Uttar Pradesh Rajarshi Tandon

Open University

Bachelor of Science

UGMM -106

(Mechanics-II (Dynamics and Hydrodynamics)

MECHANIC –II (DYNAMICS AND HYDRODYNAMICS)

UGMM - 106

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Course Introduction

I - Block

Unit 1 :

Moment of Inertia: Moment and product of some standard bodies, principle axis, Momental ellipsoid of a body.

Unit 2 :

D' Alembert Principle: The general equation of motion, motion of the center of inertia and motion relative to the center of inertia.

Unit 3 :

Motion about a fixed axis : Moment of the effective forces about the axis of rotation , moment of momentum of about the axis of rotation, kinetic energy of a body rotating about a fixed axis, equation of motion about axis of rotation.

INTRODUCTION

There are two part of the book which deals dynamics and hydrodynamics. The study of mechanics contains rigid body and fluid dynamics both.

In present book chapter 1 to 3 we studied the kinematics of rigid body. The chapter 1 and 2 give brief introduction of moment of inertia and its related theorem and quantities that is necessary to study of dynamics of rigid body. In the chapter 3 study the rotational motion of rigid body or particle.

The next part of book starts from chapter 4 which deals equation of continuity, velocity potential, stream lines. The equation of motion of fluid is discussed in chapter 5 and his two dimension application, with sink and source explained in chapter 6. The last chapter contains combine flow sink and source , which give the theory of doublet.

UNIT : 1: Moment of Inertia

Structure:

1.1 Introduction:

The discussion of moment of inertia, Principal axis, Product of inertia and momental ellipsoid goes through definition and its derivatives. After learn simple techniques, there are many theorems related to moments of inertia, product of inertia and momental ellipsoid which are necessary to calculate these objects for complex body.

In each topic explain the some important result and example to that topics. Standard example of complex body and solve problems also given for study. Moment of inertia many complex body given in chart that can be spark readers mind.

1.2 Objective:

After reading this unit students should able to:

- Define moment of inertia, product of inertia and momental ellipsoid of a body.
- Develop skills to calculate the moment of inertia, moment of product and momental ellipsoid of a body.
- Understand the application of theorems to calculate above mechanical objects for complex a body.
- Understand relation between the moment of inertia and product of moments.

1.3 Moment of inertia

Moment of Inertia : The moment of inertia of a body about an axis is given by

$$I = \Sigma m r_0^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots \dots \dots \dots \quad (1.1)$$

The expression of moment of inertia about an axis depends upon following:

(i) The mass of the body ,

(ii) The distribution of its mass about the given axis.

It means the moment of inertia of two body will be different but constant mass, because the distance of the mass from the axis of rotation alters the value of moment of inertia.

Let I is moment of inertia of a body, and a part whose moment of inertia is I_1, Then moment of inertia of the remaining part will be

$$I_2 = I - I_1 \tag{1.2}$$

1.3.1 Principal axes and principal moment of inertia: In space there are three mutually perpendicular axes and the angular momentum of any shape body is along the angular velocity when body rotated about that axis. The moment of inertia about the mutually perpendicular axis (X,Y,Z) are denoted by $I_1, I_{2,} I_3$ and are called principal moments of inertia with corresponding axes as principal axes.

Theorem of parallel axes. *According to this theorem, the moment of inertia of a body about any axis is equal to the sum of moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the distance between two axes* i.e.,

$$I = I_{cm} + Ma^2 \tag{1.3}$$

Where $I = M.I.$ of the body about any axis,

$I_{cm} = M.I.$ of a body about a parallel axis passing through its center of mass,

M = mass of the body ,

and a = the distance between two axis.

Proof. Let AB be the axis, in the plane of the paper, about which we have to determine the moment of inertia of the body. In fig. 1.1 , CD is an axis parallel to AB and at a distance a from it. Axis CD passes through the center of mass O of the body.

Let us consider a particle of body of mass m at P at a distance x from CD. Now,

Moment of inertia of m about

$$AB = m(x+a)^2$$

So moment of inertia of the whole body about AB

$$I = \Sigma m(x+a)^2$$

Or $$I = \Sigma mx^2 + \Sigma ma^2 + 2\Sigma max$$

If I_{cm} is the moment of inertia of the body about the axis CD, passing through its center of mass, then

$$I_{cm} = \Sigma mx^2 \text{ Also } \Sigma ma^2 = a^2\Sigma \mathrm{m} = Ma^2$$

Since the distance a between the two axes is constant and $\Sigma m = M$ is the mass of the body,

$$\text{So } I = I_{cm} + Ma^2 + 2a\Sigma mx$$

But $\sum$mx is the sum of moment of all the particles about the axis CD, which passes through its center of mass. So the algebraic sum of all moments about it is zero.

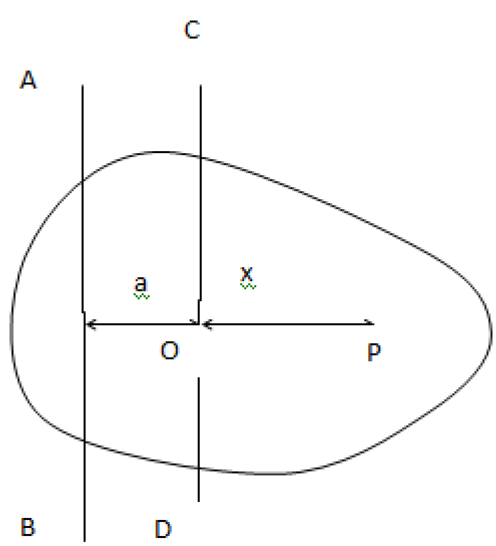


Figure:1.1

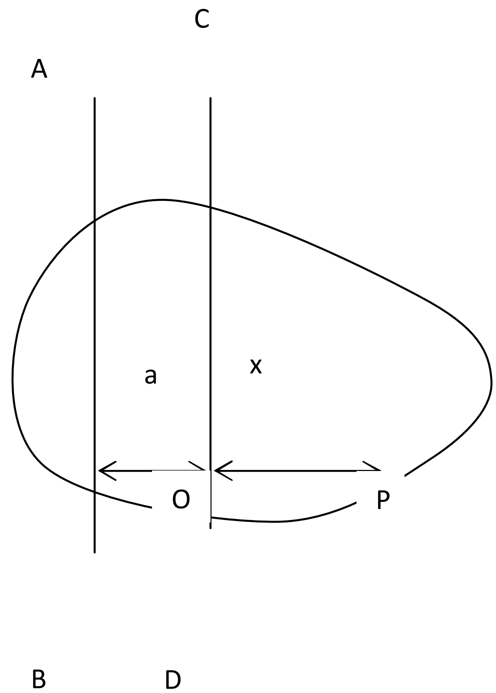


Figure: 1.1

Theorem of perpendicular axes. *The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia of a lamina about two axes at right angle to each other, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it.*

$$I = I_x + I_y \qquad (1.4)$$

Where I_x and I_y be moment of inertia of a plane lamina about OX and OY axes, which are mutually perpendicular each other in plane at the intersecting point O. The moment of inertia I about an axis, which is passing through O.

Proof: Let mass of particle P of lamina is m.

Distance P from OX and OY axes are y and x respectively and distance P from O is r. Then

$I = \Sigma mr^2$

But $\quad I_x = \Sigma my^2 \text{ and } I_y = \Sigma mx^2$

So $I_x + I_y = \Sigma \mathrm{my}^2 + \Sigma \mathrm{mx}^2 = \Sigma \mathrm{m}(\mathrm{y}^2 + \mathrm{x}^2)$

Hence $I_x + I_y = \Sigma mr^2 = I.$

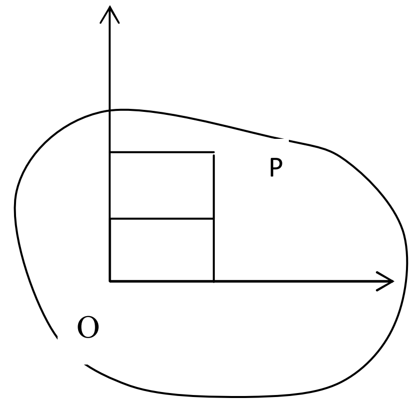
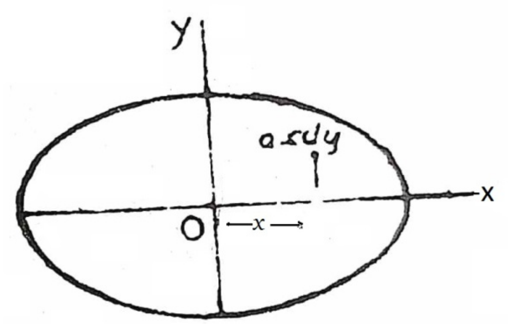


Fig. 1.2

Example- 02- *Show that at the centre of a quadrant of an ellips, the principal axes in its plane are inclined at an angle* $\frac{1}{2} tan^{-1}\left(\frac{4}{\pi}\frac{1b}{a^2 - b^2}\right)$ *to the axes.*

Sol. *The equation of the ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Consider an element $\delta x \delta y$ *surrounding the point* (x, y).

$$then \quad A = M.I. about\ OX = \int_0^a \int_0^{b/a\sqrt{(a^2-x^2)}} y^2 \rho \ \ dx \ \ dy$$

$$= \rho \int_0^a \left[\frac{y^3}{3}\right]_0^{b/a\sqrt{(a^2-x^2)}} dx$$

$$= \frac{1}{3}\rho \int_0^a \frac{b^3}{a^3}(a^2 - x^2)^{3/2}\, dx = \frac{1}{3}\frac{\rho b^3}{a^3}\int_0^{\pi/2} a^2 \ \ cos^2 \ \theta . \mathrm{a}\cos\theta \ \ d\theta.$$

$$[Put\ x = a\sin \ \theta, \quad dx = a \ \cos \ \ \theta \ \ \ d\theta]$$

$$= \frac{1}{3}\rho ab^3 \frac{3.1}{4.2}.\frac{\pi}{2} = \frac{1}{16}\rho a\pi b^3$$

$$=\frac{1}{4}Mb^2 \ \ where \ \ M = mass \ \ of \ the \ quadrant = \frac{\pi ab\rho}{4}.$$

$$Similarly \ \ B = M.I.\rho bout \ OY = \frac{1}{4}Ma^2$$

$$and \qquad B = P.I. \ about \ OX, \qquad OY$$

$$= \int_0^a \int_0^{b/a\sqrt{(a^2-x^2)}} xy.\rho dx \ \ dy = \frac{1}{2}\int_0^a x\,[y^2]_0^{b/a\sqrt{(a^2-x^2)}}$$

$$= \frac{\rho}{2}\int_0^a x\frac{b^2}{a^2}(a^2-x^2)dx = \frac{\rho}{2}.\frac{b^2}{a^2}\left[a^2\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^a$$

$$= \frac{\rho}{2}.\frac{b^2}{a^2}\frac{1}{2} \ \ a^2 = \ \frac{\rho}{8}a^2 \ b^2 = \frac{1}{2}\left(\frac{1}{4}\pi ab\rho\right)\frac{ab}{\pi} = \frac{M}{2\pi} \ \ ab.$$

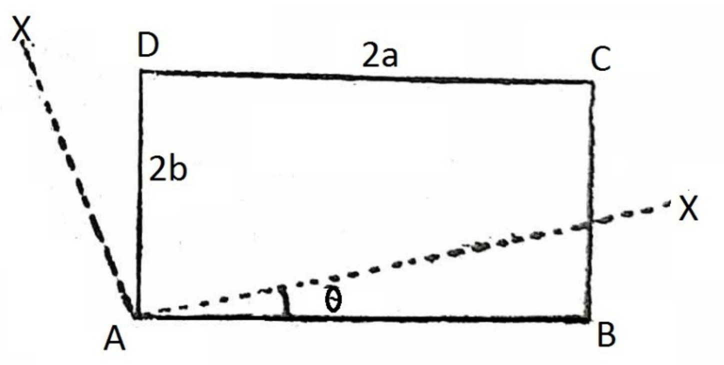
Now let θ *be the inclination of principal axis with* OX*, then*

$$\tan 2\theta = \frac{2F}{B-A} = \frac{(Mab)/\pi}{\frac{1}{4}M(a^2-b^2)} = \frac{4ab}{(a^2-b^2)\pi}$$

$$or \qquad \theta = \frac{1}{2}tan^{-1}\left(\frac{4}{\pi}.\frac{ab}{a^2-b^2}\right).$$

Example- 03

The lengths AB and CD of the sides of a rectangle ABCD are 2a and 2b, show that the inclination to AB of one of the pricipal axos at A is $\frac{1}{2} \ \ tan^{-1}\frac{3ab}{2(a^2-b^2)}$



Sol. Let AB and AD be the axes of $x \ and \ y$. Now B=M.I. of the rectangle about

$$AD = \frac{1}{2} \ \ Ma^2 + Ma^2 = \frac{4}{2}Mb^2.$$

$$Similarly \ A = \frac{4}{3}Mb^2.$$

$$F = Product\ of\ inertia\ about\ AB, AD = M.ab.$$

$Now\ if\ \theta is\ the\ inclination\ of\ one\ principal\ axis\ to\ AB, then\ we\ have$

$$\tan 2\theta = \frac{2F}{B-A} = \frac{2Mab}{\frac{4}{3}Ma^2 - \frac{4}{3}Mb^2} = \frac{3ab}{2(a^2-b^2)}$$

$$or \qquad \theta = \frac{1}{2}tan^{-1}\frac{2ab}{2(a^2-b^2)}$$

1.3.2 Moment of inertia of continuous, homogeneous structure: When the body is of continuous, homogeneous structure, the moment of inertia of body given in integration form as,

$$I = \int r^2 dm = \int r^2 \rho dV \qquad (1.5)$$

1.4 Moment of inertia some standard bodies:

1. Moment of inertia of a thin uniform rod.

(i) About an axis passing through the center of mass and perpendicular to its length.

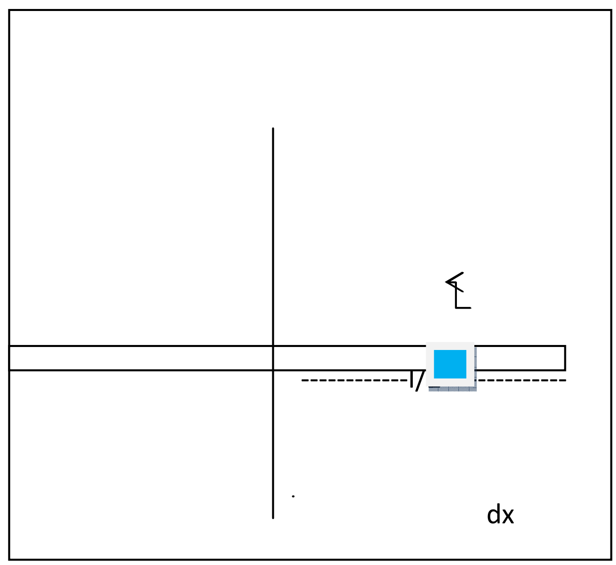


Fig. 1.3

Let PQ (Fig. 1.3) be a thin uniform rod of length $\boldsymbol{l}$ and mass M and AB the axes of rotation passing through the center of mass O of rod and perpendicular to its length.

Therefore, mass per unit length = M/*l*.

Consider a small element of thickness dx at a distance x from O. Hence the mass of the element is (M/*l*)dx and and its M.I. about the

$$x = -\frac{l}{2} \ \ at\ (P)\ and \ \ x = \frac{l}{2} \ \ at\ (Q)$$

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{M}{l} x^2 dx = \frac{2M}{l} \int_0^{\frac{l}{2}} x^2 dx$$

$$= \frac{2M}{l} \left[\left[\frac{x^3}{3} \right] \right]_0^{\frac{l}{2}} = \left[\frac{2M}{l} \frac{(\frac{l}{2})^3}{3} - 0 \right].$$

So, $$I = \frac{Ml^2}{12}. \qquad (1.6)$$

(ii) Moment of inertia about an axis passing one end of the rod and perpendicular to its length.

If *I* be the moment of inertia of the rod about the axis AB, then according to the theorem of parallel axes

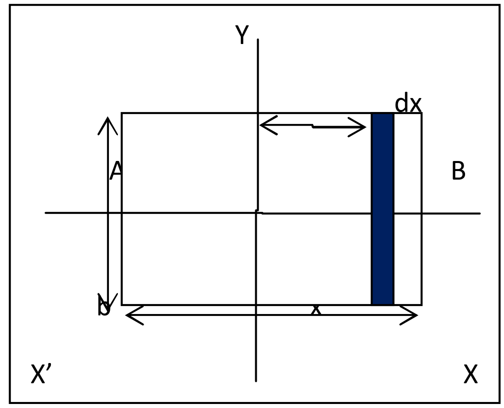
$$I = I_{cm} + Ma^2$$

$$I_{cm} = \frac{Ml^2}{12} \ \ and\ a = \frac{l}{2}$$

$$I = \frac{Ml^2}{12} + \frac{Ml^2}{4}\ or\ I = \frac{Ml^2}{3}. \qquad (1.7)$$

1. **Moment of inertia of a rectangular lamina.**

(ii)About an axis perpendicular to its plane and passing through its centre of mass.



The rectangular lamina of mass M and Length l and breadth b represented by ABCD. Let the axes YY' be an axis parallel to the side AD and passing through its center of gravity G. Consider a small strip of width dx , distant x from YY' and also parallel to it. Let σ be the mass per unit area of the lamina, then the mass of strip is equal to b.dx.σ.

Therefore, moment of inertia of the strip about YY',= $\sigma b.dx\ x^2$.

The moment of inertia of whole lamina about YY' is given by,

$$I_y = \int_{-\frac{l}{2}}^{\frac{l}{2}} \sigma b x^2 dx = 2\sigma b \int_0^{\frac{l}{2}} x^2 dx = 2\sigma b \left[\frac{x^3}{3}\right]_0^{\frac{l}{2}}$$

$$= \frac{2\sigma b l^3}{24} = \frac{Ml^2}{12}.$$

Similarly the moment of inertia of the lamina about the axis XX', an axis parallel to AB or CD and passing through its center of mass O, will be given by ,

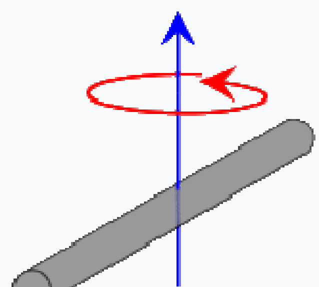
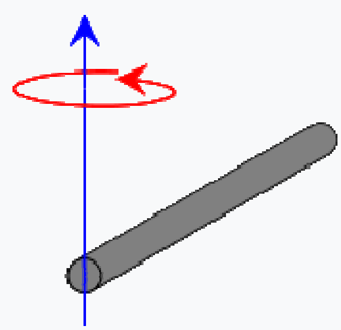
$$I_x = \frac{Mb^2}{12}.$$

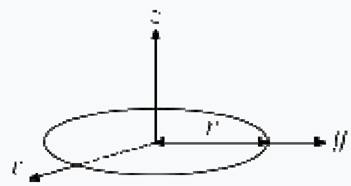
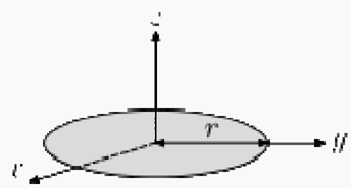
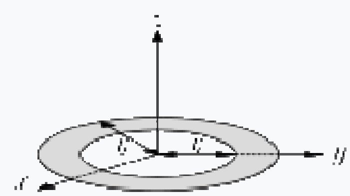
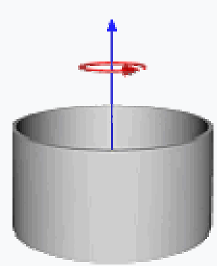
According to the perpendicular axes theorem , the moment of inertia of the rectangular lamina about an axis perpendicular to its plane and passing through its center of mass will be as,

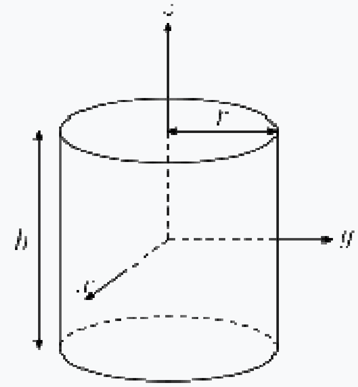
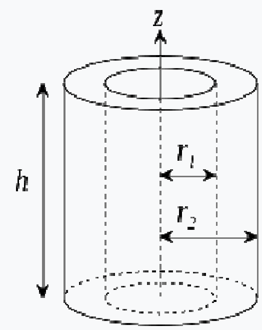
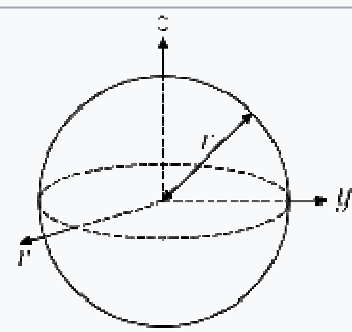
$$I = I_x + I_y$$

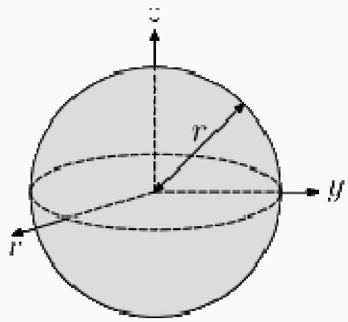
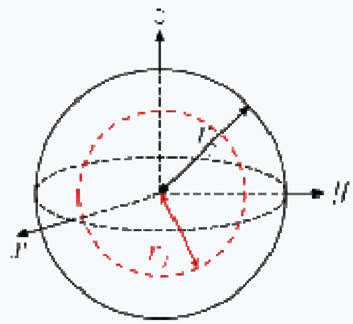
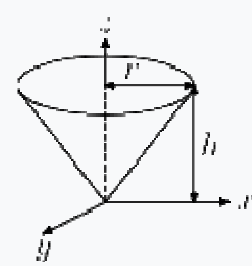
$$= \frac{Mb^2}{12} + \frac{Ml^2}{12} = M\left(\frac{b^2 + l^2}{12}\right). \qquad (1.8)$$

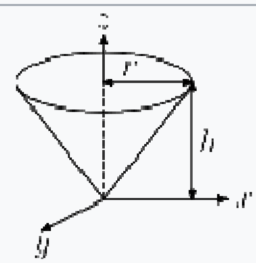
Table of Moment of Inertia

Thin rod of length L and mass m, perpendicular to the axis of rotation, rotating about its center. This expression assumes that the rod is an infinitely thin (but rigid) wire. This is a special case of the thin rectangular plate with axis of rotation at the center of the plate, with $w = L$ and $h = 0$.		$I = ML^2/12$
Thin rod of length L and mass m, perpendicular to the axis of rotation, rotating about one end. This expression assumes that the rod is an infinitely thin (but rigid) wire. This is also a special case of the thin rectangular plate with axis of rotation at the end of the plate, with $h = L$ and $w = 0$.		$\mathrm{I} = \frac{\mathrm{ML}^2}{3}$

Thin circular loop of radius r and mass m. This is a special case of a torus for $a = 0$ (see below), as well as of a thick-walled cylindrical tube with open ends, with $r_1 = r_2$ and $h = 0$.		$I_z = mr^2$ $I_x = I_y = \frac{1}{2}mr^2$
Thin, solid disk of radius r and mass m. This is a special case of the solid cylinder, with $h = 0$. That is a consequence of the perpendicular axis theorem.		$I_z = \frac{mr^2}{2}$ $I_x + I_y = \frac{1}{4}mr^2$
A uniform annulus (disk with a concentric hole) of mass m, inner radius r_1 and outer radius r_2		$I = \frac{m}{2}(r_1^2 + r_2^2)$
An annulus with a constant area density ρ_A		$I = \frac{1}{2}\pi\rho_{A(r_2^4 - r_1^4)}$
Thin cylindrical shell with open ends, of radius r and mass m. This expression assumes that the shell thickness is negligible. It is a special case of the thick-walled cylindrical tube for $r_1 = r_2$. Also, a point mass m at the end of a rod of length r has this same moment of inertia and the value r is called the radius of gyration.		$I = mr^2$

Solid cylinder of radius r, height h and mass m. This is a special case of the thick-walled cylindrical tube, with $r_1 = 0$.		$I_z = \frac{mr^2}{2}$ $I_x + I_z = \frac{1}{12}m(3r^2 - h^2)$
Thick-walled cylindrical tube with open ends, of inner radius r_1, outer radius r_2, length h and mass m.		$I_z = \frac{m}{2}(r_1^2 + r_2^2)$ $I_x = I_y = \frac{1}{12}m(3(r_2^2 + r_1^2) + h^2)$ The above formula is for the xy plane being at the middle of the cylinder. If the xy plane is at the base of the cylinder, then the following formula applies:
Hollow sphere of radius r and mass m.		$I = \frac{2}{3}mr^2$

Solid sphere (ball) of radius r and mass m.		$I = \frac{2}{5} m r^2$
Sphere (shell) of radius r_2 and mass m, with centered spherical cavity of radius r_1. When the cavity radius $r_1 = 0$, the object is a solid ball (above). When $r_1 = r_2$, , and the object is a hollow sphere.		$I = \frac{2}{5} m \left[\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right]$
Right circular cone with radius r, height h and mass m		$I_z = \frac{3}{10} m m^2$ About an axis passing through the tip: $I_x = I_y = m(\frac{3}{20} r^2 + \frac{3}{5} h^2)$ About an axis passing through the base: $I_x = I_y = m\left(\frac{3}{20} r^2 + \frac{1}{10} h^2\right)$ About an axis passing through the center of mass:

		$I_x = I_y = m(\frac{3}{20}r^2 + \frac{3}{80}h^2)$
Right circular hollow cone with radius *r*, height *h* and mass *m*		$I_z = \frac{1}{2}mr^2$ $I_x = I_y = \frac{1}{4}m(r^2 + 2h^2)$

Product of inertia about co ordinate axes:

For a particle system;

Consider a particle P of mass, ‘m’ having the co ordinate (x,y,z)

Here the perpendicular distance of particle P of mass ‘m’ from z-axis is given by d.

