(I - I_g)}}$$

The value of the shunt current is very small as compared to the supply current.

Conversion of Galvanometer into a voltmeter

The galvanometer is used as a voltmeter by connecting the high resistance in series with the circuit.

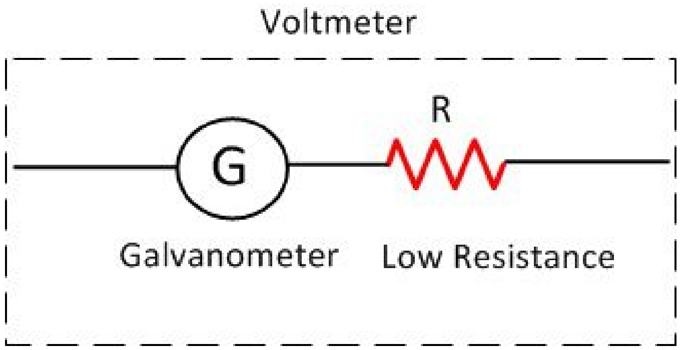


Fig 8.4 Conversion of Galvanometer into an Voltmeter

Let resistance of galvanometer be G and high resistance R_s is connected in series to it. Then total resistance equals to G + R_s. If potential between two points to be measured be V and the current passing through galvanometer be I_g. Then,

$$v = I_g(G + R_s) = I_g G + I_g . R_s$$

$$\Rightarrow \quad I_g R_s = v - I_g G$$

$$\Rightarrow \quad \boxed{R_s = \frac{v}{I_g} - G}$$

The value of potential depends on the value of high resistance which are connected in series with galvanometer.

SAQ1: -

a) What is transient phenomenon?

b) Draw the symbol of Galvanometer?

c) How Galvanometer works?

d) Can we use Galvanometer as ammeter and voltmeter?

8.4 TRANSIENT STATE AND STEADY STATE

A system is said to be in a transient state when a process variable or variables have been changed and the system has not yet reached a state. The time taken for the circuit to change from one steady state to another steady state is called the transient time.

Time constant :

The time required for a changing quantity in a circuit, as voltage or current, to rise or fall approximately 0.632 of the difference between its old and new value after an impulse has been applied that induces such a change: equal in seconds to the inductance of the circuit in Henries divided by its resistance in ohms

τ is found using the formula T = R x C in seconds

8.5 TRANSIENT RESPONSE OF RL CIRCUITS

All coils, inductors, chokes and transformers create a magnetic field around themselves consist of an Inductance in series with a Resistance forming an LR Series Circuit

We looked briefly at the time constant of an inductor stating that the current flowing through an inductor could not change instantaneously, but would increase at a constant rate determined by the self-induced emf in the inductor.

In other words, an inductor in an electrical circuit opposes the flow of current, (i) through it. While this is perfectly correct, we made the assumption in the tutorial that it was an ideal inductor which had no resistance or capacitance associated with its coil windings.

However, in the real world "ALL" coils whether they are chokes, solenoids, relays or any wound component will always have a certain amount of resistance no matter how small. This is because the actual coils turns of wire being used to make it uses copper wire which has a resistive value.

Then for real world purposes we can consider our simple coil as being an "Inductance", L in series with a "Resistance", R. In other words forming an LR Series Circuit.

$$v_L(t) = L\frac{di_L(t)}{dt}$$

where =current flowing through the inductor ; current through the capacitor

$$i_C(t) = C\frac{dv_C(t)}{dt},$$

voltage across the capacitor) or in integral form as ()C

$$i_L(t) = \frac{1}{L}\int_0^t v_L(t)dt + i_L(0) \quad or \quad v_C(t) = \frac{1}{C}\int_0^t i(t)dt + v_C(0)$$

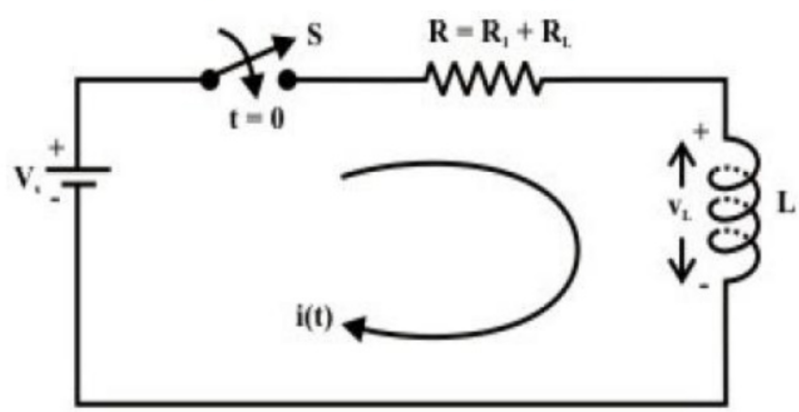


Fig 8.12 RL Series circuit

D.C Transients :

The behavior of the current and the voltage in the circuit switch is closed until it reaches its final value is called dc transient response of the concerned circuit. The response of a circuit (containing resistances, inductances, capacitors and switches) due to sudden application of voltage or current is called transient response. The most common instance of a transient response in a circuit occurs when a switch is turned on or off –a rather common event in an electric circuit.

Growth or Rise of current in R-L circuit :

To find the current expression (response) for the circuit shown in fig. 10.6(a), we can write the KVL equation around the circuit

The table shows how the current i(t) builds up in a R-L circuit.

Actual time (t) in sec	Growth of current in inductor (Eq.10.15)
$t=0$	$i(0)=0$
$t=\tau\left(=\frac{L}{R}\right)$	$i(\tau)=0.632\times\frac{V_s}{R}$
$t=2\tau$	$i(2\tau)=0.865\times\frac{V_s}{R}$
$t=3\tau$	$i(3\tau)=0.950\times\frac{V_s}{R}$
$t=4\tau$	$i(4\tau)=0.982\times\frac{V_s}{R}$
$t=5\tau$	$i(5\tau)=0.993\times\frac{V_s}{R}$

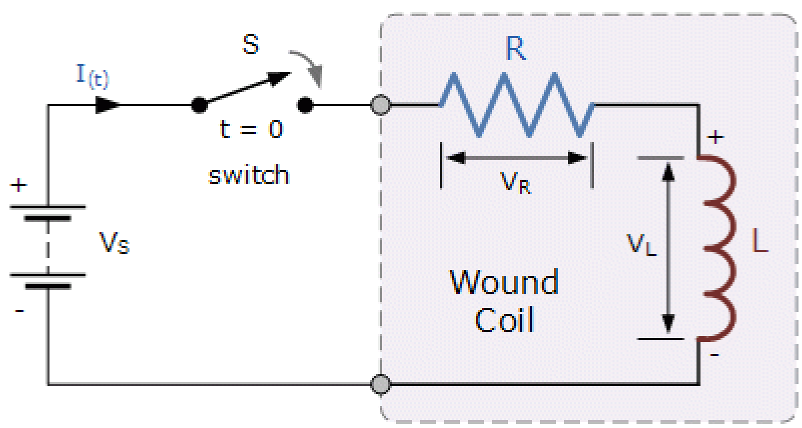


Fig 8.13 Rise of current in R-L circuit

LR Series Circuit :-

The above LR series circuit is connected across a constant voltage source, (the battery) and a switch. Assume that the switch, S is open until it is closed at a time t = 0, and then remains permanently closed producing a "step response" type voltage input. The current, i begins to flow through the circuit but does not rise rapidly to its maximum value of Imax as determined by the ratio of V / R (Ohms Law).

This limiting factor is due to the presence of the self induced emf within the inductor as a result of the growth of magnetic flux, (Lenz's Law). After a time the voltage source neutralizes the effect of the self induced emf, the current flow becomes constant and the induced current and field are reduced to zero.

We can use Kirchhoff's Voltage Law, (KVL) to define the individual voltage drops that exist around the circuit and then hopefully use it to give us an expression for the flow of current.

Kirchhoff's voltage law (KVL) gives us:

$$V_{(t)} - (V_R + V_L) = 0$$

The voltage drop across the resistor, R is I*R (Ohms Law).

$$V_R = I \times R$$

The voltage drop across the inductor, L is by now our familiar expression L(di/dt)

$$V_L = L\frac{di}{dt}$$

Then the final expression for the individual voltage drops around the LR series circuit can be given as:

$$V_{(t)} = I \times R + L\frac{di}{dt}$$

We can see that the voltage drop across the resistor depends upon the current, i, while the voltage drop across the inductor depends upon the rate of change of the current, di/dt. When the current is equal to zero, (i = 0) at time t = 0 the above expression, which is also a first order differential equation, can be rewritten to give the value of the current at any instant of time as:

Expression for the Current in an LR Series Circuit

$$I_{(t)} = \frac{V}{R}\left(1 - e^{-Rt/L}\right) \quad (A)$$

Where:

- V is in Volts
- R is in Ohms
- L is in Henries
- t is in Seconds
- e is the base of the Natural Logarithm = 2.71828

The Time Constant, (τ) of the LR series circuit is given as L/R and in which V/R represents the final steady state current value after five time constant values. Once the current reaches this maximum steady state value at 5τ, the inductance of the coil has reduced to zero acting more like a short circuit and effectively removing it from the circuit.

Therefore the current flowing through the coil is limited only by the resistive element in Ohms of the coils windings. A graphical representation of the current growth representing the voltage/time characteristics of the circuit can be presented as.

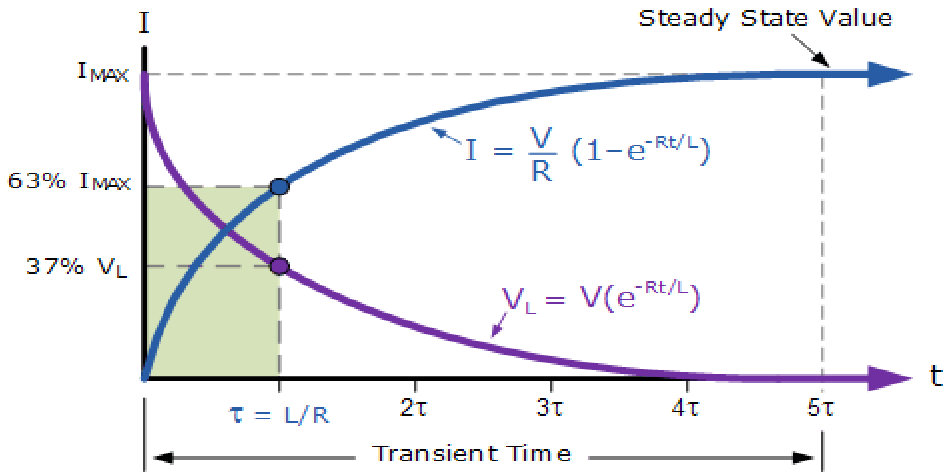


Fig 8.14 Transient Curves for an LR Series Circuit

Since the voltage drop across the resistor, V_R is equal to I*R (Ohms Law), it will have the same exponential growth and shape as the current. However, the voltage drop across the inductor, V_L will have a value equal to: $Ve^{(-Rt/L)}$. Then the voltage across the inductor, V_L will have an initial value equal to the battery voltage at time t = 0 or when the switch is first closed and then decays exponentially to zero as represented in the above curves.

The time required for the current flowing in the LR series circuit to reach its maximum steady state value is equivalent to about **5-time** constants or 5τ. This time constant τ, is measured by $\tau = L/R$, in seconds, where R is the value of the resistor in ohms and L is the value of the inductor in Henries. This then forms the basis of an RL charging circuit were 5τ can also be thought of as "5*(L/R)" or the transient time of the circuit.

The transient time of any inductive circuit is determined by the relationship between the inductance and the resistance. For example, for a fixed value resistance the larger the inductance the slower will be the transient time and therefore a longer time constant for the LR series circuit. Likewise, for a fixed value inductance the smaller the resistance value the longer the transient time.

Power in an LR Series Circuit

Then from above, the instantaneous rate at which the voltage source delivers power to the circuit is given as:

$$P = V \times I \quad \text{in Watts}$$

The instantaneous rate at which power is dissipated by the resistor in the form of heat is given as:

$$P = I^2 \times R \quad \text{in Watts}$$

The rate at which energy is stored in the inductor in the form of magnetic potential energy is given as:

$$P = Vi = Li\frac{di}{dt} \quad \text{in Watts}$$

Then we can find the total power in a RL series circuit by multiplying by i and is therefore:

$$P = i^2 R + Li\frac{di}{dt} \quad \text{(Watts)}$$

Where the first I^2R term represents the power dissipated by the resistor in heat, and the second term represents the power absorbed by the inductor, its magnetic energy.

Inductor Behavior : -

Assume the switching action takes place at t = 0. Inductor current does not change instantaneously, when the switching action takes place. That means, the value of inductor current just after the switching action will be same as that of just before the switching action.

Mathematically, it can be represented as

$$i_L(0+) = i_L(0-) i_L(0+) = i_L(0-)$$

Capacitor Behavior: -

The capacitor voltage does not change instantaneously similar to the inductor current, when the switching action takes place. That means, the value of capacitor voltage just after the switching action will be same as that of just before the switching action.

Mathematically, it can be represented as

$$V_c(0+) = V_c(0-) V_c(0+) = V_c(0-)$$

Steady state Response: -

The part of the time response that remains even after the transient response has become zero value for large values of ‘t’ is known as steady state response. This means, there won't be any transient part in the response during steady state.

RC Circuit: -

The combination of a pure resistance R in ohms and pure capacitance C in Farads is called **RC circuit.** The capacitor stores energy and the resistor connected in series with the capacitor controls the charging and discharging of the capacitor. The RC circuit is used in camera flashes, pacemaker, timing circuit etc.

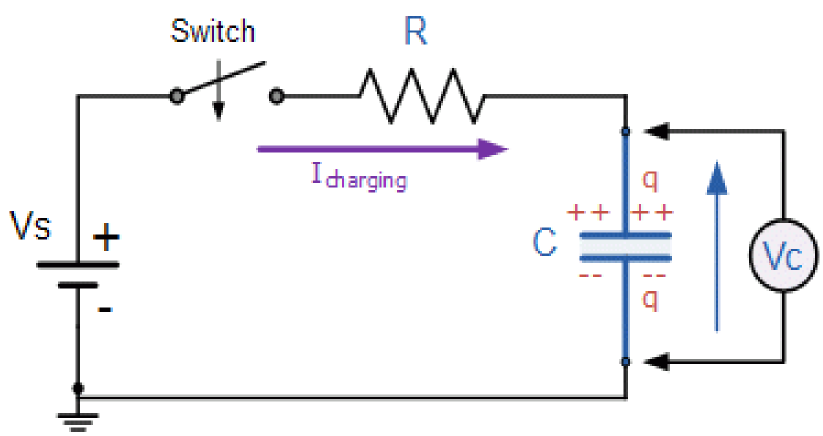


Fig 8.15 RC circuit with DC

RC Charging Circuit :-

When a voltage source is applied to an RC circuit, the capacitor, C charges up through the resistance, R

All Electrical or Electronic circuits or systems suffer from some form of "time-delay" between its input and output terminals when either a signal or voltage, continuous, (DC) or alternating (AC), is applied to it.

This delay is generally known as the circuits time delay or Time Constant which represents the time response of the circuit when an input step voltage or signal is applied. The resultant time constant of any electronic circuit or system will mainly depend upon the reactive components either capacitive or inductive connected to it. Time constant has units of, Tau – τ

When an increasing DC voltage is applied to a discharged Capacitor, the capacitor draws what is called a "charging current" and "charges up". When this voltage is reduced, the capacitor begins to discharge in the opposite direction. Because capacitors can store electrical energy they act in many ways like small batteries, storing or releasing the energy on their plates as required.

The electrical charge stored on the plates of the capacitor is given as: Q = CV. This charging (storage) and discharging (release) of a capacitors energy is never instant but takes a certain amount of time to occur with the time taken for the capacitor to charge or discharge to within a certain percentage of its maximum supply value being known as its Time Constant (τ).

If a resistor is connected in series with the capacitor forming an RC circuit, the capacitor will charge up gradually through the resistor until the voltage across it reaches that of the supply voltage. The time required for the capacitor to be fully charge is equivalent to about 5 time constants or 5T. Thus, the transient response or a series RC circuit is equivalent to 5 time constants.

This transient response time T, is measured in terms of τ = R x C, in seconds, where R is the value of the resistor in ohms and C is the value of the capacitor in Farads. This then forms the basis of an RC charging circuit were 5T can also be thought of as "5 x RC".

RC Charging Circuit :-

The figure below shows a capacitor, (C) in series with a resistor, (R) forming a RC Charging Circuit connected across a DC battery supply (Vs) via a mechanical switch. at time zero, when the switch is first closed, the capacitor gradually charges up through the resistor until the voltage across it reaches the supply voltage of the battery. The manner in which the capacitor charges up is shown below.

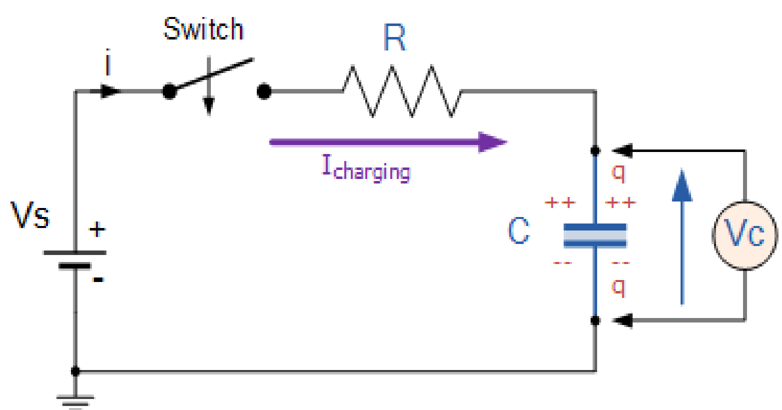


Fig 8.16 RC Charging Circuit

Let us assume above, that the capacitor, C is fully "discharged" and the switch (S) is fully open. These are the initial conditions of the circuit, then t = 0, i = 0 and q = 0. When the switch is closed the time begins at t = 0 and current begins to flow into the capacitor via the resistor.

Since the initial voltage across the capacitor is zero, (Vc = 0) at t = 0 the capacitor appears to be a short circuit to the external circuit and the maximum current flows through the circuit restricted only by the resistor R. Then by using Kirchhoff's voltage law (KVL), the voltage drops around the circuit are given as:

$$V_S - R \times i(t) - V_C(t) = 0$$

The current now flowing around the circuit is called the Charging Current and is found by using Ohms law as: i = Vs/R.

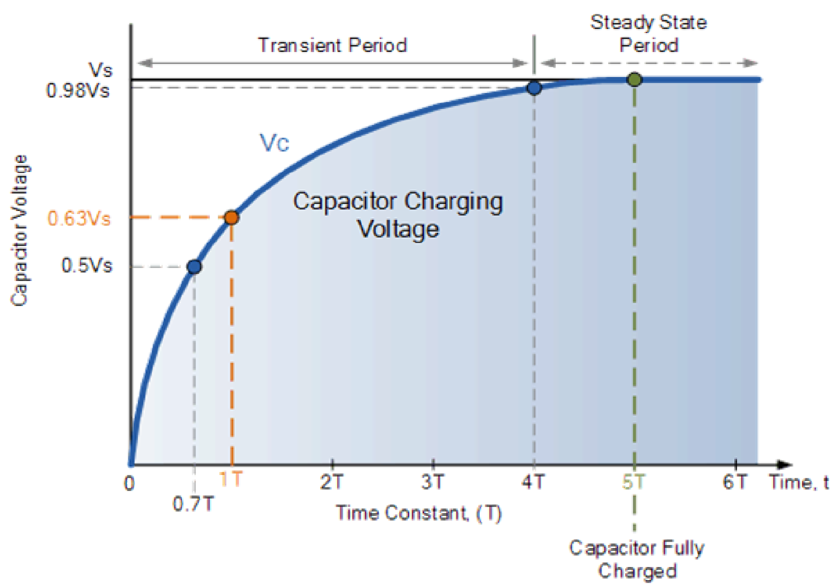


Fig 8.17 RC Charging Curves for voltage

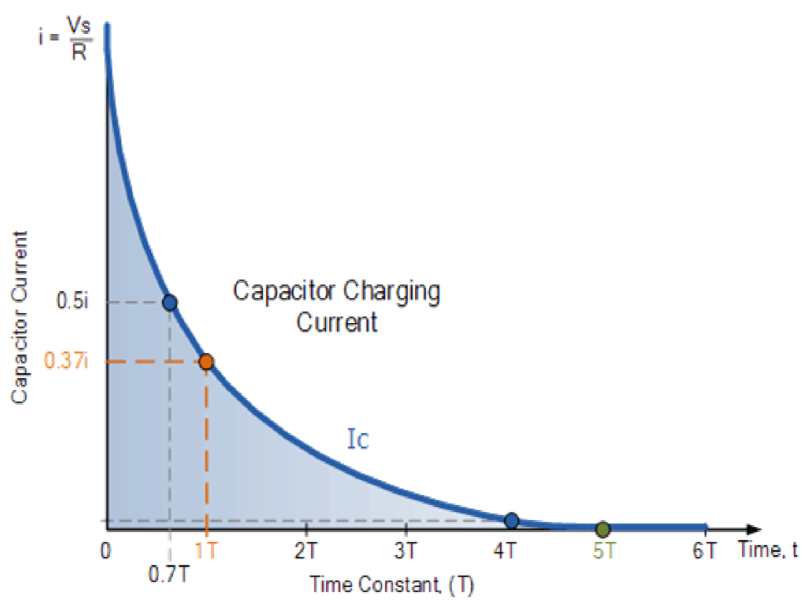


Fig 8.18 RC Charging Curves for current

As the capacitor charges up as shown, the rise in the RC charging curve is steeper at the beginning because the charging rate is fastest at the start and then tapers off as the capacitor takes on additional charge at a slower rate.

RC Time Constant (τ)**:-**

As the capacitor charges up, the potential difference across its plates slowly increases with the actual time taken for the charge on the capacitor to reach 63% of its maximum possible voltage, in our curve 0.63Vs being known as one Time Constant, (T).

This 0.63Vs voltage point is given the abbreviation of 1T, (one time constant).

The capacitor continues charging up and the voltage difference between Vs and Vc reduces, so too does the circuit current, i. Then at its final condition greater than five time constants (5T) when the capacitor is said to be fully charged, $t = \infty$, $i = 0$, $q = Q = CV$. At infinity the charging current finally diminishes to zero and the capacitor acts like an open circuit with the supply voltage value entirely across the capacitor as Vc = Vs.

So mathematically we can say that the time required for a capacitor to charge up to one time constant, (1T) is given as:

$$\tau \equiv R \times C$$

This RC time constant only specifies a rate of charge where, R is in Ω and C in Farads.

Since voltage V is related to charge on a capacitor given by the equation, Vc = Q/C, the voltage across the capacitor (Vc) at any instant in time during the charging period is given as:

$$V_C = V_S\left(1 - e^{(-t/RC)}\right)$$

Where

- Vc is the voltage across the capacitor
- Vs is the supply voltage
- t is the elapsed time since the application of the supply voltage
- RC is the time constant of the RC charging circuit

After a period equivalent to 4 time constants, (4T) the capacitor in this RC charging circuit is virtually fully charged and the voltage across the capacitor is now approx 98% of its maximum value, 0.98Vs. The time period taken for the capacitor to reach this 4T point is known as the Transient Period.

After a time of 5T the capacitor is now said to be fully charged with the voltage across the capacitor, (Vc) being equal to the supply voltage, (Vs). As the capacitor is fully charged no more current flows in the circuit. The time period after this 5T point is known as the Steady State Period.

Then we can show in the following table the percentage voltage and current values for the capacitor in a RC charging circuit for a given time constant.

RC Charging Table :-

Time Constant	RC Value	Percentage of Maximum	
		Voltage	Current
0.5 time constant	0.5T = 0.5RC	39.3%	60.7%
0.7 time constant	0.7T = 0.7RC	50.3%	49.7%
1.0 time constant	1T = 1RC	63.2%	36.8%
2.0 time constants	2T = 2RC	86.5%	13.5%
3.0 time constants	3T = 3RC	95.0%	5.0%
4.0 time constants	4T = 4RC	98.2%	1.8%
5.0 time constants	5T = 5RC	99.3%	0.7%

Notice that the charging curve for a RC charging circuit is exponential and not linear. This means that in reality the capacitor never reaches 100% fully charged. So for all practical purposes, after five time constants (5T) it reaches 99.3% charge, so at this point the capacitor is considered to be fully charged.

As the voltage across the capacitor Vc changes with time, and is therefore a different value at each time constant up to 5T, we can calculate the value of capacitor voltage, Vc at any given point, for example.

RC Discharging Circuit :-

When a voltage source is removed from a fully charged RC circuit, the capacitor, C will discharge back through the resistance, R

In the previous RC Charging Circuit tutorial, we saw how a Capacitor, C charges up through the resistor until it reaches an amount of time equal to 5 time constants known as 5T, and then remains fully charged as long as a constant supply is applied to it.

If this fully charged capacitor is now disconnected from its DC battery supply voltage, the stored energy built up during the charging process would stay indefinitely on its plates, (assuming an ideal capacitor and ignoring any internal losses), keeping the voltage stored across its connecting terminals at a constant value.

If the battery was replaced by a short circuit, when the switch is closed the capacitor would discharge itself back through the resistor, R as we now have a RC discharging circuit. As the capacitor discharges its current through the series resistor the stored energy inside the capacitor is extracted with the voltage Vc across the capacitor decaying to zero as shown below.

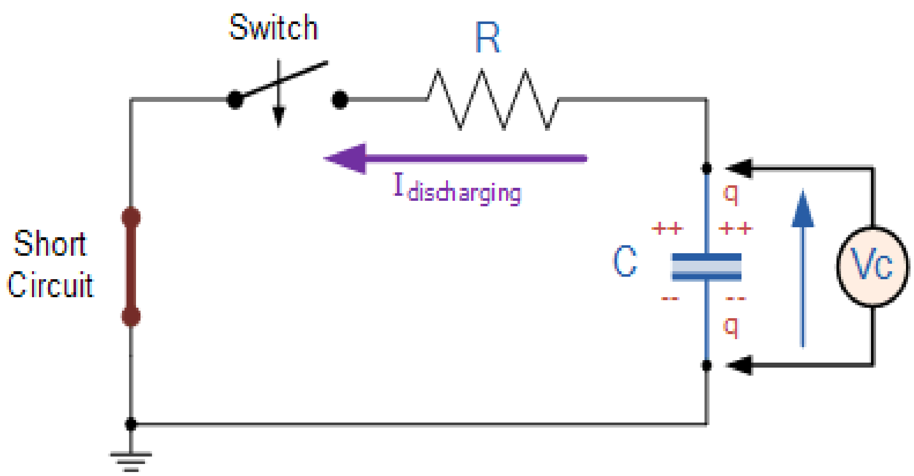


Fig 8.19 RC Discharging Circuit

As we saw in the previous tutorial, in a RC Discharging Circuit the time constant (τ) is still equal to the value of 63%. Then for a RC discharging circuit that is initially fully charged, the voltage across the capacitor after one time constant, 1T, has dropped by 63% of its initial value which is 1 – 0.63 = 0.37 or 37% of its final value.

Thus the time constant of the circuit is given as the time taken for the capacitor to discharge down to within 63% of its fully charged value. So one time constant for an RC discharge circuit is given as the voltage across the plates representing 37% of its final value, with its final value being zero volts (fully discharged), and in our curve this is given as 0.37Vs.

As the capacitor discharges, it does not lose its charge at a constant rate. At the start of the discharging process, the initial conditions of the circuit are: t = 0, i = 0 and q = Q. The voltage across the capacitors plates is equal to the supply voltage and $V_C = V_S$. As the voltage at t = 0 across the capacitors plates is at its highest value, maximum discharge current therefore flows around the RC circuit.

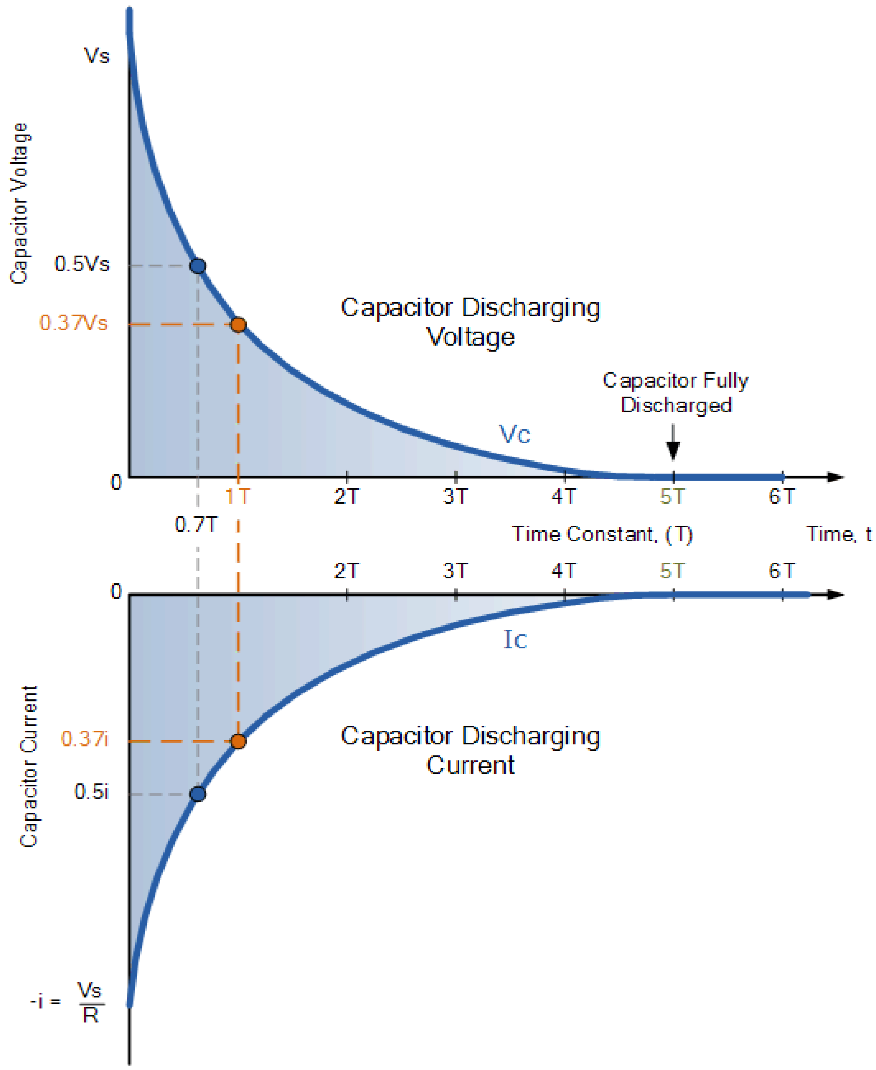


Fig 8.20 RC Discharging Circuit Curves

When the switch is first closed, the capacitor starts to discharge as shown. The rate of decay of the RC discharging curve is steeper at the beginning because the discharging rate is fastest at the start, but then tapers off exponentially as the capacitor looses charge at a slower rate. As the discharge continues, V_C reduces resulting in less discharging current.

We saw in the previous RC charging circuit that the voltage across the capacitor, C is equal to 0.5Vc at 0.7T with the steady state fully discharged value being finally reached at 5T.

For a RC discharging circuit, the voltage across the capacitor (V_C) as a function of time during the discharge period is defined as:

$$V_C = V_S \times e^{-t/RC}$$

Where:

- V_C is the voltage across the capacitor
- V_S is the supply voltage

- t is the elapsed time since the removal of the supply voltage
- RC is the time constant of the RC discharging circuit

Just like the previous RC Charging circuit, we can say that in a RC Discharging Circuit the time required for a capacitor to discharge itself down to one-time constant is given as

$$:\tau \equiv R \times C$$

Where, R is in Ω and C in Farads.

Thus, we can show in the following table the percentage voltage and current values for the capacitor in a RC discharging circuit for a given time constant.

RC Discharging Table : -

Time Constant	RC Value	Percentage of Maximum	
		Voltage	Current
0.5 time constant	0.5T = 0.5RC	60.7%	39.3%
0.7 time constant	0.7T = 0.7RC	49.7%	50.3%
1.0 time constant	1T = 1RC	36.6%	63.4%
2.0 time constants	2T = 2RC	13.5%	86.5%
3.0 time constants	3T = 3RC	5.0%	95.0%
4.0 time constants	4T = 4RC	1.8%	98.2%
5.0 time constants	5T = 5RC	0.7%	99.3%

Note that as the discharging curve for a RC discharging circuit is exponential, for all practical purposes, after five-time constants the capacitor is considered to be fully discharged.

So, an RC circuit's time constant is a measure of how quickly it either charges or discharges.

LC Circuit: -

An LC circuit, also called a resonant circuit, tank circuit, or tuned circuit, is an electric circuit consisting of an inductor, represented by the letter L, and a capacitor, represented by the letter C, connected together. The circuit can act as an electrical resonator, an electrical analogue of a tuning fork, storing energy oscillating at the circuit's resonant frequency.

LC circuits are used either for generating signals at a particular frequency, or picking out a signal at a particular frequency from a more complex signal; this function is called a bandpass filter. They are key components in many electronic devices, particularly radio equipment, used in circuits such as oscillators, filters, tuners and frequency mixers.

An LC circuit is an idealized model since it assumes there is no dissipation of energy due to resistance. Any practical implementation of an LC circuit will always include loss resulting from small but non-zero resistance within the components and connecting wires. The purpose of an LC circuit is usually to oscillate with minimal damping, so the resistance is made as low as possible. While no practical circuit is without losses, it is nonetheless instructive to study this ideal form of the circuit to gain understanding and physical intuition. For a circuit model incorporating resistance, see RLC circuit.

Terminology: -

The two-element LC circuit described above is the simplest type of inductor-capacitor network (or LC network). It is also referred to as a second order LC circuit to distinguish it from more complicated (higher order) LC networks with more inductors and capacitors. Such LC networks with more than two reactances may have more than one resonant frequency.

The order of the network is the order of the rational function describing the network in the complex frequency variable s. Generally, the order is equal to the number of L and C elements in the circuit and in any event cannot exceed this number.

Operation: -

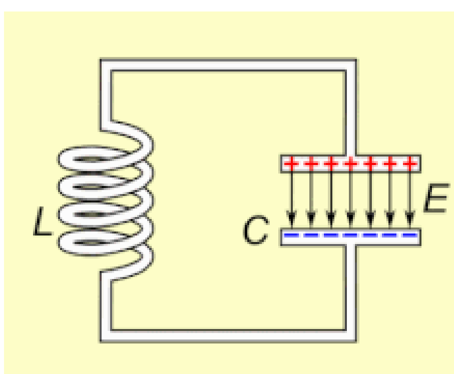


Fig 8.21 Tank circuit

The above diagram showing the operation of a tuned circuit (LC circuit). The capacitor C stores energy in its electric field E and the inductor L stores energy in its magnetic field B (green). The animation shows the circuit at progressive points in the oscillation. The oscillations are slowed down; in an actual tuned circuit the charge may oscillate back and forth thousands to billions of times per second.

