U P RAJARSHI TANDON OPEN UNIVERSITY ALLAHABAD

UGMM-104

DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATION

Course Design Committee

Prof. Ashutosh Gupta Director, School of Science, UPRTOU, Prayagraj	**Chairman**
Prof. Sudhir Srivastav Professor, Dept. of Mathematics, DDU Gorakhpur University Gorakhpur	**Member**
Prof. P. K. Singh Dept. of Mathematics, University of Allahabad, Prayagraj	**Member**
Prof. Mona Khare Dept. of Mathematics, University of Allahabad, Prayagraj	**Member**
Dr. A. K. Pandey Associate Professor, E.C.C University of Allahabad, Prayagraj	**Member**
Dr. Vikas Singh Academic Consultant School of Science, UPRTOU, Prayagraj	**Member**
Dr. S. S. Tripathi Academic Consultant School of Science, UPRTOU, Prayagraj	**Member**

Course Preparation Committee

Dr. Viask Singh Dept. Of Mathematics, Lecturer DIET, Jaunpur	**Author**
Dr. S. S. Tripathi Academic Consultant School of Science, UPRTOU, Prayagraj	**Editor**
Prof. Ashutosh Gupta Director, School of Computer and Information Science, UPRTOU, Prayagraj	

Faculty Members, School of Sciences

Prof. Ashutosh Gupta, Director, School of Science, UPRTOU, Prayagraj
Dr. Shruti, Asst. Prof., (Statistics), School of Science, UPRTOU, Prayagraj
Dr. Marisha Asst. Prof., (Computer Science), School of Science, UPRTOU, Prayagraj
Mr. Manoj K Balwant Asst. Prof., (Computer Science), School of Science, UPRTOU, Prayagraj
Dr. Dinesh K Gupta Academic Consultant (Chemistry), School of Science, UPRTOU, Prayagraj
Dr. S. S. Tripathi, Academic Consultant (Maths), Scool of Science, UPRTOU, Prayagraj
Dr. Dharamveer Singh, Academic Consultant (Bio-Chemistry), School of Science, UPRTOU, Prayagraj
Dr. R. P. Singh, Academic Consultant (Bio-Chemistry), School of Science, UPRTOU, Prayagraj
Dr. Susma Chuhan, Academic Consultant (Botany), School of Science, UPRTOU, Prayagraj
Dr. Deepa Chubey, Academic Consultant (Zoology), School of Science, UPRTOU, Prayagraj
Dr. Arvind Kumar Mishra, Academic Consultant (Physics), School of Science, UPRTOU, Prayagraj

Block-I

Differential Equations of First Order and First Degree

U P RAJARSHI TANDON
OPEN UNIVERSITY
PRAYAGRAJ

UGMM-104
DIFFERENTIAL EQUATION

Unit-1

Differential equation

Unit-2

Methods of solution of a differential equation of first order and first degree

---

Unit-3

Linear differential equation

Unit 4

Exact differential equations

Unit –01: Differential equation

Structure

1.1. Introduction

The entire field of engineering and science – heat, light, sound, gravitation, magnetism, fluid flow, population dynamics and mechanics are described by differential equations. Other modern technologies such as radio, television, cars and aircraft all depend on the mathematics of differential equations

Differential equations play an important role in modeling virtually every physical, technical or biological process, from celestial motion to bridge design to interactions between neurons. Further applications are found in fluid dynamics with the design of containers and funnels in heat conduction analysis with the design of heat spreaders in microelectronics, in rigid-body dynamic analysis, with falling objects, and in exponential growth of current in an R-L circuit. This unit introduces first order differential equations – the subject is clearly of great importance in many different areas of science and engineering.

1.2 Objectives

After reading this unit students should be able to:

- Define and understand the differential equation and its types

- Find the order and degree of a differential equation
- Form the differential equation of given function
- Solve the given differential equation
- Understand the geometrical meaning of differential equation
- Solve the initial value problems
- State the existence and uniqueness theorems

1.3 Definition of Differential Equation

An equation which contains the independent variable, dependent variable and its derivatives is called a differential equation.

Equivalently,

An equation involving dependent and independent variables and the differential coefficients (derivatives) of dependant variable with respect to one independent variables is called a Differential equation.

The general first-order differential equation for the function y = y(x) is written as $\frac{dy}{dx} = f(x, y)$, where f(x, y) can be any function of the independent variable x and the dependent variable y.

For Example:

1) $3\frac{dy}{dx} + x = \cos y$

2) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-5x=\cos ecy$

3) $2\frac{d^3y}{dx^3}-3\frac{dy}{dx}+5x=\sec y+e^x$

4) $\frac{dy}{dx}-5y=7$

5) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+\cos y=8x$

6) $m\frac{d^2y}{dx^2}=-mg$

7) $\left(\frac{dy}{dx}\right)^3-5y=7$

1.4 Types of differential equations

A differential equation is an equation for a function that relates the values of the function to the values of its derivatives. An ordinary differential equation (ode) is a differential equation for a function of a single variable, e.g., x(t) or y(t), while a partial differential equation (pde) is a differential equation for a function of several variables, e.g., v(x, y, z, t). An ode contains ordinary derivatives and a pde contains partial derivatives. Typically, pde's are much harder to solve than ode's.

There are two types of differential equations, namely

i. Ordinary Differential Equations (ODE's)

ii. Partial Differential Equations (PDE's)

i. Ordinary Differential Equations (ODE's):

Ordinary Differential Equations (ODE's): A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation (ODE's).

Equivalently,

ODE is an equation involving an unknown function y of a single variable t together with one or more of its derivatives $y^{/}, y^{//}$ etc.

Example 1.4.1: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$ is an ordinary differential equation.

ii. Partial Differential Equations (PDE's):

Many functions depend on more than one independent variable. Of course, there are differential equations involving derivatives with respect to more than one independent variables, called partial differential equations (PDE's).

Example 1.4.2: $\frac{\partial^2 y}{\partial x^2}=9\left(\frac{\partial y}{\partial t}\right)$ is partial differential equation.

1.5 Order and degree of differential equations:

1.5.1 Order of a differential equation:

A differential equation can be classified according to its order and degree.

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

i. $\frac{dy}{dx} = e^x$, this ODE involves highest derivative of first order. Therefore order is one

ii. $\frac{d^2y}{dx^2} + y = 0$, this ODE involves highest derivative of second order. Therefore order is two.

iii. $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = \sin x$, this ODE involves highest derivative of third order. Therefore order is three.

1.5.2 Degree of a differential equation:

Degree of a differential equation To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e., y′, y″, y‴ etc. Consider the following differential equations:

iv. $$\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}+y=0$$

v. $$\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}-\sin^2 y=0$$

vi. $$\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$$

We observe that, in the above equation (iv) is a polynomial equation in y‴, y″ and y′, equation (v) is a polynomial equation in y′ (not a polynomial in y though). Degree of such differential equations can be defined. But equation (vi) is not a polynomial equation in y′ and degree of such a differential equation cannot be defined. By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation. In view of the above definition, one may observe that differential equations (i), (ii), (iii) and (iv) each are of degree one, equation (v) is of degree two while the degree of differential equation (vi) is not defined.

Note: Order and degree (if defined) of a differential equation are always positive integers.

1.5 Test Your Progress

Find the order and degree, if defined, of each of the following differential equations:

i. $\frac{dy}{dx} - \cos x = 0$

ii. ii. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$

iii. $\frac{d^3y}{dx^3} + y^2 + e^{y'} = 0$

vii. iv. $\frac{d^4y}{dx^4} + \sin(y''') = 0$

viii. $y' + 5y = 0$

ix. $y^{iv} + 3yy'' = 0$

x. $(y'')^2 + \cos(y') = 0$

xi. $y'' = \cos 3x + \sin 3x$

xii. $(y'')^2 + (y'')^3 + (y')^4 + y^5 = e^{-2x}$

$y''' + 2y'' + y' = 0$.

1.6 Solution of Differential Equation

In earlier topics, we have solved the equations of the type:

$x^2 + 1 = 0$ .... (1)

sin2 x – cos x = 0 ....(2)

The solutions of equations (1) and (2) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown x in the given equation, L.H.S. becomes equal to the R.H.S.

Now consider the differential equation ,

$\frac{d^2y}{dx^2} + y = 0$ .... (3)

In contrast to the first two equations, the solution of this differential equation is a function ϕ that will satisfy it.

i.e., when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S.

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the given differential equation.

Consider the function given by

$y = \phi(x) = a\sin(x + b)$ .... (4)

where a, b ∈ R. When this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.

So it is the solution of a differential equation (3).

Let a and b be given some particular values say $a = 2$ and $b = \frac{\pi}{4}$, then we get

a function $y = \phi_1(x) = 2\sin\left(x + \frac{\pi}{4}\right)$ .... (5)

When this function and its derivative are substituted in equation (3) again L.H.S. = R.H.S. Therefore ϕ_1 is also a solution of the equation (3).

Function ϕ consists of two arbitrary constants (parameters) a and b, it is called general solution of the given differential equation.

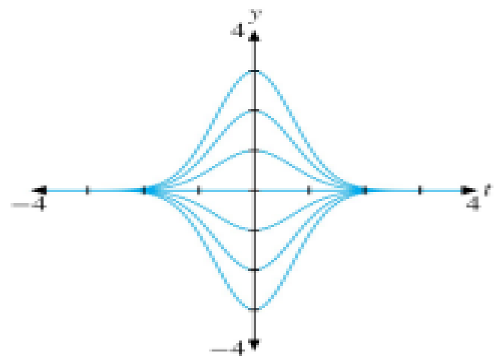
Whereas function ϕ_1 contains no arbitrary constants but only the particular values of the parameters a and b and hence is called a particular solution of the given differential equation. The solution which contains arbitrary constants is called the general solution (or primitive) of the differential equation.

Thus,

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution of the differential equation.

For example:

1. The solution formula $y = Ce^{-t^2}$, which depends on the arbitrary constant C, describes a family of solutions and is called a general solution.
2. The graphs of these solutions, drawn in the figure, are called solution curves.



Figure

3. Given the value of the solution at a point, we can determine the unique particular solution.

Example 1.6.1: Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Solution: Given function is $y = e^{-3x}$ .... (1)

Differentiating both sides of equation (1) with respect to x , we get

$$\frac{dy}{dx} = -3e^{-3x} \qquad \text{................. (2)}$$

Now, differentiating (2) with respect to x, we have

$$\frac{d^2y}{dx^2} = 9e^{-3x} \qquad \text{.....................(3)}$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from equations (3), (2) and (1) in the given differential given equation, we get

$$LHS = \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 9e^{-3x} - 3e^{-3x} - 6(e^{-3x}) = 9e^{-3x} - 9e^{-3x} = 0 = RHS .$$

Example 1.6.2: Verify that the function $y = a\cos x + b\sin x$, where a, b Є R is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution: Given function is $y = a\cos x + b\sin x$ .... (1)

Differentiating both sides of equation (1) with respect to x , we get

$$\frac{dy}{dx} = -a\sin x + b\cos x \qquad \text{................. (2)}$$

Now, differentiating (2) with respect to x, we have

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x \qquad \text{....................(3)}$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y from equations (3) and (1) in the given differential given equation, we get

$$LHS = \frac{d^2y}{dx^2} + y = -a\cos x - b\sin x + a\cos x + b\sin x = 0 = RHS \,.$$

Example 1.6.3: Verify that the function $x + y = \tan^{-1} y$, is a solution of the differential equation $y^2 \frac{dy}{dx} + y^2 + 1 = 0$.

Solution: Given function is $x + y = \tan^{-1} y$ .... (1)

Differentiating both sides of equation (1) with respect to x , we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2} \cdot \frac{dy}{dx} \;\; Or \;\; \frac{dy}{dx}\left(\frac{y^2}{1+y^2}\right) = -1 \;\; Or \;\; \frac{dy}{dx} = -\frac{1+y^2}{y^2} \qquad \text{................ (2)}$$

Substituting the values of $\frac{dy}{dx}$ and y from equations (2) and (1) in the given differential given equation, we get

$$LHS = y^2 \frac{dy}{dx} + y^2 + 1 = y^2\left(-\frac{1+y^2}{y^2}\right) + y^2 + 1 = 0 = RHS \,.$$

Note: The particular solution satisfying the initial condition $y(x_0) = y_0$ is the solution $y(x) = y$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A first-order initial value problem is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

1.6 Test Your Progress

In each of the following exercises, verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation.

1. $y = e^x + 1$; $y'' - y' = 0$

2. $y = x^2 + 2x + c$; $y' - 2x - 2 = 0$

3. $y = \cos x + c$; $y' + \sin x = 0$

4. $y = \sqrt{1 + x^2}$; $y' = \dfrac{xy}{1 + x^2}$

5. $y = Ax$; $xy' = y \ (x \neq 0)$.

6. $y = x \sin x$; $xy' = y + x\sqrt{x^2 - y^2}$; $(x \neq 0,\ x > y)$.

7. $xy = \log y + c$; $y' = \dfrac{y^2}{1 - xy}$; $(xy \neq 1)$.

8. $y - \cos y = x$; $(y \sin y + \cos y + x)y' = y$.

9. $y = \sqrt{a^2 - x^2}$; $x \in (-a,\ a)$; $(x + y)y' = 0$; $(y \neq 0)$.

1.7 Formation of Differential Equation

We know that the equation, $x^2 + y^2 + 2x - 4y + 4 = 0$ .... (1). This represents a circle of centre at (–1, 2) and radius 1 unit.

Differentiating equation (1) with respect to x, we get

$\frac{dy}{dx} = \frac{x+1}{2-y}, \quad y \neq 2$ ....(2), which is a differential equation. We will find later on that this equation represents the family of circles and one member of the family is the circle given in equation (1). This leads to the concept of formation of the differential equation.

Procedure to form a differential equation that will represent a given family of curves.

(i) If the given family F_1 of curves depends on only one parameter then it is represented by an equation of the form

$F_1(x, y, a) = 0$ ....(1).

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form f (x, y, a) : $y^2 = ax$.

Differentiating equation (1) with respect to x, we get an equation involving y′, y, x, and a, i.e., $g(x, y, y', a) = 0$ .... (2)

The required differential equation is then obtained by eliminating a from equations (1) and (2) as $F(x, y, y') = 0$ . ...(3)

(ii) If the given family F_2 of curves depends on the parameters a, b (say) then it is represented by an equation of the from

$F_2(x, y, a, b) = 0$ ... .(4)

Differentiating equation (4) with respect to x, we get an equation involving y′, x, y, a, b,

i.e., $g(x, y, y', a, b) = 0$ ... .(5)

But it is not possible to eliminate two parameters a and b from the two equations and so, we need a third equation. This equation is obtained by differentiating equation (5), with respect to x, to obtain a relation of the form

$h(x, y, y', y'', a, b) = 0$ ... .(6)

The required differential equation is then obtained by eliminating a and b from equations (4), (5) and (6) as

$F(x, y, y', y'') = 0$ ... .(7)

Note: The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Example 1.7.1: Form the differential equation representing the family of curves y = mx, where, m is arbitrary constant.

Solution: We have , y = mx .... (1)

Differentiating both sides of equation (1) with respect to x, we get $\frac{dy}{dx} = m$

Substituting the value of m in equation (1) we get $y = x\frac{dy}{dx}$ Or $x\frac{dy}{dx} - y = 0$,

which is free from the parameter m and hence this is the required differential equation.

Example 1.7.2: Form the differential equation representing the family of curves

y = a sin (x + b), where a, b are arbitrary constants.

Solution: We have y = a sin(x + b) ...... (1)

Differentiating both sides of equation (1) with respect to x, successively we get

$$\frac{dy}{dx} = a\cos(x+b) \text{........................}(2)$$

$$\frac{d^2y}{dx^2} = -a\sin(x+b) \text{....................}(3)$$

Eliminating a and b from equations (1), (2) and (3), we get

From equation (1), $\sin(x+b) = \frac{y}{a}$, substitute this in equation (3) we obtain

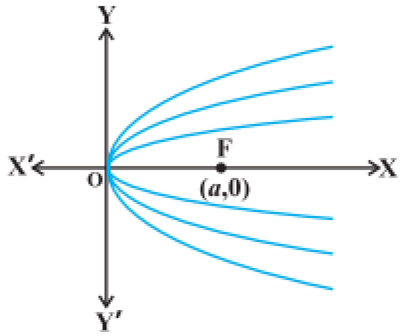
$$\frac{d^2y}{dx^2} = -a\left(\frac{y}{a}\right) = -y \text{ or } \frac{d^2y}{dx^2} + y = 0$$

Which is free from the arbitrary constants a and b and hence this is the required differential equation.

Example 1.7.3: Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

Solution: Let P denote the family of above said parabolas (see Figure) and let (a, 0) be the

focus of a member of the given family, where a is an arbitrary constant. Therefore, equation of family P is



Figure

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \quad \ldots\ \ldots\ldots\ldots\ldots\ldots(2)$$

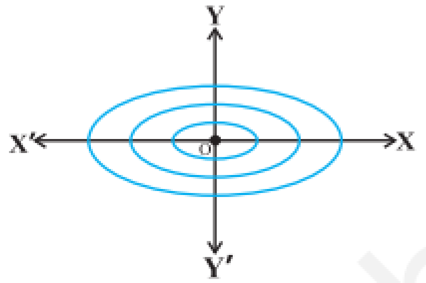
Substituting equation (2) in equation (1), we obtain the required differential equation as,

$$y^2 = 2xy\frac{dy}{dx} \quad \text{or} \quad 2xy\frac{dy}{dx} - y^2 = 0 \quad \text{or} \quad 2x\frac{dy}{dx} - y = 0\,.$$

Example 1.7.4: Form the differential equation representing the family of ellipses having foci on X-axis and centre at the origin.

Solution: We know that the equation of said family of ellipses (see Figure) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{... ....} \quad (1)$$



Figure

Differentiating equation (1) with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \;\; Or \;\; \frac{y}{b^2}\frac{dy}{dx}=-\frac{x}{a^2} \;\; Or \;\; \frac{y}{x}\frac{dy}{dx}=-\frac{b^2}{a^2} \qquad \text{... ....} \quad (2)$$

Differentiating equation (2) with respect to x, we get

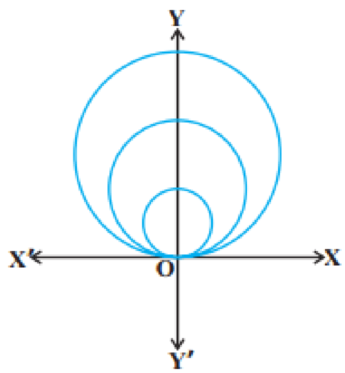
$$\frac{x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\left(\frac{dy}{dx}\right)}{x^2}=0 \;\; Or \;\; x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-y\left(\frac{dy}{dx}\right)=0$$

Or $\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{y}{x}\left(\frac{dy}{dx}\right)=0$

Example 1.7.5: Form the differential equation of the family of circles touching the x-axis at origin.

Solution: Let C denote the family of circles touching x-axis at origin. Let (0, a) be the coordinates of the centre of any member of the family (see the following Fig). Therefore, equation of family C is $x^2 + (y-a)^2 = a^2$ Or

$x^2 + y^2 = 2ay$ ....(1)



Figure

where, a is an arbitrary constant. Differentiating both sides of equation (1) with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx} \quad \text{Or} \quad x + y\frac{dy}{dx} = a\frac{dy}{dx} \quad \text{Or} \quad a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \text{..............(2)}$$

Substituting the value of a from equation (2) in equation (1), we get

$$x^2 + y^2 = 2y\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}$$

Or $\left(x^2+y^2\right)\frac{dy}{dx}=2y\left(x+y\frac{dy}{dx}\right)$

Or $\left(x^2+y^2\right)\frac{dy}{dx}-2y^2\frac{dy}{dx}=2xy$

Or $\left(x^2-y^2\right)\frac{dy}{dx}=2xy$ *Or* $\frac{dy}{dx}=\frac{2xy}{\left(x^2-y^2\right)}$

This is the required differential equation of the given family of circles.

1.7 Test Your Progress
In each of the following exercises, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b. 1. 1. $\frac{x}{a}+\frac{y}{b}=1$ 2. $y^2=a(b^2-x^2)$ 3. $y=ae^{3x}+be^{-2x}$ 4. $y=e^{2x}(a+bx)$ 5. $y=e^x(a\cos x+b\sin x)$. 6. Form the differential equation of the family of circles touching the y-axis at origin. 7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis. 8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

11. Find the differential equation, which has $y = ae^{x} + be^{-x}$ as the general solution?.

12. Find the differential equation, which has $y = x$ as its particular solution?.

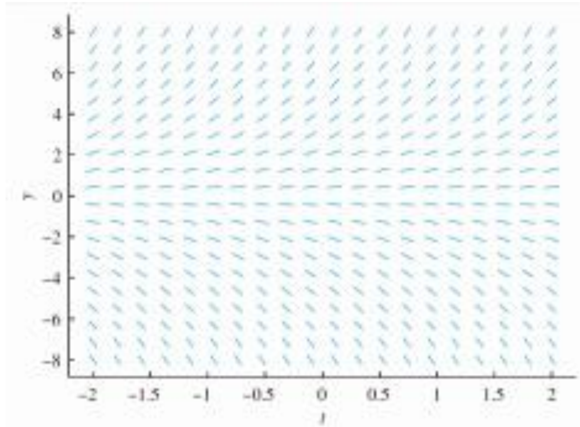
1.8 Geometrical meaning of a differential equation

Let $y(t)$ be a solution of the ODE $y = f(t, y)$. The graph of the solution $y(t)$ is called a solution curve. For any point (t_0, y_0) on the solution curve $y(t_0) = y_0$ and the differential equation says that $y(t_0) = y(t_0, y(t_0))$.

The LHS is the slope of the solution curve, and the RHS tells us what the slope is at (t_0, y_0).

1.8.1. Direction Field for $y^{/} = f(t, y)$:

Draw a line segment with slope $f(t_i, y_j)$ attached to every grid point (t_i, y_j) in a rectangle R where f (t, y) is defined R = { (t, y)| a ≤ t ≤ b and c ≤ y ≤ d }. The result is called a direction field.

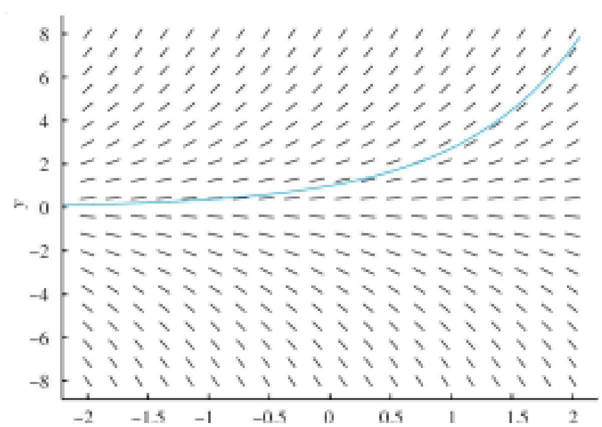


Figure

Geometric interpretation of Solution is,

Direction field provides information about qualitative form of solution curves.

Finding a solution to the differential equation is equivalent to the geometric problem of finding a curve in ty-plane that is tangent to the direction field at every point.



Figure

1.9. Initial value problems (IVP)

A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial-value problems (IVP) or boundary-value problems (BVP).

The distinction between the two classifications lies in the location where the extra conditions are specified. For an initial value problem (IVP), the conditions are given at the same value of x, whereas in the case of the boundary value problem (BVP), they are prescribed at two different values of x.

An initial value problem is a differential equation (of any order) together with initial conditions that must be satisfied by the solution of the differential equation and its derivatives at the initial point.

For example:

i) Consider the differential equation, $\frac{dy}{dx} = x^2 - y^2, \quad y(0) = 1$. Here $x = 0$ is the initial point.

ii) Consider the differential equation, $\frac{d^2y}{dx^2} = -yx, \ \ y(0) = 2, \ \ y^{/}(0) = 1$.

Here $x = 0$ is the initial point.

Equivalently,

Initial Value Problem:

The problem of finding a function y of x when we know its derivative and its value y_0 at a particular point $x = 0$ is called an initial value problem.

This problem can be solved in two steps.

1. Find the general solution of the given differential equation.
2. Using the initial data, plug it into the general solution and solve for C.

Consider the problem of finding a function $y(t)$ that satisfies the following ordinary differential equation (ODE):

$\frac{dy}{dt} = f(t, y), \ \ a \le t \le b$. The function $f(t, y)$ is given, and we denote the derivative of the sought function by $y^{/}(t) = \frac{dy}{dt}$ and refer to 't' as the independent variable. Earlier we dealt with the question of how to approximate, differentiate or integrate an explicitly known function.

Here, similarly, the function $f(t, y)$ is given and the sought result is different from $f(t, y)$ but related to it. The main difference though is that $f(t, y)$ depends on $y(t)$, and we would like to be able to compute $y(t)$ possibly for all 't' in the interval [a, b], given the ODE which characterizes the relationship between the function and some of its derivatives.

Example 1.9.1: The function $f(t, y) = -y + t$ defined for t ≥ 0 and any real $y(t)$ gives the ODE $y'(t) = \frac{dy}{dt} = -y + t, \ t \geq 0$. You can verify directly that for any scalar α the function $y(t) = t - 1 + \alpha e^{-t}$ satisfies the ODE. If it is given, in addition, that $y(0) = 1$, then $1 = 0 - 1 + \alpha e^{-0}$, hence α = 2 and the unique solution is $y(t) = t - 1 + 2e^{-t}$.

Boundary Value Problem:

It is a differential equation together with a collection of values that must be satisfied by the solution of the differential equation or its derivative at no fewer than two different points.

For example:

i) Consider the differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = \cos x, \quad y(0) = 1 \ and \ y'(1) = 2.$$

ii) Consider the differential equation, $\frac{d^2y}{dx^2} = -yx, \;\; y(0) = 2, \;\; y(1) = 1$.

Example 1.9.2: Show that the function $y = (x+1) - \frac{1}{3}e^x$ is a solution to the first-order initial value problem $\frac{dy}{dx} = y - x, \;\; y(0) = \frac{2}{3}$.

Solution: Consider, $\frac{dy}{dx} = f(x, y) = y - x$ .... (1)

From the function, $y = (x+1) - \frac{1}{3}e^x, \;\; \frac{dy}{dx} = 1 - \frac{1}{3}e^x$ .... (2)

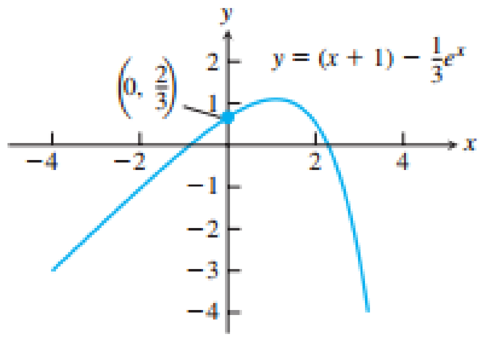
Substituting the values of $\frac{dy}{dx}$ and y from equation (2) in equation (1), we obtain

$$\frac{dy}{dx} = y - x \Rightarrow 1 - \frac{1}{3}e^x = (x+1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x$$

Thus, LHS = RHS.

The function satisfies the initial condition because $y = (x+1) - \frac{1}{3}e^x \Rightarrow y(0) = 1 - \frac{1}{3} = \frac{2}{3}$.

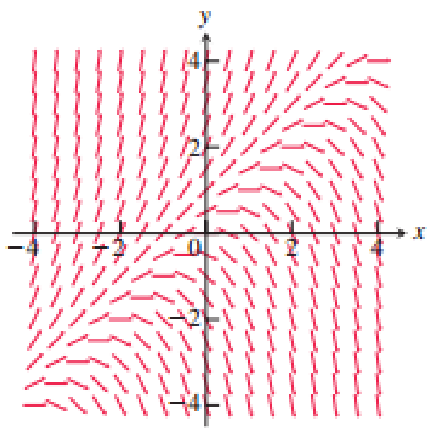
The graph of the function is shown in the following Figure.



Figure

Slope Fields, Viewing Solution Curves:

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the solution curve (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a slope field (or direction field) and gives a visualization of the general shape of the solution curves. The following Figure (a) shows a slope field, with a particular solution sketched into it in Figure (b). We see how these line segments indicate the direction the solution curve takes at each point it passes through.



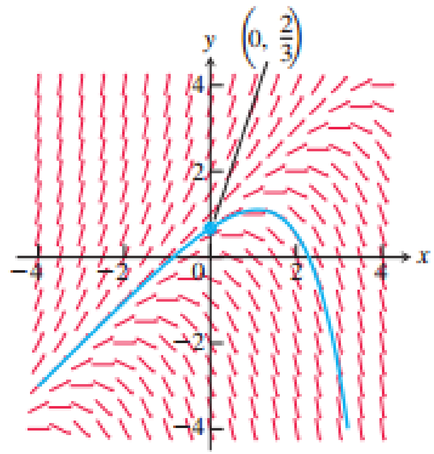


Figure (a)

Figure (b)

Slope field for $\frac{dy}{dx} = y - x$

The particular solution curve through the point $\left(0, \frac{2}{3}\right)$

1.10 Statements of Existence and Uniqueness Theorems

In ODE theory the following questions are naturally arising :

- Given an initial value theorem (IVP) is there a solution to it (question of existence)?
- If there is a solution is the solution unique (question of uniqueness)?
- For which values of x does the solution to initial value theorem (IVP) exists (the interval of existence)?

The fundamentally important question of existence and uniqueness of solution for initial value theorem (IVP) was first answered by Rudolf Lipschitz in 1876 (nearly 200 years later than the development of ODE). In 1886 Giuseppe Peano discovered that the initial value theorem IVP has a solution (it may not be unique) if f is a continuous function of (x, y). In 1890 Peano extended this theorem for system of first order ODE using method of successive approximation. In 1890 Charles Emile Picard and Ernst Leonard Lindelöf presented existence and uniqueness theorem for the solutions of initial value theorem (IVP). According to Picard Lindelöf theorem if f and $\frac{\partial f}{\partial y}$ are continuous functions of x, y in some rectangle: $\{(x, y): \alpha < x < \beta; \gamma < y < \delta\}$ containing the point (x_0, y_0) then in some interval $x_0 - \delta < x < x_0 + \delta$ $(\delta > 0)$ there exists a unique solution of initial value problem (IVP).

Equivalently,

In addition to its intrinsic mathematical interest, the theory of ordinary differential equations has extensive applications in the natural sciences, notably physics, as well as other fields. The existence and uniqueness of a solution to a first-order differential equation, given a set of initial conditions, is one of the most fundamental results of ODE.

We will investigate solutions to the differential equation,

$$\frac{dy}{dt} = f(t,\ y),\ \ y(t_0) = y_0 \text{ ....}(1)$$

where t ∈ R, y ∈ R^n , and $f(t,\ y)$ is defined and differentiable (of class C^r , r ≥ 1) in a domain U of R × R^n. A solution will be a function φ : R → R^n n, where

$$\dot{\varphi}(t) = f(t,\ \varphi(t)),\ \varphi(t_0) = y_0 \text{ ....}(2)$$

We will state the following theorems, which guarantee the existence and uniqueness of the solution for any equation of the form (1).

Theorem 1.10.1: (The Existence Theorem).

Suppose the right-hand side y of the differential equation $\frac{dy}{dt} = f(t,\ y)$ is continuously differentiable in a neighbourhood of the point (t_0, y_0) ∈ R × R^n . Then there is a neighbourhood of the point t_0 such that a solution of the differential equation is defined in this neighbourhood with the initial condition $\varphi(t_0) = y$, where y is any point sufficiently close to y_0. Moreover, this solution depends continuously on the initial point y .

Theorem 1.10.2: (The Uniqueness Theorem).

Given the above conditions, there is only one possible solution for any given initial point, in the sense that all possible solutions are equal in the neighbourhood under consideration.

1.11 Summary

In this unit, we studied the definition of differential equation and its various types. We also saw the order and degree of differential equations and its solution approaches. We also studied formation of differential equations and its geometrical interpretation. The initial value problem and boundary value problem are discussed with existence and uniqueness theorems.

1.12 Terminal Questions

1. Find the differential equation of the curve. $x^2 + y^2 = c$
2. Find the differential equation of the curve as $y^2 = x$
3. Find the order and degree of the differential equation $(\frac{d^2y}{dx^2})^2 + (\frac{dy}{dx})^2 + y = 0$

1.13. Answers to exercises

1.5Test Your Progress

i. Order 1 and degree 1

ii. Order 2 and degree 1

iii. Order 3 and degree 1

iv. Order 4 and degree 1

v. Order 1 and degree 1

vi. Order 4 and degree 1

vii. Order 2 and degree 2

viii. Order 2 and degree 1

ix. Order 2 and degree 3

x. Order 3 and degree 1

1.7Test Your Progress

1. $\frac{d^2y}{dx^2} = 0$

2 $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

3 $y'' - 5y' + 6y = 0$

4. $y'' - 4y' + 4y = 0$

5. $y'' - 2y' + y = 0$

6. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

7. $x\frac{dy}{dx}-2y=0$

8. $\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{y}{x}\left(\frac{dy}{dx}\right)=0$

9. $\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{y}{x}\left(\frac{dy}{dx}\right)=0$

10. $\frac{dy}{dx}=\frac{x}{3-y}$

11. $y^{//}-y=0$

12. $\frac{dy}{dx}=1$

Unit –02: Methods of solution of a differential equation of first order and first degree

Structure

2.1 Introduction

Ordinary differential equations find a wide range of application in biological, physical, social and engineering systems which are dynamic in character. They can be used to affectively analyze the evolutionary trend of such systems, they also aid in the formulation of these systems and the qualitative examination of this stability under and adaptability to external stimuli.

Ordinary Differential Equation: A differential equation which contains only one independent variable and the derivatives are with respect to this independent variable only is called ordinary differential Equations.

For Example:

1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + y = \sin x$ 2. $\frac{d^2y}{dx^2} + 4y = 0$ 3. $\frac{dy}{dx} + y = \log x$

The general first-order differential equation for the function $y = f(x)$ is written as $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ can be any function of the independent variable x and the dependent variable y.

OR

Any differential equation of the first order and first degree can also be written in the form

$M(x, y)dx + N(x, y)dy = 0$.

Example 2.1.1: The differential equation

$\frac{dy}{dx} = \frac{x-3y}{y-2x}$ this can also be written as $(y-2x)dx-(x-3y)dy=0$

Existence of a solution: The general solution of the equation $\frac{dy}{dx} = f(x, y)$, if it exists, has the form $f(x, y, C)=0$, where C is an arbitrary constant. Under what circumstances does a general solution exist? We have the following theorem.

Theorem 1: A general solution of $\frac{dy}{dx} = f(x, y)$ exists over some specified region R of points (x, y) if the following conditions are met:

a) $f(x, y)$ is continuous and single-valued over R

b) $\frac{\partial f}{dy}$ exists and is continuous at all points of R

The general solution f(x, y, C) = 0 of a differential equation $\frac{dy}{dx} = f(x, y)$ over some region R consists of a family of curves, called the integral curves of the differential equation, (one curve for each possible value of C, each curve representing a particular solution), such that through each point in R there passes one and only one curve of the family $f(x, y, C)=0$.

2.2. Objectives

After reading this unit students should be able to:

- Understand and apply the different methods of solution of differential equations of first order and first degree
- Solve the separable, homogeneous and non homogeneous first order and first degree ODE's

2.3. Methods of solution of differential equations of first order and first degree

i) Method of Separation of variables

ii) Method of solving the reducible to separable form

iii) Method of solving the Homogenous ODE

iv) Method of solving the Non-Homogenous (Reducible to homogeneous) ODE

2.4. Method of Separation of Variables

If $\frac{dy}{dx} = f(x, y)$ can be expressed as a product $\frac{dy}{dx} = f(x) \cdot g(y)$, where $f(x)$ is a function of x and $g(y)$ is a function of y, then the differential equation $\frac{dy}{dx} = f(x,y)$ is said

to be of variable

separable form. The differential equation $\frac{dy}{dx} = f(x,y)$ then has the form

$\frac{dy}{dx} = f(x) \cdot g(y)$, it

can be written as $\frac{dy}{g(y)} = f(x) \cdot dx$, then it becomes separable equation.

The method of solution of it is, by integrating both sides

i.e., $\int \frac{dy}{g(y)} = \int f(x) \cdot dx + C$. Where C is the arbitrary constant.

Example 2.4.1: Solve the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}, \quad y \neq 2$.

Solution: Consider,

$$\frac{dy}{dx} = \frac{x+1}{2-y},$$
$$(2-y)dy = (x+1)dx \quad (Variable\ Separable\ Form)$$
$$Integrating,\ we\ obtain$$
$$\int (2-y)dy = \int (x+1)dx + C$$
$$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + c$$

This is the required general solution.

Example 2.4.2: Solve the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution: Consider,

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\therefore \frac{dy}{1+y^2} = \frac{dx}{1+x^2} \; (Variable\ Separable\ Form)$$

Integrating, we obtain

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C$$

$$\tan^{-1} y = \tan^{-1} x + C$$

$$Or\ \tan^{-1} y - \tan^{-1} x = C$$

This is the required general solution.

Example 2.4.3: Solve the differential equation $x\frac{dy}{dx} = \cot y$

Solution: Consider,

$$x\frac{dy}{dx} = \cot y$$

$$\therefore \tan y dy = \frac{1}{x} dx \quad (Variable\ \ Separable\ Form)$$

Integrating, weobtain,

$$\int \tan y dy = \int \frac{1}{x} dx + \log C$$

$$\log(\sec y) = \log x + \log C$$

$$Or\ \frac{\sec y}{x} = C$$

This is the required general solution.

Example 2.4.4: Solve the initial value problem $\frac{dy}{dx} + (1+y^2) = 0 \ with\ \ y(0) = 0$

Solution: Consider,

$$\frac{dy}{dx}+(1+y^2)=0$$

$$\therefore \quad \frac{dy}{dx}=-(1+y^2)$$

$$\frac{dy}{(1+y^2)}=-dx \quad (Variable\ \ Separable\ \ Form)$$

Integrating, we obtain

$$\int\frac{dy}{(1+y^2)}=-\int dx+C$$

$$\therefore \ \tan^{-1}y=-x+C \quad ....(1)$$

Now, using the given initial conditions as $y(0)=0$, equation (1) becomes

$$\tan^{-1}y=-x+C \Rightarrow \tan^{-1}0=-0+C \ \ Or \ \ C=0$$

Thus, the required particular solution is $\tan^{-1}y=-x \ \ Or \ \ \tan^{-1}y+x=0$

Example 2.4.5: Solve the differential equation $\sqrt{xy}\frac{dy}{dx}=\sqrt{4-x}$

Solution: Consider,

$$\sqrt{xy}\frac{dy}{dx}=\sqrt{4-x}$$

$$\therefore \quad \sqrt{y}\,dy=\sqrt{\frac{4-x}{x}}\,dx \ \ (Variable\ \ Separable\ \ Form)$$

$$Or \quad \sqrt{y}\,dy=\left(\frac{2}{\sqrt{x}}-1\right)dx, \ \ Integrating$$

$$\int\sqrt{y}\,dy=\int\left(\frac{2}{\sqrt{x}}-1\right)dx+C$$

$$Or \qquad \frac{2}{3}y^{3/2}=2\int x^{-1/2}dx-x+C$$

$$Or \qquad \frac{2}{3}y^{3/2}=4x^{1/2}-x+C$$

This is the required general solution.