So $$d = \sqrt{x^2 + y^2}.$$

Hence moment of inertia of particle P about z-axis,

$$I = md^2 = m(x^2 + y^2)$$

M.I. of system of particle about z-axis,

$$I_{oz} = m(x^2 + y^2).$$

and standard notation of moment of inertia about z-axis is C.

Similarly moment of inertia about x-axis and y-axis are denoted as 'A' and 'B' respectively given as;

M.I. about x-axis $I_{ox} = m(y^2 + z^2) = A$,

M.I. about y-axis $I_{oy} = x^2 + z^2 = B$.

Product of inertia;

The product of inertia of a particle system defined as,

$$D = \Sigma myz$$

$$E = \Sigma mxz$$

$$F = \Sigma mxy$$

The quantizes 'D', 'E' ,and 'F' are called product of inertia w.r.t. the pair of axes (oy, oz), (ox, oz), and (ox, oy) respectively.

Parallel axis theorem for Product of Inertia:

Statement: For body of mass 'm', the co ordinate of center of mass $(\bar{x}, \bar{y}, \bar{z})$ w.r.t. co ordinate system ox, oy and oz. The co ordinate of particle P are (x,y,z) and (x', y' z') respectively the x,y,z-axis and x' ,y' ,z'-axis. Then product of inertia F= F' +M$\bar{x}\bar{y}$.

Proof:

Here product of inertia w.r.t. (ox, oy) is

$$F = \Sigma xy = \Sigma \mathrm{m}(\bar{x} + x')(\bar{y} + y')$$

$$= \Sigma m(\bar{x}\bar{y} + \bar{x}y' + x'\bar{y} + x'y')$$

$$= \bar{x}\bar{y}\Sigma m + \bar{x}\Sigma \mathrm{my'} + \bar{\mathrm{y}}\Sigma \mathrm{mx'} + \Sigma \mathrm{mx'y'}$$

$$= F' + M\bar{x}\bar{y} + 0$$

Where $\bar{x}\Sigma my' = \bar{y}\Sigma mx' = 0$.

$$\Rightarrow F = F' + M\bar{x}\bar{y} \qquad (1.9)$$

Similarly product of inertia w.r.t. (oy,oz) and (ox,oz) respectively given by,

$D = D' + M\bar{y}\bar{z}$ and $E = E' + M\bar{x}\bar{z}$

Principal axes:

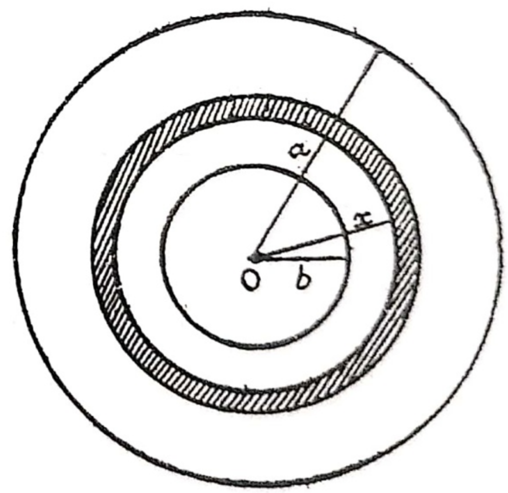
The principal axes of a rigid body are the lines along that the product of inertia of pair of line are vanish. If OX, OY and OZ are principal of axes of a rigid body then it can be find by applying the condition,

$$\Sigma mxy = 0, \Sigma mxz = 0, and\ \Sigma myz = 0.$$

1.5 Solve Example:

Example-1 *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being* a and b.

Sol. Let a and b be the external and internal radii of a hollow sphere of mass M. Consider a shell of width at a distance x from O .



$\therefore$ *Moment of inertia of this shell about a diameter*

$= \frac{2}{3}$*(Mass of the shell)*

$$\times (radius\ of\ the\ shell)^2$$

$= \frac{2}{3} 4\pi x^2 p\delta x^2 . x^2$

$$= \frac{8\pi\rho}{3} x^2 \delta x.$$

$\therefore$ *M.I. of the hollow sphere about a diameter*

$$= \int_b^a \frac{8\pi\rho}{3} x^4 \quad dx = \frac{8\pi\rho}{3} \int_b^a x^4 \; dx$$

$$= \frac{8\pi\rho}{3} \left[\frac{x^b}{5}\right]_b^a = \frac{8}{15}\pi\rho\left(a^\delta - b^\delta\right)$$

$$= \frac{8}{15\pi} \frac{3M}{4\pi(a^3 - b^3)} . \left(a^\delta - b^\delta\right)$$

$$\left[\because \qquad M = \frac{4}{8}\pi a^2 - \frac{4}{3}\pi(a^3 - b^3)\right]$$

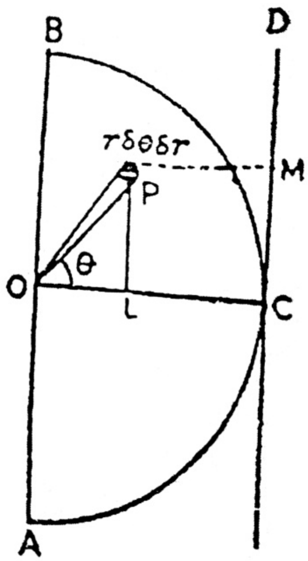
$$= \frac{2M}{5} \frac{a^\delta - b^\delta}{a^3 - b^3}$$

Example-2 *Show that the moment of inertia of a seml - clrcular lamina about a tangent parallel to the bounding diameter is*

$Ma^2\left(\frac{5}{8} - \frac{8}{3\pi}\right)$ *where a is the radius and Miss the mass of the lamina.*

Sol. *Consider a semi- circular lamina ACB with its bounding diameter as AB. Let CD be a tangent parallel to AB. Take an elementary area* $r\delta\theta\delta r$ *at* P

where OP=r and $\angle POC = \theta$.



Then distance of this elementary area from the tangent CD

=PM=LC

=OC-OL=a-r

M.I. of this element about CD

=mass$\times$ $(distance)^2$

=$r\delta\theta\delta r\rho.(a - r\cos\theta^2)$

=M.I. of the semi-circular lamina about CD

$$\int_{-\pi/2}^{\pi/2}\int_0^a r(a - r\ \cos\theta)^2$$

$$= 2\rho\int_0^{\pi/2}\int_0^a r(a - r\cos\theta)^2\,drd\theta$$

$$= 2\rho\int_0^{\pi/2}\int_0^a (a^2r - 2ar^2\cos\theta + r^2\cos^2\theta)^2\,drd\theta$$

$$= 2\rho \int_0^{\pi/2} \left\{ a^2 \left[\frac{r^2}{2}\right]_0^a - 2a\cos\theta \left[\frac{r^3}{2}\right]_0^a + cos^2\theta \left[\frac{r^4}{4}\right]_0^a \right\} d\theta$$

$$= 2p \int_0^{\pi/8} \left(\frac{a^4}{2} - \frac{2a^4}{3}\cos\theta + \frac{a^4}{4} - \cos^2\theta \right) d\theta$$

$$2\rho \left[\frac{a^4}{2} \cdot \frac{\pi}{2} - \frac{2a^4}{3} + \frac{a^4}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] = 2p \left[\frac{\pi a^2}{4} - \frac{2a^4}{3} + \frac{\pi a^4}{16} \right]$$

moment of inertia

$$= 2\rho a^4 \left[\frac{5^\pi}{16} - \frac{2}{3} \right] = \frac{1}{2}\pi\rho a^3 . a^2 \left[\frac{5}{4} - \frac{8}{3\pi} \right] = Ma^2 \left[\frac{5}{8} - \frac{8}{3\pi} \right]$$

$$\left[\because \quad M = \frac{1}{2}\pi a^2 \rho \right]$$

Example-3 *Show that the moment of inertia of the area bounded by* $r^2 = a^2 \cos 2\theta$*about its axis is* $\frac{Ma^2}{16}\left(\pi - \frac{8}{3}\right)$.

$\boldsymbol{Sol.} r^2 = a^2 \cos 2\theta$

Let OX be the axis of the curve. This curve consists of two loops. The limits of θ *for one loop are* $-\frac{\pi}{4}$ to $\frac{\pi}{4}$*. consider an elementary arer*$\delta\theta\delta r$*. It* ρ *is the mass per unit area, then mass of this element*

$= \rho r\delta\theta \;\; \theta r.$

Mass M of the whole area (both loops)

$$M = 2\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} \rho r \, d\theta \, dr$$

$$= 4\rho\int_{0}^{\pi/4}\left[\frac{r^2}{2}\right]_{o}^{a\sqrt{(\cos 2\theta)}} d\theta = 2\rho d^3\int_{4}^{\pi/4}\cos 2\theta \, d\theta$$

$$= 2\rho a^2\left[\frac{\sin 2\theta}{2}\right]_{0}^{\pi/4} = \rho a^2. \quad \therefore \quad \rho = M/a^2. \qquad \dots(1)$$

$$Distance\ of\ the\ element\ r\ \delta\theta\ \delta r\ from\ OX = r\sin\theta.$$

$$M.I\ of\ the\ elementary\ area\ about\ OX$$
$$= (r\ \sin\theta)^2.\rho r\ \delta\theta\ \delta r.$$

$\therefore M.I. of\ whole\ area\ about\ OX$

$$= 2\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta \, d\theta \, dr$$

$$= 4\rho\int_{0}^{\pi/4}\left[\frac{r^4}{4}\right]_{o}^{a\sqrt{(\cos 2\theta)}} .\sin^2\theta \, d\theta$$

$$= \rho a^4\int_{0}^{\pi/4}\cos^2 2\theta \ \sin^2\theta \, d\theta = \frac{\rho a^4}{2}\int_{4}^{\pi/4}\cos^2 2\theta . 2\sin^2\theta \ d\theta$$

$$= \frac{\rho a^4}{2}\int_{4}^{\pi/4}\cos^2 2\theta . (1-\cos 2\theta) \ d\theta$$

$$= \frac{\rho a^4}{4}\int_4^{\pi/2} \cos^2 t(1-\cos\ t)dt \qquad \left[put\ 2\theta = t, d\theta = \frac{1}{2}dt\right]$$

$$= \frac{\rho a^4}{4}\left[\frac{\pi}{4}-\frac{2}{3}\right] = \frac{Ma^2}{4}\left[\frac{\pi}{4}-\frac{2}{3}\right] = \frac{Ma^2}{16}\left[\pi-\frac{8}{3}\right].$$

Example-4.

Consider a rectangular parallelepiped of uniform density ρ, mass M with sides a, b and c. For origin O at one corner, find product of inertia of the parallelepiped by taking the coordinate axes along the edges. Also in case cube.

Solution:

We have the side of the parallelepiped a, b and c. Then the product of inertia will be,

$$I_{xy} = -\int_0^c\int_0^b\int_0^a \rho\ x\ y\ dx\ dy\ dz = -\rho c\int_0^b\int_0^a xy\ dx\ dy$$

$$= -\rho c\frac{a^2}{2}\frac{b^2}{2} = -\frac{M}{4}ab = I_{yx}.$$

Similarly,

$$I_{xz} = I_{zx} = -\frac{1}{4}Mac$$

$$I_{vz} = I_{zy} = -\frac{1}{4}Mbc.$$

If the object is cube then a = b = c. So,

$$I_{xy} = -\frac{1}{4}Ma^2 = I_{yx} = I_{zx} = I_{xz} = I_{yz} = I_{zy}.$$

1.6 Momental Ellipsiod :

We know that the moment if inertia, I_{OL} about line whose d. c, are $< \lambda, \mu, \nu >$ is

$$I_{OL} = I = A\lambda^2 + B\mu^2 + C\nu^2 - 2D\mu\nu - 2E\nu\lambda - 2F\lambda\mu \quad (2.0)$$

Let P(x,y,z) any point on OL and OP = R then R =($\lambda i, \mu j, \nu k$) =(xi,yj,zk)

$$\Rightarrow \lambda = \frac{x}{R}, \mu = \frac{y}{R}, \nu = \frac{z}{R} \quad (2.1)$$

If P moves such a way that IR^2 is constant then from (2.0) and (2.1)

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = IR^2 = constant$$

The coefficients A, B, C, are positive hence this equation represent the equation of ellipsoid known as momental ellipsoid.

1.7 Solved Example:

Example-1.

Find the two principal axes when one principal axis is given.

Solution:

Let O is point in a body. The line OZ perpendicular to the plane of paper and principal axis of body. Let OX' and OY' two principal axis of body at point O.

Let P be any point (x, y) or (x', y') and body places above and below plane of paper. The line OX' and OY' will be principal axis if $\Sigma mx'y' = 0$. Considering angle between two co-ordinate system θ. Then,

$$x' = x\cos\theta + y\sin\theta \;\; and \;\; y' = -x\sin\theta + y\cos\theta.$$

For principal axis,

$$\Sigma m\left(-x^2\cos\theta\sin\theta + y^2\sin\theta\cos\theta + xy\overline{\cos\theta^2 - \sin\theta^2}\right) = 0,$$

Which becomes,

$$\tan 2\theta = \frac{2\Sigma mxy}{\Sigma mx^2 - \Sigma my^2} = \frac{2F}{B - A}.$$

for this θ, OX' and OY' represents principal axes.

Example-2.

To construct a momental ellipsoid at one of the corner of a cube.

Solution:

Taking the edge as axes, A= B =C, D= E = F, and the equation of momental ellipsoid becomes,

$$A(x^2 + y^2 + z^2) - 2D(xy + yz + zx) = c.$$

Which on transformation would give a spheroid of the form,

$$A'x^2 + B'(y^2 + z^2) = c'.$$

It can be seen that one principal axis is the diagonal through the corner of cube, and any two lines at the right angles to one other and to the diagonal will be the other two principal axes.

Example-3.

To find the momental ellipsoid at a point on the edges of a right circular cone.

Solution:

Let OX, OY, and OZ are the axes in space and a right circular cone AOBC situated on xy plane with A is vertex and B is centre of base of cone on X-axis. By inspection it found that $D = F = 0$, and axis OY is one principal axis.

Then if,

$$AB = a, \qquad OB = b; BG = \frac{1}{4}a,$$

and

$$A = M\left(\frac{3}{20}b^2 + \frac{a^2}{10}\right), \qquad B = A + Mb^2, C = M\frac{13}{20}b^2, E = M\frac{ab}{4}.$$

So the equation of momental ellipsoid at O is,

$$(3b^2 + \ 2a^2)x^2 + (23b^2 \ + 2a^2)y^2 + 26b^2z^2 - 10abxz = c.$$

The momental ellipsoid at at the point A or any point along the axis AB is spheroid.

1.8 Summary.

Definition of moment of inertia, principal axis and product of inertia discussed. The parallel axis theorem $I = I_{cm} + Ma^2$ and perpendicular axis theorem $I = I_x + I_y$ has proven. The derivation of moment of inertia of continuous and homogeneous structure as, uniform rod and rectangular lamina done. Parallel theorem of product of moment $F = F' + M\bar{x}\bar{y}$ has been proven. Definition and explanation of momental ellipsoid also discuss in this chapter. The example with solution given to each topic for understanding concepts.

1.9 Terminal question.

Q.1. Find the principal axes at any point of a square or a rectangular plate.

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Q.2. Find the principal axes at any point within a cube or a rectangular parallelepiped.

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Q.3. Show that, the momental ellipsoid at the center of an elliptic plate is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2=c.$$

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Q.4. Show that, the momental ellipsoid at the center of a solid ellipsoid is,

$$(b^2+c^2)x^2+(a^2+c^2)y^2+(a^2+b^2)z^2=c'.$$

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Q.5. Find the moment of inertia and product s of inertia of a uniform square plate of length a and mass M about X, Y, Z axes.

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Q.6. Find moment of inertia and products of inertia of a uniform solid sphere $x^2+y^2+z^2=a^2$ in the first octant.

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Q. 7. Find the moment of inertia of a disc of mass 3 kg and radius 50 cm about the following axes.

(i) axis passing through the center and perpendicular to the plane of the disc,

(ii) axis touching the edge and perpendicular to the plane of the disc and (iii)axis passing through the center and lying on the plane of the disc.

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Q.8 Determine the moments and product of inertia of the area of the square with respect to the x'- y' axes.

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Unit-2. D'Alembert Principle

Structure:

2.1 Introduction.

The unit contains general equation of motion in three dimension space. The motion of centre of moment of inertia and motion relative to the center of inertia are discussed.

2.2 Objectives.

After reading this unit students should be able to:

1.Understand the general equation of motion in general dimensions.

2.Understand D'Alembert Principle.

3.To learn the equation of motion of moment of inertia.

4. To learn motion relative to the center of inertia.

5. To able to solve the problems related to D'Alembert Principle.

2.3 The general equation of motion:

The equation of motion of a particle or system of particle is describe the path of particle or system of particle on the basis of given system of forces, or derived the system of forces or force when particle or system of particle moves along given path.

Now we formulate the rotational mechanics to a system of particles;

i) By Newton's Second Law of motion for a system of particle given as,

$$m_i\ddot{x}_i = X_i, \quad m_i\ddot{y}_i = Y_i, \quad m_i\ddot{z}_i = Z_i, \qquad i = 1, 2, \dots\dots\dots\dots\dots..n,: \qquad (2.1)$$

ii) The equation (2.1) represent 3n relations between the 6n+1 variable $(x_i, y_i, z_{i,}X_i, Y_i, Z_i, t)$.

The system of equation (2.1) can be solved for X_i, Y_i, Z_i as the function of x_i, y_i, z_i and its derivative as,

$$\begin{cases} X_i = \varphi_j(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i, t) \\ Y_i = \psi_i(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i, t) \\ Z_i = \Omega_i(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i, t) \end{cases} \qquad (2.2)$$

So when path and velocity of particle is given then the forces can be calculated and converse also true by equation (2.2)

When these two result is obtaining the *constraints* occurs whice is eliminated by D'Alembert principle.

Let $\delta x_i, \delta y_i, \delta z_i$be any 3n quantities whatsoever. Then it got the equation,

$$\sum_{i=1}^{n}(m_i\ddot{x}_i - X_i)\delta x_i + \; (m_i\ddot{y}_i - Y_i)\delta y_i + (m_i\ddot{z}_i - Z_i)\delta z_i = 0. \tag{2.3}$$

This equation is called general equation of motion.

D'Alembert Principle:

When we sum of all the couples produced by external forces and by effective forces reversed, we must obtain equilibrium.

According to Newton's second law of motion, the force acting on the ith particle,

$$F_i = \frac{dp_i}{dt} = \dot{p}_{i,}$$

It given as,

$$F_i - \dot{p}_i = 0, \quad i = 1, 2, 3, \dots \dots \dots .., N$$

These equations indicate that nay particle in the system is in equilibrium under a force, which is equal to the actual force F_i and a reverse effective force $\dot{p}_i$. If virtual displacement δr_i, then

$$\sum_{i=1}^{N}(F_i - \dot{p}_i) \cdot \delta r_i = 0,$$

But $F_i = F_i^a + f_i$, then,

$$\sum_{i=1}^{N}(F_i^a - \dot{p}_i) \cdot \delta r_i + \sum_{i=1}^{N} f_i \cdot \delta r_i = 0.$$

The virtual work of system is zero so, $\sum_{i=1}^{N} f_i \cdot \delta r_i = 0$, then

$$\sum_{i=1}^{N}(F_i^a - \dot{p}_i) \cdot \delta r_i = 0. \qquad (2.4)$$

This is known as D'Alembert's principle.

Example.

Two heavy particle of weight $\mathbf{W_1}$ and $\mathbf{W_2}$ are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radius R, the axis of which is horizontal. Find the condition of equilibrium of the system by applying the principle of virtual work.

Solution.

According to the principle of virtual work,

$$\sum_{i=1}^{N}(F_i \cdot \delta r_i) = 0,$$

For i = 1, 2 , here

$$W_1 \sin\theta\, \delta r_1 + W_2 \sin\varphi\, \delta r_2 = 0,$$

But $\delta r_1 = Rd\theta, and\ \delta r_2 = Rd\varphi,$

Therefore

$$W_1 \sin\theta\, \delta\theta + W_2 \sin\varphi\, \delta\varphi = 0,$$

Here θ +φ = constant, therefore

$$\delta\theta + \delta\varphi = 0, or\ \delta\theta = -\delta\varphi,$$

Thus

$$W_1 \sin\theta - W_2 \sin\varphi = 0.$$

The system is in equilibrium, so condition will be (δθ≠0),

$$\frac{W_1}{W_2} = \frac{\sin\varphi}{\sin\theta}.$$

2.4 Motion of center of inertia and Motion relative to center of inertia:

When the impulsive forces are applied on system of particles then the equation of motion describe by equation (after appling D'Alembert principle),

$$\sum m\ddot{x} = \sum X,$$

for finite force it will become,

$$\sum m\{\dot{x}' - \dot{x}\} = \sum X,$$

for impulsive force, where the velocity of each particle of mass m changed from $\dot{x}$ to $\dot{x}'$ by action of impulsive force X.

Lagrange's equations from D'Alembert's principle:

Now consider a system of N particles. The transform equations for the position vectors of the particles are,

$$r_i = r_i(q_1, q_2, \dots \dots q_k, \dots \dots q_n,\ t\,), \quad (2.5)$$

Where t is the time and q_k (k=1, 2, ……..,n) are the generalized coordinates.