An LC circuit, oscillating at its natural resonant frequency, can store electrical energy. See the animation. A capacitor stores energy in the electric field (E) between its plates, depending on the voltage across it, and an inductor stores energy in its magnetic field (B), depending on the current through it.

If an inductor is connected across a charged capacitor, the voltage across the capacitor will drive a current through the inductor, building up a magnetic field around it. The voltage across the capacitor falls to zero as the charge is used up by the current flow. At this point, the energy stored in the coil's magnetic field induces a voltage across the coil, because inductors oppose changes in current. This induced voltage causes a current to begin to recharge the capacitor with a voltage of opposite polarity to its original charge. Due to Faraday's law, the EMF which drives the current is caused by a decrease in the magnetic field, thus the energy required to charge the capacitor is extracted from the magnetic field. When the magnetic field is completely dissipated the current will stop and the charge will again be stored in the capacitor, with the opposite polarity as before. Then the cycle will begin again, with the current flowing in the opposite direction through the inductor.

The charge flows back and forth between the plates of the capacitor, through the inductor. The energy oscillates back and forth between the capacitor and the inductor until (if not replenished from an external circuit) internal resistance makes the oscillations die out. The tuned circuit's action, known mathematically as a harmonic oscillator, is similar to a pendulum swinging back and forth, or water sloshing back and forth in a tank; for this reason the circuit is also called a tank circuit. The natural frequency (that is, the frequency at which it will oscillate when isolated from any other system, as described above) is determined by the capacitance and inductance values. In most applications the tuned circuit is part of a larger circuit which applies alternating current to it, driving continuous oscillations. If the frequency of the applied current is the circuit's natural resonant frequency (natural frequency below), resonance will occur, and a small driving current can excite large amplitude oscillating voltages and currents. In typical tuned circuits in electronic equipment the oscillations are very fast, from thousands to billions of times per second.

Resonance effect: -

Resonance occurs when an LC circuit is driven from an external source at an angular frequency ω_0 at which the inductive and capacitive reactances are equal in magnitude. The frequency at which this equality holds for the particular circuit is called the resonant frequency. The resonant frequency of the LC circuit where L is the inductance in henrys, and C is the capacitance in farads. The angular frequency ω_0 has units of radians per second.

Applications

- The most common application of tank circuits is tuning radio transmitters and receivers. For example, when we tune a radio to a particular station, the LC circuits are set at resonance for that particular carrier frequency.
- A series resonant circuit provides voltage magnification.
- A parallel resonant circuit provides current magnification.
- A parallel resonant circuit can be used as load impedance in output circuits of RF amplifiers. Due to high impedance, the gain of amplifier is maximum at resonant frequency.
- Both parallel and series resonant circuits are used in induction heating.

LC circuits behave as electronic resonators, which are a key component in many applications:

- Amplifiers
- Oscillators
- Filters
- Tuners
- Mixers
- Foster-Seeley discriminator
- Contactless cards
- Graphics tablets
- Electronic article surveillance (security tags)

Time domain solution

Kirchhoff's laws: -

By Kirchhoff's voltage law, the voltage across the capacitor, V_C, plus the

voltage across the inductor, V_L must equal zero:

Likewise, by Kirchhoff's current law, the current through the capacitor equals the current through the inductor:

LRC Ciruit :-

An RLC circuit is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. The name of the circuit is derived from the letters that are used to denote the constituent components of this circuit, where the sequence of the components may vary from RLC.

The circuit forms a harmonic oscillator for current, and resonates in a similar way as an LC circuit. Introducing the resistor increases the decay of these oscillations, which is also known as damping. The resistor also reduces the peak resonant frequency. In ordinary conditions, some resistance is unavoidable even if a resistor is not specifically included as a component; an ideal, pure LC circuit exists only in the domain of superconductivity, a physical effect demonstrated to this point only at temperatures far below ambient temperatures found anywhere on the Earth's surface.

RLC circuits have many applications as oscillator circuits. Radio receivers and television sets use them for tuning to select a narrow frequency range from ambient radio waves. In this role, the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter. The tuning application, for instance, is an example of band-pass filtering. The RLC filter is described as a second-order circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.

The three circuit elements, R, L and C, can be combined in a number of different topologies. All three elements in series or all three elements in parallel are the simplest in concept and the most straightforward to analyse. There are, however, other arrangements, some with practical importance in real circuits. One issue often encountered is the need to take into account inductor resistance. Inductors are typically constructed from coils of wire, the resistance of which is not usually desirable, but it often has a significant effect on the circuit.

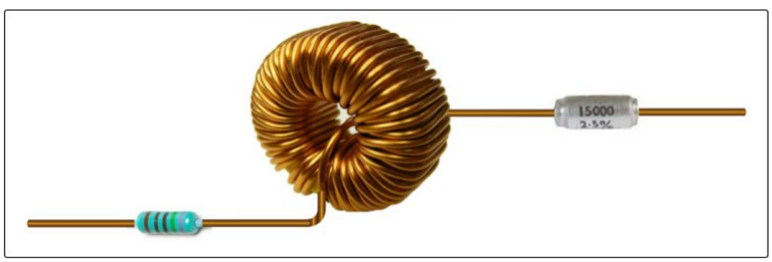


Fig 8.22 LRC circuit

In this circuit, the three components are all in series with the voltage source. The governing differential equation can be found by substituting into Kirchhoff's voltage law (KVL) the constitutive equation for each of the three elements. From the KVL,

where V_R, V_L and V_C are the voltages across R, L and C respectively and V(t) is the time-varying voltage from the source.

Substituting , and into the equation above yields:

For the case where the source is an unchanging voltage, taking the time derivative and dividing by L leads to the following second order differential equation:

This can usefully be expressed in a more generally applicable form:

α and ω_0 are both in units of angular frequency. α is called the neper frequency, or attenuation, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. Neper occurs in the name because the units can also be considered to be nepers per second, neper being a unit of attenuation. ω_0 is the angular resonance frequency.[3]

For the case of the series RLC circuit these two parameters are given by:[4]

A useful parameter is the damping factor, ζ, which is defined as the ratio of these two; although, sometimes α is referred to as the damping factor and ζ is not used.[5]

In the case of the series RLC circuit, the damping factor is given by

The value of the damping factor determines the type of transient that the circuit will exhibit.[6]

Transient response :-

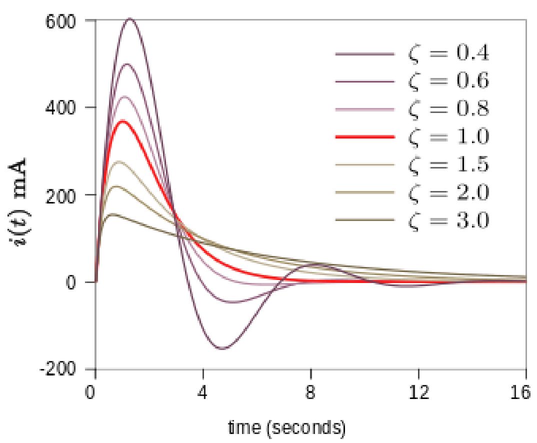


Fig 8.23 Time response of RLC circuit

Plot showing underdamped and overdamped responses of a series RLC circuit. The critical damping plot is the bold red curve. The plots are normalised for L = 1, C = 1 and $\omega_0 = 1$

The differential equation has the characteristic equation, The roots of the equation in s-domain are, The general solution of the differential equation is an exponential in either root or a linear superposition of both, The coefficients A_1 and A_2 are determined by the boundary conditions of the specific problem being analysed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.[8] The differential equation for the circuit solves in three different ways depending on the value of ζ. These are overdamped ($\zeta > 1$), underdamped ($\zeta < 1$), and critically damped ($\zeta = 1$).

Overdamped response :-

The overdamped response ($\zeta > 1$) is the overdamped response is a decay of the transient current without oscillation.

Underdamped response :-

The underdamped response ($\zeta < 1$) is by applying standard trigonometric identities the two trigonometric functions may be expressed as a single sinusoid with phase shift,

The underdamped response is a decaying oscillation at frequency ω_d. The oscillation decays at a rate determined by the attenuation α. The exponential in α describes the envelope of the oscillation. B_1 and B_2 (or B_3 and the phase shift φ in the second form) are arbitrary constants determined by boundary conditions. The frequency ω_d is given by

This is called the damped resonance frequency or the damped natural frequency. It is the frequency the circuit will naturally oscillate at if not driven by an external source. The resonance frequency, ω_0, which is the frequency at which the circuit will resonate when driven by an external oscillation, may often be referred to as the undamped resonance frequency to distinguish it.

Critically damped response :-

The critically damped response ($\zeta = 1$) is the critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation. This consideration is important in control systems where it is required to reach the desired state as quickly as possible without overshooting. D_1 and D_2 are arbitrary constants determined by boundary conditions.

8.6 THEORY OF MOVING COIL GALVANOMETER

Ballistic galvanometers are the measuring instruments which are used for measuring the quantity of electric charges obtained from magnetic flux. Its construction is similar to the moving coil galvanometer and it consists of two additional properties.

- It consists of extremely small electromagnetic damping.
- It consists of undamped oscillations.

Working Principle of Ballistic Galvanometer :-

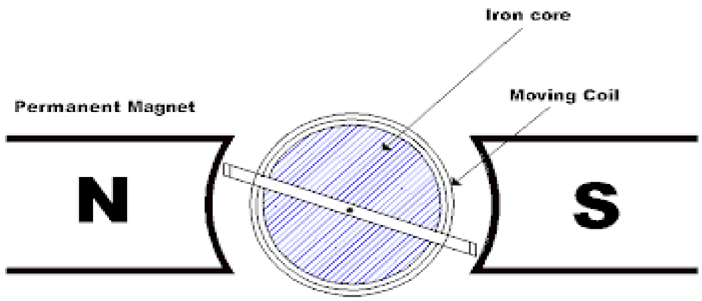


Fig 8.5 Ballistic Galvanometer

The working principle of ballistic galvanometer is that the charge measured by the ballistic galvanometer must be passed through the coil. So, the coil starts oscillating. When the charge flows through the coil, it gives rise to a current due to the torque produced in the coil. This torque acts for a short time. The product of the torque and the time period provides a force to the coil and the coil starts rotating. When the initial kinetic energy of the coil is completely used in doing work, the coil starts moving back to its original position. Thus, the coil oscillates in the magnetic field and the deflection is noted from which charge can be calculated.

Construction of Ballistic Galvanometer :-

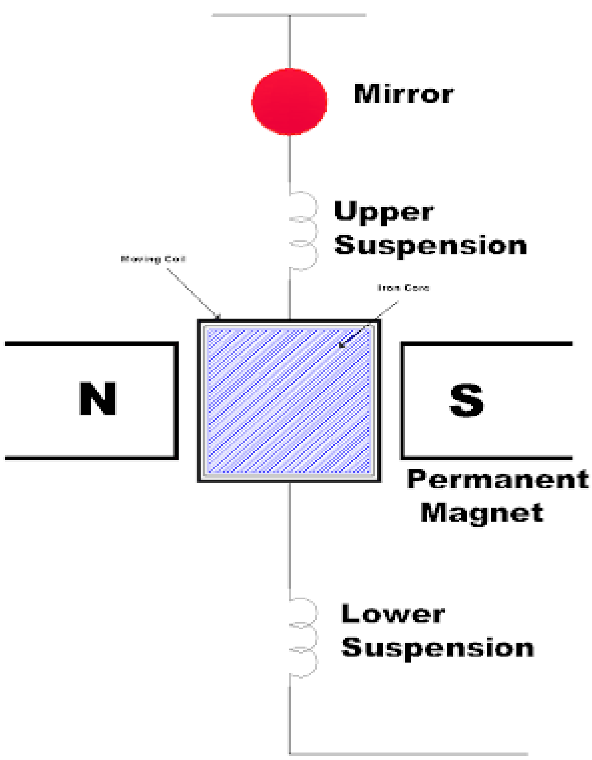


Fig 8.6 Construction of Ballistic Galvanometer

In the construction of ballistic galvanometer, a ballistic galvanometer consists of a circular or a rectangular coil of a copper wire of almost 10 to 15 turns. This coil is suspended in a radial field between the concave pole pieces of a strong magnet. when the coil rotates in the magnetic field, an EMF is induced across the coil according to the lenz's law it opposes the motion of the coil and this is known as electromagnetic damping. To minimize the electromagnetic damping the coil should be wound on a wooden frame and the whole suspension is enclosed in a metal case provided with glass faces.

Theory of Ballistic Galvanometer :-

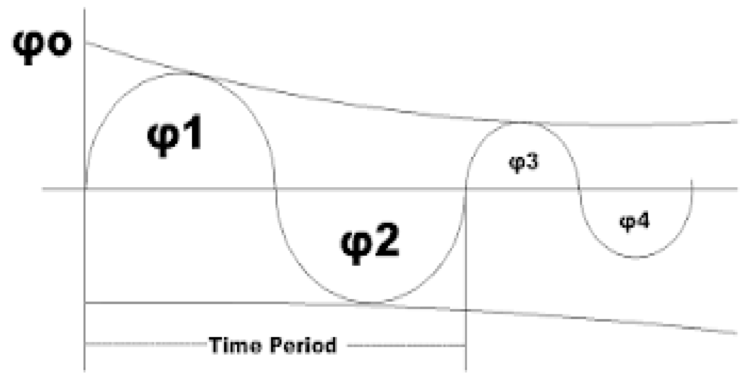


Fig 8.7 Deflection of Ballistic Galvanometer

The torque developed by the coil at any point of time is:

$$\tau_d = Bi \times 2Ln\frac{\omega}{2} = i(BLn\omega) = K_1 i \qquad \text{…………. (1)}$$

Where L is the length, W is the width, n is the number of turns of the coil and B is the air gap flux density.

The torque of acceleration is:

$$T_a = J \times Acceleration = J\frac{d\omega}{dt} \qquad \text{…………. (2)}$$

Where J is the moment of inertia of the coil and ω is the angular velocity. If the coil is closed to its zero point then the discharge takes place and the torque of suspension is zero. The value of the driving torque is equal if the damping torque is neglected. During the short discharge period:

$$J\frac{d\omega}{dt} = K_1 i \qquad \text{…………. (3)}$$

By integrating:

$$\boldsymbol{Jw}\Big|_{w=0}^{w=w_o} = \boldsymbol{K_1}\int_{t=0}^{t_o} \boldsymbol{idt} \qquad \text{--- (4)}$$

Where the subscript zero refers the conditions at the end of the discharge time. The integral form of the Eq. (4) is the amount of the charge that has passed through the coil. Therefore:

$$\boldsymbol{\omega_0 = \frac{K_1}{J} Q} \quad \text{............ (5)}$$

The above equation indicates the velocity of the coil acquires from the pulse is proportional to the quantity of charge that passed through it.

During the actual motion, the deflection torque is zero and the equation of motion is:

$$J\frac{d^2\Theta}{dt^2} + D\frac{d\Theta}{dt} + S\Theta = 0 \quad \text{--- (6)}$$

Where D is the damping constant, S is the control constant and (H) is the deflection in radians. Thus,

$$\Theta = Ae^{m_1 t} + Be^{m_2 t} \quad \text{--- (7)}$$

Where A and B are constant m_1 and m_2 are imaginary. The initial conditions are:

$$\Theta = 0 \; and \; \frac{d\Theta}{dt} = \omega_0 at \; t = 0 \quad \text{............ (8)}$$

Under this condition the solution may be written as:

$$\Theta = e^{\frac{-Dt}{2J}} \times \omega_0 \times \frac{sin\beta t}{\beta} \quad \text{............ (9)}$$

Where

$$\beta = \sqrt{\frac{S}{J}}$$

$$\Theta_1 = W_0\sqrt{\frac{J}{S}} \quad \text{--- (10)}$$

The ratio of successive swings is found by exponential multiplier for time interval

t = π/β. The ratio of successive swings is:

$$r = \frac{\Theta_1}{\Theta_2} = e^{\left(\frac{\pi}{2}\right)\left(\frac{D}{J\beta}\right)} \quad \text{--- (11)}$$

The natural logarithm of this ratio is:

$$\lambda = \ln\left(\frac{\Theta_1}{\Theta_2}\right) = \frac{\pi}{2} \times \frac{D}{J\beta} \qquad \text{---}(12)$$

The third swing in the same manner is as follows:

$$\frac{\Theta_1}{\Theta_3} = \frac{\Theta_1}{\Theta_2} \times \frac{\Theta_2}{\Theta_3} = r^2 \qquad \text{---}(13)$$

In general:

$$\frac{\Theta_1}{\Theta_n} = r^{n-1} \qquad \text{---}(14)$$

In case of critical damping, Eq. (9) will be written as:

$$\Theta = \omega_0 e^{-(D/2J)t} \qquad \text{............ (15)}$$

and

$$\omega = \frac{d\Theta}{dt} = \omega_0 e^{-\left(\frac{D}{2J}\right)t}\left(1 - \frac{D}{2J}t\right) \qquad \text{............ (16)}$$

Maximum deflection is found for or t = 2J/D. Substitute this value in Eq. 15 and call the deflection , then:

$$\Theta_1 = w_o \frac{2J}{D} \times \frac{1}{e} \qquad \text{---}(17)$$

or

$$\Theta_1 = w_o \sqrt{\frac{J}{S}} \times \frac{1}{e} \qquad \text{---}(18)$$

Summarizing the results in the following equation of the charge passing through the galvanometer:

$$Q = K_2 \times \Theta \qquad \text{---}(19)$$

The working units of the Eq. (19) are:

K2 = galvanometer sensitivity in millimeter deflection

Θ = deflection in millimeters

Q = charge in micro coulombs

Measurement of Electric Flux by Ballistic Galvanometer : -

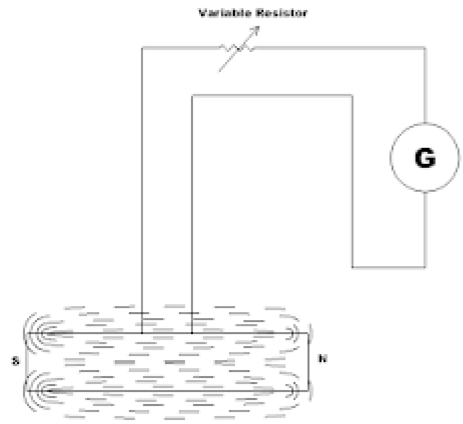


Fig 8.8 Measurement of Electric Flux by Ballistic Galvanometer

To measure the magnetic flux of a bar magnet, the bar magnet is surrounded by a coil connected in series with a variable resistor and a galvanometer. The series resistor provides critical damping and it is used to control the sensitivity of the magnetic flux. This sensitivity is controlled by adjusting the number of turns in a coil. When the magnet is suddenly withdrawn from the coil, an impulse is produced in a coil for few seconds and the deflection of the galvanometer is taken as a measure of the flux. The induced voltage in the coil are:

$$e = N\frac{d\varphi}{dt}\ volts \qquad \text{---(20)}$$

Where flux is measured in webers and N is the number of turns in a coil. If R is the total resistance of the circuit including series resistor and a galvanometer then the current flowing to the circuit is:

$$i = \frac{e}{R} = \frac{N}{R} \times \frac{d\theta}{dt} amperes \qquad \text{---(21)}$$

or

$$\int_{t=0}^{t_o} idt = \frac{N}{R} \times \int_{\varphi=\varphi}^{0} d\varphi \qquad \text{---(22)}$$

The quantity of charge passed through the galvanometer is:

$$Q = \frac{N\varphi}{R}\ coulombs \qquad \text{---(23)}$$

Deflection of the galvanometer is:

$$\varphi_1 = \frac{Q}{K_2} = \frac{N\varphi}{K_2 R} \quad \text{---}(24)$$

or

$$\varphi = \frac{K_2 R \varphi_1}{N} \; webers \quad \text{---}(25)$$

Where K2 is the sensitivity factor and it must be properly evaluated for the resistance used in the test measurements.

Calibration of Ballistic Galvanometer: -

The calibration of ballistic galvanometer can be done in so many different ways. Some methods of calibration are as follows:

By Capacitor: -

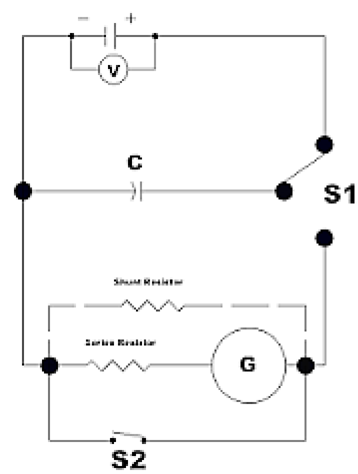


Fig 8.9 Calibration of a Ballistic Galvanometer by Capacitor

In this method, a capacitor is charged through the voltage and is discharged by the galvanometer. The resistor and a switch S_2 is used to bring the galvanometer to its zero position quickly after a deflection. The capacitor is charged through the upper position of the switch S_1 and is discharged by the contacts of this switch S1 in the lower position. The discharged quantity of electricity and the capacitance of the capacitor is calculated so the constant K2 is divided by the observed deflection. This is the undamped sensitivity because of the infinite resistance of the galvanometer. A shunt is added in the parallel to the series resistor and a galvanometer. This shunt provides damping and if the shunt is in critical value then the action is sluggish and the damping conditions are improved with the combination of shunt and series resistances.

This method is not used commonly because it is difficult to measure the exact amount of capacitance of the capacitor and the damping of the galvanometer is different during the operation of test.

By Standard Solenoid: -

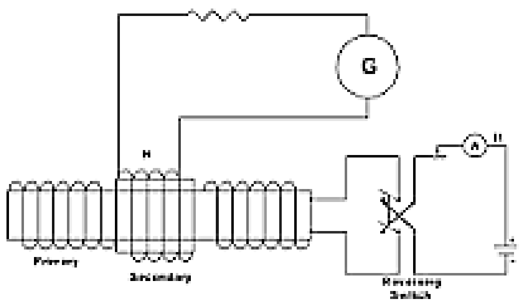


Fig 8.10 Calibration of Ballistic Galvanometer by Solenoid

This method is mostly used for calibration purposes. In this method, a standard solenoid of a long coil is wound on a cylinder of a nonmagnetic material. The length of a solenoid is at least 1 meter and the diameter is of 10cm.the winding should be uniform and its length must be that its field strength H is of 10000A/m or more when maximum current is applied on a coil. The calibration is done by means of a known flux. The flux linking to the coil is:

$$\varphi = 4\pi \times 10^{-7} N_1 I_1 A \; webers \quad \text{---(26)}$$

Where N1 are the primary turns/meter, I1 is the primary current in amperes, A is the cross-section area of the coil m^2.

This arrangement creates a flux change twice, so by substitution:

$$Q = \frac{8\pi N_1 I_1 A N_2}{R} \times 10^{-7} \; columb \quad \text{---(27)}$$

Where N2 are the turns of the coil, R is resistance of the coil and galvanometer circuit.

The calibration for flux measurements is in convenient form, once sensitivity factor K2 is evaluated. If the galvanometer is used for the measurement of unknown flux, then it will be written as:

$$\varphi_x = \frac{\varphi_{1x} K_2 R}{N_x} \quad \text{---(28)}$$

where is the unknown flux change, is the deflection in millimeters and is the number of turns in the coil.

By Mutual Inductance:

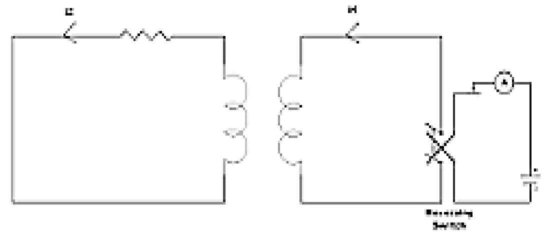


Fig 8.11 Calibration of Ballistic Galvanometer by Mutual Inductor

This method is used to measure the large range of calibration. It consists of a mutual inductor and is very small in size as compared to a solenoid. In this method, if the mutual inductance is known in the circuit then a deflection φ1 is produced by the reversal of a known primary current I is observed. By changing primary current:

$$e_2 = M\frac{di_1}{dt} \qquad \text{---(29)}$$

Let R be the total resistance of the galvanometer circuit then the galvanometer current is:

$$i_2 = \frac{M}{R} \times \frac{di_1}{dt} \qquad \text{---(30)}$$

By integration

$$\int_{t=0}^{to} 1_2 dt = \frac{M}{R}\int_{-t}^{t} di_1 \qquad \text{---(31)}$$

So,

$$Q = \frac{2MI}{R} \qquad \text{---(32)}$$

and

$$K_2 = \frac{Q}{\theta} = \frac{2MI}{R\theta_1} \qquad \text{---(33)}$$

The different type of moving coil galvanometers is

(a) **Pivoted Galvanometer:** It consists of a coil of fine insulated wire wound on a metallic frame. The coil is mounted on two jewelled pivots and is symmetrically placed between cylindrical pole pieces of a strong permanent horse-shoe magnet.

(b) **Dead beat Galvanometer:** Here coil is wound over the metallic frame to make it dead beat. On passing current the galvanometer shows a steady deflection without any oscillation. The damping is produced by eddy currents.

(c) **Ballistic Galvanometer:** This is used for measurement of charge. Here coil is wound on an insulating frame and oscillates on passing current.

The current sensitivity of a galvanometer is defined as the deflection produced when unit current passes through the galvanometer. A galvanometer is said to be sensitive if it produces large deflection for a small current.

In a galvanometer, $I = (C/nBA) \times \theta$

∴ Current sensitivity $\theta/I = nBA/C$ … (1)

The current sensitivity of a galvanometer can be increased by

(i) increasing the number of turns

(ii) increasing the magnetic induction

(iii) increasing the area of the coil

(iv) decreasing the couple per unit twist of the suspension wire.

This explains why phosphor-bronze wire is used as the suspension wire which has a small couple per unit twist.

Current sensitivity of a galvanometer is defined as the deflection produced in the galvanometer when a unit current flow through it.

$I_S = \theta/I = nBA/c$

Where n is no of turns in the coil of the galvanometer, B is Magnetic field around the coil, A is Area of the coil and c is restoring torque per unit twist.

8.7 THE SENSITIVITY OF MOVING COIL GALVANOMETER

Moving coil galvanometer is an electromagnetic device that can measure small values of current. The sensitivity of a Moving Coil Galvanometer is defined as the ratio of the change in deflection of the galvanometer to the change in current. Therefore, we write, Sensitivity = $d\theta/di$. If a galvanometer gives a bigger deflection for a little current it is said to be sensitive. The current in Moving Coil galvanometer is: $I = (C/nBA) \times \theta$

- The sensitivity of Moving Coil Galvanometer increases by:

  (i) Increasing the no. of turns and the area of the coil,

  (ii) Increasing the magnetic induction and

  (iii) Decreasing the couple per unit twist of the suspension fiber.

Voltage sensitivity is the measure of the responsiveness of an appliance to the transform of applied voltage across it. A galvanometer is a type of ammeter. It is an appliance for detecting and measuring electric current

So, the voltage sensitivity of a galvanometer is defined as the deflection per unit voltage across the galvanometer.

So, **Voltage sensitivity = θ/V = θ/IG = nBA/CG**

where G is the galvanometer resistance.

Unit: rad V^{-1} or mm V^{-1}

High voltage sensitivity is desirable in circuits of relatively low resistance.

An interesting point to note is that increasing the current sensitivity does not necessarily, increase the voltage sensitivity. When the number of turns (n) is doubled, current sensitivity is also doubled (equation). But increasing the number of turns correspondingly increases the resistance (G). Hence voltage sensitivity remains unchanged.

So, we can say, Voltage Sensitivity = θ/V = (NAB/KR)

where,

- θ is the angular displacement, i.e. the reading you see on the galvanometer
- V is, of course, the voltage across the galvanometer for which the reading is θ
- N is the number of turns of the moving-coil in the galvanometer.
- A is the length of the rectangular-coil, B is the breadth of the rectangular-coil,

Therefore, AB represents the area of the 2D-coil.

K is the torsion-constant of the galvanometer, i.e., the spring constant of the spring that’s used in the galvanometer.

R is the resistance of the coil.

Now, increasing the number of turns N of the coil will result in the same increase in the resistance of the coil R as $R \propto l$ and surface-area of the coil is kept stable.

Therefore, the increasing number of turns N of the coil does not affect the voltage-sensitivity of the galvanometer.

1. Therefore, in order to increase the voltage-sensitivity of the galvanometer,

2. You can increase the area of the coil

3. You can decrease the torsion-constant of the galvanometer.

Charge sensitivity: The charge sensitivity (the ballistic reduction factor) of a moving coil galvanometer is the charge (transient current) required to produce a deflection (throw or kick) of 1 mm on a scale kept at a distance of 1 m from the mirror. By eqn.7 and 8, E K P Q R M G 66 Optics & Electricity Practical II Charge sensitivity, K = q θ = T 2π ⎞⎛ ⎜⎜ ⎟⎟⎠⎝ C NAB ⎞⎛ ⎜⎜ ⎟⎟⎠⎝ = T ×current sensitivity

Advantages and Disadvantages of Moving Coil Galvanometer: -

Advantages: -

- Sensitivity increases as the value of n, B, A increases and value of k decreases.
- The eddy currents produced in the frame bring the coil to rest quickly, due to the coil wound over the metallic frame.

Disadvantages: -

- We cannot change the sensitivity of the galvanometer at will.
- Overloading can damage any type of galvanometer.

SAQ 2

a) Write the different types of Galvanometer?

b) Draw the neat and clean schematic diagram of Ballistic galvanometer?

c) What is critical resistance of ballistic galvanometer?

d) Explain damping in ballistic galvanometer?

e) Explain sensitivity of moving coil galvanometer?

8.8 APPLICATIONS TO MEASUREMENT OF HIGH RESISTANCE (>100KΩ)

Following are few methods used for measurement of high resistance values-

- Loss of Charge Method
- Megohm bridge Method
- Measurement of High Resistance by Leakage Method

We normally utilize very small amount of current for such measurement, but still owing to high resistance chances of production of high voltages is not surprising. Due to this we encounter several other problems such as-

1. Electrostatic charges can get accumulated on measuring instruments

2. Leakage current becomes comparable to measuring current and can cause error

3. Insulation resistance is one of the most common in this category; however a dielectric is always modeled as a resistor and capacitor in parallel. Hence while measuring the insulation resistance (I.R.) the current includes both the component and hence true value of resistance is not obtained. The capacitive component though falls exponentially but still takes very long time to decay. Hence different values of I.R. are obtained at different times.

4. Protection of delicate instruments from high fields.

Hence to solve the problem of leakage currents or capacitive currents we use a guard circuit. The concept of guard circuit is to bypass the leakage current from the ammeter so as to measure the true resistive current. Figure below shows two connections on voltmeter and micro ammeter to measure R, one without guard circuit and one with guard circuit.

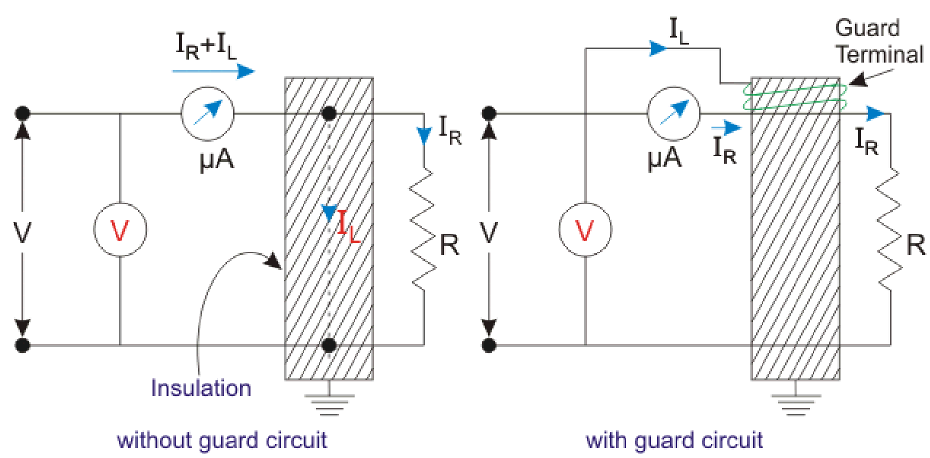


Fig 8.24 Measurement of High Resistance

In the first circuit the micro ammeter measures both capacitive and the resistive current leading to error in value of R, while in the other circuit the micro ammeter reads only the resistive current.

Loss of Charge Method

In this method we utilize the equation of voltage across a discharging capacitor to find the value of unknown resistance R. Figure below shows the circuit diagram and the equations involved are-

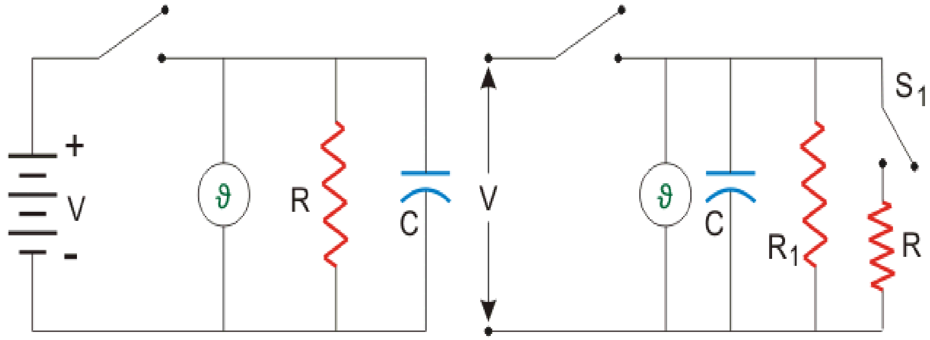


Fig 8.25 Loss of Charge Method

$$v = Ve^{\frac{-t}{RC}}$$

$$R = \frac{0.4343t}{C\ log_{10}V/v}$$

However the above case assumes no leakage resistance of the capacitor. Hence to account for it we use the circuit shown in the figure below. R_1 is the leakage resistance of C and R is the unknown resistance. We follow the same procedure but first with switch S_1 closed and next with switch S_1 open. For the first case we get

$$R' = \frac{0.4343t}{C\ log_{10}V/v}$$

$$Where,\ R' = \frac{RR_1}{R + R_1}$$

For second case with switch open we get

$$R_1 = \frac{0.4343t}{C\ log_{10}V/v}$$

Using R_1 from above equation in equation for R' we can find R.