Method of Reducible to Separation of Variables

Some differential equations may not appear in variable separable form initially but through appropriate substitutions, then it can be made variable separable form.

Example 2.4.6: Solve the differential equation $x\frac{dy}{dx}+y=2x\sqrt{1-x^2y^2}$

Solution: Consider,

$$x\frac{dy}{dx}+y=2x\sqrt{1-x^2y^2}\text{ ....(1)}$$

Substitute $xy=v$ *then differentiating with respect to* x, *we get*

$$x\frac{dy}{dx}+y=\frac{dv}{dx}$$

Equation (1) *becomes*

$$\frac{dv}{dx}=2x\sqrt{1-v^2},\ \textit{this is in variable separabe forn in terms of } x \textit{ and } v$$

$$\therefore\ \frac{dv}{\sqrt{1-v^2}}=2x\,dx,\ \textit{integrating, we obtain}$$

$$\int\frac{dv}{\sqrt{1-v^2}}=\int 2x\,dx+C$$

$$\therefore\ \sin^{-1}v=2\left(\frac{x^2}{2}\right)+C$$

Or $\sin^{-1}(xy)=x^2+C$ *is the required general solution.*

2.4 Test Your Progress

Solve the following ODE's

i) $x\frac{dy}{dx} = 2x^2 + 1$

ii) $\frac{dy}{dx} = \frac{2x}{y^2}$

iii) $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

iv) $\frac{dy}{dx} = e^{x+y}$

v) $\frac{dy}{dx} = e^x \sin x$

vi) $xy\frac{dy}{dx} = (x+2)(y+2)$

vii) $(e^x + e^{-x})dy = (e^x - e^{-x})dx$

viii) $x(x^2 - 1)\frac{dy}{dx} = 1 \ \ with \ \ y(2) = 0$

ix) $\frac{dy}{dx} = y\tan x \ \ with \ \ y(0) = 1$

x) $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \ \ with \ \ y(0) = 1$

xi) $\frac{dy}{dx} = (x+y)^2$

2.5. Solution of Homogeneous equations

A function $f(x, y)$ is said to be homogeneous function of degree n if

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for any nonzero constant λ.

Consider the following functions;

i) $f(x, y) = y^2 + 2xy$

By the definition of homogeneous functions;

$$f(\lambda x, \lambda y) = \lambda^2 y^2 + 2(\lambda x)(\lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 f(x, y)$$

This implies that the given function $f(x, y)$ is a homogeneous function of degree 2.

ii) $f(x, y) = 2x - 3y$

By the definition of homogeneous functions;

$$f(\lambda x, \lambda y) = 2\lambda x - 3\lambda y = \lambda(2x - 3y) = \lambda f(x, y)$$

This implies that the given function $f(x, y)$ is a homogeneous function of degree 1.

iii) $f(x, y) = \cos\frac{y}{x}$

By the definition of homogeneous functions;

$$f(\lambda x, \lambda y) = \cos\frac{y}{x} = \cos\frac{\lambda y}{\lambda x} = \cos\frac{y}{x} = \lambda^0 f(x, y)$$

This implies that the given function $f(x, y)$ is a homogeneous function of degree 0.

iv) $f(x, y) = \sin x + \cos y$

By the definition of homogeneous functions;

$f(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n f(x, y)$

This implies that the given function $f(x, y)$ is not an homogeneous function.

Equivalently,

Therefore, a function $f(x, y)$ is a homogeneous function of degree n if

$f(x, y) = x^n f(y/x)$ *Or* $f(x, y) = y^n f(x/y)$.

Consider the following functions;

i) $f(x, y) = y^2 + 2xy = x^2\left[(y/x)^2 + 2(y/x\right] = x^2 f(y/x)$

Or $f(x, y) = y^2 + 2xy = y^2\left[1 + 2(x/y)\right] = y^2 f(x/y)$.

Therefore, $f(x, y)$ is a homogeneous function of degree 2.

ii) $f(x, y) = 2x - 3y = x\left[2 - 3(y/x)\right] = xf(y/x)$.

Or $f(x, y) = 2x - 3y = y\left[2(x/y) - 3\right] = xf(x/y)$

Therefore, $f(x, y)$ is a homogeneous function of degree 1.

iii) $f(x, y) = \cos\frac{y}{x} = x^0 f(y/x)$

Therefore, $f(x, y)$ is a homogeneous function of degree 0.

iv) $f(x, y) = \sin x + \cos y \neq x^n f(y/x)$ Or $f(x, y) = \sin x + \cos y \neq y^n f(x/y)$

Therefore, $f(x, y)$ is not a homogeneous function.

2.5.1. Method of Solution of Homogeneous equations:

If the differential equation $\frac{dy}{dx} = f(x, y) = x^n f(y/x)$ .... (1) is homogeneous of degree zero. Then,

i. Substitute $y = vx,$ *on differentiating w.r.t* x, *we obtain* $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

ii. Put the value of $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (1)

iii. Simplified equation reduces to the variable separable form in terms of v and x

iv. Integration leads to the general solution in terms of v and x

v. Substitute $v = \frac{y}{x}$ in the solution obtained in step (iii) gives the required general solution of the equation (1).

Note: If the homogeneous differential equation is in the form $\frac{dx}{dy} = f(x, y) = y^n f(x/y)$ where, $f(x, y)$ is homogeneous of degree n., then we make a substitution $v = \frac{x}{y}$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above.

Example 2.5.1: Show that the differential equation $\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}$ is homogeneous and solve it

Solution: Consider,

$$\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \quad \frac{dy}{dx}=\frac{x^2\left[1+(y/x)^2\right]}{x^2\left[1+y/x\right]}=x^0 f(y/x)$$

RHS is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $y=vx$. Then,

$$\textit{on differentiating w.r.t } x, \textit{ we get } \frac{dy}{dx}=v+x\frac{dv}{dx} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Substitute equation (2) in equation (1) , we obtain,

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{x^2+x^2v}$$

$$Or \qquad x\frac{dv}{dx}=\frac{1+v^2}{1+v}-v=\frac{1+v^2-v-v^2}{1+v}$$

$$x\frac{dv}{dx}=\frac{1-v}{1+v}$$

This is in variable separable form

$$\therefore \quad \frac{1+v}{1-v}dv=\frac{dx}{x}, \ \text{integrating}$$

$$\int\frac{1+v}{1-v}dv=\int\frac{dx}{x}+\log C$$

$$\int \frac{1}{1-v}\,dv + \int \frac{v}{1-v}\,dv = \log x + \log C$$
$$-\log(1-v) + \log \qquad = \log x + \log C$$

Example 2.5.2: Show that the differential equation $(x-y)dy-(x+y)dx=0$ is homogeneous and solve it

Solution: Consider,

$$(x-y)dy-(x+y)dx=0$$

RHS is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $y = vx$. Then,

$$\textit{on differentiating w.r.t } x, \textit{ we get } \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ....(2)}$$

Substitute equation (2) in equation (1) , we obtain,

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-xv}$$
$$Or \qquad x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v+v^2}{1-v}$$
$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

This is in variable separable form

$$\therefore \quad \frac{1-v}{1+v^2}dv = \frac{dx}{x}, \ \text{integrating}$$

$$\int \frac{1-v}{1+v^2}dv = \int \frac{dx}{x} + \log C$$

$$\int \frac{1}{1+v^2}dv - \int \frac{v}{1+v^2}dv = \log x + \log C$$

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + \log C$$

$$Or \ \tan^{-1} v + \log \frac{1}{\sqrt{1+v^2}} = \log Cx$$

Now, substitute $v = \frac{y}{x}$ we get

$$\tan^{-1}\left(\frac{y}{x}\right) + \log \frac{1}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \log Cx$$

$$Or \ \tan^{-1}\left(\frac{y}{x}\right) + \log \frac{x}{\sqrt{x^2+y^2}} = \log Cx$$

This is the required general solution of the given homogeneous differential equation (1).

Example 2.5.3: Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it

Solution: Consider,

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$Or \ \frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \quad \text{............................(1)}$$

$$\frac{dy}{dx} = \frac{x\left(y/x\cos\left(\frac{y}{x}\right)+1\right)}{x\cos\left(\frac{y}{x}\right)} = \frac{\left(y/x\cos\left(\frac{y}{x}\right)+1\right)}{\cos\left(\frac{y}{x}\right)} = x^0 f(y/x)$$

RHS is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $y = vx$. Then,

$$\textit{on differentiating w.r.t } x, \textit{ we get } \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ....(2)}$$

Substitute equation (2) in equation (1) , we obtain,

$$v + x\frac{dv}{dx} = \frac{vx\cos v + x}{x\cos v}$$

$$Or \qquad x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v = \frac{v\cos v + 1 - v\cos v}{\cos v}$$

$$x\frac{dv}{dx} = \frac{1}{\cos v}$$

This is in variable separable form

$$\therefore \quad \cos v\, dv = \frac{dx}{x}, \text{ integrating}$$

$$\int \cos v\, dv = \int \frac{dx}{x} + \log C$$

$$\therefore \ \sin v = \log x + \log C$$

$$\sin v = \log(C x)$$

Now, substitute $v = \frac{y}{x}$ we get

$$\sin\left(\frac{y}{x}\right) = \log(C x)$$

This is the required general solution of the given homogeneous differential equation (1).

Example 2.5.4: Show that the differential equation $2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$ is homogeneous and solve it

Solution: Consider,

$$2ye^{x/y}dx + (y - 2xe^{x/y})dy = 0$$

$$Or \quad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad ....(1)$$

$$= \frac{y\left(2(x/y)e^{x/y} - 1\right)}{2ye^{x/y}} = y^0 f(x/y)$$

RHS is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $x = vy$. Then,

$$on\ differentiating\ w.r.t\ x,\ we\ get\ \frac{dx}{dy} = v + y\frac{dv}{dy} \quad ....(2)$$

Substitute equation (2) in equation (1) , we obtain,

$$v + y\frac{dv}{dy} = \frac{2vye^v - y}{2ye^v} = \frac{2ve^v - 1}{2e^v}$$

$$\therefore\ v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$Or \quad y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v = \frac{2ve^v - 1 - 2ve^v}{2e^v}$$

$$Or \quad y\frac{dv}{dy} = \frac{-1}{2e^v}$$

This is in variable separable form

$$\therefore \quad y\frac{dv}{dy} = \frac{-1}{2e^{v}},$$

$$2e^{v}dv = -\frac{dy}{y}, \text{ int}\textit{egrating}$$

$$2\int e^{v}dv = -\int \frac{dy}{y} + \log C$$

$$\therefore \; 2e^{v} = -\log y + \log C$$

$$2e^{v} = \log(C/y)$$

Now, substitute $v = \frac{x}{y}$ we get

$$2e^{x/y} = \log(C/y)$$

This is the required general solution of the given homogeneous differential equation (1).

Example 2.5.5: Show that the differential equation

$ydx + x\log(y/x)dy - 2xdy = 0$ is homogeneous and solve it.

Solution: Consider,

$$ydx + x\log(y/x)dy - 2xdy = 0$$

$$Or \quad ydx = [2x - x\log(y/x)]dy$$

$$Or \quad \frac{dy}{dx} = \frac{y}{2x - x\log(y/x)} \quad ....(1)$$

$$\therefore \frac{dy}{dx} = \frac{y}{x(2 - \log(y/x))} = x^{0} f(y/x)$$

RHS is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $y = vx$. Then,

$$on\ differentiating\ w.r.t\ x,\ we\ get\ \frac{dy}{dx} = v + x\frac{dv}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substitute equation (2) in equation (1) , we obtain,

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v} = \frac{v}{2 - \log v}$$

$$\therefore\ x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$Or\ \ x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v} = \frac{v\log v - v}{2 - \log v}$$

$$Or\ x\frac{dv}{dx} = \frac{v(\log v - 1)}{2 - \log v}$$

This is in variable separable form

$$\therefore\ \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$$

$$Or\ \ \frac{1 + (1 - \log v)}{v(1 - \log v)}dv = -\frac{dx}{x},\ \text{int}egrating$$

$$\int\frac{dv}{v(1 - \log v)} + \int\frac{dv}{v} = -\int\frac{dy}{x} + \log C$$

$$\therefore\ \int\frac{dv}{v(1 - \log v)} + \log v + \log x = \log C$$

Now, substitute $v = \frac{y}{x}$ we get

This is the required general solution of the given homogeneous differential equation (1).

Example 2.5.6: For the differential equation

$$\frac{dy}{dx}-\frac{y}{x}+\cos ec(y/x)=0;\ y=0\ when\ x=1,\ \text{find}$$

the particular solution satisfying the given condition.

Solution: Consider,

$$\frac{dy}{dx}-\frac{y}{x}+\cos ec(y/x)=0$$

$$Or\ \ \frac{dy}{dx}=\frac{y}{x}-\cos ec(y/x)=x^0 f(y/x)......................(1)$$

RHS of equation (1) is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve equation (1), let's substitute $y=vx$. Then, Substitute equation (2) in equation (1) , we obtain,

$$v+x\frac{dv}{dx}=v-\cos ecv$$

$$\therefore\ \ x\frac{dv}{dx}=v-\cos ecv-v$$

$$Or\ \ x\frac{dv}{dx}=\cos ecv$$

This is in variable separable form

$$\therefore\ \ \sin v\,dv=\frac{dx}{x}\ ,\ \text{int}\,egrating$$

$$\int\sin v\,dv=\int\frac{dx}{x}+\log C$$

$$\therefore\qquad -\cos v=\log xC$$

Now, substitute $v = \frac{y}{x}$ we get

$$-\cos y/x = \log xC \text{....................}(3)$$

This is the required general solution of the given homogeneous differential equation (1).

Now, to find the particular solution, using the given initial conditions;

$$y = 0 \ \ when \ \ x = 1.$$

From equation (3), we get $\log C = -1 \ Or \ C = 1/e$.

Therefore, the require particular solution is given by

$$-\cos y/x = \log x + \log \frac{1}{e}$$

$$Or \quad -\cos y/x = \log x - 1$$

2.5 Test Your Progress
1. Which of the following are the homogeneous differential equations? (i) $(4x+6y+5)dy-(3y+2x+4)dx=0$ (ii) $(xy)dx-(x^3+y^3)dy=0$ (iii) $(x^3+2y^2)dx+2xydy=0$ (iv) $y^2dx+(x^2-xy-y^2)dy=0$

2. Show that the given differential equation is homogeneous and solve each of them.

(i) $\frac{dy}{dx} = \frac{x+y}{x}$

(ii) $(x-y)\frac{dy}{dx} = x+2y$

(iii) $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

(iv) $[x\cos(y/x) + y\sin(y/x)]ydx = [y\sin(y/x) - x\cos(y/x)]xdy$

(v) $x\frac{dy}{dx} - y + x\sin(y/x) = 0$

(vi) $(2xy+y^2) - 2x^2\frac{dy}{dx} = 0;\ y = 2\ when\ x = 1$

(vii) $x\frac{dy}{dx} - y = \frac{y}{\log y - \log x}$

2.6. Equations reducible to Homogeneous form (Non-Homogeneous ODE's of first order)

The differential equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$......................(1) is called non homogeneous equation.

The method of solution involves following two types

Type 1: If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, if however $\frac{a_1}{a_2} = \frac{b_1}{b_2} = m(say)$ then the differential equation becomes of the form $\frac{dy}{dx} = \frac{m(ax+by)+c_1}{ax+by+c_2}$......................(2). To solve this equation, we substitute the common expression $ax+by=v$, on differentiation we get $a+b\frac{dy}{dx} = \frac{dv}{dx}$. The transformed equation will be solved by the method of variable separable.

Type 2: If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then we substitute $x = X+h$ and $y = Y+k$, where $h \ and \ k$ are arbitrary constant to be so chosen as to make the given equation homogeneous. With the above substitutions, we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$. This reduces the equation to homogeneous form.

Hence, the given equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \text{......................(3)}$$

Now, choose $h \ and \ k$ such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2$ =0.

Then, the differential equation becomes

$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$, which is homogeneous.

Now, this equation can be solved as in case of homogeneous equations by substituting $Y = VX$. Finally, by replacing X by $(x-h)$ and Y by $(y-k)$ we shall get the solution in original variables x and y .

Illustrations on Type-1, non homogeneous ODE's of first order:

Example 2.6.1: Find the solution of the differential equation $\frac{dy}{dx} = \frac{x+y+3}{2x+2y+1}$.

Solution: We have, $\frac{dy}{dx} = \frac{x+y+3}{2x+2y+1}$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 1

Here, $\frac{a_1}{a_2} = \frac{1}{2} = \frac{b_1}{b_2} = \frac{1}{2}$

$$\therefore \frac{dy}{dx} = \frac{x+y+3}{2(x+y)+1}$$

Put $x + y = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$ Or $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Equation (1) gives, $\frac{dv}{dx} - 1 = \frac{v+3}{2v+1}$ Or $\frac{dv}{dx} = \frac{v+3}{2v+1} + 1$

$$Or \ \frac{dv}{dx} = \frac{v+3}{2v+1} + 1$$

$$= \frac{v+3+2v+1}{2v+1}$$

$$\frac{dv}{dx} = \frac{3v+4}{2v+1}$$

This is in variable separable form

$$\frac{v+1}{3v+4}dv = dx$$

$$Or \left(\frac{2}{3}-\frac{5/3}{3v+4}\right)dv = dx, \text{ integrating}$$

$$\therefore \int\left(\frac{2}{3}-\frac{5/3}{3v+4}\right)dv = \int dx + C$$

$$\frac{2}{3}v-\frac{5}{3}\times\frac{1}{3}\log(3v+4) = x + C$$

$$Or \ \frac{2}{3}v-\frac{5}{9}\log(3v+4) = x + C$$

Substitute $x+y=v$, we get

$\frac{2}{3}(x+y)-\frac{5}{9}\log[3(x+y)+4] = x+C$ is the required general solution.

Example 2.6.2: Find the solution of the differential equation $\frac{dy}{dx}=\frac{2x+y-1}{4x+2y-4}$.

Solution: We have, $\frac{dy}{dx}=\frac{2x+y-1}{4x+2y-4}$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 1

Here, $\frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}=\frac{b_1}{b_2}=\frac{1}{2}$

$$\therefore \frac{dy}{dx}=\frac{2x+y-1}{2(2x+y)-4}$$

Put $2x+y=v \Rightarrow 2+\frac{dy}{dx}=\frac{dv}{dx} \ Or \ \frac{dy}{dx}=\frac{dv}{dx}-2$

Equation (1) gives, $\frac{dv}{dx}-2=\frac{v-1}{2v-4} \ \ Or \ \frac{dv}{dx}=\frac{v-1}{2v-4}+2$

$$\therefore \quad \frac{dv}{dx} = \frac{v-1+4v-8}{2v-4}$$

$$Or \quad \frac{dv}{dx} = \frac{5v-9}{2v-4}$$

This is in variable separable form

$$\therefore \quad \frac{2v-4}{5v-9}\, dv = dx$$

$$\left(\frac{2}{5} - \frac{2/5}{5v-9}\right)dv = dx, \text{ integrating}$$

$$\int\left(\frac{2}{5} - \frac{2/5}{5v-9}\right)dv = \int dx + C$$

$$\frac{2}{5}v - \frac{2}{5}\times\frac{1}{5}\log(5v-9) = x + C$$

$$Or \quad \frac{2}{5}v - \frac{2}{25}\log(5v-9) = x + C$$

Substitute $2x + y = v$, we get

$\frac{2}{5}(2x+y) - \frac{2}{25}\log[5(2x+y)-9] = x + C$ is the required general solution.

Example 2.6.3: Find the solution of the differential equation $\frac{dy}{dx} = \frac{x-y+6}{3x-3y+4}$

Solution: We have, $\frac{dy}{dx} = \frac{x-y+6}{3x-3y+4}$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 1

Here, $\frac{a_1}{a_2} = \frac{1}{3} = \frac{b_1}{b_2} = \frac{-1}{-3} = \frac{1}{3}$

$$\therefore \frac{dy}{dx} = \frac{x-y+6}{3(x-y)+4}$$

Put $x - y = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx}$ Or $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

Equation (1) gives, $1-\frac{dv}{dx}=\frac{v+6}{3v+4}$ Or $\frac{dv}{dx}=1-\frac{v+6}{3v+4}$

$$\therefore \quad \frac{dv}{dx}=\frac{3v+4-v-6}{3v+4}$$

$$Or \quad \frac{dv}{dx}=\frac{2v-2}{3v+4}$$

This is in variable separable form

$$\therefore \quad \frac{3v+4}{2v-2}\,dv=dx$$

$$\frac{1}{2}\left(3+\frac{7}{v-1}\right)dv=dx, \text{ int} egrating$$

$$\frac{1}{2}\int\left(3+\frac{7}{v-1}\right)dv=\int dx+C$$

$$\frac{1}{2}\left(3v+7\log(v-1)\right)=x+C$$

$$Or \quad \frac{3}{2}v+\frac{7}{2}\log(v-1)=x+C$$

Substitute $x-y=v$, we get

$\frac{3}{2}(x-y)+\frac{7}{2}\log\left[(x-y)-1\right]=x+C$ is the required general solution.

Illustrations on Type-2, non homogeneous ODE's of first order:

Example 2.6.4: Find the solution of the differential equation $\frac{dy}{dx}=\frac{x-3y-7}{x-4}$

Solution: We have, $\frac{dy}{dx}=\frac{x-3y-7}{x-4}$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 2, since

$$\frac{a_1}{a_2} = \frac{1}{1} \neq \frac{b_1}{b_2} = \frac{-3}{0}$$

Let’s substitute $x = X + h$ and $y = Y + k$, where $h \; and \; k$ are arbitrary constants.

Then,

we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$.

Thus, the given equation (1) becomes

$$\frac{dY}{dX} = \frac{(X+h) - 3(Y+k) - 7}{(X+h) - 4} = \frac{X - 3Y + (h - 3k - 7)}{X + (h-4)} ....(2)$$

Now, choose h and k such that $h - 3k - 7 = 0$ and $h - 4 = 0$, this implies that h = 4 and k = -1

Then, the differential equation (2) becomes,

$$\frac{dY}{dX} = \frac{X - 3Y}{X} = \frac{X\left(1 - 3\left[\frac{Y}{X}\right]\right)}{X} = X^0 f(Y/X) ....(3),$$ which is homogeneous in terms of X and Y of degree zero.

Now, substituting $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$.

Therefore, Equation (3) reduces to

$$V + X\frac{dV}{dX} = 1 - 3V \; Or \; X\frac{dV}{dX} = 1 - 4V$$

This is in variable separable form

$$\therefore \ \frac{dV}{1-4V} = \frac{dX}{X}, \ Integrating$$

$$\int \frac{dV}{1-4V} = \int \frac{dX}{X} + \log C$$

$$\therefore \ -\frac{1}{4}\log(1-4V) = \log CX$$

$$Or \ \log CX = \log \frac{1}{(1-4V)^{1/4}}$$

$$Or \ X = \frac{C}{(1-4V)^{1/4}}$$

Now, Substitute $V = \frac{Y}{X}$ then put $X = x-4$ and $Y = y+1$, we get

$$X = \frac{C}{\left(1-4\frac{Y}{X}\right)^{1/4}} \ Or \ x-4 = \frac{C}{1-4\left(\frac{y+1}{x-4}\right)}$$

$$Or \ x-4 = \frac{C(x-4)}{x-4y-8} \ Or \ C = x-4y-8$$

This is the required general solution of the equation (1).

Example 2.6.5: Find the solution of the differential equation $\frac{dy}{dx} = \frac{2y+6}{x+y+1}$

Solution: We have, $\frac{dy}{dx} = \frac{2y+6}{x+y+1}$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 2, since

$$\frac{a_1}{a_2} = \frac{0}{1} \neq \frac{b_1}{b_2} = \frac{2}{1}$$

Let's substitute $x = X + h$ and $y = Y + k$, where $h \ and \ k$ are arbitrary constants.

Then,

we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$.

Thus, the given equation (1) becomes

$$\frac{dY}{dX} = \frac{2(Y+k)+6}{(X+h)+(Y+k)+1} = \frac{2Y+(2k+6)}{X+Y+(h+k+1)} \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, choose h and k such that $2k+6=0$ and $h+k+1=0$, this implies that h = 2 and k = -3

Then, the differential equation (2) becomes,

$$\frac{dY}{dX} = \frac{2Y}{X+Y} = \frac{X(2Y/X)}{X\left(1+\frac{Y}{X}\right)} = X^0 f(Y/X) \ldots\ldots\ldots\ldots\ldots\ldots(3),$$ which is homogeneous in terms of X and Y of degree zero.

Now, substituting $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$.

Therefore, Equation (3) reduces to

$$V + X\frac{dV}{dX} = \frac{2V}{1+V}$$

$$Or\ X\frac{dV}{dX} = \frac{2V}{1+V} - V = \frac{2V - V - V^2}{V+1} = \frac{V(1-V)}{(1+V)}$$

$$\therefore\ X\frac{dV}{dX} = \frac{V(1-V)}{(1+V)} = -\frac{V^2 - V}{V+1}$$

$$i.e.,\ X\frac{dV}{dX} = -\frac{V^2 - V}{V+1}$$

This is in variable separable form

$$\therefore \quad \frac{(V+1)}{V^2-V}dV = -\frac{dX}{X}, \; Integrating$$

$$\int \frac{(V+1)dV}{V^2-V} = -\int \frac{dX}{X} + \log C$$

$$\therefore \quad \int\left(\frac{2}{V-1} - \frac{1}{V}\right)dV = \log\left(\frac{C}{X}\right)$$

$$\therefore \quad 2\log(V-1) - \log V = \log\left(\frac{C}{X}\right)$$

$$Or \; \log\left[\frac{(V-1)^2}{V}\right] = \log\left(\frac{C}{X}\right)$$

$$Or \;\; C = X\left[\frac{(V-1)^2}{V}\right]$$

Now, Substitute $V = \frac{Y}{X}$ then put $X = x-2$ and $Y = y+3$, we get

$$C = X\left[\frac{(Y/X-1)^2}{Y/X}\right] = X\left[\frac{(Y-X)^2}{XY}\right]$$

$$Or \;\; C = (x-2)\left[\frac{[(y+3)-(x-2)]^2}{(x-2)(y+3)}\right]$$

This is the required general solution of the equation (1).

Example 2.6.6: Find the solution of the differential equation

$$\frac{dy}{dx} = \left(\frac{2x+y-1}{x-2}\right)^2$$

Solution: We have, $\frac{dy}{dx} = \left(\frac{2x+y-1}{x-2}\right)^2$ ....(1)

Equation (1) is a non-homogeneous ordinary differential equation of Type 2, since

$$\frac{a_1}{a_2} = \frac{2}{1} \neq \frac{b_1}{b_2} = \frac{1}{0}$$

Let's substitute $x = X + h$ and $y = Y + k$, where $h \text{ and } k$ are arbitrary constants.

Then,

we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$.

Thus, the given equation (1) becomes

$$\frac{dY}{dX} = \left(\frac{2(X+h)+(Y+k)-1}{(X+h)-2}\right)^2 = \left(\frac{2X+Y+(2h+k-1)}{X+(h-2)}\right)^2 \text{.....................(2)}$$

Now, choose h and k such that $2h + k - 1 = 0$ and $h - 2 = 0$, this implies that h = 2 and k = -3

Then, the differential equation (2) becomes,

$$\frac{dY}{dX} = \left(\frac{2X+Y}{X}\right)^2 = \left(\frac{X(2+Y/X}{X}\right)^2 = X^0 f(Y/X) \qquad \text{.......................(3)},$$ which is homogeneous in terms of X and Y of degree zero.

Now, substituting $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$.

Therefore, Equation (3) reduces to

$$V + X\frac{dV}{dX} = (2+V)^2 \; Or \; X\frac{dV}{dX} = V^2 + 3V + 4$$
$$Or \; X\frac{dV}{dX} = \left(V + \frac{3}{2}\right) \pm i\frac{\sqrt{7}}{2}$$

This is in variable separable form

$$\therefore \quad \frac{dV}{\left(V+\frac{3}{2}\right)\pm i\frac{\sqrt{7}}{2}}=\frac{dX}{X}, \ \textit{Integrating}$$

$$\int \frac{dV}{\left(V+\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}=\int \frac{dX}{X}+\log C$$

$$Or \ \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2(V+3/2}{\sqrt{7}}\right)=\log XC$$

Now, Substitute $V=\frac{Y}{X}$ then put $X=x-2$ and $Y=y+3$, we get

$$\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2(Y/X+3/2}{\sqrt{7}}\right)=\log XC$$

$$Or \ \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{(2Y+3X)}{X\sqrt{7}}\right)=\log XC$$

$$Or \ \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{(2(y+3)+3(x-2))}{(x-2)\sqrt{7}}\right)=\log C(x-2)$$

OR

$$Y(X)=\frac{1}{2}\left[\sqrt{7}X\tan\left(\frac{\sqrt{7}}{2}(\log|X|+C\right)-3X\right]$$

$$\textit{Substitute } X=x-2 \textit{ and } Y=y+3, \textit{we get}$$

$$y(x)=\frac{1}{2}\left[\sqrt{7}(x-2)\tan\left(\frac{\sqrt{7}}{2}(\log|x-2|+C\right)-3(x-2)\right]-3$$

This is the required general solution of the equation (1).

2.6 Test Your Progress

Find the solutions of the following differential equations.

(i) $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$

(ii) $\dfrac{dy}{dx} = \dfrac{x+y+3}{2x+2y+1}$

(iii) $\dfrac{dy}{dx} = \dfrac{6x-4y+3}{3x-2y+1}$

(iv) $(4x+6y+5)\dfrac{dy}{dx} = 3y+2x+4$

(v) $(12x+5y-9)dx+(5x+2y-4)dy = 0$

(vi) $\dfrac{dy}{dx} = \dfrac{2x-y+1}{x+2y-3}$

(vii) $(2x+y+1)dx+(4x+2y-1)dy = 0$

(viii) $\dfrac{dy}{dx} = \dfrac{3y+2x+4}{4x+6y+5}$

(ix) $(2x+4y+3)dy = (2y+x+1)dx$

2.7 Summary

In this unit, we studied the method of solving the differential equation in which variables are separable and homogeneous equation and method of solving them. We also studied the equation which are reductive to the homogeneous from.

2.8 Terminal Questions

1. Solve the differential equations.

   a) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

   b) $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y - \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x.\frac{dy}{dx} = 0$

   c) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

   d) $(x^2 - y^2dx + 2xydy = 0)$

2.9. Answers to exercises

2.4 Test Your Progress

i) $y = x^2 + \log x + C$

ii) $\frac{y^3}{3} = x^2 + C$

iii) $y = \tan x - x + C$

iv) $e^x + e^{-y} = C$

v) $y = \frac{e^x}{2}(\sin x - \cos x) + C$

vi) $(y+2) - \log(y+2)^2 = x + \log x^2 + C$

vii) $y = \log \sec hx + C$

viii) $y = \log\left[\frac{9(x+1)^2(x-1)^2}{2x}\right]$

ix) $y^2 = \log \sec^2 x + 1$

x) $\log y + y^2 = \sin x + 1$

xi) $y = \tan(x+c) - x$

2.5 Test Your Progress

1. (iv) is the only homogeneous differential equation

2. (i) $y = \log(C\,x)^x$

(ii) $\log(x^2 + xy + y^2) = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C$

(iii) $x^2 - y^2 = C\,x$

(iv) $C = \frac{\sec\frac{y}{x}}{xy}$

(v) $Cx = \cos ec\frac{y}{x} - \cot\frac{y}{x}$

(vi) $General\ Solution: C = \frac{1}{x}e^{-2x/y}$ $and\ Particular\ Solution: xe^{\frac{2x}{y}-1} = 1$

(vii) $C = \frac{y}{x^2}$

2.6 Test Your Progress

(i) $e^{y-x} = C(x+y)$

(ii) $\frac{2}{3}y - \frac{5}{9}\log(3x+3y+4) = \frac{x}{3} + C$

(iii) $Ce^{2x-y} = 3x - 2y + 3$

(iv) $14(2x+3y) - 9\log|(14x+21y+22)| = 49x + C$

(v) $6x^2 + 5xy + y^2 - 9x - 4y = C$

(vi) $\left(y-\frac{7}{5}\right)^2 + \left(x-\frac{1}{5}\right)\left(y-\frac{7}{5}\right) - \left(x-\frac{1}{5}\right)^2 = C$

(vii) $x + 2y + \log|2x + y - 1| = C$

(viii) $14(2x+3y) - 9\log|(14x+21y+22)| = 49x + C$

(ix) $4x + 8y + 5 = Ce^{4x-8y}$

(x) $Ce^{2x-y} = 3x - 2y + C$

2.7 Test Your Progress

i) $y(x^2-1) = x + C$

ii) $y\frac{x}{x-1} = \frac{x^3}{3} + C$

iii) $yx^{\frac{1}{2}\log x} = e^x + C$

iv) $y(x^2 - 1) = x + C$

v) $ye^{x^2} = x + C$

vi) $yx^2 = \frac{x^4 \log x}{4} - \frac{x^4}{16} + C$

vii) $xe^y = \tan y + C$

viii) $y = \tan x + C\sqrt{\tan x}$

ix) $2y = x^2 \log x^2 + x^2 \cos\left(\frac{1}{x^2}\right) + Cx^2$

x) $2y \log x + \cos 2x = C$

xi) $y \sec^2 x = \sec x - 2$

xii) $y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}$

2.8 Test Your Progress

i) $\frac{1}{x^5 y^5} = \frac{5}{2x^2} + C$

ii) $\frac{1}{xy} = \frac{-\log x}{x} - \frac{1}{x} + C$

iii) $y^3(1+x)^2 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + C$

iv) $\tan^{-1} y \cdot e^{x^2} = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

v) $y^2 \cos^2 x = \frac{-2}{5}\cos^5 x + C$

$vi)$ $\frac{\sin y^2}{(x+1)^2} = \frac{1}{2}(x+1)^2 + C$

$vii)$ $e^y e^{e^x} = e^{e^x}(e^x - 1) + C$

$viii)$ $x \log y = xe^x - e^x + C$

$ix)$ $\frac{y^3}{x^6} = \frac{-2}{x^3} + 3$

$x)$ $x^3 - y^3 = 3y^3 \sin x$

Unit –03: Linear differential equation

Structure

3.1. Introduction

3.2. Objectives

3.3. Linear differential equations of first order and first degree

3.4. Bernoulli's differential equation (Or Non linear equations of first order and first degree)

3.5. Summary

3.6 Terminal Questions

3.7 Answers to exercises

3.1. Introduction

The ordinary differential equations may be divided into two large classes, namely, linear equations and non-linear equations. Whereas non-linear equations are difficult in general, linear equations are much simpler because then solutions have general properties that facilitate working with them, and there are standard methods for solving many particularly important linear differential equations.

In the previous unit, we had learnt how to solve variable separable form, homogeneous and non homogeneous differential equations of first order and first degree. In this lesson we will learn how to solve linear, non linear (i.e., Bernoulli's Equation) , exact and non exact ODE's which are very useful in various physical and Engineering applications.

3.2. Objectives

After reading this unit students should be able to:

- Understand and find the solution of linear differential equations of first order and first degree
- Solve the Bernoulli's equation (Or Non linear differential equations) of first order and first degree

- Solve the exact differential equations of first order and first degree
- Identify the integrating factor to Solve the non-exact first order and first degree ODE's

3.3. Linear differential equations of first order and first degree

A first-order **linear** differential equation is one that can be written in the form $\frac{dy}{dx}+P(x)y=Q(x)$.

Where *P* and *Q* are continuous functions of *x*. Equation (1) is the linear equation's **standard**

form.

Equation (1) is *linear* (in *y*) because *y* and its derivative $\frac{dy}{dx}$ occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function (such as sin *y*, e^y, or $\sqrt{\frac{dy}{dx}}$).

Example 3.3.1: Put the following equation in standard form:

$$x\frac{dy}{dx}=x^2+3y$$

Solution: Consider, $x\frac{dy}{dx}=x^2+3y$

$$Dividing \quad by \quad x$$

$$\therefore \frac{dy}{dx}=x+\frac{3}{x}y$$

$$\frac{dy}{dx}-\frac{3}{x}y=x$$

Which is in the standard form $\frac{dy}{dx}+P(x)y=Q(x)$ with $P(x)=-\frac{3}{x}$ and

$Q(x)=x$, so the

minus sign is part of the formula for $P(x)$.

3.3.2. Solving Linear Equations:

We solve the equation $\frac{dy}{dx}+Py=Q$

by multiplying both sides by a *positive* function $v(x)$ that transforms the

left-hand side into

the derivative of the product $v(x)\cdot y$. We will show how to find $v(x)$ in a

moment, but first we

want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

Original equation in standard form is $\frac{dy}{dx}+P(x)y=Q(x)$ ....(1)

Multiply by positive $v(x)$, we obtain

$$v(x)\frac{dy}{dx}+P(x)v(x)y=Q(x)v(x)$$

$v(x)$ is chosen to make $v\frac{dy}{dx}+Pvy=\frac{d}{dx}(v.y)$

$$\frac{d}{dx}\left(v(x)\cdot y\right)=v(x)Q(x)$$

Integrating w. r. t 'x'

$$\left(v(x)\cdot y\right)=\int v(x)Q(x)\,dx$$

$$y=\frac{1}{v(x)}\int v(x)Q(x)\,dx \quad \text{.......................(2)}$$

Equation (2) expresses the solution of Equation (1) in terms of the functions $v(x)$ and

$Q(x)$. We call $v(x)$ an **integrating factor** for Equation (1) because its presence makes the

equation integrable.

Remark 3.3.1:

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly,

in the construction of the positive function $v(x)$. We have

The condition imposed on $v(x)$ is $\frac{d}{dx}\left(v(x)\cdot y(x)\right)=v(x)\frac{dy}{dx}+P(x)v(x)y(x)$

By using the product rule of derivatives

$$\therefore v(x)\frac{dy}{dx} + y\frac{dv}{dx} = v(x)\frac{dy}{dx} + P(x)v(x)y(x)$$

$$Or\ y\frac{dv}{dx} = P(x)v(x)\,y(x)$$

This last equation will hold if $\frac{dv}{dx} = P(x)v(x)$, which is in variable separable form

$$\therefore \frac{dv}{v(x)} = P(x)\ dx\ ;\ \ v(x) > 0$$

Integrate both sides

$$\int \frac{dv}{v(x)} = \int P(x)\ dx$$

Or $\log v(x) = \int P\,dx$

Taking exponentiation on both sides to solve for $v(x)$.

$$e^{\log v} = e^{\int p dx}\ \ Or\ \ v = e^{\int P dx}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Thus a formula for the general solution to Equation (1) is given by Equation (2), where $v(x)$ is given by Equation (3). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified.