Differentiating equation (2.5) with respect to t, we get,

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \dots . + \frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt} + \cdots \dots . + \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}.$$

$$v_i = \dot{r}_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}. \qquad (2.6)$$

Where $\dot{q}_k$ are generalized velocities.

The virtual displacement is given by,

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \cdots \dots . + \frac{\partial r_i}{\partial q_k}\delta q_k + \cdots \dots . + \frac{\partial r_i}{\partial q_n}\delta q_n,$$

$$\delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k. \qquad (2.7)$$

According to D'Alembert's principle,

$$\sum_{i=1}^{N}(F_i - \dot{p}_i)\cdot\delta r_i = 0, \quad (2.8)$$

$$\sum_{i=1}^{N} F_i\cdot\delta r_i = \sum_{i=1}^{N} F_i\cdot\sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[F_i\cdot\frac{\partial r_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{n} G_k\delta q_k. \quad (2.9)$$

Where G_k are called the components of generalized force associated with general coordinates q_k.

Next

$$\sum_{i=1}^{N}\dot{p}_i\cdot\delta r_i = \sum_{i=1}^{N} m_i\ddot{r}_i\cdot\sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\delta q_k$$

$$= \sum_{k=1}^{n}\left[\sum_{i=1}^{N} m_i\ddot{r}_i\cdot\frac{\partial r_i}{\partial q_k}\right]\delta q_k. \quad (2.10)$$

Now

$$\sum_{i=1}^{N} m_i\ddot{r}_i\cdot\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{r}_i\cdot\frac{\partial r_i}{\partial q_k}\right) - m_i\dot{r}_i\cdot\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)\right]. \quad (2.11)$$

Therefore from equation (2.10),

$$\sum_{i=1}^{N} m_i\ddot{r}_i\cdot\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left[m_i v_i\cdot\frac{\partial v_i}{\partial q_k}\right] - m_i v_i\cdot\frac{\partial v_i}{\partial q_k}\right]. \quad (2.12)$$

Now put the value in equation (2.9),

$$\sum_{i=1}^{N}\dot{p}_i\cdot\delta r_i = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i v_i\cdot\frac{\partial v_i}{\partial q_k}\right) - m_i v_i\cdot\frac{\partial v_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{q}_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(v_i\cdot v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\frac{1}{2}m_i(v_i\cdot v_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k. \quad (2.13)$$

Substituting the value of $\sum_{i=1}^{N} F_i\cdot\delta r_i$ from (2.9) and $\sum_{i=1}^{N}\dot{p}_i\cdot\delta r_i$ from (2.13) in the equation (2.8), then D'Alembert's principle becomes,

$$\sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}-G_k\right]\delta q_k=0, \qquad (2.14)$$

As the equation (2.14) are holonomic, it means that any virtual displacement δq_k is independent of δq_j.

Therefore the coefficients in the square bracket for each δq_k must be zero.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}-G_k=0,$$

Or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=G_k. \qquad (2.15)$$

This represents the general form of Lagrange's equations.

2.5 Solved Example:

Example-1.

A rough uniform board, of length 2a and mass m, rest on a smooth horizontal plane. A man of mass M walks from one end to other. Determine the motion.

Solution:

The motion of the center of inertia of the system will be same as if we apply all the external forces to the system, acting in its proper direction. All the forces at the center of inertia are downwards, and as the center of inertia cannot moves to downwards. It must therefore be at rest. When man walks along board, he will therefore moves relatively to the fixed horizontal plane to a distance $\frac{2ma}{M+m}$ and board will recede to a distance $\frac{2Ma}{M+m}$.

Now, The motion is performed along horizontal direction, while there are no horizontal external forces to the system, so equation of motion,

$$\sum m\ddot{x}=0,$$

$$\therefore \sum M\ddot{x} = 0,$$

$$\therefore \dot{\bar{x}} = 0 \; or \; constant.$$

The man and board start from rest as we have supposed, then

$$\dot{\bar{x}} = 0,$$

$$\therefore \bar{x} = constant.$$

It means the position of the center of inertia remains unaltered throughout the motion of two part of the system.

Example-2.

Obtain the equation of motion of a simple pendulum by using Lagrangian methods and deduce the formula for time period for small amplitude oscillations.

Solution.

Let θ be the angular displacement of the simple pendulum from the equilibrium position. If *l* be the effective length of the pendulum and m be the mass of the bob, then the displacement along arc OA =s is given by,

$$s = l\theta,$$

Kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2.$$

If the potential energy of the system, when the bob is at O (middle position), is zero, then the potential energy , when the bob is at A, (end position), is given by

$$V = mg(l - l\cos\theta) = mgl(1 - \cos\theta),$$

Hence

$$L = T - V \; or \; L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta).$$

Now $\frac{\partial L}{\partial \theta} = -mgl\sin\theta \;\; and \;\; \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

Substitute these values in the Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

We get,

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] + mgl\sin\theta = 0, or\; ml^2\ddot{\theta} + mgl\sin\theta = 0.$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This represent the equation of motion of a simple pendulum.

For small amplitude oscillations, $\sin\theta \cong \theta$, and therefore the equation of motion of a simple pendulum is,

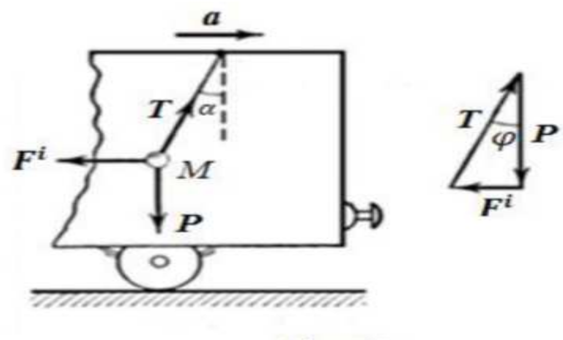
$$\ddot{\theta} + \frac{g}{l}\theta = 0.$$

This represents the simple harmonic motion of period, given by,

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Example-3

When a train accelerates, a load attached to a string hanging from the ceiling of a carriage deflects by an angle α from the vertical (figure 1). Determine the acceleration of the carriage.



Solution.

Acting on the load is the force of gravity *P* and the reaction of the thread *T*. Applying D'Alembert's principle, add to these forces the inertia force F^i directed to the opposite of acceleration a of the carriage. Magnitude of the inertia force $F^i = ma = \frac{P}{g}a$. The forces *P, T* and F^i are balanced. Constructing a closed force triangle and taking into account that $\varphi = \alpha$, we get

$$F^i = P\tan\alpha \text{ or } \frac{P}{g}a = P\tan\alpha.$$

Hence, the acceleration of the carriage is $a = g\tan\alpha$.

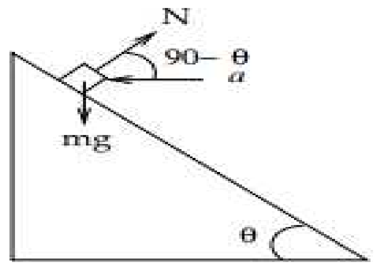
Example-4

Consider a mass resting on a frictionless incline. The mass slides down with an acceleration when released. A horizontal acceleration is applied on the mass to keep the mass from sliding. Find the acceleration.

Solution:

The problem is reasonably simple and can be solved by Newton's law also. From figure 2, we have

$$N = mg\cos\theta + ma\sin\theta\,,\ ma\cos\theta = mg\sin\theta$$



which gives, $a = g\tan\theta$.

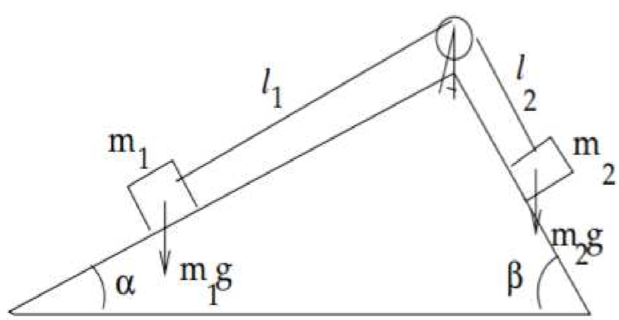
Now, let us solve the problem by applying D'Alembert's principle. Suppose the mass has an instantaneous (virtual) displacement δl along the incline. We have $\delta x = \delta l\cos\theta$ and $\delta y = -\delta l\sin\theta$. The only applied force is mg along the –y axis. Therefore, $\vec{F} = -mg\hat{y}$. From the principle of virtual work it follows that

$$F_x\delta x + F_y\delta y - ma_x\delta x - ma_y\delta y = 0\,.$$

We need to apply only horizontal acceleration so that we have, $a_y = 0$. Since $F_x = 0$, we get $mg\delta y - ma_x\delta x = 0$ i.e., $mg\delta l\sin\theta - ma\delta l\cos\theta = 0$. Thus, we get the acceleration $a = g\tan\theta$.

Example-5

Consider the arrangement shown in the figure. The pulley is fixed on the fixed wedge. Find the acceleration of the masses when released.



Solution. From D'Alembert's Principle, we have

$$(\overrightarrow{F_1}-\overrightarrow{p_1}).\delta\overrightarrow{l_1}+(\overrightarrow{F_2}-\overrightarrow{p_2}).\delta\overrightarrow{l_2}=0. \qquad (1)$$

Since $l_1 + l_2$ = constant, we have

$$\delta l_1 = -\delta l_2 \text{ and } \ddot{\imath}_1 = \ddot{\imath}_2. \qquad (2)$$

The inertial forces are $\dot{p}_1 = m_1\ddot{\imath}_1$ and $\dot{p}_2 = m_2\ddot{\imath}_2 = -m_2\ddot{\imath}_1$, and the only applied forces the weight of the masses. Taking the components of (1) along the incline, we have

$$(m_1 g \sin\alpha - m_1\ddot{\imath}_1)\delta l_1 + (m_2 g \sin\beta - m_2\ddot{\imath}_2)\delta l_2 = 0.$$

Using (2), we get

$$(m_1 g \sin\alpha - m_1\ddot{\imath}_1 - m_2 g \sin\beta - m_2\ddot{\imath}_1)\delta l_1 = 0$$

so that

$$\ddot{\imath}_1 = \frac{m_1 g \sin\alpha - m_2 g \sin\beta}{m_1 + m_2}.$$

Example. 6.

Two equal bars of length *l* and weight *p* each are welded at right angle to a vertical shaft of length *b* at distance h from each other. Determine the dynamical pressures acting on the shaft if it rotates with a constant angular velocity ω.

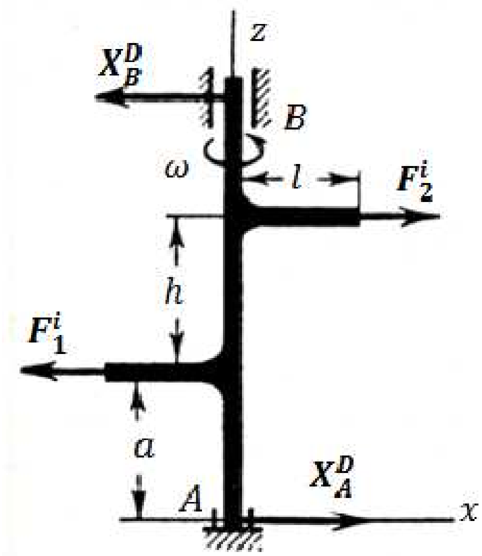
Solution. The centrifugal inertia forces in each rod are equal in mangnitude:

$$F_1^i = F_2^i = \frac{P}{2g}\omega^2, \qquad (1)$$

They make a couple which, apparently, is balance by the couple $\boldsymbol{X}_A^D, X_B^D$. The moments of these couples are equal in magnitude. Consequently, $X_A^D b = F_1^i h$, whence,

$$X_A^D = X_B^D = \frac{F_1^i h}{b} = \frac{plh}{2gb}\omega^2. \qquad (2)$$

The couple is continuously in the A_{xz} plane, which rotates with the body.



Example. 7. Find the relation between the moment **M** of the couple acting on the crankshaft mechanism in Fig. and pressure P on the piston when the system is in equilibrium. The crank is of length OA=r and the connecting rod of length AB=*l*.

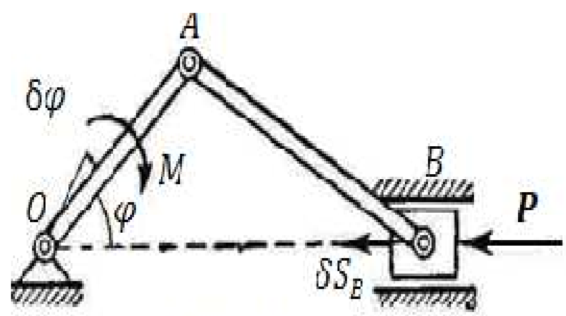


Fig. 85

Solution. Equilibrium conditions give,

$M\delta\varphi - P\delta s_B = 0, or\ M\omega_{OA} = Pv_B,$ (1)

Since $\delta\varphi = \omega_{OA}dt$ and $\delta s_B = \omega_{OA}$ can be found by the methods of kinematics:

$v_B = \omega_{OA} r\left(1 + \frac{r\cos\varphi}{\sqrt{l^2 - r^2\sin.^2\varphi}}\right)\sin\varphi.$ (2)

Referring to this result, we find

$M = Pr\left(1 + \frac{r\cos\varphi}{\sqrt{l^2 - r^2\sin.^2\varphi}}\right)\sin\varphi.$ (3).

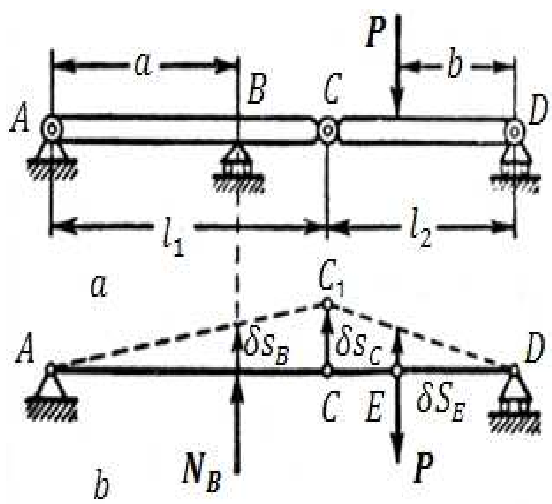
Example. 8. Two beams are hinged together at C and loaded as shown in Fig. Neglecting the weight of the beams, determine the pressure on support B.

Solution. Replace the support at B by a force N_B, which is equal in magnitude to the required pressure. For a virtual displacement of the system gives,

$N_B\delta S_B - P\delta S_E = 0.$ (1)

The relation between δS_B and δS_E is found from the proportions

$\frac{\delta s_B}{a} = \frac{\delta s_C}{l_1}; \frac{\delta S_E}{b} = \frac{\delta s_C}{l_2},$ (2)



Whence

$$\delta S_E = \frac{bl_1}{al_2}\delta S_{B,}$$

and consequently,

$$N_B = \frac{bl_1}{al_2}P.$$

Example. 9. Determine the relation between forces Q and P at which the press is in equilibrium if angles α and β are known. Neglect the weight of the rods.

Solution. Placing the origin of a coordinate system in the fixed point A and drawing the x axes as shown,

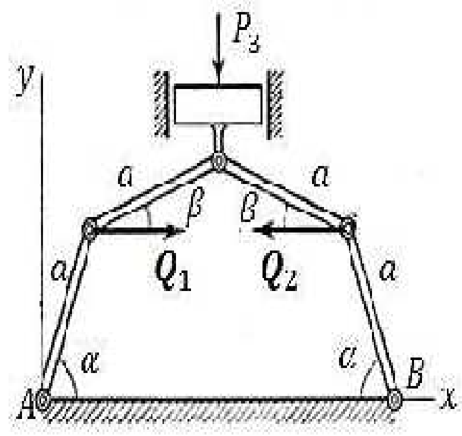


Fig. 90

We obtain

$Q_{1x}\delta x_1 + Q_{2x}\delta x_2 + Q_{3y}\delta y_3 = 0,$ (1)

Since all the other projections of the forces vanish.

To find δx_1, δx_2, δy_3 compute the coordinates x_1, x_2, y_3 of the points of application of the forces, expressing them in terms of the angle α and β. Denoting the length of each rod by a, we obtain

$$x_1 = a\cos\alpha\,, x_2 = a\cos\alpha + 2a\cos\beta\,, y_3 = a(\sin\beta + \sin\alpha),$$

Differentiating which we find,

$$\delta x_1 = -a\sin\alpha\,\delta\alpha, \delta x_2 = -a(\sin\alpha\,\delta\alpha + 2\sin\beta\,\delta\beta),$$

$$\delta y_3 = a(\cos\beta\,\delta\beta + \cos\alpha\,\delta\alpha).$$

Substituting these expressions and taking into account that

$Q_{1x} = Q, Q_{2x} = -Q$, and $P_{3y} - P$, we have,

$$2Q\sin\beta\,\delta\beta - P(\cos\beta\,\delta\beta + \cos\alpha\,\delta\alpha) = 0.$$

To find the relation between δα and δβ we make use of the fact that AB= *constant.* Therefore, $2a(\cos\alpha + \cos\beta) = constant$. Differentiating this equation, we obtain

$\sin\alpha\,\delta\alpha + \sin\beta\,\delta\beta = 0$ and $\delta\alpha = -\frac{\sin\beta}{\sin\alpha}\delta\beta$.

Substituting the expression for δα, we have,

$$2Q\sin\beta - P(\cos\beta - \cot\alpha\sin\beta) = 0,$$

Whence

$$P = \frac{2Q}{\cot\beta - \cot\alpha}.$$

At an angle β very close to α the pressure P will be very large.

2.6 Summary.

The statement of D'Alembert Principle is given to understand next theory of motion of center of inertia. The general equation of motion for 3n quantities are derived in general forms. The motion of center of inertia and motion relative to center of inertia also discussed. The equation for motion $\sum m\{\dot{x}' - \dot{x}\} = \sum X$, derived. The general Lagrange's equation equation from

D'Alembert's principle derived as $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k$. There are many solved and unsolved problems given.

2.7 Terminal question.

Q.1. A person is placed on a perfectly smooth surface. How may he get off?

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Q.2. How is a person able to increase his amplitude in swinging without touching the ground with his feet.

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Q.3. Two coins, a large and a small one, are spun together on a ordinary table about an axis nearly vertical. Which will comes to rest first, and why?

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Q.4. An inextensible string of negligible mass hanging overa smooth peg connects one mass m_1 on a frictionless inclined plane of angle θ to another mass m_2. Using D'Alembert's principle, prove that the mass will be in equilibrium, if $\sin\theta = \frac{m_2}{m_1}$.

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Q.5. Obtain the equation of motion of a system of two masses, connected by an inextensible string over a small smooth pulley.

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Q.6. Use Lagrange's equations to find the equation of motion of a compound pendulum in a vertical plane about a fixed horizontal axis. Hence find the period of small amplitude oscillation of the compound pendulum.

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Q. 7. One end of a thread is wound on a uniform cylinder of weight P_1. The thread passes over a pulley O, and its other end is attached to a load A of weight P_2 which slides on a horizontal plane, the coefficient of friction being f. Neglecting the mass of the pulley, and the string, determine the acceleration of the load and of the centre C of the cylinder.

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Unit-3. Motion about a fixed Axis

Structure:

3.1 Introduction.

3.2 Objectives.

3.3 Moment of effective forces about the axis of rotation..

3.4 Kinetic energy of the body rotating about a fixed axis.

3.5 Solved Examples.

3.6 Self-learning questions.

3.7 Equation of motion about axis of rotation.

3.8 Solved Examples.

3.9 Summary.

3.9 Terminal Questions.

3.1 Introduction.

The rotation of any rigid body about fixed axis contains the moment and moment of momentum that is studies in this unit. To explain the motion of rotation need the moment of inertia that has been study in first unit. In this unit contains the explanation of kinetic energy only attained through rotation. The rotation experience from two type of forces as impulsive and relative. The motion of rigid body under these forces is discuss in the unit 3.

3.2 Objective.

After reading this unit students should be able to:

- Understand the difference between motion of rotation and translation.
- To derive many results of motion of rigid body under influence of impulsive and relative forces.
- To energy equation of motion rotation about fixed axis.
- Develop the concept to solve word problems base on rotation of a body about fixed axis.
- Develop the concept to explain daily life problems.

3.3 Moment of effective forces about the axis of rotation.

The motions of all system, the momentum balance equation apply to any system or part of a system that has fixed axis of rotation. This type of motion based on following equations:

Linear momentum balance: $\sum \overrightarrow{F_i} = \overrightarrow{\dot{L}},$

Angular momentum balance: $\sum \overrightarrow{M}_{i/c} = \dot{\vec{H}}_c.$

Power balance: $P = \dot{E}_K + \dot{E}_P + \dot{E}_{int}.$

The quantities $\dot{\vec{L}}$ and $\dot{\vec{H}}_c$ are defined as position and acceleration of the system's mass.

Angular velocity: The rate of change of orientation rigid body is ω, that is known as angular velocity.

For fixed-axis rotation $\vec{\omega} = \omega\hat{\lambda}$ and $\dot{\vec{\omega}} = \dot{\omega}\hat{\lambda}$ with $\hat{\lambda}$ a constant unit vector along the axis of rotation.

Example-1.

The round flat uniform disk is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{k}$ about its center. It has mass 'm' and radius R_0. What force required to cause this motion? What torque? What power?

Solution:

From linear momentum balance we have:

$$\sum \vec{F}_i = \dot{\vec{L}} = m\vec{a}_{cm} = 0,$$

Which calculated by evaluating integral $\dot{\vec{L}} \equiv \int \vec{a}\, dm$instead of using the general result that $\dot{\vec{L}} = m\vec{a}_{cm}$.

From angular momentum balance: $\sum \vec{M}_{i/o} = \dot{\vec{H}}_{/o}$

$$\Rightarrow \vec{M} = \int \vec{r}_{/o} \times \vec{a} dm$$

$$= \int_0^{R_0} \int_0^{2\pi} (R\hat{e}_R) \times (-R\omega^2\hat{e}_R) \frac{m}{\pi R^2{}_o} R d\theta dR$$

$$\int \int \vec{0} d\theta dR = 0.$$

So no force or torque acting on the disk. The power is also zero.

3.3.1. Moment of momentum:

The moment of momentum of a particle with respect to a point is defined as a vector,

$$\sigma = r \times mv, \qquad (3.1)$$

Where **r** is the vector drawn from the point to the particle, and **v** is the vector velocity of the particle.

The moment of momentum of a system of particles with respect to a point is define as the vector,

$$\sigma = \sum_{k=1}^{n} \boldsymbol{r_k} \times m_k \boldsymbol{V}_{k,} \qquad (3.2)$$

Where $\mathbf{r}_k$ is drawn from the point in question to m_k, and $\mathbf{v}_k$ is the vector velocity of m_k.

Fundamental Theorem of Moments:

The rate of change of the vector moment of momentum of any system of particle is equal to vector moment of the applied forces, provided that the internal forces between each pair of particles are equal and opposite and in the line through the particle:

$$\frac{d\sigma}{dt} = \sum_{k=1}^{n} \boldsymbol{r_k} \times \boldsymbol{F}_{k,} \qquad (3.3)$$

The equation (3.3) known as ***The Equation of Moment of momentum*** .

3.3.2. Motion of the center of mass:

Let O be an arbitrary fixed point in space, and let **r** be the vector drawn from O to the center of gravity, G, of a material system. Let $\mathbf{F}_1, \ldots\ldots, \mathbf{F}_n$ be the forces that act or applied or external, forces. Then the Principle of the Motion of the Center of mass is expressed by the equation:

$$M \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \sum_k \boldsymbol{F_k}, \qquad (3.4)$$

Where $\mathbf{v} = d\mathbf{r}/dt$.