Megohm Bridge Method: -

In this method we use the famous Wheatstone bridge philosophy but in a slightly modified way. A high resistance is represented as in the figure below.

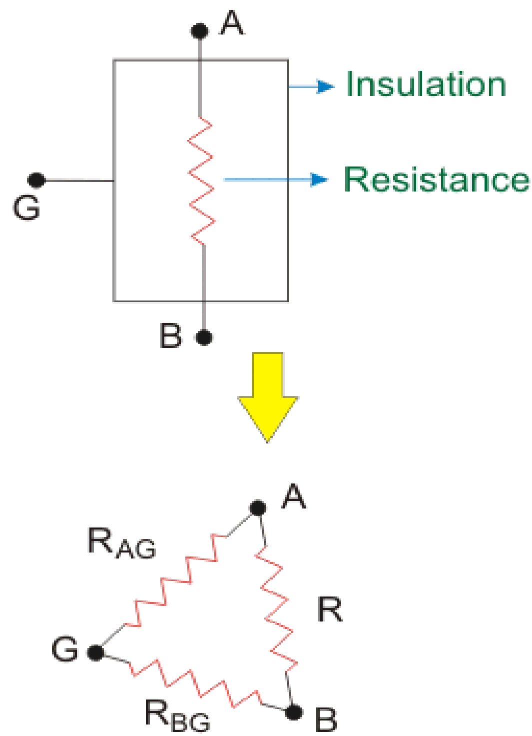


Fig 8.26 High resistance material

G is the guard terminal. Now we can also represent the resistor as shown in the adjoining figure, where R_{AG} and R_{BG} are the leakage resistances. The circuit for measurement is shown in the figure below.

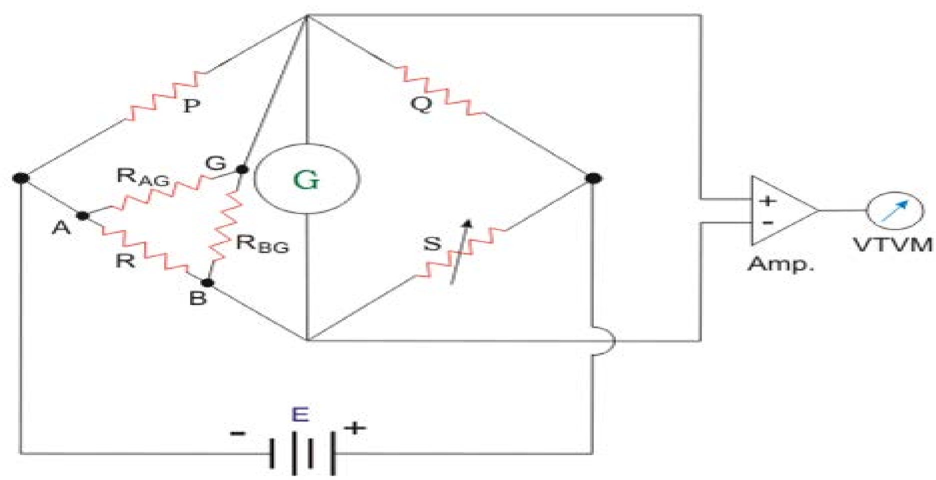


Fig 8.26 Megohm Bridge

It can be observed that we actually obtain the resistance which is parallel combination of R and R_{AG}. Although this causes very insignificant error.

Measurement of High Resistance by Leakage Method

When a capacitor of capacitance C and initial charge Q_0 is allowed to discharge through a resistance R for a time t, the charge remaining on the capacitor is given by

$$Q = Q_0 e^{-t/CR}$$

$$Q_0/Q = e^{t/CR}$$

$$\log_e Q_0/Q = t/CR$$

$$\therefore \quad R = t/C \log_e (Q_0/Q) = t / 2.3026C \log_{10} (Q_0/Q)$$

If R is high, CR will be high and the rate of discharge of capacitor will be very slow. Thus if we determine Q_0/Q from experiment, then R can be calculated.

C is a capacitor of known capacitance, R is the high resistance to be measured, B.G. is a ballistic galvanometer, E is a cell, and K_1, K_2, K_3 are tap keys.

Keeping K_2 and K_3 open, the capacitor is charged by depressing the key K_1. K_1 is then opened and at once K_3 is closed. The capacitor discharges through the galvanometer which records a throw θ0 is proportional to Q_0. The capacitor is again charged to the maximum value keeping K_2 and K_3 open and closing K_1. K_1 is the open and K_2 is closed for a known time t. Some of the charge leaks through R. K_2 is opened and at once K_3 is closed. The charge Q remaining on the capacitor then discharges through the galvanometer. The resulting throw θ is noted. Then $Q \infty \theta$

$$\text{Now,} \quad Q_0/Q = \theta_0/\theta$$

$$\therefore \quad R = t/ 2.3026\, C \log_{10} (\theta_0/\theta)$$

A series of values of t and θ are obtained. A graph is plotted between t and $\log_{10} (\theta_0/\theta)$ which is a straight line. Its slope gives the mean value of t / $\log_{10} (\theta_0/\theta)$. As C is known, the value of R can be calculated.

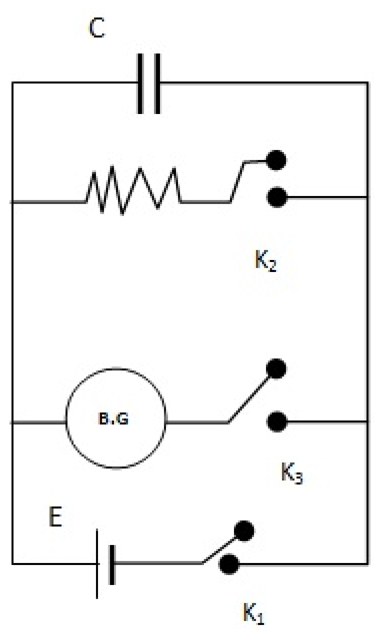


Fig 8.27 Leakage resistance method

SAQ 3: -

a) What is Transient response?

b) Differentiate between transient state and steady state?

c) What is time constant also write the expression for time constant in RL and RC circuit?

d) Explain LC transient circuit?

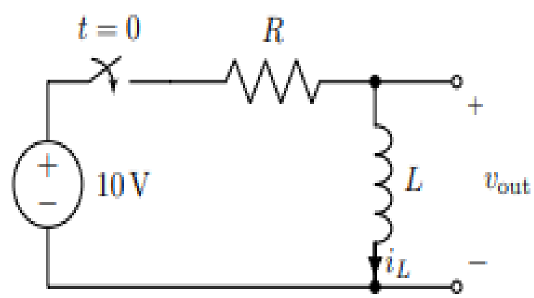
e) Write the different method for measurement of high resistance?

f)) What is the characteristic time constant for a 7.50 mH inductor in series with a 3.00 Ω resistor? (b) Find the current 5.00 ms after the switch is moved to position 2 to disconnect the battery, if it is initially 10.0 A.

g) In the below circuit, the switch closes at time t = 0, before which it had been open for a long time.

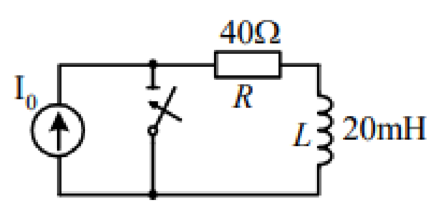
(a) Find and plot $i_L(t)$.

(b) Find and plot $v_{out}(t)$.



Examples: -

Q1. Consider a numerical example. The RL circuit in Fig. is fed by a d.c. current source, 0I = 5A. At instant t = 0 the switch is closed and the circuit is short-circuited. Find:1) the current after switching, by separating the variables and applying the definite integrals, 2) the voltage across the inductance.



Solution :-

1) First, we shall write the differential equation:

$V_L + V_R = L\,(di/dt) + Ri = 0$

or after separating the variables

$(di/i) = (R/L)dt$

Since the current changes from 0I at the instant of switching to i(t), at any instant of t, which means that the time changes from t = 0 to this instant, we may perform the integration of each side of the above equation between the corresponding limits

$$\int_{Io}^{i(t)} \frac{di}{i} = \int_{0}^{t} -\frac{R}{L}dt$$

Therefore,

$$\text{In}\left|\frac{i(t)}{I0} = -\frac{R}{L}t\right|\begin{matrix} t \\ 0 \end{matrix}$$

And

$\text{In}\,i(t) - \ln I_0 = (R/L)t$

Or

$$\text{In}\,\frac{i(t)}{\text{Io}} = -\frac{R}{L}t$$

Which result in

$$\frac{i(t)}{Io} = e^{-\frac{R}{L}t}$$

Thus,

$$i(t) = Io\,e^{-\frac{R}{L}t} = 5e^{-2000t}$$

or

$$i(t) = I_0 e \frac{t}{\tau} = 5e^{-\frac{t}{0.5*10^{-3}}}$$

Where

$$\frac{R}{L} = \frac{40}{20*10^{-3}} = 2000s^{-1}$$

which result in time constant

$$\tau = \frac{L}{R} = 0.5ms.$$

Note that by applying the definite integrals we avoid the step of evaluating the constant of the integration.

2) The voltage across the inductance is

$$v_l(t) = L\frac{di}{dt}(5e^{-2000t}) = -200e^{-\frac{t}{0.5}} \text{ , V}$$

Note that the voltage across the resistance is

$$v_R = Ri = 40.5e^{\frac{-t}{0.5}} ,$$

i.e., it is equal in magnitude to the inductance voltage, but opposite in sign, so that the total voltage in the short-circuit is equal to zero

Q 2: A 100 V de is applied to a circuit consisting of a resistance of 100 Ω in series with an inductance of 10 H through the switch. If the switch is closed at t = 0, fing out

(i) the expressions for $i(t)$, VR(t) and VL(t),

(ii) the value of $i(t)$ for t = 0.2 seconds and

(iii) time at which VR(t) = VL(t).

Solution :

Resistance. R = 100 n

Inductance, L = 10 H

Voltage, V = 100 V

Time constant $\tau = \frac{L}{R} = \frac{10}{100} = 0.1 second.$

(i) The expression for the current $i(t)$ is given as

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$= \frac{100}{100}\left(1 - e^{-\frac{t}{0.1}}\right) = (1 - e^{-10t})\text{Ans}$$

The expression for the voltage drop across resistor R is given as

$V_R(t) = R \times i(t) = 100(1 - e^{-10t})$ Volts Ans.

The expression for voltage drop across inductor L is given as

$$V_L(t) = -L\frac{di(t)}{dt}$$

$$= -10 \times \frac{d}{dt}(1 - e^{-10t}) \text{ VoltsAns.}$$

(ii) When t = 0.2 Seconds.

Current $i(t)$ = (1-$e^{-10*0.2}$) =1 – e^{-2} = 0.8647A Ans.

(*iii*) Let the voltage drop across resistance and inductance be equal t seconds after closure of switch, then

$100(1-e^{-10t}) = 100e^{-10t}$

$t = \frac{-1}{10} log_e 0.5 = 0.0693$ Seconds Ans.

Q3 : The switch in fig A has been in position 1 for a long time, it is moved to 2 at $t = 0$ obtain

the expression for i, for $t > 0$.

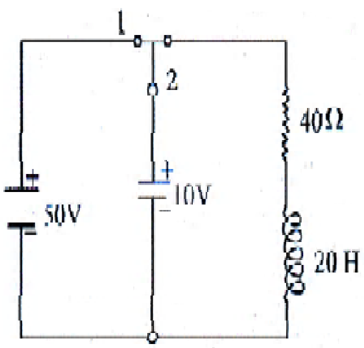


Fig. (A)

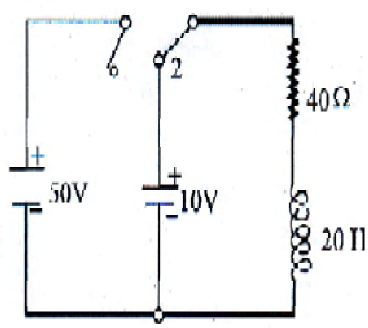


Fig. (B)

Solution : With switch in position 1

$i(0^-) = \frac{50}{40} A = 1.25A$

∵ Because inductance of 20 H act as a short during steady state with switch in position 2 (circuit

shown in fig B)

$i(e^+) = i(0^-)$ =1.25A

.: Current in an inductor does not change instantaneously. Applying Kirchhoffs voltage law in closed

loop of circuit diagram shown in fig (B) we have.

$10 = 40i(t) + 20\frac{di(t)}{dt}$

Or

$\frac{di(t)}{dt} + 2i(t) = 0.5$

$i(t) = \frac{0.5}{2} + Ae^{-2t}$

A = 1.00

The equation for i is given by

i(t) = (0.25 + e^{-2t})A

Q 4 : In the circuit shown in fig A the switch S is initially in position 1. Find the voltage across

the coil at the instant at which the switch is changed to position 2.

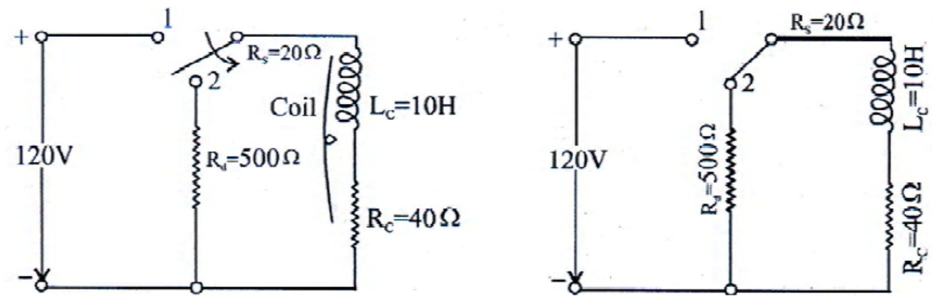


Fig. (A) Fig. (B)

Solution : With switch in position *l*

$$I(o) = \frac{V}{R_s + R_C} = \frac{120}{20+40} = 2A$$

∵ Inductor Lc acts as a short during steady state

At positions 2.

Time constant of the circuit is

$$\tau = \frac{L_C}{R_d + R_s + R_c} = \frac{10}{500+20+40} s = \frac{1}{56} s$$

The equation for i becomes

$$i(t) = i(o)e^{\frac{-t}{\tau}} = 2e^{-56t}$$

Voltage across the coil

$$V = R_c i(t) + L_c \frac{di(t)}{dt} = 40 \times 2e^{-56t} + 10 \times 2 \times (-56)e^{-56t}$$

$$= -1040e^{-56t}$$

Q5 : A resistance Rand 5μ F Capacitor are connected in series across a 200 V supply. Calculate

the value of R such that the voltage across the capacitor becomes 100 V in 5 seconds after the circuit is switched on.

Solution : Voltage across the capacitor is given as

$V_c(t) = V(1\text{-}e^{t/CR})$

Given $t = 5, c = 5 \times 10^{-6}F, v = 200v, v_c(t) = 100v$

$$100 = 200\left(1 - e^{-\frac{5}{5\times10^{-6}\times R}}\right) = 200\left(1 - e^{-\frac{10^6}{R}}\right)$$

Q6 : In fig (A) the switch S is closed at *t* = 0 Determine the time when the current drawn from

the battery attains the value of 0.5A.

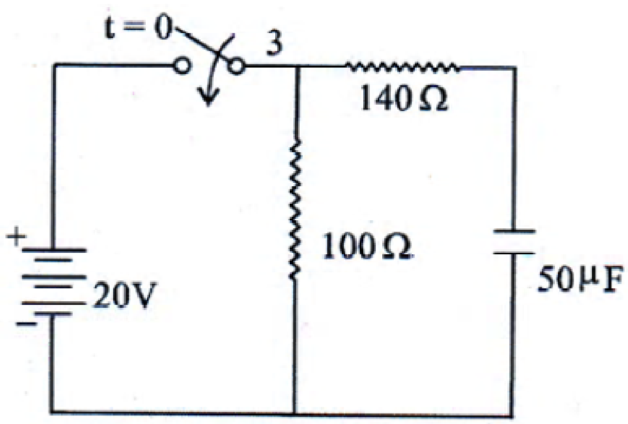


Fig (A)

Solution: Let the current through 100 Ω resistor be I_1 and through R–C branch be I_2 after the switch S is closed.

Now

$I_I = \frac{20}{100} = 0.2$ A

Current

$I_2 = \frac{V}{R} e^{-1/CR}$

$= \frac{20}{140} e^{\frac{-t}{140X50X10^{-6}}} = \frac{1}{7} e^{-142.86t}$

and since

$I = I_1 + I_2$

So

$\frac{1}{7} e^{-142.86t} = 0.5 - 0.2 = 0.3$ A

$t = 5.2$ ms

8.9 SUMMARY

- Transient phenomena are Rapidly changing actions occurring in a circuit during the interval between closing of a switch and settling to a steady-state condition, or any other temporary actions occurring after some change in a circuit.
- The galvanometer is the device used for detecting the presence of small current and voltage or for measuring their magnitude.
- The galvanometer is used as an ammeter by connecting the low resistance wire in parallel with the galvanometer. The potential difference between the voltage and the shunt resistance are equal.
- Ballistic galvanometers are the measuring instruments which are used for measuring the quantity of electric charges obtained from magnetic flux.
- The sensitivity of a Moving Coil Galvanometer is defined as the ratio of the change in deflection of the galvanometer to the change in current

The current in Moving Coil galvanometer is: $I = (C/nBA) \times \theta$

- The time required for a changing quantity in a circuit, as voltage or current, to rise or fall approximately 0.632 of the difference between its old and new value after an impulse has been applied that induces such a change: equal in seconds to the inductance of the circuit in henries divided by its resistance in ohms
- The Time Constant, (τ) of the LR series circuit is given as L/R
- As the capacitor charges up, the potential difference across its plates slowly increases with the actual time taken for the charge on the capacitor to reach 63% of its maximum possible voltage, in our curve 0.63Vs being known as one Time Constant, (T).
- Mathematically we can say that the time required for a capacitor to charge up to one time constant, (1T) is given as:

$$\tau \equiv R \times C$$

- Resonance occurs when an LC circuit is driven from an external source at an angular frequency ω_0 at which the inductive and capacitive reactances are equal in magnitude. The frequency at which this equality holds for the particular circuit is called the resonant frequency.
- By Kirchhoff's voltage law, the voltage across the capacitor, V_C, plus the voltage across the inductor, V_L must equal zero:

❖ Likewise, by Kirchhoff's current law, the current through the capacitor equals the current through the inductor:

❖ The overdamped response (***ζ* > 1**) is the overdamped response is a decay of the transient current without oscillation.

- The underdamped response is a decaying oscillation at frequency ω_d. The oscillation decays at a rate determined by the attenuation α.

- The critically damped response (ζ = 1) is the critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation.

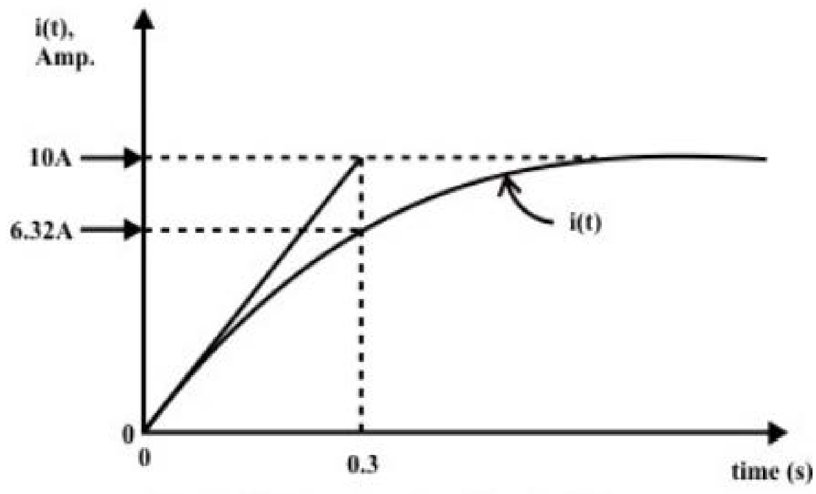
- In Loss of charge method we utilize the equation of voltage across a discharging capacitor to find the value of unknown resistance R.

$$v = Ve^{\frac{-t}{RC}}$$

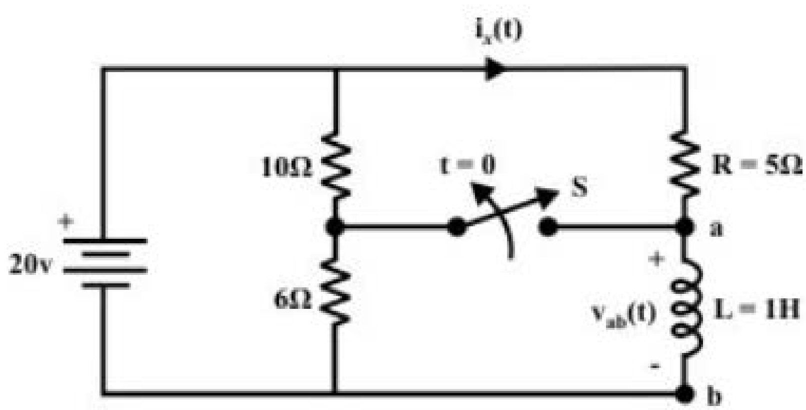
$$R = \frac{0.4343t}{C\ log_{10}V/v}$$

8.10 TERMINAL QUESTION

1. Explain the construction details and the principle of operation of galvanometer?
2. Write the different types of galvanometer?
3. Explain the principle of operation of deadbeat and Ballistic type galvanometer?
4. Explain the critical resistance and damping in moving coil galvanometer?
5. Explain the theory of moving coil galvanometer?
6. What is sensitivity of moving coil galvanometer and write their types?
7. Explain the transient response in RL series circuit?
8. Explain the transient response in RC series circuit?
9. What is time constant explain with the help of curve?
10. Explain the theory of measurement of high resistance by leakage method?
11. Write the application of measurement of high resistance?
12. Fig. shows the plot of current i (t) through a series R − L circuit when a constant forcing function of magnitudes = 50 V is applied to it. Calculate the values of resistance R and inductance L.



13. For the circuit shown in Fig, the switch ‘S’ has been closed for a long time and then opens at. t = 0



Find,

(i) $v_{ab}(0^-)$ (ii) $i_x(0^-), i_L(0^-)$ (iii) $i_x(0^+)$ (iv) $v_{ab}(0^+)$ (v) $i_x(t=\infty)$ (vi) $v_{ab}(t=\infty)$

(vii) $i_x(t)$ *for* $t>0$

UNIT-9 ALTERNATING CURRENT

Structure

9.1 INTRODUCTION

In this topic, we will understand the basic properties of Alternating Current (AC) and cover topics such as Current through AC circuits, Power Factor, Wattless Current, and wattfull current In electrical engineering this type of number is called an "imaginary number" and to distinguish an imaginary number from a real number the letter " j " known commonly in electrical engineering as the **j-operator. We will study about resistance** phasor notations, reactance, impedance, susceptance, admittance. We able to learn about Instaneous, Peak, RMS and Average value of alternating voltage and current, Form factor.In this chapter we learn about power used in electrical engineering like active power, reactive power and apparent power. We will study about series and parallel ac circuit like CR, LR, LCR series, LCR parallel. We will also learn about resonance and their types.

9.2 OBJECTIVES

After studying this unit you should be able to

- Study and identify J-Operator and phasor notations, reactance, impedance, susceptance, admittance.
- Explain and identify Instaneous, Peak, RMS and Average value of alternating voltage and current, Form factor.
- Study and identify Angle of lag and lead, wattful and wattless current, average power consumed (active, reactive and apparent), power factor.
- Study and identify Phasor and vector diagram of CR, LR, LCR series, LCR parallel, LR in series with C in parallel circuits.
- Explain and identify Parallel and series resonance, sharpness of resonance, Quality factor, Bandwidth Resonance frequency.

9.3 THE J OPERATOR

In electrical circuit this type of number is called an “imaginary number” and to distinguish an imaginary number from a real number the letter “ j ” known commonly in electrical engineering as the **j-operator**, is used. Thus the letter “j” is placed in front of a real number to signify its imaginary number operation.

The j-operator has a value exactly equal to $\sqrt{-1}$, so successive multiplication of “ j “, (j x j) will result in j having the following values of, -1, -j and +1. As the j-operator is commonly used to indicate the anticlockwise rotation of a vector, each successive multiplication or power of “ j “, j^2, j^3 etc, will force the vector to rotate through a fixed angle of 90^o in an anticlockwise direction as shown below. Likewise, if the multiplication of the vector results in a -j operator then the phase shift will be -90^o, i.e. a clockwise rotation.

Vector Rotation of the j-operator

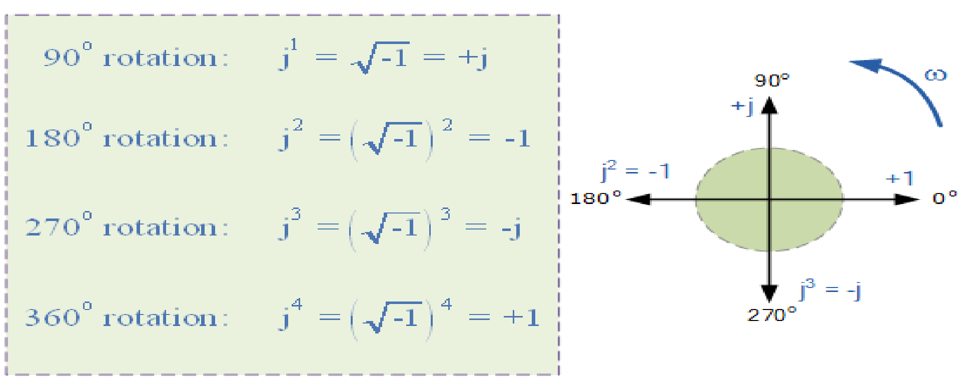


Fig 9.1 Vector Rotation of the j-operator

So by multiplying an imaginary number by j^2 will rotate the vector by 180^o anticlockwise, multiplying by j^3 rotates it 270^o and by j^4 rotates it 360^o or back to its original position. Multiplication by j^{10} or by j^{30} will cause the vector to rotate anticlockwise by the appropriate amount. In each successive rotation, the magnitude of the vector always remains the same.

In Electrical Engineering there are different ways to represent a complex number either graphically or mathematically. One such way that uses the cosine and sine rule is called the Cartesian or Rectangular Form.

Phasor Notation :

Phasor notation can be used to represent the phase relationship between two sinusoidal waveforms

Reactance :

"Reactance is a form of opposition that electronic components exhibit to the passage of AC (alternating current) because of capacitance or inductance" It is denoted by X. It is expressed in ohms. It is observed for AC (alternating current), but not for DC (direct current).

Types of Reactance :

Inductive Reactance :

When AC (alternating current) passes through a component that contains reactance, energy might be stored and released in the form of a magnetic field which is known as inductive reactance. It is denoted by $+jX_L$

Capacitive Reactance :

When AC (alternating current) passes through a component that contains reactance, energy might be stored and released in the form of an electric field which is known as capacitive reactance.It is denoted by $-jX_C$

Reactance is conventionally multiplied by the positive square root of -1, which is the unit imaginary number called the j operator, to express Z as a complex number of the form $R + jX_L$ (when the net reactance is inductive) or $R - jX_C$ (when the net reactance is capacitive).

Impedance :

"Impedance is the total resistance/opposition offered by the circuit elements to the flow of alternating or direct current!"

OR

"The impedance of a circuit is the ratio of the phasor voltage (V) to the phasor current (I)"

It is denoted by Z.

$Z=V/I$

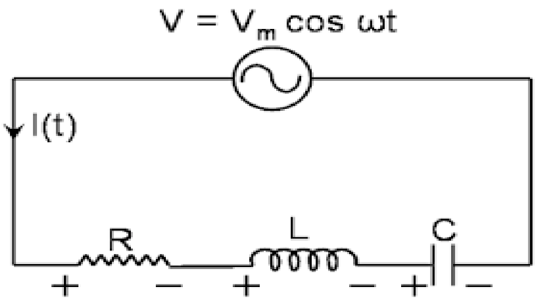


Fig 9.2 Impedance Circuit

As complex quantity, we can write as:

Z=R+jX

Susceptance :

"Susceptance is an expression of the readiness with which an electronic component, circuit, or system releases stored energy as the current and voltage fluctuate"

OR

"It is a reciprocal of reactance" .It is denoted by B.

B=1/X

Susceptance is expressed in imaginary number Siemens. Susceptance is observed with AC, but not for DC.

Types of Susceptance :

Inuductive Susceptance :

When AC (alternating current) passes through a component that contains susceptance, energy might be stored and released in the form of a magnetic field which is known is inductive susceptance..It is denoted by -jB $_L$

Capacitive Susceptance:

When AC (alternating current) passes through a component that contains susceptance, energy might be stored and released in the form of an electric field which is known is capacitive susceptance. It is denoted by + jB $_C$

Admittance :

"Admittance is the allowance of circuit elements to the flow of alternating current or direct current ".

OR

"It is the inverse of impedance". It is denoted by Y.

We can write as:

Y=1/Z=I/V

As complex quantity, we can write as:

Y=G+jB

Admittance is a vector quantity comprised of two Independent scalar phenomena : conductance and sustenance.

9.4 INSTANTANEOUS, AVERAGE, AND RMS VALUES

Instantaneous value :

The instantaneous value is “the value of an alternating quantity (it may ac voltage or ac current or ac power) at a particular instant of time in the cycle”. There are uncountable number of instantaneous values that exist in a cycle.

Average value :

The average value is defined as “the average of all instantaneous values during one alternation”. That is, the ratio of the sum of all considered instantaneous values to the number of instantaneous values in one alternation period.

Whereas the average value for the entire cycle of alternating quantity is zero. Because the average value obtained for one alteration is a positive value and for another alternation is a negative value. The average values of these two alternations (for entire cycle) cancel each other and the resultant average value is zero.

Consider the single cycle alternating current wave in Figure.

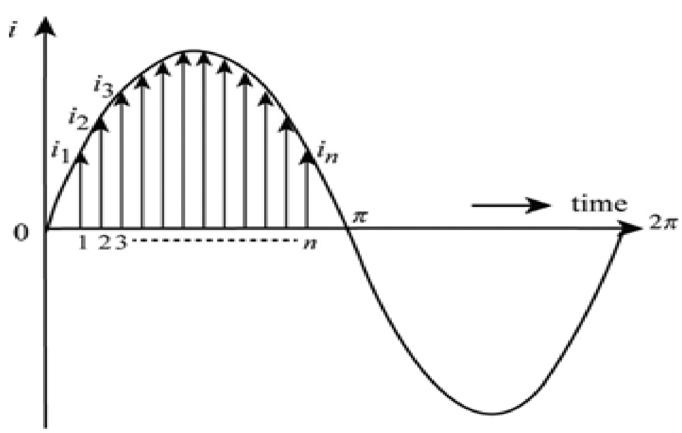


Fig 9.3 Average value

The instantaneous value at t=2 is i_2

and so on at t = n, i is i_n

The average value for one alternation (0 to π) is

$$i_{avg} = \frac{i_1 + i_2 + i_3 + \ldots\ldots + i_n}{n}$$

RMS (Root Mean Square) value:

The Root Mean Square (RMS) value is “the square root of the sum of squares of means of an alternating quantity”.

It can also express as “the effect that produced by a certain input of AC quantity which is equivalent to an effect produced by the equal input of DC quantity”.

Consider one example, the heat produced by a resistor when one ampere direct current (DC) passed through it, is not an equal amount of heat produced when one ampere of alternating current (AC) passed through the same resistor. Since the AC current is not constant value rather than it is varying with the time. The heat produced by AC quantity (equal amount of DC quantity) is nothing but RMS value of an alternating parameter or quantity.

$$i_{rms} = \sqrt{i_1^2 + i_2^2 + \ldots . i_n^2}$$

Here, $i_1, i_2, \ldots i_n$ are mean values

$$i_{rms} = \frac{i_{max}}{\sqrt{2}}$$

Peak Value :

The maximum value attained by an alternating quantity during one cycle is called its P**eak value**. It is also known as the maximum value or amplitude or crest value. The sinusoidal alternating quantity obtains its peak value at 90 degrees as shown in the figure below.

The peak values of alternating voltage and current is represented by E_m and I_m respectively.

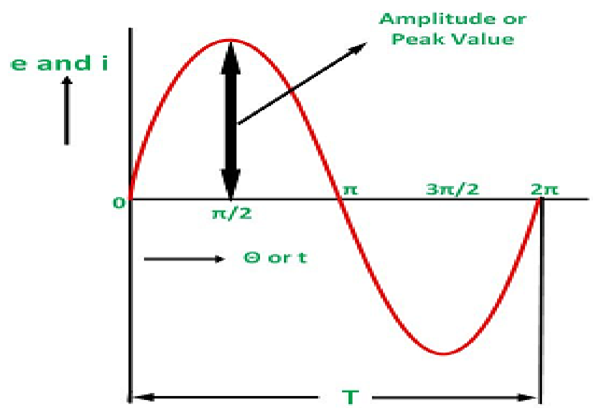


Fig 9.4 Peak Value

Form Factor :

Form Factor is the ratio between the average value and the RMS value and is given as.

$$\text{Form Factor} = \frac{I_{r.m.s}}{I_{av}} = \frac{I_m/\sqrt{2}}{2I_m/\pi} = \frac{\pi I_m}{2\sqrt{2}I_m} = 1.11$$

SAQ 1 :

a) What is j operator and where it used ?

b) Derive the expression for rms and average value of alternating current ?

c) Explain form factor and what is the ideal value of it for sinusoidal wave?

d) Define the following : (i) Reactance (ii) Maximum Value (iii) Susceptance

e) An AC circuit carries an rms current of 7.0 Amps. The current travels through a 12 Ohm resistor. Calculate the peak current?

9.5 PHASE

The phase is defined as the position of the waveform at a fraction of time period. Phase is expressed in angle or radian. Phase can also be an expression of relative displacement between two corresponding features (for example, peaks or zero crossings) of two waveforms having the same frequency.

Phase Difference:

Phase difference is the difference, between two waves is having the same frequency and referenced to the same point in time. It is expressed in degrees or radians. Let's consider two sinusoidal wave, both have same frequency, Example: R phase and B phase (in our three-phase circuit.)