Any anti derivative of P works for Equation (3).

To solve the linear equation $\frac{dy}{dx}+P(x)y=Q(x)$, multiply both sides by the integrating factor $v(x)=e^{\int P(x)dx}$ and integrate both sides.

When you integrate the product on the left-hand side in this procedure, you always obtain

the product $v(x)y(x)$ of the integrating factor and solution function $y(x)$ because of the way $v(x)$ is defined.

Remark 3.3.2: Here, it is observed that if the function $Q(x)$ is identically zero in the standard form given by the equation (1), the linear equation is separable and can be solved by the method of variable separable form.

$$\frac{dy}{dx}+P(x)y=Q(x)$$

If $Q(x)=0$, then we get

$$\frac{dy}{dx}+P(x)y=0$$

$$Or\ \frac{dy}{y}=-P(x)dx$$

This is in variable separable form.

Example 3.3.1: Solve the equation $x\frac{dy}{dx} = x^2 + 3y, \quad x > 0.$

Solution: Let's first put the given equation $x\frac{dy}{dx} = x^2 + 3y, \quad x > 0$ in the standard form as $\frac{dy}{dx} - \frac{3}{x} y = x$ ....(1) so that it is linear and $P(x) = -\frac{3}{x}$ is identified.

The integrating factor is

$$v(x) = e^{\int P(x)dx} = e^{\int -(3/x)dx}$$
$$= e^{-3\log x} = \frac{1}{x^3}$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$\frac{1}{x^3} \cdot \left(\frac{dy}{dx} - \frac{3}{x} y \right) = \frac{1}{x^3} \cdot x$$
$$Or\ \frac{1}{x^3} \cdot \frac{dy}{dx} - \frac{3}{x^4} y = \frac{1}{x^2}$$
$$Or\ \frac{d}{dx}\left(\frac{1}{x^3} y \right) = \frac{1}{x^2}, \text{ int} egrating$$
$$\frac{1}{x^3} y = \int \frac{1}{x^2} dx$$
$$\frac{1}{x^3} y = -\frac{1}{x} + C$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \quad x > 0$$

Example 3.3.2: Solve the equation $x^2 \frac{dy}{dx} + (x-2)y = x^2 e^{-2/x}$.

Solution: Let's first put the given equation $x^2 \frac{dy}{dx} + (x-2)y = x^2 e^{-2/x}$ in the standard form as $\frac{dy}{dx} + \left(\frac{x-2}{x^2}\right)y = e^{-2/x}$ ....(1) so that it is linear and $P(x) = \frac{x-2}{x^2}$ is identified.

The integrating factor is

$$v(x) = e^{\int \frac{x-2}{x^2}dx} = e^{\int\left(\frac{1}{x} - \frac{2}{x^2}\right)dx}$$

$$= e^{\log|x| + \frac{2}{x}}$$

$$= e^{\log|x|} e^{\frac{2}{x}} = xe^{\frac{2}{x}}$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$xe^{2/x}\cdot\left(\frac{dy}{dx}+\frac{x-2}{x^2}y\right)=xe^{2/x}e^{-2/x}$$

$$Or\ xe^{2/x}\cdot\left(\frac{dy}{dx}+\frac{x-2}{x^2}y\right)=x$$

$$Or\,\frac{d}{dx}\left(xye^{2/x}\right)=\int x\,dx,\ \text{int}\,egrating$$

$$x\cdot y\cdot e^{2/x}=\frac{x^2}{2}+C$$

Solving this last equation for y gives the general solution:

$$y=e^{-2/x}\left(\frac{x}{2}+\frac{C}{x}\right).$$

Example 3.3.3: Solve the equation $x\log x\frac{dy}{dx}+y=2\log x.$

Solution: Let's first put the given equation $x\log x\frac{dy}{dx}+y=2\log x$ in the

standard form as $\frac{dy}{dx}+\left(\frac{1}{x\log x}\right)y=\frac{2}{x}$ ....(1) so that it is linear and

$P(x)=\frac{1}{x\log x}$ is identified.

The integrating factor is

$$v(x)=e^{\int\frac{1}{x\log x}dx}=e^{\int\left(\frac{1/x}{\log x}\right)dx}$$

$$=e^{\log(\log x)}=\log x$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$\log x\left[\frac{dy}{dx}+\left(\frac{1}{x\log x}\right)y\right]=\frac{2}{x}\log x$$

$$Or\ \frac{d}{dx}[y(\log x)]=\frac{2}{x}\log x,\ \text{int}\, egrating$$

$$y(\log x)=\int\frac{2}{x}\log x\,dx+C$$

$$Or\ \ y(\log x)=2\int\frac{1}{x}\log x\,dx+C$$

$$\therefore\quad y(\log x)=\frac{2(\log x)^2}{2}+C$$

$$Or\ \ y(\log x)=(\log x)^2+C$$

Solving this last equation for y gives the general solution:

$$y=\log x+\frac{C}{\log x}.$$

Example 3.3.4: Solve the equation $(x+y+1)\frac{dy}{dx}=1.$

Solution: Let's first put the given equation $(x+y+1)\frac{dy}{dx}=1$ in the standard form as $\frac{dy}{dx}=\frac{1}{(x+y+1)}$ Or $\frac{dx}{dy}=x+y+1$ Or $\frac{dx}{dy}-x=y+1$ ....(1) so that it is linear and $P(y)=-1$ is identified.

The integrating factor is

$$v(x) = e^{\int P(y)dy} = e^{\int -1dy}$$
$$= e^{-y}$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$e^{-y}\left(\frac{dx}{dy} - x\right) = e^{-y}(y+1)$$
$$Or \ \frac{d}{dy}\left[xe^{-y}\right] = e^{-y}(y+1), \text{ int}\,egrating$$
$$\left[xe^{-y}\right] = \int e^{-y}(y+1)dx + C$$
$$Or \ \ xe^{-y} = -(y+1)e^{-y} + \int e^{-y}dy + C$$
$$\therefore \quad xe^{-y} = -(y+1)e^{-y} - e^{-y} + C$$

Solving this last equation for x gives the general solution:

$x = -y - 2 + Ce^{y}$.

Example 3.3.5: Solve the equation $(1+y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0.$

Solution: Let's first put the given equation $(1+y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0$ in the standard form as $(1+y^2)\dfrac{dx}{dy} + (x - e^{\tan^{-1}y}) = 0$ Or $(1+y^2)\dfrac{dx}{dy} + x = e^{\tan^{-1}y}$

$Or \dfrac{dx}{dy} + \dfrac{x}{(1+y^2)} = \dfrac{e^{\tan^{-1}y}}{(1+y^2)}$ ....(1) so that it is linear and $P(y) = -1$ is identified.

The integrating factor is

$$v(x) = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy}$$
$$= e^{\tan^{-1} y}$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$e^{\tan^{-1} y}\left(\frac{dx}{dy} + \frac{x}{(1+y^2)}\right) = e^{\tan^{-1} y}\left(\frac{e^{\tan^{-1} y}}{(1+y^2)}\right)$$

$$\frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = e^{\tan^{-1} y}\left(\frac{e^{\tan^{-1} y}}{(1+y^2)}\right), \text{ integrating}$$

$$\therefore \quad xe^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{(1+y^2)} e^{\tan^{-1} y}\, dy + C$$

$$Or \ xe^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{(1+y^2)}\, dy + C$$

$$\therefore \quad xe^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + C$$

Solving this last equation for x gives the general solution:

$$x = \frac{1}{2} e^{\tan^{-1} y} + Ce^{-\tan^{-1} y}.$$

Example 3.3.6: Find the particular solution of $3x\frac{dy}{dx} - y = \log x + 1, \quad x > 0$, satisfying $y(1) = -2$.

Solution: Let's first put the given equation $3x\frac{dy}{dx}-y=\log x+1,\ \ x>0$ in the standard form as $\frac{dy}{dx}-\frac{1}{3x}y=\frac{\log x+1}{3x}$ ....(1) so that it is linear and $P(x)=-\frac{1}{3x}$ is identified.

The integrating factor is

$$v(x)=e^{\int P(x)dx}=e^{\int -\frac{1}{3x}dx}$$
$$=e^{-\frac{1}{3}\log x}=\frac{1}{x^{1/3}}$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$x^{-1/3}\left(\frac{dy}{dx}-\frac{1}{3x}y\right)=x^{-1/3}\left(\frac{\log x+1}{3x}\right)$$
$$Or\ \frac{d}{dx}\left(yx^{-1/3}\right)=\frac{1}{3}(\log x+1)x^{-4/3},\ \text{integrating}$$
$$yx^{-1/3}=\frac{1}{3}\int(\log x+1)x^{-4/3}dx+C$$
$$Or\ yx^{-1/3}=\frac{1}{3}\left[(\log x+1)\frac{x^{-1/3}}{-1/3}+3\int\frac{x^{-1/3}}{x}dx\right]+C$$
$$Or\ yx^{-1/3}=\left[-(\log x+1)x^{-1/3}+\int x^{-4/3}dx\right]+C$$
$$Or\ yx^{-1/3}=\left[-(\log x+1)x^{-1/3}-3x^{-1/3}\right]+C$$

Solving this last equation for y gives the general solution:

$$y=\left[-(\log x+1)-3\right]+Cx^{1/3}\ \ Or\ \ y=\left[-(\log x+4\right]+Cx^{1/3}\ \ Or\ \ y=Cx^{1/3}-\log x-4$$

.... (2)

Now, let's use the given initial conditions as; when $x = 1$ and $y = -2$ in the equation (2), we get

$-2 = -(0+4) + C \Rightarrow C = 2$. Substituting the value of C in equation (2) gives the particular solution given by $y = 2x^{1/3} - \log x - 4$.

Example 3.3.7: Find the particular solution of $\frac{dy}{dx} + y\cot x = 4x\cos ecx$,

satisfying $y\left(\frac{\pi}{2}\right) = 0$.

Solution: Consider , $\frac{dy}{dx} + y\cot x = 4x\cos ecx$ ....(1) this is linear

and $P(x) = \cot x$ is identified.

The integrating factor is

$$v(x) = e^{\int P(x)dx} = e^{\int \cot x dx}$$
$$= e^{\log\sin x} = \sin x$$

Now, we multiply both sides of the equation (1) by the integrating factor viz., $v(x)$ and then integrate

$$\sin x\left(\frac{dy}{dx} + y\cot x\right) = (4x\cos ecx)\sin x$$

$$Or \ \sin x\frac{dy}{dx} + y\cos x = 4x$$

$$Or \quad \frac{d}{dx}(y\sin x) = 4x, \ \text{int}\, egrating$$

$$y\sin x = 4\int x\,dx + C$$

$$y\sin x = 2x^2 + C$$

Solving this last equation for y gives the general solution:

$$y = \frac{2x^2 + C}{\sin x} \;\; Or \;\; y = (2x^2 + C)\cos ecx \quad .... \quad (2)$$

Now, let's use the given initial conditions as; when $x = \frac{\pi}{2}$ and $y = 0$ in the equation (2), we get

$0 = \frac{\pi^2}{2} + C \Rightarrow C = -\frac{\pi^2}{2}$. Substituting the value of C in equation (2) gives the particular solution given by $y = (2x^2 - \frac{\pi^2}{2})\cos ecx$.

3.3 Test Your Progress
Solve the following differential equations: i) $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$ ii) $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$ iii) $x\frac{dy}{dx} + y\log x = e^x x^{1-\frac{1}{2}\log x}$ iv) $(x^2 - 1)\frac{dy}{dx} + 2xy = 1$ v) $\frac{dy}{dx} + 2xy = e^{-x^2}$

vi) $x\frac{dy}{dx}+2y-x^2\log x=0$

vii) $dx+xdy=e^{-y}\sec^2 ydy$

viii) $\sin 2x\frac{dy}{dx}-y=\tan x$

ix) $\frac{dy}{dx}-\left(\frac{2}{x}\right)y=x+\frac{1}{x}\sin\left(\frac{1}{x^2}\right)$

x) $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$

xi) Solve $\frac{dy}{dx}+2y\tan x=\sin x.\ Given\ that\ \ y=o\ \ when\ \ x=\frac{\pi}{3}$

xii) Solve $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}.\ Given\ that\ \ y=o\ \ when\ \ x=1$

3.4. Non-linear (or Bernoulli's) differential equations

Definition 3.4.1:

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ ....(1), where P and Q are functions of x only and 'n' is a real number.

Case 1: If $n=1$ then equation (1) can be written as

$$\frac{dy}{dx}+\left[P(x)-Q(x)\right]y=0 \text{ ....(2)}$$

This is of variable separable form, therefore its general solution is given by

$$\int \frac{dy}{y} + \int (P-Q)dx = C.$$

Case 2: If $n \neq 1$ then divide the equation (1) by y^n, we obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \quad \text{................................(3)}$$

Put $y^{1-n} = u$

Differentiate with respect to x; $(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$.

Equation (3) gives; $\frac{du}{dx} + (1-n)P \cdot u = (1-n)Q$. This is linear differential equation in u and x

Therefore, Integrating factor = $e^{\int (1-n)Pdx}$

Its general solution is given by $ue^{\int (1-n)Pdx} = \int (1-n)Q \cdot e^{\int (1-n)Pdx} dx$ ....(4)

To get the general solution of equation (1), substitute $y^{1-n} = u$.

Example 3.4.1: Solve the equation $\frac{dy}{dx} + y\cot x = y^2 \sin^2 x \cos^2 x$.

Solution: Given equation $\frac{dy}{dx} + y\cot x = y^2 \sin^2 x \cos^2 x$ ....(1) is Bernoulli's equation.

Dividing equation (1) by y^2, we obtain $y^{-2}\frac{dy}{dx}+\frac{1}{y}\cot x=\sin^2 x\cos^2 x$

.....................(2)

$$\text{Put}\ \frac{1}{y}=u \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{du}{dx}\ Or\ \frac{1}{y^2}\frac{dy}{dx}=-\frac{du}{dx}$$

Equation (2) gives,

$$-\frac{du}{dx}+u\cot x=\sin^2 x\cos^2 x$$

$$Or\quad \frac{du}{dx}-u\cot x=-\sin^2 x\cos^2 x \qquad ....(3)$$

This is linear equation in u and x with $P(x)=-\cot x$ is identified

The integrating factor is

$$v(x)=e^{\int P(x)dx}=e^{-\int \cot x dx}$$
$$=e^{-\log\sin x}=\cos ecx$$

Now, we multiply both sides of the equation (3) by the integrating factor viz., $v(x)$ and then integrate

$$\cos ecx\left(\frac{du}{dx}-u\cot x\right)=\left(-\sin^2 x\cos^2 x\right)\cos ecx$$
$$Or\ \cos ecx\frac{du}{dx}-u\cot x\cos ecx=-\sin x\cos^2 x$$
$$Or\quad \frac{d}{dx}\left(u\cos ecx\right)=-\cos^2 x\sin x,\ \ \text{int}\, egrating$$
$$u\cos ecx=\int \cos^2 x(-\sin x)dx+C$$
$$u\cos ecx=\frac{\cos^3 x}{3}+C$$

Substitute $\frac{1}{y}=u$, we get

$\frac{1}{y}\cos ecx=\frac{\cos^3 x}{3}+C$ is the required general solution of the given differential equation (1)

Example 3.4.2: Solve the equation $\frac{dy}{dx}+\frac{xy}{1-x^2}=xy^{1/2}$

Solution: Given equation $\frac{dy}{dx}+\frac{xy}{1-x^2}=xy^{1/2}$.............(1) is Bernoulli's equation.

Dividing equation (1) by $y^{1/2}$, we obtain $y^{-1/2}\frac{dy}{dx}+\frac{x}{1-x^2}y^{1/2}=x$ ....(2)

Put $y^{1/2}=u \Rightarrow \frac{1}{2}y^{-1/2}\frac{dy}{dx}=\frac{du}{dx}$ *Or* $y^{-1/2}\frac{dy}{dx}=2\frac{du}{dx}$

Equation (2) gives,

$$2\frac{du}{dx}+\frac{x}{1-x^2}u=x$$

Or $\frac{du}{dx}+\frac{x}{2(1-x^2)}u=\frac{x}{2}$ ....(3)

This is linear equation in u and x with $P(x)=\frac{x}{2(1-x^2)}$ is identified

The integrating factor is

$$v(x) = e^{\int P(x)dx} = e^{\int \frac{x}{2(1-x^2)}dx}$$

$$= e^{\frac{1}{2}\int \frac{x}{(1-x^2)}dx}$$

$$= e^{-\frac{1}{4}\int \frac{1}{t}dt} = e^{-\frac{1}{4}\log(1-x^2)}$$

$$= \frac{1}{(1-x^2)^{1/4}}$$

Now, we multiply both sides of the equation (3) by the integrating factor viz., $v(x)$ and then integrate

$$\frac{1}{(1-x^2)^{1/4}}\left[\frac{du}{dx} + \frac{x}{2(1-x^2)}u\right] = \frac{x}{2}\left(\frac{1}{(1-x^2)^{1/4}}\right)$$

$$Or \quad \frac{1}{(1-x^2)^{1/4}}\frac{du}{dx} + \frac{x}{2(1-x^2)^{5/4}}u = \frac{x}{2(1-x^2)^{1/4}}$$

$$Or \quad \frac{d}{dx}\left[\frac{u}{(1-x^2)^{1/4}}\right] = \frac{1}{2}\frac{x}{(1-x^2)^{1/4}},\ Integrating$$

$$\therefore \quad \left[\frac{u}{(1-x^2)^{1/4}}\right] = \frac{1}{2}\int \frac{x}{(1-x^2)^{1/4}}dx + C$$

$$\therefore \quad \left[\frac{u}{(1-x^2)^{1/4}}\right] = \frac{1}{2} \times \frac{-1}{2} \times \frac{(1-x^2)^{31/4}}{3/4} + C$$

$$Or \quad \frac{u}{(1-x^2)^{1/4}} = -3(1-x^2)^{31/4} + C$$

Substitute $y^{1/2} = u$, we get

$\frac{y^{1/2}}{(1-x^2)^{1/4}} = -3(1-x^2)^{3/4} + C \quad Or \quad y^{1/2} = -3(1-x^2) + C(1-x^2)^{1/4}$ is the required general solution of the given differential equation (1).

Example 3.4.3: Solve the equation $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Solution: Given equation $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$ ....(1) is

Bernoulli's equation.

Dividing equation (1) by $\cos^2 y$, we obtain $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$

.....................(2)

Put $\tan y=u \Rightarrow \sec^2 y\frac{dy}{dx}=\frac{du}{dx}$

Equation (2) gives,

$$\frac{du}{dx}+2xu=x^3 \qquad ....(3)$$

This is linear equation in u and x with $P(x)=2x$ is identified.

The integrating factor is

$$v(x)=e^{\int P(x)dx}=e^{2\int xdx}$$
$$=e^{2\frac{(x^2)}{2}}=e^{x^2}$$

Now, we multiply both sides of the equation (3) by the integrating factor

viz., $v(x)$ and then integrate

$$e^{x^2}\left(\frac{du}{dx}+2xu\right)=e^{x^2}x^3$$

$$Or \quad \frac{d}{dx}\left(ue^{x^2}\right)=e^{x^2}x^3, \; Integrating$$

$$\therefore \quad ue^{x^2}=\int e^{x^2}x^2\,x\,dx+C$$

$$\therefore \quad ue^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+C$$

Substitute $\tan y = u$, we get

$$\tan y e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1)+C \ Or \ \tan y = \frac{x^2-1}{2}+Ce^{-x^2} \ Or \ y=\tan^{-1}\left[\frac{x^2-1}{2}+Ce^{-x^2}\right]$$ is

the required general solution of the given differential equation (1).

Example 3.4.4: Solve the equation $\frac{dy}{dx}\left(x^2y^3+xy\right)=1$

Solution: Given equation $\frac{dy}{dx}\left(x^2y^3+xy\right)=1$ can be written as

$$\frac{dx}{dy}=\left(x^2y^3+xy\right)$$

Or $\frac{dx}{dy}-xy=x^2y^3$ ....(1) is Bernoulli's equation.

Dividing equation (1) by x^2, we obtain $\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}y=y^3$ ....(2)

Put $\frac{1}{x}=u \Rightarrow \frac{-1}{x^2}\frac{dx}{dy}=\frac{du}{dy}$

Equation (2) gives,

$$-\frac{du}{dy}-yu=y^3 \ \ Or \ \ \frac{du}{dy}+yu=-y^3 \qquad ....(3)$$

This is linear equation in u and y with $P(y)=-y$ is identified.

The integrating factor is

$$v(x)=e^{\int P(y)dy}=e^{\int ydy}=e^{y^2/2}$$

Now, we multiply both sides of the equation (3) by the integrating factor viz., $v(x)$ and then integrate.

$$e^{y^2/2}\left(\frac{du}{dy}+yu\right)=-e^{y^2/2}y^3$$

$$Or\ \ \frac{d}{dy}\left(e^{y^2/2}u\right)=-e^{y^2/2}y^3, Integrating$$

$$u\,e^{y^2/2}\ =-\int e^{y^2/2}y^3dy+C$$

$$\therefore\qquad u\,e^{y^2/2}\ =-y^2e^{y^2/2}+2e^{y^2/2}+C$$

Substitute $\frac{1}{x}=u$, we get

$\frac{1}{x}e^{y^2/2}\ =-y^2e^{y^2/2}+2e^{y^2/2}+C\ \ Or\ \ \frac{1}{x}\ =-y^2+2+Ce^{-y^2/2}$ is the required general solution of the given differential equation (1).

Example 3.4.5: Solve the equation $\frac{dy}{dx}=(\sin x-\sin y)\frac{\cos x}{\cos y}$

Solution: Given equation $\frac{dy}{dx}=(\sin x-\sin y)\frac{\cos x}{\cos y}$ can be written as

$\cos y\frac{dy}{dx}=(\sin x\cos x-\sin y\cos x)$ Or $\cos y\frac{dy}{dx}+\sin y\cos x=\sin x\cos x$

....................(1) is Bernoulli's equation.

Put $\sin y=u\Rightarrow\cos y\frac{dy}{dx}=\frac{du}{dx}$

Equation (1) gives,

$$\frac{du}{dx}+\cos x\,u=\sin x\cos x \qquad ....(2)$$

This is linear equation in u and x with $P(x)=\cos x$ is identified.

The integrating factor is

$$v(x)=e^{\int P(x)dx}=e^{\int \cos x dx}=e^{\sin x}$$

Now, we multiply both sides of the equation (2) by the integrating factor viz., $v(x)$ and then integrate.

$$e^{\sin x}\left(\frac{du}{dx}-\cos x u\right)=e^{\sin x}(\sin x\cos x)$$

$$Or \quad \frac{d}{dx}\left(u e^{\sin x}\right)=e^{\sin x}(\sin x\cos x), \text{ integrating}$$

$$\therefore \quad u e^{\sin x}=\int e^{\sin x}(\sin x\cos x)dx+C$$

$$u e^{\sin x}=\int \sin x\cdot e^{\sin x}\cdot\cos x\,dx+C$$

$$Or\ u e^{\sin x}=e^{\sin x}(\sin x-1)+C$$

Substitute $\sin y=u$, we get

$$\sin y e^{\sin x}=e^{\sin x}(\sin x-1)+C\ Or\ \sin y=(\sin x-1)+Ce^{-\sin x}\ Or\ y=\sin^{-1}\left[\sin x-1+Ce^{-\sin x}\right]$$

is the required general solution of the given differential equation (1).

Example 3.4.6: Solve the equation

$$\frac{dy}{dx}+\frac{y}{x}=y^2x\sin x,\ x>0\ \textit{with initial condition}\ y(\pi)=1.$$

Solution: Given equation $\frac{dy}{dx}+\frac{y}{x}=y^2x\sin x$ ....(1) is

Bernoulli's equation.

Dividing equation (1) by y^2, we obtain $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=x\sin x$

.....................(2)

Put $\frac{1}{y}=u \Rightarrow \frac{-1}{y^2}\frac{dy}{dx}=\frac{du}{dx}$

Equation (2) gives,

$-\frac{du}{dx}+\frac{1}{x}u=x\sin x \; Or \; \frac{du}{dx}-\frac{1}{x}u=-x\sin x$ ....(3)

This is linear equation in u and x with $P(x)=\frac{-1}{x}$ is identified.

The integrating factor is

$$v(x)=e^{\int P(x)dx}=e^{-\int \frac{1}{x}dx}$$
$$=e^{-\log x}=\frac{1}{x}$$

Now, we multiply both sides of the equation (3) by the integrating factor viz., $v(x)$ and then integrate.

$$\frac{1}{x}\left(\frac{du}{dx}-\frac{1}{x}u\right)=-\sin x$$
$$Or\ \frac{d}{dx}\left(\frac{u}{x}\right)=-\sin x, Integrating$$
$$\frac{u}{x}=-\int \sin x\,dx+C$$
$$\frac{u}{x}=\cos x+C$$

Substitute $\frac{1}{y}=u$, we get

$\frac{1/y}{x}=\cos x+C$ Or $\frac{1}{xy}=\cos x+C$ ....(4) is the required general solution of the given differential equation (1).

Now, let's use the given initial conditions as; when $x=\pi$ and $y=1$ in the equation (4), we get

$\frac{1}{\pi}=\cos\pi+C \Rightarrow C=\frac{1}{\pi}+1$. Substituting the value of C in equation (4) gives the particular solution given by $\frac{1}{xy}=\cos x+\frac{1}{\pi}+1$.

3.4 Test Your Progress
Solve the following differential equations: i) $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$ ii) $x\frac{dy}{dx}+y=y^2\log x$ iii) $3\frac{dy}{dx}+\frac{2y}{1+x}=\frac{x^3}{y^2}$ iv) $\frac{dy}{dx}+\left(2x\tan^{-1}y-x^3\right)\left(1+y^2\right)=0$ v) $\frac{dy}{dx}-y\tan x=\frac{\sin x\cos^2 x}{y}$

vi) $2y\cos y^2\frac{dy}{dx}-\frac{2}{x+1}\sin y^2=(x+1)^3$

vii) $\frac{dy}{dx}=e^{x-y}\left(e^x-e^y\right)$

viii) $x\frac{dy}{dx}+y\log y=xye^x$

ix) $xy^2\frac{dy}{dx}-2y^3=2x^3.\ Given\ that\ \ y(1)=1$

$$x^2y-x^3\frac{dy}{dx}=y^4\cos x.\ Given\ that\ \ y(\pi)=\pi$$

3.5 Summary

In this unit, we studied the liner differential equation of first order and first degree and method of solving them. We have also seen the Bernoulli's differential equation i.e. equation which are reducible to linear from and its solution.

3.6 Terminal Questions

1. Solve the following differential equations

   A. $\frac{dy}{dx}=mx+ny+q$

B. $\frac{dy}{dx} = x^3y^2 - xy$

C. $sin.\frac{dy}{dx} + 3y = \cos x$

D. $(\cos x)dy = y(sin - y)dx.$

3.7 Answers to exercises

3.3 Test Your Progress

i) $y(x^2 - 1) = x + C$

ii) $y\frac{x}{x-1} = \frac{x^3}{3} + C$

iii) $yx^{\frac{1}{2}\log x} = e^x + C$

iv) $y(x^2 - 1) = x + C$

v) $ye^{x^2} = x + C$

vi) $yx^2 = \frac{x^4 \log x}{4} - \frac{x^4}{16} + C$

vii) $xe^y = \tan y + C$

viii) $y = \tan x + C\sqrt{\tan x}$

ix) $2y = x^2 \log x^2 + x^2 \cos\left(\frac{1}{x^2}\right) + Cx^2$

x) $2y \log x + \cos 2x = C$

xi) $y\sec^2 x = \sec x - 2$

xii) $y(1+x^2) = \tan^{-1} x - \dfrac{\pi}{4}$

3.4 Test Your Progress

i) $\dfrac{1}{x^5 y^5} = \dfrac{5}{2x^2} + C$

ii) $\dfrac{1}{xy} = \dfrac{-\log x}{x} - \dfrac{1}{x} + C$

iii) $y^3(1+x)^2 = \dfrac{x^6}{6} + \dfrac{2x^5}{5} + \dfrac{x^4}{4} + C$

iv) $\tan^{-1} y \cdot e^{x^2} = \dfrac{x^2 e^{x^2}}{2} - \dfrac{e^{x^2}}{2} + C$

v) $y^2 \cos^2 x = \dfrac{-2}{5} \cos^5 x + C$

vi) $\dfrac{\sin y^2}{(x+1)^2} = \dfrac{1}{2}(x+1)^2 + C$

vii) $e^y e^{e^x} = e^{e^x}(e^x - 1) + C$

viii) $x \log y = xe^x - e^x + C$

ix) $\dfrac{y^3}{x^6} = \dfrac{-2}{x^3} + 3$

x) $x^3 - y^3 = 3y^3 \sin x$

Unit –04: Exact differential equations

Structure

4.1. Introduction

The ordinary differential equations first order and first degree has different forms according to their nature, namely, exact differential equations and non-exact differential equations. Whereas non-exact equations are difficult in general, exact equations are much simpler because the solutions of exact differential equations are solved by using direct testing of necessary condition of exactness and formula for general solution, and there are standard methods (i.e., integrating factors) for solving many particularly important non exact differential equations.

In the previous unit, we had learnt how to solve variable separable form, homogeneous, non homogeneous and linear and non linear differential equations of first order and first degree. In this unit we will learn the techniques for solving analytically some special forms of ODE's namely exact and non exact, which are useful in various applications.

4.2. Objectives

After reading this unit students should be able to:

- Solve the exact differential equations of first order and first degree

- Find the integrating factors in different forms
- Identify the suitable integrating factor to solve the given non-exact first order and first degree ODE

4.3. Exact differential equations of first order and first degree

Definition 4.3.1: A differential equation $M(x,y)dx + N(x,y)dy = 0$. Where M and N are functions of x, y is called an exact differential equation if there exists a function $f(x, y)$ having continuous first order partial derivatives such that $M = \frac{\partial f}{\partial x}$ and $N == \frac{\partial f}{\partial y}$ for such a function f, we write $Mdx + Ndy = df$ where df stands for $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

Equivalently,

A differential expression $M(x,y)dx + N(x,y)dy$ is an exact differential in a region R of the $xy - plane$ if it corresponds to the differential of some function $f(x, y)$ defined on R.

A first order differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation if the expression on the left hand side is an exact differential.

The criterion for an exact differential:

Let $M(x,\ y) and\ N(x,\ y)$ continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b,\ c < y < d$. Then a necessary and sufficient condition for $M(x, y)dx + N(x, y)dy$ to be exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Theorem 4.3.1: The Necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: By the definition of an exact differential equation, there exists a function $f(x,\ y)$ having continuous partial derivatives such that $M = \frac{\partial f}{\partial x}$ ....(1) and $N == \frac{\partial f}{\partial y}$ ....(2)

Differentiate equation (1) partially with respect to y and equation (2) partially with respect to x, we obtain, $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \cdot \partial y}$.

For the functions having continuous first order partial derivatives it is true that $\frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial^2 f}{\partial x \cdot \partial y}$

$$\therefore\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now, let us suppose that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

To prove: $Mdx + Ndy = 0$ is an exact differential equation.

Let $F(x, y) = \int_y M\,dx$ [where $\int_y M\,dx$ means while integrating keep 'y' constant].

Consider $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \cdot \partial x} = \frac{\partial^2 F}{\partial x \cdot \partial y}$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial^2 F}{\partial x \cdot \partial y} = 0$$

$$Or\ \frac{\partial}{\partial x}\left(N - \frac{\partial F}{\partial y}\right) = 0$$

As $N - \frac{\partial F}{\partial y} = \phi(y)$ (say) as it does not contain terms of x

$$\therefore\ N = \frac{\partial F}{\partial y} + \phi(y)$$

Now,

$$\begin{aligned} M\,dx + N\,dy &= \frac{\partial F}{\partial x}dx + \left(\frac{\partial F}{\partial y} + \phi(y)\right)dy \\ &= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \phi(y)dy \\ &= dF + \phi(y)dy = d\left(F + \psi(y)\right)\ where\ d[\psi(y)] = \phi(y)dy \end{aligned}$$

Therefore, $Mdx + Ndy = 0$ is an exact differential equation.

Thus, the general solution of it is given by,

$F(x, y) + \int \phi(y)dy = c$ where $F(x, y) = \int_{Keeping\ y\ constant} M\,dx$

$\phi(y) = N - \dfrac{\partial F}{\partial y}$, where $\phi(y)$ is independent of x (i.e., free from x).

Hence, the general solution of an exact differential equation is given by

$$\int\limits_{Keeping\, y\, constant} M\, dx + \int \phi(y) dy = C$$

Equivalently, $$\int\limits_{Keeping\, y\, constant} M\, dx + \int \left(Terms\ of\ \ N\ \ free\ \ from\, x\right) dy = C$$

Note 4.3.1: The general solution of an exact differential equation can also be given by

$$\int\limits_{Keeping\, x\, constant} N\, dy + \int \phi(x) dx = C\ .$$

Equivalently, $$\int\limits_{Keeping\, x\, constant} N\, dy + \int \left(Terms\ of\ \ M\ \ free\ \ from\, y\right) dx = C$$

Example 4.3.1: Solve the differential equation

$(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution: Given equation is $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$$Here, \quad M = (x^2 - 4xy - 2y^2) \qquad and \qquad N = y^2 - 4xy - 2x^2$$

$$\therefore \frac{\partial M}{\partial y} = -4x - 4y \qquad\qquad \therefore \frac{\partial N}{\partial x} = -4y - 4x$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ Given equation is an exact differential equation.

$\therefore$ The general solution of it is given by

$$\int_{Keeping\, y\, constant} M\,dx + \int \left(Terms\ of\ \ N\ \ free\ \ from\, x\right) dy = C$$

$$\Rightarrow \quad \int_{Keeping\, y\, constant} (x^2 - 4xy - 2y^2)\,dx + \int y^2\,dy = C$$

$$\therefore \quad \frac{x^3}{3} - 4y\left(\frac{x^2}{2}\right) - 2y^2 x + \left(\frac{y^3}{3}\right) = C$$

$$Or \quad \frac{x^3}{3} - 2x^2 y - 2y^2 x + \left(\frac{y^3}{3}\right) = C$$

Example 4.3.2: Solve the differential equation

$$(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$$

Solution: Given equation is $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

$$Here, \quad M = y^2 e^{xy^2} + 4x^3 \qquad and \qquad N = 2xye^{xy^2} - 3y^2$$

$$\therefore \frac{\partial M}{\partial y} = y^2 e^{xy^2}(2xy) + 2ye^{xy^2} \qquad \therefore \frac{\partial N}{\partial x} = 2xye^{xy^2}(y^2) + 2ye^{xy^2}$$

$$= 2ye^{xy^2}(xy^2 + 1) \qquad\qquad = 2ye^{xy^2}(xy^2 + 1)$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ Given equation is an exact differential equation.

$\therefore$ The general solution of it is given by

$$\int_{Keeping\, y\, constant} M\,dx + \int \left(Terms\ of\ \ N\ \ free\ \ from\, x\right) dy = C$$

$$\Rightarrow \quad \int_{Keeping\, y\, constant} (y^2 e^{xy^2} + 4x^3)\,dx - 3\int y^2\,dy = C$$

$$\therefore \quad y^2\left(\frac{e^{xy^2}}{y^2}\right) + 4\left(\frac{x^4}{4}\right) - 3\left(\frac{y^3}{3}\right) = C$$

$$Or \quad e^{xy^2} + x^4 - y^3 = C$$

Example 4.3.3: Solve the differential equation

$(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

Solution: Given equation is $(2xy + y - \tan y)dx + (x^2 + x - x\sec^2 y)dy = 0$

$$Here, \quad M = 2xy + y - \tan y \qquad and \qquad N = x^2 + x - x\sec^2 y$$

$$\therefore \frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y \qquad \therefore \frac{\partial N}{\partial x} = 2x + 1 - \sec^2 y$$

$$= 2ye^{xy^2}(xy^2 + 1) \qquad = 2ye^{xy^2}(xy^2 + 1)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Given equation is an exact differential equation.

$\therefore$ The general solution of it is given by

$$\int_{Keeping\, y\, constant} M\, dx + \int \left(Terms\ of\ N\ free\ from\, x\right) dy = C$$

$$\Rightarrow \int_{Keeping\, y\, constant} (2xy + y - \tan y)\, dx = C$$

$$\therefore\ 2y\left(\frac{x^2}{2}\right) + xy - x\tan y = C$$

$$Or \quad x^2 y + xy - x\tan y = C$$

4.3 Test Your Progress
Solve the following differential equations
i) $x(1 + y^2)dx + y(1 + x^2)dy = 0$
ii) $(x^2 - ay)dx = (ax - y^2)dy$
iii) $y\sin 2x dx - (y^2 + \cos^2 x)dy = 0$

iv) $(1+e^{x/y})dx+e^{x/y}(1-\frac{x}{y})dy=0$

v) $(1+e^{y})\cos xdx+e^{y}\sin xdy=0$

vi) $(2x^2+6xy-y^2)dx+(3x^2-2xy+y^2)dy=0$

vii) $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

viii) $(xe^{xy}+2y)dx+ye^{xy}dy=0$

ix) $xdx+ydy=\frac{xdy-ydx}{x^2+y^2}$

x) $(2xy+y-\tan y)dx+(x^2-x\tan^2 y+\sec^2 y)dy=0$

4.4. Non exact differential equations and Integrating factors

It is sometimes possible that even though the original first order differential equation $M(x,y)dx+N(x,y)dy=0$ is not exact, but we can multiply both sides of this differential equation by some function [say, $f(x,y)$] so that the resulting differential equation $f(x,y)M(x,y)dx+f(x,y)N(x,y)dy=0$ becomes exact. Such a function (or factor) $f(x,y)$ is known as an integrating factor for the original differential equation $M(x,y)dx+N(x,y)dy=0$.

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an integrating factor.

Remark: It is possible that we lose or gain solutions while multiplying an ordinary differential equation by an integrating factor.

For Example: consider the first order differential equation $ydx - xdy = 0$, which is clearly non-exact. But observe that if we multiply both sides of this differential equation by the factor $\frac{1}{y^2}$, the resulting ODE becomes $\frac{dx}{y} - \frac{x}{y^2}dy = 0$ which is exact.

The rules for finding integrating factors of the equation $M(x, y)dx + N(x, y)dy = 0$ are as follows.

4.4.1 Integrating factors found by Inspection:

In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful.

i) $xdy + ydx = d(xy)$

ii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

iii) $$\frac{xdy - ydx}{xy} = d\left(\log\left[\frac{y}{x}\right]\right)$$

iv) $$\frac{ydx - xdy}{y^2} = -d\left(\frac{x}{y}\right)$$

v) $$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

vi) $$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)$$

Example 4.4.1 (i): Solve the differential equation

$(ye^x dx - e^x dy) + 2xy^2 dx = 0$.

Solution: We have $(ye^x dx - e^x dy) + 2xy^2 dx = 0$. Here, we can observe that the terms ye^x and $e^x dy$ should be put together.