Let ρ denoted the momentum,

$$\rho = M\boldsymbol{v}.$$

$$\Rightarrow \frac{d\rho}{dt} = \sum_{k=1}^{n} \boldsymbol{F}_k.$$

Hence equation (3.4) converts in the forms:

$$\frac{d\sigma}{dt} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{n}} \boldsymbol{r}_k \times \boldsymbol{F}_k. \quad (3.5)$$

3.4 Kinetic energy of a body rotating about fixed axis:

Let a rigid body be rotating about an axis passing through a fixed point in the body with an velocity ω.The velocity $\mathbf{v_i}$ of ith particle of mass m_i of body is given by,

$$\boldsymbol{v}_i = \omega \times \boldsymbol{r}_i \qquad (3.6)$$

The kinetic energy of this particle is given by,

$$T_i = \frac{1}{2} m_i {v^2}_i = \frac{1}{2} m_i v_i \cdot v_i.$$

Total kinetic energy of entire body is given by,

$$T = \sum_i \frac{1}{2} m_i {v_i}^2 = \sum_i \frac{1}{2} m_i v_i \cdot v_i = \sum_i \frac{1}{2} v_i \cdot m_i v_i$$

$$= \frac{1}{2} \sum_i (\omega \times r_i) \cdot m_i v_i = \frac{1}{2} \sum_i \omega \cdot (r_i \times m_i v_i)$$

$$= \frac{1}{2} \omega \cdot \sum_i (r_i \times m_i v_i) = \frac{1}{2} \omega \cdot J. \quad (3.7)$$

Where **J** defined as, $J = (r_i \times m_i v_i)$.

3.5 SolveExample:

Example-1.

A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft.lbf. Assume the wheel to be a thin uniform disk.

(a) Find the rate of rotation of the wheel.

(b) Find the average power required to bring the wheel to a complete stop in 5 s.

Solution:

(a) Let ω be the rate of rotation of the wheel. When the rotation occurs without slipping then speed of center of mass $v_{cm}= \omega r$. The wheel has both translation and rotational kinetic energy. The total kinetic energy is,

$$\begin{aligned} E_k &= \frac{1}{2}mv^2{}_{cm} + \frac{1}{2}I^{cm}\omega^2 \\ &= \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}I^{cm}\omega^2 \\ &= \frac{1}{2}\left(mr^2 + \frac{1}{2}mr^2\right)\omega^2 \\ &= \frac{3}{4}mr^2\omega^2. \end{aligned}$$

$$\Rightarrow \omega^2 = \frac{4E_K}{3mr^2}$$

$$= \frac{4 \times 1700}{3 \times 20} = 3649.33\ \frac{1}{s^2},$$

$$\Rightarrow \omega = 60.4\frac{rad}{s}.$$

(b) Power is the rate of work done on a body or the rate of change of kinetic energy. In the problem initial kinetic energy given, and the final kinetic energy zero to the time achieve the final state. Therefore the average power is,

$$P = \frac{E_{K_1} - E_{K_2}}{\triangle t} = \frac{1700 - 0}{5},$$

$$= 340\, ft.\frac{lbf}{s} = 340ft.\frac{lbf}{s}.\frac{1hp}{550ft.\frac{lbf}{s}}$$

$$= 0.62hp.$$

So power require to stop body is 0.62hp.

Example-2.

Consider the disk rolling down the incline plane. Suppose the disk starts rolling from rest. Find the speed of the center of mass when the disk is 2 m down the incline plane.

Solution:

We are given that the disk rolls down, starting with zero initial velocity. To find the speed of center of mass after traveled 2 m the incline, we use the equation of motion. Let ω_1 and ω_2 be the initial and final angular speeds of the disk, respectively. we know that in rolling, the kinetic energy is given by,

$$E_K = \frac{1}{2}mv_{cm}{}^2 + \frac{1}{2}I^{cm}{}_{zz}\omega^2 = \frac{1}{2}(mR^2 + I^{cm}{}_{zz})\omega^2. \quad (1)$$

Therefore,

$$\triangle E_K = E_{K_1} - E_{K_2} = \frac{1}{2}(mR^2 + I_{zz}{}^{cm})(\omega^2{}_2 - \omega^2{}_1) \qquad (2)$$

Let rolling of disk is ideal rolling, then work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\jmath}) \cdot \big(d\hat{\lambda}\big) = -mgd\big(\hat{\jmath} \cdot \hat{\lambda}\big) = mgd \sin\alpha \quad (3)$$

Where α is angle of inclination of incline plane from horizontal, and λ is a vector along inclination of plane.

From work – energy principle, we know that $W = \triangle E_K$. Therefore from equation (2) and (3), we get

$$mgd \sin\alpha = \frac{1}{2}(mR^2 + I_{zz}{}^{cm})(\omega^2{}_2 - \omega^2{}_1),$$

$$\Rightarrow \quad {\omega^2}_2 = {\omega^2}_1 + \frac{2mgd\sin\alpha}{mR^2 + {I^{cn}}_{zz}} = {\omega_1}^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{{I^{cm}}_{zz}}{mR^2}\right)}$$

$$= {\omega^2}_1 + \frac{4gd\sin\alpha}{3R^2}.$$

Substituting the value of g, d, α, R,etc., and letting $\omega_1 = 0$, we get

$${\omega^2}_2 = \frac{4 \cdot (9.8) \cdot 2 \cdot \sin(30^0)}{3 \cdot (0.4)^2} = 81.67/s^2$$

$$\Rightarrow \quad \omega_2 = 9.04\frac{rad}{s}.$$

The corresponding speed of the center of mass is

$$v_{cm} = \omega_2 R = \frac{9.04rad}{s} \cdot 0.4m = 3.61\frac{m}{s}.$$

Example-3

The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where θ is in radians. If t is in seconds, (a) what are the angular velocities at $t = 2s$ and $t = 4s$? (b) what is the average angular acceleration for the time interval that begins at $t = 2s$ and ends at $t = 4s$?

Solution:

(a) The angular position θ is given as a function of time. To find the angular velocity at any time, we use the relation $\omega = \frac{d\theta}{dt}$ and find:

$$\omega(t) = \frac{d\theta}{dt} = \frac{d}{dt}(4t - 3t^2 + t^3)$$

$$= 4 - 6t + 3t^2.$$

Therefore, the angular velocity at the given times are

$\omega(2) = 4(2) - 3(2)^2 + (2)^3 = 4$ rad/s,
$\omega(4) = 4(4) - 3(4)^2 + (4)^3 = 28$ rad/s.

(b) Average angular acceleration

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{28 - 4}{4 - 2}$$
$$= 12 \text{ rad/s2.}$$

Example-4

What is the angular speed in radians per second of the earth in its orbit about the sun?

Solution:

The earth goes around sun in a (nearly) circular path with a period of one year. In seconds this is:

$$1 \text{ year} = (365.25)(24)(3600) = 3.156 \times 10^7 \text{seconds.}$$

In one year its angular displacement is 2π radians. So, its angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{3.156 \times 10^7} = 1.99 \times 10^{-7} \text{rad/s.}$$

Example-5

What is the angular speed of the car travelling at 50 km/hr and rounding a circular turn of radius 110m?

Solution:

Given that the linear velocity of the car

$$= 50\frac{\text{km}}{\text{hr}} = (50)\left(\frac{1000}{3600}\right)\frac{\text{m}}{\text{s}} = 13.9\frac{\text{m}}{\text{s}}.$$

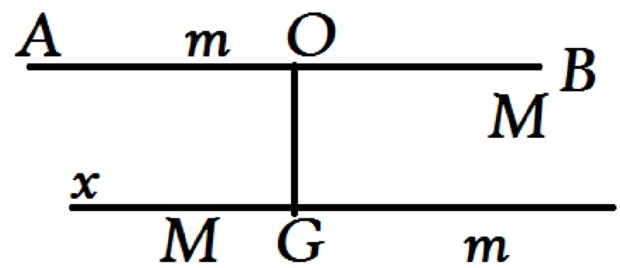
The relation between the linear speed v and its angular speed ω is $v = r\omega$. This gives:

$$\omega = \frac{v}{r} = \frac{13.9}{110} = 0.126 \text{ rad/s.}$$

Example-6- *A rough uniform board of mass m and length 2a, resta on a smooth horizontal plane, and a boy, of mass M, walks on it from one end to the other, show that the distance through which the board moves in this time is 2Ma/(m+M).*

D' Alembert's principal and motion About a Fixed Axis.

Sol. The weight of the boy and the board are acting downwards, the actions and reactions between the boy and the board vanish for the system. The reaction of the smooth plane is acting vertically upwards. Thus there are no external forces on the system in the horizontal direction. Thus by D' Alembert's principal the C>G. of the system does not move. As the boy goes to left, the board comes to the right.



Let x be the distance of the C.G. of the system from *A and* x be the distance (from *A*) through which the board moves, when the boy goes from one end to the other.

Now in the initial position $(M+m)\bar{x} = M.2a + ma$

Therefore $M2a + ma = Mx + m(a+x)\ or\ \ x = 2Ma/(M+m)$.

3.6 Self-learning questions.

Q.1. A block of mass m= 2.5kg slids down a frictionless incline from a 5m height. The block encounters a frictional bed AB of lenth 1 m on the ground. If the speed of the block is 9 m/s at point B, find the coefficient of friction between the block and the frictional surface AB.

Q.2. A marble and a bowling ball, made of same material, are launched on a horizontal platform with the same initial velocity, say v_0. The initial velocity large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the ratius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

Q.3. A straight uniform rod can turn freely about one end O, hangs from O vertically. Find the least angular velocity with which it must begin to move to so that it may perform complete revolution in a vertical plane.

Q. 4. A uniform rod of mass m and length 2a can turn freely about one end which is fixed, it is started with angular velocity w from the position in which it hangs vertically; find its angular velocity at any instant

Q.5. A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{1}{5}h\left(4+\tan^2\alpha\right)$

3.7. Equation of Motion about axis of rotation.

If a rigid body is rotating under the action of a torque τ with one point fixed, then the torque is expressed as,

$$\tau = \left[\frac{dJ}{dt}\right]_s \qquad (3.8)$$

Where **J** is the angular momentum and its time derivative w.r.t. to the space set of axes, represented by the subscripts s, because the equation holds in an inertial frame.

When the body co-ordinate system is rotating with an instantaneous angular velocity ω. The time derivatives of angular momentum **J** in the body coordinate and space co-ordinate system are related as;

$$\left[\frac{dJ}{dt}\right]_s = \left[\frac{dJ}{dt}\right]_b + \omega \times \boldsymbol{J} \qquad (3.9)$$

Then the torque in co-ordinate system x, y, z, and components respectively these axis given by,

$$\begin{cases} \tau_1 = I_1\omega_1 + (I_3 - I_2)\omega_2\omega_3 \\ \tau_2 = I_2\omega_2 + (I_1 - I_3)\omega_3\omega_1 \\ \tau_3 = I_3\omega_3 + (I_2 - I_3)\omega_1\omega_2 \end{cases} \qquad (3.10)$$

Equation (3.10) are known as Euler's equations for the motion of a rigid body with one point fixed under action of a torque.

When a rigid body is not subjected to any net torque, the Euler's equation of motion of the body with one point fixed reduced to,

$$\begin{cases} I_1\omega_1 = (I_2 - I_2)\omega_2\omega_3 \\ \mathrm{I}_2\omega_2 = (I_3 - I_1)\omega_3\omega_1 \\ I_3\omega_3 = (I_1 - I_2)\omega_1\omega_2 \end{cases} \qquad (3.11)$$

Here $\tau = 0$, because no net torque subjected on the body.

3.8 Solve example.

Example-1.

Find the kinetic energy of motion of a rigid body with respect to principle axes in terms of Euler,s angle and interpret the result when $I_1 = I_2$.

Solution.

The kinetic energy is given by,

$$T = \frac{1}{2}[I_1\omega^2{}_1 + I_2\omega^2{}_2 + I_3\omega^2{}_3],$$

Where I_1, I_2, I_3 are the principal moment of inertia and ω_1, ω_2, ω_3 are the components of angular velocity along these axes.

Substituting for $\omega_1, \omega_2, \omega_3$ in terms of Euler's angle φ, θ, ψ, we obtained

$$T = \frac{1}{2}I_1\left[\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right]^2 + \frac{1}{2}I_2\left[\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\sin\psi\right]^2$$
$$+\frac{1}{2}I_3\left[\dot{\phi}\cos\theta + \dot{\psi}\right]^2.$$

If $I_1 = I_2$, then

$$T = \frac{1}{2}I_1\left[\dot{\phi}^2\sin\theta^2 + \dot{\theta}^2\right] + \frac{1}{2}I_2\left[\dot{\phi}\cos\theta + \dot{\psi}\right]^2.$$

Example-2

Calculate the rotational inertia of a wheel that has a kinetic energy of 24,400J when rotating at 602rev/min.

Solution. The angular speed ω of the wheel

$$= 602\,\frac{\text{rev}}{\text{min}}\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\text{s}}\right) = 63.0\,\frac{\text{rad}}{\text{s}}.$$

Now, the rotational kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$ which implies that

$I = \frac{2K_{\text{rot}}}{\omega^2} = \frac{2(24{,}400)}{(63)^2} = 12.3\text{kg.m}^2$.

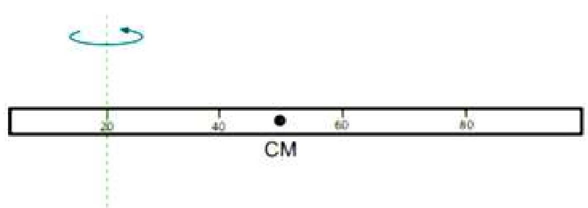
Thus, the moment of inertia of the wheel is 12.3kg.m^2.

Example-3

Calculate the rotational inertia of a meter stick with mass 0.56kg, about an axis perpendicular to the stick and located at 20cm from one end.

Solution.

The stick is one meter long (being a meter stick and all that) and we take it to be uniform so that its center of mass is at the 50 cm mark. But the axis of rotation goes through the 20 cm mark.



Now, if the axis did pass through the center of mass (perpendicular to the stick), the rotational inertia

$$I_{\text{CM}} = \frac{1}{12}ml^2 = \frac{1}{12}(0.56)(1)^2 = 4.7 \times 10^{-2}\text{kg.m}^2.$$

The rotational inertia about our axis will not be the same.

We note that our axis is displaced from the one through the CM by 30 cm. Then the Parallel Axis Theorem tells us that the moment of inertia about our axis is given by

$$I = I_{\mathrm{CM}} + md^{d},$$

where, d is the distance the axis is displaced (parallel to itself), namely 30 cm. Calculating the value, we get that $I = 9.7 \times 10^{-d}$kg.m^2.

So the rotational inertia of the stick about the given axis is 9.7×10^{-d}kg.m^2.

Example. 4.

A disk, initially rotating at 120 rad/s, is slow down with a constant angular acceleration of magnitude 4.0 rad/s (a) How much time elapse before the disk stop? (b) Through what angle dose the disk rotate in coming to rest?

Solution. (a) We are the initial angular velocity of the disk, ω_0=120 rad/s. We are given the magnitude of the disk's angular acceleration as it shows, but then we must write,

$$\alpha = -4.0 \frac{rad}{s}.$$

The final angular velocity is $\omega = 0$. Then we can find,

$$\omega = \omega_0 + \alpha t \quad \Rightarrow t = \frac{\omega - \omega_0}{\alpha}$$

We get :

$$t = \frac{(0 - 120\, rad/s)}{(-4.0\, rad/s)} = 30.0s.$$

(a) We'll let the initial angle be θ_0=0. We can now use any of the constant-α equation containing θ to solve it,

$$\omega^2 = \omega_0^2 + 2\alpha(\theta),$$

$$\theta = \frac{(\omega^2 - \omega_0^2)}{2\alpha}$$

and we get,

$$\theta = \frac{(0^2 - (120\,rad/s)^2)}{2(-4.0\,rad/s)} = 1800\,rad.$$

The disk turns through an angle of 1800 radians before coming to rest.

3.9 Summary.

The study of rotation of a body about a axis start from moment of effective forces about the axis of rotation and moment of momentum ($\sigma = r \times mv$.). The fundamental theorem of momentum given as, $\frac{d\sigma}{dt} = \sum_{k=1}^{n} r_k \times F_k$. Motion of centre of mass and kinetic energy of a body rotating about fixed axis describes as, $T = \frac{1}{2}\omega \cdot J$. Equation of motion about axis of rotation discussed with example.

3.10 Terminal questions.

Q.1. If T be the kinetic energy, **G** be the external torque about the instantaneous axis of rotation and ω the angular velocity, then prove that $\frac{dT}{dt} = \boldsymbol{G}.\boldsymbol{\omega}$.

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Q.2. Show that the kinetic energy of a rigid body can be represented as,

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{J}.$$

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Q.3. Show that the angular momentum **J** of a rotating body is given by,

$$\boldsymbol{J} = \boldsymbol{I\omega}.$$

Where ω is the angular velocity.

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Q.4. If I is the moment of inertia about the axis of rotation, prove that the kinetic energy can be expressed as,

$$T = \frac{1}{2} I\omega^2.$$

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Q.5. A solid homogeneous cone of height h and vertical angle α oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\frac{1}{5}h\left(2+3\tan^2\alpha\right)$.

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Q. 6. Two masses M and m are connected by a rigid rod of length L and negligible mass. For an axis perpendicular to the rod, show that the system has the minimum moment of the inertia when the axis passes through the center of mass. Show that this moment of inertia is $I = \mu L^2$, where $\mu = \frac{mM}{(m+M)}$.

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Uttar Pradesh Rajarshi Tandon

Open University

Bachelor of Science

UGMM -106

(Mechanics-II (Dynamics and Hydrodynamics)

Block –II Hydrodynamics

UGMM - 106

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Course Introduction

II - Block

Unit 4 :

Equation of Motion : Equation of continuity in different coordinate system and boundary surface, velocity potential, stream lines.

Unit 5 :

Some standard equation of Motion : Euler's equation of motion, steady motion, Bernaullies equation, Helmholtz equation, Impulsive equation.

Unit 6 :

Motion of Liquid in two dimensions : Motion in two dimensions, stream function, irrotational motion, complex potential, sources and sinks.

Unit 7 :

Objects in Hydrodynamics : Doublets, image system of a simple source with respect a plane, a circle, a sphere. Image system of a doublet with respect to a plane, a circle and a sphere, circle theorem.

INTRODUCTION

There are two part of the book which deals dynamics and hydrodynamics. The study of mechanics contains rigid body and fluid dynamics both.

In present book chapter 1 to 3 we studied the kinematics of rigid body. The chapter 1 and 2 give brief introduction of moment of inertia and its related theorem and quantities that is necessary to study of dynamics of rigid body. In the chapter 3 study the rotational motion of rigid body or particle.

The next part of book starts from chapter 4 which deals equation of continuity, velocity potential, stream lines. The equation of motion of fluid is discussed in chapter 5 and his two dimension application, with sink and source explained in chapter 6. The last chapter contains combine flow sink and source , which give the theory of doublet.

Unit 4: Equation of continuity in different coordinate system and boundary surface, velocity potential, stream lines.

Structure:

4.1 Introduction.

The second section of the book is fluid mechanics, that starts from unit 4. Every student of science has been familiar with equation of continuity, but in this unit the equation of continuity has discussed in different coordinate system. The equation of continuity derived in vector form after that it simply written in different coordinate systems. The velocity potential that determines stream lines explained with suitable example. Definition of velocity potential and stream line have been given very simple language.

4.2 Objective.

After reading this unit students should be able to:

- To derive the equation of continuity in vector form and write in different coordinate system.
- Develop the skills to solve the problems based on equation of continuity.
- To introduce the velocity potential and stream lines.
- To find stream lines from given velocity potential function.
- To introduce the physical condition in problems with solve examples.

4.3 Equation of continuity in different coordinate system and boundary surfaces.

Consider a fixed mass 'm' of fluid contained in an arbitrary region *R(t).* In general move of fluid we consider, the boundary *S(t)* to *R(t)* with the time.

If the fluid of density ρ, mass of system 'm' contained in region R(t).

Then,

$$m = \int_{R(t)} \rho dV. \qquad (4.1)$$

Differentiating of equation (4.1) w.r.t.

$$\frac{dm}{dt} = \frac{d}{dt}\int_{R(t)}^{\cdot} \rho dV = 0. \quad (4.2)$$

By applying transport theorem, we obtained

$$\int_{R(t)}^{\cdot} \frac{\partial \rho}{\partial t} dV + \int_{S(t)}^{\cdot} \rho \boldsymbol{W} \cdot \boldsymbol{n} dA = 0. \quad (4.3)$$

Where the W is velocity field of fluid. For locally in region R(t), as arbitrary fluid elements, the equation (4.3) becomes,

$$\int_{R(t)}^{\cdot} \frac{\partial \rho}{\partial t} dV + \int_{S(t)}^{\cdot} \rho \boldsymbol{U} \cdot \boldsymbol{n} dA = 0. \quad (4.4)$$

By using Gauss's theorem, in a part of equation (4.4), we get

$$\int_{S(t)}^{\cdot} \rho \boldsymbol{U} \cdot \boldsymbol{n} dA = \int_{R(t)}^{\cdot} \nabla \cdot \rho U dV,$$

Substitute these value in equation (4.4), the equation becomes,

$$\int_{R(t)}^{\cdot} \frac{\partial \rho}{\partial t} + \nabla \cdot \rho U dV = 0. \quad (4.5)$$

In equation (4.5) volume integral over the arbitrary region R(t) is zero. Thus,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{U} = 0. \quad (4.6)$$

The differential equation (4.6) is called equation of continuity in differential form.

4.3.1. Equation of Continuity in Cartesian coordinate system:

From (4.6), equation of continuity in Cartesian coordinate system becomes as;

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0. \quad (4.7)$$

When flow is planer, the velocity and the derivatives in one direction (z-axis) becomes zero, so equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0,$$

4.3.2. Continuity Equation in Cylindrical Polar Coordinates:

We have derived the Continuity Equation, (4.7) using Cartesian Coordinates. It is possible to use the same system for all flows. But sometimes the equations may become cumbersome. So depending upon the flow geometry it is better to choose an appropriate system. Many flows which involve rotation or radial motion are best described in Cylindrical Polar Coordinates. Let us now write equations for such a system. In this system coordinates for a point P are $\boldsymbol{r}, \boldsymbol{\theta}$ and z, which are indicated in Fig.4.1. The velocity components in these directions respectively are v_r, v_θ and z. Transformation between the Cartesian and the polar sytems is provided by the relations,

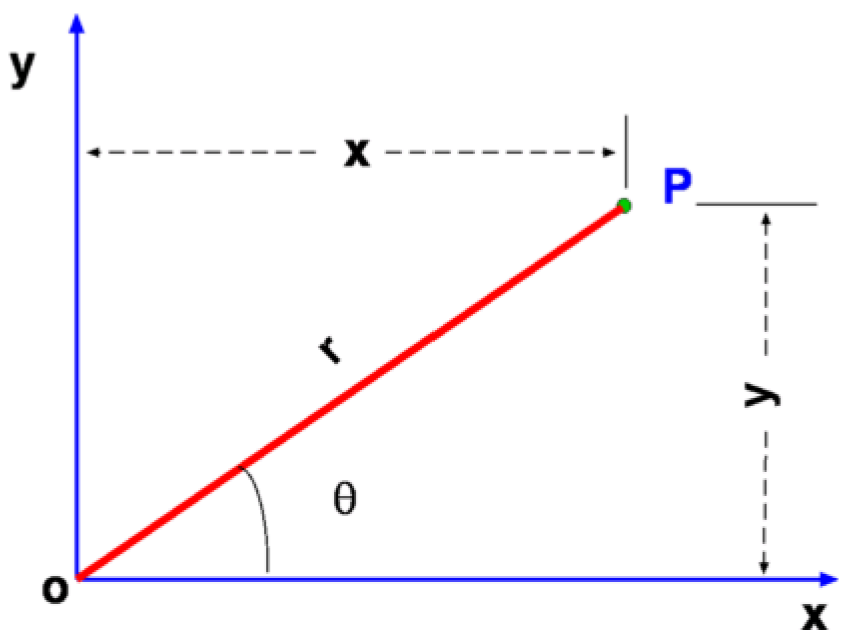


Figure 4.1: Cylindrical Polar Coordinate System

$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \tan^{-1}\frac{y}{x}\,, \qquad z = z \tag{4.8}$$

The gradient operator is given by,

$$\nabla P = \frac{1}{r}\frac{\partial}{\partial r}(rP) + \frac{1}{r}\frac{\partial}{\partial \theta}(P) + \frac{\partial}{\partial z}(z) \qquad (4.9)$$

In view of (4.7) and (4.9), the general three-dimensional flow, the continuity of equation in cylindrical coordinates (r, θ, z) is,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0, \qquad (4.10)$$

Where v_r, v_θ and v_z are the velocity in the r, θ and z directions of the cylindrical coordinate system.