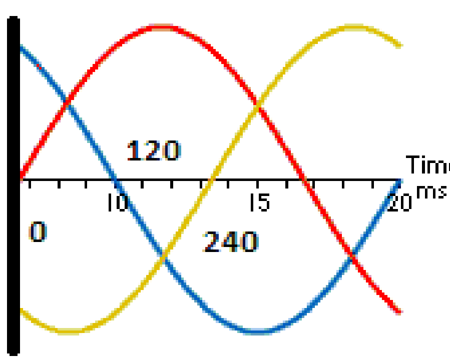


Fig 9.5 Phase difference

phase difference = R phase starting point (angle) – B phase starting point (angle)

$$Phase\ difference = 0deg - 120\ deg$$

$$Hence\ the\ phase\ difference = -120\ deg$$

Phase angle :

What is phase angle refers to the angular component of the complex number representation of the function. The notation of the phase angle is defined as

$$Phase\ angle = A\angle\theta$$

A is the magnitude and phase angle θ, is called angle notation. This notation is mostly used in electrical circuit to represent an electrical impedance (vector sum of resistance and reactance) and the apparent power (vector sum of real power and reactive power). Here the phase angle theta is the phase difference between the voltage applied to the impedance and the current flow through the impedance.

In Phase:

Here the sign indicates the leading or lagging nature of both waves. Generally – symbol indicates leading nature (the wave leads the angle from the reference wave) + symbol indicates lagging nature (the wave lags the angle from the reference wave)

The two waveform said to be in phase where the two wave should reach maximum, minimum and zero values simultaneously at the same time.

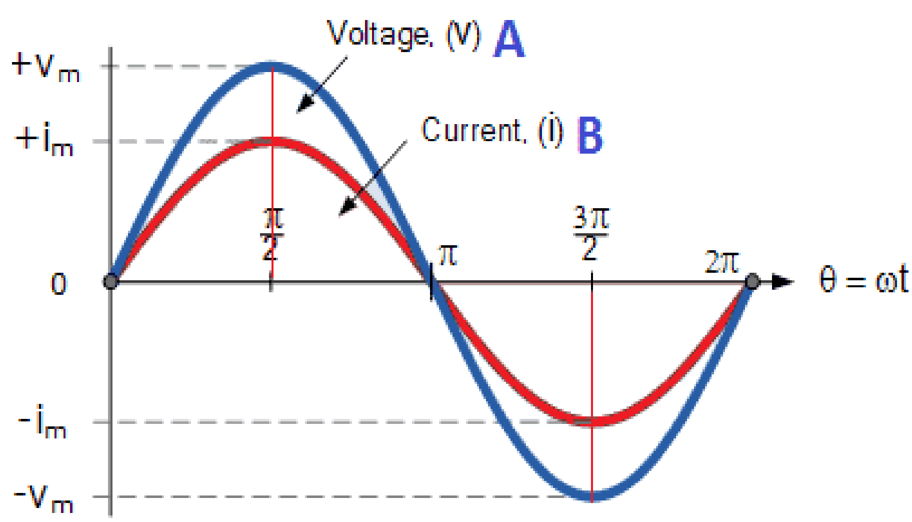


Fig 9.6 In Phase for Resistive circuit

Out Phase :

Two sinusoidal signals are said to be out of phase when they do not reach maximum or zero values at the same time.

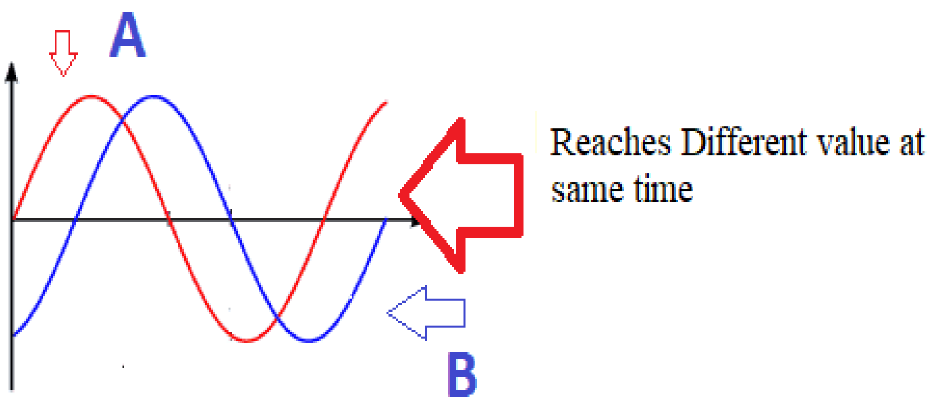


Fig 9.7 Out of Phase

Leading Power Factor

The leading power factor in an ac electrical circuit is attained by the use of capacitive load in the circuit. As in the presence of purely capacitive load or combination of resistive-capacitive load, the current leads supplied voltage. This gives rise to power factor generally said to be leading in nature.

As it is known that power factor is the ratio of true to the apparent power. And generally for sinusoidal waveform power factor is the cosine function of the phase angle existing between voltage and current.

Consider the wave shapes of voltage supplied to the ac circuit and the current through the purely capacitive load:

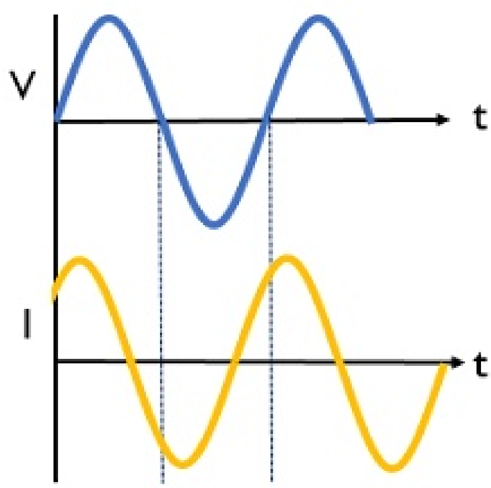


Fig 9.8 Leading Power Factor

As it is clear from the above figure that current, I encounters the 0 crossings of the time axis some phase earlier than that of voltage, V. This is referred as leading power factor.

The figure below represents the leading power factor triangle:

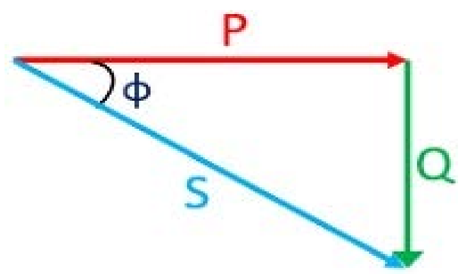


Fig 9.9 Power factor triangle

Where, S= Apparent Power

P=Real Power

Q=Reactive Power

Lagging Power Factor :

In ac circuits lagging power factor, is achieved when the load is capacitive in nature. This is so because when a purely capacitive or resistive capacitive load is present then there exists a phase difference between voltage and current in which the current lags the voltage.Thus the power factor of such circuits is of lagging nature.

Let us consider the waveforms of supplied voltage to an ac circuit and the current through the purely capacitive load as:

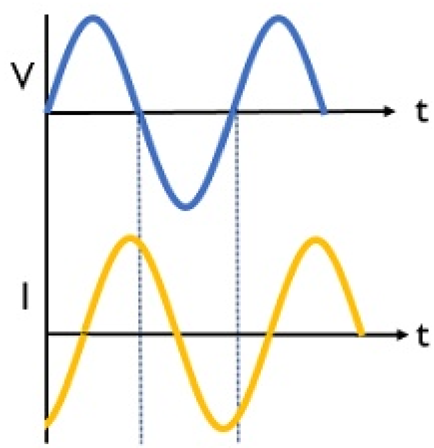


Fig 9.10 Lagging Power Factor

Here the current encounters the 0 crossings at some phase after the voltage. Thereby giving rise to lagging power factor.

The lagging power factor triangle is given below:

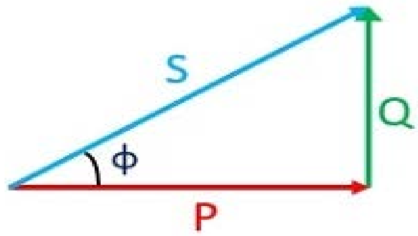


Fig 9.11 Power factor triangle for lagging

Where, S= Apparent Power

P=Real Power

Q=Reactive Power

Wattless and Wattful Current:

The average power over a cycle of AC is given by,

P = Vrms X Irms X Cos ϕ

In pure inductor or an ideal capacitor $\phi = 90^0$. So average power consumed in a pure inductor or ideal capacitor is,

P = Vrms X Irms X Cos 90 = 0

Therefore, Current through pure 'L' or pure 'C' which consumes no power for its maintenance in the circuit is called Wattless Current.

At resonance $X_L = X_C$ and $\phi = 0^0$

$\therefore$ Cos ϕ = Cos 0 = 1

Therefore Maximum power is dissipated in a circuit at resonance.

The Current through resistance (R), which consumes power for its maintenance in the circuit is called Wattful Current.

Active Power

The power which is actually consumed or utilised in an AC Circuit is called **True power** or **Active power** or **Real power**. It is measured in kilowatt (kW) or MW. It is the actual outcomes of the electrical system which runs the electric circuits or load.

Active power P = V x I cosϕ = V I cosϕ

Reactive Power

The power which flows back and forth that means it moves in both the directions in the circuit or reacts upon itself, is called **Reactive Power**.

The reactive power is measured in kilo volt-ampere reactive (kVAR) or MVAR.

Reactive power $Q = V \times I \sin\phi = V I \sin\phi$

Apparent Power

The product of root mean square (RMS) value of voltage and current is known as **Apparent Power**. This power is measured in kVA or MVA.

Apparent power P_a or $S = V \times I = VI$

Power Triangle :

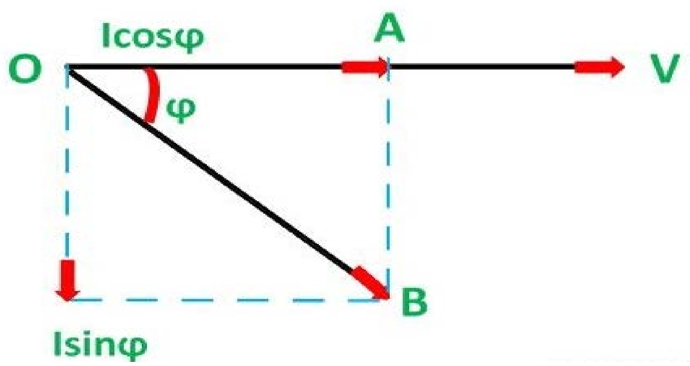
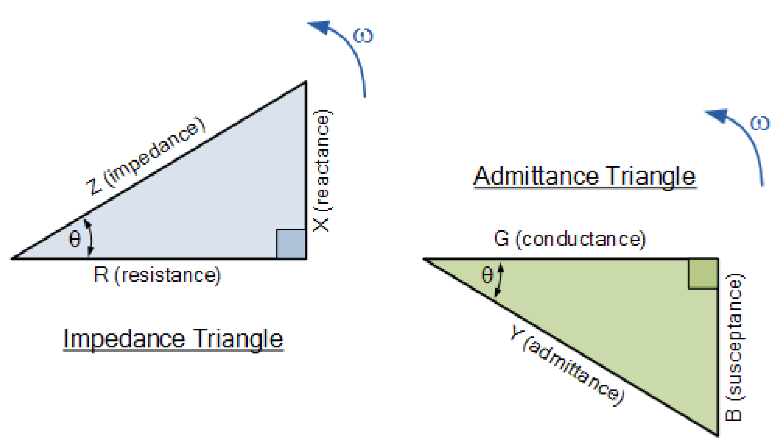


Fig 9.11 Power triangle

Taking voltage V as reference, the current I lags behind the voltage V by an angle ϕ. The current I is divided into two components:

- I Cos ϕ in phase with the voltage V
- I Sin ϕ which is 90 degrees out of phase with the voltage V

Impedance Triangle :

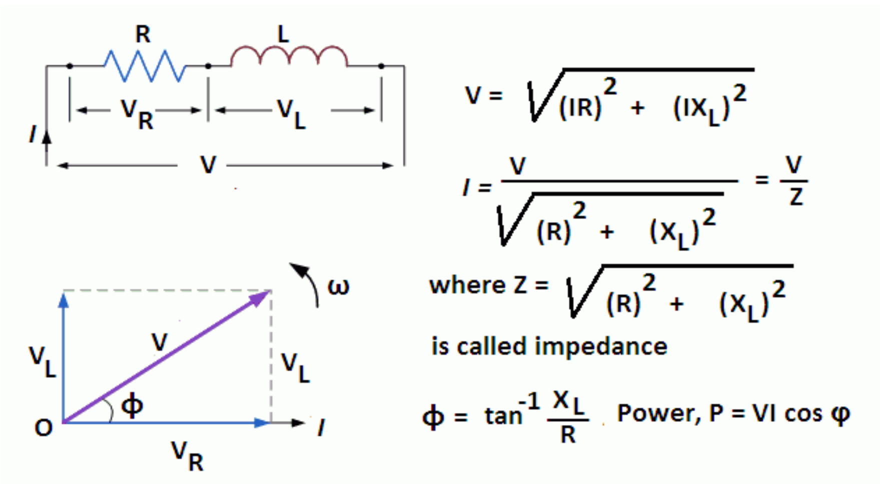


Power Factor :

Power factor is a crucial property of AC electrical systems. It is dimensionless in nature. It is used for both single and three-phase AC circuits. It is the ratio of true or actual power to the apparent power in the ac systems. More simply, the power factor is the cosine of phase difference existing between V and I. The power factor of ac circuits with linear loads lies between -1 to 1. Generally, it is considered that if a system exhibits power factor closer to 1, then such systems are said to be stable.

9.6 RL SERIES CIRCUIT

In actual practice, AC circuits contain two or more than two components connected in series. In a series circuit, each component carries the same current. An AC series circuit may be classified as under:



In an RL series circuit, a pure resistance (R) is connected in series with a coil having the pure inductance (L). To draw the phasor diagram of RL series circuit, the current I (RMS value) is taken as reference vector because it is common to both elements. Voltage drop V_R is in phase with current vector, whereas, the voltage drop in inductive reactance V_L leads the current vector by 90^o since current lags behind the voltage by 90^o in the purely inductive circuit. The vector sum of these two voltage drops is equal to the applied voltage V (RMS value).

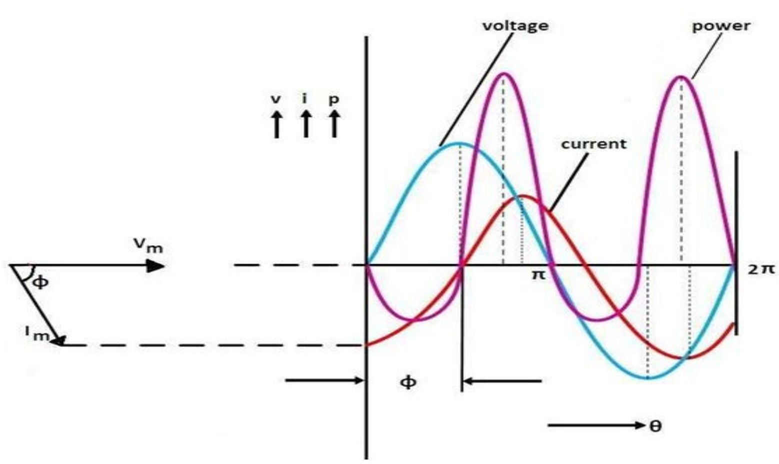
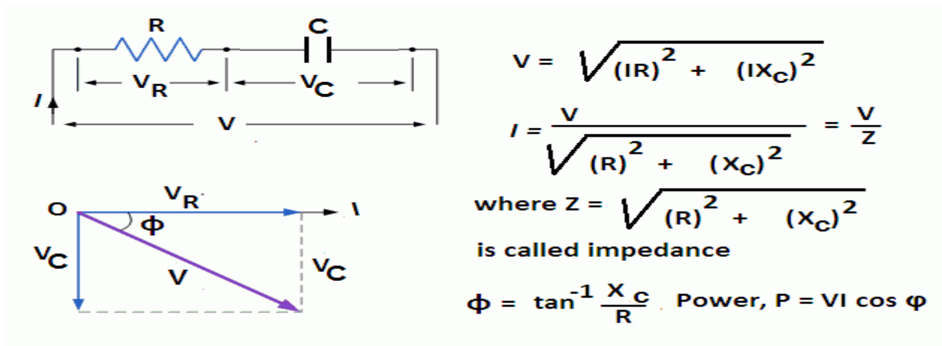


Fig 9.12 Wave form of RL circuit

The power waveform for RL series circuit is shown in the figure. In this figure, voltage wave is considered as a reference. The points for the power waveform are obtained from the product of the corresponding instantaneous values of voltage and current. It is clear from the power waveform that power is negative between 0 and φ and between 180° and (180° + φ). The power is positive during rest of the cycle. Since the area under the positive loops is greater than that under the negative loops, the net power over a complete cycle is positive. Hence a definite quantity of power is consumed by the RL series circuit. But power is consumed in resistance only; inductance does not consume any power.

RC Series Circuit



In an RC series circuit, a pure resistance (R) is connected in series with a pure capacitor (C). To draw the phasor diagram of RC series circuit, the current *I* (RMS value) is taken as reference vector. Voltage drop V_R is in phase with current vector, whereas, the voltage drop in capacitive reactance V_C lags behind the current vector by 90°, since current leads the voltage by 90° in the pure capacitive circuit. The vector sum of these two voltage drops is equal to the applied voltage V (RMS value).

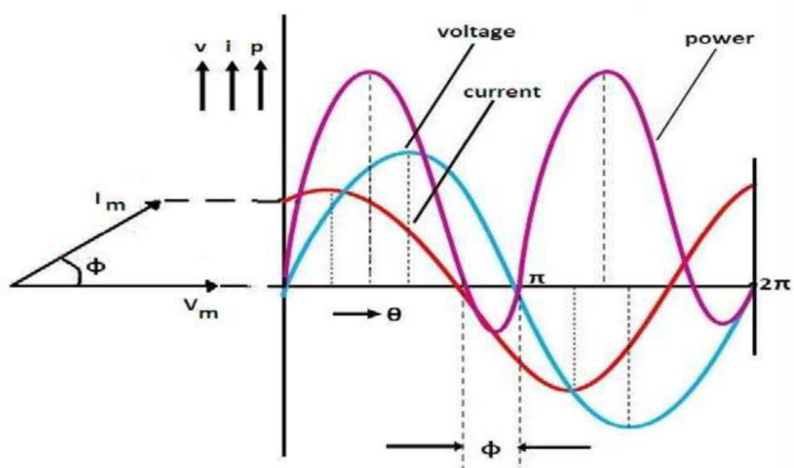
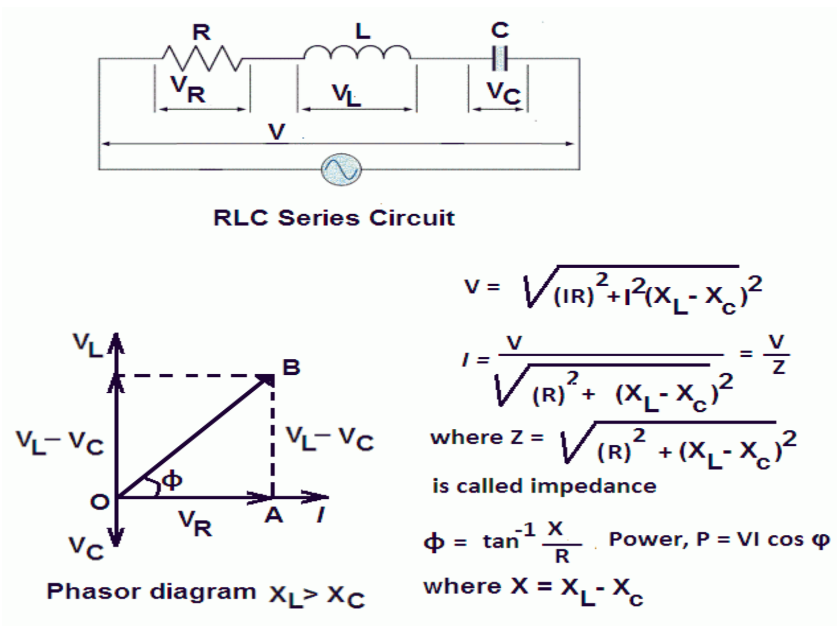


Fig 9.13 Wave form for RC circuit

The power waveform for RC series circuit is shown in the figure. In this figure, voltage wave is considered as a reference. The points for the power waveform are obtained from the product of the corresponding instantaneous values of voltage and current. It is clear from the power waveform that power is negative between (180^o – φ) and 180^o and between (360^o – φ) and 360^o. The power is positive during rest of the cycle.

Since the area under the positive loops is greater than that under the negative loops, the net power over a complete cycle is positive. Hence a definite quantity of power is consumed by the RC series circuit. But power is consumed in resistance only; capacitor does not consume any power. RLC Series Circuit :



In an RLC series circuit a pure resistance (R), pure inductance (L) and a pure capacitor (C) are connected in series. To draw the phasor diagram of RLC series circuit, the current I (RMS value) is taken as the reference vector. The voltages across three components are represented in the phasor diagram by three phasors V_R, V_L and V_C respectively.

The voltage drop V_L is in phase opposition to V_C. It shows that the circuit can either be effectively inductive or capacitive. In the figure, phasor diagram is drawn for the inductive circuit. There can be three cases of RLC series circuit.

- When $X_L > X_C$, the phase angle φ is positive. In this case, RLC series circuit behaves as an RL series circuit. The circuit current lags behind the applied voltage and power factor is lagging. In this case,

  if the applied voltage is represented by the equation;

  $v = V_m \sin \omega t$

then, the circuit current will be represented by the equation;

$i = I_m \sin(\omega t - \varphi)$.

When $X_L < X_C$, the phase angle φ is negative. In this case, the RLC series circuit behaves as an RC series circuit. The circuit current leads the applied voltage and power factor is leading. In this case, the circuit current will be represented by the equation:

$i = I_m \sin(\omega t + \varphi)$.

When $X_L = X_C$, the phase angle φ is zero. In this case, the RLC series circuit behaves like a purely resistive circuit. The circuit current is in phase with the applied voltage and power factor is unity. In this case, the circuit current will be represented by the equation:

$i = I_m \sin(\omega t)$.

Parallel RLC Circuit Analysis :

The **Parallel RLC Circuit** is the exact opposite to the series circuit we looked at in the previous tutorial although some of the previous concepts and equations still apply.

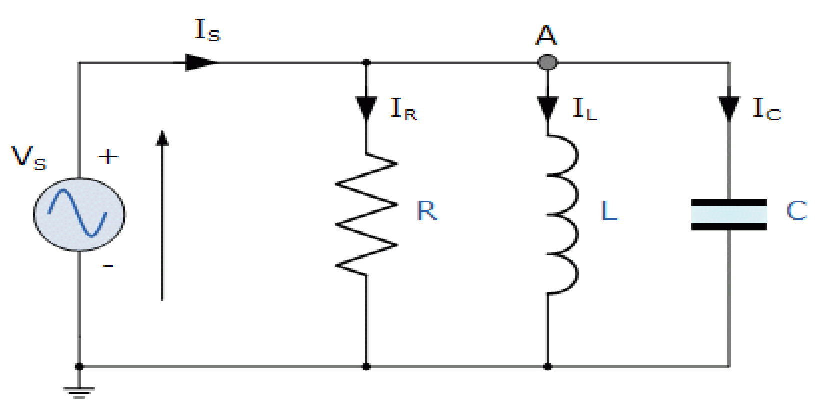


Fig 9.14 Parallel circuit

The applied voltage is now common to all so we need to find the individual branch currents through each element. The total impedance, Z of a parallel RLC circuit is calculated using the current of the circuit similar to that for a DC parallel circuit, the difference this time is that admittance is used instead of impedance. Consider the parallel RLC circuit below.

In the above parallel RLC circuit, we can see that the supply voltage, V_S is common to all three components whilst the supply current I_S consists of three parts. The current flowing through the resistor, I_R, the current flowing through the inductor, I_L and the current through the capacitor, I_C.

But the current flowing through each branch and therefore each component will be different to each other and also to the supply current, I_S. The total current drawn from the supply will not be the mathematical sum of the three individual branch currents but their vector sum.

Like the series RLC circuit, we can solve this circuit using the phasor or vector method but this time the vector diagram will have the voltage as its reference with the three current vectors plotted with respect to the voltage. The phasor diagram for a parallel RLC circuit is produced by combining together the three individual phasors for each component and adding the currents vectorially.

Since the voltage across the circuit is common to all three circuit elements we can use this as the reference vector with the three current vectors drawn relative to this at their corresponding angles. The resulting vector current I_S is obtained by adding together two of the vectors, I_L and I_C and then adding this sum to the remaining vector I_R. The resulting angle obtained between V and I_S will be the circuits phase angle as shown below.

Phasor Diagram for a Parallel RLC Circuit :

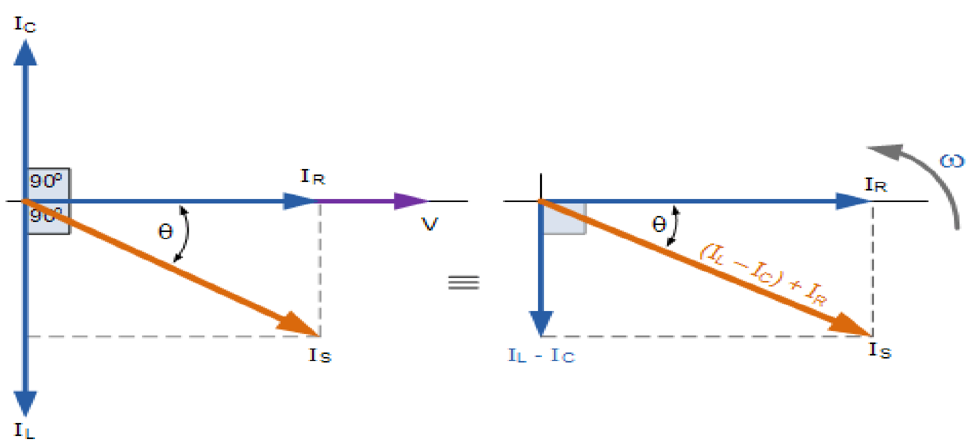


Fig 9.16 Phasor Diagram for a Parallel RLC Circuit

We can see from the phasor diagram on the right hand side above that the current vectors produce a rectangular triangle, comprising of hypotenuse I_S, horizontal axis I_R and vertical axis $I_L – I_C$ Hopefully you will notice then, that this forms a Current Triangle. We can therefore use Pythagoras's theorem on this current triangle to mathematically obtain the individual magnitudes of the branch currents along the x-axis and y-axis which will determine the total supply current I_S of these components as shown.

Current Triangle for a Parallel RLC Circuit

$$I_S^2 = I_R^2 + (I_L - I_C)^2$$

$$I_S = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$\therefore\ I_S = \sqrt{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_L} - \frac{V}{X_C}\right)^2} = \frac{V}{Z}$$

$$\text{where:}\quad I_R = \frac{V}{R},\quad I_L = \frac{V}{X_L},\quad I_C = \frac{V}{X_C}$$

Since the voltage across the circuit is common to all three circuit elements, the current through each branch can be found using Kirchhoff's Current Law, (KCL). Rember that Kirchhoff's current law or junction law states that "the total current entering a junction or node is exactly equal to the current leaving that node". Thus the currents entering and leaving node "A" above are given as:

$$\text{KCL: } I_S - I_R - I_L - I_C = 0$$

$$I_S - \frac{V}{R} - \frac{1}{L}\int v\,dt - C\frac{dv}{dt} = 0$$

Taking the derivative, dividing through the above equation by C and then re-arranging gives us the following Second-order equation for the circuit current. It becomes a second-order equation because there are two reactive elements in the circuit, the inductor and the capacitor.

$$I_S - \frac{d^2V}{dt^2} - \frac{dV}{RCdt} - \frac{V}{LC} = 0$$

$$\therefore\ I_{S(t)} = \frac{d^2V}{dt^2} + \frac{dV}{dt}\frac{1}{RC} + \frac{1}{LC}V$$

The opposition to current flow in this type of AC circuit is made up of three components: X_L X_C and R with the combination of these three values giving the circuits impedance, Z. We know from above that the voltage has the same amplitude and phase in all the components of a parallel RLC circuit.

SAQ 2

a) Explain Power in Electrical Engineering and write their types?

b) Differentiate between Wattless and Wattfull power?

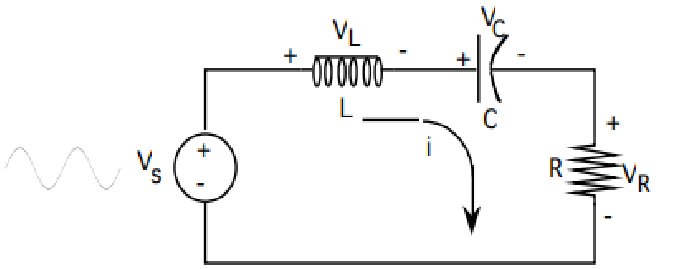
c) What is power factor?

d) Explain with diagram about legging and leading power factor?

e) In a series *RLC*, circuit $R = 30\ \Omega$, $L = 15$ mH, and $C = 51\ \mu$F. If the source voltage and frequency are 12 V and 60 Hz, respectively, what is the current in the circuit?

9.7 SERIES RESONANCE

The basic series-resonant circuit is shown in fig. Of interest here in how the steady state amplitude and the phase angle of the current vary with the frequency of the sinusoidal voltage source. As the frequency of the source changes, the maximum amplitude of the source voltage (Vm) is held constant.



$V_s = V_m\cos(\omega t)$ $\qquad$ $i = I_m\cos(\omega t + \theta)$

Fig 9.17 series-resonant circuit

The frequency at which the reactances of the inductance and the capacitance cancel each other is the resonant frequency (or the unity power factor frequency) of this circuit. This occurs at

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Since i = VR /R, then the current i can be studied by studying the voltage across the resistor. The current i has the expression

$$i = I_m\cos(\omega t + \theta)$$

where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and

$$\theta = -\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

The bandwidth of the series circuit is defined as the range of frequencies in which the amplitude of the current is equal to or greater than its maximum amplitude, as shown in fig. This yields the bandwidth B = R/L

Where

$$\omega_{2,1} = \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \pm \frac{R}{2L}$$

are called the half power frequencies or the 3 dB frequencies, i.e the frequencies at which the value of Im equals the maximum possible value divided by 2 equal to 1.414 .

The quality factor $$Q = \frac{\omega_o}{B} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Then the maximum value of :

1- V_R occurs at $\omega = \omega_o$

2- V_L occurs at $$\frac{\omega_o}{\sqrt{1 - \frac{R^2C}{2L}}}$$

3- V_C occurs at $$\omega_o\sqrt{1 - \frac{R^2C}{2L}}$$

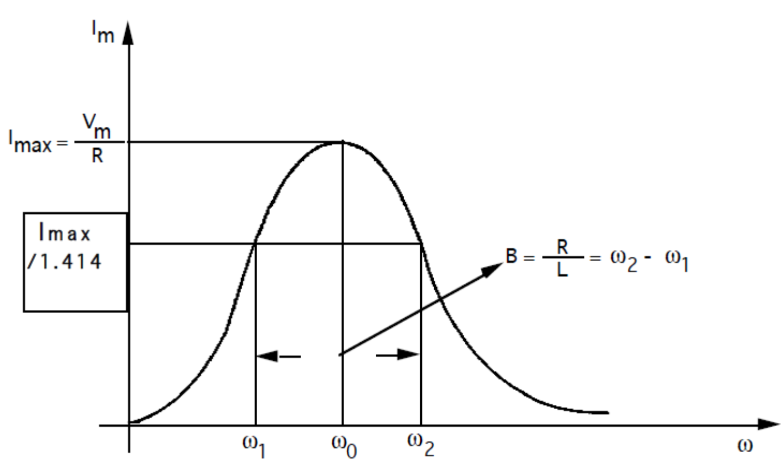
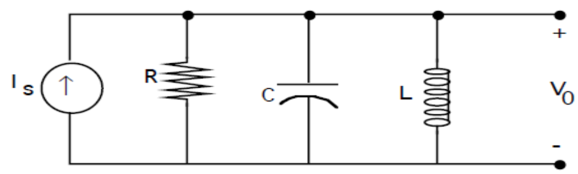


Fig 9.18 Frequency Response of a Series - Resonant Circuit

Parallel Resonance:

The basic parallel-resonant circuit is shown in fig. Amplitude and the phase angle of the output voltage V_0 vary with the frequency of the sinusoidal voltage source in study state also



$I_s = I_m\cos(\omega t)$ $V_o = V_m\cos(\omega t+\theta)$

If $I_s = I_m\cos(\omega t)$, then $V_o = V_m\cos(\omega t+\theta)$ where

$$V_m = \frac{I_m}{\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}}$$

and

$$\theta = -\tan^{-1}\left(R\left(\omega C-\frac{1}{\omega L}\right)\right)$$

The resonant frequency is $\omega_o = \frac{1}{\sqrt{LC}}$

The 3 dB frequencies are: $\omega_{2,1} = \sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}} \pm \frac{1}{2RC}$

The bandwidth $B = \omega_2 - \omega_1 = 1/RC.$

The quality factor $Q = \frac{\omega_o}{B} = R\sqrt{\frac{C}{L}}$

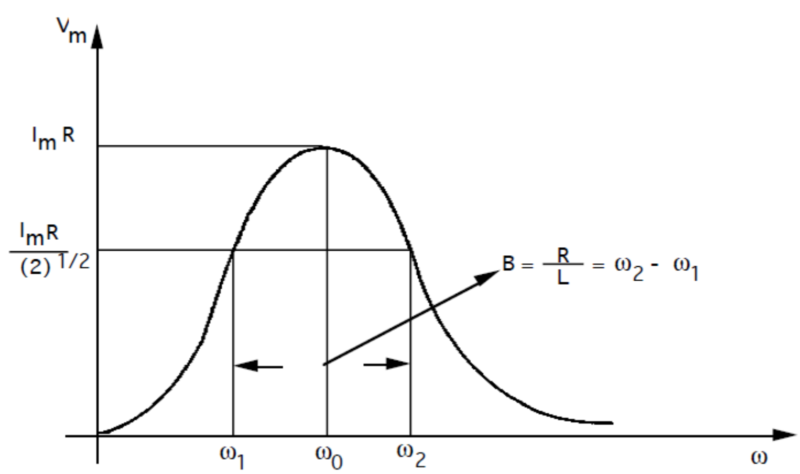


Fig 9.19 Frequency Response of the Parallel - Resonant Circuit

Quality Factor :

The Q, or quality, factor of a resonant circuit is a measure of the "goodness" or quality of a resonant circuit. A higher value for this figure of merit corresponds to a more narrow bandwidth, which is desirable in many applications. More formally, Q is the ratio of power stored to power dissipated in the circuit reactance and resistance.

Sharpness of resonance :

Sharpness of resonance in a series resonant circuit is defined by the Q factor . This can be defined as how quickly the energy of the oscillating system decays.

Sharpness of resonance depend upon two factors -Damping and Amplitude .Thus, we can say that when damping increase the sharpness also increase and when damping decrease sharpness of resonance also decrease.Also, the sharpness increase when amplitude decreases .