$\therefore \frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0$ Or $d\left(\frac{e^e}{y}\right) + 2xdx = 0$. Integrating, we obtain

$\frac{e^x}{y} + x^2 = C$ is the required general solution.

Example 4.4.1 (ii): Solve the differential equation

$xdy - ydx + a(x^2 + y^2)dx = 0$.

Solution: We have $xdy - ydx + a(x^2 + y^2)dx = 0$. Here, we can observe that the terms $xdy - ydx$ and $(x^2 + y^2)$ should be put together.

$\therefore\ \dfrac{xdy - ydx}{x^2 + y^2} + adx = 0$ Or $d\left(\tan^{-1}\dfrac{y}{x}\right) + adx = 0$ Integrating, we obtain

$\left(\tan^{-1}\dfrac{y}{x}\right) + ax = C$ is the required general solution.

Example 4.4.1 (iii): Solve the differential equation

$xdy + ydx = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$.

Solution: We have $xdy + ydx = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$. Here, we can observe that the all terms are already combined together.

$\therefore\ d(xy) = a^2 d\left(\tan^{-1}\dfrac{y}{x}\right)$. Integrating, we obtain

$xy = a^2 \tan^{-1}\left(\dfrac{y}{x}\right) + C$ is the required general solution.

4.4.2 Integrating factor of a Homogeneous Equation:

If $M(x, y)dx + N(x, y)dy = 0$ be a homogeneous equation in x and y, then $\dfrac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0$.

Example 4.4.2(i): Solve the differential equation $x + y\dfrac{dy}{dx} = y - x\dfrac{dy}{dx}$.

Solution: Given equation is $x + y\dfrac{dy}{dx} = y - x\dfrac{dy}{dx}$ ....(1)

$$Or \quad x\frac{dy}{dx} + y\frac{dy}{dx} = y - x$$

$$Or \quad (x+y)\frac{dy}{dx} = y - x$$

$$Or \quad (y-x)dx - (x+y)dy = 0 \quad ....(2)$$

Here, $M = y - x \;\; and \;\; N = -(x+y)$

$$\therefore \frac{\partial M}{\partial y} = 1 \;\; and \;\; \frac{\partial N}{\partial x} = -1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is homogeneous differential equation.

Now, Consider $Mx + Ny = x^2 - xy + xy + y^2 = x^2 + y^2 \neq 0$

$$\therefore Integrating \;\; factor = \frac{1}{Mx + Ny} = \frac{1}{x^2 + y^2}$$

Multiplying equation (1) by the integrating factor $\frac{1}{x^2+y^2}$ we get

$\frac{y-x}{x^2+y^2}dx - \frac{x+y}{x^2+y^2}dy = 0$ ....(3) is exact, because as

$$M_1 = \frac{y-x}{x^2+y^2} \;\; and \;\; N_1 = -\frac{x+y}{x^2+y^2}$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{(x^2+y^2)(1) - (y-x)(2y)}{(x^2+y^2)^2} = \frac{x^2 + 2xy - y^2}{(x^2+y^2)^2}$ and

$$\frac{\partial N_1}{\partial x} = \frac{(x^2+y^2)(-1) + (x+y)(2y)}{(x^2+y^2)^2} = \frac{x^2 + 2xy - y^2}{(x^2+y^2)^2}$$

$$\therefore \ \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Hence, the general solution of the equation (3) is given by,

$$\int_{y\,constant} M_1 dx + \int (Terms\ \ of\ \ N_1\ \ free\ \ from\ x)\ dy = C$$

$$\therefore \ \int_{y\,constant} \left(\frac{y-x}{x^2+y^2}\right) dx = C$$

$$Or\ \ y \int_{y\,constant} \left(\frac{1}{x^2+y^2}\right) dx - \int \frac{x}{x^2+y^2} dx = C$$

$$\therefore \ \ y \cdot \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) - \frac{1}{2}\log\left|x^2+y^2\right| = C$$

$$Or\ \tan^{-1}\left(\frac{x}{y}\right) - \frac{1}{2}\log\left|x^2+y^2\right| = C$$

This is the required general solution of the given equation (1).

Example 4.4.2(ii): Solve the differential equation $xydx-(x^2+2y^2)dy=0$.

Solution: Given equation is $xydx-(x^2+2y^2)dy=0$ ....(1)

Here, $M = xy\ \ and\ \ N = -(x^2+2y^2)$

$$\therefore \frac{\partial M}{\partial y} = x\ \ and\ \ \frac{\partial N}{\partial x} = -2x$$

$$\therefore \ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is homogeneous differential equation.

Now, Consider $Mx + Ny = x^2y - x^2y - 2y^3 = -2y^3 \neq 0$

$$\therefore \textit{Integrating factor} = \frac{1}{Mx+Ny} = -\frac{1}{2y^3}$$

Multiplying equation (1) by the integrating factor $\frac{-1}{2y^3}$ we get

$-\frac{1}{2y^3}(xydx) + \frac{1}{2y^3}(x^2+2y^2)dy = 0$ ....(3) is exact, because as

$$M_1 = -\frac{xy}{2y^3} \quad and \quad N_1 = -\frac{x^2+2y^2}{2y^3}$$

$$\therefore \frac{\partial M_1}{\partial y} = \frac{(2y^3)(-x) + xy(6y^2)}{(2y^3)^2} = \frac{-2xy^3 + 6xy^3}{4y^6} = \frac{x}{y^3}$$

and

$$\frac{\partial N_1}{\partial x} = \frac{(2y^3)(2x) - (x^2+2y^2)(0)}{(2y^3)^2} = \frac{4xy^3}{4y^6} = \frac{x}{y^3}$$

$$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Hence, the general solution of the equation (3) is given by,

$$\int_{y\,constant} M_1 dx + \int (Terms\ of\ N_1\ free\ from\ x)\, dy = C$$

$$\therefore \int_{y\,constant} \left(\frac{-xy}{2y^3}\right) dx + \int \frac{1}{y} dy = C$$

$$Or\ -\frac{1}{2y^2} \int_{y\,constant} x\,dx + \int \frac{1}{y} dy = C$$

$$\therefore \quad -\frac{1}{2y^2}\frac{x^2}{2}+\log y = C$$

$$Or \ \log y - \frac{x^2}{4y^2} = C$$

This is the required general solution of the given equation (1).

Example 4.4.2(iii): Solve the differential equation

$(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$.

Solution: Given equation is $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

.............................(1)

Here, $M = x^2y-2xy^2 \ \ and \ \ N = -(x^3-3x^2y)$

$$\therefore \frac{\partial M}{\partial y} = x^2-4xy \ \ and \ \ \frac{\partial N}{\partial x} = -3x^2+6xy$$

$$\therefore \ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is homogeneous differential equation.

Now, Consider $Mx+Ny = x^3y-2x^2y^2-x^3y+3x^2y^2 = x^2y^2 \neq 0$

$$\therefore Integrating \ \ factor = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiplying equation (1) by the integrating factor $\frac{1}{x^2y^2}$ we get

$$\frac{1}{x^2y^2}(x^2y-2xy^2)dx-\frac{1}{x^2y^2}(x^3-3x^2y)dy=0$$ ....(3) is exact, because

as $M_1=\frac{1}{y}-\frac{2}{x}$ *and* $N_1=-\frac{x}{y^2}+\frac{3}{y}$

$\therefore \frac{\partial M_1}{\partial y}=-\frac{1}{y^2}$ and

$$\frac{\partial N_1}{\partial x}=-\frac{1}{y^2}$$

$$\therefore \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

Hence, the general solution of the equation (3) is given by,

$$\int_{y\,constant} M_1 dx+\int (Terms\ of\ N_1\ free\ from\ x)\ dy=C$$

$$\therefore \int_{y\,constant}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=C$$

$$Or\ \frac{x}{y}-2\log x+3\log y=C$$

This is the required general solution of the given equation (1).

Note: If the given differential equation $M(x,y)dx+N(x,y)dy=0$ is homogeneous and exact then solution of differential equation is factor of $Mx+Ny=C$.

4.4.3 Integrating factor for an Equation of the type $f_1(x,y)ydx + f_2(x,y)xdy = 0$:

If the equation $M(x,y)dx + N(x,y)dy = 0$ be of this form, then $\frac{1}{Mx - Ny}$ is an integrating factor (Provided $M\,dx - N\,dy \neq 0$).

Example 4.4.3(i): Solve the differential equation

$(1+xy)ydx + (1-xy)xdy = 0$.

Solution: Given equation is $(1+xy)ydx + (1-xy)xdy = 0$ ....(1)

Here, $M = y + xy^2 \text{ and } N = x - x^2 y$

$$\therefore \frac{\partial M}{\partial y} = 1 + 2xy \text{ and } \frac{\partial N}{\partial x} = 1 - 2xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is of the form $f_1(x,y)ydx + f_2(x,y)xdy = 0$.

Now, Consider $Mx - Ny = yx + x^2y^2 - xy + x^2y^2 = 2x^2y^2 \neq 0$

$$\therefore Integrating\ \ factor = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

Multiplying equation (1) by the integrating factor $\frac{1}{2x^2y^2}$ we get

$$\frac{1}{2x^2y^2}(y+xy^2)dx+\frac{1}{2x^2y^2}(x-x^2y)dy=0$$ ....(3) is exact, because as

$$M_1=\frac{1}{2x^2y}+\frac{1}{2x} \quad and \quad N_1=\frac{1}{2xy^2}-\frac{1}{2y}$$

$$\therefore \frac{\partial M_1}{\partial y}=-\frac{1}{2x^2y^2} \text{ and } \frac{\partial N_1}{\partial x}=-\frac{1}{2x^2y^2}$$

$$\therefore \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

Hence, the general solution of the equation (3) is given by,

$$\int_{y\,constant} M_1dx+\int(Terms\ of\ N_1\ free\ from\ x)\,dy=C$$

$$\therefore \int_{y\,constant}\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx-\int\frac{1}{2y}dy=C$$

$$\therefore \left(\frac{1}{2y}\times-\frac{1}{x}\right)+\frac{1}{2}\log x-\frac{1}{2}\log y=C$$

$$Or\quad \frac{1}{2}\left(\frac{-1}{xy}+\log x-\log y\right)=C$$

$$Or\quad -\frac{1}{xy}+\log\frac{x}{y}=C_1$$

$$Or\quad \log\frac{x}{y}-\frac{1}{xy}=C_1$$

This is the required general solution of the given equation (1).

Example 4.4.3(ii): Solve the differential equation

$(x^3y^2+x)dy+(x^2y^3-y)dx=0$.

Solution: Given equation is

$(x^3y^2+x)dy+(x^2y^3-y)dx=0 \; Or \; (x^2y^2-1)ydx+(x^2y^2+1)xdy=0$
............................(1)

Here, $M=x^2y^3-y \; and \; N=x^3y^2+x$

$$\therefore \frac{\partial M}{\partial y}=3x^2y^2-1 \; and \; \frac{\partial N}{\partial x}=3x^2y^2+1$$

$$\therefore \; \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is of the form $f_1(x,y)ydx+f_2(x,y)xdy=0$.

Now, Consider $Mx-Ny=x^3y^3-xy-x^3y^3-xy=-2xy\neq 0$

$$\therefore Integrating \; factor=\frac{1}{Mx-Ny}=-\frac{1}{2xy}$$

Multiplying equation (1) by the integrating factor $-\frac{1}{2xy}$ we get

$\frac{-1}{2xy}(x^2y^3-y)dx-\frac{1}{2xy}(x^3y^2+x)dy=0$ ....(3) is exact, because as

$$M_1=\frac{-xy^2}{2}+\frac{1}{2x} \; and \; N_1=\frac{-x^2y}{2}-\frac{1}{2y}$$

$\therefore \frac{\partial M_1}{\partial y}=-xy$ and $\frac{\partial N_1}{\partial x}=-xy$

$$\therefore \; \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

Hence, the general solution of the equation (3) is given by,

$$\int\limits_{y\,constant} M_1 dx + \int (Terms\ of\ N_1\ free\ from\ x)\, dy = C$$

$$\therefore \quad \int\limits_{y\,constant} \left(\frac{-xy^2}{2} + \frac{1}{2x} \right) dx - \int \frac{1}{2y} dy = C$$

$$\therefore \quad \left(-\frac{y^2}{2} \times \frac{x^2}{2} \right) + \frac{1}{2}\log x - \frac{1}{2}\log y = C$$

$$Or \quad \left(\frac{-x^2y^2}{4} + \frac{1}{2}\log x - \frac{1}{2}\log y \right) = C$$

$$Or \quad \frac{1}{2}\left(-\frac{x^2y^2}{2} + \log\frac{x}{y} \right) = C$$

$$Or \quad \left(\log\frac{x}{y} - \frac{x^2y^2}{2} \right) = C_1$$

This is the required general solution of the given equation (1).

Example 4.4.3(iii): Solve the differential equation

$(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$.

Solution: Given equation is $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$

.............................(1)

Here, $M = x^2y^3 + xy^2 + y$ *and* $N = x^3y^2 - x^2y + x$

$$\therefore \frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \quad and \quad \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation is non-exact differential equation.

But equation (1) is of the form $f_1(x, y)ydx + f_2(x, y)xdy = 0$.

Now, Consider $Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy = 2x^2y^2 \neq 0$

$\therefore Integrating\ \ factor = \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2y^2}$

Multiplying equation (1) by the integrating factor $\dfrac{1}{2x^2y^2}$ we get

$M = x^2y^3 + xy^2 + y \ \ and \ \ N = x^3y^2 - x^2y + x$

$\dfrac{1}{2x^2y^2}(x^2y^3 + xy^2 + y)dx + \dfrac{1}{2x^2y^2}(x^3y^2 - x^2y + x)dy = 0$ ....(3) is exact,

because as $M_1 = \dfrac{y}{2} + \dfrac{1}{2x} + \dfrac{1}{2x^2y}$ and $N_1 = \dfrac{x}{2} - \dfrac{1}{2y} + \dfrac{1}{2xy^2}$

$\therefore \dfrac{\partial M_1}{\partial y} = \dfrac{1}{2} - \dfrac{1}{2x^2y^2}$ and $\therefore \dfrac{\partial N_1}{\partial y} = \dfrac{1}{2} - \dfrac{1}{2x^2y^2}$

$\therefore \ \dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$

Hence, the general solution of the equation (3) is given by,

$$\int_{y\,constant} M_1 dx + \int (Terms\ \ of\ \ N_1\ \ free\ \ from\ x)\ dy = C$$

$$\therefore \quad \int_{y\,constant} \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right) dx - \int \frac{1}{2y} dy = C$$

$$\therefore \quad \left(\frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy} \right) - \frac{1}{2}\log y = C$$

$$Or \quad \frac{1}{2}\left(xy + \log x - \frac{1}{xy} - \log y \right) = C$$

This is the required general solution of the given equation (1).

4.4.4 (a) If the differential equation $M(x,y)dx+N(x,y)dy=0$ is non-exact and there exists a continuous single variable function $f(x)$ such that $f(x)=\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ then $e^{\int f(x)dx}$ is an integrating factor of $M(x,y)dx+N(x,y)dy=0$.

Equivalently,

In the equation $M(x,y)dx+N(x,y)dy=0$**,**

if $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$ **(say)[i.e., function of x only], then** $e^{\int f(x)dx}$ **is an integrating factor.**

Example 4.4.4(a) (i) Solve the differential equation

$(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0$

Solution: Given equation is $(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0$......................(1)

$Here,\quad M=xy^2-e^{\frac{1}{x^3}}\quad and \quad N=-x^2y$

$\therefore\ \dfrac{\partial M}{\partial y}=2xy \quad and \quad \dfrac{\partial N}{\partial x}=-2xy$

$\therefore\ \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2xy+2xy}{-x^2y}=\frac{4xy}{-x^2y}=-\frac{4}{x}=f(x).\ \left(i.e.,\ function\ of\ only\ x\right)$$

Therefore, $Integrating\ Factor\,(I\,F)=e^{\int f(x)dx}=e^{-4\int\frac{1}{x}dx}=e^{-4\log x}=\frac{1}{x^4}$

Multiplying the given differential equation by the Integrating Factor (I F)

$=\frac{1}{x^4}$, we get

$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})dx-\frac{1}{x^4}(x^2y)dy=0\ \ Or\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx-\left(\frac{y}{x^2}\right)dy=0$ is an exact differential equation. Because as $M_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}$ and $N_1=-\frac{y}{x^2}$ gives

$$\frac{\partial M_1}{\partial y}=\frac{2y}{x^3}\ \ and\ \ \frac{\partial N_1}{\partial x}=-y\left(\frac{-2}{x^3}\right)=\frac{2y}{x^3}\Rightarrow\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int\limits_{y\,constant}M_1\,dx+\int(Terms\ of\ N_1\ free\ from\ x)\ dy=C$$

$$\therefore\ \int\limits_{y\,constant}\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)dx=C$$

$$\therefore\ \left(-\frac{y^2}{2x^2}+\frac{e^{\frac{1}{x^3}}}{3}\right)=C$$

$$Or\ \left(\frac{e^{\frac{1}{x^3}}}{3}-\frac{y^2}{2x^2}\right)=C$$

This is required solution of given differential equation (1).

Example 4.4.4(a) (ii) Solve the differential equation

$$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}\left(x+xy^2\right)dy=0$$

Solution: Given equation is

$$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}\left(x+xy^2\right)dy=0 \text{ ....(1)}$$

$$Here,\quad M=y+\frac{y^3}{3}+\frac{x^2}{2}\quad and\quad N=\frac{1}{4}\left(x+xy^2\right)$$

$$\therefore\ \frac{\partial M}{\partial y}=1+y^2\quad and\quad \frac{\partial N}{\partial x}=\frac{1}{4}\left(1+y^2\right)$$

$$\therefore\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{1+y^2-\dfrac{1}{4}(1+y^2)}{\dfrac{1}{4}(x+xy^2)}=\frac{\dfrac{3}{4}(1+y^2)}{\dfrac{x}{4}(1+y^2)}=\frac{3}{x}=f(x).\ \left(i.e.,\ function\ of\ only\ x\right)$$

Therefore, $Integrating\ Factor\,(I\,F)=e^{\int f(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\log x}=x^3$

Multiplying the given differential equation by the Integrating Factor (I F) $=x^3$, we get

$$x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}\times x^3\left(x+xy^2\right)dy=0$$ is an exact differential equation.

Because as $M_1=x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}$ and $N_1=\frac{x^4}{4}+\frac{x^4y^2}{4}$ gives

$$\frac{\partial M_1}{\partial y}=x^3+x^3y^2 \; and \quad \frac{\partial N_1}{\partial x}=x^3+x^3y^2 \;\Rightarrow \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int_{y\,constant} M_1\,dx+\int (Terms\;\;of\;\;N_1\;\;free\;\;from\;x)\;dy=C$$

$$\therefore \int_{y\,constant}\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx=C$$

$$\therefore \quad \left(\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}\right)=C$$

$$Or\;\; 3x^4y+x^4y^3+x^6=C_1$$

$$Or\;\; x^4(3y+y^3+x^2)=C_1$$

This is required solution of thegiven differential equation (1).

Example 4.4.4(a) (iii): Solve the differential equation

$$\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$$

Solution: Given equation is

$$\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0 \text{ ....(1)}$$

$$Here,\quad M=3xy-2ay^2\;\; and \quad N=x^2-2axy$$

$$\therefore \;\frac{\partial M}{\partial y}=3x-4ay \;\; and \quad \frac{\partial N}{\partial x}=2x-2ay$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x - 4ay - 2x + 2ay}{x^2 - 2axy} = \frac{x - 2ay}{x(x - 2ay)} = \frac{1}{x} = f(x). \left(i.e., \ function \ of \ only \ x\right)$$

Therefore, $Integrating \ Factor\,(I\,F) = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Multiplying the given differential equation by the Integrating Factor (I F) $= x$, we get

$x\left(3xy - 2ay^2\right)dx + x\left(x^2 - 2axy\right)dy = 0$ is an exact differential equation.

Because as $M_1 = 3x^2 y - 2axy^2$ and $N_1 = x^3 - 2ax^2 y$ gives

$$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy \; and \quad \frac{\partial N_1}{\partial x} = 3x^2 - 4axy \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int_{y\,constant} M_1\,dx + \int (Terms \ of \ N_1 \ free \ from \ x)\ dy = C$$

$$\therefore \quad \int_{y\,constant} \left(3x^2 y - 2axy^2\right)dx = C$$

$$\therefore \quad \left(\frac{3x^3 y}{3} + \frac{2ax^2 y^2}{2}\right) = C$$

$$Or \quad 6x^3 y + 6ax^2 y^2 = C_1$$

$$Or \quad x^2 y(x + ay) = C_1$$

This is required solution of the given differential equation (1).

Example 4.4.4(a) (iv): Solve the differential equation

$(x^2 + y^2 + 2x)dx + (2y)dy = 0$

Solution: Given equation is $(x^2 + y^2 + 2x)dx + (2y)dy = 0$...............(1)

$Here, \quad M = x^2 + y^2 + 2x \; and \quad N = 2y$

$$\therefore \frac{\partial M}{\partial y} = 2y \; and \quad \frac{\partial N}{\partial x} = 0$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = 1$$

Therefore, $Integrating\ Factor (I\,F) = e^{\int f(x)dx} = e^{\int 1dx} = e^x$

Multiplying the given differential equation by the Integrating Factor (I F)

$= e^x$, we get

$e^x(x^2 + y^2 + 2x)dx + e^x(2y)dy = 0$ is an exact differential equation. Because

as $M_1 = e^x(x^2 + y^2 + 2x)$ and $N_1 = 2ye^x$ gives

$$\frac{\partial M_1}{\partial y} = 2ye^x \; and \quad \frac{\partial N_1}{\partial x} = 2ye^x \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int_{y\,constant} M_1\, dx + \int (Terms\;\; of\;\; N_1\;\; free\;\; from\;\; x)\; dy = C$$

$$\therefore \quad \int_{y-constant} \left(x^2 + y^2 + 2x\right) e^x dx = C$$

$$\therefore \quad \int_{y-constant} x^2 e^x dx + \int_{y-constant} y^2 e^x dx + 2 \int_{y-constant} x e^x dx = C$$

$$Or \quad x^2 e^x - 2 \int_{y-constant} x e^x dx + y^2 e^x + 2(xe^x - e^x) = C_1$$

$$Or \quad x^2 e^x - 2(xe^x - e^x) + y^2 e^x + 2(xe^x - e^x) = C_1$$

$$Or \quad x^2 e^x + y^2 e^x = C_1 \quad Or \quad e^x (x^2 + y^2) = C_1$$

This is required solution of the given differential equation (1).

4.4.4 (b) If the differential equation $M(x, y)dx + N(x, y)dy = 0$ is non-exact and there exists a continuous single variable function $f(y)$ such that $f(y) = \dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{N}$ then $e^{\int f(y)dy}$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$.

Equivalently,

In the equation $M(x, y)dx + N(x, y)dy = 0$**,**

If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ **(say) [i.e., function of y only], then** $e^{\int f(y)dy}$ **is an integrating factor.**

Example 4.4.4(b) (i) Solve the differential equation

$$(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$$

Solution: Given equation is

$$(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0 \text{ ....(1)}$$

$$Here, \quad M=3x^2y^4+2xy \quad and \quad N=2x^3y^3-x^2$$

$$\therefore \ \frac{\partial M}{\partial y}=12x^2y^3+2x \quad and \quad \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\therefore \ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{6x^2y^3-2x-12x^2y^3-2x}{xy(3xy^3+2)}$$

$$=\frac{-6x^2y^3-4x}{xy(3xy^3+2)}=-\frac{2x(3xy^3+2)}{xy(3xy^3+2)}=\frac{-2}{y}=f(y).\ \left(i.e.,\ function\ of\ only\ y\right)$$

Therefore, $Integrating\ Factor(I\,F)=e^{\int f(y)dy}=e^{-\int\frac{2}{y}dy}=e^{-2\log y}=\dfrac{1}{y^2}$

Multiplying the given differential equation by the Integrating Factor (I F)

$=\dfrac{1}{y^2}$, we get

$$\frac{1}{y^2}(3x^2y^4+2xy)dx+\frac{1}{y^2}(2x^3y^3-x^2)dy=0 \;\; Or \;\; (3x^2y^2+\frac{2x}{y})dx+(2x^3y-\frac{x^2}{y^2})dy=0$$

is an exact differential equation. Because as $M_1=3x^2y^2+\frac{2x}{y}$ and

$N_1=2x^3y-\frac{x^2}{y^2}$ gives $\frac{\partial M_1}{\partial y}=6x^2y-\frac{2x}{y^2}$ *and* $\frac{\partial N_1}{\partial x}=6x^2y-\frac{2x}{y^2} \;\; \Rightarrow \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

The general solution is given by

$$\int_{y\,constant} M_1\,dx+\int (Terms\;\; of\;\; N_1\;\; free\;\; from\;\; x)\;dy=C$$

$$\therefore \int_{y\,constant}\left(3x^2y^2+\frac{2x}{y}\right)dx=C$$

$$\therefore \quad \left(\frac{3x^3y^2}{3}+\frac{2x^2}{2y}\right)=C$$

$$Or \;\; \left(x^3y^2+\frac{x^2}{y}\right)=C$$

This is required solution of given differential equation (1).

Example 4.4.4(b) (ii) Solve the differential equation

$$(2x^2y+e^x)ydx-(e^x+y^3)dy=0$$

Solution: Given equation is

$$(2x^2y+e^x)ydx-(e^x+y^3)dy=0 \;\text{..............................}(1)$$

$$Here, \;\; M=2x^2y^2+e^xy \;\; and \;\; N=-(e^x+y^3)$$

$$\therefore \frac{\partial M}{\partial y}=4x^2y+e^x \;\; and \;\; \frac{\partial N}{\partial x}=-e^x$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-e^x - 4x^2 y - e^x}{2x^2 y^2 + e^x y}$$

$$= \frac{-2e^x - 4x^2 y}{2x^2 y^2 + e^x y} = -\frac{-2(e^x + 2x^2 y)}{y(e^x + 2x^2 y)} = \frac{-2}{y} = f(y). \left(i.e., \ function \ of \ only \ y\right)$$

Therefore, $Integrating \ Factor (I\ F) = e^{\int f(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$

Multiplying the given differential equation by the Integrating Factor (I F)

$= \frac{1}{y^2}$, we get

$\frac{1}{y^2}[(2x^2 y^2 + e^x y)]dx - \frac{1}{y^2}(e^x + y^3)dy = 0 \ Or \ (2x^2 + \frac{e^x}{y})dx - (\frac{e^x}{y^2} + y)dy = 0$ is an

exact differential equation. Because as $M_1 = (2x^2 + \frac{e^x}{y})$ and $N_1 = -(\frac{e^x}{y^2} + y)$

gives $\frac{\partial M_1}{\partial y} = -\frac{e^x}{y^2} \ \ and \ \ \frac{\partial N_1}{\partial x} = -\frac{e^x}{y^2} \ \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

The general solution is given by

$$\int_{y\,constant} M_1 \, dx + \int (Terms \ of \ N_1 \ free \ from \ x) \ dy = C$$

$$\therefore \int_{y\,constant} (2x^2 + \frac{e^x}{y})dx - \int y dy = C$$

$$\therefore \quad \left(\frac{2x^3}{3} + \frac{e^x}{y}\right) - \frac{y^2}{2} = C$$

This is required solution of given differential equation (1).

Example 4.4.4(b) (iii) Solve the differential equation

$$(x + y + 1)ydx + (x + 3y + 2)xdy = 0$$

Solution: Given equation is

$$(x + y + 1)ydx + (x + 3y + 2)xdy = 0 \text{ ....}(1)$$

$$Here, \quad M = xy + y^2 + y \quad and \quad N = x^2 + 3xy + 2x$$

$$\therefore \frac{\partial M}{\partial y} = x + 2y + 1 \; and \quad \frac{\partial N}{\partial x} = 2x + 3y + 2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now, consider

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x + 3y + 2 - x - 2y - 1}{xy + y^2 + y}$$

$$= \frac{x + y + 1}{y(x + y + 1)} = \frac{1}{y} = f(y). \left(i.e., \; function \; of \; only \; y\right)$$

Therefore, $Integrating\ Factor\,(I\,F) = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Multiplying the given differential equation by the Integrating Factor (I F) $= y$, we get

$$(x+y+1)y^2dx+(x+3y+2)xydy=0 \; Or \; (xy^2+y^3+y^2)dx+(x^2y+3xy^2+2xy)dy=0$$

is an exact differential equation. Because as $M_1 = xy^2 + y^3 + y^2$ and $N_1 = x^2y + 3xy^2 + 2xy$ gives

$$\frac{\partial M_1}{\partial y} = 2xy+3y^2+2y \quad and \quad \frac{\partial N_1}{\partial x} = 2xy+3y^2+2y \quad \Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int_{y\,constant} M_1\,dx + \int (Terms\;\; of\;\; N_1\;\; free\;\; from\;\; x)\;dy = C$$

$$\therefore \quad \int_{y\,constant} (xy^2+y^3+y^2)dx = C$$

$$\therefore \quad \left(\frac{x^2y^2}{2} + y^3x + y^2x\right) = C$$

This is required solution of given differential equation (1).

4.4.5 For the equation of the type

$x^a y^b (m\,y\,dx + n\,x\,dy) + x^{a^1} y^{b^1} (m^1\,y\,dx + n^1\,x)\,dy$ **, then an integrating factor is** $x^h y^k$ **. Where** $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ **and** $\frac{a^1+h+1}{m^1} = \frac{b^1+k+1}{n^1}$

Example 4.4.5 (i) Solve the differential equation

$$(xy+2x^2y^2)ydx+(xy-x^2y^2)xdy=0$$

Solution: Given equation is

$$(xy+2x^2y^2)ydx+(xy-x^2y^2)xdy=0 \;......................(1)$$

$$Here, \quad M=xy^2+2x^2y^3 \quad and \quad N=x^2y-x^3y^2$$

$$\therefore \; \frac{\partial M}{\partial y}=2xy+6x^2y^2 \; and \quad \frac{\partial N}{\partial x}=2xy-3x^2y^2$$

$$\therefore \; \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now,

Equation (1) can be written in the form

$x^a y^b(m\,y\,dx+n\,x\,dy)+x^{a^1}y^{b^1}(m^1\,y\,dx+n^1\,x)\,dy$ as

$$xy(ydx+xdy)+x^2y^2(2ydx-xdy)=0$$

$$with \;\; a=b=1,\; m=n=1 \;\; and \;\; a^1=b^1=2,\; m^1=2,\; n^1=-1$$

Where $\frac{a+h+1}{m}=\frac{b+k+1}{n}$ **and** $\frac{a^1+h+1}{m^1}=\frac{b^1+k+1}{n^1}$

$\Rightarrow \frac{1+h+1}{1}=\frac{1+k+1}{1}$ **and** $\frac{2+h+1}{2}=\frac{2+k+1}{-1}$ on solving we obtain

$$h-k=0 \;\; and \;\; h+2k+9=0 \Rightarrow h=k=-3$$

$$\therefore Integrating \;\; factor = x^h y^k = x^{-3}y^{-3}=\frac{1}{x^3y^3}\,.$$

Multiplying the given differential equation (1) by the Integrating Factor (I F) = $\frac{1}{x^3y^3}$, we get

$$\frac{1}{x^3y^3}(xy+2x^2y^2)ydx+\frac{1}{x^3y^3}(xy-x^2y^2)xdy=0 \;\; Or \left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$$

is an exact differential equation. Because as $M_1=\left(\frac{1}{x^2y}+\frac{2}{x}\right)$ and

$N_1=\left(\frac{1}{xy^2}-\frac{1}{y}\right)$ gives $\frac{\partial M_1}{\partial y}=\frac{-1}{x^2y^2}$ *and* $\frac{\partial N_1}{\partial x}=\frac{-1}{x^2y^2}$ $\Rightarrow \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$

The general solution is given by

$$\int\limits_{y\,constant} M_1\,dx+\int(Terms\;\; of\;\; N_1\;\; free\;\; from\;\; x)\;dy=C$$

$$\therefore \int\limits_{y\,constant}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx-\int\frac{1}{y}dy=C$$

$$\therefore \left(\frac{-1}{xy}+2\log x\right)-\log y=C$$

$$Or\;\;\log\left(\frac{x^2}{y}\right)-\frac{1}{xy}=C$$

This is required solution of given differential equation (1).

Example 4.4.5 (ii) Solve the differential equation

$$(x^2y+y^4)dx+(2x^3+4xy^3)dy=0$$

Solution: Given equation is

$$(x^2y+y^4)dx+(2x^3+4xy^3)dy=0\;\;\text{........................}(1)$$

$Here, \quad M = x^2 y + y^4 \quad and \quad N = 2x^3 + 4xy^3$

$$\therefore \frac{\partial M}{\partial y} = x^2 + 4y^3 \; and \; \frac{\partial N}{\partial x} = 6x^2 + 4y^3$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, given equation (1) is non-exact differential equation.

Now,

Equation (1) can be written in the form

$x^a y^b (m\, y\, dx + n\, x\, dy) + x^{a^1} y^{b^1} (m^1\, y\, dx + n^1\, x)\, dy$ as

$x^2 y^0 (ydx + 2xdy) + x^0 y^3 (ydx + 4xdy) = 0$

$with \;\; a = 2,\ b = 0,\ m = 1,\ n = 2 \;\; and \;\; a^1 = 0,\ b^1 = 3,\ m^1 = 1,\ n^1 = 4$

Where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ **and** $\frac{a^1+h+1}{m^1} = \frac{b^1+k+1}{n^1}$

$\Rightarrow \frac{2+h+1}{1} = \frac{0+k+1}{2}$ **and** $\frac{0+h+1}{1} = \frac{3+k+1}{4}$ on solving we obtain

$$2h - k = -5 \;\; and \;\; 4h - k = 0 \Rightarrow h = \frac{5}{2} \;\; and \;\; k = 10$$

$\therefore Integrating \;\; factor = x^h y^k = x^{5/2} y^{10}$.

Multiplying the given differential equation (1) by the Integrating Factor (I F) $= x^{5/2} y^{10}$, we get

$$x^{5/2} y^{10} (x^2 y + y^4) dx + x^{5/2} y^{10} (2x^3 + 4xy^3) dy = 0 \;\; Or \;\; (x^{9/2} y^{11} + x^{5/2} y^{14}) dx + (2x^{11/2} y^{10} + 4x^{7/2} y^{13}) dy = 0$$

is an exact differential equation.

Because as $M_1 = (x^{9/2}y^{11} + x^{5/2}y^{14})$ and $N_1 = (2x^{11/2}y^{10} + 4x^{7/2}y^{13})$ gives

$$\frac{\partial M_1}{\partial y} = 11x^{9/2}y^{10} + 14x^{5/2}y^{13} \quad and \quad \frac{\partial N_1}{\partial x} = 2\times\frac{11}{2}x^{\frac{11}{2}-1}\ y^{10} + 4\times\frac{7}{2}x^{\frac{7}{2}-1}y^{13}$$

$$= 11x^{9/2}y^{10} + 14x^{5/2}y^{13}$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

The general solution is given by

$$\int_{y\,constant} M_1\, dx + \int (Terms\ of\ N_1\ free\ from\ x)\ dy = C$$

$$\therefore \int_{y\,constant} (x^{9/2}y^{11} + x^{5/2}y^{14})dx = C$$

$$\therefore \quad y^{11}\left(\frac{x^{11/2}}{11/2}\right) + y^{14}\left(\frac{x^{7/2}}{7/2}\right) = C$$

$$Or\ \frac{2}{11}y^{11}x^{11/2} + \frac{2}{7}y^{14}x^{7/2} = C$$

$$Or\ 2y^{11}x^{7/2}\left(\frac{x^2}{11} + \frac{y^3}{7}\right) = C$$

This is required solution of given differential equation (1).

4.4 Test Your Progress
Solve the following differential equations. **i)** $ydx - xdy + \log x\, dx = 0$ ii) $ydx - xdy + 3x^2y^2e^{x^3}\ dx = 0$ iii) $2ydx + x(2\log x - y)dy = 0$

iv) $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$

v) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

vi) $(2y^3 + 2)dx + 3xy^2 dy = 0$

vii) $(x^3 - 2y^2)dx + 2xydy = 0$

viii) $2xydy - (x^2 + y^2 + 1)dx = 0$

ix) $(y + y^2)dx + xydy = 0$

x) $(x^3 + xy^4)dx + 2y^3 dy = 0$

xi) $(xy^3 + y)dx + 2(x^2 y^2 + x + y^4)dy = 0$

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dx = 0$$

4.5 Summary

In this unit, we studied criteria for an exact differential equation and non exact differential equation and to find Integrating factors in different cases to make the equation exact. We also studied how to find the integrating factor by inspection of the differential equation.

4.6 Terminal Questions

1. Solve the following differential equation.

A. $(x^2 - ay)dx = (ax - y^2)dy.$

B. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

C. $(e^y + 1) \cos x \; dx + e^y \sin dx = 0$

D. $y(axy + e^x)dx - e^x dy = 0$

E. $\cos(x + y) \; dy = dx \; (hint \; put \; x + y = t)$

F. $\frac{dy}{dx} = (4x + y + 1)^2.$

4.7. Answers to exercises

4.3. Test Your Progress

i) $(1+x^2)(1+y^2) = C$

ii) $x^3 - 3axy + y^3 = C$

iii) $3y\cos 2x + 2y^3 + 3y = C$

iv) $x + ye^{x/y} = C$

v) $(e^y + 1)\sin x = C$

vi) $2x^3 - 9x^2 y - 3xy^2 + y^3 = C$

vii) $e^{xy^2} + x^4 - y^3 = C$

viii) $e^{x/y} + y^2 = C$

ix) $x^2 + y^2 + 2\tan^{-1}(x/y) = C$

x) $x^2 y + (y - \tan y)x + \tan y = C$

4.4. Test Your Progress

i) $y + cx + \log x + 1 = 0$

ii) $\frac{x}{y} + e^{x^3} = c$

iii) $4y\log x = y^2 + c$

iv) $3\log x - (y/x)^3 = c$

v) $\left(y + \frac{2}{y^2}\right)x + y^2 = c$

vi) $x^2 y^3 + x^2 = c$

vii) $x^3 + y^2 = cx^2$

viii) $x^3 - y^2 - 1 = cx$

ix) $x + xy = c$

x) $(x^2 + y^4 - 1)e^{x^2} = c$

xi) $(x^2 + y^4 - 1)e^{x^2} = c$

xii) $\left(xy + \frac{2x}{y^2} + y^2\right) = c$

Unit –05: Differential equation of the first order but not of the first degree

Structure

5.1 Introduction

As $\frac{dy}{dx}$ will occur in higher degrees, it is convenient to denote $\frac{dy}{dx}$ by p.

Such equations are of the form $f(x, y, p) = 0$, which is not of first degree, is called a differential equation of first order and higher degree, the general form of first order and nth degree differential equation is

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + ... + a_{n-1} p + a_n = 0 \ (n > 1)$$

Where $a_1, a_2, ..., a_n$ are functions in x and y. Now we shall discuss the solution of the above differential equation in the following three cases.