So for polar coordinate system (r, θ), continuity of equation becomes,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} = 0. \qquad (4.11)$$

4.3.3. Continuity Equation for steady flow:

For a steady flow the time derivative vanishes. Equation (4.7) becomes,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \qquad (4.12)$$

The equation in polar coordinates also undergoes the same simplification.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad (4.13)$$

These equations are the ones that are to be used for a compressible flow as we have kept density, ρ still variable.

4.3.4: Equation Continuity in Spherical Coordinates:

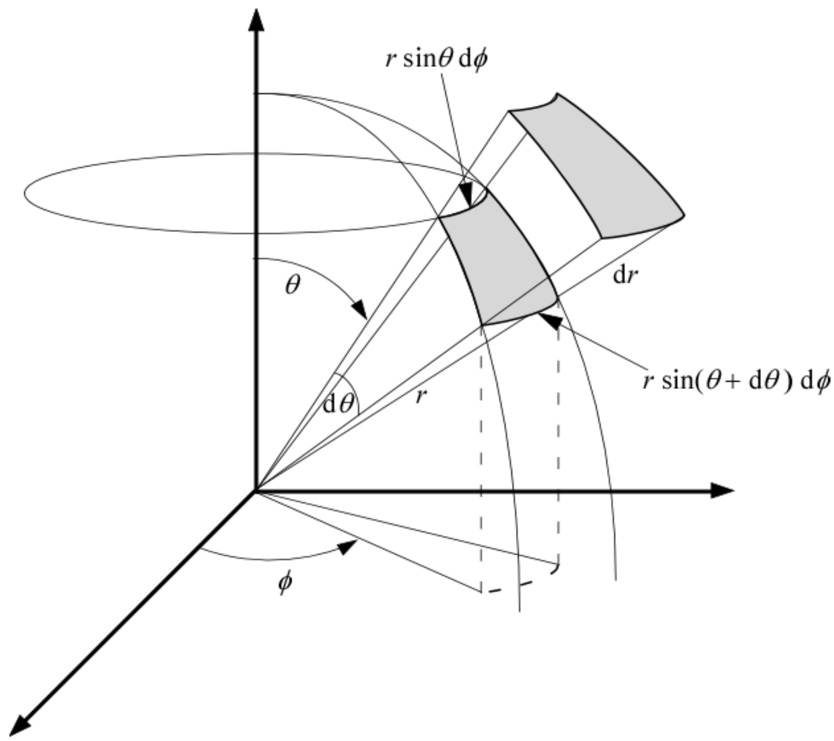
We start by selecting a spherical control volume dV. As shown in the figure below, this is given by

$$\mathrm{d}\mathcal{V} = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi,$$

where r, θ, and φ stand for the radius, polar, and azimuthal angles, respectively. The azimuthal angle is also referred to as the zenith or colatitude angle.

The differential mass is

$$\mathrm{d}\mathcal{M} = \rho r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi$$



We will represent the velocity field via

$$\mathbf{u} = u\,\mathbf{e}_r + v\,\mathbf{e}_\theta + w\,\mathbf{e}_\phi$$

In an Eulerian reference frame mass conservation is represented by accumulation, net flow, and source terms in a control volume.

Accumulation:

The accumulation term is given by the time rate of change of mass. We therefore have

$$\frac{\partial \rho}{\partial t} r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi$$

The net flow through the control volume can be divided into that corresponding to each direction.

Radial Flow:

Starting with the radial direction, we have

$$\dot{m}_{\text{in}} = \rho\, u\, A_{\text{in}}$$

The inflow area Ain is a trapezoid whose area is given by

$$A_{\text{in}} = \text{mid segment} \times \text{height} = \frac{1}{2}\left[r\sin\theta d\phi + r\sin\left(\theta + d\theta\right)d\phi\right] \times r d\theta$$

The key term here is the sine term. Note that the mid segment is the average of the bases (parallel sides). Upon expansion of A_{in}, and in the limit of vanishing dθ, we have

$$\sin\left(\theta + d\theta\right) = \sin\theta\cos d\theta + \cos\theta\sin d\theta \approx \sin\theta + \cos\theta d\theta\,,$$

substitution into A_{in} yields

$$A_{\text{in}} = r^2\sin\theta d\theta d\phi + \frac{1}{2}r^2\cos\theta d\theta^2 d\phi \approx r^2\sin\theta d\theta d\phi\,,$$

where high order terms have been dropped.

The outflow in the radial direction is

$$\dot{m}_{\text{out}} = \left(\rho u + \frac{\partial \rho u}{\partial r}\mathrm{d}r\right) A_{\text{out}}$$

but

$$A_{\text{out}} = \text{mid segment} \times \text{height},$$

Where

$$\text{mid segment} = \frac{1}{2}\left[\left(r + dr\right)\sin\theta d\phi + \left(r + dr\right)\sin\left(\theta + d\theta\right)d\phi\right]$$

And

$$\text{height} = (r + \mathrm{d}r)\mathrm{d}\theta$$

By only keeping the lowest (second & third) order terms in the resulting expression, we have

$$A_{\text{out}} = r^2 \sin\theta d\theta d\phi + 2r\sin\theta dr d\theta d\phi$$

Note, that in the expression for A_{out}, we kept both second order and third order terms. The reason for this is that this term will be multiplied by dr and therefore, the overall order will be three. In principle, one must carry all those terms until the final substitution is made, and only then one can compare terms and keep those with the lowest order. At the outset, the net flow in the radial direction is given by

$$\dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 2\rho ur\sin\theta dr d\theta d\phi + \frac{\partial\rho u}{\partial r} r^2 \sin\theta dr d\theta d\phi$$

Polar Flow (θ):

The inflow in the polar direction is

$$\dot{m}_{\text{in}} = \rho\, v\, A_{\text{in}},$$

where

$$A_{\text{in}} = r\sin\theta\,\mathrm{d}r\,\mathrm{d}\phi$$

The outflow in the θ direction is

$$\dot{m}_{\text{out}} = \left(\rho\, v + \frac{\partial\rho\, v}{\partial\theta}\mathrm{d}\theta\right) A_{\text{out}}$$

Where

$$A_{\text{out}} = \frac{1}{2}\left[r\sin(\theta + d\theta)d\phi + (r + dr)\sin(\theta + d\theta)d\phi\right]dr\,.$$

Upon expansion, and keeping both second and third order terms, we get

$$A_{\text{out}} \approx r\cos\theta dr d\theta d\phi + r\sin\theta dr d\phi .$$

Finally, the net flow in the polar direction is

$$\dot{m}_{\text{out}} - \dot{m}_{\text{in}} = \rho v r\cos\theta dr d\theta d\phi + \frac{\partial \rho v}{\partial \theta} r\sin\theta dr d\theta d\phi .$$

Azimuthal Flow (φ):

The inflow in the azimuthal direction is given by

$$\dot{m}_{\text{in}} = \rho\, w A_{\text{in}}$$

with

$$A_{\text{in}} = r\, \mathrm{d}r\, \mathrm{d}\theta$$

while the outflow is

$$\dot{m}_{\text{out}} = \left(\rho w + \frac{\partial \rho w}{\partial \phi}\mathrm{d}\phi\right) A_{\text{out}}$$

and

$$A_{\text{out}} = r\, \mathrm{d}r\, \mathrm{d}\theta$$

At the outset, the net flow in the polar direction is

$$\dot{m}_{\text{out}} - \dot{m}_{\text{in}} = \frac{\partial \rho w}{\partial \phi} r dr d\theta d\phi .$$

Continuity Equation:

Now, by collecting all mass fluxes we have

$$\frac{\partial \rho}{\partial t} dv + 2\rho u \frac{dv}{r} + \frac{\partial \rho u}{\partial r} dv + \rho v \cos\theta \frac{dv}{r\sin\theta} + \frac{\partial \rho v}{\partial \theta}\frac{dv}{r} + \frac{\partial \rho w}{\partial \phi}\frac{dv}{r\sin\theta} = 0,$$

which, upon dividing by dv and combining terms, reduces to

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial \rho r^2 u}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial \rho v\sin\theta}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \rho w}{\partial \phi}=0,$$

which is the continuity equation in spherical coordinates.

4.4 Solve examples.

Example-1.

An incompressible fluid of density, ρ, flows through a porous pipe of radius, R. The flow enters the pipe with a uniform velocity, V_1, and leaks out radially (axis-symmetrically) with the profile $V = V_0[1-(x/L)^2]$. Calculate the mass outflow at x= L.

Solution.

$$-\oiint_A \rho\vec{u}\cdot d\vec{A} = -\left(\iint_{left}\rho\vec{u}\cdot d\vec{A}+\iint_{right}\rho\vec{u}\cdot d\vec{A}+\iint_{bottom}\rho\vec{u}\cdot d\vec{A}\right)=0. \quad (1)$$

The velocity and surface at the left

$$\vec{u} = V_1\hat{x}, d\hat{A} = -rdrd\theta\hat{x}, \qquad (2)$$

thus the flux is

$$-\iint_{A_1} -V_1\rho rdrd\theta(\hat{x}\cdot\hat{x}) = V_1\rho\int_0^R\int_0^{\pi} rdrd\theta = \pi R^2 V_1\rho. \qquad (3)$$

At the bottom surface, or more exact, at the circumference of the pipe, the velocity, surface and flux are,-

$$\vec{u} = V_0\left(1-\left(\frac{x}{L}\right)^2\right)\hat{r}, d\hat{A} = Rd\theta dx\hat{r}, \qquad (4)$$

$$-\iint_{A_2} V_0\left(1-\left(\frac{x}{L}\right)^2\right)\rho Rd\theta dx\hat{r}(\hat{r}\cdot\hat{r}) = -\int_0^L\int_0^{2\pi} V_0\left(1-\left(\frac{x}{L}\right)^2\right)\rho Rd\theta dz = -\frac{4}{3}\pi RLV_0\rho. \quad (5)$$

After integrating equation (3) and (5) put in equation (1), which gives,

$$\pi\rho R^2 V_1 - \frac{4}{3}\pi R L V_0 \rho - \dot{m}_{out} = 0,$$

$$\Rightarrow \ \dot{m}_{out} = \pi\rho R^2 V_1 - \tfrac{4}{3}\pi R L \rho V_0. \quad (6)$$

Example-2.

A tank initially holds within it salt water with a density $\rho_i > \rho_{water}$. At t=0 water is pumped through an entrance located at the left, with a volumetric flux Q. Assuming that the volume of the fluid inside, V, remains constant:

a. Write the equation governing the dynamics of the density.

b. Solve for the time it takes the fluid to reach density ρ_f.

Solution.

Suppose that the mixing time of the fresh water and salt water is instantaneous $\rho(x, y, z, t) = \rho(t)$.

Considering the fluid has incompressible and non-moving volume.

The volume of the fluid inside does not change indicates that volumetric flux at the same as the entrance. Hence,

$$\dot{m}_{in} - \dot{m}_{out} = (\rho_{water} - \rho)Q = -(\rho - \rho_{water})Q. \qquad (1)$$

The accumulation term is,

$$\frac{\partial}{\partial t}\iiint_V \rho(t)dV = \frac{\partial}{\partial t}\rho(t)V = V\frac{\partial\rho(t)}{\partial t}. \qquad (2)$$

Demanding equality gives,

$$V\frac{\partial\rho(t)}{\partial t} = V\frac{d\rho(t)}{dt} = -(\rho - \rho_{water})Q. \quad (3)$$

Solving for the time gives,

$$dt = -\frac{V}{Q}\frac{d\rho}{\rho - \rho_{water}}$$

$$\Rightarrow \quad t = -\frac{V}{Q}\log(\rho - \rho_{water})|_{\rho_i}{}^{\rho_f} = -\frac{V}{Q}\log\left(\frac{\rho_f - \rho_{water}}{\rho_i - \rho_{water}}\right). \ (4)$$

Example-3.

Blood travels through an artery at velocity v. If a vasoconstricting chemical is consumed and the artery constricts to half the original diameter, what is the new velocity of the blood?

Solution.

The continuity equation states that:

$$A_1 v_1 = A_2 v_2$$

In other words, the volumetric flow rate stays constant throughout a pipe of varying diameter. If the diameter decreases (constricts), then the velocity must increase.

To establish the change in cross-sectional area, we need to find the area in terms of the diameter:

$$A = \pi r^2 = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}.$$

If the diameter is halved, the area is quartered.

$$A_2 = \frac{\pi\left(\frac{D}{2}\right)^2}{4} = \frac{\frac{\pi D^2}{4}}{4} = \frac{1}{4}\frac{\pi D^2}{4},$$

$$A_2 = \frac{1}{4}A.$$

To keep the volumetric flow constant, the velocity would have to increase by a factor of 4.

$$A_1 v_1 = \left(\frac{1}{4} A_1\right) v_2,$$

$$v_2 = 4 v_1.$$

Example-4.

A pipe with a diameter of 4 centimeters is attached to a garden hose with a nozzle. If the velocity of flow in the pipe is 2*m/s*, what is the velocity of the flow at the nozzle when it is adjusted to have a diameter of 8 millimeters?

Solution.

Flow rate in a pipe must be constant in order to create linear flow. This flow rate is given by the product of the cross-sectional area and the velocity of the fluid.

$$A_1 v_1 = A_2 v_2$$

The cross-sectional areas of the pipe and nozzle can be found using their radii. Note that you were given dimensions in terms of diameter, so be sure to divide by 2 to get the radius.

$$A = \pi r^2,$$

$$A_1 = \pi (0.02m)^2 = 0.0004\pi m^2,$$

$$A_2 = \pi (0.004m)^2 = 0.000016\pi m^2$$

Use these areas and the initial velocity to calculate the final velocity in the nozzle.

$$(0.0004\pi m^2)(2m/s) = (0.000016\pi m^2) v_2,$$

$$v_2 = \frac{(0.0004\pi m^2)(2m/s)}{0.000016\pi m^2},$$

$$v_2 = 50m/s.$$

Example-5. A liquid flows through a pipe with a diameter of 10cm at a velocity of 9cm/s. If the diameter of the pipe then decreases to 6cm, what is the new velocity of the liquid?

Solution.

Rate of flow, $A*v$, must remain constant. Use the continuity equation,

$$A_1 v_1 = A_2 v_2.$$

Solving the initial cross-sectional area yields:

$$A_1 = \pi r^2 = 25\pi cm^2.$$

The initial radius is 5cm. Then find the final area of the pipe:

$A_2 = \pi r^2 = 9\pi cm^2$. The final radius is 3cm.

Using these values in the continuity equation allows us to solve the final velocity.

$$\left(25\pi cm^2\right)\left(9cm/s\right) = \left(9\pi cm^2\right)v2.$$

$$v_2 = 25cm/s$$

Example 6. A mass of fluid moves in such a way that each particle describes a circle in one plane about axis ; show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\omega)}{\partial \theta} = 0,$$

Where ω is he angular velocity of a particle whose azimuthal angle is θ at time t.

Solution. Here the motion is in a plane. Consider a particle moving in a circle of radius r. At this point consider an element PABQ, such that PA = , PQ = $r\delta\theta$. There is no motion along PA.

There excess of flow-in over flow-out along

$PQ = r\delta\theta \frac{\partial}{r\partial\theta}[\rho r\omega\ \delta r]$ per unit time.

Also mass of the fluid inside the element

$$= \rho\delta r\ r\ \delta\theta.$$

$\therefore$ change in the mass of the element

$$= \frac{\partial}{\partial r}(\rho\delta r\ r\ \delta\theta) \text{ in unit time.}$$

The equation of continuity is

$$\frac{\partial}{\partial t}(\rho\delta r\ r\ \delta\theta) = -r\delta\theta \frac{\partial}{r\partial\theta}[\rho r\omega\ \delta r],$$

i.e. $\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\omega)}{\partial\theta} = 0,$

cancelling $\delta r\ r\ \delta\theta$ throughout.

Example. 7. The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0,$$

Where u is the velocity at distance r.

Solution. If the fixed centre be the origin, then in this example there is motion only along PQ and no motion along other edges of the element. $\therefore$ excess of flow-in over flow-out along PQ

$$= -\delta r \frac{\partial}{\partial r}[\rho u. r\delta\theta. r \sin\theta \, \delta\omega] \text{per unit time.}$$

Excess of flow-in over flow-out along $PR = 0$ and excess of flow-in over flow-out along

$$PS = 0,$$

there being no motion of the fluid along these directions. Now the mass of the fluid inside the element

$$= \rho\delta r \, r \, \delta\theta \, r \, \sin\theta \, \delta\omega.$$

∴ change in the mass of the element

$$= \frac{\partial}{\partial t}(\rho\delta r \, r \, \delta\theta \, r \sin\theta \, \delta\omega) \text{ per unit time.}$$

Therefore the equation of continuity is

$$\frac{\partial}{\partial t}(\rho\delta r \, r \, \delta\theta \, r \sin\theta \, \delta\omega) = -\delta r \frac{\partial}{\partial r}(\rho u r \, \delta\theta \, r \, \sin\theta \, \delta\omega)$$

i.e. $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2) = 0,$$

i.e. $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}[u r^2 \frac{\partial\rho}{\partial r} + \rho\frac{\partial(u r^2)}{\partial r}] = 0,$$

i.e. $$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial\,(r^2 u)}{\partial r} = 0.$$ This gives the result.

Example. 8. A mass of fluid is in motion so that the lines of motion lie on the surface of coaxial cylinder; show that the equation of continuity is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial\,(\rho v_e)}{\partial\theta} + \frac{\partial\,(\rho v_z)}{\partial z} = 0,$$

Where v_e, v_z are the velocities perpendicular and parallel to z.

Solution. Consider a point P, Whose cylindrical co-ordinate are (r, θ, z). Construct an element at P with edges

$$PQ = r\delta\theta, PR = \delta r \; and \; PS = \delta z.$$

The fluid moves on the surface of coaxial cylinder; hence is no motion along PR. Excess of flow-in over flow-out along PQ

$$= -r\ \delta\theta \frac{\partial}{r\ \partial\theta}[\rho v_e\, \delta r\ \delta z]$$

and excess of flow-in over flow-out along PS

$$= -\delta z \frac{\partial}{\partial z}[\rho v_z\, \delta r. r\ \delta\theta].$$

Excess of flow-in over flow-out along $PR = 0$,There being no motion of the fluid in this direction.

Also mass of the fluid in the element

$$= \rho r\ \delta\theta. \delta r. \delta z.$$

∴ change in mass of the element

$$= \frac{\partial}{\partial r}(\rho r\ \delta\theta\ \delta r\ \delta z) = r\delta\theta\ \delta r\ \delta z\ \frac{\partial \rho}{\partial t}.$$

The equation of continuity is

$$(r\delta\theta\ \delta r\ \delta z\,)\frac{\partial \rho}{\partial t} \qquad = -r\ \delta\theta \frac{\partial}{r\ \partial\theta}[\rho v_e\, \delta r\ \delta z] -$$

$$\delta z \frac{\partial}{\partial z}[\rho v_z\, \delta r. r\ \delta\theta].$$

i.e. $\quad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial\,(\rho v_e)}{\partial\theta} + \frac{\partial\,(\rho v_z)}{\partial z} = 0,$ dividing by $\quad r\delta\theta\ \delta r\ \delta z.$

Example 9. If the lines of motion are curves on the surface of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho q_r)}{\partial r}+\frac{2\rho q_r}{r}+\frac{cosec\,\theta}{r}\frac{\partial(\rho q_\omega)}{\partial \omega}=\mathbf{0}.$$

Solution. Let O, the common vertex, be the origin and OZ the common axis, the axis of z. consider a cone OAB of semi-vertical angle θ. Let $p(r,\theta,\omega)$ be a point on the surface of the cone, PQ, PR, PS being edges of the elementary parallelepiped. Since the line of the motion are curves on the surface of cones there will be no motion perpendicular to the surface of the cone,

i.e., there is no velocity along the edge PR. Here $PQ=\delta r, C=r\delta\theta$ and $PS=r\sin\theta\ \delta\omega$.

Also volume of the elementary parallelepiped

$$=r\delta\theta\ .r\sin\theta\ \delta\omega\ \delta r$$

and q_r and q_ω are the velocity components along PQ and PS (direction in which r and ω increase). Therefore excess of the flow-in over the flow-out in the direction PQ

$$=-\frac{\partial}{r\sin\theta\,\partial\omega}[\rho q_\omega\, r\ \delta\theta\ \delta r)r\ sin\ \theta\ \delta\omega.$$

Also the increase in the mass of the element

$$=\frac{\partial}{\partial t}(\rho r\ \delta\theta\ r\sin\theta\ \delta\omega.\delta r)\quad \text{per unit time.}$$

∴ The equation of continuity is given by

$$\frac{\partial}{\partial r}(\rho r\ \delta\theta\ r\sin\theta\ \delta\omega.\delta r)$$

$$= -\frac{\partial}{\partial r}[\rho q_r.r\ \delta\theta.r\sin\theta\ \delta\omega]\ \delta r - \frac{\partial}{r\sin\theta\ \partial\omega}[\rho q_\omega\ r\ \delta\theta.\delta r)r\ sin\ \theta\ \delta\omega.$$

The factor $r\ sin\ \theta\ \delta\omega.r\ \delta\theta\ \ \delta r$ can be cancelled and we get

$$\frac{\partial\rho}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}[\rho q_, r^2] - \frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}[\rho q_\omega]$$

Or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}[r^2\frac{\partial}{\partial r}(\rho q_r) + \rho q_r\frac{\partial}{\partial r}(r^2)] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}[\rho q_\omega] = 0$$

Or
$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho q_r)}{\partial r} + \frac{2\rho q_r}{r} + \frac{cosec\ \theta}{r}\ \frac{\partial(\rho q_\omega)}{\partial\omega} = 0.$$

Example 10. If ω is the area of cross-section of a stream filament, prove that the equation oc continuity is

$$\frac{\partial}{\partial r}(\rho\omega) + \frac{\partial}{\partial s}(\rho\omega q) = 0$$

Where ds is an element of arc of the filament in the direction of flow and q is the speed.

Solution. Let P be the point. Consider a volume bounded by the cross section through P and at a distance ds from P.

Excess of flow in over flow out in this volume = $-\frac{\partial}{\partial s}[\rho\omega q]\ ds$ per unit time.