Quality factor is defined as ratio of the flowing branch currents to the supply current .Thus, q-factor of circuit = $R/(2\pi * f * L)$

where, R = resistance, L = inductor, f = frequency of resonance circuit = $1/2\pi\sqrt{LC}$,

C = capacitance

q-factor of circuit = $R/(2\pi * f * L)$

$= 2\pi * f * C * R$

$$= 2\pi*(1/2\pi\sqrt{1/CL})*C*R$$

$$= R * (\sqrt{C/L})$$

SAQ 3

a) What is resonance in electrical engineering?

b) Drive the formula of for resonant frequency in series resonance circuit?

c) What is quality factor and derive its expression?

d) Draw resonance curve and show the bandwidth and resonance frequency in the curve?

e) A circuit tuned to a frequency of 1.5 MHz and having an effective capacitance of 150 pF. In this circuit, the current falls to 70.7 % of its resonant value. The frequency deviates from the resonant frequency by 5 kHz. Q factor is?

Examples :

Q1.A sinusoidal voltage supply defined as: $V(t) = 100 \cos(\omega t + 30^o)$ is connected to a pure resistance of 50 Ohms. Determine its impedance and the peak value of the current flowing through the circuit. Draw the corresponding phasor diagram.

Solution

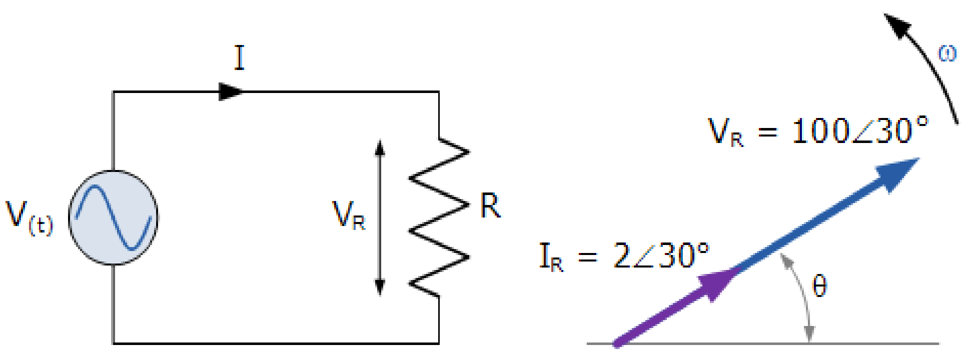
The sinusoidal voltage across the resistance will be the same as for the supply in a purely resistive circuit. Converting this voltage from the time-domain expression into the phasor-domain expression gives us:

$$V_{R(t)} = 100\cos(\omega t + 30^{\circ}) \quad \Rightarrow \quad V_R = 100\angle 30^{\circ} \text{ volts}$$

Applying Ohms Law gives us:

$$I_R = \frac{V_R}{R} = \frac{100\angle 30^{\circ}}{50\Omega} = 2\angle 30^{\circ} \text{ Amps}$$

The corresponding phasor diagram will therefore be:



Q2. An ac generator produces an emf of amplitude 10 V at a frequency f=60Hz. Determine the voltages across and the currents through the circuit elements when the generator is connected to (a) a 100Ω resistor, (b) a 10μF capacitor, and (c) a 15-mH inductor.

Solution

The voltage across the terminals of the source is

$v(t)=V_0\sin\omega t=(10V)\sin 120\pi t$,

where $\omega=2\pi f=120\pi$ rad/s

is the angular frequency. Since *v*(*t*) is also the voltage across each of the elements, we have

$v(t)=v_R(t)=v_C(t)=v_L(t)=(10V)\sin 120\pi t$.

a. When R=100Ω the amplitude of the current through the resistor is

$I0=V_0/R=10V/100\Omega=0.10A$,

so

$i_R(t)=(0.10A)\sin 120\pi t$.

b. From Equation the capacitive reactance is

$X_C=1/\omega C=1/(120\pi rad/s)(10\times10-6F)=265\Omega$,

so the maximum value of the current is

$I0=V_0/X_C=10/265=3.8\times10^{-2}A$

and the instantaneous current is given by

$i_C(t)=(3.8\times10-2A)\sin(120\pi t+\pi 2)$.

c. From Equation the inductive reactance is

$X_L=\omega L=(120\pi rad/s)(15\times10-3H)=5.7\Omega$

The maximum current is therefore

$I0=10/5.7=1.8A$

and the instantaneous current is

$i_L(t)=(1.8A)\sin(120\pi t-\pi/2)$.

Q3. The equation for an alternating current is given by i = 77 sin 314t. Find the peak value, frequency, time period and instantaneous value at t = 2 ms.

Solution

Given,

i = 77 sin 314t ; t = 2 ms = 2×10-3 s

The general equation of an alternating current is $i = I_m \sin \omega t$. On comparsion,

(i) Peak value, I_m = 77 A

(ii) Frequency, $f = \omega/2\pi$ = 314 / 2 ×3.14 = 50 Hz

Time period, T = 1/f = 150 = 0 .02 s

(iii) At t = 2 m s,

Instantaneous value,

$i = 77\sin(314\times2\times10-3\ )$

i = 45.24 A

Q4. Find the impedance of a series RLC circuit if the inductive reactance, capacitive reactance and resistance are 184 Ω, 144 Ω and 30 Ω respectively. Also calculate the phase angle between voltage and current.

Solution

X_L = 184 Ω; X_C = 144 Ω R = 30 Ω

(i) The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{30^2 + (184 - 144)^2}$$
$$= \sqrt{900 + 1600}$$

Impedance, Z = 50 Ω

(ii) Phase angle is

$$\tan\phi = \frac{X_L - X_C}{R}$$

$$= \frac{184 - 144}{30} \quad = 1.33$$

$$\phi = 53.1^\circ$$

Q5. The current in an inductive circuit is given by 0.3 sin (200t – 40°) A. Write the equation for the voltage across it if the inductance is 40 mH.

Solution

Given,

L = 40 × 10-3 H; i = 0.3 sin (200t – 40º)

X_L = ωL = 200 × 40 × 10-3 = 8 Ω

V_m = Im X_L = 0.3 × 8 = 2.4 V

In an inductive circuit, the voltage leads the current by 90º Therefore,

V = Vm sin (ωt +90º)

V = 2 . 4 sin(200t −40 + 90 º)

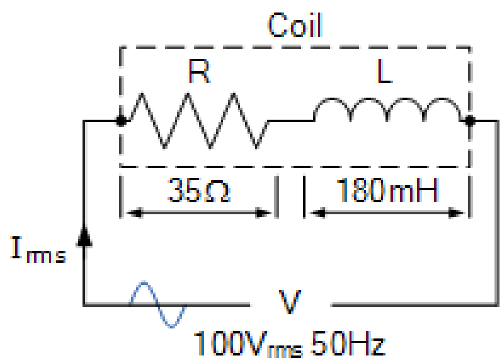
V = 2 . 4 sin(200t +50 º)volt

Q6. A wound coil that has an inductance of 180mH and a resistance of 35Ω is connected to a 100V and 50Hz supply. Calculate: a) the impedance of the coil, b) the current, c) the power factor, and d) the apparent power consumed.

Also draw the resulting power triangle for the above coil.

Solution:

Data given: R = 35Ω, L = 180mH, V = 100V and *f* = 50Hz.



(a) Impedance (Z) of the coil:

$$R = 35\Omega$$

$$X_L = 2\pi f L = 2\pi \times 50 \times 0.18 = 56.6\Omega$$

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{35^2 + 56.6^2} = 66.5\Omega$$

(b) Current (I) consumed by the coil:

$$V = I \times Z$$

$$\therefore I = \frac{V}{Z} = \frac{100}{66.5} = 1.5\ A$$

(c) The power factor and phase angle, Φ:

$$\cos\phi = \frac{R}{Z}, \quad \text{or} \quad \sin\phi = \frac{X_L}{Z}, \quad \text{or} \quad \tan\phi = \frac{X_L}{R}$$

$$\therefore \cos\phi = \frac{R}{Z} = \frac{35}{66.5} = 0.5263$$

$$\cos^{-1}(0.5263) = 58.2^{o}\ (\text{lagging})$$

(d) Apparent power (S) consumed by the coil:

$$P = V \times I\cos\phi = 100 \times 1.5 \times \cos\left(58.2^o\right) = 79W$$

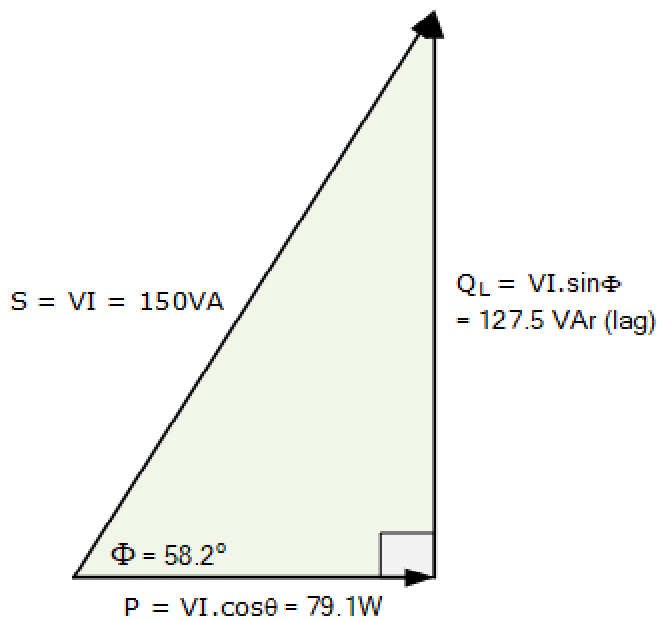
$$Q = V \times I\sin\phi = 100 \times 1.5 \times \sin\left(58.2^o\right) = 127.5VA$$

$$S = V \times I = 100 \times 1.5 = 150VA$$

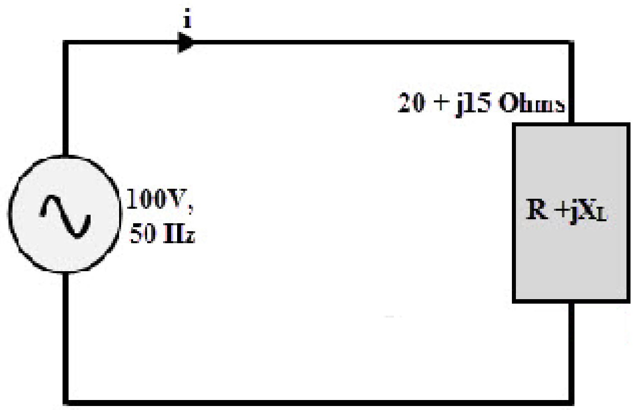
$$\text{or} \quad S^2 = P^2 + Q^2$$

$$\therefore S = \sqrt{P^2 + Q^2} = \sqrt{79^2 + 127.5^2} = 150VA$$

(e) Power triangle for the coil:



Q8. If an AC power supply of 100V, 50Hz is connected across a load of impedance, 20 + j15 Ohms. Then calculate the current flowing through the circuit, active power, apparent power, reactive power and power factor.



Solution

Given that, Z = R + jXL = 20 + j 15 Ω

Converting the impedance to polar form, we get

Z = 25 ∠36.87 Ω

Current flowing through the circuit,

I = V/Z = $100\angle 0^0$ /25 ∠36.87

I = 4 ∠–36.87

Active power, P = I^2R = 42 × 20 = 320 watts

Or P = VI cos ϕ = 100 × 4 × cos (36.87) = 320.04 ≈ 320 W

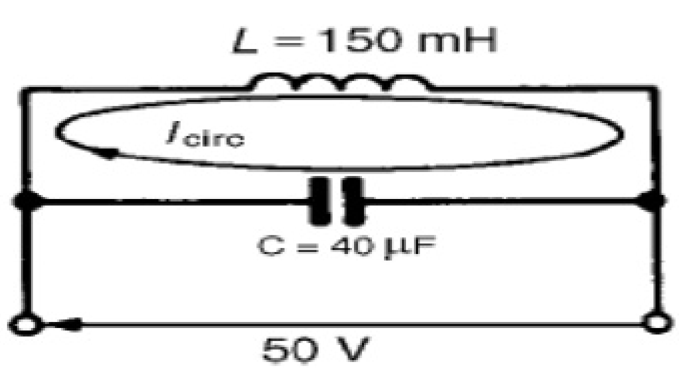
Apparent power, S = VI = 100 × 4 = 400 VA

Reactive power, Q = √ $(S^2 – P^2)$

= √ $(400^2 – 320^2)$ = 240 VAr

Power factor, PF = cos ϕ = cos 36.87 = 0.80 lagging.

Q9. A pure inductance of 150 mH is connected in parallel with a 40 μF capacitor across a 50 V, variable frequency supply. Determine (a) the resonant frequency of the circuit and (b) the current circulating in the capacitor and inductance at resonance.



The circuit diagram is shown in Figure

(a) Parallel resonant-R/L frequency,

However, resistance R = 0. Hence,

$f = 1/2\pi\sqrt{1/LC}$

$= 1/2\pi\sqrt{(1/15 \times 10^{-3}(40 \times 10^{-6})}$

$= 1/2\pi\sqrt{15 \times 4 \times 10^{-7}}$

$= \frac{1}{2\pi.10^{-3}\sqrt{6}} = \frac{10^3}{2\pi\sqrt{6}}$

= 64.97 Hz

(b) Current circulating in L and C at resonance, $= V/X_C$

$= V /1/2\pi fC$

$= 2\pi\ f\ C.V$

Hence $= 2\pi \times 64.97 \times 40 \times 10^{-6} \times 50 = 0.816$ A

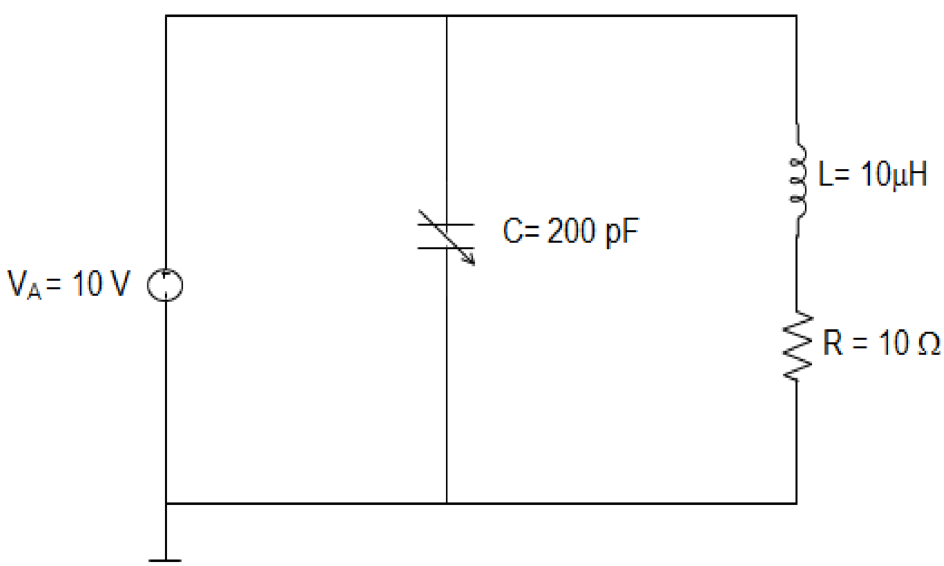
Alternatively, ICIRC = V/XL

$= \frac{V}{2\pi f L}$

$= \frac{50}{2\pi \times 64.97 \times 0.15}$

= **0.817 A**

Q10. For the resonant circuit given below, find the value of the quality factor of the circuit?



Solution

Given $L = 10 \times 10^{-6} H, C = 200 \times 10^{-12} F, R = 10\Omega$

Explanation: f = 1/2π√LC
= 16.28√ (10×10^{-6}) (200×10^{-12})
= 16.282×10^{-15}
= 1888×10^{-9} = 1.13 MHz
Inductive Reactance, $X_L = 2\pi fL = (6.28)(1.13 \times 10^6)(10 \times 10^{-6})$
= 70.96 Ω

$$\therefore \quad Q = \frac{XL}{R} = \frac{70.96}{10} = 7.096 \cong 7.1$$

9.8 SUMMARY

1. The letter " j " known commonly in electrical engineering as the **j-operator**
2. Phasor notation can be used to represent the phase relationship between two sinusoidal waveforms.
3. "Reactance is a form of opposition that electronic components exhibit to the passage of AC (alternating current) because of capacitance or inductance"
4. When AC (alternating current) passes through a component that contains reactance, energy might be stored and released in the form of a magnetic field which is known as inductive reactance. It is denoted by $+jX_L$
5. When AC (alternating current) passes through a component that contains reactance, energy might be stored and released in the form of an electric field which is known as capacitive reactance.It is denoted by $-jX_C$
6. "Impedance is the total resistance/opposition offered by the circuit elements to the flow of alternating or direct current
7. "Susceptance is an expression of the readiness with which an electronic component, circuit, or system releases stored energy as the current and voltage fluctuate"
8. The instantaneous value is "the value of an alternating quantity (it may ac voltage or ac current or ac power) at a particular instant of time in the cycle".
9. Average value is the ratio of the sum of all considered instantaneous values to the number of instantaneous values in one alternation period.
10. The Root Mean Square (RMS) value is "the square root of the sum of squares of means of an alternating quantity".

11. The maximum value attained by an alternating quantity during one cycle is called its **Peak value.**

12. Form Factor is the ratio between the average value and the RMS value

13. Phase difference is the difference, between two waves is having the same frequency and referenced to the same point in time

14. The power which is actually consumed or utilised in an AC Circuit is called **True power** or **Active power** or **Real power**. It is measured in kilowatt (kW) or MW

15. The power which flows back and forth that means it moves in both the directions in the circuit or reacts upon itself, is called **Reactive Power**. The reactive power is measured in kilo volt-ampere reactive (kVAR) or MVAR

16. The product of root mean square (RMS) value of voltage and current is known as **Apparent Power**. This power is measured in kVA or MVA.

17. Power factor is the ratio of true or actual power to the apparent power in the ac systems.

18. The frequency at which the reactances of the inductance and the capacitance cancel each other is the resonant frequency (or the unity power factor frequency) of this circuit.

19. Quality factor is the ratio of power stored to power dissipated in the circuit reactance and resistance

9.9 TERMINAL QUESTION

1. Derive the expression for RMS value and average value of alternating voltage and current?

2. Define the active, reactive and apparent power in ac circuit and also draw the power triangle?

3. Draw the figure diagram of RC,RL and RLC series ac circuit?

4. Explain series and parallel resonance and drive the expression for resonant frequency for both type of resonance?

5. Write sort notes on

   i) Phasor notation

   ii) Reactance, suspectance

   iii) From Factor

   iv) Power Factor

   v) Quality factor

6. A circuit tuned to a frequency of 1.5 MHz and having an effective capacitance of 150 pF. In this circuit, the current falls to 70.7 % of its resonant value. The frequency deviates from the resonant frequency by 5 kHz. Effective resistance of the circuit is?

7. A sinusoidal voltage $v = 50\sin\omega t$ is applied to a series RL circuit. The current in the circuit is given by $I = 25\sin(\omega t - 53^0)$. Determine the apparent power (VA)

8. Calculate the quality factor of the coil for a series circuit having $R = 10\Omega$, $L = 0.1H$, $C = 10\mu F$?

UNIT-10 NETWORK ANALYSIS (FOR BOTH AC AND DC)

Structure

10.1 INTRODUCTION

Generally speaking, *network analysis* is any structured technique used to mathematically analyze a circuit (a "network" of interconnected components). Quite often the technician or engineer will encounter circuits containing multiple sources of power or component configurations that defy simplification by series/parallel analysis techniques. In those cases, he or she will be forced to use other means. This chapter presents a few techniques useful in analyzing such complex circuits. Network theory is the study of solving the problems of electric circuits or electric networks. In this introductory chapter, let us first discuss the basic terminology of electric circuits and the types of network elements The types of active circuit elements that are most important to us are those that supply electrical energy to the circuits or network connected to them. These are called "electrical sources" with the two types of electrical sources being the voltage source and the current source. The current source is usually less common in circuits than the voltage source, but both are used and can be regarded as complements of each other.

one of the interesting characteristic of an electrical source, is that they are also capable of converting non-electrical energy into electrical energy and vice versa. For example, a battery converts chemical energy into electrical energy, while an electrical machine such as a DC generator or an AC alternator converts mechanical energy into electrical energy.

10.2 OBJECTIVES

After studying this unit you should be able to

- Study and identify Circuit elements and various networks circuits.
- Explain and identify T and π networks and their equivalence.
- Study and identify Kirchoff's current and voltage laws. Mesh and nodal analysis of electrical circuits. (Matrics and determinant methods).
- Explain and identify Concept of constant current and constant voltage source. Thevenin and Norton's theorem.
- Study and identify Maximum power transfer theorem, superposition theorem, reciprocity theorem.

10.3 CIRCUIT ELEMENTS

A circuit element is an idealised mathematical model of a tw o-terminal electrical device that is completely characterised by its voltage-current relationship. Although ideal circuit elements are not "off-the-shelf" circuit components, their importance lies in the fact that they can be interconnected to approximate actual circuits that are composed of nonideal elements and assorted electrical components – thus allowing for the analysis of such circuits.

Circuit elements can be categorized as either active or passive.

Active Circuit Elements :

Active circuit elements can deliver a non-zero average power indefinitely. There are four types of active circuit element, and all of them are termed an ideal source. They are:

- Independent voltage source
- Independent current source
- Dependent voltage source
- Dependent current source

Passive Circuit Elements :

Passive circuit elements cannot deliver a non-zero average power indefinitely. Some passive elements are capable of storing energy, and therefore delivering power back into a circuit at some later time, but they cannot do so indefinitely.

There are three types of passive circuit element. They are:

- Resistor
- Inductor
- Capacitor

Network circuits :

Types of Circuits :

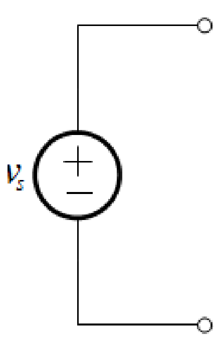
The interconnection of two or more circuit elements forms an electrical network. If the network contains at least one closed path, it is also an electrical circuit. A network that contains at least one active element, i.e. an independent or dependent source, is an active network. A network that does not contain any active elements is a passive network.

Independent Sources:

Independent sources are ideal circuit elements that possess a voltage or current value that is independent of the behaviour of the circuits to which they belong.

The Independent Voltage Source:

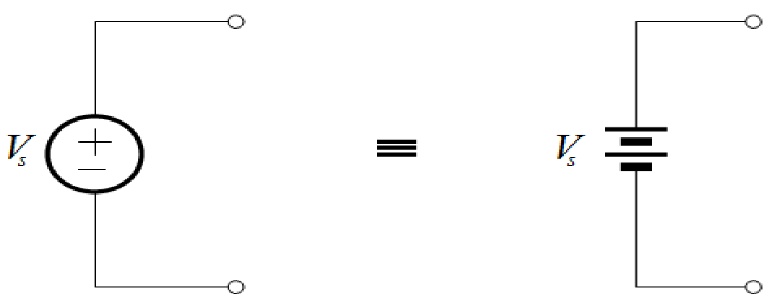
An independent voltage source is characterised by a terminal voltage which is completely independent of the current through it. The representation of an independent voltage source is shown below:



Independent Voltage Source

If the value of the voltage source is constant, that is, does not change with time, then we can also represent it as an ideal battery :

Independent Voltage Source

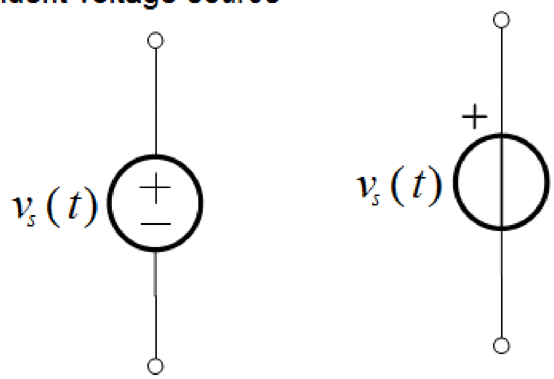


Although a "real" battery is not ideal, there are many circumstances under which an ideal battery is a very good approximation.

In general, however, the voltage produced by an ideal voltage source will be a function of time. In this case we represent the voltage symbolically as v(t) .

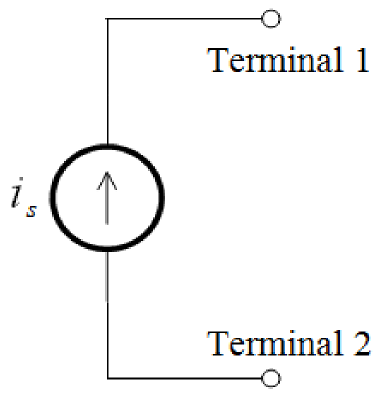
Since the voltage produced by a source is in general a function of time, then the most general representation of an ideal voltage source is as shown below:

ideal independent voltage source



The Independent Current Source:

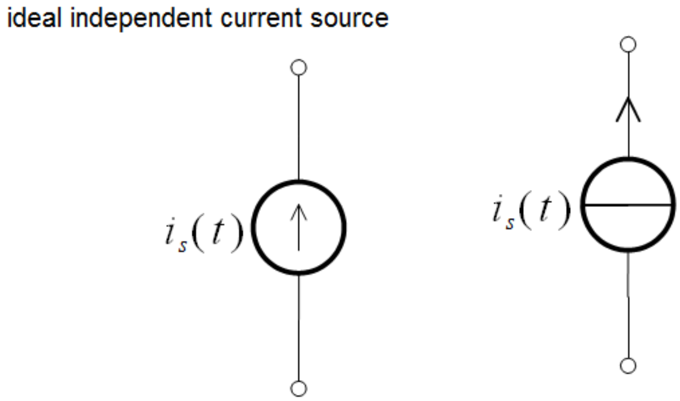
An independent current source establishes a current which is independent of the voltage across it. The representation of an independent current source is shown below:



Independent current source

In other words, an ideal current source is a device that, when connected to anything, will always push current (is) out of terminal 1 and pull is into terminal 2

Since the current produced by a source is in general a function of time, then the most general representation of an ideal current source is as shown below:



Types of Electrical Circuits :

DC Circuits:

In DC Circuits, the excitation applied is a constant source. Based on the type of connection of active and passive components with the source, a circuit can be classified into Series and Parallel circuits.

Series Circuits:

When several passive elements are connected in series with an energy source, such a circuit is known as a series circuit. For a series circuit, same amount of current flows through each element and voltage is divided. In series circuit, as the elements are connected in a line,if there is faulty element among them ,complete circuit acts as open circuit.

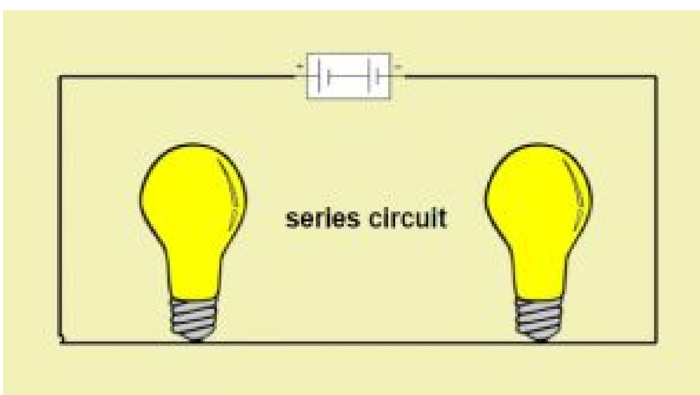


Fig 10.1 Series circuit

- For a resistor connected in DC circuits, the voltage across its terminals is directly proportional to the current passing through it, thus maintaining a linear relationship between the voltage and current. For resistors connected in series, the total resistance is equal to the sum of all resistance values.
- For capacitors connected in series, the total capacitance is equal to the sum of reciprocals of all capacitance values.

- For inductors connected in series, total inductance is equal to the sum of all inductance values.

Parallel Circuits:

In a parallel circuit, one terminal of all the elements is connected to the one terminal of the source and the other terminal of all elements is connected to the other terminal of the source.

In parallel circuits, the voltage remains the same in the parallel elements while the current changes. If there is any faulty element among parallel elements there is no effect on the circuit.

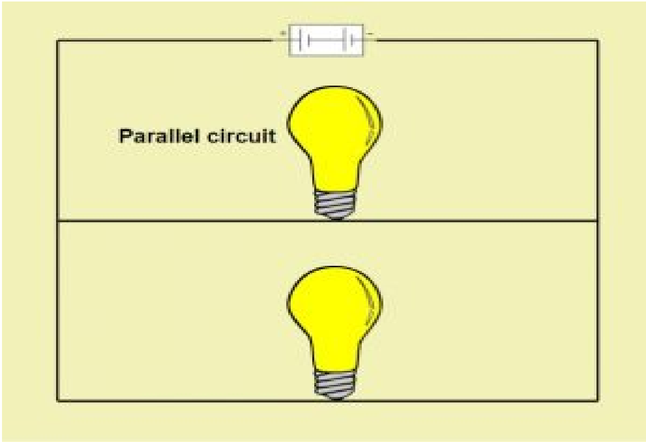


Fig 10.2 Parallel circuit

- For resistors connected in parallel, the total resistance is equal to the sum of reciprocals of all resistance values.
- For capacitors connected in series, the total capacitance is equal to the sum of all capacitance values.
- For inductors connected in series, total inductance is equal to the sum of all reciprocals of inductance values.

AC circuits :

Ac circuits are those circuits, Whose excitation element is an AC source. Unlike DC source which is constant AC source has variable current and voltage at regular intervals of time. Generally, for high power applications, AC circuits are used.

Simple AC Circuit using resistance :

For alternating current passing through the resistor, the ratio of current and voltage depends upon the phase and frequency of the supply. The applied voltage will change constantly with time and Ohm's law can be used to calculate current passing through the resistor at any instant of time.

In other words, if at time t seconds, the value of voltage is v volts, current will be:

i = v/R

where the value of R is always constant.

Above equation shows that polarity of current depends upon that of the voltage. Also, both current and voltage reach their maximum and zero points at the same time. Thus, for a resistor, voltage is in phase with the applied current.

Consider the below circuit diagram

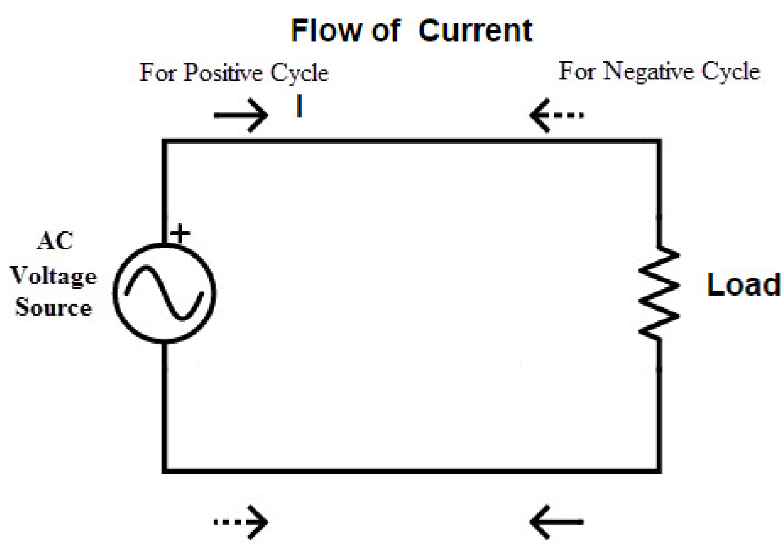


Fig 10.3 Simple AC Circuit

When the switch is closed, current passes through the resistor and is given by the below equation

i=Im cos(ωt+Φ)

Voltage,**V=IR=RIm cos(ωt+Φ)**

For a resistor, both voltage and current values will rise and fall at the same time. Hence, the phase difference between voltage and current is zero.

AC Circuit using pure inductance:

A coil of thin wire wrapped on a cylindrical core is known as an Inductor. The core can be an air core (hollow laminated) or an iron core. As alternating current flows through the inductor, the magnetic field also changes. This change in magnetic field results in an induced voltage across the inductor. As per Lenz law, the induced voltage is such that it opposes the flow of current through it.

During the first half cycle of the source voltage, the inductor stores energy in form of magnetic field and in the next half, it releases energy.

The induced EMF is given as below

e=Ldi/dt

Here, L is the self-inductance.

Now, Input AC voltage applied is given as

v(t)=Vm Sinωt

Current through the inductor is:

I(t)=Im Sinωt

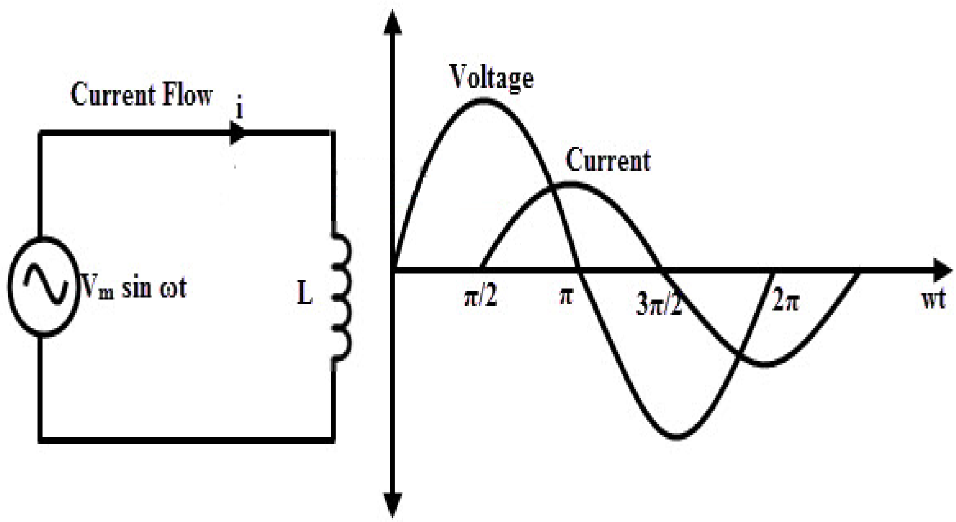


Fig 10.4 AC Circuit using pure inductance

So, the voltage across the inductor would be

e=L di/dt= ωLIm cos ωt

= ωLIm sin(ωt+90)

Thus, for an inductor, voltage leads the current by 90 degrees.

Now, resistance by an inductor is termed as Reactance and given by

Thus, impedance or resistance is proportional to rate of change of current for an inductor.