(i.) Equations solvable for p

(ii.) Equations solvable for x

(iii.) Equations solvable for y

5.2. Objectives

After reading this unit students should be able to:

- Solve the Differential equations of the first order but not of the first degree,

- Recognize and solve the equations solvable for x, y and p
- Identify and solve the Clairaut's equation
- Find the singular solutions of the given ODE's

5.3 Equations solvable for p

Let $p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots\dots\dots\dots\dots\dots + a_{n-1} p + a_n = 0 \ (n > 1)$

..........................(1) be the differential equation of first order and n[th] degree. If it can be solved for 'p' then equation (1) can be resolved into 'n' linear factors in 'p'. Then, we have

$$[p - f_1(x, y)][p - f_2(x, y)]\dots\dots\dots\dots[p - f_n(x, y)] = 0$$

$$\Rightarrow p - f_1(x, y) = 0,\ p - f_2(x, y) = 0,\ \dots\dots\dots\dots p - f_n(x, y) = 0,\ where\ \ p = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = f_1(x, y),\ \frac{dy}{dx} = f_2(x, y),\ \dots\dots\dots\dots\dots\dots \frac{dy}{dx} = f_n(x, y)$$

Solving each of n differential equations, we get 'n' solutions. Let them be

$$F_1(x, y, c_1) = 0,\ F_2(x, y, c_2) = 0 \dots\dots\dots\dots F_n(x, y, c_n) = 0$$

Thus, the solution of equation (1) is given by

$$F_1(x, y, c_1) \times F_2(x, y, c_2) \times \dots\dots\dots\dots \times F_n(x, y, c_n) = 0$$

But equation (1) is of first order differential equation.

Therefore, the solution of equation (1) is given by

$F_1(x,y,c)\times F_2(x,y,c)\times\text{.............}\times F_n(x,y,c)=0$ *by taking* $c_1=c_2=\text{........}=c_n=c$.

Example 5.3.1: Solve the differential equation $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$.

Solution: Given equation is $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$..................................(1)

Using $\frac{dy}{dx}=p$ gives $p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}$ Or $p^2+p\left(\frac{y}{x}-\frac{x}{y}\right)-1=0$

Factorising leads $\left(p+\frac{y}{x}\right)\left(p-\frac{x}{y}\right)=0$

$\Rightarrow\left(p+\frac{y}{x}\right)=0$ ....(2) *and* $\left(p-\frac{x}{y}\right)=0$......................(3)

From equation (2), we have

$$\frac{dy}{dx}+\frac{y}{x}=0 \;\; Or \;\; x\,dy+y\,dx=0$$

$Or\;\; d(xy)=0,$ integrating we obtain

$$xy=c$$

From equation (3), we have

$$\frac{dy}{dx}-\frac{x}{y}=0 \;\; Or \;\; x\,dx-y\,dy=0$$

Integrating, we obtain

$$x^2-y^2=c$$

Thus, $xy=c$ Or $x^2-y^2=c$ constitute the required solution.

Example 5.3.2: Solve the differential equation $\frac{dy}{dx}\left(\frac{dy}{dx}+y\right)=x(x+y).$

Solution: Given equation is $\frac{dy}{dx}\left(\frac{dy}{dx}+y\right)=x(x+y)$ ....(1)

Using $\frac{dy}{dx}=p$ gives $p(p+y)=x(x+y)$

Factorising leads to $p^2+py-x^2-xy=0$ Or $p^2-x^2+py-xy=0$

Or $(p+x)(p-x)+y(p-x)=0$

Or $(p-x)(p+x+y)=0$

$\Rightarrow (p-x)=0$ ....(2) *and* $(p+x+y)=0$ ....(3)

From equation (2), we have

$\frac{dy}{dx}-x=0$ *Or* $dy=xdx$, *integrating , we obtain* $y=\frac{x^2}{2}+c$

From equation (3), we have $\frac{dy}{dx}+x+y=0$ *Or* $\frac{dy}{dx}+y=-x$

This represents the linear differential equation of first order and first degree.

$\therefore$ *Integrating factor* $=I.F=e^{\int dx}=e^x$

It's general solution is given by

$$ye^{x} = \int (-x)e^{x}dx + c \quad Or \; ye^{x} = -xe^{x} + e^{x} + c \quad Or \; y = (1-x) + ce^{-x}$$

Thus, $y = \frac{x^2}{2} + c$ and $y = (1-x) + ce^{-x}$ constitutes the required solution.

Example 5.3.3: Solve the differential equation $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

Solution: Given equation is $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0 ....(1)$
.

Using $\frac{dy}{dx} = p$ gives $yp^2 + (x-y)p - x = 0$

Factorising leads to $yp^2 - py + xp - x = 0$

Or $py(p-1) + x(p-1) = 0$

Or $(p-1)(py+x) = 0$

$\Rightarrow (p-1) = 0$ ....(2) *and* $(py+x) = 0$ ....(3)

From equation (2), we have

$$\frac{dy}{dx} - 1 = 0 \; Or \; dy = dx, \text{ integrating}, we \; obtain \; y = x + c \; Or \; y - x = c$$

From equation (3), we have $y\frac{dy}{dx} + x = 0 \quad Or \quad \frac{dy}{dx} = -\frac{x}{y}$

Or $ydy + xdx = 0, \; Integrating, we \; obtain \; \frac{y^2}{2} + \frac{x^2}{2} = c$

Thus, $y - x = c$ and $\frac{y^2}{2} + \frac{x^2}{2} = c$ constitutes the required solution.

5.3 Test Your Progress
Solve the following differential equations (Equations Solvable for 'p') i) $p^2 + 2py\cot x - y^2 = 0$ ii) $xy(p^2 + 1) = (x^2 + y^2)p$ iii) $xyp^2 + (x^2 + xy + y^2)p + (x^2 + xy) = 0$ iv) $p^3 + (2x - y^2)p^2 = 2xy^2 p$ v) $xy^2(p^2 + 2) = 2py^3 + x^3$ vi) $xp^2 - 2yp + x = 0$

5.4. Equations Solvable for y

Let $f(x, y, p) = 0$ ....(1) be given differential equation. If equation (1) cannot be resolved into linear factors in 'p' and if it can be put in the form $y = f_1(x, p)$ ....(2) then we say that equation (1) can be solved for y. Differentiating equation (2) with respect to $'x'$ we get a differential equation in two variables x and p of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \text{........................(3)}$$

Now, it may be possible to solve this new differential equation in $'x'$ and $'p'$.

Let its solution be $F(x, p, c) = 0$ ....(4) .

Eliminating 'p' from equations (2) and (4), we obtain the required general solution of the equation (1) in the form $F(x, y, c) = 0$. Where 'c' is arbitrary constant.

Note 5.4.1:

(i) In case elimination of p is not possible from equations (1) and (2) for x and y and obtain $x = F_1(p, c),\ y = F_2(p, c)$ as the required solution, where $'p'$ is the parameter.

It is called a solution in the parametric form.

(ii) The solution which does not contain an arbitrary constant is called singular solution.

(iii) The solution which does not contain $'x'$ i.e., in the form $f(y, p) = 0$ and if it is solvable for $'p'$ we get $p = \phi(y)$ which can be solved by using variables separable method. If it is solvable for $'y'$ it can be written in the form which can also be solved by using variables separable method.

(iv) If given differential equation is homogeneous in $'x'$ and $'y'$ then it can be written in the form $f(p,\ y/x)=0$, this equation can be solved using method of solving homogeneous equation.

Example 5.4.1: Solve the differential equation $y+px=x^4p^2$.

Solution: Given equation is $y+px=x^4p^2\ \ Or\ y=x^4p^2-px$ ....(1)

Equation (1) is a differential equation solvable for $'y'$

Differentiating equation (1) with respect to $'x'$

$$\frac{dy}{dx}=\left[2p\frac{dp}{dx}x^4+p^2(4x^3)\right]-\left[\frac{dp}{dx}(x)+p\right],\ \text{using}\ p=\frac{dy}{dx}$$

$$\Rightarrow p-4p^2x^3+p+x\frac{dp}{dx}-2px^4\frac{dp}{dx}=0$$

$$\Rightarrow 2p(1-2px^3)+x\frac{dp}{dx}(1-2px^3)=0$$

$$\Rightarrow (1-2px^3)+\left(2p+x\frac{dp}{dx}\right)=0$$

$$\Rightarrow (1-2px^3)=0 \text{ ....(2) } and \left(2p+x\frac{dp}{dx}\right)=0 \text{ ....(3)}$$

Here, equation (2) is discarded as it does not contain $\frac{dp}{dx}$ and it gives singular solution.

From equation (3), we obtain $2p = -x\frac{dp}{dx}$ Or $\frac{2dx}{x} = -\frac{dp}{p}$, integrating,

$$2\int \frac{dx}{x} = -\int \frac{dp}{p} + \log c \Rightarrow 2\log x + \log p = \log c \; Or \; x^2 p = c \; Or \; p = \frac{c}{x^2} \; ....(4)$$

Eliminating 'p' from equations (1) and (4) gives the general solution of the equation (1).

$i.e., \; y = x^4 p^2 - px$

Using , $p = \frac{c}{x^2} \Rightarrow y = x^4\left(\frac{c}{x^2}\right)^2 - x\left(\frac{c}{x^2}\right) = c^2 - \frac{c}{x}$

Thus, $y = c^2 - \frac{c}{x}$. Where 'c' is arbitrary constant.

Example 5.4.2: Solve the differential equation $y = x + a\tan^{-1} p$

.

Solution: Given equation is $y = x + a\tan^{-1} p$ ....(1)

Equation (1) is a differential equation solvable for $'y'$

Differentiating equation (1) with respect to $'x'$

$\frac{dy}{dx} = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$, using $p = \frac{dy}{dx}$

$$\Rightarrow p = 1 + \frac{a}{1+p^2}\frac{dp}{dx} \Rightarrow p + p^3 = 1 + p^2 + a\frac{dp}{dx}$$

$$\Rightarrow Or \quad 1+p^2+a\frac{dp}{dx}-p-p^3=0 \;\; Or \;\; (1+p^2)+a\frac{dp}{dx}-p(1+p^2)$$

$$\Rightarrow (1+p^2)+\left(1-p+a\frac{dp}{dx}\right)=0$$

$$\Rightarrow (1+p^2)=0 \;\text{.................}(2) \;\; and \;\; \left(1-p+a\frac{dp}{dx}\right)=0 \;\text{...................}(3)$$

Here, equation (2) is discarded as it does not contain $\frac{dp}{dx}$ and it gives singular solution.

From equation (3), we obtain,

$(1-p)+a\frac{dp}{dx} \;\; Or \; -\frac{adp}{dx}=(p-1) \;\; Or \;\; \frac{dp}{(p-1)}=\frac{-1}{a}dx$, integrating,

$$\int\frac{dp}{(p-1)}=-\frac{1}{a}\int dx+c \Rightarrow \log(p-1)+\frac{x}{a}=c \;Or\; (p-1)=e^{c-\frac{x}{a}} \;\; Or \;\; p=e^{c-\frac{x}{a}}+1 \;\text{............}(4)$$

Eliminating ‘p’ from equations (1) and (4) gives the general solution of the equation (1).

$i.e., \;\; y=x+a\tan^{-1}p$

Using , $p=e^{c-\frac{x}{a}}+1$

Thus, $y=x+a\tan^{-1}\left(e^{c-\frac{x}{a}}+1\right)$. Where ‘c’ is arbitrary constant.

Example 5.4.3: Solve the differential equation $(8p^3-27)x=12p^2y$

.

Solution: Given equation is $(8p^3-27)x=12p^2y$ *Or* $y=\frac{(8p^3-27)x}{12p^2}$

$\therefore\ y=\frac{2px}{3}-\frac{9x}{4p^2}$ ....(1)

Equation (1) is a differential equation solvable for $'y'$

Differentiating equation (1) with respect to $'x'$

$$\frac{dy}{dx}=\frac{2}{3}\left[p+x\frac{dp}{dx}\right]-\frac{9}{4}\left[\frac{p^2-2xp\frac{dp}{dx}}{p^4}\right],\ \text{using}\ p=\frac{dy}{dx}$$

$$p=\frac{2}{3}\left[p+x\frac{dp}{dx}\right]-\frac{9}{4}\left[\frac{p^2-2xp\frac{dp}{dx}}{p^4}\right]$$

Multiplying by $12p^4$

$$12p^5=8p^4\left[p+x\frac{dp}{dx}\right]-27\left[p^2-2xp\frac{dp}{dx}\right]$$

$$\Rightarrow 2xp\frac{dp}{dx}(4p^3+27)-p^2(4p^3+27)=0$$

$$Or\ \ (4p^3+27)(2xp\frac{dp}{dx}-p^2)=0$$

$\Rightarrow\ (4p^3+27)=0$ ....(2) $\quad 2xp\frac{dp}{dx}-p^2=0$ ....(3)

Here, equation (2) is discarded as it does not contain $\frac{dp}{dx}$ and it gives singular solution.

From equation (3), we obtain $2xp\frac{dp}{dx} - p^2 = 0$, separating the variables and integrating,

$$2\int \frac{dp}{p} = \int \frac{dx}{x} + \log c \;\Rightarrow\; 2\log p - \log x = \log c$$

$$Or\ \log\left(\frac{p^2}{x}\right) = \log c \ \ Or\ \frac{p^2}{x} = c \ \ Or\ p = \sqrt{xc} \ \text{.........................(4)}$$

Eliminating 'p' from equations (1) and (4) gives the general solution of the equation (1).

$$i.e.,\ y = \frac{2px}{3} - \frac{9x}{4p^2}$$

Using , $p = \sqrt{xc} \Rightarrow y = \frac{2x\sqrt{xc}}{3} - \frac{9x}{4xc}\ \ Or\ y = \frac{2x\sqrt{xc}}{3} - \frac{9}{4c}$

Thus, $y = \frac{2x\sqrt{xc}}{3} - c_1$. Where '$c_1$' is arbitrary constant.

Example 5.4.4: Solve the differential equation $y = 2px + p^n$.

Solution: Given equation is $y = 2px + p^n$ .... (1)

Equation (1) is a differential equation solvable for $'y'$

Differentiating equation (1) with respect to $'x'$

$$\frac{dy}{dx} = \left[2\left(p + x\frac{dp}{dx}\right) + np^{n-1}\frac{dp}{dx}\right], \text{ using } p = \frac{dy}{dx}$$

$$\Rightarrow\ p = \left[2\left(p + x\frac{dp}{dx}\right) + np^{n-1}\frac{dp}{dx}\right]$$

$$\Rightarrow\ p + \left(2x + np^{n-1}\right)\frac{dp}{dx} = 0$$

$$Or\ \ p\frac{dx}{dp} + 2x = -np^{n-1}.\ \ Or\ \ \frac{dx}{dp} + \frac{2}{p}x = -np^{n-2}$$

This represents linear equation in terms of x and p.

$$\therefore I.F = e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2.$$

Its general solution is given by

$$xp^2 = \int -np^{n-2}p^2 dp + c\ Or\ xp^2 = -n\frac{p^{n+1}}{n+1} + c\ Or\ \ x = -\frac{np^{n-1}}{(n+1)} + cp^{-2}$$

...............(2)

Substituting the value of x from equation (2) in equation (1), we get

$$y = 2p\left(-\frac{np^{n-1}}{n+1} + cp^{-2}\right) + p^n\ \ Or\ \ y = 2\left(\frac{-np^n}{n+1} + \frac{c}{p}\right) + p^n\ \ Or\ y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$$

...............(3)

An equation (2) and (3) constitutes the required general solution.

5.4 Test Your Progress
Solve the following differential equations (Equations Solvable for ‘y’)

i)	$xp^2 - 2py + x = 0$
ii)	$xp^3 - 2p^2y + 4x^2 = 0$
iii)	$y = xp^2 + p$
iv)	$y - 2xp = \tan^{-1}(xp^2)$
v)	$y = p\sin p + \cos p$
vi)	$x^2\left(\frac{dy}{dx}\right)^4 + 2x\left(\frac{dy}{dx}\right) - y = 0$

5.5. Equations solvable for x

Let $f(x, y, p) = 0$ ....(1) be given differential equation. If equation (1) cannot be resolved into linear factors in 'p' and if it can be put in the form $x = f_1(y, p)$ ....(2) .

Then we say that equation (1) can be solved for x. Differentiating equation (2) with respect to $'x'$ we get a differential equation in two variables x and p of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right) \text{.........................(3)}$$

Now, it may be possible to solve this new differential equation in $'y'$ and $'p'$.

Let its solution be $F(y, p, c) = 0$ ....(4) .

Eliminating ‘p’ from equations (2) and (4), we obtain the required general solution of the equation (1) in the form $F(x, y, c) = 0$. Where ‘c’ is arbitrary constant.

Note 5.5.1:

i) This method is especially useful for equations which do not contain y.

ii) If it is not possible to eliminate ’p’ from equations (2) and (3)[i.e., elimination is not feasible) then equations (2) and (3)together represent the general solution of equation (1) in terms of p. Where ‘p’ may be regarded as a parameter.

Suppose that the given differential equation does not contain ‘y’ i.e., in the form $f(x, p) = 0$. If it is solvable for ‘p’ then it may be written as $p = \phi(x)$ which can be solved by variables separable method. If it is solvable for $'x'$, it may be written as $x = \phi(p)$ which can also be solved as explained above.

Example 5.5.1: Solve the differential equation $p^3 - 4xyp + 8y^2 = 0$

Solution: Given equation is $p^3 - 4xyp + 8y^2 = 0$

Equation (1) is a differential equation solvable for $'x'$

$\therefore \; x = \frac{1}{4}\left[\frac{p^2}{y} + \frac{8y}{p}\right]$ .... (1)

Differentiating equation (1) with respect to $'y'$

$$\frac{dx}{dy} = \frac{1}{4}\left[\frac{2p\frac{dp}{dy}y - p^2}{y^2} + 8.\frac{p - y\frac{dp}{dy}}{p^2}\right], \text{ using } p = \frac{dy}{dx}$$

$\Rightarrow \frac{1}{p} = \frac{1}{4}\left[\frac{2p\frac{dp}{dy}y - p^2}{y^2} + 8.\frac{p - y\frac{dp}{dy}}{p^2}\right]$, multiplying by $4y^2p^2$, we obtain

$$4y^2p = p^2\left(2py\frac{dp}{dy} - p^2\right) + 8y^2\left(p - y\frac{dp}{dy}\right)$$

$$\Rightarrow 2y\frac{dp}{dy}(p^3 - 4y^2) + p(4y^2 - p^3) = 0$$

$$Or \left(2y\frac{dp}{dy} - p\right)(p^3 - 4y^2) = 0$$

$\Rightarrow (p^3 - 4y^2) = 0$ ....(2) and $\left(2y\frac{dp}{dy} - p\right) = 0$ ....(3)

Here, equation (2) is discarded as it does not contain $\frac{dp}{dy}$ and it gives singular solution.

From equation (3), we obtain $2y\frac{dp}{dy} - p = 0 \ \ Or \ 2\frac{dp}{p} = \frac{dy}{y}$, integrating,

$$2\int\frac{dp}{p} = \int\frac{dy}{y} + \log c \Rightarrow 2\log p - \log y = \log c \ \ Or \ \ \frac{p^2}{y} = c \ \ Or \ \ p^2 = cy \ ....(4)$$

Eliminating 'p' from equations (1) and (4) gives the general solution of the equation (1).

$$i.e., \ \ x = \frac{1}{4}\left[\frac{p^2}{y} + \frac{8y}{p}\right] \ p^3 - 4xyp + 8y^2 = 0$$

Using , $p^2 = cy \Rightarrow \ x = \frac{1}{4}\left(c + \frac{8y}{\sqrt{cy}}\right) = \frac{1}{4}\left(c + 8\sqrt{\frac{y}{c}}\right) = \frac{c}{4} + 2\sqrt{\frac{y}{c}}$

Thus, $x = \frac{c}{4} + 2\sqrt{\frac{y}{c}}$. Where 'c' is arbitrary constant.

Example 5.5.2: Solve the differential equation $p = \tan\left(x - \frac{p}{1+p^2}\right)$

Solution: Given equation is $p = \tan\left(x - \frac{p}{1+p^2}\right)$

$$\Rightarrow \tan^{-1} p = x - \frac{p}{1+p^2} \ \ Or \ \ x = \tan^{-1} p + \frac{p}{1+p^2} \ ....(1)$$

Equation (1) is a differential equation solvable for $'x'$

Differentiating equation (1) with respect to $'y'$

$$\frac{dx}{dy}=\left[\frac{1}{1+p^2}\frac{dp}{dy}+\frac{(1+p^2)\frac{dp}{dy}-2p^2\frac{dp}{dy}}{(1+p^2)^2}\right],\ \text{using}\ p=\frac{dy}{dx}$$

$$\frac{1}{p}=\left[\frac{1}{1+p^2}\frac{dp}{dy}+\frac{(1+p^2)\frac{dp}{dy}-2p^2\frac{dp}{dy}}{(1+p^2)^2}\right]$$, multiplying by $p(1+p^2)^2$, we obtain

$$\therefore\quad (1+p^2)^2=p(1+p^2)\frac{dp}{dy}+p(1+p^2-2p^2)\frac{dp}{dy}$$

$Or\quad 2p\frac{dp}{dy}=(1+p^2)^2$. This is in variable separable form

$$\therefore\quad \frac{2pdp}{(1+p^2)^2}=dy,\ Integrating, we\ obtain$$

$$\therefore\quad \int\frac{2p}{(1+p^2)^2}dp=\int dy$$

$$-(1+p^2)^{-1}=y+c$$

$$Or\ \ y=c-(1+p^2)^{-1}$$

The general solution of equation (1) is given by $x=\tan^{-1}p+\frac{p}{1+p^2}$ and

$y=c-(1+p^2)^{-1}$.

Example 5.5.3: Solve the differential equation $p^3x-bp=a$

Solution: Given equation is $p^3x - bp = a \;\; Or \;\; p^3x = a + bp \;\; Or \;\; x = \frac{a}{p^3} + \frac{b}{p^2}$

Equation (1) is a differential equation solvable for $'x'$

$$\therefore \;\; x = \frac{a}{p^3} + \frac{b}{p^2} \;\; .... (1)$$

Differentiating equation (1) with respect to $'y'$

$$\frac{dx}{dy} = \left[\frac{-3a}{p^4} - \frac{2b}{p^3}\right]\frac{dp}{dy}, \text{ using } p = \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{p} = \left[\frac{-3a}{p^4} - \frac{2b}{p^3}\right]\frac{dp}{dy} \;\; Or \;\; \left[\frac{-3a}{p^3} - \frac{2b}{p^2}\right]\frac{dp}{dy} = 1$$

$Or \; \left(-3ap^{-3} - 2bp^{-2}\right)dp = dy$. This is in variable separable form

Integrating, $\int \left(-3ap^{-3} - 2bp^{-2}\right)dp = \int dy + c$

$$Or \;\; -3a\int p^{-3}dp - 2b\int p^{-2}dp = y + c$$

$$Or \;\; y = \frac{3a}{2p^2} + \frac{2b}{p} + c$$

This gives the general solution of the equation (1).

5.5 Test Your Progress
Solve the following differential equations (Equations Solvable for ‘x’) i) $p^3 - (y+3)p + x = 0$ ii) $y = 2px + y^2p^3$

iii) $p^3y+2px=y$

iv) $x-yp=ap^2$

v) $4xp^2+4yp-y^4=0$

5.6. Clairaut's equation

Differential equation of the form $y=px+f(p)$ ....(1) is called Clairauts equation.

Clairauts equation is solvable for 'y'.

Differentiating with respect to 'x', we obtain

$$\frac{dy}{dx}=x\frac{dp}{dx}+p+f^{/}(p)\frac{dp}{dx} \quad Or \quad \frac{dy}{dx}=\left(x+f^{/}(p)\right)\frac{dp}{dx}+p$$

Using $p=\frac{dy}{dx}$, we obtain $\left(x+f^{/}(p)\right)\frac{dp}{dx}=0$

$$\Rightarrow \frac{dp}{dx}=0 \text{ ....(2)} \quad and \quad x+f^{'}(p)=0 \text{ ....(3)}$$

Here, Equation (3) is discarded as it gives singular solution. Therefore to find the general solution of equation (1), let's solve equation (2).

$\frac{dp}{dx}=0 \Rightarrow dp=0$, integrating $\int dp = c \; Or \; p = c$ ....(4), where ‘c’ is arbitrary constant.

Now, eliminating ‘p’ from equations (1) and (4), we get $y = cx + f(c)$, which is the general solution of the Clairaut’s equation given by (1).

Note 5.6.1:

i) It can be observed that, the general solution of the Clairaut’s equation $y = px + f(p)$ will be obtained by replacing ‘p’ with ‘c’.

ii) Many equations of the first order but of higher degree can be easily reduced to the Clairaut’s form by making suitable substitutions.

Example 5.6.1: Solve the differential equation; $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.

Solution: Given differential equation is $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.................(1)

Using $p=\frac{dy}{dx}$, we get $y+2p^2=(x+1)p$ Or $y=(x+1)p-2p^2=p(x+1-p)$.

This is in Clairaut's equation, therefore it's general solution is

$y=c(x+1-c).$

Example 5.6.2: Solve the differential equation; $x^2(y-px)=yp^2$

Solution: Given differential equation is $x^2(y-px)=yp^2$ ....(1)

Put $x^2=u$ and $y=v \Rightarrow 2xdx=du$ and $dy=dv$

$$\therefore\ p=\frac{dy}{dx}=\frac{dv}{du/2x}=2x\frac{dv}{du}=2xP \ \ with\ \ P=\frac{dv}{du}=\frac{dy}{2xdx}=\frac{p}{2x}.$$

$$\therefore\ x^2(y-px)=yp^2 \Rightarrow u(v-2Pu)=4uvP^2$$

$$\therefore (v-2Pu)=4vP^2 \ \ Or\ \ 2pu=v-4vP^2$$

$$Or\ v=\frac{2Pu}{1-4P^2}$$

This is in Clairaut's equation, therefore its general solution is

$$v=\frac{2cu}{1-4c^2}\ \ Or\ \ y=\frac{2cx^2}{1-4c^2}\ .$$

Example 5.6.3: Solve the differential equation; $(px-y)(py+x)=2p$

Solution: Given differential equation is $(px-y)(py+x)=2p$ ....(1) Put $x^2=u$ and $y^2=v \Rightarrow 2xdx=du$ and $2ydy=dv$

$$\therefore\ p=\frac{dy}{dx}=\frac{dv/2y}{du/2x}=\frac{xdv}{ydu}=\frac{x}{y}P \ \ with\ \ P=\frac{dv}{du}=\frac{2ydy}{2xdx}=\frac{yp}{x}$$

$$\therefore (px-y)(py+x)=2p \Rightarrow \left(\frac{uP}{\sqrt{v}}-\sqrt{v}\right)\left(P\sqrt{u}+\sqrt{u}\right)=\frac{2P\sqrt{u}}{\sqrt{v}}$$

$$Or \ \left(\frac{uP-v}{\sqrt{v}}\right)(P+1)\sqrt{u}=\frac{2P\sqrt{u}}{\sqrt{v}} \ Or \ (uP-v)(P+1)=2P$$

$$\therefore \ (uP-v)=\frac{2P}{P+1} \ Or \ v=uP-\frac{2P}{P+1}$$

This is in Clairaut's equation, therefore its general solution is

$$v=cu-\frac{2c}{c+1} \ Or \ y^2=cx^2-\frac{2c}{c+1}.$$

5.6 Test Your Progress
Solve the following differential equations (Clairaut's Equation) i) $x^2(y-px)=yp^2$ ii) $(px+y)^2=py^2$ iii) $y=2px+p^2y$ iv) $y=px+\sqrt{1+p^2}$ v) $\sin(y-px)=p$ vi) $y^2-2pxy+p^2x^2-p^2=a^2$

5.7. Singular Solutions

If we eliminate ‘p’ from equation (3) given by $x + f^{'}(p) = 0$ and the Clairaut’s equation in (1) given by $y = px + f(p)$, we obtain an equation involving no constant. This is the singular solution of equation (1) which gives the envelope of the family of straight lines given by the equation (4).

We need to proceed as below to obtain the singular solution.

i) Find the general solution by replacing ‘p’ by ‘c’, we obtain the equation (4)

ii) Differentiate equation (4) with respect to ‘c’, we obtain $x + f^{'}(c) = 0$ ....(5).

iii) Eliminating ‘c’ from equations (4) and (5) leads to the singular solution.

Example 5.7.1: Find the general and singular solution of the differential equation $xp^2 - yp + a = 0$

Solution: Given differential equation is $xp^2 - yp + a = 0$ ....(1)

$$\Rightarrow yp = a + xp^2 \;\; Or \;\; y = xp + \frac{a}{p}$$

This is in the form of Clairaut's equation given by $y = px + f(p)$.

Therefore it's general solution is given by

$$y = cx + \frac{a}{c} \text{ ....}(2)$$

Now, to find the singular solution differentiating equation (2) with respect to 'c', we obtain

$$0 = x - \frac{a}{c^2} \;\; Or \;\; c^2 = \frac{a}{x} \;\; Or \;\; c = \sqrt{\frac{a}{x}} \text{ ....}(3)$$

Eliminating 'c' from equations (2) and (3), we get

$$y = \sqrt{\frac{a}{x}} + \frac{a}{\sqrt{\frac{a}{x}}} \;\; Or \;\; y = \sqrt{\frac{a}{x}} + \sqrt{ax} \;\; Or \;\; y = \frac{\sqrt{a}(x+1)}{\sqrt{x}}$$

This is the desired singular solution.

Example 5.7.2: Find the general and singular solution of the differential equation $p = \log(px - y)$

Solution: Given differential equation is $p = \log(px - y)$

$$\Rightarrow (px - y) = e^p \;\; Or \;\; y = px + e^p \text{ ....}(1)$$

This is in the form of Clairaut's equation given by $y = px + f(p)$.

Therefore its general solution is given by

$$y = cx + e^{c} \qquad \text{...............................}(2)$$

Now, to find the singular solution differentiating equation (2) with respect to 'c', we obtain

$$0 = x + e^{c} \quad Or \quad e^{c} = -x \quad Or \quad c = \log\left(\frac{1}{x}\right) \text{................}(3)$$

Eliminating 'c' from equations (2) and (3), we get

$$y = x\log\left(\frac{1}{x}\right) + e^{\log(1/x)} \quad Or \quad y = x\log\left(\frac{1}{x}\right) + \frac{1}{x}$$

This is the desired singular solution.

Example 5.7.3: Find the general and singular solution of the differential equation $p = \sin(y - xp)$

Solution: Given differential equation is $p = \sin(y - xp)$

$$\Rightarrow \sin^{-1} p = y - xp \quad Or \quad y = px + \sin^{-1} p \text{ ....}(1)$$

This is in the form of Clairaut's equation given by $y = px + f(p)$.

Therefore its general solution is given by

$$y = cx + \sin^{-1} c \qquad \text{...............................}(2)$$

Now, to find the singular solution differentiating equation (2) with respect to 'c', we obtain

$$0 = x + \frac{1}{\sqrt{1-c^2}} \quad Or \quad x\sqrt{1-c^2} = -1 \quad Or \quad (1-c^2) = \frac{1}{x^2}$$

$$c^2 = 1 - \frac{1}{x^2} \quad Or \quad c = \frac{\sqrt{x^2-1}}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Eliminating 'c' from equations (2) and (3), we get

$$y = \frac{\sqrt{x^2-1}}{x} + \sin^{-1}\left[\frac{\sqrt{x^2-1}}{x}\right]$$

This is the desired singular solution.

5.7 Test Your Progress
Find the general and singular solutions of the following differential equations i) $y = px + \sqrt{(a^2 p^2 + b^2)}$ ii) $\sin px \cos y = \cos px \sin y + p$ iii) $y^2 - 2pxy + p^2 x^2 - p^2 = a^2$ iv) $y = px + \sqrt{1+p^2}$

5.8 Summary

In this unit, we have studied the diffential equation of the first order but not of first degree which are (1) Solvable for y (2) Solvable for x. (3) Solvable for p. We also studied Clairaut's equation and how to determine singular solution.

5.9 Terminal Questions

1. Find the general and singular solution of the differential equation$(xp-y)^2 = p^2 - 1$.
2. Solve

   $$y(\frac{dy}{dx})^2 + (x+y)\frac{dy}{dx} - y = 0$$

3. Solve

   $$y^2.\log y = xyp + p^2$$

4. Solve

   $$y = px + (1+p^2)^{\frac{1}{2}}.$$

5.10 Answers to Exercises

5.3 Test Your Progress (Solvable for ‘p’)

i) $c^2 y^2 \sin^2 x - \tan^2\left(\frac{x}{2}\right) = 0$

ii) $(y^2 - x^2 - c)(y - cx) = 0$

iii) $(y^2 + x^2 - c)(2xy + x^2 - c) = 0$

iv) $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$

v) $(y^2 - x^2 - x^4 c)(y^2 - x^2 - c) = 0$

vi) $2cxy = c^2 x^3 + x$

5.4 Test Your Progress (Solvable for ‘y’)

i) $2cxy = c^2 x^3 + x$

ii) $2c^2 y = c^3 x^2 + 4$

iii) $y = xp^2 + p \;\; and \;\; e^{x(p-1)^2 + p} = pc$

iv) $y = 2\sqrt{cx} + \tan^{-1} c$

v) $x = \sin p + c$

vi) $y = 2\sqrt{cx} + c^2$

5.5 Test Your Progress (Solvable for ‘y’)

i) $x=(y+3)p-p^3 \;\; and \;\; y=(p^2-1)+c(p^2-1)^{-1/2}$

ii) $y^2=2cx+c^3$

iii) $y^2=2cx+c^3$

iv) $(y+ap)\sqrt{p^2-1}+a\cosh^{-1}p=c$

v) $4x(cy^2)^2+4y(cy^2)-y^4=0 \;\; Or \;\; 4c(1+cxy)-y=0$

5.6 Test Your Progress (Clairaut’s Equation)

i) $y^2=cx^2+c^2$

ii) $xy=cy-c^2$

iii) $y=\dfrac{2cx}{1-c^2}$

iv) $y=cx+\sqrt{1+c^2}$

v) $y=cx+\sin^{-1}c$

vi) $y=cx\pm\sqrt{a^2+c^2}$

5.7 Test Your Progress

i) General solution is $y=cx+\sqrt{a^2c^2+b^2}$; Singular solution is

$y+\sqrt{1-x^2}=0$

ii) General solution is $y = cx - \sin^{-1} c$; Singular solution is

$$y = \sqrt{1-x^2} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$$

iii) General solution is $y = cx + \sqrt{1+c^2}$; Singular solution is

$$y = \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1+\frac{x^2}{1-x^2}}$$

iv) General solution is $y = cx \pm \sqrt{a^2+c^2}$; Singular solution is

$$y = \frac{ax^2}{\sqrt{1-x^2}} \pm \sqrt{a^2+\frac{a^2x^2}{1-x^2}}$$

Block-II

Applications of differential equation

U P RAJARSHI TANDON
OPEN UNIVERSITY
PRAYAGRAJ

UGMM-104
DIFFERENTIAL EQUATION

Unit-6
Geometrical Applications of Differential Equations

Unit-7 Physical applications of differential equations of first order and first degree-I

Unit-8
Physical applications of differential equations of first order and first degree-II

Unit –06: Geometrical Applications of Differential Equations

Structure

6.1. Introduction

Differential equations are very important mathematical subject from both theoretical and practical perspectives. The theoretical importance is given by the fact that most pure mathematical theories have applications in differential equations.

The practical importance is given by the fact that the most important time dependent scientific, social and economical problems are described by differential equations. The bridge between nature (or universe) and us is provided by mathematical modelling, which is the process of finding the correct mathematical equations describing a certain problems. This process might start with experimental measurements and analysis, which leads to differential equations.

Many real-world problems, when formulated mathematically, lead to differential equations. We encountered a number of these equations in previous units when studying phenomena such as the motion of an object moving along a straight line, the simple harmonic motion of moving object, simple electrical circuits, heat flow of an object, the decay of a radioactive material, the growth of a population, and the cooling of a heated object placed within a medium of lower temperature.

In the previous units we introduced differential equations of the form $\frac{dy}{dx} = f(x)$, where f

is given and y is an unknown function of x. When f is continuous over some interval, we

learned that the general solution $y(x)$ was found directly by integration, $y = \int f(x)\,dx$. And we also investigated differential equations of the form $\frac{dy}{dx} = f(x, y)$, where f is a function of both the independent variable x and the dependent variable y. There we learned how to find the general solution when the differential equation is separable.

In this unit we further extend our study to include other commonly occurring *first-order*

differential equations. They involve only first derivatives of the unknown function $y(x)$, and

model phenomena such as simple electrical circuits, or the resulting concentration of a

chemical being added and mixed with some other fluid in a container.

In this unit, we shall consider only such practical problems which give rise to differential equations of the first order and first degree.

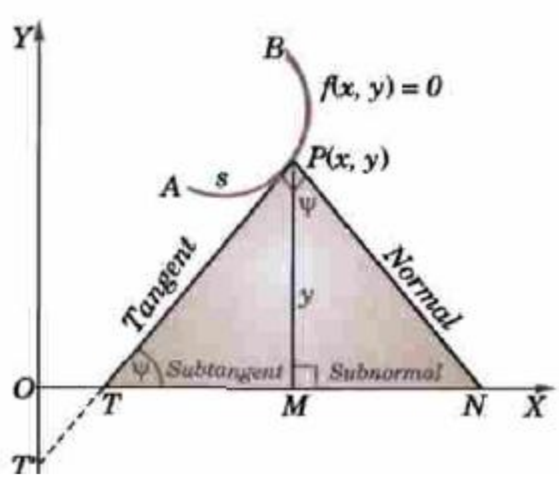
6.2. Objectives

After reading this unit students should be able to:

- Understand the geometrical applications of ODE's of first order and degree
- Understand the orthogonal trajectories
- Understand the oblique or isogonal trajectories

6.3. Geometrical Applications of Differential Equations of First Order and First Degree

(i) Cartesian Coordinates:



Let P(x, y) be any point on the curve $f(x, y) = 0$ (as shown in the figure).

Then, we have

1. Slope of the tangent at the point P(x, y) is $\tan\psi = \frac{dy}{dx}$

2. Equation of the tangent at the point P(x, y) is $Y - y = \frac{dy}{dx}(X - x)$ so that its

   x-intercept is $OT = x - y.\frac{dy}{dx}$ and y-intercept is $OT^{/} = y - x.\frac{dy}{dx}$.