Also the mass of the fluid in this volume = $\rho.\omega.ds$

∴ rate of change of mass = $\frac{\partial}{\partial t}(\rho\omega.ds)$

∴ equation of continuity is

$$-\frac{\partial}{\partial s}(\rho\omega q)ds = \frac{\partial}{\partial t}(\rho\omega\ ds)$$

$$\text{i.e}\quad \frac{\partial}{\partial t}(\rho\omega) + \frac{\partial}{\partial s}(\rho\omega q) = 0$$

This proves the result.

4.5 Self-learning Questions.

Q.1. An incompressible fluid is emptying out from a small aperture at the bottom of a conic tank. Assuming that the initial height of the fluid was y_0, the angle of cone is , θ, the radius of aperture is, R, and the exit velocity is given by $\sqrt{2gy}$, find time when tank is empty.

Q.2. An incompressible fluid flows through a long circular channel of radius R. Given that the entrance velocity is uniform, U, and that the exit velocity has a parabolic profile $u = u_{max}\left(1 - \frac{r^2}{R^2}\right)$. Find the maximum velocity of exit profile.

Q. 3. If blood flows through the aorta with velocity, v_a, with what velocity would blood flow through the capillaries in the body?

Q. 4. If a pipe with flowing water has a cross-sectional area nine times greater at point 2 than at point 1, what would be the relation of flow speed at the two points?

Q. 5. As water is traveling from a water tower, to somone's home, the pipes it travels in frequently change size. Water is traveling at 5m/s in a tube with a diameter of 0.5m. The tube gradually increases in size to a diameter of 1.5m, and then gradually decreases to a diameter of 1m. Neglecting any energy losses due to friction and pressure changes, what is the speed of the water when it reaches the tube diameter of 1m?

4.6 Potential Flow Theory.

We can define a potential function, φ(x, z, t), as a continuous function that satisfies the basic laws of fluid mechanics; conservation of mass and momentum, assuming incompressible, invicid and irrotational flow.

Then for any vector identity for any scalar function φ,

$$\nabla \times \nabla\varphi = 0. \qquad (4.11)$$

So for irrotational flow,

$$\nabla \times \vec{V} = 0. \qquad (4.12)$$

Therefore

$$\vec{V} = \nabla\varphi. \quad (4.13)$$

Where $\varphi = \varphi(x, y, z, t)$ is the velocity potential function. Hence velocity component in Cartesian coordinate, as functions of space and time, are

$$u = \frac{\partial\varphi}{\partial x}, v = \frac{\partial\varphi}{\partial y}, w = \frac{\partial\varphi}{\partial z}. \qquad (4.14)$$

The velocity must still satisfy the conservation of mass equation. We can substitute in the relationship between potential and velocity, which gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\Rightarrow \quad \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} + 0. \quad (4.15)$$

Here potential function satisfy the Laplace equations, so this gives potential flow.

Potential lines:

Lines of constant φ are called potential lines of the flow. In two dimensions

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy,$$

$$d\varphi = udx + vdy.$$

Since $d\varphi = 0$ along a potential line, we have

$$\frac{dy}{dx} = -\frac{u}{v}. \qquad (4.16)$$

Stream line:

Streamlines are lines everywhere tangent to the velocity, $\frac{dy}{dx} = \frac{v}{u}$, so potential lines are perpendicular to streamlines.

Or

The streamline is everywhere tangent to the velocity.

Streamline function is represented by ψ. Line of constant ψ are perpendicular to lines of constant φ, except at a stagnation point.

Relation between streamline and potential function;

The φ and ψ are related mathematically through the velocity components:

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (4.17)$$

$$v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (4.18)$$

These equations (4.16) and (4.17) known as the Cauchy-Riemann equations.

4.7 Solve Examples:

Example-1.

Find potential and stream line function for uniform, free stream flow, in 1, 2 and 3 dimension.

Solution.

Let the velocity vector in 1 dimension of uniform, free stream flow is

$$\boldsymbol{\vec{V} = U\vec{\imath} + 0\vec{\jmath} + 0\vec{k}} \qquad \textbf{(1)}$$

Therefore,

$$u = U = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (2)$$

$$v = 0 = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad (3)$$

After integration the equation (2) and (3) we get velocity field, as

$$\varphi = Ux \; and \; \psi = Uy \qquad (4)$$

So the streamlines are horizontal straight lines for all values of y and equipotential lines are vertical straight lines perpendicular to the streamlines.

Similarly for 2D and 3D, potential and stream function given by,

a) For 2D uniform flow: $\boldsymbol{V} = (U, V, 0)\ \varphi = Ux + Vy;\ \psi = Uy - Vx$

b) For 3D uniform flow: $\boldsymbol{V} = (U, V, 0); \varphi = Ux + Vy + Wz;$ no stream function in 3D.

Example-2.

The velocity components in a two-dimensional velocity field for an incompressible fluid are expressed as

$$u = \frac{y^3}{3} + 2x - x^2 y$$

$$v = xy^2 - 2y - \frac{x^3}{3}$$

Show that these functions represent a possible case of an irrotational flow.

Solution:

The functions given satisfy the continuity equation for their partial derivatives are

$$\frac{\partial u}{\partial x} = 2 - 2xy \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2xy - 2$$

so that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2-2xy+2xy-2=0$$

Therefore they represent a possible case of fluid flow. The rotation w of any fluid element in the flow field is,

$$w=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(xy^2-2y-\frac{x^3}{3}\right)-\frac{\partial}{\partial y}\left(\frac{y^3}{3}+2x-x^2y\right)\right]$$

$$=\frac{1}{2}\left[\left(y^2-x^2\right)-\left(y^2-x^2\right)\right]=0.$$

Example -3. A stream function is given by

$$\psi=3x^2-y^3$$

Determine the magnitude of velocity components at the point (3, 1) **Solution.**

x-component: $u=\frac{\partial\psi}{\partial y}=\frac{\partial}{\partial y}\left(3x^2-y^3\right)=-3y^2.$

y-component: $v=-\frac{\partial\psi}{\partial x}=-\frac{\partial}{\partial x}\left(3x^2-y^3\right)=-6x$

At the point (3,1)

$$u=-3 \quad \text{and } v=-18.$$

and the total velocity is the vector sum of the two components

$$\vec{V}=-3\overline{i}-18\overline{j}$$

Note that ∂u/∂x=0 and ∂v/∂y=0, so that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Therefore the given stream function satisfies the continuity equation.

The equation for vorticity,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

May also be expressed in term ψ

$$\zeta = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

However, for irrotational flow $\zeta = 0$, and the classic Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

This means that the stream functions of all irrotational flows must satisfy the Laplace equation and that such flows may be identified in this manner; conversely, flows whose ψ does not satisfy the Laplace equation are rotational ones. Since both rotational and irrotational flow fields are physically possible, the satisfaction of the Laplace equation is no criterion of the physical existence of a flow field.

Example-4.

A flow field is described by the equation $\psi = y - x^2$. Derive an expression for the velocity v at any point in the flow field. Calculate the vorticity.

Solution.

From the equation for ψ, the flow field is a family of parabolas symmetrical about the y-axis with the streamline ψ = 0 passing through the origin of coordinates.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(y - x^2\right) = 1$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(y - x^2\right) = 2x$$

Which allows the directional arrows to be placed on streamlines as shown. The magnitude V of the velocity may be calculated from

$$V=\sqrt{u^2+v^2}=\sqrt{1+4x^2}$$

and the vorticity

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}(2x)-\frac{\partial}{\partial y}(1)=2\ \sec^{-1} \qquad \text{(Counter clockwise)}$$

Since $\zeta \neq 0$, this flow field is seen to be rotational one.

Example-5.

A stream function in a two-dimensional flow is ψ = 2xy. Show that the flow is irrotational (potential) and determine the corresponding velocity potential function ϕ.

Solution.

The given stream function satisfies the condition of irrotationality, that is,

$$w=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$

$$=\frac{1}{2}\left[\frac{\partial^2}{\partial x^2}(2xy)+\frac{\partial^2}{\partial y^2}(2xy)\right]=0$$

which shows that the flow is irrotational. Therefore, a velocity potential function φ will exist for this flow.

By using

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}=\frac{\partial}{\partial y}(2xy)=2x$$

Therefore,

$$\phi=\int 2x\partial x=x^2+f_1(y) \qquad \text{(a)}$$

From Equ.

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(2xy) = -2y$$

From this equation,

$$\phi = \int -2y \partial y = y^2 + f_2(x) \qquad \text{(b)}$$

The velocity potential function,

$$\phi = x^2 - y^2 + C.$$

satisfies both φ functions in Equations (a) and (b).

Example 6. Give u= -ωy , = ωx , $w = 0$; show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z axis, although the velocity potential does not exist.

Solution: $$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial y} = 0 \text{ ,}$$

so the equation of continuity namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied. Therefore, the motion is quite possible.

The differential equation of lines of flow are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, i.e. \quad \frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0}$$

Integrating, $x^2 + y^2 = const. and\ z = const.$

Now,

u dx + v dy + w dz = $-\omega y\ dx + \ \omega x\ dy,$

Which is not a perfect differential? Therefore velocity potential does not exist.

However u dx + v dy + w dz = 0 gives

$$-\omega y\, dx + \omega x\, dy = 0, i.e. \quad \frac{dx}{x} - \frac{dy}{y} = 0$$

or, $\frac{x}{y} = const.$ or, y=kx (planes through z axis)

which are the surfaces that cut the stream lines orthogonally.

4.8. Summary.

The unit starts with derivation of equation of continuity in vector form after that it is defined in different coordinate systems. The potential flow equation given in the form $\bar{V} = \nabla\emptyset$. The potential line and stream line also describe with suitable examples that helps to understand the concepts of potential flow theory.

4.9. Terminal questions.

Q.1. Find the expression of potential and stream function for line source or sink**.**

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Q.2. Calculate the steady-state velocity field for the flow of an incompressible, invicid fluid around a solid sphere of diameter 2R. The Fluid approaches the sphere with a uniform upstream velocity v_∞. The geometry is the same as in the viscous, creeping-flow calculation; however, in this problem, the fluid inviscid($\mu = 0$) and inertia is not neglected.

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Q.3. Calculate the drag on a sphere in steady potential flow around a sphere (high Reynolds number, inviscid fluid).

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Q.4. What is the pressure distribution around a cylinder in potential flow? The flow field is irrotational.

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Q.5. In a two-dimensional, incompressible flow the fluid velocity components are given by: $u = x - 4y$ and $v = -y - 4x$. Show that the flow satisfies the continuity equation and obtain the expression for the stream function. If the flow is potential (irrotational) obtain also the expression for the velocity potential.

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Unit-5 Euler’s equation of motion, steady motion, Bernaullies equation, Helmholtz equation, Impulsive motion.

Structure:

5.1 Introduction.

The Bernaullie equation is an important equation in fluid mechanics that helps to deal many problems. The Euler's equation is another equation which is a general equation of fluid flow. After integrating the Euler's equation we derived the Bernaullie's equation. In the hydrodynamics the Hemholtz's equation play an significance role which is also govern the motion of fluid influence under impulsive motions.

5.2 Objectives.

After reading this unit students should be able to:

1. To learn the derivation of the Euler's equation.
2. To learn the derivation of the Bernaullie's equation.
3. To learn the derivation of Hemholtz's equation.
4. Understand the impulsive motion.

5.3 Euler's equation of motion:

Consider a stream line in which flow is taking place in direction of flow. Consider a cylindrical element of cross-section dA and length ds. The force acting on the cylindrical element are:

1) Pressure force in the direction of flow pdA.
2) Pressure force in the opposite direction of flow $\left(p + \frac{\partial p}{\partial s} ds\right)$.
3) Weight of element ρgdAds.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of of flow must be equal to the mass of fluid element×acceleration in the direction of flow.

$$\therefore \quad pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dAds \cos\theta = \rho dAds \times a_s. \quad (5.1)$$

Where a_s is the acceleration in the direction of s(direction of flow).

Now

$$a_s = \frac{dv}{dt},$$

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t}. \qquad (5.2)$$

Where v is the function of s and t.

If the flow is study,

$$\frac{\partial v}{\partial t} = 0,$$

$$\therefore\ a_s = \frac{v\partial v}{\partial s}. \qquad (5.3)$$

Substituting the value of a_s in equation (5.1) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} dsdA - \rho g dAds \cos\theta = \rho dads \times \frac{\partial v}{\partial s}$$

Dividing by ρdsdA, we get

$$-\frac{\partial p}{\rho\partial s} - g\cos\theta = \frac{v\partial v}{\partial s},$$

Or

$$\frac{\partial p}{\rho\partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0. \quad (5.4)$$

We have, $\cos\theta = \frac{dz}{d\theta}$.

$$\therefore \qquad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + v\frac{dv}{ds} = 0,$$

$$\Rightarrow \quad \frac{dp}{\rho} + gdz + vdv = 0. \quad (5.5)$$

The equation (5.5) known as Euler's equation.

5.4 Bernoulli;s equation :

Now we integrate the Euler's equation,

$$\int \frac{dp}{\rho} + \int gdz + \int vdv = constant,$$

If flow is incompressible, ρ is constant and

$$\therefore \qquad \frac{p}{\rho} + gz + \frac{v^2}{2} = constant, \quad (5.6)$$

Or

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = constant. \quad (5.7)$$

The equation (5.7) is a Bernoulli's equation in which,

a) $\frac{p}{\rho g}$ is pressure energy per unit weight of fluid or pressure head.
b) $\frac{v^2}{2g}$ is kinetic energy per unit weight or kinetic head.
c) z is potential energy per unit weight or potential head.

The Bernoulli's equation is follow under the following conditions,

i) The fluid is ideal, it means viscosity is zero.
ii) The flow is study.
iii) The flow is incompressible.
iv) The flow is irrotational.

Now in real world the fluid are not inviscid, so they offer resistance to flow. Hence Bernoulli equation will be the form,

$$\frac{p_1}{\rho g} + \frac{v^2{}_1}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2{}^2}{2g} + z_2 + h_L. \qquad (5.8)$$

Where h_L is loss of energy between points 1 and 2.

5.5 Solve examples:

Example-1.

Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm^2 and with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of the water at a cross-section, which is 5m above the datum line.

Solution:

We have given that,

Diameter of pipe = 5cm = 0.05 m,

Pressure p = 29.43 N/cm^2 = 29.43*10^4 n/m^2

Velocity, v = 2.0 m/s

Datum head z= 5m

Total head = pressure head + kinetic head + datum head

Pressure head = $\frac{p}{\rho g} = \frac{29.43\times10^4}{1000\times9.81} = 30,$

Kinetic head = $\frac{v^2}{2g} = \frac{2\times2}{2\times9.81} = 0.204m,$

So total head = $\frac{p}{\rho g} + \frac{v^2}{2g} + z = 30 + 0.204 + 5 = 35.205m.$

Example-2.

Water is flowing through a pipe having diameter 300mm and 200mm at the bottom and upper and respectively. The intensity of pressure at the bottom end is 24.525N/cm^2 and the pressure at the upper end is 9.81 N/cm^2. Determine the difference in datum head if the rate of flow through pipe is 40lit/s.

Solution.

According to the question, we have

Section-1,

D_1 = 300mm = 0.3m.

p_1 = 24.525 N/cm^2 = 24.525 * 10^4 N/m^2

Section-2,

D_2 = 200mm = 0.2m.

p_2 = 9.81 N/cm^2 = 9.81*10^4 N/m^2.

Rate of flow = 40lit/s,

Or

$$Q = \frac{40}{1000} = 0.04\, m^3/s$$

Now $A_1V_1 = A_2V_2 = rate\ of\ flow = 0.04.$

$$\therefore \quad V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D^2{}_1} = \frac{.04}{\frac{\pi}{4}(0.3)^2} = 0.5658\,\frac{m}{s}.$$

$$V_2 = \frac{.04}{V_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{.04}{\frac{\pi}{4}(0.2)^2} = 1.274\, m/s$$

Applying Bernoulli's equation at section (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V^2{}_1}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V^2{}_2}{2g} + z_2,$$

Or $\frac{24.525\times10^4}{1000\times9.81} + \frac{.566}{2\times9.81} + z_1 = \frac{9.81\times10^4}{2\times9.81} + \frac{(1.274)^2}{2\times9.81} + z_2$

Or $25 + .32 + z_1 = 10 + 1.623 + z_2,$

$$z_1 - z_2 = 25.32 - 11.623 = 13.697m.$$

Hence difference in datum head = $z_1 - z_2 = 13.70m.$

Example-3.

A pipe of diameter 400m carries water at a velocity of 25 m/s. The pressure at the point A and B are given as 29.43 N/cm^2 and 22.563 N/cm^2 respectively while the datum head at A and B are 28m and 30m. Find the loss of head between A and B.

Solution:

We have given that,

Dia. Of pipe, $D = 400mm = 0.4m,$

Velocity $V = \frac{25m}{s}.$

So total energy at the point A, $E_A = \frac{p_A}{\rho g} + \frac{V^2{}_A}{2g} + z_A,$

$$= \frac{29.43 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 28,$$

$$= 30 + 31.85 + 28 = 89.85,$$

Total energy at point B,

$$E_B = \frac{p_B}{\rho g} + \frac{{V_B}^2}{2g} + z_{B,}$$

$$= \frac{22.563 \times 10^4}{1000 \times 9.81} + \frac{25^2}{2 \times 9.81} + 30,$$

$$= 23 + 31.85 + 30 = 84.85.$$

Therefore loss of total energy $= E_A - E_B = 89.85 - 84.85 = 5.0m.$

Example-4

Air, obeying Boyle's law, is in motion in a uniform tube of small section; prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t,

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left\{\left(v^2 + k\right)\rho\right\}.$$

Solution: Air obeys Boyle's law

Therefore $\quad p = k\rho \qquad (5.9)$

The equation of continuity in this case

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \qquad (5.10)$$

And the equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

i.e. $\quad \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{k}{\rho}\dfrac{\partial \rho}{\partial x} \qquad$ (from 5.9) $\qquad$ (5.11)

In the result, we require $\dfrac{\partial^2 \rho}{\partial t^2}$ which can be obtained by differentiating (5.10) partially with respect to t, we get

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\frac{\partial(\rho v)}{\partial x} = 0,$$

i.e. $\quad \dfrac{\partial^2 \rho}{\partial t^2} + \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x}(\rho v) = 0,$

i.e. $\quad \dfrac{\partial^2 \rho}{\partial t^2} + \dfrac{\partial}{\partial x}\left[\rho\dfrac{\partial v}{\partial t} + v\dfrac{\partial \rho}{\partial t}\right] = 0,$

In View of (5.10) and (5.11), we have

i.e. $\quad \dfrac{\partial^2 \rho}{\partial t^2} + \dfrac{\partial}{\partial x}\left[\rho\left\{-v\dfrac{\partial v}{\partial x} - \dfrac{k}{\rho}\dfrac{\partial \rho}{\partial x}\right\} + v\left\{-\dfrac{\partial(\rho v)}{\partial x}\right\}\right] = 0,$

i.e. $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\rho v \frac{\partial v}{\partial x} + k\frac{\partial \rho}{\partial x} + v\frac{\partial(\rho v)}{\partial x}\right] = 0,$$

i.e. $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial(\rho v.v)}{\partial x} + k\frac{\partial \rho}{\partial x}\right] = 0,$$

ie. $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left[\rho v^2 + k\rho\right] = 0,$$

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left[\left(v^2 + k\right)\rho\right] = 0.$$

Example-5

An elastic fluid, the weight of which is neglected obeying Boyle's law, is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time at a distance r from a fixed point in the tube is defined by the equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t} + v^2\frac{\partial v}{\partial r}\right) = k\frac{\partial^2 v}{\partial r^2}.$$

Solution:

The fluid obeys Boyle's law

Therefore $p = k\rho$. (5.12)

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial r} = 0, \qquad (5.13)$$

and the equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial \rho}{\partial r},$$

i.e.
$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{k}{\rho}\frac{\partial \rho}{\partial r} \qquad \text{(from 5.12)} \qquad (5.14)$$

In the result, we require $\frac{\partial^2 v}{\partial t^2}$ which can be obtained by differentiating (5.14) partially with respect to t, we get

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial r}+\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right]=0,$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right]=0$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}+\frac{k}{\rho}\left\{-\frac{\partial(\rho v)}{\partial r}\right\}\right]=0,$$

i.e.
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}-\frac{k}{r}\left\{\rho\frac{\partial v}{\partial r}+v\frac{\partial \rho}{\partial r}\right\}\right]=0,$$

i.e.
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}+v\left\{\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}\right\}\right]=0,$$

ie.
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}-k\frac{\partial v}{\partial r}\right]=0,$$

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right]=k\frac{\partial^2 v}{\partial r^2}.$$

5.6 Self-learning questions.

Q.1. A pipe through which water is flowing, is having diameter, 20 cm and 10 cm at the cross-section 1 and 2 respectively of water at section 1 given 4.0 m/s. Find the velocity head at section 1 and 2 and also rate of discharge.

Q.2. The water is flowing through a taper of length 100 m having diameter 600 mm at the upper end and 300 mm at the lower end, at the rate of 50 liters/s. The pipe has a slope of 1 in 30. Find the pressure at the lower at end if the pressure at the higher level is 19.62 N/cm^2.

Q.3. A conical tube of length 2.0 m is fixed vertically with its smaller end upwards. The velocity of flow at the smaller end is 5 m/s while the lower end it is 2 m/s. The pressure head at the smaller end is 2.5 m of liquid. The loss of head in the tube is $\frac{0.35(v_1-v_2)^2}{2g}$, where v_1 is the velocity at the smaller end and v_2 at the lower end respectively. Determine the pressure head at the lower end. Flow take place in the downward direction.

Q.4. State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's equation from first principle and state the assumption made for such a derivation.

Q. 5. Obtain the equations of motion of a liquid and show that, in absence of external forces, the pressure is least where the velocity is greatest.

5.7 Steady flow:

The flow which is characterized on the basis of velocity, pressure, density etc and at any point during the flow, the change of these quantities is zero. This type of flow is called steady flow.

The mathematically this type of flow is express as;

$$\left(\frac{\partial V}{\partial t}\right)_{x_0,y_0,z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0,y_0,z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0,y_0,z_0} = 0. \quad (5.9)$$

Where (x_0, y_0, z_0) is a fixed point in fluid field.

Now the Euler's equation for steady flow we derived on basis of equation (5.9) as equation (5.5).

5.8 Helmholtz's equation.

From Euler's equation for an incompressible fluid in a conservative force field.

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p - \nabla\Omega,$$

Or $\quad \frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - u \times \omega = -\nabla\left(\frac{p}{\rho} + \Omega\right)$, (5.10)

Taking the curl,

$$\nabla \times \frac{\partial u}{\partial t} - \nabla \times (u \times \omega) + \nabla \times \left[\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \Omega\right)\right] = 0$$

We know that,

$$\nabla \times (u \times \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.$$

As ω is always solenoidal and **u** is solenoidal in an incompressible fluid, we obtained

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \qquad (5.11)$$

Which is Helmholtz vorticity equation.

5.9 Impulsive motion.

Viscosity is responsible for retarding or damping forces which can not begin to act until the motion has started. Hence any flow will be initially irrotational everywhere except at actual boundaries.