AC Circuit with a capacitor:

For a constant DC supply, the capacitor plates charge up to the applied voltage, stores this charge temporarily and then starts discharging. Once a capacitor is fully charged, it blocks the flow of current as the plates get saturated.

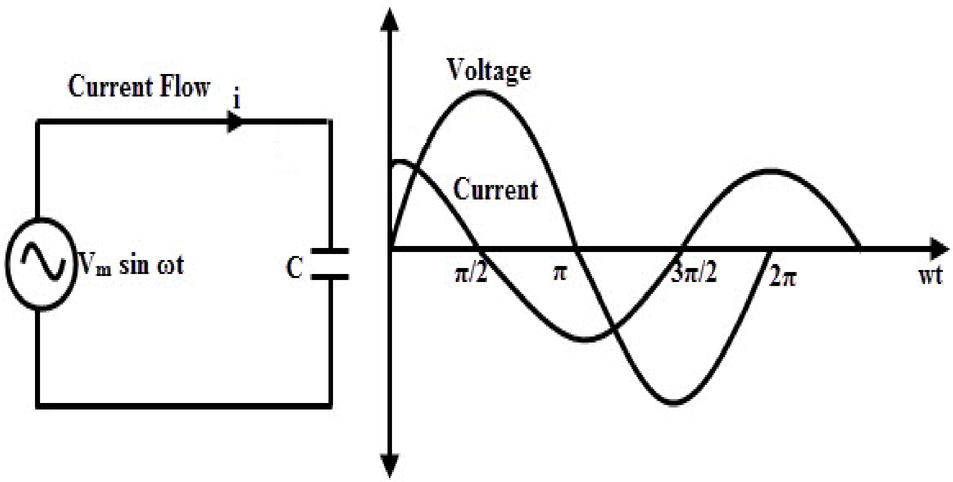


Fig 10.5 AC Circuit with a capacitor

When AC supply voltage is applied to a capacitor, the rate of charging and discharging depends upon the supply frequency. Voltage across the capacitor lags the current flowing through it by 90 degrees.

Current through the capacitor is given as

e = Ldi/dt

The capacitive reactance is given as:

e = Ld/idt

Thus, impedance or reactance to AC supply is inversely proportional to the frequency of supply.

T-connected and Equivalent Star Network :

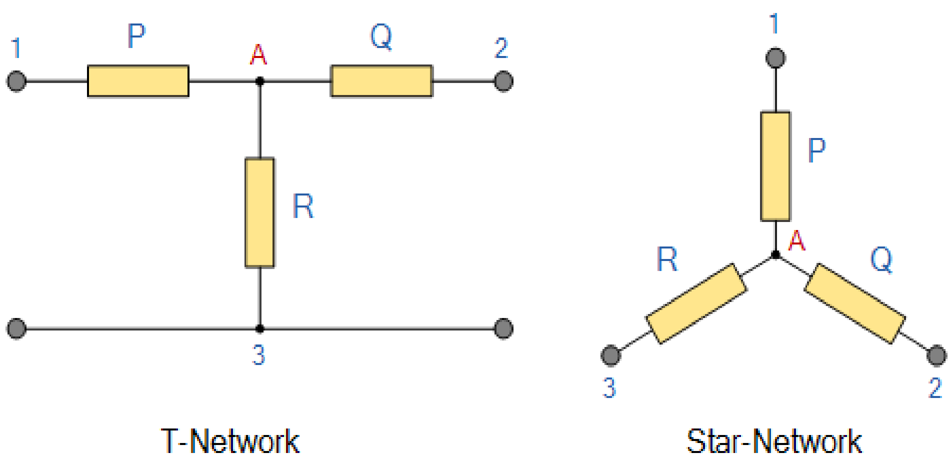


Fig 10.6

As we have already seen, we can redraw the T resistor network above to produce an electrically equivalent **Star** or Y type network. But we can also convert a Pi or π type resistor network into an electrically equivalent **Delta** or Δ type network as shown below.

Pi-connected and Equivalent Delta Network:

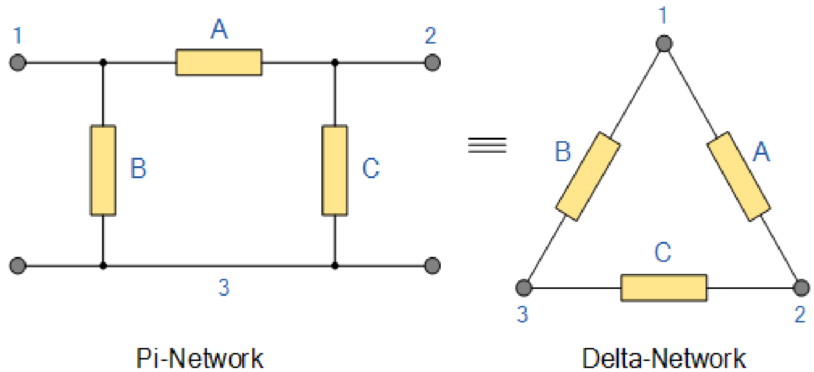


Fig 10.7

Having now defined exactly what is a Star and Delta connected network it is possible to transform the Y into an equivalent Δ circuit and also to convert a Δ into an equivalent Y circuit using a the transformation process.

This process allows us to produce a mathematical relationship between the various resistors giving us a **Star Delta Transformation** as well as a **Delta Star Transformation.**

These circuit transformations allow us to change the three connected resistances (or impedances) by their equivalents measured between the terminals 1-2, 1-3 or 2-3 for either a star or delta connected circuit. However, the resulting networks are only equivalent for voltages and currents external to the star or delta networks, as internally the voltages and currents are different but each network will consume the same amount of power and have the same power factor to each other.

Delta to Star Transformation:

To convert a delta network to an equivalent star network we need to derive a transformation formula for equating the various resistors to each other between the various terminals. Consider the circuit below.

Delta to Star Network:

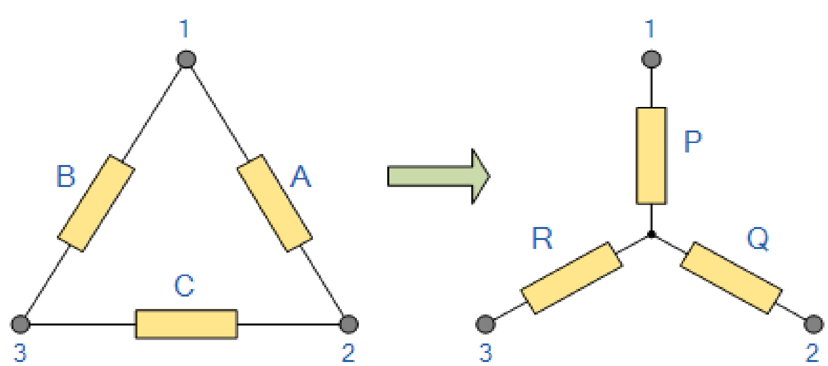


Fig 10.8 Delta to Star Network

Compare the resistances between terminals 1 and 2.

$$P + Q = A \text{ in parallel with } (B + C)$$

$$P + Q = \frac{A(B + C)}{A + B + C} \quad \ldots EQ1$$

Resistance between the terminals 2 and 3.

$$Q + R = C \text{ in parallel with } (A + B)$$

$$Q + R = \frac{C(A + B)}{A + B + C} \quad \ldots EQ2$$

Resistance between the terminals 1 and 3.

$$P + R = B \text{ in parallel with } (A + C)$$

$$P + R = \frac{B(A + C)}{A + B + C} \quad \ldots EQ3$$

This now gives us three equations and taking equation 3 from equation 2 gives:

$$EQ3 - EQ2 = (P + R) - (Q + R)$$

$$P + R = \frac{B(A + C)}{A + B + C} - Q + R = \frac{C(A + B)}{A + B + C}$$

$$\therefore P - Q = \frac{BA + CB}{A + B + C} - \frac{CA + CB}{A + B + C}$$

$$\therefore P - Q = \frac{BA - CA}{A + B + C}$$

Then, re-writing Equation 1 will give us:

$$P + Q = \frac{AB + AC}{A + B + C}$$

Adding together equation 1 and the result above of equation 3 minus equation 2 gives:

$$(P - Q) + (P + Q)$$

$$= \frac{BA - CA}{A + B + C} + \frac{AB + AC}{A + B + C}$$

$$= 2P = \frac{2AB}{A + B + C}$$

From which gives us the final equation for resistor P as:

$$P = \frac{AB}{A + B + C}$$

Then to summarize a little about the above maths, we can now say that resistor P in a Star network can be found as Equation 1 plus (Equation 3 minus Equation 2) or Eq1 + (Eq3 – Eq2).

Similarly, to find resistor Q in a star network, is equation 2 plus the result of equation 1 minus equation 3 or Eq2 + (Eq1 – Eq3) and this gives us the transformation of Q as:

$$Q = \frac{AC}{A + B + C}$$

and again, to find resistor R in a Star network, is equation 3 plus the result of equation 2 minus equation 1 or Eq3 + (Eq2 – Eq1) and this gives us the transformation of R as:

$$R = \frac{BC}{A + B + C}$$

When converting a delta network into a star network the denominators of all of the transformation formulas are the same: A + B + C, and which is the sum of ALL the delta resistances. Then to convert any delta connected network to an equivalent star network we can summarized the above transformation equations as

Delta to Star Transformations Equations:

$$P = \frac{AB}{A + B + C}$$

$$Q = \frac{AC}{A + B + C}$$

$$R = \frac{BC}{A + B + C}$$

If the three resistors in the delta network are all equal in value then the resultant resistors in the equivalent star network will be equal to one third the value of the delta resistors. This gives each resistive branch in the star network a value of: $R_{STAR} = 1/3 * R_{DELTA}$ which is the same as saying:

Star to Delta Transformation:

Star Delta transformation is simply the reverse of above. We have seen that when converting from a delta network to an equivalent star network that the resistor connected to one terminal is the product of the two delta resistances connected to the same terminal, for example resistor P is the product of resistors A and B connected to terminal 1.

By rewriting the previous formulas a little we can also find the transformation formulas for converting a resistive star network to an equivalent delta network giving us a way of producing a star delta transformation as shown below.

Star to Delta Network :

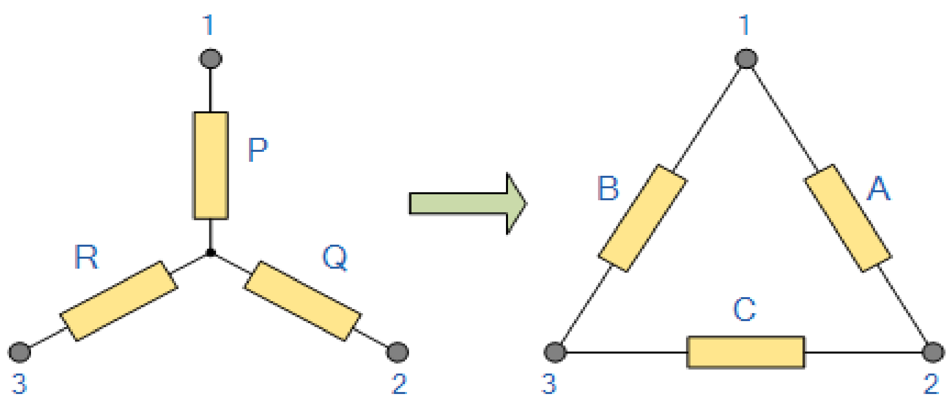


Fig 10.9 Star to Delta Transformation

The value of the resistor on any one side of the delta, Δ network is the sum of all the two-product combinations of resistors in the star network divide by the star resistor located “directly opposite” the delta resistor being found. For example, resistor A is given as:

$$A = \frac{PQ + QR + RP}{R}$$

with respect to terminal 3 and resistor B is given as:

$$B = \frac{PQ + QR + RP}{Q}$$

with respect to terminal 2 with resistor C given as:

$$C = \frac{PQ + QR + RP}{P}$$

with respect to terminal 1.

By dividing out each equation by the value of the denominator we end up with three separate transformation formulas that can be used to convert any Delta resistive network into an equivalent star network as given below.

Star to Delta Transformation Equations :

$$A = \frac{PQ}{R} + Q + P$$

$$B = \frac{RP}{Q} + P + R$$

$$C = \frac{QR}{P} + Q + R$$

One final point about converting a star resistive network to an equivalent delta network. If all the resistors in the star network are all equal in value then the resultant resistors in the equivalent delta network will be three times the value of the star resistors and equal, giving:

$R_{DELTA} = 3*R_{STAR}$

SAQ 1

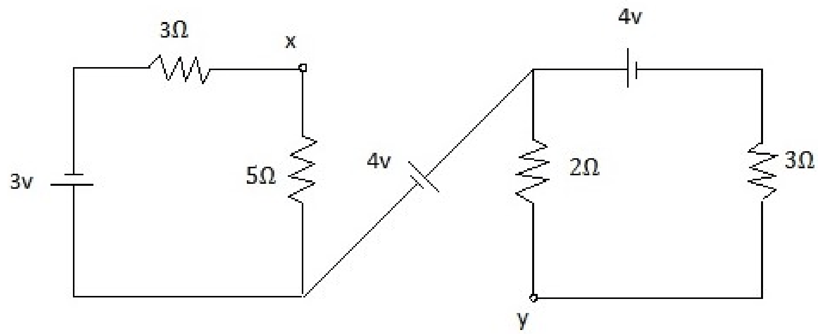
a) Explain the various circuit element which are used in Electrical Circuit?

b) Explain and draw the symbol of various sources of voltage and current?

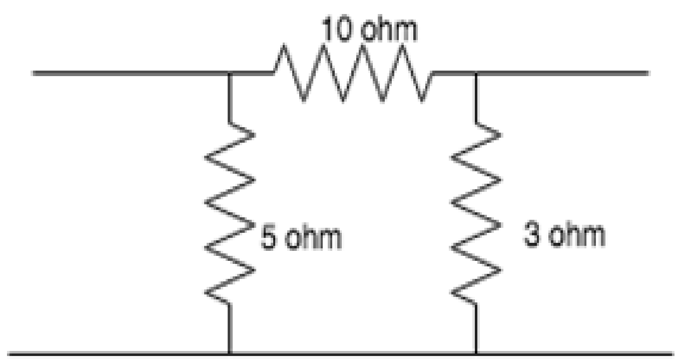
c) Campare the dependent and independent sources?

d) Define the constant current source and constant voltage source?

e) Calculate potential difference between x and y



f) The value of the 3 resistances when connected in star connection is?



10.5 KIRCHHOFF'S CURRENT LAW

Kirchhoff's Current Law states that" the algebraic sum of all the currents at any node point or a junction of a circuit is zero".

$\Sigma I = 0$

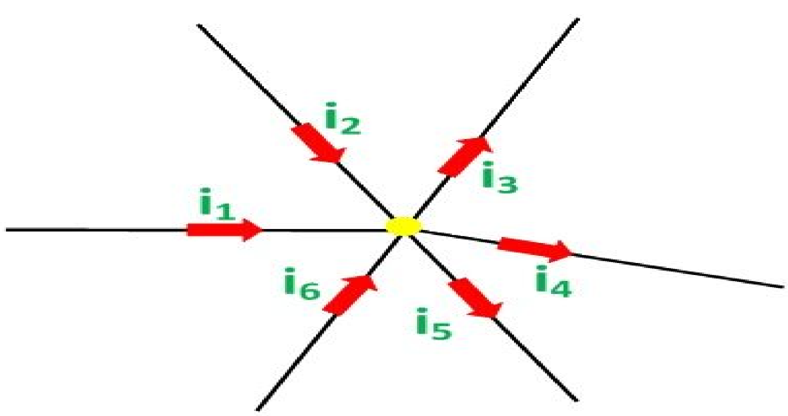


Fig 10.10. Kirchhoff's Current Law

Considering the above figure as per the Kirchhoff's Current Law:

$i_1 + i_2 - i_3 - i_4 - i_5 + i_6 = 0$ ……… (1)

The direction of incoming currents to a node is taken as positive while the outgoing currents are taken as negative. The reverse of this can also be taken, i.e. incoming current as negative or outgoing as positive. It depends upon your choice.

The equation (1) can also be written as:

$i_1 + i_2 + i_6 = i_3 + i_4 + i_5$

Sum of incoming currents = Sum of outgoing currents

According to the **Kirchhoff's Current Law**, The algebraic sum of the currents entering a node must be equal to the algebraic sum of the currents leaving the node in an electrical network.
Kirchhoff's Voltage Law:

Kirchhoff's Voltage Law states that the algebraic sum of the voltages (or voltage drops) in any closed path of a network that is transverse in a single direction is zero. In other words, in a closed circuit, the algebraic sum of all the EMFs and the algebraic sum of all the voltage drops (product of current (I) and resistance (R)) is zero.

$\Sigma E + \Sigma V = 0$

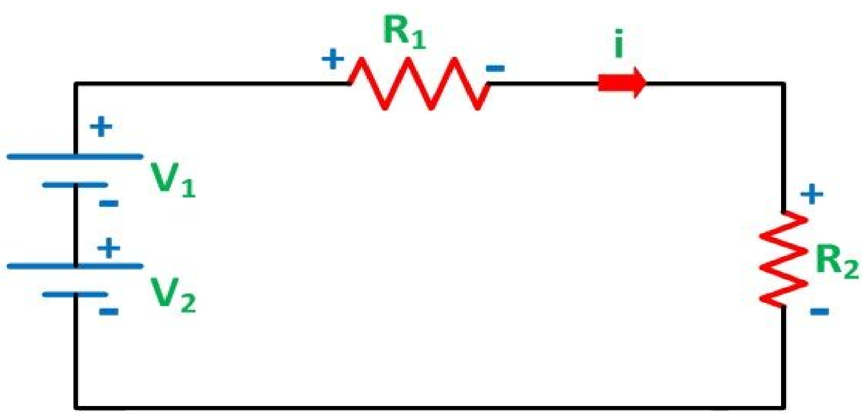


Fig 10.11 Kirchhoff's Voltage Law

The above figure shows closed-circuit also termed as a mesh. As per the Kirchhoff's Voltage Law:

$$-V_1 + (-V_2) + iR_1 + iR_2 = 0$$

Here, the assumed current I causes a positive voltage drop when flowing from the positive to negative potential while negative potential drop when the current flowing from negative to the positive potential.

$$i(R_1 + R_2) = V_1 + V_2 \quad \text{or}$$

$$i = \frac{V_1 + V_2}{(R_1 + R_2)}$$

Considering the other figure shown below and assuming the direction of the current i

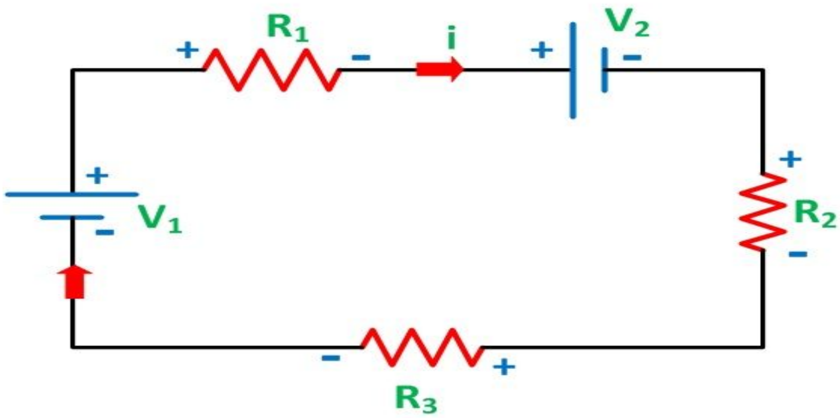


Fig 10.12 Kirchhoff's Voltage Law

Therefore,

$$-V_1 + iR_1 + V_2 + iR_2 + iR_3 = 0$$

$$i(R_1 + R_2 + R_3) = V_1 - V_2 \quad \text{or}$$

$$i = \frac{V_1 - V_2}{R_1 + R_2 + R_3}$$

In figure 10.11 the current in both the source V_1 and V_2 flows from negative to positive polarity while in figure 10.12 the current in the source V_1 is negative to positive but for V_2 is positive to negative polarity.

For the dependent sources in the circuit, KVL can also be applied. In case of the calculation of the power of any source, when the current enters the source, the power is absorbed by the sources while the source delivers the power if the current is coming out of the source.

It is important to know some of the terms used in the circuit while applying KCL and KVL like node, Junction, branch, loop, mesh. They are explained with the help of a circuit shown below:

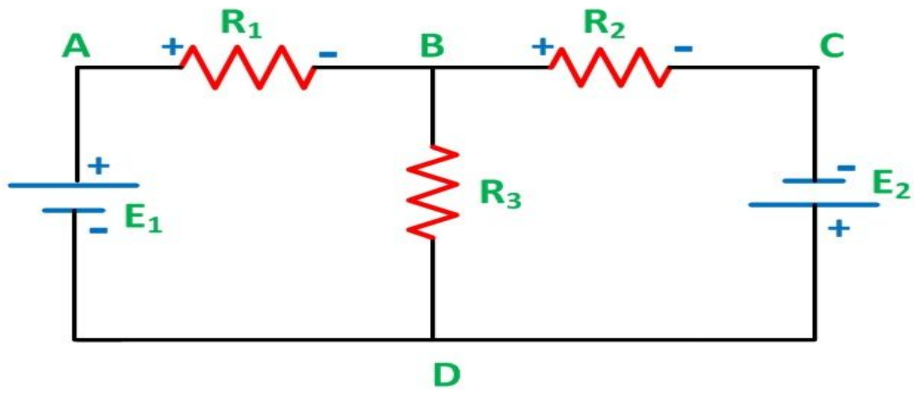


Fig 10.13 Kirchhoff's Voltage Law

Node :

A node is a point in the network or circuit where two or more circuit elements are joined. For example, in the above circuit diagram, A and B is the node points.

Junction :

A junction is a point in the network where three or more circuit elements are joined. It is a point where the current is divided. In the above circuit, B and D are the junctions.

Branch:

The part of a network, which lies between the two junction points is called a Branch. In the above circuit DAB, BCD and BD are the branches of the circuit.

Loop:

A closed path of a network is called a loop. ABDA, BCDB are loops in the above circuit diagram shown.

Mesh :

The most elementary form of a loop which cannot be further divided is called a mesh.

Nodal Voltage Analysis Method :

The **Nodal Voltage Analysis** is a method to solve the electrical network. It is used where it is essential to compute all branch currents.The nodal voltage analysis method determines the voltage and current by using the nodes of the circuit.

A node is a terminal or connection of more than two elements. The nodal voltage analysis is commonly used for networks having many parallel circuits with a common terminal ground.

This method requires less number of the equation for solving the circuit.

In Nodal Voltage Analysis, Kirchhoff's Current Law (KCL) is used, which states that the algebraic sum of all incoming currents at a node must be equal to the algebraic sum of all outgoing currents at that node.

It is the method of finding the potential difference between the elements or branches in an electric circuit. This method defines the voltage at each node of the circuit. This method has two types of nodes. These are the non-reference node and the reference node.

The non-reference nodes have a fixed voltage, and the reference node is the reference points for all other nodes.

In the nodal method, the number of independent node pair equations needed is one less than the number of junctions in the network. That is if n denotes the number of independent node equations and j is the number of junctions.

n = j – 1

In writing the current expression, the assumptions are made that the node potentials are always higher than the other voltages appearing in the equations.

Let us understand the Nodal Voltage Analysis Method with the help of an example shown below:

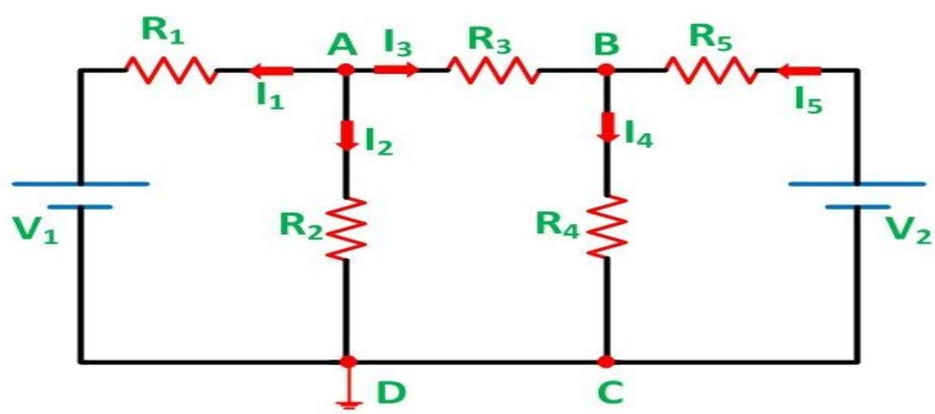


Fig 10.14 Circuit for nodal analysis

Steps for Solving Network by Nodal Voltage Analysis Method:

Considering the above circuit diagram, the following steps are explained below

Step 1 – Identify various nodes in the given circuit and mark them in the given circuit, we have marked the nodes as A and B.

Step 2 – Select one of the nodes as the reference or zero potential nodes at which a maximum number of elements are connected, is taken as reference. In the above figure, node D is taken as the reference node. Let the voltages at nodes A and B be V_A and V_B respectively.

Step 3 – Now apply KCL at the different nodes.

Applying KCL at node A, we have

$$I_1 + I_2 + I_3 = 0$$

Where,

$$\frac{(V_A - V_1)}{R_1} + \frac{V_A}{R_2} + \frac{(V_A - V_B)}{R_3} = 0$$

$$V_A\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right] - \frac{V_B}{R_3} = \frac{V_1}{R_1}$$

Applying KCL at the node B, we have

$$-I_3 + I_4 - I_5 = 0$$

$$\frac{-(V_A - V_B)}{R_3} + \frac{V_B}{R_4} - \frac{(V_B - V_2)}{R_5} = 0 \quad \text{or}$$

$$V_B\left[\frac{1}{R_3} + \frac{1}{R_4} - \frac{1}{R_5}\right] - \frac{V_A}{R_3} = -\frac{V_2}{R_5}$$

After simulating circuits for some time, I began to ask myself - how does this SPICE program work? What mathematical tricks does the code execute to simulate complex electrical circuits described by non-linear differential equations? After some searching and digging, some answers were uncovered. At the core of the SPICE engine is a basic technique called Nodal Analysis. It calculates the voltage at any node given all resistances (conductances) and current sources of the circuit. Whether the program is performing DC, AC, or Transient Analysis, SPICE ultimately casts its components (linear, non-linear and energy-storage elements) into a form where the innermost calculation is Nodal Analysis.

Kirchoff discovered this: the total current entering a node equals the total current leaving a node! And, these currents can be described by an equation of voltages and conductances. If you have more than one node, then you get more than one equation describing the same system (simultaneous equations). The trick now is finding the voltage at each node that satisfies all of the equations simultaneously.

Mesh Current Analysis Method:

Mesh Current Analysis Method is used to analyze and solve the electrical network having various sources or the circuit consisting of several meshes or loop with a voltage or current sources. It is also known as the **Loop Current Method.**

In the Mesh Current method, a distinct current is assumed in the loop and the polarities of drops in each element in the loop are determined by the assumed direction of loop current for that loop.

The unknown in mesh current analysis is the current in different meshes, and the law which is applicable to solve the circuit by the mesh current method is known as **Kirchhoff's Voltage Law** (KVL) which states that .

In any closed circuit, the net voltage applied is equal to the sum of the product of current and resistance or in another word in any closed circuit, the sum of the voltage rise is equal to the sum of voltage drop, in the direction of current flow.

Contents:

- Steps for Solving Network by Mesh Current Method
- Matrix Form

KVL is already discussed in the topic **ALSO SEE:** Kirchhoff's Current Law and Kirchhoff's Voltage Law

Let us understand the Mesh Current method with the help of the circuit shown below

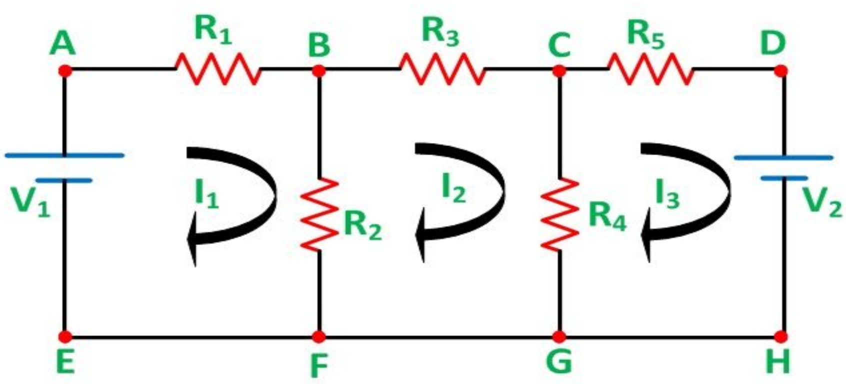


Fig 10.15 Mesh Current Analysis Circuit In the above network

- R_1, R_2, R_3, R_4 and R_5 are the various resistances
- V_1 and V_2 are the voltage source
- I_1 is the current flowing in the mesh ABFEA
- I_2 is the current flowing in the mesh BCGFB
- I_3 is the current flowing in the mesh CDHGC

The direction of the current is assumed in the clockwise for simplicity in solving the network.

Steps for Solving Network by Mesh Current Method :

Considering the above circuit diagram, the following steps are given below to solve the circuit by the Mesh Current method.

Step 1 – First of all, identify the independent circuit meshes or loop.As there is three mesh in the circuit diagram shown above which are considering.

Step 2 – Assign a circulating current to each mesh as shown in circuit diagram where I_1, I_2 and I_3 are flowing in each mesh.

It is preferable to assign the same direction of all the currents and in a clockwise direction for making the calculation easier.

Step 3 – Now, write the KVL equation for each mesh.As there are three meshes in the circuit, there will be three KVL equations as shown below

Applying KVL in the mesh ABFEA

$$I_1 R_1 + (I_1 - I_2) R_2 = V_1$$

By rearranging the equation, we will get an equation (1)

$$I_1 (R_1 + R_2) + I_2(-R_2) + I_3(0) = V_1$$

Applying KVL in the mesh BCGFB

Applying KVL in the mesh CDHGC

$$I_3 R_5 + V_2 + (I_3 - I_2) R_4 = 0 \quad \text{or}$$

$$I_1(0) + I_2(-R_4) + I_3(R_4 + R_5) = -V_2$$

Step 4 – Now solve equations simultaneously to get the value of current I_1, I_2 and I_3.

By knowing the mesh currents, we can determine the various voltages and currents in the circuit.

Matrix Form:

The above circuit can be solved by the Matrix method also, as shown below.

The above equations (1), (2) and (3) in matrix form can be expressed as

$$\begin{bmatrix} R_1 + R_2 & -R_2 & 0 \\ -R_2 & R_2 + R_3 + R_4 & -R_4 \\ 0 & -R_4 & R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ -V_2 \end{bmatrix}$$

Thus, the equation can be solved to get the values it the various currents.

It is seen from the equation that the resistance matrix [R] is symmetric, i.e.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \pm V_1 \\ \pm V_2 \\ \pm V_3 \end{bmatrix}$$

Equation (5) can be written as:

$$[R][I] = [V]$$

Where,

[R] is the mesh resistance

[I] is the column vector of mesh currents and

[V] is the column vector of the algebraic sum of all the source voltages around the mesh.

This is all about the mesh current analysis method.

Constant Voltage Source :

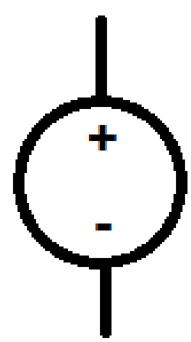


Fig 10.16 Constant Voltage Source symbol

A constant voltage source is a power source which provides a constant voltage to a load, even despite changes and variance in load resistance. In other words, the voltage which a constant voltage source provides is steady, even if the resistance of the load varies.

A constant voltage source is, thus, a very valuable component because it can supply steady voltage even if there are changes in resistance, even a wide variance in the resistance. This comes in use when a circuit needs a steady voltage supply, without fluctuations.

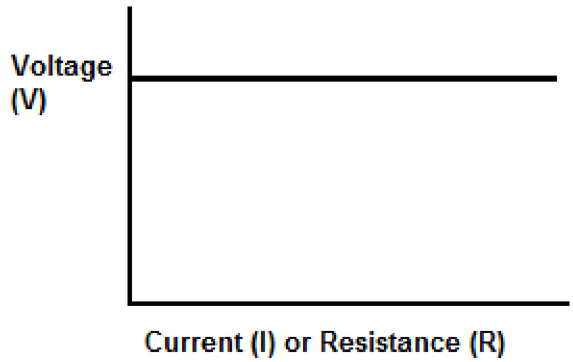


Fig 10.17 Constant voltage

You can see that the voltage is constant all throughout despite changes in current or resistance.

A Constant Voltage Source Work :

A constant voltage source is a power generator whose internal resistance is very low compared with the load resistance it is giving power to. Because its internal resistance is so low, it dumps most of its voltage across the higher resistance load. Remember that according to ohm's law, voltage is equal to current x resistance (V=IR). So voltage is dropped across the higher resistance component. If the resistance of the voltage source is practically zero, then instead of dropping its voltage across itself, it will drop it across the load entirely instead.

Thus, a constant voltage source follows the rules of voltage division. Being that it has very low internal resistance and the load resistance is much higher, the voltage will practically drop entirely across the load.

Look at the following voltage divider circuit below:

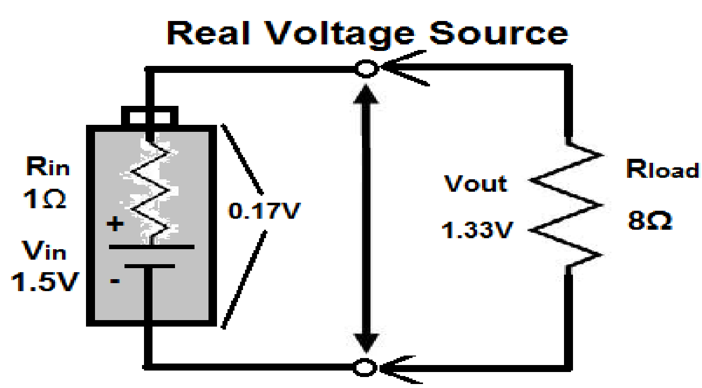


Fig 10.18

Notice how this voltage source, shown above, supplies 1.5V, in total, from out of it. The majority of this 1.5 volts drops across the resistor of greater resistance, which is 8Ω; 1.33V of the 1.5V drops across the load. The remaining 0.17V drops across the battery which has a resistance of 1Ω.

Now let's decrease the resistance of the voltage source so that now it has a resistance of 0Ω. The below voltage source represents a voltage source which has zero internal resistance.