3. Equation of the normal at the point P(x, y) is $Y - y = -\frac{dx}{dy}(X - x)$

4. Length of the tangent is $PT = y.\sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}$

5. Length of the normal is $PN = y.\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$

6. Length of the sub-tangent is $TM = y.\frac{dx}{dy}$

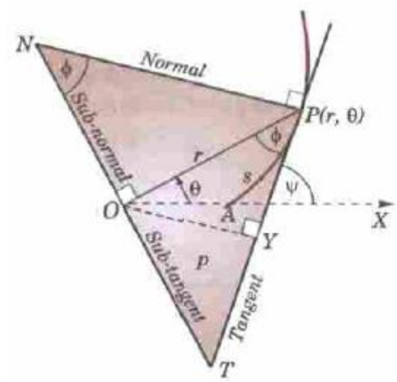
7. Length of the sub-normal is $MN = y.\frac{dy}{dx}$

8. $\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$ and $\frac{ds}{dy} = \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}$

9. Differential of the area = y.dx Or x.dy

10. Radius of the curvature at the point P(x, y) is $P = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

(ii)



Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$ (as shown in the figure). Then, we have

1. $\psi = \theta + \phi$

2. $\tan\phi = r.\dfrac{d\theta}{dr}$, $p = r\sin\phi$

3. $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

4. Polar sub-tangent is $OT = r^2 \cdot \dfrac{d\theta}{dr}$

5. Polar sub-normal is $ON = \dfrac{dr}{d\theta}$

6. $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]}$; $\dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

6.3.1. Some Basic illustrations to understand/study the importance of further geometrical applications:

Example 6.3.1: Find the equation of the curve which passes through the point (3, -4) and has the slope $\frac{2y}{x}$ at the point (x, y) on it.

Solution: It is given that, slope of the curve is $\frac{dy}{dx}=\frac{2y}{x}$

$$Or\ \ \frac{dy}{y}=\frac{2}{x}dx\ \ (Variable\ Seperable\ Form)$$

$$Integrating,\ \ we\ obtain$$

$$\log y = 2\log x + \log C$$

Where, C is the constant of integration.

$\therefore\ y = Cx^2$ …………. (1) is the equation of the curve

Since, this curve passes through the point (3, -4). Equation (1) gives

$$-4 = 9C\ Or\ C = -\frac{4}{9}$$

Therefore required equation of the curve is given by $y=-\frac{4}{9}x^2\ \ Or\ \ 4x^2+9y=0$.

Example 6.3.2: Find the equation of the curve which passes through the origin and has the slope $x+3y-1$.

Solution: It is given that, slope of the curve is $\frac{dy}{dx}=x+3y-1$ …………… (1)

$$Put\ \ x+3y-1=v$$

$$\Rightarrow 1+3\frac{dy}{dx}=\frac{dv}{dx}$$

$$Or\ \ \frac{dy}{dx}=\frac{1}{3}\left[\frac{dv}{dx}-1\right]$$

Substituting in equation (1), we obtain

$$\frac{1}{3}\left[\frac{dv}{dx}-1\right]=v$$

$$Or\ \frac{dv}{dx}=3v+1\ \ (Variable\ Separable\ form)$$

$$\therefore \frac{dv}{3v+1}=dx,\ Integrating$$

$$\frac{1}{3}\log(3v+1)=x+\log c$$

Where, ‘c’ is the constant of integration.

$$\therefore \quad \log\left[\frac{(3v+1)}{c^3}\right]=3x$$

$$Or\ \frac{3v+1}{c^3}=e^{3x}$$

$$Put\ v=x+3y-1$$

$$\frac{3x+9y-3}{c^3}=e^{3x} \text{..........................}(1)$$

Since, this curve passes through origin. Equation (1) gives $-3=c^3\ Or\ c_1=-3$

Therefore required equation of the curve is given by

$3x+9y-3=-3e^{3x}\ \ Or\ x+3y-1=-e^{3x}$.

Example 6.3.3: At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa and the curve passes through (0, 1). Find the equation of the curve. **Solution:** It is given that, slope of the curve is $\frac{dy}{dx}=x+xy\ \ Or\ \frac{dy}{dx}=x(1+y)$

$$Or \quad \frac{dy}{1+y} = xdx \quad (Variable\ Seperable\ Form)$$

$Integrating, \ we\ obtain$

$$\log(1+y) = \frac{x^2}{2} + \log C$$

Where, C is the constant of integration.

$\therefore \ \frac{(1+y)}{C} = e^{\frac{x^2}{2}}$ ………… (1) is the equation of the curve

Since, this curve passes through the point (0, 1). Equation (1) gives

$$\frac{2}{C} = 1 Or\ C = 2$$

Therefore required equation of the curve is given by

$$y+1 = 2e^{x^2/2} \quad Or \quad y = 2e^{x^2/2} - 1.$$

Example 6.3.4: A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

Solution: Let the equation of the tangent at P(x, y) is given by

$$Y - y = \frac{dy}{dx}(X - x).$$

Since the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P.

$$\Rightarrow \quad \frac{x\frac{dy}{dx}-y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=x$$

$$Or \quad \left(x\frac{dy}{dx}-y\right)^2=x^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$Or \quad x^2\left(\frac{dy}{dx}\right)^2+y^2-2xy\frac{dy}{dx}=x^2+x^2\left(\frac{dy}{dx}\right)^2$$

$$Or \quad y^2-x^2-2xy\frac{dy}{dx}=0$$

This is the required differential equation of the curve.

Now, to find the equation of the curve, consider $y^2-2xy\frac{dy}{dx}-x^2=0$

$\therefore \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$ ....(1), it is homogeneous ODE of first order and first degree.

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$, then equation (1) gives

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2x^2v}$$

$$\therefore x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{v^2-2v^2-1}{2v}=\frac{-v^2-1}{2v}$$

$$\therefore \frac{2v}{v^2+1}dv=-\frac{dx}{x} \ (Variable \ Seperable \ form)$$

Integrating, we obtain

$$\log(v^2+1)=-\log x+\log c$$

Where, ‘c’ is the constant of integration.

$$Or\ \log(v^2+1)=\log\left(\frac{c}{x}\right)$$

$$Or\ (v^2+1)=\frac{c}{x}\ Or\ x(v^2+1)=c$$

$$Substitute\ v=\frac{y}{x}$$

$$\therefore\left(\frac{y^2}{x^2}+1\right)x=c\ Or\ x^2+y^2=cx$$

This is the required equation of the curve.

Example 6.3.5: A plane curve has the property that the tangents from any point on the Y-axis to the curve are of constant length 'a'. Find the differential equation of the family to which the curve belongs and hence obtain the curve.

Solution: Equation of the tangent at the point P(x, y) is $Y-y=\frac{dy}{dx}(X-x)$.

Since $X=0$.

$$\therefore Y-y=\frac{dy}{dx}(0-x)\Rightarrow Y=y-x\frac{dy}{dx}$$

The point on Y-axis is $\left(0,\ y-x\frac{dy}{dx}\right)$.

The tangents from any point on the Y-axis to the curve are of constant length 'a'. This implies that

$$\sqrt{(0-x)^2+\left(y-x\frac{dy}{dx}-y\right)^2}=a\Rightarrow\left(x^2+x^2\left(\frac{dy}{dx}\right)^2\right)=a^2$$

$$\therefore \quad x^2\left(\frac{dy}{dx}\right)^2 = a^2 - x^2$$

$$Or\ \frac{dy}{dx} = \frac{\sqrt{a^2 - x^2}}{x} \ (Variable\ Seperable\ \ form)$$

$$\therefore\ dy = \frac{\sqrt{a^2 - x^2}}{x} dx,\ \text{int}\, egrating$$

$$y\ = \sqrt{a^2 - x^2} + a\log\frac{a - \sqrt{a^2 - x^2}}{x} + c$$

Example 6.3.6: Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point (1, 2).

Solution: Let P(x, y) be the point on the curve. The sub tangent at P(x, y) is

$$y.\frac{dx}{dy}$$

$$y.\frac{dx}{dy} = 2x\ (Variable\ Seperable\ \ form)$$

Since $\therefore \frac{dx}{x} = 2\frac{dy}{y}$, $\text{int}\, egrating$

$$\therefore \log x = 2\log y + \log c\ \ Or\ \ x = cy^2\ \ ....(1)$$

This is the required equation of the curve. Since, this curve passes through the point (1, 2).

From equation (1), we get $1 = 4c\ Or\ c = \frac{1}{4}$.

Hence, the equation of the curve is $y^2 = 4x$.

Example 6.3.7: Determine the curve in which the length of the subnormal is proportional to the square of the ordinate.

Solution: Let P(x, y) be the point on the curve. The sub normal at P(x, y) is $y.\frac{dy}{dx}$.

Since $y.\frac{dy}{dx} \alpha\, y^2 \Rightarrow y.\frac{dy}{dx} = k\, y^2$, where 'k' is constant of proportionality.

$$\therefore \frac{dy}{y} = k\, dx \;\; (Variable\;\; Seperable\;\; form)$$

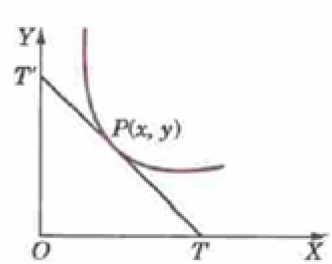
Integrating, we obtain

$$\log y = kx + \log c \;\; Or \;\; y = ce^{kx}$$

Where, 'c' is the constant of integration.

Example 6.3.8: Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.

Solution: Let the tangent at any point P(x, y) of a curve cut the axes at T and T' as shown in the following figure.



Since, its x-intercept is $OT = x - y.\frac{dy}{dx}$

And its y-intercept is $OT^{/} = y - x.\frac{dy}{dx}$

Therefore, the co-ordinates of T and $T^{/}$ are $\left(x - y\cdot\frac{dx}{dy}, 0\right)$ *and* $\left(0, y - x\cdot\frac{dy}{dx}\right)$

Since P is the mid-point of $TT^{/}$.

$$\therefore \frac{\left(x - y\cdot\frac{dx}{dy}\right) + 0}{2} = x$$

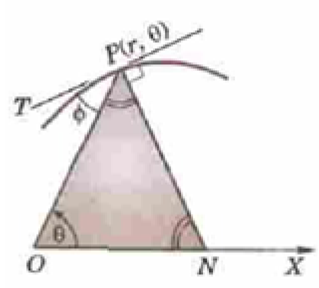
$$Or\ x - y\cdot\frac{dx}{dy} = 2x\ Or\ x\cdot dy + y\cdot dx = 0$$

$$Or\ d(xy) = 0. Integrating,\ xy = c$$

This represents the equation of the rectangular hyperbola, having 'x' and'y' axes as its asymptotes.

Example 6.3.9: Find the curve for which the normal makes equal angles with the radius vector and the initial line.

Solution: Let PT and PN be the tangent and normal at the point $P(r,\theta)$ of the curve so that $\tan\phi = r\cdot\frac{d\theta}{dr}$. Since $\angle OPN = 90^0 - \phi = \angle ONP$ (as shown in the following figure).



$\therefore \quad \theta = \angle PON = 180^0 - (180^0 - 2\phi) = 2\phi$

$Or \ \frac{\theta}{2} = \phi \ \ \therefore \tan\frac{\theta}{2} = \tan\phi = r \cdot \frac{d\theta}{dr}$ (*Variable Seperable Form*)

$\therefore \ \frac{dr}{r} = \cot\theta \cdot d\theta$. *Integrating, we obtain*

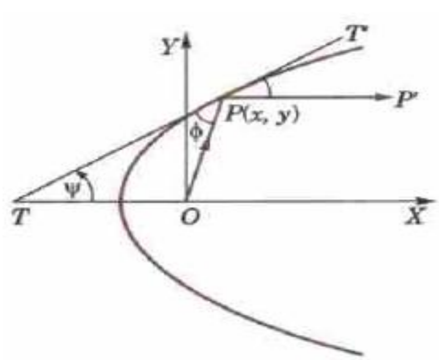
$$\log r = 2\log\sin\frac{\theta}{2} + \log c$$

$Or \ \ r = c\sin^2\frac{\theta}{2} = \frac{1}{2}c(1-\cos\theta)$

Thus, the curve is the cardioid $r = a(1-\cos\theta)$.

Example 6.3.10: Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.

Solution: In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.



If $TPT^{/}$ be the tangent at P, then angle of incidence = angle of reflection.

$$\therefore\ \phi = \angle OPT = \angle P^{/}PT^{/} = \angle OTP = \psi$$

$$i.e.,\ p = \frac{dy}{dx} = \tan\angle XOP = \tan 2\phi$$

$$= \frac{2\tan\phi}{1-\tan^2\phi} = \frac{2p}{1-p^2}$$

$$Or\ \ 2x = \frac{y}{p} - yp\ \ which\ is\ solvable\ for\ x\ ....(1)$$

Now, differentiating equation (1) w. r.t. ‘y’, we obtain

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - p - y\cdot\frac{dp}{dy}$$

$$i.e.,\ \left(\frac{1}{p} + p\right) + \left(\frac{1}{p^2} + 1\right)y\frac{dp}{dy} = 0$$

$$Or\ \ \left(\frac{1}{p^2} + 1\right)y\frac{dp}{dy} = -\left(\frac{1}{p} + p\right)$$

$$Or\ y\frac{dp}{dy} = -\frac{(1+p^2)/p}{(1+p^2)/p^2} = p$$

$$\therefore\ \ \frac{dp}{p} = -\frac{dy}{y},\ integrating,\ we\ obtain$$

$$\log p = -\log y + \log c$$

$$Or\ \ p = \frac{c}{y}\ \ ....(2)$$

Now, eliminating ‘p’ from equations (1) and (2), we obtain

$$2x = \frac{y}{p} - yp \Rightarrow 2x = \frac{y^2}{c} - c\ \ Or\ \ y^2 - c^2 = 2cx\ \ Or\ \ y^2 = c^2 + 2cx$$

Hence the reflector is the member of the family of paraboloids of revolution

$y^2 + z^2 = c^2 + 2cx$.

6.3. Test Your Progress

(i) The tangent at any point of a certain curve forms with the ordinate axes a triangle of constant area A. Find the equation to the curve.

(ii) Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the X-axis is twice the cube of that ordinate.

(iii) Find the curve whose

(a) Polar sub-tangent is constant

(b) Polar sub-normal is proportional to the sine of the victorial angle.

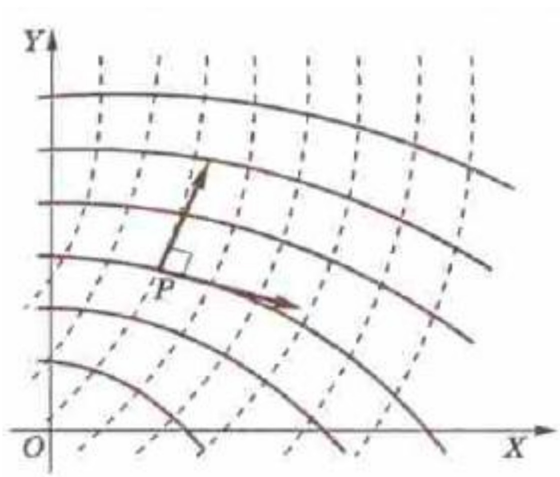
(iv) Determine the curve for which the angle between the tangent and the radius vector is twice the victorial angle.

(v) Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.

(vi) Find the curve for which the tangent, the radius vector 'r' and the perpendicular from the origin on the tangent form a triangle of area kr^2.

6.4. Orthogonal Trajectories

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and vice versa. In fluid flow, the stream lines and equipotential lines (line of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.



Definition 6.4.1:

An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family at right angles, or orthogonally (as shown in the following figure 1).

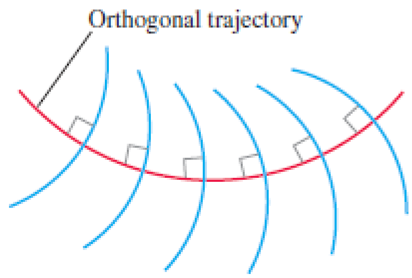


Figure 1: An orthogonal trajectory intersects the family of curves at right angles, or orthogonally

For instance, each straight line through the origin in an orthogonal trajectory of family of circles $x^2 + y^2 = a^2$, centred at the origin is an orthogonal trajectory (as shown in the following figure 2).

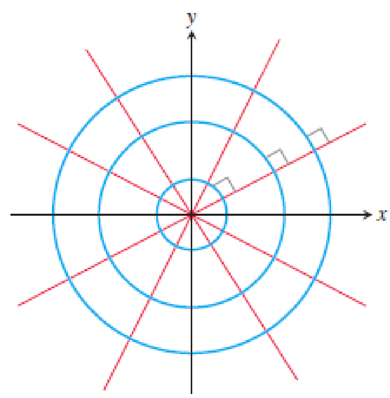


Figure 2: Every straight line through the origin is orthogonal to the family of circles centred at the origin.

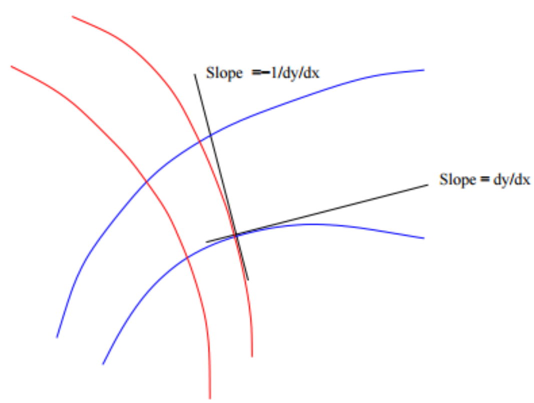
Such mutually orthogonal system of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to strength of an electric field and those in the other family correspond to constant electric potential. They also occur in hydrodynamics and heat-flow problems.

Equivalently,

Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical, then we say that the family is self-orthogonal.

Note: Orthogonal trajectories have important applications in the field of physics.

For example: The equipotential lines and the streamlines in an irrotational two dimensional flow are orthogonal.



Orthogonal Trajectories

6.4.1. To find the orthogonal trajectories of the family of curves F(x, y, c)=0

[i.e., Cartesian Form]

i. Form its differential equation in the form $f\left(x, y, \frac{dy}{dx}\right) = 0$ by eliminating 'c'.

ii. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation, (so that the product of their slopes at each point of intersection is -1).

iii. Solve the differential equation of the orthogonal trajectories of the form $f\left(x, y, -\frac{dx}{dy}\right) = 0$.

Example 6.4.1: Find the orthogonal trajectories of family of straight lines through the origin.

Solution: The family of straight lines through the origin is given by,

$$y = mx \quad \text{............. ...} \quad (1)$$

Differentiating equation (1) w. r. t $'x'$, we obtain

$$\frac{dy}{dx} = m \quad \text{..................} \quad (2)$$

Eliminating 'm' from equations (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{x} \;\; Or \;\; xy' - y = 0 \quad \text{........................} \quad (3)$$

This represents the ODE for the family of straight lines represented by the equation (1).

Now, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation represented by the equation (3).

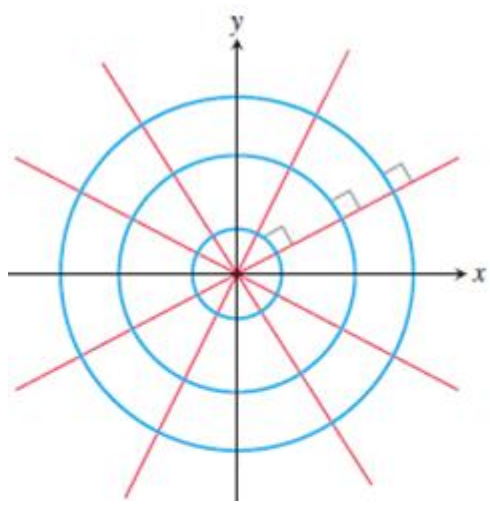
$$-\frac{dx}{dy} = \frac{y}{x} \quad Or \quad xdx = -ydy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

This represents the ODE for the orthogonal family of straight lines represented by the equation (1).

Integrating equation (4), we obtain

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c^2}{2} \quad Or \quad x^2 + y^2 = c^2$$

This represents the family of circles with centre at the origin (as shown in the following figure).



Example 6.4.2: Find the orthogonal trajectories of family of curves $xy = a$,

$a \neq 0$ is an arbitrary constant.

OR

If the stream lines (paths of fluid particles) of a flow around a corner are

$xy = constant$, find their orthogonal trajectories (called equipotential lines).

Solution: The given family of curves $xy = a$ represents the family of parabolas

,

$$xy = a \quad \text{............ ...} \quad (1)$$

Differentiating equation (1) w. r. t $'x'$, we obtain

$$x\frac{dy}{dx} + y = 0 \quad \text{.................} \quad (2)$$

This represents the ODE for the family of straight lines represented by the equation (1).

Now, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation represented by the equation (2).

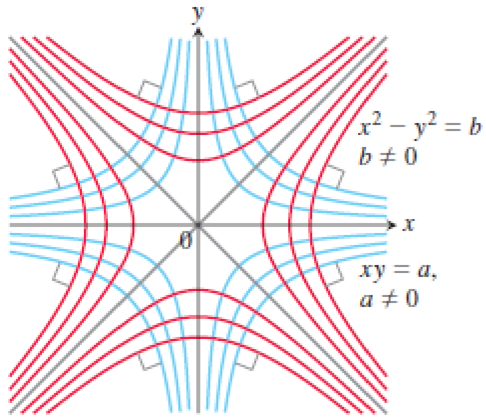
$$-x\frac{dx}{dy} + y = 0 \;\; Or \;\; xdx = ydy \quad \text{.....................} \quad (3)$$

This represents the ODE for the orthogonal family of straight lines represented by the equation (1).

Integrating equation (3), we obtain

$$\frac{x^2}{2}-\frac{y^2}{2}=\frac{c^2}{2} \quad Or \quad x^2-y^2=c$$

This represents the family of hyperbolas (as shown in the following figure).



Example 6.4.3: Find the orthogonal trajectories of family of semicubical parabola $ay^2 = x^3$.

Solution: The given family of curves $ay^2 = x^3$ represents the family of semi cubical parabolas ,

$$ay^2 = x^3 \quad .... \quad (1)$$

Differentiating equation (1) w. r. t $'x'$, we obtain

$$2ay\frac{dy}{dx} = 3x^2 \quad .... \quad (2)$$

Eliminating ‘a’ from equations (1) and (2), we get

$$2y\left(\frac{x^3}{y^2}\right)\frac{dy}{dx} = 3x^2 \quad Or \quad \frac{2x}{y}\frac{dy}{dx} = 3 \quad .... \quad (3)$$

This represents the ODE for the family of semi cubical parabolas represented by the equation (1).

Now, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation represented by the equation (3).

$$-\frac{2x}{y}\frac{dx}{dy}=3 \quad (Variable\ Seperable\ Form)$$
$$\therefore \quad -2xdx=3ydy.\ Integrating,\ we\ obtain,$$
$$-2\left(\frac{x^2}{2}\right)+\frac{c^2}{2}=3\left(\frac{y^2}{2}\right)$$

$$\text{Or} \quad 2x^2+3y^2=c^2 \quad \text{............................} \quad (4)$$

This represents the family of circles with centre at the origin (as shown in the following figure).

Example 6.4.4: Find the orthogonal trajectories of system of confocal and coaxial parabolas.

Solution: The equation of family of confocal parabolas having X-axis as their axis is given by,

$$y^2=4a(x+a) \quad \text{............. ...} \quad (1)$$

Differentiating equation (1) w. r. t $'x'$, we obtain

$$2y\frac{dy}{dx}=4a \quad Or \quad y\frac{dy}{dx}=2a \quad \text{..................} \quad (2)$$

Eliminating ‘a’ from equations (1) and (2), we get

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right) \; Or \;\; y = 2\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$$Or \;\; 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 - y = 0 \; \text{........................} \;\; (3)$$

This represents the ODE for the family of confocal parabolas represented by the equation (1).

Now, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation represented by the equation (3).

$$2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2 - y = 0 \;\; Or \;\; -2x\frac{dx}{dy} + y\left(\frac{dx}{dy}\right)^2 - y = 0$$

$$Or \;\; y\left(\frac{dx}{dy}\right)^2 - 2x\frac{dx}{dy} - y = 0 \; \text{......................} \;\; (4)$$

This is same as equation (3)

Thus, we see that a system of confocal and coaxial parabolas is self-orthogonal, each member of the family (1) cuts every other member of the same family orthogonally.

Example 6.4.5: Find the orthogonal trajectories of family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, where λ is the parameter.

Solution: We have $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$ .... (1)

Differentiating equation (1), we get

$$\frac{2x}{a^2}+\frac{2y}{a^2+\lambda}\frac{dy}{dx}=0 \quad Or \quad \frac{x}{a^2}+\frac{y}{a^2+\lambda}\frac{dy}{dx}=0$$

$$Or \;\; \frac{y}{a^2+\lambda}=-\frac{x}{a^2(dy/dx)} \;\; Or \;\; \frac{y^2}{a^2+\lambda}=\frac{-xy}{a^2(dy/dx)} \;\; \text{.....................} \; (2)$$

To eliminate 'λ' , Substituting this in the equation (1), we get

$$\frac{x^2}{a^2}-\frac{xy}{a^2(dy/dx)}=1 \quad Or \quad \frac{x^2}{a^2}=1+\frac{xy}{a^2}\frac{dx}{dy}$$

$$Or \;\; x^2=a^2+xy\frac{dx}{dy} \qquad Or \quad (x^2-a^2)\frac{dy}{dx}=xy \;\; \text{.......................} \; (3)$$

This is the differential equation of the given family of curves represented by the equation (1).

Now, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in this differential equation represented by the equation (3), we get

$$-(x^2-a^2)\frac{dx}{dy}=xy \;\; Or \;\; (a^2-x^2)\frac{dx}{dy}=xy \;\; (Variable\ Seperable\ Form)$$

$$\therefore \; \frac{(a^2-x^2)}{x}dx=y\,dy. \; Integrating, We\ Obtain$$

$$a^2\log x-\frac{x^2}{2}=\frac{y^2}{2}+c$$

This is the equation of the required orthogonal trajectories.

6.4 Test Your Progress	
i.	Find the orthogonal trajectories of family of parabolas $y^2=4ax$.

ii. Find the orthogonal trajectories of family of parabolas $y = ax^2$.

iii. Find the orthogonal trajectories of family of Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, 'λ' being the parameter.

iv. Find the orthogonal trajectories of family of Confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, 'λ' being the parameter.

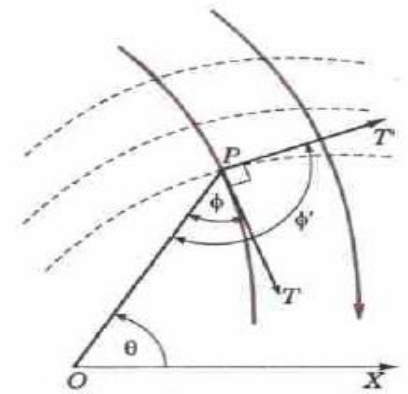
v. Show that the family of parabolas $x^2 = 4a(y+a)$ is self orthogonal.

vi. The electric lines of force of two opposite charges of the same strength at (±1, 0) are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their equipotential lines (orthogonal trajectories).

6.5. To find the orthogonal trajectories of the family of curves $F(r,\theta,c) = 0$ [i.e., Polar Form]

i. Form its differential equation in the form $f\left(r,\theta,\frac{dr}{d\theta}\right) = 0$ by eliminating 'c'.

ii. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in this differential equation.



$$\left[\begin{array}{l} \because \text{ for the given curve through } P(r,\theta),\ \tan\phi = r.\dfrac{d\theta}{dr} \\ \text{and for the orthogonal trajectory through } P\ \tan\phi^{/} = \tan(90^0+\phi) = -\cot\phi = -\dfrac{1}{r}\dfrac{dr}{d\theta} \\ \text{Thus fo getting the differenti al equation of the orthogonal trajectory} \\ r\cdot\dfrac{d\theta}{dr} \text{ is to be replaced by } -\dfrac{1}{r}\cdot\dfrac{dr}{d\theta} \text{ Or } \dfrac{dr}{d\theta} \text{ is to be replaced by } -r^2\cdot\dfrac{d\theta}{dr} \end{array}\right]$$

iii. Solve the differential equation of the orthogonal trajectories of the form $f\left(r,\theta,-r^2\dfrac{d\theta}{dr}\right)=0$.

Example 6.5.1: Find the orthogonal trajectories of family of straight lines through the origin in polar form.

Solution: The family of straight lines through the origin is given by,

$$\theta = A \quad .... \quad (1)$$

Differentiating equation (1) w. r. t $'r'$, we obtain

$$\frac{d\theta}{dr}=0 \ Or \ d\theta=0 \quad .... \quad (2)$$

This represents the ODE for the family of straight lines represented by the equation (1).

Now, replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in this differential equation represented by the equation (2).

$$-\frac{1}{r^2 \frac{d\theta}{dr}} = 0 \;\; Or \;\; dr = 0 \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; (3)$$

This represents the ODE for the orthogonal family of straight lines represented by the equation (1).

Integrating equation (3), we obtain, $r = c$.

This represents the family of circles with centre at the origin.

Example 6.5.2: Find the orthogonal trajectories of the cardioides $r = a(1 - \cos\theta)$.

Solution: Given equation of the cardioide is,

$$r = a(1 - \cos\theta) \;\; \ldots\ldots\ldots\ldots \ldots \;\; (1)$$

Differentiating equation (1) w. r. t $'\theta'$, we obtain

$$\frac{dr}{d\theta} = a\sin\theta \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; (2)$$

This represents the ODE for the family of straight lines represented by the equation (1).

Now, eliminating ‘a’ from equations (1) and (2), we obtain

From equation (1), $r = a(1-\cos\theta)\ \ Or\ a = \dfrac{r}{1-\cos\theta}$

Equation (2) gives,

$$\frac{dr}{d\theta} = \frac{r\sin\theta}{1-\cos\theta} = \frac{2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = r\cot\frac{\theta}{2}$$

$$\therefore\ \frac{dr}{d\theta} = r\cot\frac{\theta}{2}$$

Now, replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in this differential equation represented by the equation (2).

$$-r^2\frac{d\theta}{dr} = r\cot\frac{\theta}{2}\ \ Or\ -r\frac{d\theta}{dr} = \cot\frac{\theta}{2}\ (Variable\ Separable\ Form)$$

This represents the ODE for the orthogonal family of cardioides represented by the equation (1).

$$\therefore\ \tan\frac{\theta}{2}\,d\theta = -\frac{dr}{r} \quad .... \ (3)$$

Integrating equation (3), we obtain,

$$2\log\cos\frac{\theta}{2} + \log r = \log c$$

$$Or\ \ \log r = 2\log\cos\frac{\theta}{2} + \log c$$

$$Or\ \ r = c\cos^2\theta/2 = \frac{c}{2}(1+\cos\theta)$$

$$Or\ \ r = a^{/}(1+\cos\theta)$$

This represents the required orthogonal trajectory.

Example 6.5.3: Find the orthogonal trajectories of the family of $r^n = a\sin n\theta$.

Solution: Given equation of the family of curves is,

$$r^n = a\sin n\theta$$

Taking 'log' on both sides, we get

$$n\log r = \log a + \log\sin n\theta \quad \ldots\ldots\ldots\ldots\ \ldots \quad (1)$$

Differentiating equation (1) w. r. t $'\theta'$, we obtain

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta} = \cot n\theta$$

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

This represents the ODE for the family of the curves represented by the equation (1).

Now, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in this differential equation represented by the equation (2).

$$\therefore \quad \frac{1}{r}\left(-\frac{r^2 d\theta}{dr}\right) = \cot n\theta$$

$$Or\ -r\frac{d\theta}{dr} = \cot n\theta \ \ (Variable\ Separable\ Form)$$

$$\therefore \tan n\theta\, d\theta = -\frac{dr}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Integrating equation (3), we obtain,

$$-\frac{1}{n}\log\cos n\theta + \log r = \log c$$

$$Or \quad -\log\cos n\theta + n\log r = n\log c$$

$$Or \quad \frac{r^n}{\cos n\theta} = c^n$$

$$Or \; r^n = c^n \cos n\theta$$

This represents the required orthogonal trajectory.

Example 6.5.4: Find the orthogonal trajectories of the curves

$r = 2a(\cos\theta + \sin\theta)$.

Solution: Given equation of the cardioide is,

$$r = 2a(\cos\theta + \sin\theta) \quad .... \; ... \quad (1)$$

Differentiating equation (1) w. r. t $'\theta'$, we obtain

$$\frac{dr}{d\theta} = 2a(\cos\theta - \sin\theta) \quad .... \quad (2)$$

This represents the ODE for the family of straight lines represented by the equation (1).

Now, eliminating 'a' from equations (1) and (2), we obtain

From equation (1), $r = 2a(\cos\theta + \sin\theta) \; Or \; 2a = \dfrac{r}{(\cos\theta + \sin\theta)}$

Equation (2) gives,

$$\frac{dr}{d\theta} = \frac{r(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)} \quad .... \quad (3)$$

Now, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in this differential equation represented by the equation (3).

$$-r^2\frac{d\theta}{dr}=\frac{r(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}$$
$$Or\quad -r\frac{d\theta}{dr}=\frac{(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}\ (Variable\ Separable\ Form)$$

This represents the ODE for the orthogonal family of cardioides represented by the equation (1).

$$\therefore\quad \frac{(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)}d\theta=-\frac{dr}{r}\quad ....\ (3)$$

Integrating equation (3), we obtain,

$$-\log(\cos\theta-\sin\theta)=-\log r+\log c$$
$$Or\quad \log\left(\frac{r}{\cos\theta-\sin\theta}\right)=\log c$$
$$Or\quad r=c(\cos\theta-\sin\theta)$$

This represents the required orthogonal trajectory.

Example 6.5.5: Find the orthogonal trajectories of $r=\frac{2a}{1+\cos\theta}$.

Solution: Given equation of the curve is,

$$r=\frac{2a}{1+\cos\theta}\quad ....\ (1)$$

Differentiating equation (1) w. r. t 'θ', we obtain

$$\frac{dr}{d\theta}=-\frac{2a}{(1+\cos\theta)^2}\cdot(-\sin\theta)=\frac{2a\sin\theta}{(1+\cos\theta)^2}\quad ....\ (2)$$

This represents the ODE for the family of curves represented by the equation (1).

Now, eliminating ‘a’ from equations (1) and (2), we obtain,

From equation (1), $r = \frac{2a}{1+\cos\theta} \quad Or \quad 2a = r(1+\cos\theta)$

Equation (2) gives,

$$\frac{dr}{d\theta} = \frac{r(1+\cos\theta)\sin\theta}{(1+\cos\theta)^2} = \frac{r\sin\theta}{1+\cos\theta}$$

$$Or \quad \frac{dr}{d\theta} = \frac{2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = r\tan\frac{\theta}{2} \quad \text{............} \quad (3)$$

Now, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in this differential equation represented by the equation (3).

$$-r^2\frac{d\theta}{dr} = r\tan\frac{\theta}{2}$$

$$Or \quad -r\frac{d\theta}{dr} = \tan\frac{\theta}{2} \quad (Variable\ Separable\ Form)$$

This represents the ODE for the orthogonal family of curves represented by the equation (1).

$$\therefore \quad \cot\frac{\theta}{2}d\theta = -\frac{dr}{r} \quad \text{.....................} \quad (4)$$

Integrating equation (4), we obtain,

$$2\log\sin\frac{\theta}{2} = -\log r + \log c$$

$$Or \quad \sin^2\frac{\theta}{2} = \frac{c}{r}$$

$$Or \quad \frac{(1-\cos\theta)}{2} = \frac{c}{r}$$

$$Or \;\; r = \frac{2c}{1-\cos\theta}$$

This represents the required orthogonal trajectory.

6.5 Test Your Progress
i. Find the orthogonal trajectories of family of cardioides $r = a(1+\cos\theta)$.
ii. Find the orthogonal trajectories of the family of curves $r^2 = a^2\cos 2\theta$.
iii. Find the orthogonal trajectories of the family of curves $r^n \cos n\theta = a^n$.
iv. Show that the family of curves $r^n = a\sec n\theta$ and $r^n = b\cos ecn\theta$ are orthogonal.

6.6. Isogonal Trajectories (Or Oblique Trajectories)

Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (say), are called isogonal or oblique trajectories of each other. The slopes m_1 and m_2 of the tangents to the corresponding curves at each point are connected by the relation

$\frac{m_1 - m_2}{1 + m_1 m_2} = \tan\alpha = constant$ (as shown in the figure).

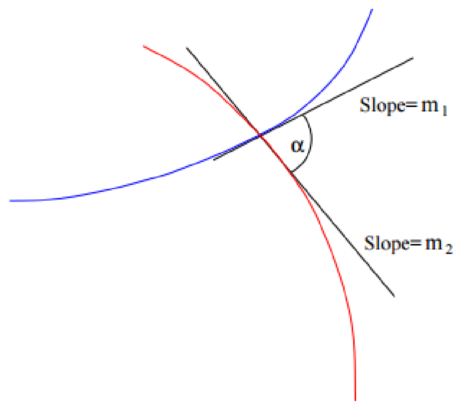


Figure: Oblique (Or Isogonal)Trajectories

In other words,

Here the two families of curves intersect at an arbitrary angle α 6= π/2.

Suppose the first family be

$f(x, y, c) = 0$ .... (1).

To find the oblique trajectories of this family we proceed as follows. First,

differentiate (1) w.r.t. 'x' to find $f_1(x, y, y', c) = 0$ .... (2).

Eliminate 'c' between the equations (1) and (2) to find the differential

equation ,

$f_2(x, y, y') = 0$ .... (3).

If m_1, is the slope of this family, then we write equation (3) as

$f_2(x, y, m_1) = 0$ .... (4).

Let m_2 be the slope of the second family. Then,

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \pm \tan\alpha .$$

Thus, we find $m_1 = \dfrac{m_2 \pm \tan\alpha}{1 \pm m_2 \tan\alpha}$.

Hence, from equation (4), the ODE for the second family satisfies

$$f_2\left(x,\ y,\ \frac{m_2 \pm \tan\alpha}{1 \mp m_2 \tan\alpha}\right) = 0$$

Replacing m_2 by y^1, the ODE for the second family is written as

$$f_2\left(x,\ y,\ \frac{y' \pm \tan\alpha}{1 \mp y' \tan\alpha}\right) = 0 \quad \text{...........} \quad (5)$$

General solution of equation (5) gives the required oblique trajectories.

Note: If we let $\alpha \to \dfrac{\pi}{2}$, we obtain the orthogonal trajectories.

Example 6.6.1: Find the oblique trajectories of the family of circles

$x^2 + y^2 = a^2$, which intersect at 45^0.

Solution: Given family of circles is $x^2 + y^2 = a^2$ ……………….(1)

Differentiate the equation (1) w. r. t 'x'. We obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad Or \quad \frac{dy}{dx} = -\frac{x}{y} \quad Or \quad y' = -x/y .$$

For the oblique (Or isogonal) trajectories, we replace

$$y' = \frac{y' \pm \tan(\pi/4)}{1 \mp y' \tan(\pi/4)} = \frac{y' \pm 1}{1 \mp y'}.$$

Thus, the ODE for the oblique (Or isogonal) trajectories is given by

$$\frac{y' \pm 1}{1 \mp y'} = -\frac{x}{y}$$

$$\therefore \frac{y' \pm 1}{1 \mp y'} = -\frac{x}{y} \Rightarrow y(y' \pm 1) = -x(1 \mp y')$$

$$\therefore (x+y)y' = -(x+y) \quad Or \quad (x+y)y' = (y-x)$$

$$Thus,\ y' = -1 \qquad Or \quad y' = \frac{y-x}{x+y}$$

Integrating, we obtain

$$y = -x + c_1 \qquad Or \quad y' = y' = \frac{y-x}{x+y} \text{ (homogeneous ODE)}$$

Example 6.6.2: Find the oblique trajectories that intersects the family

$y = x + A$ at an angle of 60^0.