When a motion started from rest by an instantaneous impulse must be irrotational. Next integrate the Euler's equation over the time interval (t, t+δt),

$$\int_t^{t+\delta t} \frac{Du}{Dt}\,dt = \int_t^{t+\delta t} \boldsymbol{F} dt - \int_t^{t+\delta t} \frac{1}{\rho}\nabla p\,dt.$$

Or

$$[u]\int_{t}^{t+\delta t}. = \int_{t}^{t+\delta t} Fdt - \frac{1}{\rho}\nabla \int_{t}^{t+\delta t} pdt \qquad (5.12)$$

When the limit $limit\ \delta t\ \rightarrow 0$ for start-up by an instantaneous impulse, the impulse of the body force →0 and

$$u - u_0 = -\frac{1}{\rho}\nabla P,$$

Where the fluid responds instantaneously with the impulsive pressure field,

$$P = \int_0^{\delta t} pdt,$$

And the impulse on a fluid element is $-\nabla P$ per unit volume, producing a velocity from rest is,

$$u_0 = -\frac{1}{\rho}\nabla P.$$

This is irrotational as;

$$\nabla \times v = -\frac{1}{\rho}\nabla \times (\nabla \mathrm{P}) \equiv 0. \qquad (5.13)$$

This equation (5.13) represent the impulsive motion produce irrotational motion.

5.10 Solve example

Example-1- *A Sphere of radius R, whose centre is at rest, vibrates radically in an in incompressible fluid of density ρ, which is at rest infinity. If the pressure at infinity is II show that the pressure at the surface of the sphere at time t is*

$$II + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}.$$

$If\ R = a(2 + cos\ \ nt), show\ that, prevent\ cavitation\ in$

$the\ fluid\ will\ take\ place\ in\ such\ a\ manner\ so\ that\ each\ element$

the fluid moves towards the centre. Hence the free surface would be spherical. Thus the flu velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin), and time t only. Let p be pressure at a distance r'. Let be the pressure on the surface of the sphere of radius R and V be the velocity there. Then the equation of continuity is

$$r'^2{}_{v'} = R^2 V = F\ (t) \qquad \ldots\ldots..(1)$$

$$From\ , \qquad \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots\ldots..(2)$$

$$Again\ equation\ of\ motion\ is \qquad \frac{\partial v'}{\partial t} + v'\tfrac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial \rho}{\partial r'}$$

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial \rho}{\partial r'} \qquad using\ (2) \qquad \ldots\ldots\ldots\ldots(3)$$

$Integrating\ with\ respect\ to\ r',\quad (3)\ reduces\ to$

$$\frac{F'(t)}{r'^2} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, C\ being\ an\ arbitrary\ conctant$$

When r', ∞, $then\ v' = 0\ and\ p = II\ so\ that\ C = II/\rho.\ Then, we\ get$

$$\frac{F'(t)}{r'^2} + \frac{1}{2}v'^2 = \frac{II - p}{\rho} \qquad or \qquad p = +\frac{1}{2}\rho\left[2\frac{F'(t)}{r'^2} - v'^2\right]. \qquad \ldots..(4)$$

$But\ p = P\ and\ v' = V\ when\ r' = R.\ Hence\ (4)\ gives$

$$P = II + \frac{1}{2}\rho\left[\frac{2}{R}\{F'(t)\}_{r'=R} - V^2\right] \qquad \ldots\ldots..(5)$$

$Also\ V = \dfrac{dR}{dt}. Henceusing\ (1), we\ have$

$$\{F'(t)\}_{r'=R} = \frac{d}{dt}(R^2V) = \frac{d}{dt}\left(R^2\ \frac{dR}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}.\frac{dR^2}{dt}\right)$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + \frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt} = \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V *and* $\{F'(t)\}_{r'=R,}$ (5) *reduces to*

$$P = II + \frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

$$P = II + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right] \qquad \dots\dots\dots\dots(6)$$

Example-2- *Prove that in the steady motion of an incompressible liquid, under the action of conservative forces . we have* $\xi(\partial u/\partial x) + n(\partial u/\partial x) + \zeta(\partial u/\partial z) = 0$ *and two similar equation in* v *and w.*

Sol. *Helmholtz equation are given by*

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z}$$

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z}$$

For the steady incompressible liquid, $\quad \frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right)\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = 0$

$(1a) \Rightarrow \qquad \xi(\partial u/\partial x) + \eta(\partial u/\partial y) + \zeta(\partial u/\partial z) = 0$

$(1b) \Rightarrow \qquad \xi(\partial v/\partial x) + \eta(\partial v/\partial y) + \zeta(\partial v/\partial z) = 0$

$(1c) \Rightarrow \qquad \xi(\partial w/\partial x) + \eta(\partial w/\partial y) + \zeta(\partial w/\partial z) = 0$

5.11 Summary.

There are many equation that govern the motion of fluid as; Euler's equation, Bernoullie's equation and Helmholtz's equation. The derivation of these equation, $\frac{dp}{\rho} + gdz + vdv = 0, \frac{p}{\rho g} + \frac{V^2}{2g} + z = constant.$ and $\frac{D\omega}{Dt} = (\omega \cdot \nabla)u.$ are described. The equation for impulsive motion given as $\nabla \times v = 0$. The steady flow theory and other topic illustrated with suitable wxamples.

5.12 Terminal Questions.

Q.1. A pipeline carrying oil of specific gravity 0.87, change in diameter from 200 mm diameter at a position A to 500 mm diameter at a position B which is 4 meter at a higher level. If the pressure at A and B are 9.81 N/cm^2 and 5.886 N/cm^2 respectively and the discharge is 200 liters/s determine loss of head and direction of flow.

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Q.2. Find the discharge of water flowing through a pipe 30 cm diameter placed in an inclined position where a venturimeter is inserted, having a throat diameter of 15 cm. The difference of pressure between the main and throat is measure by a liquid of sp. Gr. 0.6 in an inverted U-tube which gives a reading of 30 cm. The loss of head between the main throat is 0.2 times the kinetic head of the pipe.

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Q.3. A nozzle of diameter 20 mm is fitted to a pipe of diameter 40 mm. Find the force excerted by the nozzle on the water which is flowing through the pipe at the rate of 1.2 m^3/minute.

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Q.4. State Bernoulli's theorem for steady flow of an incompressible fluid. Derive an expression for Bernoulli's equation from first principle and state the assumptions made for such a derivation.

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Q.5. Derive Euler's equation of motion along a stream line for an ideal fluid stating clearly the assumptions. Explain how this is integrated to get Bernoulli's equation along a stream-line.

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Unit 6: Motion in two dimensions, stream function, irrotational motion, complex potential, sources and sink.

Structure:

6.1 Introduction.

6.2 Objectives.

6.3 Motion in two dimensions.

6.4 Irrotational motion.

6.5 Stream function.

6.6 Solve examples.

6.7 Self-learning Questions.

6.8 Source and sink.

6.9 Solve example.

6.10 Summary.

6.11 Self-learning Questions.

6.1 Introduction.

The present unit contains theory of motion in two dimension. There are many equation related to the motion of fluid as Elure's equation, stream function, irrotational motion, complex potential etc. The motion for source and sink also discuss in the unit.

6.2 Objectives.

After reading this unit students should be able to:

1. Understand the Euler's equation for two dimensions.
2. Understand stream function and irrotational motion.
3. To stream function and velocity potential for source and sink.
4. To learn, find out velocity from velocity potential.
5. To learn, solve problems in two dimension flow.

6.3 Motion in two dimension.

Consider the homogeneous, incompressible, invicid fluid in two dimension (x, z).

Then the Euler's equation of motion are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad (6.1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \quad (6.2)$$

and the equation of continuity is,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (6.3)$$

In this fluid the non zero vorticity component will only one along y-components as: ω=(0, η, 0) where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad (6.4)$$

Now from equations (6.1)-(6.2) and using (6.3), get

$$\frac{D\eta}{dt} = 0. \qquad (6.5)$$

This is a powerful and useful constraint. Which state that fluid particles conserve there vorticity. When η=0, for all particles, such flow are called ***irrotational.***

6.4 Irrotational flow:

A flow field with velocity vector **U** is said to be rotational if curl**U**≠0; otherwise it is irrotational.

So the governing equation of irrotational flow is given by equation (6.4), considering as η=0.

Consider a steady, uniform flow U past a cylinder of radius a. All fluid particles originated from far upstream ($x \to -\infty$) where u=0, w=0, and therefore η=0. It follows that fluid particle particles have zero vorticity for all time.

6.5 Stream function:

It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ. Mathematically, for steady flow defined by equation (6.6).

From equation of continuity we can introduce a stream function ψ, define by equations

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}. \qquad (6.6)$$

Then from equation of continuity we get,

$$\eta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}. \qquad (6.7)$$

So for irrotational flow η=0, and ψ satisfies Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \qquad (6.8)$$

Relation between stream function and velocity potential function.

We have the velocity potential function,

$$u = -\frac{\partial \varphi}{\partial x} \ \ and\ v = -\frac{\partial \varphi}{\partial z} \quad (6.9)$$

Hence from equation (6.6) and (6.9) we get,

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial z} \\ \frac{\partial \varphi}{\partial z} = -\frac{\partial \psi}{\partial x} \end{aligned}\right\}. \quad (6.10)$$

6.6 Solve example:

Example-1. Find stream function on a solid boundary and normal velocity must be zero.

Solution:

If normal velocity on a solid boundary is zero, then $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, on the boundary. If $\boldsymbol{n} = (n_1, 0, n_2)$, Then

$$n_1 \frac{\partial \psi}{\partial z} - n_2 \frac{\partial \psi}{\partial x} = 0,$$

Or

$$\boldsymbol{n} \wedge \boldsymbol{\nabla} \psi = 0.$$

It show that $\nabla\psi$ in the direction of **n,** where ψ is a constant on the boundary.

Example-2.

Find stream function in the region outside the cylinder (as r > a) and subject to the boundary conditions,

$$(U, 0, 0)\ as\ r \to \infty.$$

Solution:

The boundary condition of cylindrical flow is given as,

$$u = \left(\frac{\partial\psi}{\partial z}, 0, \frac{\partial\psi}{\partial x}\right) \rightarrow (U, 0, 0)\ as\ r \rightarrow \infty,$$

and

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0\ on\ r = a,$$

Where $r = (x^2 + y^2)^{\frac{1}{2}}$.

Now the solution of Laplace equation on given boundary conditions, in cylindrical polar coordinate is,

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta.$$

Which is the required stream function for given boundary conditions.

Example-3.

A stream function is given by $\psi = 5x - 6z$. Calculate the velocity component and also magnitude and direction of the resultant velocity at any point.

Solution.

We have

$$\psi = 5x - 6z$$

$$\therefore\ \frac{\partial\psi}{\partial x} = 5\ and\ \frac{\partial\psi}{\partial z} = -6.$$

Here the velocity component u and v in terms of stream function are given by,

$$u = -\frac{\partial\psi}{\partial z} = -(-6) = 6\frac{unit}{sec}.$$

$$v = \frac{\partial\psi}{\partial x} = 5\ unit/sec$$

Resultant velocity $=\sqrt{u^2 + v^2} = \sqrt{6^2 + 5^2} = \sqrt{61} = 7.81\frac{unit}{sec}$.

Direction is given by, $\tan\theta = \frac{v}{u} = \frac{5}{6} = 0.833$.

Example-4.

The velocity potential for a flow is given by $\phi(x, y, t) = (-3x + 5y)\cos\omega t$ where ω is a constant. Determine the stream function for the flow.

Solution:

Given $\qquad \phi(x, y, t) = (-3x + 5y)\cos\omega t$

Thus, $\qquad \psi_y = \phi_x = -3\cos\omega t$

$$\Rightarrow \quad \psi = -3y\cos\omega t + f(x)$$

Further, $\psi_x = -\dfrac{\partial\phi}{\partial y} = 5\cos\omega t = f'(x)$

$\Rightarrow \quad f(x) = 5x\cos\omega t + K$ (K is arbitrary and is chosen as zero)

Thus, the stream function is given by

$$\psi = (5x - 3y)\cos\omega t \ .$$

Example-5

Suppose the stream function are given by $\psi(x, y) = xy$ which represent flow around a rectangular corner. Find the velocity potentials for the flow if exist.

Solution:

Given

$$\psi(x, y) = xy$$

Therefore

$$\nabla^2\psi = 0 .$$

Thus, the flow is irrotational. Thus, there exists a velocity potential ϕ which will satisfy

$$u = \phi_x = \psi_y = x$$

$$\Rightarrow \phi = \frac{x^2}{2} + f(y) \qquad \text{(A)}$$

$$v = \phi_y = -\psi_x = -y$$

$$\Rightarrow \phi = -\frac{y^2}{2} + g(x) \qquad \text{(B)}$$

From (A) and (B),

$$f(y) = -\frac{y^2}{2}, \qquad g(x) = \frac{x^2}{2}.$$

Thus

$$\phi = \frac{1}{2}\left(x^2 - y^2\right)$$

6.7 Self-learning questions.

Q.1. Find the Laplace equation of stream function example (2), when r is very large.

Q.2. If for a two-dimensional potential flow, the velocity potential is given by $\varphi = x(2z - 1)$. Determine the velocity at the point P(4,5). Determine also the value of stream function ψ at the point P.

Q.3. The stream function for a two-dimensional flow is given by ψ=2xz, calculate the velocity at the point p(2,3). Find the velocity potential function φ.

Q.4. The velocity components in a two-dimensional flow field for an incompressible fluid are as follow:

$$u = \frac{z^3}{3} + 2x - x^2 z \text{ and } v = xz^2 - 2z - \frac{x^3}{3},$$

Obtain an expression for the stream function ψ.

Q. 5. Determine the flow near a stagnation point in the xy plane

6.8 Source and Sink.

Sourcs: The source flow is the flow coming from a point and out radially in all direction of a plane at uniform rate.

Let u_r = radial velocity of flow at a radius r from the any source O.

q = volume flow rate per unit depth,

r = radius.

The radial velocity u_r at any radius r is given by,

$$u_r = \frac{q}{2\pi r} \qquad (6.11),$$

The equation (6.11) show that with increase of r, the radial velocity decrease, and at a large distance away from the source, the velocity will become to zero. The flow is in radial direction, hence tangential velocity u_θ =0.

Next we discuss the equation of stream function and velocity potential function for the source flow.

Equation of stream function

The radial velocity and tangent velocity components in terms of stream function are given by,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \text{ and } u_\theta = -\frac{\partial\psi}{\partial r} \qquad (6.12)$$

But here

$$u_r = \frac{q}{2\pi r},$$

$$\therefore \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{q}{2\pi r},$$

Or

$$d\psi = r\frac{q}{2\pi r}d\theta = \frac{q}{2\pi}d\theta,$$

Integrating the equation the equation w.r.t θ, we get

$$\psi = \frac{q}{2\pi} \times \theta + C_{1,}$$

Where C_1 is integrating constant.

Let ψ=0, when θ=0, then C_1=0.

So function of stream function will be,

$$\psi = \frac{q}{2\pi}.\theta \qquad (6.13)$$

where q is constant.

The equation (6.13) represents the stream function for source.

Equation of Potential function.

According definitions, the radial and tangential components in term of velocity function are given by,

$$u_r = \frac{\partial\varphi}{\partial r} \text{ and } u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta},$$

We know that

$$u_r = \frac{q}{2\pi r},$$

From two value of u_r we get,

$$\frac{\partial\varphi}{\partial r} = \frac{q}{2\pi r}, or\ d\varphi = \frac{q}{2\pi r}dr,$$

Integrating the above equations,

$$\int d\varphi = \int \frac{q}{2\pi r}.dr,$$

Or

$$\varphi = \frac{q}{2\pi}\int \frac{1}{r}dr = \frac{q}{2\pi}\log_e r. \qquad (6.14)$$

The equation (6.14) show that the velocity potential is a function of r.

Sink flow,

When the fluid flow toward a point along a radial direction and disappear at a point with constant rate called sink flow. The sink flow just opposite to the source flow. The pattern of stream line and equipotential line of a sink flow is the same as that source flow. All the equations derived for a source flow shall hold to good for sink flow, when q is replaced by (-q).

6.9 Solve Examples.

Example-1.

Find equation of stream lines for a uniform flow of:

(i) 5m/s parallel to the positive direction of the x-axis and

(ii) 10m/s parallel to the positive direction of the y-axis.

Solution.

(i) The stream function for a uniform flow parallel to the positive direction of x-axis given by as,

$$\psi = U \times y,$$

The above equation show that stream line are straight lines parallel to the x-axis at a distance y from the x-axis. Here U=5m/s and hence above equation becomes as,

$$\psi = 5y,$$

For y = 0, stream function $\psi = 0$,

For y = 0.2, stream function $\psi = 1$ unit,

For y = 0.4, stream function $\psi = 2$ unit, The other values of stream function can be obtained by substituting the different value of y.

(ii) The stream function for a uniform flow parallel to the positive direction of the y-axis is given by equation as,

$$\psi = -U \times x,$$

The above equation show that the stream lines are parallel to the y-axis at a distance x from the y-axis. Here U = 10 m/s and equation becomes as,

$$\psi = -10x.$$

The negative sign show that the stream lines are in the downward direction.

For x = 0, the stream function ψ = 0,

For x = 0.1, the stream function ψ = -1 unit,

For x = 0.2, the stream function ψ = -2 unit,

For x = 0.3, the stream function ψ = -3 unit,

The other value of the stream function can be obtained by substituting the different value of x.

Example-2.

Determine the velocity of flow at radii of 0.2 m, 0.4 m, and 0.8 m, when the water is flowing radially downward in a horizontal plane from a source at a strength of $12 m^2/s$.

Solution. Given:

Strength of source, q = 12 m^2/s,

The radial velocity u_r at any radius is given by,

$$u_r = \frac{q}{2\pi r},$$

When r = 0.2 m, $u_r = \frac{12}{2\pi \times 0.2} = 9.55 \frac{m}{s}$.

When r = 0.4 m, $u_r = \frac{12}{2\pi \times 0.4} = 4.77\frac{m}{s}$.

When r = 0.8 m, $u_r = \frac{12}{2\pi \times 0.8} = 2.38\frac{m}{s}$.

Example-3

Consider a source and a sink each of strength m are located at distance c in either side of the origin.

Solution:

Assuming that the source of strength m is located (c, 0) and sink of strength m is located at (-c, 0) , the complex potential given by
$w(z) = -m\ln(z+c) + m\ln(z-c)$ which yields
$\phi + i\psi = -m\ln(x+c+iy) + m\ln(x-c+iy)$

Therefore the stream function ψ is obtained as

$$\psi = m\tan^{-1}\frac{y}{x-c}\, m\tan^{-1}\frac{y}{x+c}$$

$$= m\tan^{-1}\left(\frac{\frac{y}{x-c} - \frac{y}{x+c}}{1+\frac{y^2}{x^2-c^2}}\right) = m\tan^{-1}\frac{2cy}{x^2-c^2}$$

Thus, the streamlines are given by $\frac{2cy}{x^2+y^2-c^2} = \tan\frac{\psi}{m}$ which is rewritten as

$x^2 + y^2 - 2cy\cot\frac{\psi}{m} - c^2 = 0$, where ψ is assume to be constant.

The above streamline represent a circle with radius $c\sqrt{1+\cot^2\frac{\psi}{m}}$ and Centre $\left(0, c\cot\frac{\psi}{m}\right)$ lying on the y-axis. Each value of ψ will give a stream line.

Example-4

For the complex potential $w(z)=ua(z/a)^{\pi/\alpha}$, Find the stagnation point.

Solution.

For stagnation point

$$\frac{dw}{dz}=0,$$

Thus, for $w(z)=ua(z/a)^{\pi/\alpha}$, the stagnation points are for $z^{\frac{\pi}{\alpha}-1}=0$.

Therefore, if $\pi<\alpha$, the stagnation point is at infinity. On the other hand, if $\pi>\alpha$, the stagnation point is at z=0.

Example-5

Discuss the flow pattern for the complex potential $w(z)=z^2$.

Solution.

Given

$$w(z)=z^2,$$

$$\Rightarrow \phi+i\psi=x^2-y^2+2ixy,$$

$$\Rightarrow \quad \psi=2xy,$$

Which yield xy=constant as the streamlines.

Thus, streamlines are rectangular hyperbolas representing flow around a rectangular corner.

6.10 Summary.

The equation of motion in two dimension give the result $\frac{D\eta}{dt}=0$. The irrotational equation of motion can be find out putting as $\eta=0$. There are defined the terms stream function, velocity potential function, source and sink. The equation of stream function and velocity potential derived for source and sink as, $\psi = \frac{q}{2\pi}\theta, and\ \phi = \frac{q}{2\pi}\log_e r$.

6.11 Terminal questions.

Q.1. Two discs are placed in a horizontal plane, one over the other. The water enters at the center of the lower disc and flow radially outward from a source of strength 0.62 m^2/s. The pressure at a radius 50 mm, is 200 kN/m^2. Find:

(i) pressure in kN/m^2 at a radius of 500mm and

(ii) stream function at angle of 30^0 and 60^0 if $\psi = 0$ at $\theta = 0^0$.

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Q.2. A source and a sink of strength 4 m^2/s and 8 m^2/s are located at (-1, 0) and (1, 0) respectively. Determine the velocity and stream function at a point P(1, 1) which is lying on the flownet of the resultant stream line.

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Q.3. State if the flow represented by $u = 3x + 4y$ and $v = 2x - 3y$ is rotational or irrotational.

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Q.4. A open circular cylinder of 20 cm diameter and 100 cm long contains water upto a height of 80 cm. It rotated about its vertical axis. Find the speed of rotation when:

(i) no water spills, (ii) axial depth is zero.

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Q.5. A fluid is given by: $V = 10x^3 i - 8x^3 yj$. Find the shear strain rate and state whether the flow is rotational or irrotational.

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Unit-7: Doublets, image system of a simple source with respect a plane, a circle, a sphere, image system of a doublet with respect to a plane, a circle and a sphere, circle theorem.

Structure:

7.1 Inroduction.

7.2 Objectives.

7.3 Doublet.

7.4 Solve Examples.

7.5 Self-learning Questions.

7.6 Equation of solid sphere in a plane source.

7.8 Solve Examples.

7.5 Summary.

7.6 Terminal Questions.

7.1 Introduction.

This unit contains the theory of doublet. It stars from definition of doublet. Next stream function and potential function has been discussed for doublet. The equation of circle and sphere of doublet given for study of uniform flow.

7.2 Objective.

After reading this unit students should be able to:

Understand the concept of doublet.

To learn, calculation of stream function using doublet.

To learn, calculation of potential function using doublet.

Develop skills to solve the problems of combination source and sink.

To learn, equation of image circle.

7.3 Doublet:

The doublet is a special case of a source and sink pair with equal strength when the two approach each other in such a way that the distance 2a between them approach zero and product 2a.q remain constant. This product 2a.q is known as doublet strength and denoted by μ.

Let q and (-q) is the strength of source and sink respectively. Let 2a be the distance between them and P be any point in combined field of source and sink.

Let θ is the angle made by P at A where (θ+δθ) is the angle at B.

$$\psi = \frac{q}{2\pi}\theta - \frac{q}{2\pi}(\theta + \delta\theta) = -\frac{q}{2\pi}\delta\theta. \quad (7.1)$$

From B, draw BC $\perp$ on AP. Let AC=δr, CP=r+δr, also angle BPC=δθ. The distance BC can be taken equal to r.δθ. In triangle ABC, angle BCA=90^0 and hence distance BC is also equal to $2a \cdot \sin\theta$.We get,

$$r \times \delta\theta = 2a.\sin\theta$$

$$\delta\theta = \frac{2a \cdot \sin\theta}{r}. \quad (7.2)$$

Substituting the value of δθ in equation (7.1), we get

$$\psi = \frac{q}{2\pi}\frac{2a\sin\theta}{r} = \frac{\mu}{2\pi}\frac{\sin\theta}{r},$$

Where 2a.q=μ.