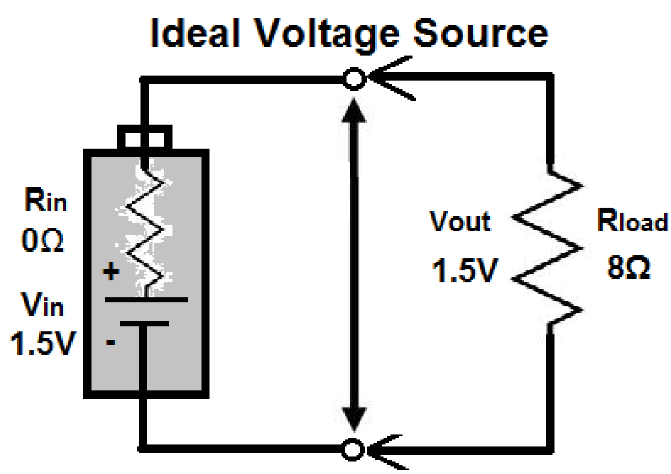


Fig 10.19

Because the resistance is 0Ω and the load is 8Ω, all of the voltage drops across the 8Ω load resistor. Greater voltage will always drop across the component with the higher resistance.

This is how constant voltage sources work.

Constant Voltage Source Circuit:

A constant voltage source circuit is just a constant voltage source connected to the load which it powers.

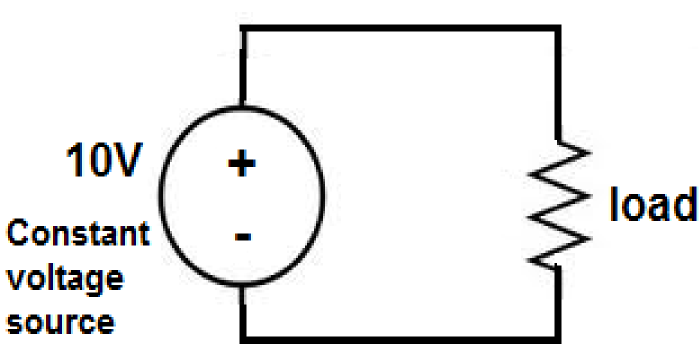


Fig 10.20 Constant Voltage Source Circuit

This load above will have a constant voltage of 10V supplied to it regardless of whether the load resistance varies.

Constant Current Source :

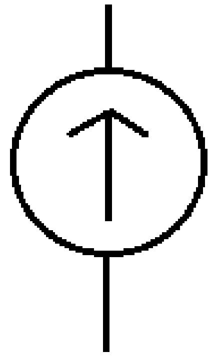


Fig 10.21 Constant Current Source symbol

A constant current source is a power source which provides a constant current to a load, even despite changes and variance in load resistance.

In other words, the current which a constant current source provides is steady, even if the resistance of the load varies.

A constant current source is, thus, a very valuable component because it can supply steady current even if there are changes in resistance, even a wide variance in the resistance. This comes in use when a circuit needs a steady current supply, without fluctuations.

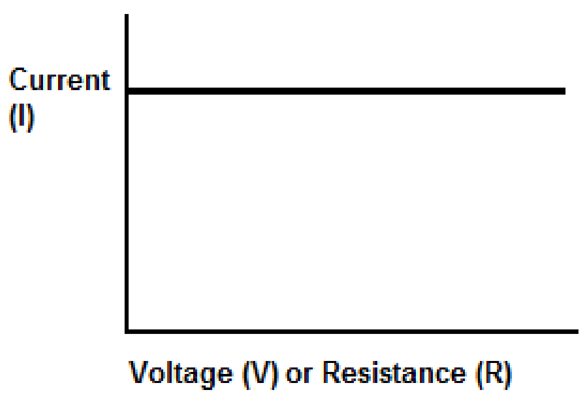


Fig 10.22 Constant current

You can see that the current is constant all throughout despite changes in voltage or resistance.

A Constant Current Source Work:

A constant current source is a power generator whose internal resistance is very high compared with the load resistance it is giving power to. Because its internal resistance is so high, it can supply a constant current to a load whose resistance value varies, even over a wide range.

Thus, a constant current source follows the rules of current division. Being that it has very high internal resistance and the load resistance is much lower, current takes the path of least resistance, flowing out of the (high internal resistance) current source and into the load resistance, since it is of much lower resistance.

If you know current division, current takes the path of least resistance. Look at the following current divider circuit below:

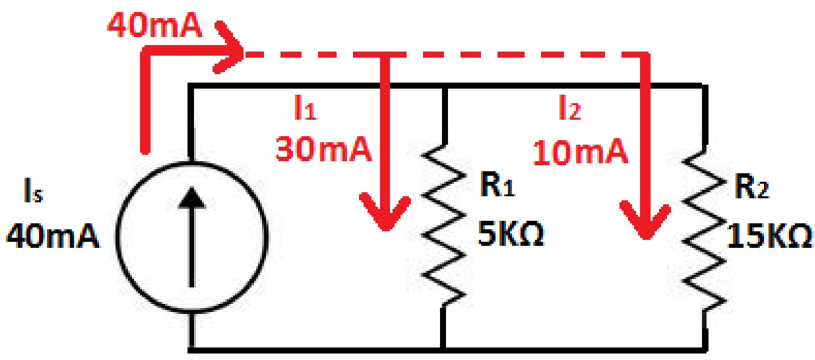


Fig 10.23 Constant Current Source circuit

Notice how this current source, shown above, supplies 40mA of total current from out of it. The majority of this 40mA of current takes the path of least resistance, the 5KΩ resistor, and the other 10mA of current goes through the larger resistance, 15KΩ.

Now let's increase the resistance again. The below current source represents a current source which has infinite internal resistance.

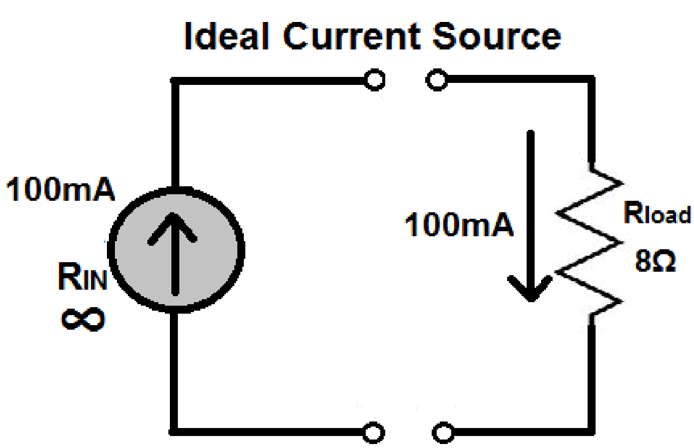


Fig 10.24

Because the resistance is infinite and the load is only 8Ω, most of the current goes through the 8Ω resistor, which is the path of least resistance. Again, current always take the path of least resistance. Since the load has infinite internal resistance, current will always seek to escape from it to a lower resistance path.

This is how constant current sources work.

Constant Current Source Circuit :

A constant current source circuit is just a constant current source connected to the load which it powers.

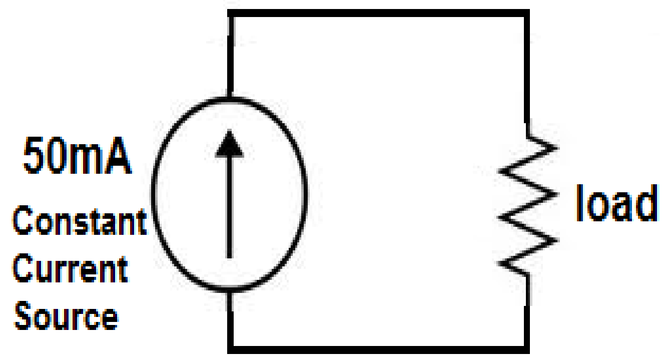


Fig 10.25 Constant Current Source Circuit

This load above will have a constant current of 50mA supplied to it regardless of whether the load resistance varies.

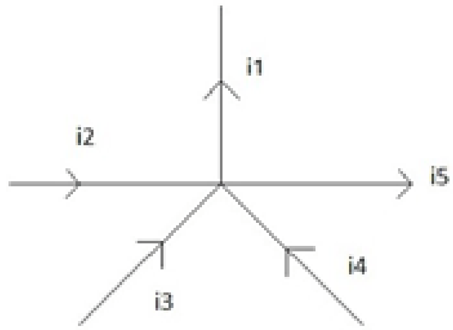
SAQ 2:

a) Define Kirchoff's current and voltage laws?

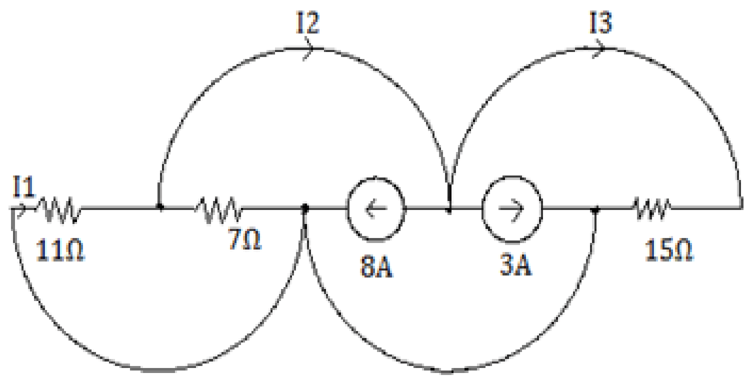
b) Define the following: (i) Mesh (ii) Loop (iii)Active and Passive circuit

c) Differencetiate between mesh and Nodal analysis?

d) Relation between currents according to KCL is



e) Determine currents I_1 , I_2 and I_3.



10.6 THEVENIN'S THEOREM

Thevenin's Theorem states that any complicated network across its load terminals can be substituted by a voltage source with one resistance in series. This theorem helps in the study of the variation of current in a particular branch when the resistance of the branch is varied while the remaining network remains the same.

For example in designing electrical and electronics circuits.

A more general statement of Thevenin's Theorem is that any linear active network consisting of independent or dependent voltage and current source and the network elements can be replaced by an equivalent circuit having a voltage source in series with a resistance.

Where the voltage source being the open-circuited voltage across the open-circuited load terminals and the resistance being the internal resistance of the source.

In other words, the current flowing through a resistor connected across any two terminals of a network by an equivalent circuit having a voltage source E_{th} in series with a resistor R_{th}. Where E_{th} is the open-circuit voltage between the required two terminals called the Thevenin voltage and the R_{th} is the equivalent resistance of the network as seen from the two-terminal with all other sources replaced by their internal resistances called Thevenin resistance.

Explanation of Thevenin's Theorem:

The Thevenin's statement is explained with the help of a circuit shown below:

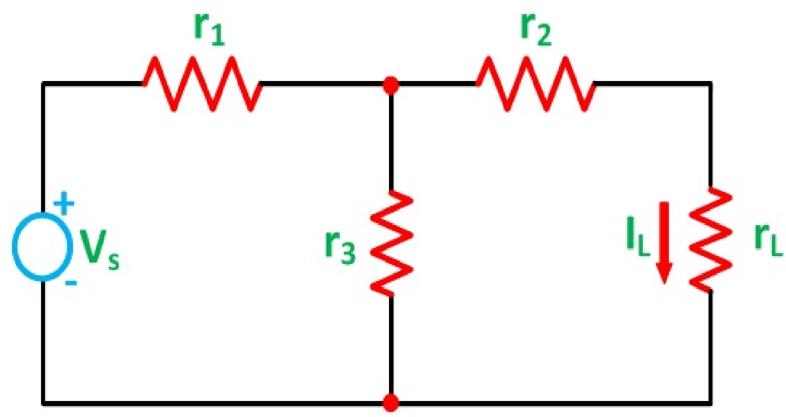


Fig 10.26 Thevenin's Theorem circuit

Let us consider a simple DC circuit as shown in the figure above, where we have to find the load current $\mathbf{I_L}$ by the Thevenin's theorem.

In order to find the equivalent voltage source, $\mathbf{r_L}$ is removed from the circuit as shown in the figure below and $\mathbf{V_{oc}}$ or $\mathbf{V_{TH}}$ is calculated.

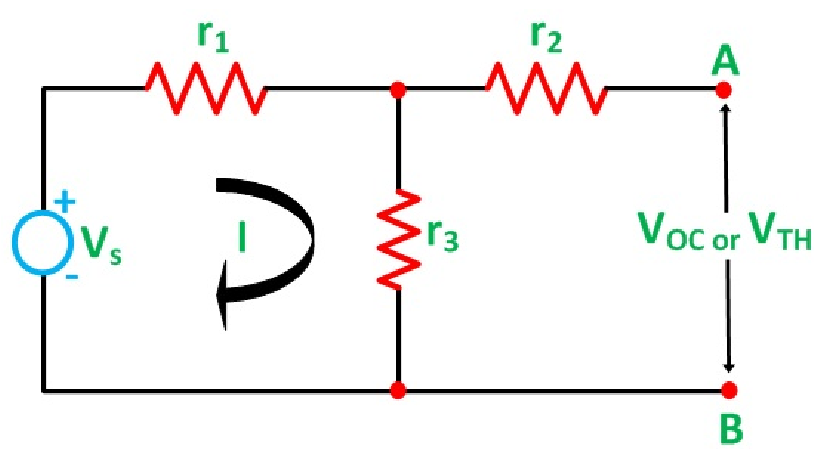
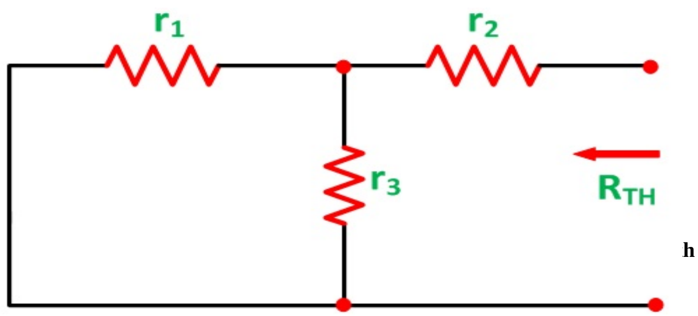


Fig 10.27 Thevenin's Theorem circuit for $\mathbf{V_{oc}}$

$$V_{OC} = I\,r_3 = \frac{V_S}{r_1 + r_3}\,r_3$$

Now, to find the internal resistance of the network (Thevenin's resistance or equivalent resistance) in series with the open-circuit voltage $\mathbf{V_{OC}}$, also known as Thevenin's voltage $\mathbf{V_{TH}}$, the voltage source is removed or we can say it is deactivated by a short circuit (as the source does not have any internal resistance) as shown in the figure below:



$$R_{TH} = r_2 + \frac{r_1 r_3}{r_1 + r_3}$$

Therefore,

So,

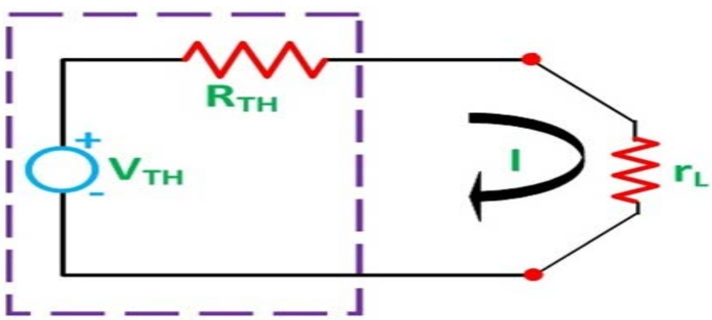


Fig 10.29 Equivalent Circuit of Thevenin's Theorem

Equivalent Circuit of Thevenin's Theorem:

As per Thevenin's Statement, the load current is determined by the circuit shown above and the equivalent Thevenin's circuit is obtained.

The load current $\mathbf{I_L}$ is given as:

$$I_L = \frac{V_{TH}}{R_{TH} + r_L}$$

Where,

$\mathbf{V_{TH}}$ is the Thevenin's equivalent voltage. It is an open circuit voltage across the terminal AB known as **load terminal** $\mathbf{R_{TH}}$ is the Thevenin's equivalent resistance, as seen from the load terminals where all the sources are replaced by their internal impedance $\mathbf{r_L}$ is the **load resistance.**

Steps for Solving Thevenin's Theorem

Step 1 – First of all remove the load resistance $\mathbf{r_L}$ of the given circuit.

Step 2 – Replace all the sources by their internal resistance.

Step 3 – If sources are ideal then short circuit the voltage source and open circuit the current source.

Step 4 – Now find the equivalent resistance at the load terminals, known as Thevenin's Resistance (R_{TH}).

Step 5 – Draw the Thevenin's equivalent circuit by connecting the load resistance and after that determine the desired response.

This theorem is possibly the most extensively used networks theorem. It is applicable where it is desired to determine the current through or voltage across any one element in a network.

Norton's Theorem :

Norton's Theorem states that – A linear active network consisting of the independent or dependent voltage source and current sources and the various circuit elements can be substituted by an equivalent circuit consisting of a current source in parallel with a resistance. The current source being the short-circuited current across the load terminal and the resistance being the internal resistance of the source network.

The Norton's theorems reduce the networks equivalent to the circuit having one current source, parallel resistance and load**. Norton's theorem** is the converse of Thevenin's Theorem. It consists of the equivalent current source instead of an equivalent voltage source as in Thevenin's theorem.

The determination of internal resistance of the source network is identical in both the theorems.

In the final stage that is in the equivalent circuit, the current is placed in parallel to the internal resistance in Norton's Theorem whereas in Thevenin's Theorem the equivalent voltage source is placed in series with the internal resistance.

Explanation of Norton's Theorem: To understand Norton's Theorem in detail, let us consider a circuit diagram given below

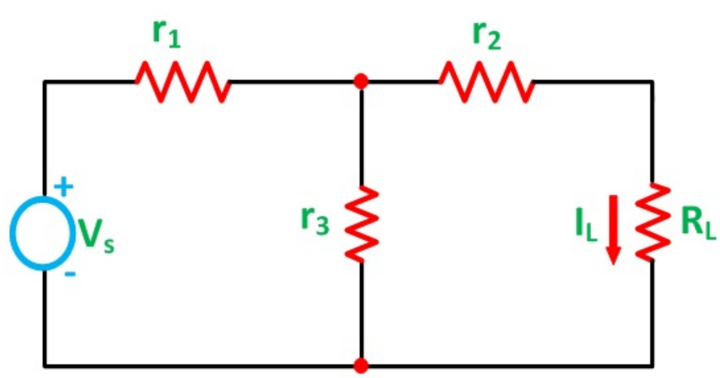


Fig 10.30 Norton's Theorem circuit

In order to find the current through the load resistance I_L as shown in the circuit diagram above, the load resistance has to be short-circuited as shown in the diagram below:

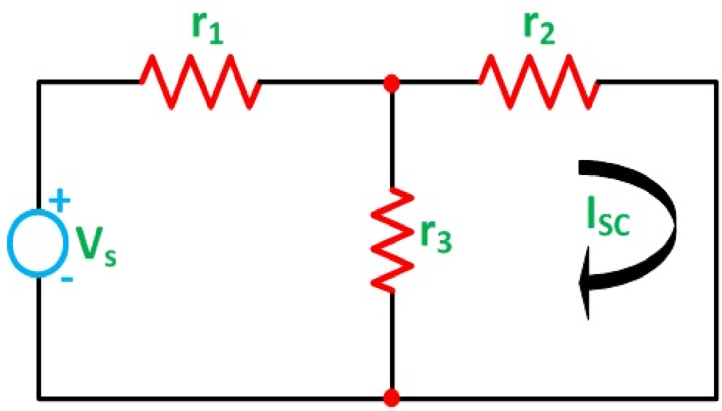


Fig 10.31 Fig 10.32 Circuit for calculating Isc

Now, the value of current I flowing in the circuit is found out by the equation

$$I = \frac{V_S}{r_1 + \frac{r_2 r_3}{r_2 + r_3}}$$

And the short-circuit current I_{SC} is given by the equation shown below:

$$I_{sc} = I \frac{r_3}{r_3 + r_2}$$

Now the short circuit is removed, and the independent source is deactivated as shown in the circuit diagram below and the value of the internal resistance is calculated by:

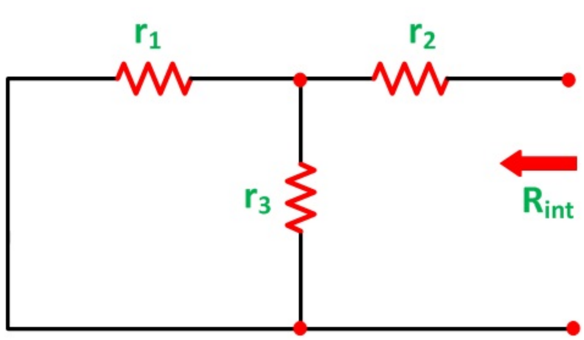


Fig 10.32 Circuit for calculating Rint

$$R_{int} = r_2 + \frac{r_1 r_3}{r_1 + r_3}$$

As per Norton's Theorem, the equivalent source circuit would contain a current source in parallel to the internal resistance, the current source being the short-circuited current across the shorted terminals of the load resistor. The Norton's Equivalent circuit is represented as

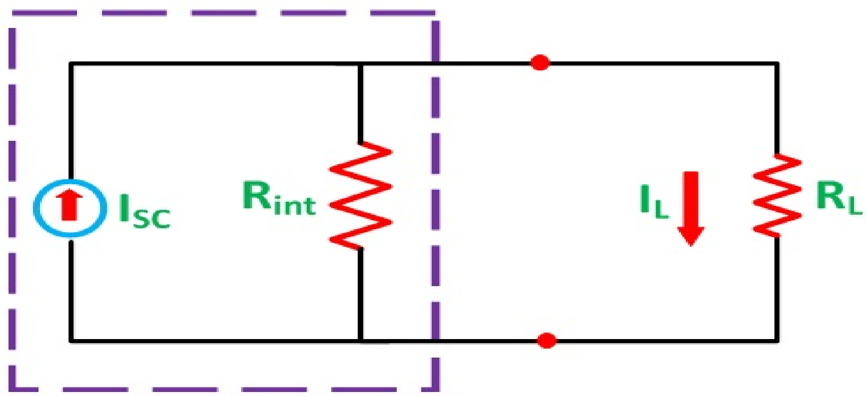


Fig 10.33 Norton's Equivalent circuit

Finally, the load current I_L calculated by the equation shown below

$$I_L = I_{sc} \frac{R_{int}}{R_{int} + R_L}$$

Where,

- I_L is the load current
- I_{sc} is the short circuit current
- R_{int} is the internal resistance of the circuit
- R_L is the load resistance of the circuit

Steps for Solving a Network Utilizing Norton's Theorem

Step 1 – Remove the load resistance of the circuit.

Step 2 – Find the internal resistance R_{int} of the source network by deactivating the constant sources.

Step 3 – Now short the load terminals and find the short circuit current I_{SC} flowing through the shorted load terminals using conventional network analysis methods.

Step 4 – Norton's equivalent circuit is drawn by keeping the internal resistance R_{int} in parallel with the short circuit current I_{SC}.

Step 5 – Reconnect the load resistance R_L of the circuit across the load terminals and find the current through it known as load current I_L.

This is all about Norton's Theorem.

Maximum Power Transfer Theorem :

Maximum Power Transfer Theorem states that – A resistive load, being connected to a DC network, receives maximum power when the load resistance is equal to the internal resistance known as (Thevenin's equivalent resistance) of the source network as seen from the load terminals. The Maximum Power Transfer theorem is used to find the load resistance for which there would be the maximum amount of power transfer from the source to the load.

The maximum power transfer theorem is applied to both the DC and AC circuit. The only difference is that in the AC circuit the resistance is substituted by the impedance.

The maximum power transfer theorem finds their applications in communication systems which receive low strength signal. It is also used in speaker for transferring the maximum power from an amplifier to the speaker.

Explanation of Maximum Power Transfer Theorem:

A variable resistance R_L is connected to a DC source network as shown in the circuit diagram in figure A below and the figure B represents the Thevenin's voltage V_{TH} and Thevenin's resistance R_{TH} of the source network.

The aim of the Maximum Power Transfer theorem is to determine the value of load resistance R_L, such that it receives maximum power from the DC source.

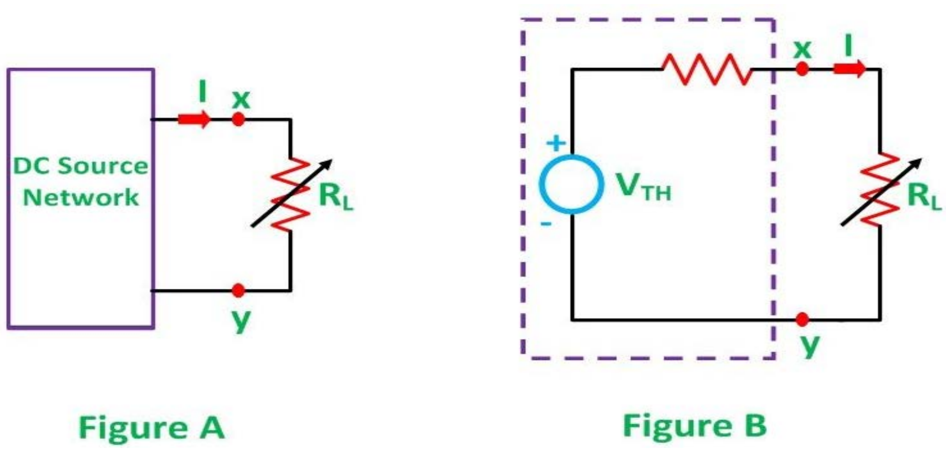


Fig 10.34 (A) Circuit diagram (B) Thevenin's voltage V_{TH} and Thevenin's resistance R_{TH}

Considering figure B the value of current will be calculated by the equation shown below

$$I = \frac{V_{TH}}{R_{TH} + R_L}$$

While the power delivered to the resistive load is given by the equation

$$P_L = I^2 R_L$$

Putting the value of I from the equation (1) in the equation (2) we will get

$$P_L = \left(\frac{V_{TH}}{R_{TH} + R_L}\right)^2 x\, R_L$$

P_L can be maximized by varying R_L and hence, maximum power can be delivered when $(dP_L/dR_L) = 0$

However,

$$\frac{dP_L}{dR_L} = \frac{1}{[(R_{TH} + R_L)^2]^2}\left[(R_{TH} + R_L)^2 \frac{d}{dR_L}(V_{TH}^2 R_L) - V_{TH}^2 R_L \frac{d}{dR_L}(R_{TH} + R_L)^2\right]$$

$$\frac{dP_L}{dR_L} = \frac{1}{(R_{TH} + R_L)^4}[(R_{TH} + R_L)^2 V_{TH}^2 - V_{TH}^2 R_L\, x\, 2(R_{TH} + R_L)]$$

$$\frac{dP_L}{dR_L} = \frac{V_{TH}^2\,(R_{TH} + R_L - 2R_L)}{(R_{TH} + R_L)^3} = \frac{V_{TH}^2\,(R_{TH} - R_L)}{(R_{TH} + R_L)^2}$$

But as we know, $(dP_L/dR_L) = 0$

Therefore,

$$\frac{V_{TH}^2\,(R_{TH} - R_L)}{(R_{TH} + R_L)^2} = 0$$

Which gives

$$(R_{TH} - R_L) = 0 \quad \text{or} \quad R_{TH} = R_L$$

Hence, it is proved that power transfer from a DC source network to a resistive network is maximum when the internal resistance of the DC source network is equal to the load resistance.

Again, with $R_{TH} = R_L$, the system is perfectly matched to the load and the source, thus, the power transfer becomes maximum, and this amount of power Pmax can be obtained by the equation shown below:

$$P_{max} = \frac{V_{TH}^2 R_{TH}}{(R_{TH} + R_{TH})^2} = \frac{V_{TH}^2}{4R_{TH}}$$

Equation above gives the power which is consumed by the load. The power transfer by the source will also be the same as the power consumed by the load, i.e. above equation as the load power and the source power being the same.

Thus, the total power supplied is given by the equation

$$P = 2\,\frac{V_{TH}^2}{4R_{TH}} = \frac{V_{TH}^2}{2R_{TH}}$$

During Maximum Power Transfer the efficiency η becomes:

$$\eta = \left(\frac{P_{max}}{P}\right) x\,100 = 50\%$$

The concept of Maximum Power Transfer theorem is that by making the source resistance equal to the load resistance, which has wide application in communication circuits where the magnitude of power transfer is sufficiently small. To achieve maximum power transfer, the source and the load resistance are matched and with this, efficiency becomes 50% with the flow of maximum power from the source to the load.

In the Electrical Power Transmission system, the load resistance being sufficiently greater than the source resistance, it is difficult to achieve the condition of maximum power transfer.

In power system emphasis is given to keep the voltage drops and the line losses to a minimum value and hence the operation of the power system, operating with bulk power transmission capability, becomes uneconomical if it is operating with only **50%** efficiency just for achieving maximum power transfer.

Hence, in the electrical power transmission system, the criterion of maximum power transfer is very rarely used.

Steps for Solving Network Using Maximum Power Transfer Theorem

Following steps are used to solve the problem by Maximum Power Transfer theorem

Step 1 – Remove the load resistance of the circuit.

Step 2 – Find the Thevenin's resistance (R_{TH}) of the source network looking through the open-circuited load terminals.

Step 3 – As per the maximum power transfer theorem, this R_{TH} is the load resistance of the network, i.e., $R_L = R_{TH}$ that allows maximum power transfer.

Step 4 – Maximum Power Transfer is calculated by the equation shown below

$$P_{max} = \frac{V_{TH}^2}{4R_{TH}}$$

This is all about Maximum Power Transfer Theorem.

Superposition Theorem:

Superposition theorem states that in any linear, active, bilateral network having more than one source, the response across any element is the sum of the responses obtained from each source considered separately and all other sources are replaced by their internal resistance. The superposition theorem is used to solve the network where two or more sources are present and connected

In other words, it can be stated as if a number of voltage or current sources are acting in a linear network, the resulting current in any branch is the algebraic sum of all the currents that would be produced in it when each source acts alone while all the other independent sources are replaced by their internal resistances.

It is only applicable to the circuit which is valid for the ohm's law (i.e., for the linear circuit).

Explanation of Superposition Theorem:

Let us understand the superposition theorem with the help of an example. The circuit diagram is shown below consists of two voltage sources V_1 and V_2.

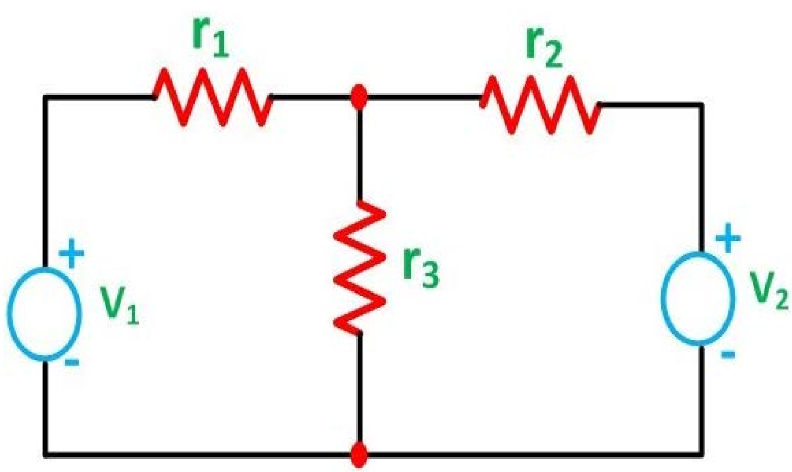


Fig 10.35 Circuit for Superposition theorem

First, take the source V_1 alone and short circuit the V_2 source as shown in the circuit diagram below:

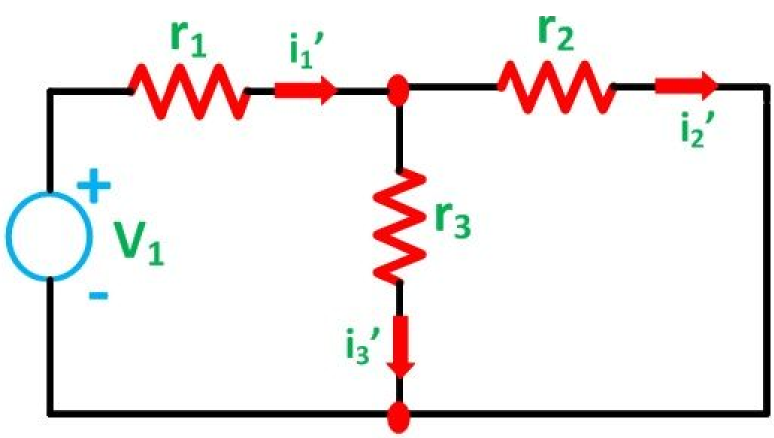


Fig 10.36 Circuit for Superposition theorem with source V_1

Here, the value of current flowing in each branch, i.e. i_1, i_2 and i_3 is calculated by the following equations.

$$i_1' = \frac{V_1}{\frac{r_2 r_3}{r_2 + r_3} + r_1}$$

$$i_2' = i_1' \frac{r_3}{r_2 + r_3}$$

The difference between the above two equations gives the value of the current i3'

$$i_3' = i_1' - i_2'$$

Now, activating the voltage source V_2 and deactivating the voltage source V_1 by short-circuiting it, find the various currents, i.e. i_1, i_2, i_3 flowing in the circuit diagram shown below:

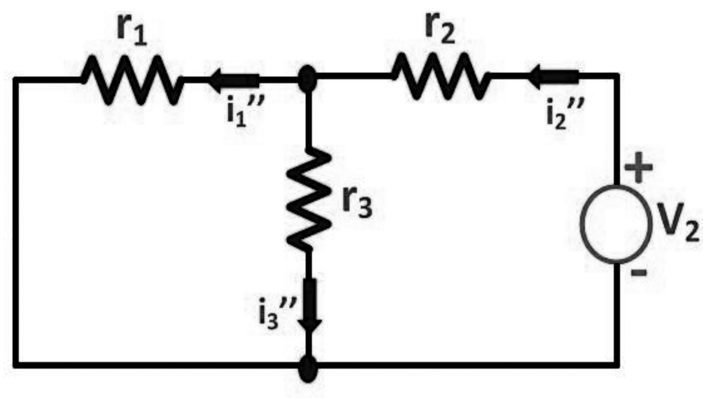


Fig 10.37 Circuit for Superposition theorem with source V_2

Here,

$$i_2'' = \frac{V_2}{\frac{r_1 r_3}{r_1 + r_3} + r_2} \quad \text{and} \quad i_1'' = i_2'' \frac{r_3}{r_1 + r_3}$$

And the value of the current i_3 will be calculated by the equation shown below:

$$i_3'' = i_2'' - i_1''$$

As per the superposition theorem, the value of current i_1, i_2, i_3 is now calculated as:

$$i_3 = i_3' + i_3''$$

$$i_2 = i_2' - i_2''$$

$$i_1 = i_1' - i_1''$$

The direction of the current should be taken care of while finding the current in the various branches.