Solution: Given family if curves is $y = x + A$ .... (1)

Differentiate the equation (1) w. r. t ‘x’. We obtain

$$\frac{dy}{dx} = 1 \quad Or \quad y' = 1.$$

For the oblique (Or isogonal) trajectories, we replace

$$y' = \frac{y' \pm \tan(\pi/3)}{1 \mp y' \tan(\pi/3)} = \frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'}.$$

Thus, the ODE for the oblique (Or isogonal) trajectories is given by

$$\frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'} = 1$$

$$\therefore \; \frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'} = 1 \Rightarrow \; y' \pm \sqrt{3} = 1 \mp \sqrt{3}y'$$

$$\therefore (1+\sqrt{3})y' = 1-\sqrt{3} \quad Or \quad (1-\sqrt{3})y' = 1+\sqrt{3}$$

$$Thus, \; y' = \frac{1-\sqrt{3}}{1+\sqrt{3}} \qquad Or \quad y' = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Integrating, we obtain

$$y = \frac{1-\sqrt{3}}{1+\sqrt{3}}x + c_1 \qquad Or \quad y' = \frac{1+\sqrt{3}}{1-\sqrt{3}}x + c_2$$

6.7 Summary

In this unit, we studied Geometrical application of differential equation based on tangent, normal, subtangent etc. to find orthogonal trajectories in Cartesian and polar form, also to find oblique trajectories.

6.8 Terminal Questions

1) Find the Cartesian equation of the curve whose subtangent is constant.
2) Show that the parabola is the only curve in which the subnormal is constant.
3) Find the orghugant trajectories of the system of curve $\left(\frac{dy}{dx}\right) = \sqrt{\frac{a}{x}}$.

4) Find the orthogonal trajectories of $r = ae^{m\theta}$.

5) Find the equation of the family of oblique trajectories which cut the line $y = mx \ at \ 45^0$.

6.7 Answers to Exercises

6.3. Test Your Progress

(i) $y = ax + b$ **(ii)** $x = 3y^2$ **(iii) (a)** $r(\theta - \alpha) = c$ **(b)** $r = a + b\cos\theta$

(iv) $r^2 = a^2 \sin 2\theta$ **(v)** $c^2x^2 = 2cy + 1$ **(vi)** $r = ae^{\theta\cot\alpha}$

6.4. Test Your Progress

(i) $2x^2 + y^2 = c$ **(ii)** $x^2 + 2y^2 = c^2$ **(iii)** $x^2 + y^2 + 2\mu y - c = 0$

(iv) This system is self orthogonal **(vi)** $x^2 + y^2 + cx + 1 = 0$

6.5. Test Your Progress

(i) $r = c(1 - \cos\theta)$ **(ii)** $r^2 = c^2 \sin 2\theta$ **(iii)** $r^n n \sin\theta = c$

Unit –07: Physical applications of differential equations of first order and first degree

Structure

7.1. Introduction

Applied mathematics involves the relationships between mathematics and its applications. Often the type of mathematics that arises in applications is differential equations. Thus, the study of differential equations is an integral part of applied mathematics.

Differential equations are found in many areas of mathematics, science, and engineering. One may be surprised to see the way in which differential equations dominate the study of many aspects of science and engineering.

Differential equations are very important mathematical subject from both theoretical and practical perspectives. The theoretical importance is given by the fact that most pure mathematical theories have applications in differential equations.

Applied mathematics is said to have three fundamental aspects;

Firstly, the modelling process by which physical objects and processes are described by physical laws and mathematical formulations. Since so many physical laws involve rates of change (or the derivative), differential equations are often the natural language of science and engineering.

Secondly, the analysis of the mathematical problems that are posed. This involves the complete investigation of the differential equation and its solutions.

Thirdly, however, the mathematical solution of the differential equation does not complete the overall process. The interpretation of the solution of the differential equation in the contest of the original physical problem must be given, and the implications further analysed.

The practical importance is given by the fact that the most important time dependent scientific, social and economical problems are described by differential equations. The bridge between nature (or universe) and us is provided by mathematical modelling, which is the process of finding the correct mathematical equations describing a certain problems. This process might start with experimental measurements and analysis, which leads to differential equations.

Many real-world problems, when formulated mathematically, lead to differential equations. We encountered a number of these equations in previous units when studying phenomena such as the motion of an object moving along a straight line, the simple harmonic motion of moving object, simple electrical circuits, heat flow of an object, the decay of a radioactive material, the growth of a population, and the cooling of a heated object placed within a medium of lower temperature.

In the previous units we introduced differential equations of the form $\frac{dy}{dx} = f(x)$, where f

is given and y is an unknown function of x. When f is continuous over some interval, we

learned that the general solution $y(x)$ was found directly by integration, $y = \int f(x)\,dx$. And we also investigated differential equations of the form $\frac{dy}{dx} = f(x, y)$, where f is a function of both the independent variable x and the dependent variable y. There we learned how to find the general solution when the differential equation is separable.

In this unit we further extend our study to include other commonly occurring *first-order*

differential equations. They involve only first derivatives of the unknown function $y(x)$, and

model phenomena such as simple electrical circuits, or the resulting concentration of a

chemical being added and mixed with some other fluid in a container.

In this unit, we shall consider only such practical problems which give rise to differential equations of the first order and first degree.

Also, we present a sufficient number of applications to enable the students to understand how differential equations are used and to develop some feeling for the physical information they convey.

7.2. Objectives

After reading this unit students should be able to:

- Understand the physical applications of ODE's of first order and degree
- Apply Newton's law of cooling
- Understand to model the electrical circuits and solve the first order ordinary differential equations which arise.
- Understand the Chemical applications of ODE's of first order and degree

7.3. Physical Applications of Differential Equations of First Order and First Degree

In applications, the dependent variables are frequently functions of time, which we denote by 't'. Some applications such as Newton's Law of Cooling, Kirchhoff's Laws of Electric Circuits, Motion under Gravity, Rectilinear Motion, Simple Harmonic Motion, Rate of Growth or Decay, Heat flow are discussed here. In all these cases, modelling, analysis and interpretation are important.

7.4. Newton's Law of Cooling

Modelling the Newton's Law of Cooling, according to this law, the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

If 'θ_0' is the constant temperature of the surrounding medium and 'θ' is the temperature of an object at time 't'. Then,

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where 'k' is a constant.

Note: In case of process of heating, Newton's law is given by, $\frac{d\theta}{dt} = k(\theta - \theta_0)$, where 'k' is a constant.

Example 7.4.1: If the temperature of the air is 30^0 C and the substance cools from 100^0 C to 70^0 C in 15 minutes, find when the temperature will be 40^0 C.

Solution: If 'θ' be the temperature of the body at any time 't', then

$\frac{d\theta}{dt} = -k(\theta - 30)$, where 'k' is a constant.

$\frac{d\theta}{(\theta-30)} = -k\,dt$, Integrating, we obtain

$\log(\theta-30) = -kt + \log c, \quad \textit{where 'c' is the constant of integration}$

$Or \;\; (\theta-30) = ce^{-kt}$ .... (1)

When t = 0, θ = 100^0 C and when t=15, θ = 70^0.

From equation (1), we get $(100-30) = ce^{-k(0)} \Rightarrow c = 70$

And again from equation (1), $(70-30) = 70e^{-15k} \Rightarrow k = 0.0373077191$

Thus, equation (1) becomes, $(\theta-30) = 70e^{-(0.0373077191)t}$ .... (2)

Now, when θ = 40^0C, equation (2) gives,

$(40-30) = 70e^{-(0.0373077191)t} \Rightarrow t = 52.15 \text{ mins}$

Example 7.4.2: If the air is maintained at 30^0 C and the temperature of the body cools from 80^0 C to 60^0 C in 12 minutes. Find the temperature of the body after 24 minutes.

Solution: If ‘θ’ be the temperature of the body at any time ‘t’, then

$\frac{d\theta}{dt} = -k(\theta-30)$. Where ‘k’ is a constant.

$\frac{d\theta}{(\theta-30)} = -k\,dt$, Integrating, we obtain

$\log(\theta-30) = -kt + \log c, \quad \textit{where 'c' is the constant of integration}$

$Or \;\; (\theta-30) = ce^{-kt}$ .... (1)

When t = 0, θ = 80^0 C and when t=12, θ = 60^0.

From equation (1), we get $(80-30) = ce^{-k(0)} \Rightarrow c = 50$

Again from equation (1), $(60-30) = 50e^{-12k} \Rightarrow k = 0.0425688019$

Thus, equation (1) becomes, $(\theta - 30) = 50e^{-(0.0425688019)t}$ .... (2)

Now, when t = 24 mins, equation (2) gives,

$(\theta - 30) = 50e^{-(0.0425688019)24} \Rightarrow \theta = 48^0 C$

Example 7.4.3: A body originally at 80^0 C cools down to 60^0C in 20 minutes, the temperature of the air being 40^0 C. What will be the temperature of the body after 40 minutes from the original?.

Solution: If 'θ' be the temperature of the body at any time 't', then

$\frac{d\theta}{dt} = -k(\theta - 40)$, where 'k' is a constant.

$\frac{d\theta}{(\theta - 40)} = -k\,dt$, Integrating, we obtain

$\log(\theta - 40) = -kt + \log c, \quad where\ 'c'\ is\ the\ constant\ of\ integration$

$Or\ (\theta - 40) = ce^{-kt}$ .... (1)

When t = 0, θ = 80^0 C and when t=20, θ = 60^0.

From equation (1), we get $(80-40) = ce^{-k(0)} \Rightarrow c = 40$

Again from equation (1), $(60-40) = 40e^{-20k} \Rightarrow k = 0.034657359$

Thus, equation (1) becomes, $(\theta - 40) = 40e^{-(0.034657359)t}$ .... (2)

Now, when t = 40 mins, equation (2) gives,

$$(\theta - 40) = 40e^{-(0.034657359)40} \Rightarrow \theta = 50^0 C.$$

Example 7.4.4: A body is exposed to a constant temperature of 280 K. After 1 minute the temperature of the body is 350 K and after 5 minutes it is 310K. Find an expression for the temperature θ at time t. Sketch the graph of θ against t for t≥0.

Solution: If 'θ' be the temperature of the body at any time 't', then

$\frac{d\theta}{dt} = k(\theta - 280)$, where 'k' is a constant.

$\therefore \frac{d\theta}{(\theta - 280)} = k\,dt$, Integrating, we obtain

$$\log(\theta - 280) = kt + \log c, \quad \textit{where 'c' is the constant of integration}$$

$$Or\ (\theta - 280) = c\,e^{kt} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

When t = 60 Secs, θ = 350K and when t=300 Secs, θ = 310K.

From equation (1), we get $(350 - 280) = c\,e^{60k} \Rightarrow c = 70e^{-60k}$

Again from equation (1),

$$(310 - 280) = 70e^{-60k}\left(e^{300k}\right) \ Or \ e^{240k} = \frac{30}{70} \Rightarrow k = -0.003530407$$

Thus, equation (1) becomes, $(\theta - 280) = 70e^{(t-60)k} \ Or \ \theta = 280 + 86.49e^{-0.003530407t}$

Next, to sketch the graph of θ against t for t≥0, we need to find the value of θ at t = 0 :

$\theta = 280 + 86.49e^{0} = 280 + 86.49 = 366.49$.

The graph in the following figure shows that the temperature of the body will eventually reach very close to the temperature of the surroundings at 280 K. This is because 86.49 $86.49 \times e^{-(3.53\times10^{-3})}$ is the transient term and decays to zero as t gets large.

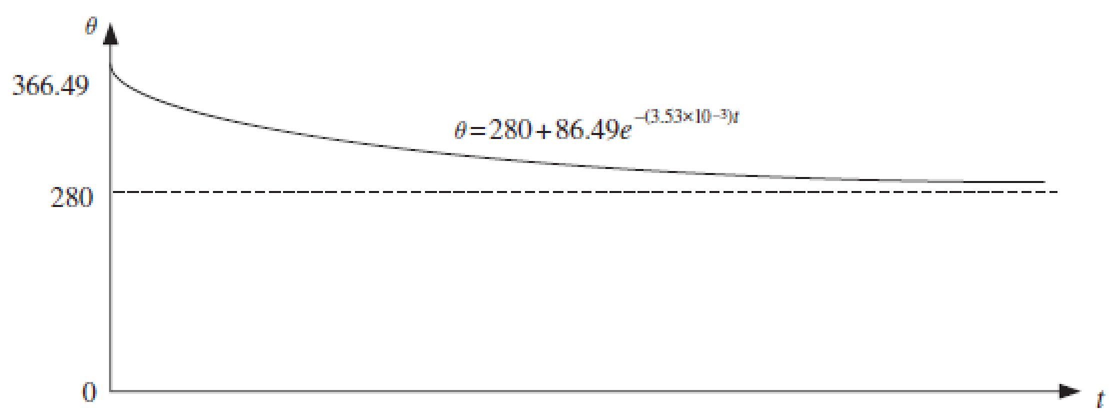


Figure 7.4.4

7.4. Test Your Progress

1) An object is initially at 400 K, and the constant surrounding temperature is 300 K. Determine an expression that gives the temperature $\theta = \theta(t)$ at time t.

2) Newton's law of cooling gives $\frac{d\theta}{dt} = k(\theta - T)$, where θ is the temperature at time t, T is the constant surrounding temperature and k is a constant. Given that $\theta(0) = T_0$. Show that $\theta = (T_0 - T)e^{kt} + T$.

3) A body is at a temperature of 373 K. After 5 minutes the temperature of the body is 330 K. Find an expression for $\theta = \theta(t)$ given that the constant surrounding temperature is 300 K. Sketch the graph of θ against t for $t \geq 0$. What does θ tends to as t tends to ∞.

4) By applying Newton's law of cooling to an object we obtain

$\frac{d\theta}{dt} = k(\theta - 320)$ where θ is the temperature at time t and k is a constant. Given that when t = 0, θ = 348 K, find an expression for θ.

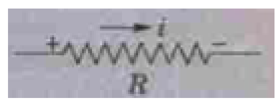
7.5. Simple Electrical Circuits (Kirchhoff's Laws of Electrical Circuits)

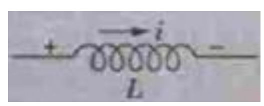
Here, we examine electrical engineering applications. In electrical principles *R*, *L* and *C* are constants representing resistance, inductance and capacitance respectively.

E represents the e.m.f. and $v = v(t),\ I = I(t)\ Or\ i = i(t)$ represents voltage and current respectively at time *t*.

We shall consider circuits made up of three passive elements: resistance, inductance and capacitance. An active element: voltage source which may be a battery or a generator.

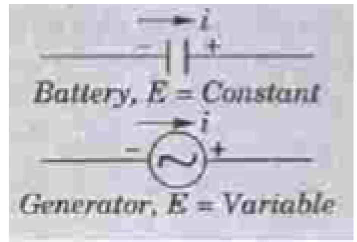
7.5.1. Symbols:

- Quantity of electricity : Symbol is Q (Or q): Unit of measurement is Coulomb.
- Current (" time rate flow of electricity"): Symbol is I (Or i) : Unit of measurement is ampere (A).
- Resistance (R) : Symbol is 

: Unit of measurement is ohm (Ω).
- Inductance (L) : Symbol is 

: Unit of measurement is henry (H).
- Capacitance (C) : Symbol is 

: Unit of measurement is farad (F).

- Electromotive force (e. m. f) Or voltage, E: 

: Unit of measurement is volt (V).
- Loop is any closed path formed by passing through two or more elements in series.
- Nodes are the terminals of any of these elements.

7.5.2. Ohm's Law: Current is the rate of flow of electricity.

- $V = IR\ (Ohm's\ Law)$. The voltage v, across an inductor of inductance L (figure 1), is given by

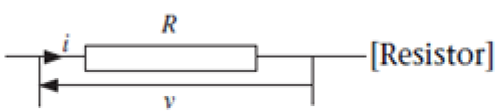


Figure. 1

- $V = L \cdot \frac{di}{dt}$, the voltage v, across an inductor of inductance L (Figure. 2), is given by

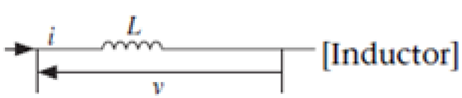


Figure. 2

- $i = C \cdot \frac{dv}{dt}$, the voltage v, across a capacitor of capacitance C (Figure 3), is related to the current.

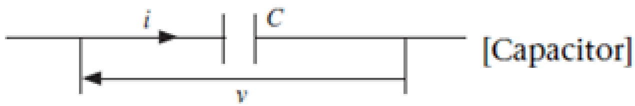


Figure. 3

Note: Remember that R, L and C are positive constant.

7.5.3. Basic Relations:

- $i = \frac{dq}{dt}$ Or $q = \int i \cdot dt$
- Voltage drop across the resistance $(R) = Ri$.
- Voltage drop across the inductance $(L) = L\frac{di}{dt}$
- Voltage drop across the inductance $(C) = \frac{q}{C}$

7.5.4. Kirchhoff's Laws:

1. The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.

2. The algebraic sum of the currents flowing into (or from) any node is zero.

Note: The formulation of differential equations for an electrical circuit depends on these two Kirchoff's laws which are of cardinal importance.

Below is the explanation of Kirchhoff's Law.

Sum of the voltage rises = sum of the voltage drops

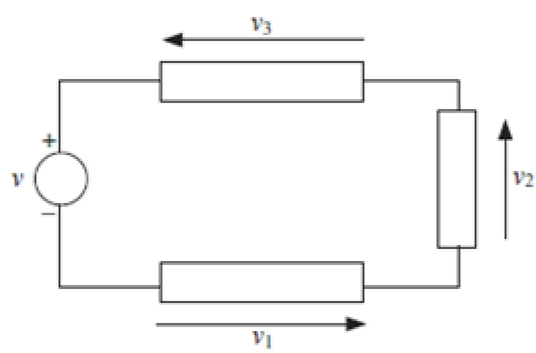
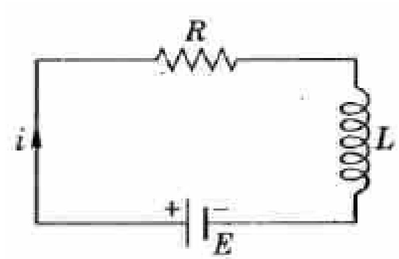


Figure. 4: Applied voltage to the circuit is v and v_1, v_2 and v_3 are voltage drops

With reference to the just above figure, Kirchhoff's law says; $v = v_1 + v_2 + v_3$

Now, we use these rules to form differential equations of electrical circuits.

7.5.5. Modelling of Electric circuit containing the resistance R and inductance L in series with a voltage source (battery) E.

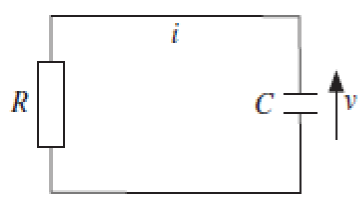


Let 'I' be the current flowing in the circuit at any time 't'. Then by Kirchhoff's first law, we have sum of voltage drops across R and L = E.

$$i.e., \; RI + L\frac{dI}{dt} = E \quad Or \quad L\frac{dI}{dt} + RI = E \quad Or \quad \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

This represents Leibnitz's linear equation.

7.5.6. Modelling of Electric circuit containing the resistance R and capacitance C in series with a voltage source (battery) E.

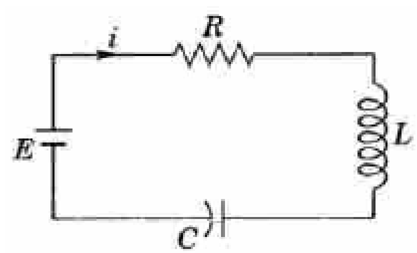


Let 'I' be the current flowing in the circuit at any time 't'. Then by Kirchhoff's first law, we have sum of voltage drops across R and C = E.

$$i.e., \; RI + \frac{1}{C}Q = E \quad Or \quad R\frac{dQ}{dt} + \frac{1}{C}Q = E \quad Or \quad \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E}{R}$$

This also represents Leibnitz's linear equation.

7.5.7. Modelling of Electric circuit containing the resistance R , inductance L and capacitance C in series with a voltage source (battery) E.

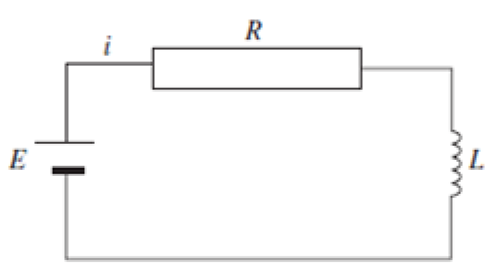


Let 'I' be the current flowing in the circuit at any time 't'. Then by Kirchhoff's first law, we have sum of voltage drops across R, L and C = E.

i.e., $RI + L\frac{dI}{dt} + \frac{1}{C}Q = E$ Or $L\frac{dI}{dt} + RI + \frac{1}{C}Q = E$ Or $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{E}{L}$

This represents the LDE of second order.

Example 7.5.1: With reference to the following figure, which consists of a resistor of resistance R, connected in series with an inductor of inductance L, and an applied constant voltage E.?



i. Obtain a first order differential equation for the current i at time ‘t’

ii. Solve this differential equation for the initial condition, when t = 0, i = 0.

iii. What is the value of i as t→∞.

iv. Sketch the graph of ‘i’ versus ‘t’ for t≥0.

Solution: By applying Kirchhoff’s law to the circuit, we have

i. Voltage drop across the resistance $(R) = Ri$.

Voltage drop across the inductance $(L) = L\frac{di}{dt}$

By applying the Kirchhoff’s law to the circuit, we obtain,

$$E = Voltage\ drop\ across\ resis\tan ce(R) + Voltage\ drop\ across\ induc\tan ce(L)$$

$$\therefore E = RI + L\cdot\frac{dI}{dt} \quad Or \quad L\cdot\frac{dI}{dt} + RI = E \quad Or \quad \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

This represents Leibnitz’s linear equation.

ii. To solve this equation,

$\therefore I.F = e^{\frac{R}{L}t}$, it’s general solution is given by,

$$I\cdot e^{\frac{R}{L}t} = \int \frac{E}{L}e^{\frac{R}{L}t}dt + c \Rightarrow I\cdot e^{\frac{R}{L}t} = \frac{E}{L}\cdot e^{\frac{R}{L}t}\cdot\left(\frac{L}{R}\right) + c$$

$$Thus,\ I = \frac{E}{R} + ce^{-\frac{R}{L}t} \quad .... (1)$$

Since, when t = 0, I = 0. Equation (1) gives, $c = \frac{E}{R}$.

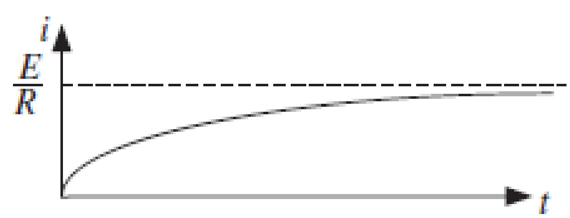
Thus, the general solution of the above differential equation is given by

$$I = \frac{E}{R}\left(1 + e^{-\frac{R}{L}t}\right).$$

iii. Since R and L are positive, we have

$$\frac{R}{L} > 0,\ e^{-\frac{R}{L}t} \to 0 \ as \ t \to \infty \ and \ so \ I \to \frac{E}{R}.$$

iv. At t = 0, I = 0 therefore the graph goes through the origin and is asymptotic to the line $I = \frac{E}{R}$ (as shown in the below figure).



Example 7.5.2: With reference to the following figure, which consists of a resistor of resistance R = 3 Ω, connected in series with an inductor of inductance L = 5 H, and an applied constant voltage $E = 240\ Volts$.

i. Obtain a differential equation giving the current I at time t.

ii. Solve the differential equation for the initial condition, when t = 0, I = 0.

Solution: By applying Kirchhoff's law to the circuit, we have

i. Voltage drop across the resistance $(R) = Ri$.

Voltage drop across the inductance $(L) = L\frac{di}{dt}$

By applying the Kirchhoff's law to the circuit, we obtain,

$$E = Voltage\ drop\ across\ resistance(R) + Voltage\ drop\ across\ inductance(L)$$

$$\therefore E = RI + L\cdot\frac{dI}{dt} \quad Or \quad L\cdot\frac{dI}{dt} + RI = E \quad Or \quad \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L} \quad \text{....................} \ (1)$$

Since, $R = 3\,\Omega$ and $L = 5H$, $E = 240\ Volts$. (Given)

Now, Equation (1) gives,

$$\frac{dI}{dt} + \frac{3}{5}I = 240$$

$$Or \quad \frac{dI}{dt} + 0.6I = 240$$

This represents Leibnitz's linear equation in terms of 'I' and 't'.

ii. To solve this equation,

$\therefore I.F = e^{0.6t}$, it's general solution is given by,

$$I\cdot e^{0.6t} = \int 240\,e^{0.6t}dt + c \Rightarrow I\cdot e^{0.6t} = 240\left(\frac{e^{0.6t}}{0.6}\right)\cdot + c$$

$$Thus,\ I\cdot e^{0.6t} = 400.e^{0.6t}\cdot + c \quad Or \quad I = 400 + c\,e^{-0.6t}$$

Since, when t = 0, I = 0. Equation (1) gives, $c = -400$.

Thus, the general solution of the above differential equation is given by,

$$I = 400 - 400e^{-0.6t} = 400(1 - e^{-0.6t}).$$

Example 7.5.3: Show that the differential equation for the current I in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E\sin\omega t$ satisfies the equation

$$L\frac{dI}{dt}+RI=E\sin\omega t .$$

Find the value of the current at any time 't', if initially there is no current in the circuit.

Solution: By Kirchhoff's first law, we have sum of voltage drops across $R \; and \; L \, is \, E\sin\omega t$.

i.e., $RI+L\frac{dI}{dt}=E\sin\omega t$

This is the required differential equation which can be written as

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}\sin\omega t$$

This represents Leibnitz's linear equation in terms of 'I' and 't'.

To solve this equation,

$\therefore I.F=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$, it's general solution is given by,

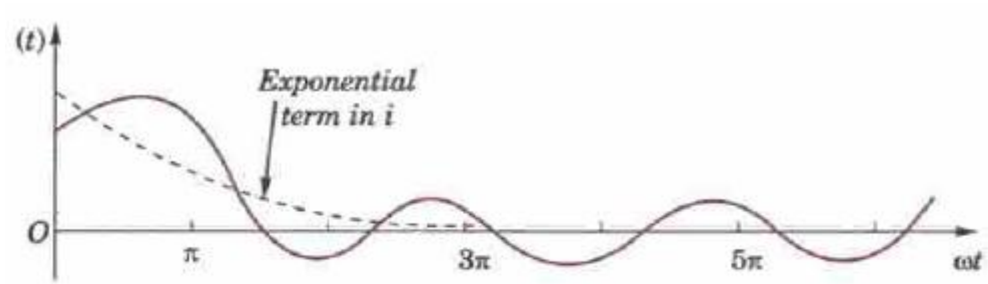
$$I\cdot e^{\frac{R}{L}t}=\int\frac{E}{L}\sin\omega t\cdot e^{\frac{R}{L}t}dt+c \Rightarrow I\cdot e^{\frac{R}{L}t}=\frac{E}{L}\int e^{\frac{R}{L}t}\sin\omega t\cdot dt+c$$

$$Thus,\; I\cdot e^{\frac{R}{L}t}=\frac{E}{L}\frac{e^{\frac{R}{L}t}}{\sqrt{\left[(R/L)^2+\omega^2\right]}}\sin\left(\omega t-\tan^{-1}\frac{L\omega}{R}\right)+c$$

$$I\cdot e^{\frac{R}{L}t}=\frac{E}{\sqrt{\left[R^2+\omega^2L^2\right]}}\sin(\omega t-\phi)+c\,e^{-\frac{R}{L}t} \quad where \;\tan\phi=\frac{L\omega}{R}$$

This is the required general solution.

Observation: As ‘t’ increases indefinitely, the exponential term will approach zero. This implies that after sometime the current I(t) will execute nearly harmonic oscillations only (as shown in the following figure).



Figure

7.5. Check Your Progress

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current I build up at a rate given by $L\frac{dI}{dt} + RI = E.$ Find I as a function of t. How long will it be, before the current has reached one half its final value if E = 6 volts, R = 100 Ohms and L = 0.1 Henry.

2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f of E volts, the current ‘I’ amperes at time t is given by

$L\frac{dI}{dt}+RI=E.$ If E = 10 sint volts and I = 0 when t = 0, find I as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 Henry are connected in series with a battery of 20 volts. Find the current in the circuit at t = 0.5 sec, if I = 0 at t = 0.

4. The equation of electromotive force in terms of current I for an electrical circuit having resistance R and condenser of capacity C in series is $E=Ri+\int\frac{idt}{C}$. Find the current I at any time t when $E=E_m\sin\omega t$.

5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. E sinpt produces currents I_1 and I_2 in two branches. If initially there is no current, determine I_1 and I_2 from the equations. $L\frac{dI_1}{dt}+RI_1=E\sin pt\ and\ \frac{I_2}{C}+R\frac{dI_2}{dt}=pE\cos pt.$ Verify that if $R^2C=L$, the total current $i_1+i_2\ will\ be\ \frac{E\sin pt}{R}$.

7.6. Heat flow

The fundamental principles involved in the problems of heat conduction are:

i. Heat flows from a higher temperature to the lower temperature.

ii. The quantity of heat in a body is proportional to its mass and temperature.

Heat: The form of energy that can be transferred from one system to another as a result of temperature difference.

◉ Thermodynamics is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another.

◉ Heat Transfer deals with the determination of the rates of such energy transfers as well as variation of temperature.

◉ The transfer of energy as heat is always from the higher-temperature medium to the lower-temperature one. Heat transfer stops when the two mediums reach the same temperature.

◉ Heat can be transferred in three different modes: conduction, convection, and radiation.

Conduction heat transfer

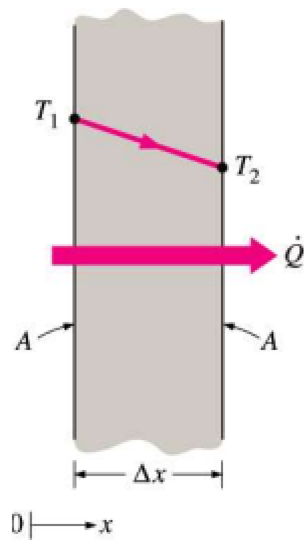
◉ Conduction: The transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles.

◉ In gases and liquids, conduction is due to the collisions and diffusion of the molecules during their random motion.

◉ In solids, it is due to the combination of vibrations of the molecules in a lattice and the energy transport by free electrons.

Rate of conduction

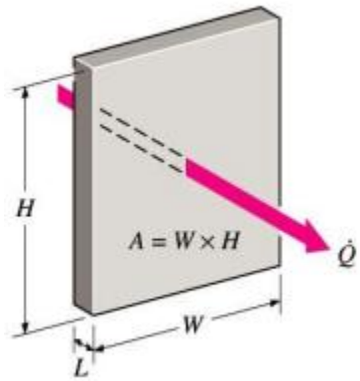
◉ The rate of heat conduction through a plane layer is proportional to the temperature difference across the layer and the heat transfer area, but is inversely proportional to the thickness of the layer.



Heat conduction through a large plane wall of thickness Δx and area A

$$Rate\ of\ heat\ conduction\ \alpha\ \frac{(Area)(Temperature\ Difference)}{Thickness}$$

$$\dot{Q}_{Conduction} = KA \cdot \frac{T_1 - T_2}{\Delta x} = -KA\frac{\Delta T}{\Delta x} \quad \text{........................} \quad (1)$$



7.6.1. Fourier Law of Heat Conduction:

"The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area".

$$\dot{Q}_{Conduction} = KA \cdot \frac{T_1 - T_2}{\Delta x} = -KA\frac{\Delta T}{\Delta x}$$

$$When\ x \to 0,\ Q_{cond} = -KA\frac{dT}{dx}$$

In heat conduction analysis, A represents the area normal to the direction of heat transfer.

Equivalently,

If q (cal/sec) be the quantity of heat that flows across a slab of area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, Then, by the statement of Fourier Law of heat conduction,

$$q = -k\alpha \cdot \frac{dT}{dx} \text{ .... } (1)$$

Where ‘k’ is a constant depending upon the material of the body and is called the thermal conductivity.

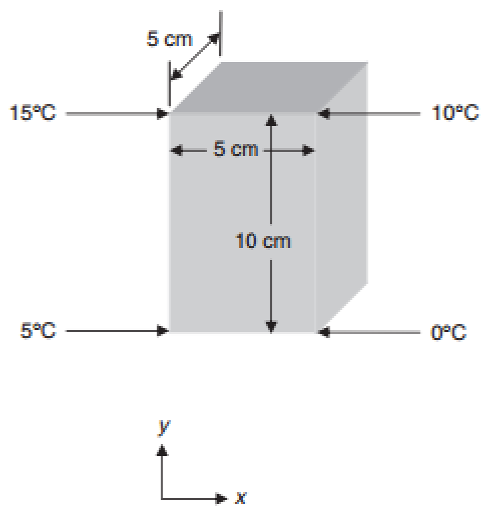
- Heat is conducted in the direction of decreasing temperature, and the temperature gradient becomes negative when the temperature decreases with increasing x.
- The negative sign in the equation ensures that heat transfer in the positive ‘x’ direction is a positive quantity.
- Thermal conductivity ‘k’ is a measure of the ability of a material to conduct heat.
- Temperature gradient $\frac{dT}{dx}$ is the slope of the temperature curve on a T-x diagram.
- The rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference.
- The thermal conductivity of a material is a measure of the ability of the material to conduct heat.
- A high value for thermal conductivity indicates that the material is a good heat conductor, and a low value indicates that the material is a poor heat conductor or insulator.

Example 7.6.1: The block of 304 stainless steel shown below, is well insulated on the front and back surfaces, and the temperature in the block varies linearly in both the X-axis and Y-directions, find the heat fluxes and heat flows in the x-and y-directions.

Solution: The thermal conductivity of 304 stainless steel is 14.4 W/m.K.

The cross sectional areas are:

$$A_x = 10 \times 5 = 50 cm^2 = 0.0050 m^2$$
$$A_y = 5 \times 5 = 25 cm^2 = 0.0025 m^2$$



Since the temperature variation is linear, replacing the partial derivatives with finite differences, the heat fluxes are:

$$\hat{q}_x = -k\frac{\partial T}{\partial x} = -k\frac{\Delta T}{\Delta x} = -14.4\left(\frac{-5}{0.05}\right) = 1440 W/m^2$$
$$\hat{q}_y = -k\frac{\partial T}{\partial x} = -k\frac{\Delta T}{\Delta x} = -14.4\left(\frac{10}{0.1}\right) = -1440 W/m^2$$

The heat flows are obtained by multiplying the fluxes by the corresponding cross-sectional areas:

$$q_x = \hat{q}_x A_x = 1440 \times 0.005 = 7.2\ W$$

$$q_y = \hat{q}_y A_y = 1440 \times 0.0025 = -3.6\ W$$

Example 7.6.2: Apply the conduction equation to the situation illustrated in the following Figure.

In order to make the mathematics conform to the physical situation, the following conditions are imposed:

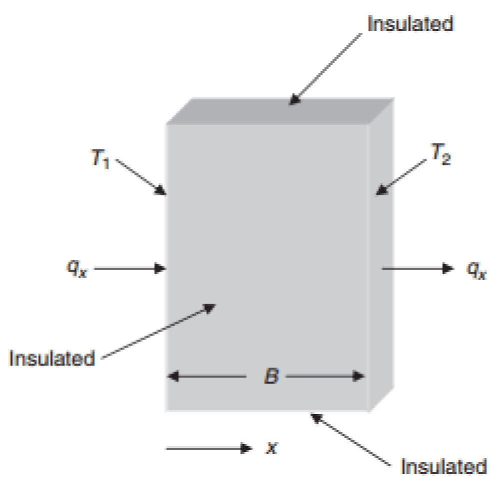


Figure 7.6.2: One-dimensional heat conduction in a solid.

i. Conduction only in x-direction $\Rightarrow T = T(x)$, so $\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial z} = 0$

ii. No heat source $\Rightarrow \dot{q} = 0$

iii. Steady state $\Rightarrow \frac{\partial T}{\partial t} = 0$

iv. Constant ‘k’

Solution: With respect to the given figure and given conditions, the conduction equation in Cartesian coordinates then becomes:

$k\frac{\partial^2 T}{\partial x^2} = 0 \;\; Or \;\; \frac{d^2 T}{dx^2} = 0$.

(The partial derivative is replaced by a total derivative because x is the only independent variable in the equation.) Integrating on both sides of the equation gives: $\frac{dT}{dx} = C_1$, integrating again, gives: $T = C_1 x + C_2$.

Thus, it is seen that the temperature varies linearly across the solid. The constants of integration can be found by applying the boundary conditions:

$At \;\; x = 0 \;\; T = T_1$ and $At \;\; x = B \;\; T = T_2$.

The first boundary condition leads to $T_1 = C_2$ and the second the gives:

$T_2 = C_1 B + T_1$.

Solving for C_1 we find: $C_1 = \frac{T_2 - T_1}{B}$

The heat flux is obtained from Fourier’s Law:

$$\hat{q}_x = -k\frac{dT}{dx} = -kC_1 = -k\frac{(T_2 - T_1)}{B} = k\frac{(T_1 - T_2)}{B}$$

Multiplying by the area gives the heat flow:

$$q_x = \hat{q}_x\, A = \frac{kA(T_1 - T_2)}{B}.$$

Example 7.6.3: A pipe 20 cm in diameter contains steam at 150^0C and is protected with a covering 5 cm thick for which the thermal conductivity is 0.0025. If the temperature of the outer surface of the surrounding is 40^0C, find the temperature half-way through the covering under steady state conditions.

Solution: Let q cal/sec. Be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. And length is 1 cm (as shown in the following Figure 7.6.3). Then the area of the lateral surface = 2πx.

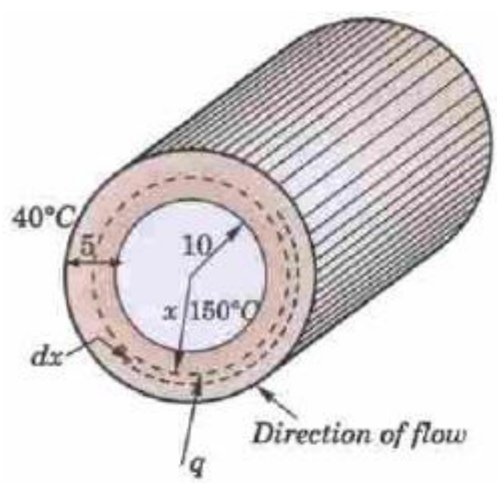


Figure 7.6.3

Therefore, by Fourier Law of heat conduction;

$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad Or \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$. Integrating, we obtain

$T = -\frac{q}{2\pi k} \log_e x + c$.