When $2a \to 0$, the angle δθ subtended by point with A and B becomes very small. Also $\delta r \to 0$ and AP becomes equal to r. Then

$$\sin\theta = \frac{PD}{AP} = \frac{y}{r},$$

Also $r^2 = x^2 + y^2$,

Substituting the value of sin θ in equation (7.2), we get

$$\psi = -\frac{\mu}{2\pi}\frac{y}{r}\frac{1}{r} = -\frac{\mu y}{2\pi r^2} = -\frac{\mu y}{2\pi(x^2+y^2)},$$

$$x^2 + y^2 + \frac{\mu y}{2\pi\psi} = 0,$$

The above equation can be written as,

$$x^2 + \left(y + \frac{\mu}{4\pi\psi}\right)^2 = \left(\frac{\mu}{4\pi\psi}\right)^2. \qquad (7.3)$$

The above is the equation of a circle with center $\left(0, \frac{\mu}{4\pi\psi}\right)$, and radius $\frac{\mu}{4\pi\psi}$.

Potential function at P:

The potential function at P is given by,

$$\phi = \frac{q}{2\pi}\log_e(r+\delta r) + \left(-\frac{q}{2\pi}\right)\log_e r,$$

$$= \frac{q}{2\pi}[\log_e(r+\delta r) - \log_e r] = \frac{q}{2\pi}\log_e\left(\frac{r+\delta r}{r}\right),$$

$$= \frac{q}{2\pi}\left[\frac{\delta r}{r} + \left(\frac{\delta r}{r}\right)^2\frac{1}{2} + \cdots \ldots\right],$$

$$= \frac{q}{2\pi} \cdot \frac{\delta r}{r}.$$

(as $\frac{\delta r}{r}$ is a small quantity. So next terms are becomes negligible)

From triangle ABC, We get $\frac{\delta r}{2a}\mu = \cos\theta$, so $\delta r = 2a\cos\theta$.

Substituting these value of δr we get,

$$\phi = \frac{q}{2\pi} \times \frac{2a\cos\theta}{r} = \frac{\mu}{2\pi}\frac{\cos\theta}{r}, \qquad (7.4)$$

When $2a \to 0$, the angle δθ becomes very small. Also $\delta r \to 0$, and AP becomes equal to r . Then,

$$\cos\theta = \frac{AD}{AP} = \frac{x}{r},$$

Where $r^2 = x^2 + y^2$,

Substituting the value of cos θ in equation (7.4), we get

$$\phi = \frac{\mu}{2\pi}\frac{x}{r}\frac{1}{r} = \frac{\mu}{2\pi}\frac{x}{r^2},$$

$$= \frac{\mu}{2\pi}\frac{x}{(x^2 + y^2)}.$$

Or

$$x^2 + y^2 - \frac{\mu}{2\pi}\frac{x}{\phi} = 0,$$

The above equation can be written as,

$$\left(x - \frac{\mu}{2\pi\phi}\right)^2 + y^2 = \left(\frac{\mu}{4\pi\phi}\right)^2. \qquad (7.5)$$

The above equation represent circle.

7.4 Solve Examples

Example-1. A point P(0.5, 1) is situated in the flow field of a doublet of strength 5 m^2/s. Calculate velocity at this point and also the value of the stream function.

Solution. Given: Point P(0.5, 1). This means x= 0.5 and y=1.0,

Strength of doublet, $\mu = 5\ m^2/s$.

(i) Velocity at point P,

The velocity at the given point can be obtained if we know the stream function. But the stream function is given by equation as,

$$\psi = -\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)},$$

The velocity component u and v are obtained from the stream function as,

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{\mu}{2\pi} \times \frac{y}{(x^2 + y^2)}\right],$$

$$= -\frac{\mu}{2\pi}\frac{\partial}{\partial y}\left[\frac{y}{x^2 + y^2}\right],$$

$$= -\frac{\mu}{2\pi}\left[\frac{x^2 - y^2}{(x^2 + y^2)^2}\right].$$

$$\text{And } v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[-\frac{\mu}{2\pi} \times \frac{y}{(x^2+y^2)}\right],$$

$$= -\frac{\mu}{2\pi}\left[\frac{2xy}{(x^2 + y^2)^2}\right].$$

Substituting the values of $\mu = 5\frac{m^2}{s}, x = 0.5\ and\ y = 1.0,$ we get the velocity components as,

$$u = -\frac{5}{2\pi}\left[\frac{0.5^2 - 1^2}{(0.5^2 + 1^2)^2}\right] = -0.382,$$

$$v = -\frac{5}{2\pi}\left[\frac{2 \times 0.5 \times 1}{(0.5^2 + 1^2)^2}\right] = -509,$$

The resultant velocity,

$$V = \sqrt{u^2 + v^2} = \sqrt{(-0.382)^2 + (-0.509)^2} = 0.636\ m/s$$

(ii) Value of stream function at point P,

$$\psi = -\frac{\mu}{2\pi}\frac{y}{(x^2+y^2)} = -\frac{5}{2\pi}\frac{1.0}{(0.5^2+1^2)} = -\frac{5}{2\pi}\frac{1}{1.25}$$

$$= -0.636\frac{m^2}{s}.$$

Example-2

Determine the function for $= c(x^2 - y^2)$, which represents a possible flow phenomenon.

Solution:

We have given that, potential function

$$\varphi = c(x^2 - y^2) \qquad (1)$$

Suppose stream function is given by,

$$\psi(x, y) = c_1 \qquad (2)$$

Then by using Cauchy-Riemann equations,

$$\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = 2cx \qquad (3)$$

Integrating (3) with respect to y,

$$\psi(x, y) = 2cxy + f(x) \qquad (4)$$

Where $f(x)$ is arbitrary function of x, and equation (4) gives the required stream function.

Example-3

Find the equation of stream line for the flow:

$q = -(3y^2)\hat{\imath} + (-6x)\hat{\jmath}$ at the point (1, 1).

Solution:

The equation of stream lines are given by definition of stream lines

$$q \times dr = 0$$

$$\frac{dx}{u} = \frac{dy}{v}$$

Here we have $u = -(3y^2)$ *and* $v = -6x$

So,

$$\frac{dx}{-3y^2} = \frac{dy}{-6x}$$

$$\frac{2dx}{y^2} = \frac{dy}{x}$$

or $$2xdx = y^2dy$$

Integrating both side we have,

$$x^2 = \frac{1}{3}y^3 + c\ , where\ c\ is\ integrarting\ constant.$$

So at the point (1, 1)

$$c = \frac{2}{3}$$

Hence the equation of stream lines for flow field is given by,

$$3x^2 = y^3 + 2.$$

Example-4

What is the arrangement of sources and sink will give rise for complex potential function

$$w = \log\left(z - \frac{1}{z^2}\right).$$

Solution:

we have given that a complex potential ,
$$w = \log\left(z - \frac{1}{z^2}\right) = \log\left(\frac{z^3 - 1}{z^2}\right)$$

$$w = \log(z^3 - 1) - 2\log z \qquad (1)$$

Since

$$z^3 - 1 = 0 \ \Rightarrow z^3 = 1$$

$$z = (1)^{\frac{1}{3}} = (\cos 2n\pi + i\, sin\ 2n\pi)^{\frac{1}{3}}$$

Using De Moivre's Theorem

$$z = (\cos\frac{2n\pi}{3} + i\, sin\ \frac{2n\pi}{3})$$

at, n=-1, 0, 1 we have,

$$at \quad n = -1, \qquad z = (x + iy) = (\cos\frac{-2\pi}{3} - i\, sin\ \frac{2\pi}{3})$$

$$x = \frac{-1}{2}, y = \frac{-\sqrt{3}}{2} \qquad i.e.\ \ (x, y) = \left(\frac{-1}{2},\ \frac{-\sqrt{3}}{2}\right).$$

and

$$at \quad n = 0, \qquad z = (x + iy) = (\cos 0 - i\, sin\ 0)$$

$$x = 1, \quad y = 0 \qquad i.e.\ \ (x, y) = (1, 0).$$

and

$$at \quad n = 1, \qquad z = (x + iy) = (\cos\frac{2\pi}{3} + i\, sin\ \frac{2\pi}{3})$$

$$x = \frac{-1}{2}, y = \frac{\sqrt{3}}{2} \qquad i.e. \ (x, y) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right).$$

Thus the complex potential due to sink of strength -1 is placed at (1,0) , $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$ and due to source having strength 2 is placed at origin.

Example-5 Find the velocity potential and stream function of the two dimensional fluid motion due two equal sources and equal sinks midway between them.

Solution:

The complex potential of the fluid motion at any point P(z) is given by

$$w = -m\log(z-a) - m\log(z+a) + m\log(z)$$

$$w = -m\log(z^2 - a^2) + m\log(z)$$

$$= -m\log(x^2 - y^2 - a^2 + 2xyi) + m\log(x+iy),$$

So,

$w = \phi + i\psi$= velocity potential + i stream function

Hence the velocity potential,

$\phi = \frac{m}{2}\left(\ln(x^2+y^2) - ln\left((x^2-y^2-a^2) + 4x^2y^2\right)\right)$

And the stream function is given by

$$\psi = m\left(tan^{-1}\left(\frac{y}{x}\right) - tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)\right).$$

And the fluid speed is given by $v = \left|\frac{dw}{dz}\right|$.

Example-6 *Find the stream function of the two-dimencial motion due to equal sources sink situated midway between them.*

Sol. Let there be two sources of strength m at the points $z = a \; and \; z = a \; and \; a \; sink \; at \; of \; same \quad z = 0$ *(origin). Then complex potential w due to these sources and sink is given by*

$$w = -m \; log \; (z - a) - m \; log \; (z + a) + m \; log \; (z - 0)$$

$$\phi + i\Psi = m \; log \; (x + iy) - m \; log \; (x + iy - a) - m \; log \; (x + iy + a)$$

$$\phi + i\Psi = m \; log \; (x + iy) - m \; log \; \{(x - a) + iy\} - m \; log \; \{(x + a) + iy\}$$

$$\phi + i\Psi = m\{(1/2) \times log(x^2 + y^2) + i \; tan^{-1}(y/x)\} - m[(1/2) \times log\{(x - a)^2 + y^2\}$$

$$+ i \; tan^{-1}\{y/(x - a)\}] - m[(1/2) \times log\{(x - a)^2 + y^2\} + i \; tan^{-1}\{y/(x + a)\}]$$

imaginers parts on both sides, we get

$$\Psi = m \; tan^{-1}(y/x) - m[tan^{-1}\{y/(x - a)\} + tan^{-1}\{y/(x + a)\}]$$

$$\frac{\Psi}{m} = tan^{-1}\frac{y}{x} - tan^{-1}\frac{\{y/(x - a)\} + \{y/(x + a)\}}{1 - \{y/(x - a)\}\{y/(x + a)\}} + tan^{-1}\frac{y}{x} - tan^{-1}\frac{2xy}{x^2 - y^2 - a^2}$$

$$\frac{\Psi}{m} = tan^{-1}\frac{(y/x) - \{2xy/(x^2 - y^2 - a^2)\}}{1 + (y/x)\{2xy/(x^2 - y^2 - a^2)\}} \qquad or \qquad \Psi = m \; tan^{-1}\frac{y(x^2 + y^2 + a^2)}{x(x^2 - y^2 - a^2)}$$

Example-7- *Use the method of images to prove that if there be a source m at the point z_0 in a pounded be the lines* $\theta = 0 \; and \; \theta = \pi/3$, *the solution is*

$\phi + i\Psi = -m\log\{(z^3 - z_0^3)(z^3 - z'^3_0)\}$ *where* $z_0 = x_0 + iy_0$

and $z_0' = x_0 - iy_0$.

Sol. Consider the following conformal transformation from $z - plane$ $(xy - plane)$ *to* $\zeta - plane)$*:*

$\zeta = z^3$ *where* $z = re^{i\theta}$ *and* $\zeta = Re^{i\Theta}$ ……… (1)

$This \Rightarrow Re^{i\Theta} = r^3e^{3i\theta} \Rightarrow R = r^3$ *and* $\Theta = 3\theta$

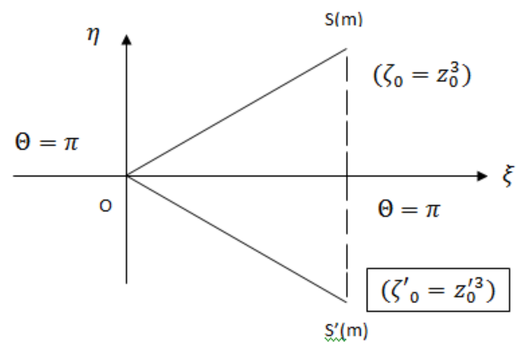
Hence the boundaries $\theta = 0$ *and* $\theta = \pi/3$ *in* $z - plane$ *transform* $\theta = 0$ *and* $\Theta = \pi$ *i.e., real axis in* $\zeta - plane$. *The point* z_0 *in* $z - plane$ *forms to point* ζ_0 *in* $z - plane$ *such that* $\zeta_0 = z_0^3$. *Hence the image with respect to real axis in* $\zeta - plane$ *consists of*

(i) *a source m at* $\zeta_0 = z_0^3$ (ii) *a source m at* $\zeta_0^3 = z_0'^3$

Hence, $w = -m\, log(\zeta - \zeta_0) - m\, log\,(\zeta - \zeta_0')$

$w = -m\, log(z^3 - z_0^3) - m\, log(z^3 - z_0'^3)$

$\phi + i\Psi = -m\log\{(z^3 - z_0^3)(z^3 - z'^3_0)\}$.



7.5 Self-learning questions.

Q.1. A uniform flow with velocity of 3 m/s is flowing over a plane source of strength 30 m^2/s. The uniform flow and source flow are in the same plane. A point P is situated in the flow field. The distance of the point P from the source is 0.5 m and it is at the angle of 30^0 to the uniform flow. Determine: (i) stream function at point P, (ii) resultant velocity of flow at P and (iii) location of stagnation point from the source.

Q.2. A uniform flow with a velocity of 2 m/s is flowing over a source placed at the origin. The stagnation point occurs at (-0.389, 0).

Determine the strength of source.

Q.3. A uniform flow of 12 m/s is flowing over a doublet of strength 18 m^2/s. The doublet is in the line of the uniform flow. Determine:

(i) shape of the Rankine oval,

(ii) Value of stream line function at Rankine circle.

Q. 4. Prove that for liquid circulating irrotationally in part of the plane between two non-intersecting circles the curve of constant velocity are Cassini's ovals.

Q.5. if a homogeneous liquid is acted on by a repulsive force from the origin, the magnitude of which at distance r from the origin is μr per unit mass, shew that it is possible for the liquid to move steadily, without being constrained by any boundaries, in the space between one branch of the hyperbola $x^2 - y^2 = a^2$ and the asymptotes, and find the velocity potential.

7.6 Equation of solid sphere in a plane source with uniform flow.

Let the source is placed on the origin O. Consider a point P(x,y) lying in the resultant flow with polar co-ordinate r and θ.

The stream function ψ and potential function φ for the resultant flow obtained as given:

Ψ = Stream function due to uniform flow + stream function due to source,

$$\psi = U \cdot y + \frac{q}{2\pi}\theta,$$

$$= U \cdot r \sin\theta + \frac{q}{2\pi}\theta. \qquad (7.6)$$

And

Φ = Velocity potential function due to uniform low + Velocitypotential function due to source.

$$\emptyset = U \cdot x + \frac{q}{2\pi}\log_e r = U \cdot r\cos\theta + \frac{q}{2\pi}\log_e r. \qquad (7.8)$$

At the stagnation point θ=180^0 and net velocity is zero. So the at stagnation point free stream velocity U and source strength q. Hence at the stagnation point, the value of stream function is obtained as:

$$\psi = U \cdot r \sin\theta + \frac{q}{2\pi} \cdot \theta$$

For stagnation point,

$$\psi_s = U \cdot r_s \sin 180^0 + \frac{q}{2\pi} \cdot \theta,$$

$$= 0 + \frac{q}{2} = \frac{q}{2} \qquad (7.7)$$

This equation of stream line passing through the stagnation point. We know that no fluid mass cross a stream line. Hence a stream line is a ***solid sphere***.

7.7 Solve Example.

Example-1.

A uniform flow with a velocity of 3 m/s is flowing over a plane source of strength 30 m^2/s. The uniform flow and source flow are same plane. A point P is situated in the flow field. The distance of the point P from the source is 0.5 m and it is at angle of 30^0 to the uniform flow. Determine: (i) stream function at the point P, (ii) resultant velocity of flow at P .

Solution.

Given: Uniform velocity U = 3 m/s; source strength, q = 30 m^2/s; co-ordinate of point P are r = 0.5 m and θ = 30^0.

(i) Stream function at point P.

The stream function at any point in the combined flow is given by,

$$\psi = U \cdot r \sin\theta + \frac{q}{2\pi} \cdot \theta,$$

At point P, r = 0.5 m and θ=30^0.

So stream function at point P,

$$\psi = 3 \times 0.5 \times \sin 30^0 + \frac{30}{2\pi} \times \left(\frac{30}{180} \times \pi\right),$$

$$= 0.75 + 2.5 = 3.25 \frac{m^2}{s}.$$

(ii) Resultant velocity at P

The velocity components any where in the flow are given by,

$$u_r = U \cdot \cos\theta + \frac{q}{2\pi r},$$

$$= 3 \times \cos 30^0 + \frac{30}{2\pi \times 0.5},$$

$$= 2.598 + 9.55 = 12.14.$$

Example-2

Use the method of images to show that if there be a source of strength m at point z_0 in a fluid bounded by the lines $\theta = 0 \; and \; \theta = \frac{\pi}{2}$, the solution is,

$$\varphi + i\psi = -m \log\{(z^2 - z_0^3)(z^2 - z_0'^3)\}$$

Where $z_0 = x_0 + i\, y_0 \quad and \; z_0' = x_0' + i\, y_0'.$

Solution:

Let us transform the z-plane to $\zeta - plane$ by using the transformation

$\zeta = z^3, where \; z = re^{i\theta} \; and \; \zeta = Re^{i\phi}$

so that $Re^{i\phi} = r^3 e^{3i\theta} \quad i.e. R = r^3 \; and \; \varphi = 3\theta.$

Therefore, the boundaries

$$\theta = 0, \;\; \theta = \frac{\pi}{3} \; in\; z - plane\; transform\; to\; \phi = 0 \; and \; \varphi = \pi$$

$i.e. real\; axis\; in\;\; \zeta - plane.$

The image system of sources m at the point z_0 in z-plane i.e. $\zeta_0 = z_0^3 \; in\; \zeta - plane\; with\; respect\; to\; real\; axis\; (\varphi = 0, \varphi = \pi)$ consists of

(i) a source m at $\zeta_0 = z_0^3$, and

(ii) a source m at $\zeta_0' = {z_0'}^3$

Hence the complex potential is given by

$$w = -m\log(\zeta - \zeta_0) - m\log(\zeta - \zeta_0')$$

$$= -m\log(z^3 - z_0^3) - m\log(z^3 - {z_0'}^3),$$

$$= \phi + i\,\psi = \; -m\log\{(z^3 - z_0^3)(z^3 - {z_0'}^3)\}$$

Example-3

Within a rigid boundary in the form circle $(x+\alpha)^2 + (y-4\alpha)^2 = 8\alpha^2$, there is liquid motion due to a doublet of strength μ at the point (0, 3α) with its axis along the axis of y. Then find the velocity potential for this motion.

Solution:

The circle has (-α, 4α) for its centre and $2\sqrt{2\alpha}$ for its radius. The given doublet A is at A(0, 3α).

So the gradient of = $\frac{3\alpha-4\alpha}{0-(-\alpha)} = -1 = \tan\left(\frac{3\pi}{4}\right)$, i.e. AC makes an angle $\frac{\pi}{4}$ OY .

Thus if B is the image of A , then axis of B will make an angle of 45^0 with AB and will therefore be parallel to x-axis .

For this we know that B is the inverse of A with respect to the circle ,

i.e. $CA.CB = 8\,\alpha^2$

$$CA(CA + AB) = 8\alpha^2$$

$$\alpha\sqrt{2}\left(\alpha\sqrt{2} + AB\right) = 8\alpha^2$$

$$AB = 4\sqrt{2}\alpha - \sqrt{2}\alpha = 3\alpha\sqrt{2}$$

$$= 3\alpha \sec 45^0 = OA\, sec\ \ 45^0$$

Which shows that B is on the x-axis , the point B $(3\alpha, 0)$. Now since μ is the strength of A. The strength of B is

$$= \mu\frac{radius^2}{(AC)^2} = \frac{8\mu\alpha^2}{2\alpha^2} = 4\mu$$

Therefore the complex potential of motion is given by

$$w = \frac{\mu e^{\pi i/_2}}{z-3i\alpha} + \frac{4\mu e^{i\theta}}{z-3\alpha}, \quad i.e.$$

$$w = \phi + i\psi = \mu\left[\frac{4}{x + iy - 3\alpha} + \frac{i}{x + iy - 3i\alpha}\right]$$

$$= \mu\left[\frac{4(x - 3\alpha) - iy}{(x - 3\alpha)^2 + y^2} + \frac{ix + (y - 3\alpha)}{x^2 + (y - 3\alpha)^2}\right]$$

$$(multiplying\ the\ denominators\ by\ their\ complex\ conjugate\ )$$

Thus

$$\phi = \mu\left[\frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2}+\frac{(y-3\alpha)}{x^2+(y-3\alpha)^2}\right]$$

is the required complex potential.

Example-4

Determine image of a line doublet parallel to the axis of a right circular cylinder.

Solution:

Let there be a uniform line doublet of strength μ per unit length through the point z = c > a. Furthermore let the axis of the line doublet be inclined at an angle α to the *x*- axis. Then the complex potential at point *z* is given by

$$f(z) = \frac{\mu e^{i\alpha}}{z-c}$$

$$then\quad \bar{f}(z) = \frac{\mu e^{-i\alpha}}{z-c}$$

and so

$$\bar{f}\left(\frac{a^2}{z}\right) = \frac{\mu e^{-i\alpha}}{(a^2/z)-c}.$$

Let a circular cylinder of section $|z| = a\ be\ introduced$. Then by using Circle Theorem (Milne- Thomson's circle theorem) the new complex potential is given by

$$w = \bar{f}(z) + \bar{f}\left(\frac{a^2}{z}\right),$$

$$w = \frac{\mu e^{-i\alpha}}{z-c} + \frac{\mu e^{-i\alpha}}{\left(\frac{a^2}{z}\right)-c}.$$

7.8 Summary.

The definition of doublet and stream function, potential function are defined. The equation of stream function and potential function derived that is a circle equation. The equation of a solid sphere in a plane source discussed as, $\psi_s = \frac{q}{2}$. The example of each topic discussed with unsolved questions which help understand the concept of problems.

7.9 Terminal questions.

Q.1. A uniform flow with a velocity of 2 m/s is flowing over a source placed at the origin. The stagnation point occurs at (-0.398, 0). Determine: (i) strength of the source, (ii) Maximum with of Rnkine half-body.

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Q.2. A uniform flow of velocity 6 m/s is flowing along x-axis over a source and a sink which are situated along x-axis. The strength of source and sink is 15 m^2/s and they are at a distance os 1.5 m apart. Determine:

(i) Location of stagnation points,

(ii) Equation of profile of the Rankine body.

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Q. 3. Prove that in the two-dimensional liquid motion due to any number of sources at points on a circle, the circle is a stream line provided that there is no boundary and that the algebraic sum of the strengths of the source is zero.

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Q.4. Find the velocity potential when there is a source and an equal sink inside a circular cavity and show that one of the stream line is an arc of the circle which passes through the source and sink and cut orthogonally the boundary of the cavity.

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Q-5. In a two dimensional liquid motion ϕ and ψ are the velocity potential and current function; show that a second fluid motion exist in which ψ is the velocity potential and $-\phi$ the current function ; and prove that if the first motion be due to sources and sinks, the second motion can be built by replacing a source and an equal sink by a line of doublets uniformly distributed along any curve joining them.

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End