Steps for Solving network by Superposition Theorem :

Considering the circuit diagram A, let us see the various steps to solve the superposition theorem:

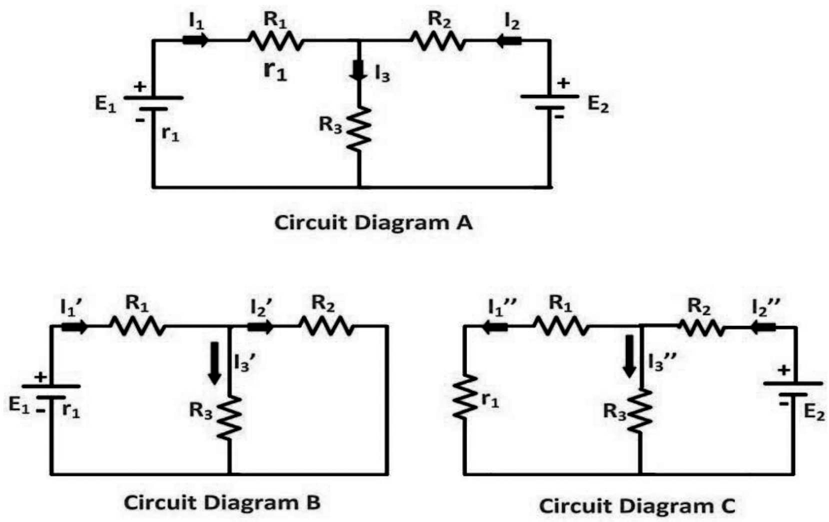


Fig 10.38 (A) Circuit diagram with both source (B) with E_1 (C) with E_2

Step 1 – Take only one independent source of voltage or current and deactivate the other sources.

Step 2 – In the circuit diagram B shown above, consider the source E_1 and replace the other source E_2 by its internal resistance. If its internal resistance is not given, then it is taken as zero and the source is short-circuited.

Step 3 – If there is a voltage source than short circuit it and if there is a current source then just open circuit it.

Step 4 – Thus, by activating one source and deactivating the other source find the current in each branch of the network. Taking the above example find the current I_1, I_2 and I_3.

Step 5 – Now consider the other source E_2 and replace the source E_1 by its internal resistance r_1 as shown in the circuit diagram C.

Step 6 – Determine the current in various sections, I_1, I_2 and I_3.

Step 7 – Now to determine the net branch current utilizing the superposition theorem, add the currents obtained from each individual source for each branch.

Step 8 – If the current obtained by each branch is in the same direction then add them and if it is in the opposite direction, subtract them to obtain the net current in each branch.

The actual flow of current in the circuit C will be given by the equations shown below:

$$I_1 = I_1' - I_1''$$

$$I_2 = I_2' - I_2''$$

$$I_3 = I_3' - I_3''$$

Thus, in this way, we can solve superposition theorem.

Reciprocity Theorem:

Reciprocity Theorem states that – In any branch of a network or circuit, the current due to a single source of voltage (V) in the network is equal to the current through that branch in which the source was originally placed when the source is again put in the branch in which the current was originally obtained. This theorem is used in the bilateral linear network which consists of bilateral components.

In simple words, we can state the reciprocity theorem as when the places of voltage and current source in any network are interchanged the amount or magnitude of current and voltage flowing in the circuit remains the same.

This theorem is used for solving many DC and AC network which have many applications in electromagnetism electronics. These circuits do not have any time-varying element.

Explanation of Reciprocity Theorem

The location of the voltage source and the current source may be interchanged without a change in current. However, the polarity of the voltage source should be identical with the direction of the branch current in each position.

The Reciprocity Theorem is explained with the help of the circuit diagram shown below

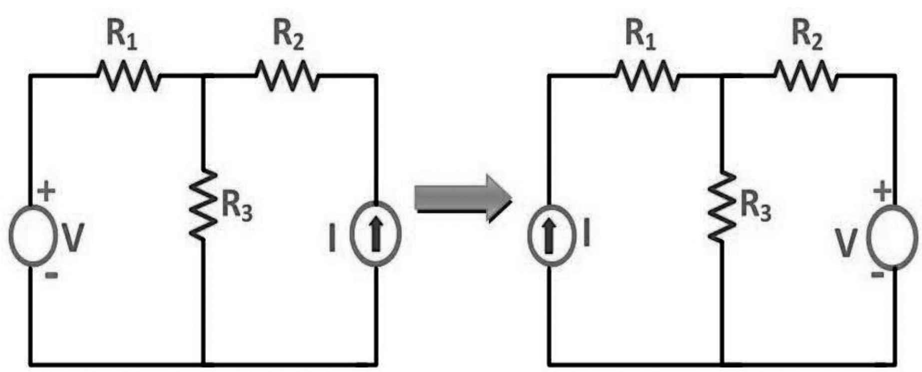


Fig 10.39 Circuit diagram for Reciprocity Theorem

The various resistances R_1, R_2, R_3 is connected in the circuit diagram above with a voltage source (V) and a current source (I). It is clear from the figure above that the voltage source and current sources are interchanged for solving the network with the help of Reciprocity Theorem.

The limitation of this theorem is that it is applicable only to single-source networks and not in the multi-source network. The network where reciprocity theorem is applied should be linear and consist of resistors, inductors, capacitors and coupled circuits. The circuit should not have any time-varying elements.

Steps for Solving a Network Utilizing Reciprocity Theorem

Step 1 – Firstly, select the branches between which reciprocity has to be established.

Step 2 – The current in the branch is obtained using any conventional network analysis method.

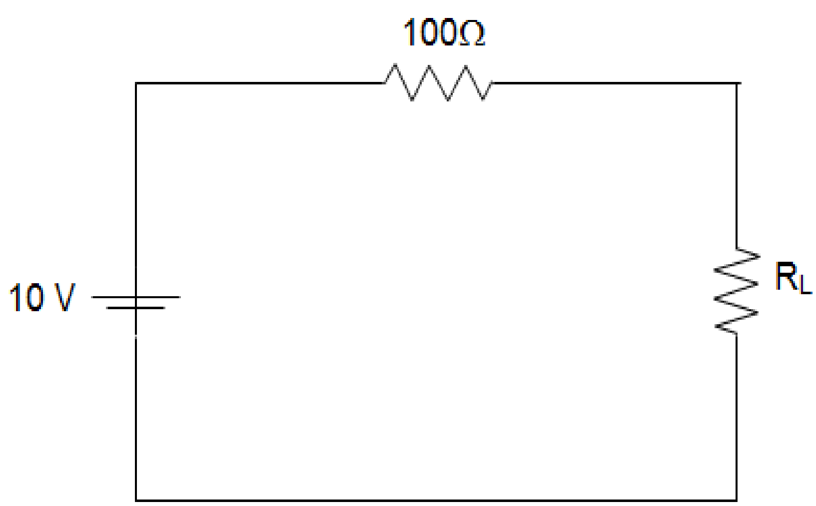
Step 3 – The voltage source is interchanged between the branch which is selected.

Step 4 – The current in the branch where the voltage source was existing earlier is calculated.

Step 5 – Now, it is seen that the current obtained in the previous connection, i.e., in step 2 and the current which is calculated when the source is interchanged, i.e., in step 4 are identical to each other.

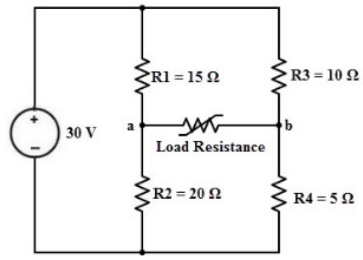
SAQ 3:

- **a)** **State the Thevenin theorum and Nortan theorem?**
- **b)** **Write the duality between Thevenin theorum and Nortan theorem?**
- **c)** **State the maximum power transfer theorem and write where its used?**
- **d)** **State the Superposition and Recipeocity theorem?**
- **e)** In the circuit given below the maximum power that can be transferred from the source voltage is



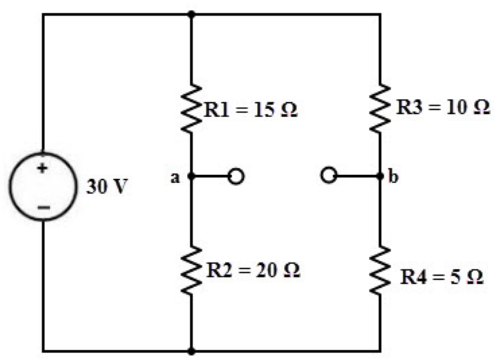
Examples:

Q1 Consider the below circuit in which determine the value of the load resistance that receives the maximum power from the supply source and the maximum power under the maximum power transfer condition.



Solution:

Disconnect the load resistance from the load terminals a and b. To represent the given circuit as Thevenin's equivalent, we are to determine the Thevenin's voltage V_{TH} and Thevenin's equivalent resistance R_{TH}.



The Thevenin's voltage or voltage across the terminals ab is $V_{ab} = V_a – V_b$

$V_a = V \times R_2 / (R_1 + R_2)$

$= 30 \times 20 /\times(20 + 15)$

$= 17.14$ V

$V_b = V \times R_4/ (R_3 + R_4)$

$= 30 \times 5 /(10 + 5)$

$= 10$ V

$V_{ab} = 17.14 – 10$

$= 7.14$ V

$V_{TH} = V_{ab} = 7.14$ Volts

Calculate the Thevenin’s equivalent resistance R_{TH} by replacing sources with their internal resistances (here assume that voltage source has zero internal resistance so it becomes a short circuited).

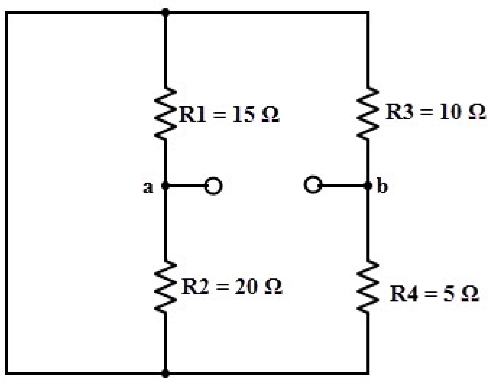
Thevenin’s equivalent resistance or resistance across the terminals ab is

$R_{TH} = R_{ab} = [R_1R_2 / (R_1 + R_2)] + [R3R4 /(R3 + R4)]$

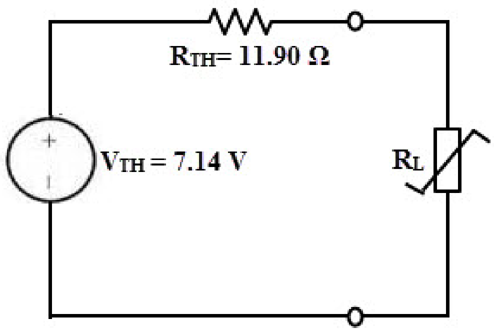
$= [(15 \times 20) / (15 + 20)] + [(10 \times 5) / (10+ 5)]$

$= 8.57 + 3.33$

$R_{TH} = 11.90$ Ohms



The Thevenin’s equivalent circuit with above calculated values by reconnecting the load resistance is shown below.



From the maximum power transfer theorem, RL value must equal to the RTH to deliver the maximum power to the load.

Therefore, $R_L = R_{TH} = 11.90$ Ohms

And the maximum power transferred under this condition is,

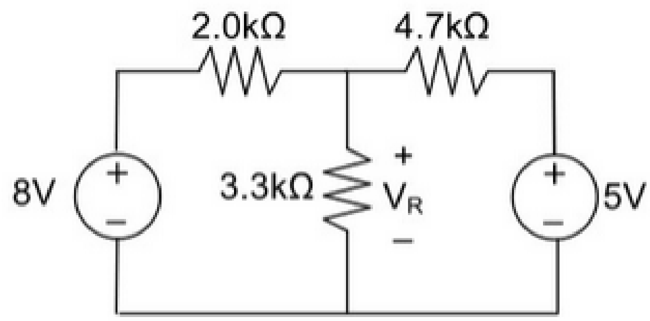
$P_{max} = V^2_{TH} / 4 R_{TH}$

$= (7.14)^2 / (4 \times 11.90)$

$= 50.97 / 47.6$

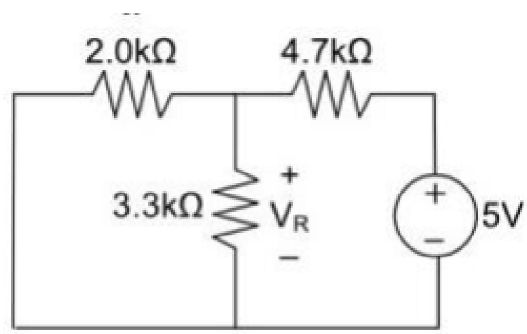
$= 1.07$ Watts

Q2 Using the superposition theorem, determine the voltage drop and current across the resistor 3.3K as shown in figure below.



Solution:

Step 1: Remove the 8V power supply from the original circuit, such that the new circuit becomes as the following and then measure voltage across resistor.

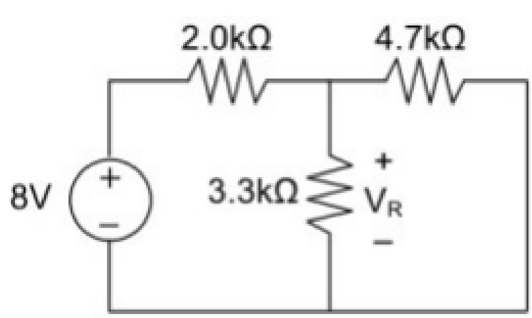


Here 3.3K and 2K are in parallel, therefore resultant resistance will be 1.245K.

Using voltage divider rule voltage across 1.245K will be

$$V_1 = [1.245/(1.245+4.7)]*5 = 1.047V$$

Step 2: Remove the 5V power supply from the original circuit such that the new circuit becomes as the following and then measure voltage across resistor.



Here 3.3K and 4.7K are in parallel, therefore resultant resistance will be 1.938K.

Using voltage divider rule voltage across 1.938K will be

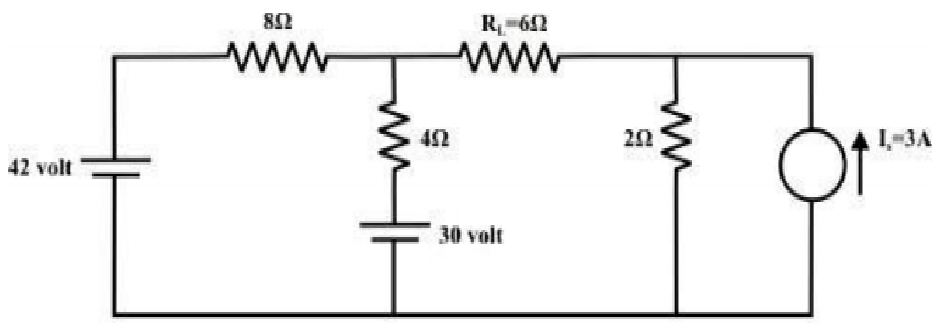
$$V_2 = [1.938/(1.938+2)]*8 = 3.9377V$$

Therefore voltage drop across 3.3K resistor is V_1+V_2 = 1.047+3.9377=4.9847V.

Q3 For the circuit shown in fig.8.4 (a), find the current I_L through 6 Ω resistor using Thevenin's theorem.

Solution:

Step-1: Disconnect 6 Ω from the terminals 'a' and 'b' and the corresponding circuit diagram. Consider point 'g' as ground potential and other voltages are measured with respect to this point.

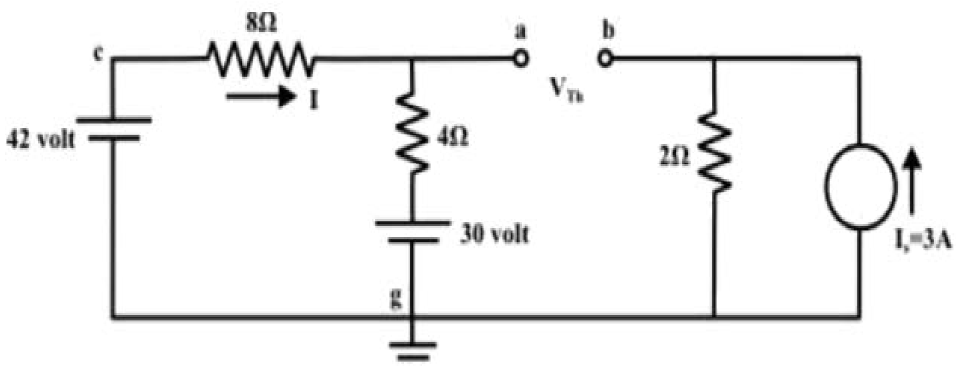


Step-2: Apply any suitable method to find the Thevenin's voltage (VTh) (or potential between the terminals 'a' and 'b'). KVL is applied around the closed path 'gcag' to compute Thevenin's voltage.

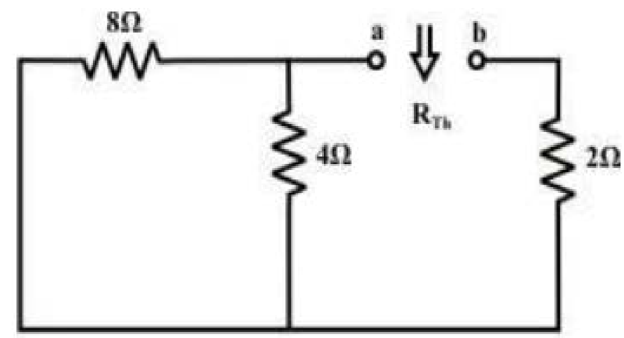
$$42 - 8I - 4I - 30 = 0 \;\Rightarrow I = 1\,A$$

$$Now,\; V_{ag} = 30 + 4 = 34\; volt\;;\; V_{bg} = 2 \times 3 = 6\; volt.$$

$$V_{Th} = V_{ab} = V_{ag} - V_{bg} = 34 - 6 = 28 volt$$



Step-3: Thevenin's resistance RTh can be found by replacing all sources by their internal resistances (all voltage sources are short-circuited and current sources are just removed or open circuited)

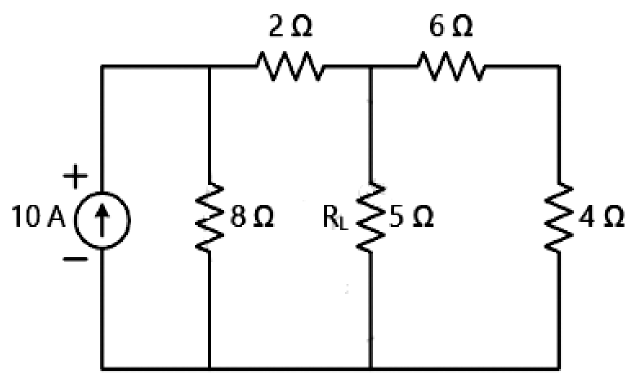


$$R_{Th} = (8 \& 4) + 2 = \frac{8 \times 4}{12} + 2 = \frac{14}{3} = 4.666\ \Omega$$

Step-4: Thevenin's equivalent circuit is now equivalently represents the original circuit

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{28}{4.666 + 6} = 2.625\ A$$

Q4. For the given circuit, calculate the current flows through the 5Ω resistor using Norton's theorem.

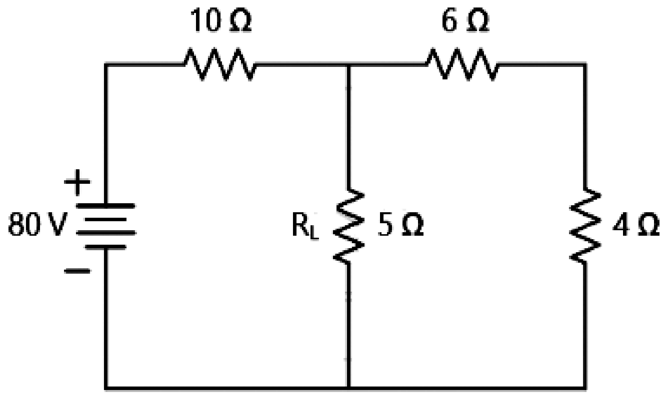


Step 1

To simplify the difficulty of the problem, replace the given current source into its equivalent voltage source. So, redraw the circuit with the equivalent voltage source and consider it for the analysis.

While converting the source, perform addition of 2Ω resistor with 8Ω (they are in series) and get 10Ω resistor for the circuit.

The circuit after the transformation of the current source to a voltage source is shown below.



We know that current select a path with low resistance and a short circuit path is considered as zero resistance.

Since a short circuit appears before the 6Ω and 4Ω resistors, all the current will flow to the short circuit only and no current will flow to the 6Ω and 4Ω resistors. So Norton's current can be calculated as follows.

$$I_N = I_{SC} = \frac{80}{10} = 8\ A$$

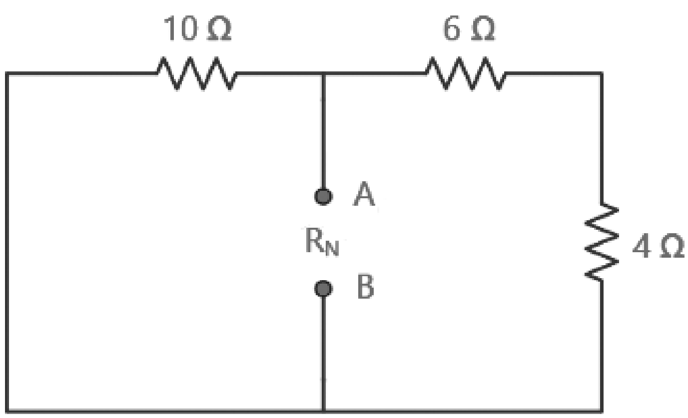
Hence, the norton's current for the given circuit is 8 Amperes.

Step 3

The next step is to find the Norton's or thevenin's equivalent resistance of the circuit.

To find the thevenin's resistance of the network, remove the load resistor and replace the 80V source by a short circuit. Now apply network reduction techniques and find the network resistance.

Simply, you can add the 6Ω and 4Ω resistors to have a 10Ω resistor in the circuit.



The norton's resistance of the network is given as

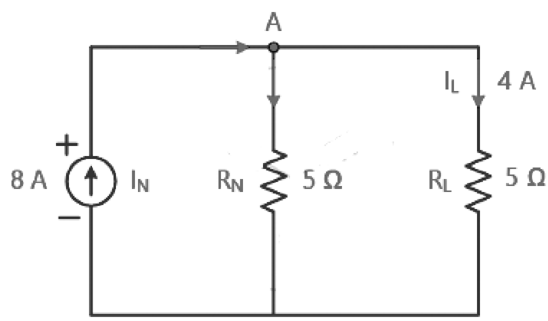
$$R_N = R_{TH} = \frac{10 \times 10}{10+10} = 5\ \Omega$$

Step 4

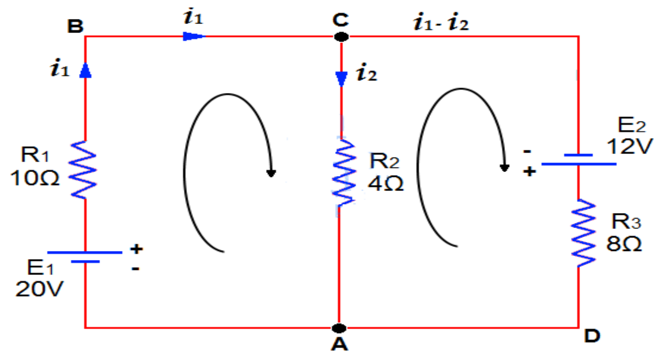
Now draw Norton's equivalent circuit with norton's current source in parallel with thevenin's resistance. Add the load resistor in parallel with the above circuit and apply current division rule to find the load current.

The load current is calculated as

$$I_L = I_N \times \frac{R_N}{R_N + R_L} = 8 \times \frac{5}{5+5} = 4\ Amperes$$



Q5.Resistors of R_1= 10Ω, R_2 = 4Ω and R_3 = 8Ω are connected up to two batteries (of negligible resistance) as shown. Find the current through each resistor.



Solution:
Assume currents to flow in directions indicated by arrows. Apply KCL on Junctions C and A.
Therefore, current in mesh ABC = i_1

Current in Mesh CA = i_2

Then current in Mesh CDA = $i_1 - i_2$

Now, Apply KVL on Mesh ABC, 20V are acting in clockwise direction. Equating the sum of IR products, we get;

$10i_1 + 4i_2 = 20$ ……………. (1)

In mesh ACD, 12 volts are acting in clockwise direction, then:

$8(i_1 - i_2) - 4i_2 = 12$

$8i_1 - 8i_2 - 4i_2 = 12$

$8i_1 - 12i_2 = 12$ ……………. (2)

Multiplying equation (1) by 3;

$30i_1 + 12i_2 = 60$

Solving for i_1

$30i_1 + 12i_2 = 60$

$8i_1 - 12i_2 = 12$

____

$38i_1 = 72$

The above equation can be also simplified by Elimination or Cramer's Rule.

$i_1 = 72/38 = 1.895$ Amperes = Current in 10 Ohms resistor

Substituting this value in (1), we get:

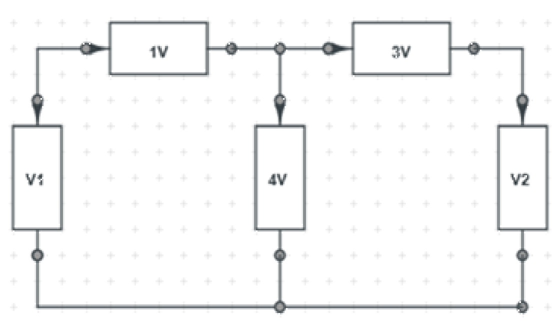
$10(1.895) + 4i_2 = 20$

$4i_2 = 20 - 18.95$

$i_2 = 0.263$ Amperes = Current in 4 Ohms Resistors.

Now,
$i_1 - i_2 = 1.895 - 0.263 = 1.632$ Amperes

Q6. Find v1 and v2 in the following circuit (note: the arrows are signifying the positive position of the box and the negative is at the end of the box)



Loop 1

$$-V_1 + 1 + 4 = 0$$

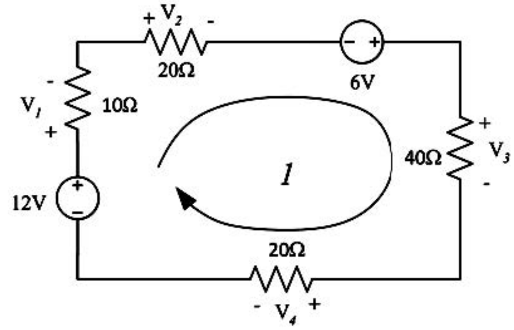
$$V_1 = 5V$$

Loop 2

$$-4 + 3 + V_2 = 0$$

$$V_2 = 1V$$

Q7. Find the current i and voltage v over the each resistor.



Solution:

KVL equations for voltages

$$v_1 + v_2 + v_3 + v_4 = 18(1)$$

Using Ohm's Law

$$v_1 = 10\Omega, v_2 = 20\Omega, v_3 = 40\Omega, v_4 = 20\Omega$$

Substituting into KVL equation

$$10i + 20i + 40i + 20i = 18$$

$$(90)i = 18$$

$$i = \frac{18}{90} = 0.2A$$

$$v_1 = R_1 i = 10(0.2) = 2V, v_2 = R_2 i = 20(0.2) = 4V$$

$$v_3 = R_3 i = 40(0.2) = 8V,$$

$$v_4 = R_4 i = 20(0.2) = -4V$$

10.8 SUMMARY

1. Ac circuits are those circuits, Whose excitation element is an AC source. Unlike DC source which is constant AC source has variable current and voltage at regular intervals of time.

2. For alternating current passing through the resistor, the ratio of current and voltage depends upon the phase and frequency of the supply

3. A coil of thin wire wrapped on a cylindrical core is known as an Inductor. The core can be an air core (hollow laminated) or an iron core.

4. When AC supply voltage is applied to a capacitor, the rate of charging and discharging depends upon the supply frequency. Voltage across the capacitor lags the current flowing through it by 90 degrees.

5. **Kirchhoff's Current Law** states that" the algebraic sum of all the currents at any node point or a junction of a circuit is zero".

$\Sigma I = 0$

6. **Kirchhoff's Voltage Law** states that the algebraic sum of the voltages (or voltage drops) in any closed path of a network that is transverse in a single direction is zero

$\mathbf{\Sigma E + \Sigma V = 0}$

7. A closed path of a network is called a loop, The most elementary form of a loop which cannot be further divided is called a mesh.

8. The **Nodal Voltage Analysis** is a method to solve the electrical network. It is used where it is essential to compute all branch currents.The nodal voltage analysis method determines the voltage and current by using the nodes of the circuit.

9. **Mesh Current Analysis Method** is used to analyze and solve the electrical network having various sources or the circuit consisting of several meshes or loop with a voltage or current sources. It is also known as the **Loop Current Method.**

10. A constant voltage source is a power source which provides a constant voltage to a load, even despite changes and variance in load resistance. In other words, the voltage which a constant voltage source provides is steady, even if the resistance of the load varies.

11. A constant current source is a power generator whose internal resistance is very high compared with the load resistance it is giving power to. Because its internal resistance is so high, it can supply a constant current to a load whose resistance value varies, even over a wide range.

12. statement of Thevenin's Theorem is that any linear active network consisting of independent or dependent voltage and current source and the network elements can be replaced by an equivalent circuit having a voltage source in series with a resistance.

13. **Norton's Theorem** states that – A linear active network consisting of the independent or dependent voltage source and current sources and the various circuit elements can be substituted by an equivalent circuit consisting of a current source in parallel with a resistance.

14. **Maximum Power Transfer Theorem** states that – A resistive load, being connected to a DC network, receives maximum power when the load resistance is equal to the internal resistance known as (Thevenin's equivalent resistance) of the source network as seen from the load terminals.

15. **Superposition theorem** states that in any linear, active, bilateral network having more than one source, the response across any element is the sum of the responses obtained from each source considered separately and all other sources are replaced by their internal resistance.

16. **Reciprocity Theorem** states that – In any branch of a network or circuit, the current due to a single source of voltage (V) in the network is equal to the current through that branch in which the source was originally placed when the source is again put in the branch in which the current was originally obtained.

10.9 TERMINAL QUESTION

1. Explain various type of electrical network?

2. Explain T and Pi network and their network equivalent?

3. Explain Kirchoff's voltage and current laws and explain with suitable diagram?

4. Explain the concept of constant current and constant voltage source?

5. Write and state Thevenin theorem and draw Thevenin equivalent circuit also write the steps to calculate Thevenin's voltage and Thevenin's resistance?

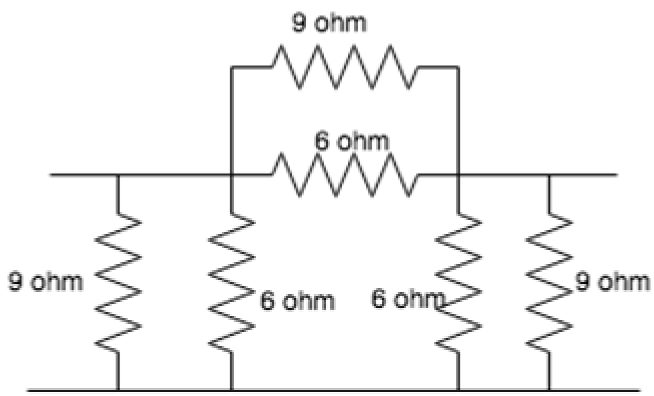
6. Write and state Norton theorem and draw Norton equivalent circuit also write the steps to calculate sort circuit current and Norton's resistance?

7. State the maximum power transfer theorem and drive the expersion for maximum power?

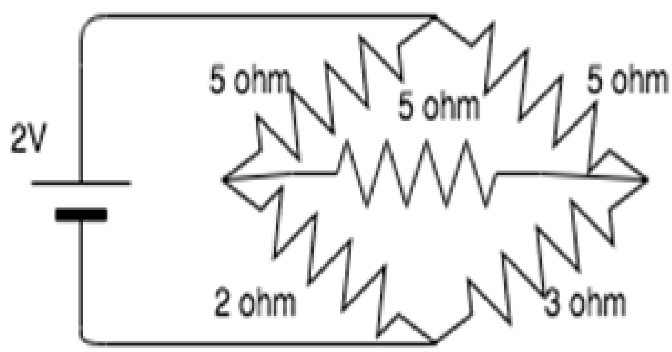
8. Explain why the efficiency of maximum power transfer theorem is 50%?

9. State super position Theorem and Reciprocity theorem?

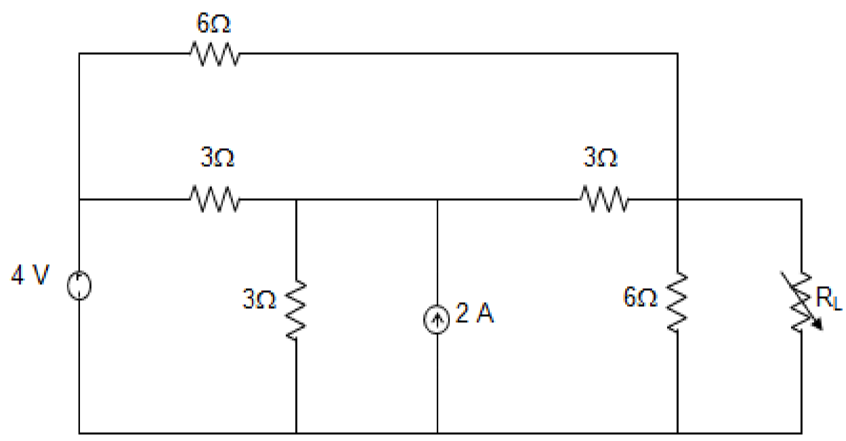
10. Find the equivalent star network.



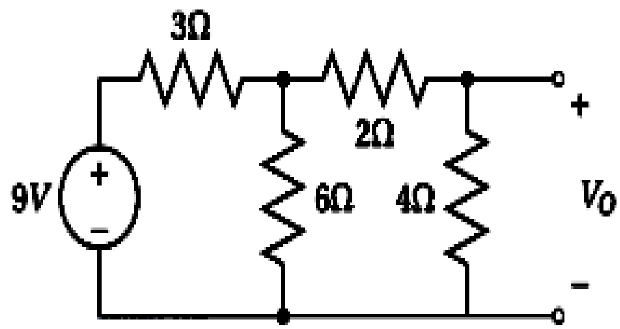
11. Find the current in the circuit.



12. In the circuit given below, the value of RL for maximum power transfer is...



13. Use Thévenin's theorem to determine V_O.



14. Find Ix using superposition theorem