Since T = 150, when x = 10.

$$\therefore 150 = -\frac{q}{2\pi k} \log_e 10 + c \text{ .... } (1)$$

$Again \text{ since } T = 40, \ when \ x = 15.$

$$40 = -\frac{q}{2\pi k} \log_e 15 + c \quad \text{ .... } (2)$$

Subtracting equation (2) from equation (1), we get

$$110 = \frac{q}{2\pi k} \log_e 1.5 \text{ .... } \quad (3)$$

Since T = t, when x = 12.5

$$\therefore \ t = -\frac{q}{2\pi k} \log_e 12.5 + c \quad \text{ .... } \quad (4)$$

Subtracting (1) from (4), $t - 150 = -\frac{q}{2\pi k} \log_e 1.25$ .... (5)

Dividing (5) by (3), $\frac{t-150}{110} = -\frac{\log_e 1.25}{\log_e 1.5}$, $whence \ t = 89.5^0 C$.

Test Your Progress 7.6
1) A pipe 20 cm in diameter contains steam at 200^{0}C. It is covered by a

layer of insulating material 6 cm thick and the thermal conductivity is 0.0003. If the temperature of the outer surface is 30^0C, find the heat loss per hour from two metre length of the pipe.

2) A steam pipe 20 cm in diameter contains steam at 150^0C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60^0C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25%?.

7.7. Chemical Reactions

Suppose a chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula,

$$\left.\begin{matrix} Rate\ of\ change\ of \\ amount\ in\ container \end{matrix}\right\} = \begin{pmatrix} Rate\ at\ which\ the \\ chemical\ arrives \end{pmatrix} - \begin{pmatrix} Rate\ at\ which\ the \\ chemical\ departs \end{pmatrix}$$

.... (1)

If y(t) is the amount of chemical in the container at time t and V(t) is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$Departure\ Rate = \frac{y(t)}{V(t)} \cdot (outflow\ rate) = \begin{pmatrix} concentration\ in \\ container\ at\ time\ t \end{pmatrix} \cdot (outflow\ rate)$$
.... (2)

Accordingly, equation (1) becomes;

$$\frac{dy}{dt} = \begin{pmatrix} Rate\ of\ \inf low \\ of\ the\ chemical \end{pmatrix} - \frac{y(t)}{V(t)} \cdot (outflow\ rate) \quad .... \quad (3)$$

Suppose y is measured in pounds, V in gallons, and t in minutes, the units in equation (3) are given by,

$$\frac{pounds}{\min utes} = \frac{pounds}{\min utes} - \frac{pounds}{gallons} \cdot \frac{gallons}{\min utes}.$$

Example 7.7.1: A tank initially contains 50 gallons of fresh water. Brine. Containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds?

Solution: Let the salt content at time t be y lb, so that its rate of change is $\frac{dy}{dt}$

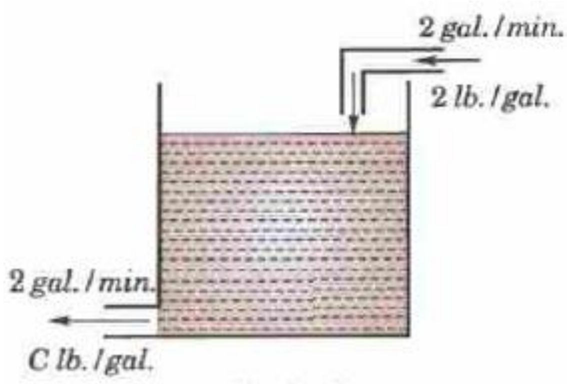


Figure 7.7.1

$Rate\ of\ \inf low\ of\ \ fresh\ water = 2\ gal \times 2\ lb = 4\ lb / \min$.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow $= 2\ gal \times C\ lb\ = 2\ lb / \min$.

$\therefore\ \frac{dy}{dt} = 4 - 2C$ .... (1)

Also, since there is no increase in the volume of the liquid, the concentration $C = \frac{y}{50}$.

Therefore, equation (1) becomes $\frac{dy}{dt} = 4 - 2\frac{y}{50} = \frac{100 - y}{25}$.

Separating the variables and integrating, we have

$\int dt = 25\int \frac{dy}{100 - y} + k$. Where k is the constant of integration.

$t = -25\log_e(100 - y) + k$ .... (2)

Initially when t = 0, y = 0; equation (2) gives,

$0 = -25\log_e 100 + k\ \ Or\ k = 25\log_e 100$ .... (3)

Eliminating k from equation (3) using equation (2), we obtain $t = 25\log_e\left(\frac{100}{100-y}\right)$.

Now, taking t = t_1 when y = 40 and t = t_2 when y = 80, we have

$$t_1 = 25\log_e\left(\frac{100}{60}\right) \; and \; t_1 = 25\log_e\left(\frac{100}{20}\right)$$

$$\therefore The\ required\ time\ (t_2 - t_1) = 25\log_e 5 - 25\log_e 5/3$$
$$= 25\log_e 3 = 25 \times 1.0986 = 27 \text{ min}.28 \text{ sec}.$$

Example 7.7.2: In an oil refinery, a storage tank contains 2000 gallons of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins?.

Solution: Let y be the amount (in pounds) of additive in the tank at time t. We know that y = 100 when t = 0. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$V(t) = 2000\ gal + \left(40\frac{gal}{\text{min}} - 45\frac{gal}{\text{min}}\right)(t \text{ min})$$
$$= (2000 - 5t)\ gal.$$

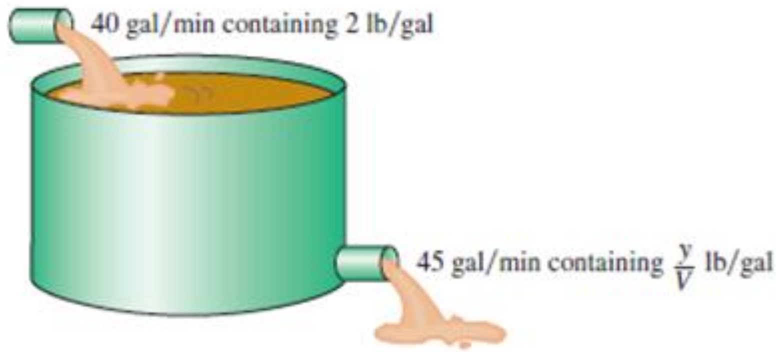


Figure 7.7.2: The storage tank mixes input liquid with stored liquid to produce an output liquid.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow $= 45\ gal/\min$.

$\therefore\ V(t) = 2000 - 5t$

Therefore,

$$Rate\ out = \frac{y(t)}{V(t)} \cdot outflow\ rate$$
$$= \left(\frac{y}{2000-5t}\right)45 = \frac{45y}{2000-5t}\ lb/\min$$

Also,

$$Rate\ in = \left(2\frac{lb}{gal}\right)\cdot\left(40\frac{gal}{\min}\right) = 80\ lb/\min$$

The differential equation modelling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t} \text{ pounds/min.}$$

$Or\ \frac{dy}{dt}+\frac{45}{2000-5t}y=80$ .... (1)

Here, $P(t)=\frac{45}{2000-5t}$, $Q(t)=80$. The integrating factor is given by

$I.F=e^{\int P\,dt}=e^{\int\left(\frac{45}{2000-5t}\right)dt}=e^{-9\log_e(2000-5t)}=(2000-5t)^{-9}$. $\quad\because\ 2000-5t>0$

The general solution of the equation (1) is given by

$y\times(2000-5t)^{-9}=\int 80\times(2000-5t)^{-9}dt+k$. Where K is the constant of integration.

$$Or\ \ y\times(2000-5t)^{-9}=80\int(2000-5t)^{-9}dt+k$$
$$\therefore\ \ (2000-5t)^{-9}y=80\cdot\frac{(2000-5t)^{-8}}{(-8)(-5)}+C$$
$$Or\ \ y=2(2000-5t)+C(2000-5t)^{9}\quad .... (1)$$

To find the value of C, using $y=100\ \ when\ \ t=0$ (Given).

Equation (1) gives;

$$100=2(2000-0)+C(2000-0)^{9}$$
$$\Rightarrow C=-\frac{3900}{(2000)^{9}}$$

Now equation (1) becomes; $y=2(2000-5t)-\frac{3900}{(2000)^{9}}(2000-5t)^{9}$ ....

(2)

Equation (2) gives the particular solution of given initial value problem.

Further, the amount of additive in the tank 20 min after the pumping begins is i.e., substitute t = 20 in equation (2); we get

$$y(20)=2[2000-5(20)]-\frac{3900}{(2000)^9}\left[2000-5(20)\right]^9 \cong 1342\ lb\ .$$

Test Your Progress 7.7
1) A tank contains 1000 gallons of brine in which 500 lt. Of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons/minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. Of salt is left in the tank?
2) A tank is initially filled with 100 gallons of salt solution containing 1 lb. Of salt per gallon. Fresh brine containing 2 lb. Of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.
3) In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation $\frac{dx}{dt}$ at any time t is known to be equal to the product $(a-x)(b-x)$ of the amounts of the two substances then remaining un

transformed. Find t in terms of x if $a = 0.7,\ b = 0.6\ and\ x = 0.3\ when$

$t = 300\ seconds$.

7.8. Rate of Growth or Decay

7.8.1: Rate of Growth

"The rate of growth of substance at time t is directly proportional to the substance present at that time".

Let y(t) be the substance present at time t. Then, the natural growth equation is the differential equation given by,

$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = ky$. Where 'k' is a constant of proportionality.

Its general solution is given by

$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\, dt$, This is in variable separable form

Integrating, we obtain

$\log_e y = kt + \log_e C\ \ Or\ \ y = Ce^{kt}$

If the initial conditions given are $y(0) = y_0$, then $C = y_0$

Therefore, the particular solution is given by $y = y_0 e^{kt}$

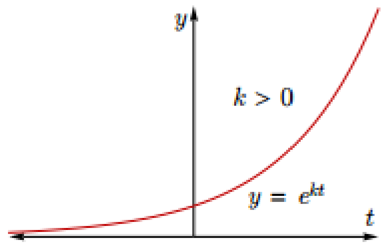


Figure 7.8.1: Exponential Growth

7.8.2: Rate of Decay

"The rate of decay of radioactive substance at time t is directly proportional to the mass of the substance present at that time".

Let y(t) be the substance present at time t. Then, the equation of decay is the differential equation given by,

$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = -ky$. Where 'k' is a constant of proportionality.

Its general solution is given by

$\frac{dy}{dt} = -ky \Rightarrow \frac{dy}{y} = -k\,dt$, This is in variable separable form

Integrating, we obtain

$\log_e y = -kt + \log_e C \;\; Or \;\; y = Ce^{-kt}$

If the initial conditions given are $y(0) = y_0$, then $C = y_0$

Therefore, the particular solution is given by $y = y_0 e^{-kt}$

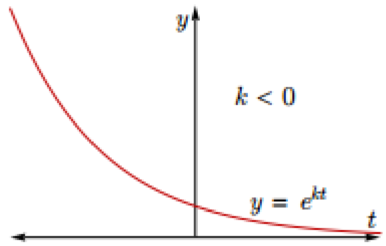


Figure 7.8.2: Exponential Decay

Example 7.8.1: The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours?

Solution: By the law of rate of growth, we have

$\frac{dN}{dt} = kN \Rightarrow \frac{dN}{N} = kdt$. Integrating, we obtain

$\log_e N = kt + \log_e C \;\; Or \;\; N = Ce^{kt}$ .... (1)

Initially, N = 100 when t = 0, equation (1) gives C = 100

Now equation (1) becomes $N = 100e^{kt}$ .... (2)

And N = 332 when t = 1 Hr, from equation (2), we get

$i.e., \;\; N = 100e^{kt} \Rightarrow 332 = 100e^{k} \Rightarrow k = 1.199965$

Now, if $t = 1\frac{1}{2} = \frac{3}{2} = 1.5\,Hrs$, then from equation (2), we obtain

$N = 100e^{(1.199965)\cdot(1.5)} = 604.9647$.

Example 7.8.2: Radium decomposes at the rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will it remain at the end of 21 years?

Solution: By the law of rate of decay, we have

Let y(t) be the radium present at time t. Then, the equation of decay is the differential equation given by,

$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = -ky$. Where 'k' is a constant of proportionality.

Its general solution is given by

$\frac{dy}{dt} = -ky \Rightarrow \frac{dy}{y} = -k\,dt$, This is in variable separable form

Integrating, we obtain

$\log_e y = -kt + \log_e C \;\; Or \;\; y = Ce^{-kt}$ .... (1)

Initially, y = Y (say) when t = 0, equation (1) gives C = Y

Now equation (1) becomes $y = Y\,e^{-kt}$ .... (2)

And y = Y/p when t = 1 Year, from equation (2), we get

$i.e., \;\; y = Y\,e^{-kt} \;\Rightarrow\; \frac{Y}{p} = Y\,e^{-k} \;\Rightarrow\; -k = \log_e\left(\frac{1}{p}\right) \;\; Or \;\; k = \log_e p$

Now, if $t = 21 years$, let $y = y_1$, then from equation (2), we obtain

$y_1 = Y\,e^{-[\log_e p](21)} \;\; Or \;\; y_1 = Y\left(\frac{1}{p}\right)^{21}$

Example 7.8.3: A 30% of radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear?

Solution: By the law of rate of decay, we have

Let y(t) be the radioactive substance present at time t. Then, the equation of decay is the differential equation given by,

$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = -ky$. Where 'k' is a constant of proportionality.

Its general solution is given by

$\frac{dy}{dt} = -ky \Rightarrow \frac{dy}{y} = -k\,dt$, This is in variable separable form

Integrating, we obtain

$\log_e y = -kt + \log_e C \;\; Or \;\; y = Ce^{-kt}$ .... (1)

Initially, y = Y (say) when t = 0, equation (1) gives C = Y

Now equation (1) becomes $y = Y\,e^{-kt}$ .... (2)

And $y = Y - \frac{30}{100}Y = 0.7Y$ when t = 10 days, from equation (2), we get

$0.7Y = Y\,e^{10k} \Rightarrow k = -0.03567$

Now, if $y = Y - \frac{90}{100}Y = 0.1Y$, then again from equation (2), we obtain

$0.1Y = Y\;e^{(0.03567)\cdot t} \Rightarrow t = 64.5\;days$

Test Your Progress 7.7
1) A rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple? 2) Under certain conditions cane sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If it is of 75 gm. At time t = 0.8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours

7.8 Summary

In this unit we have studied the physical applications of differential equations of first order and first degree like problems based on Newton's Law cooling , simple electric circuit, Kirchhoff's law, Modelling of electric circuit, heat flow, Fourier law of heat conduction , chemical reaction, growth and decay of radioactive substances.

7.9 Terminal Questions

1. Assuming v_1 as the original temperature of the body at $t = 0$ and that of the surroundings is u_0 find the temperature u of the body at time t.

2. A glass of ice cold water is kept in a room at temperature $30^0 c$. If the initial rate of temperature increase is $1.5^0 c$ per minute, find the temperature of water ofter 10 minutes.
3. In a culture of bacteria the rate of increase is proportional to the number present. If their number are 3000 and 5000 at the end of 3 and 4 hours, find their number in the beginning.

7.8. Answers to Exercises

Test Your Progress 7.4:

(1) $\theta(t) = 100(3 + e^{kt})$ (3) $\theta = 300 + 73e^{-(2.96\times 10^{-3})t}$ (4) $\theta(t) = 320 + 28e^{kt}$

Test Your Progress 7.5:

(1) 0.0006931 secs (2) $\frac{10}{L^2 + R^2}\left(R\sin t - L\cos t + Le^{-\frac{R}{L}t}\right)$ (3)

$I = \frac{1}{5}(1 - e^{-100})$

(4) $I = ke^{-\frac{1}{RC}t} + \frac{\omega CE_m}{\sqrt{1 + R^2C^2\omega^2}}\sin(\omega t + \theta)$ $where\ \theta = \cot^{-1}(RC\omega)$.

Test Your Progress 7.6:

(1) 490, 000 cal (2) 2.16 cm

Test Your Progress 7.7:

(1) 3Hrs. 50 mins. 16 Secs (2) $100(1 - e^{-t/20})$; 13.9 min s

(3) $t = 300 - 5\log_2 2 + 5\log_e\left(\frac{0.7 - x}{0.5 - x}\right)$

Test Your Progress 7.8:

(1) $2 \cdot \frac{\log_e 3}{\log_e 2}$

(2) 21.5 grams

Unit –08: Physical applications of differential equations of first order and first degree

Structure

8.1. Introduction

Differential equations are found in many areas of mathematics, science, and engineering. One may be surprised to see the way in which differential equations dominate the study of many aspects of science and engineering.

Differential equations are very important mathematical subject from both theoretical and practical perspectives. The theoretical importance is given by the fact that most pure mathematical theories have applications in differential equations.

In the previous units we introduced differential equations of the form $\frac{dy}{dx} = f(x)$, where f

is given and y is an unknown function of x. When f is continuous over some interval, we

learned that the general solution $y(x)$ was found directly by integration, $y = \int f(x)\,dx$. And we also investigated differential equations of the form

$\frac{dy}{dx} = f(x, y)$, where f is a function of both the independent variable x and the dependent variable y. There we learned how to find the general solution when the differential equation is separable.

Many real-world problems, when formulated mathematically, lead to differential equations. We encountered a number of these equations in previous units when studying phenomena such as the, simple electrical circuits, heat flow of an object, the decay of a radioactive material, the growth of a population, and the cooling of a heated object placed within a medium of lower temperature.

In this unit, we shall consider only such practical problems which give rise to differential equations of the first order and first degree.

Also, we present a sufficient number of applications to enable the students to understand how differential equations are used and to develop some feeling for the physical information they convey.

We further extend our study to include other commonly occurring *first-order* differential equations. They involve only first derivatives of the unknown function $y(x)$, and

model phenomena such as motion of an object moving along a straight line, the simple harmonic motion of moving object.

We shall consider only such practical problems which give rise to differential equations of the first order and first degree.

Also, we present a sufficient number of applications to enable the students to understand how differential equations are used and to develop some feeling for the physical information they convey.

8.2. Objectives

After reading this unit students should be able to:

- Understand the physical applications of ODE's of first order and degree
- Apply Newton's second law of motion and modeling the motion of particles in mechanics
- Understand to model the motion under gravity and solve the first order ordinary differential equations which arise.

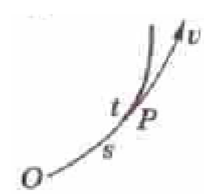
- Understand and apply the simple harmonic motion in modelling the ODE's of first order and degree

8.3. Physical Applications of Differential Equations of First Order and First Degree

In applications, the dependent variables are frequently functions of time, which we denote by't'. Some applications such as Newton's Law of Cooling, Kirchhoff's Laws of Electric Circuits, Rate of Growth or Decay are already discussed in the previous unit and Motion under Gravity, Rectilinear Motion, Simple Harmonic Motion, Heat flow are discussed here. In all these cases, modelling, analysis and interpretation are important.

8.4. Rectilinear Motion

Let a body of mass 'm' start moving from the point O along the straight line OX under the action of a force F. After any time t, let it be moving at P where OP = x, then



i. $its\ velocity\ (v) = \frac{dx}{dt}$

ii. $its\ acceleration\ (a) = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}$

If, however, the body be moving along a curve, then

i. $its\ velocity\ (v) = \frac{ds}{dt}$

ii. $its\ acceleration\ (a) = \frac{d^2s}{dt^2}\ Or\ \frac{dv}{dt}\ Or\ v \cdot \frac{dv}{ds}$

iii. The quantity mv is called the momentum.

Newton's Second Law of Motion:

The net force F (say) acting on the body is directly proportional to rate of change of momentum of the body.

i.e., Net force = mass$\times$ acceleration

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma\ \ Or\ \ F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$, if 'm' is constant.

Now, consider the example of resisted motion

Example 8.4.1: A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

Solution: By Newton's second law of motion, the equation of the motion of the body is given by

$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = -m(cx + bv^2)$. (Since the force is opposite)

$\therefore\ m\frac{dv}{dt} = -m(cx + bv^2)\ \ Or\ \ v\frac{dv}{dx} = -cx - bv^2$

$\therefore\ v\frac{dv}{dx} + bv^2 = -cx$ …………….. (1). This represents the Bernoulli's equation.

$Put\ \ v^2 = u \Rightarrow 2v\frac{dv}{dx} = \frac{du}{dx}$ so that equation (1) yields to

$\frac{1}{2}\frac{du}{dx} + bu = -cx \quad Or \quad \frac{du}{dx} + 2bu = -2cx$ ………………. (2)

This represents the Linear ODE of first order and first degree.

$I.F = e^{\int 2b dx} = e^{2bx}$.

General solution of equation (2) is given by

$u \times e^{2bx} = -\int 2cx \times e^{2bx} dx + c_1$, integrating by parts

$$u \cdot e^{2bx} = -2c\left[x \cdot \frac{e^{2bx}}{2b} - \int 1 \cdot \frac{e^{2bx}}{2b} dx \right] + c_1$$
$$= -\frac{cx}{b} e^{2bx} + \frac{c}{2b^2} e^{2bx} + c_1$$

Substituting $v^2 = u$, we get; $v^2 = \frac{c}{2b^2} + c_1 e^{-2bx} - \frac{cx}{b}$ ……………… (3)

Initially v = 0 when x = 0; equation (3) gives $0 = \frac{c}{2b^2} + c_1$.

Thus, substituting $c_1 = -\frac{c}{2b^2}$ in equation (3), we get

$$v^2 = \frac{c}{2b^2} - \frac{c}{2b^2} e^{-2bx} - \frac{cx}{b}$$

Or $v^2 = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}$

Consider the example of resisted vertical motion

Example 8.4.2: A particle falls under gravity is a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.

Solution: After falling a distance 's' in time t from rest, let 'v' be the velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance mλv upwards.

$\therefore$ Equation of motion is $m\frac{dv}{dt} = mg - m\lambda v$

$Or \quad \frac{dv}{dt} = g - \lambda v \quad Or \quad \frac{dv}{g - \lambda v} = dt$. Integrating,

$$\int \frac{dv}{dtg - \lambda v} = \int dt + c \quad Or \quad -\frac{1}{\lambda}\log(g - \lambda v) = t + c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Since v = 0 when t = 0, equation (1) gives $c = -\frac{1}{\lambda}\log g$

Thus,

$$\frac{1}{\lambda}\log_e\left[\frac{g}{g - \lambda v}\right] = t \quad Or \quad \log_e\left[\frac{g}{g - \lambda v}\right] = \lambda t$$

$$Or \quad \frac{g}{g - \lambda v} = e^{\lambda t} \quad \Rightarrow e^{-\lambda t} = \frac{g - \lambda v}{g}$$

$$\therefore \; e^{-\lambda t} = \frac{g - \lambda v}{g} \;\Rightarrow\; ge^{-\lambda t} = g - \lambda v \quad Or \quad ge^{-\lambda t} - g = \lambda v \;\; or \;\; v = \frac{g}{\lambda}(1 - e^{-\lambda t})$$

Thus, $v = \frac{ds}{dt} = \frac{g}{\lambda}(1 - e^{-\lambda t})$ ………………………… (1)

Integrating, we obtain,

$$s = \frac{g}{\lambda}\int(1-e^{-\lambda t})\,dt + c_1 \quad Or \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c_1$$

Since s = 0 when t = 0, $c_1 = -\frac{g}{\lambda^2}$.

Thus, $s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}e^{-\lambda t} - \frac{g}{\lambda^2}$ Or $s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}\left(e^{-\lambda t} - 1\right)$ …………………... (2)

Eliminating 't' from equations (1) and (2), we get

From equation (1) ,

$$v = \frac{ds}{dt} = \frac{g}{\lambda}(1-e^{-\lambda t}) \Rightarrow 1-e^{-\lambda t} = \frac{v\lambda}{g}$$

$$Or \;\; e^{-\lambda t} = 1-\frac{v\lambda}{g} \;\; Or \;\; e^{\lambda t} = \frac{g}{g-v\lambda} \;\; Or \;\; t = \frac{1}{\lambda}\log_e\left(\frac{g}{g-v\lambda}\right)$$

$$\therefore \; Equation \;\; (2)\, gives; \;\; s = \frac{g}{\lambda}\cdot\frac{1}{\lambda}\log_e\left(\frac{g}{g-v\lambda}\right) + \frac{g}{\lambda^2}\left(\frac{-v\lambda}{g}\right)$$

$$Or \;\; s = \frac{g}{\lambda^2}\log_e\left(\frac{g}{g-v\lambda}\right) - \frac{v}{\lambda}$$

This is a desired relation between s and v.

Example 8.4.3: A body of mass m, falling from rest is subject to the force of gravity and air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$, where $mg = ka^2$.

Solution: If the body is moving with the velocity 'v' after having fallen through a distance x, then its equation of motion, by Newton's law of motion is given by

$$Net\ \ Force = F = m\frac{d^2x}{dt^2} = ma = m\frac{dv}{dt} = mv\frac{dv}{dx}$$

$$\therefore\ mv\frac{dv}{dx} = mg - kv^2 \quad Or \quad mv\frac{dv}{dx} = k(a^2 - v^2) \ \ldots\ldots\ldots.. \quad (1)$$

$$\because\ mg = ka^2$$

Separating the variables and integrating, we get

$$\int \frac{v\,dv}{a^2 - v^2} = \frac{k}{m}\int dx + c;\ \ put\ a^2 - v^2 = u \Rightarrow -2vdv = du$$

$$\therefore \int -\frac{1}{2u}du = \frac{k}{m}x + c \quad Or \quad -\frac{1}{2}\log_e(a^2 - v^2) = \frac{k}{m}x + c \ \ldots\ldots\ldots\ldots\ldots\ldots.. \ (2)$$

Initially, when x = 0, v = 0. Then, $-\frac{1}{2}\log a^2 = c \quad \ldots\ldots\ldots\ldots\ldots.. \quad (3)$

To eliminate ‘c’ from equations (2) and (3): Subtracting equation (3) from equation (2), we obtain

$$\frac{1}{2}\left[\log_e a^2 - \log_e (a^2 - v^2)\right] = \frac{k}{m}x \quad Or \quad \frac{2k}{m}x = \log_e\left(\frac{a^2}{a^2 - v^2}\right).$$

Test Your Progress 8.4
1) A particle of mass ‘m’ moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\frac{m}{k}\log_e\left(1+\frac{kv}{mg}\right)$. **2)** A body of mass m falls from rest under gravity and air resistance is proportional to square of velocity. Find velocity as function of time. **3)** A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to aquire one half of its limiting speed. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of

the velocity. Show that the distance it has described in time is

$$\frac{1}{\mu v}\left(\sqrt{1+2\mu v^2 t}-1\right).$$

8.5. Motion under Gravity

Elementary motions of a particle are frequently described by differential equations. Simple integration can sometimes be used to analyze these elementary motions. For the one-dimensional vertical motion of a particle, we recall from calculus that,

Newton's Second Law of Motion:

The net force F (say) acting on the body is directly proportional to rate of change of momentum of the body.

i.e., Net force = mass$\times$ acceleration

$$F=\frac{d}{dt}(mv)=m\frac{dv}{dt}=ma \;\; Or \;\; F=m\frac{dv}{dt}=m\frac{d^2x}{dt^2}$$, if 'm' is constant

Equivalently, $F\left(x,\frac{dx}{dt},t\right)=m\frac{d^2x}{dt^2}$ .... (1)

Here we have allowed the forces to depend on position, velocity, and time.

Equation (1) is a second order differential equation, we have not yet studied the techniques to solve the equation (1). However, it can be solved by simple integration if the force F does not depend on $x \text{ and } \frac{dx}{dt}$

Consider the following situation;

Suppose that the only force on the mass is due to gravity. Then, it is known that F = -mg, where g is the acceleration due to gravity. The minus sign is introduced because gravity acts downward, toward the surface of the earth. Here we are taking the coordinate system so that x increases toward the sky. The magnitude of the force due to gravity mg is called the weight of the body near the surface of the planet earth, g is approximately g=9.8 m/s^2. If we assume that, we are interested in a mass that is located sufficiently near the surface of the earth, then g can be approximated by this constant. With the only force being gravity, equation (1) becomes

$m\frac{d^2x}{dt^2} = -mg \;\; Or \;\; \frac{d^2x}{dt^2} = -g$. Integrating yields

$\frac{dx}{dt} = -gt + C_1$, where C_1 is an arbitrary constant of integration

$Or\ v(t) = -gt + C_1$ .... (2)

If we assume that the velocity at t = 0 is given and let $v = v_0$.

Equation (2) gives; $C_1 = v_0$,this implies $\frac{dx}{dt} = -gt + v_0$ .... (3)

Now, the position can be determined by integrating the velocity in the equation (3).

$x = -\frac{1}{2}gt^2 + v_0 t + C_2$ .... (4)

Where C_2 is second constant of integration. Again we assume that the position at t = 0 is x_0. Then equation (4) gives $C_2 = x_0$, so that $x = -\frac{1}{2}gt^2 + v_0 t + x_0$.

Note: If the applied force depends only on time and is not constant, then the formulas for velocity and position may be obtained by integration. If the applied force depends on other quantities, then solving the differential equation is not so simple.

Consider the another examples;

Example 8.5.1: Suppose a ball is thrown upward from ground level with velocity v_0 and the only force is gravity. How high does the ball go before falling back toward the ground?

Solution: As in the above situation, the corresponding differential equation is given by

$$\frac{d^2x}{dt^2} = -g \quad \text{.................} \ (1)$$

The successive integrations with initial conditions, at t = 0, x = 0 and $\frac{dx}{dt} = v_0$ yields to

$$\frac{dx}{dt} = -gt + v_0 \quad \text{..................} \ (2)$$

and $$x = -\frac{1}{2}gt^2 + v_0 t \quad \text{.............} \ (3)$$

From equation (3), the height is known as a function of time. To determine the maximum height, we must first determine the time at which the ball reaches this height. From calculus, the maximum of a function x = x(t) occurs at a critical point where $\frac{dx}{dt} = 0$. At the maximum height the ball has stopped rising and has not started to fall, so the velocity is zero. Thus, the time of the maximum height is determined from equation (2):

$$-gt + v_0 = 0 \quad \text{Or} \quad \text{equivalently,} \quad t = \frac{v_0}{g} \quad \text{................ (4)}$$

When this time t in equation (4) is substituted into (3, a formula for the maximum height y (say) is obtained:

$$\therefore y = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right)\left(-\frac{1}{2}+1\right) = \frac{v_0^2}{2g}.$$

In the following example we consider the motion of a boat across a stream;

Example 8.5.2: A boat is rowed with a velocity ‘v’ directly across a stream of width ‘a’. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance downstream to the point where it lands.

Solution: Taking the origin at the point from where the boat starts, let the axes be chosen as in the following figure 8.5.2.

At any time t after its start from O, let the boat be at P(x, y), so that

$$\frac{dx}{dt} = velocity\ of\ the\ current = ky(a-y)$$

$$\frac{dy}{dt} = velocity\ with\ which\ the\ boat\ is\ being\ rowed = v$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{v}{ky(a-y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

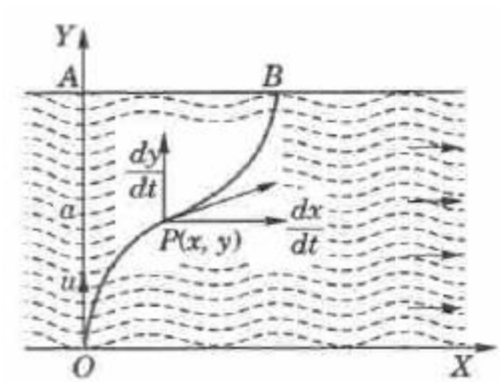


Figure 8.5.2

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Equation (1) is of variable separable form and we can write it can be written as

$y(a-y)dy = \frac{v}{k}dx$. Integrating, we obtain $\frac{ay^2}{2} - \frac{y^3}{3} = \frac{v}{k}x + C$

Since, initially y = 0 when x = 0 implies C = 0.

Hence the equation to the path of the boat is given by

$$\frac{ay^2}{2} - \frac{y^3}{3} = \frac{v}{k}x \;\; Or \;\; \frac{3ay^2 - 2y^3}{6} = \frac{v}{k}x$$

$$Or \;\; ky^2(3a - 2y) = 6vx \;\; Or \;\; x = \frac{ky^2(3a - 2y)}{6v}$$

By putting y = a, we get the distance AB, downstream where the boat lands is equal to

$$\frac{ka^2(3a - 2a)}{6v} = \frac{ka^3}{6v}.$$

Test Your Progress 8.5

1) When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand tank with velocity v_0.
2) A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch ‘x’ feet.

8.6. Simple Harmonic Motion

Differential equations of the type $\frac{d^2y}{dx^2} + k^2 y = 0$, where 'k' is a constant, arise in vibration problems. One of the simplest cases is known as simple harmonic motion (SHM).

Consider a particle P, moving to and fro about its equilibrium position O as shown in figure 8.6.

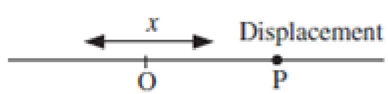


Figure 8.6

The equation of motion at any time 't' is given by

$\frac{d^2x}{dt^2} + k^2 x = 0$ ………….. (1). Where 'x' is displacement, 'k' is a constant and 't' is time.

Equation (1) is a second order differential equation, we have not yet studied the techniques to solve the equation (1). However, it can be solved by separating the variables and by simple integration if the force F does not depend on $x \text{ and } \frac{dx}{dt}$.

Equation (1) can also be written as, $v\frac{dv}{dx}+k^2x=0 \;\; Or \;\; vdv=-k^2xdx$.

Integrating, we get

$$\frac{v^2}{2}=-k^2\frac{x^2}{2}+\frac{c^2}{2} \;\; Or \;\; v^2=-k^2x^2+c_1.$$

Note: The motion of an object or weight bobbing freely up and down with no resistance on the end of a spring is an example of simple harmonic motion. The motion is periodic and repeats indefinitely. So we represent it using trigonometric functions.

Consider the following example, which describes a case in which there are no opposing forces such as friction to slow the motion.

Example 8.6.1: A weight hanging from a spring is stretched down 5 units beyond its rest position and released at time t = 0 to bob up and down. Its position at any later time t is 5cost. Find its velocity and acceleration at any time t.

Solution: We have, $Position: \ s = 5\cos t$

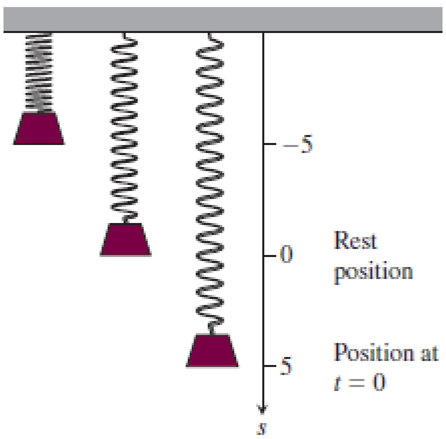


Figure 8.6.1(a): A weight hanging from a vertical spring and then displaced oscillates above and below its rest position

$$\therefore \ Velocity: \ v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = -5\sin t$$
$$and \ \ Acceleration: \ a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t$$

We can notice the following points from the above situation

1. As time passes, the weight moves down and up between s = -5 and s = 5 on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of the cosine function.
2. The velocity v = -5sint attains its greatest magnitude, 5, when cost = 0, as the graphs shown in figure 8.6.1(b). Hence, the speed of the weight,

$|v| = 5|\sin t|$, is greatest when cost = 0, that is, s = 0 (the rest position). The speed of the weight is zero when sint = 0. This occurs when s= 5cost = ±5, at the end point of the interval of motion.

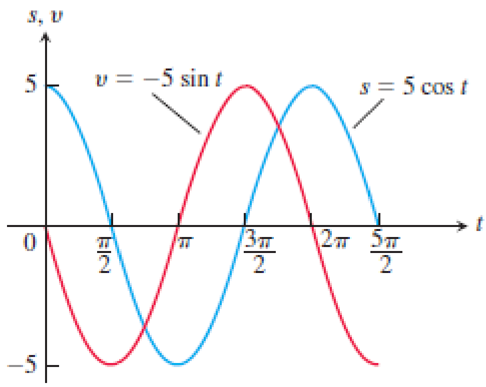


Figure 8.6.1(b): The graphs of the position and velocity of the weight

3. The weight is acted upon by the spring and by gravity. When the weight is below the rest position, the combined forces pull it up, and when it is above the rest position, they it down. The weight's acceleration is always proportional to the negative of its displacement. This property of springs is called Hooke's law. It says that "the force required to hold a stretched or compressed spring x units from its natural length (unstressed) length is proportional to x". In symbols, F = kx. Where 'k' is the spring constant or force constant.

4. The acceleration, a = -5cost, is zero only at the rest position, where cost = 0 and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest magnitude at the points farthest from the rest position, where cost = ±1.

Example 8.6.2: The motion of the spring-mass system (as shown in figure 8.6.2) is described by

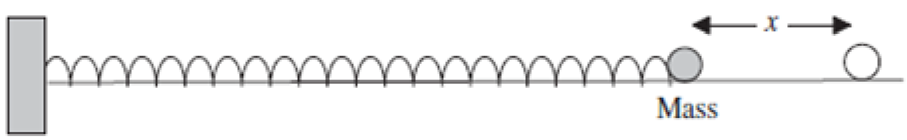


Figure 8.6.2

$\frac{d^2x}{dt^2}+25x=0$. Where x is displacement and t is time. Deternine the particular solution for this differential equation with initial conditions, when t = 0, both x = 1 and $\frac{dx}{dt}=10$.

Solution: We have $\frac{d^2x}{dt^2}+25x=0$. This equation is of second order, so we will write it as

$v \cdot \frac{dv}{dx} = -25x \Rightarrow v \cdot dv = -25dx$, this is in variable separable form.

Integrating

$\int v \cdot dv = -25\int dx + c$. Where 'c' is the constant of integration.

$\frac{v^2}{2} = -25x + c \;\; Or \;\; v^2 = -50x + 2c$ …………….. (1)

Since, when t = 0, both x = 1 and $\frac{dx}{dt} = 10$. Equation (1) gives

$100 = -50 + 2c \; Or \; c = 75$.

Thus, $v^2 = -50x + 150$.

8.7 Summary

In this unit we have studied application of differential equation of first order and first degree in (1) Rectilinear motion, (2) Motion under gravity (3) simple harmonic equation.

8.8 Terminal Questions

1) A particle is moving under gravity from rest in a medium whose resistance varies as the velocity of the particle. Find the velocity and distance of the particle after time t.

2) A particle is moving in straight line from rest with constant acceleration f. Find the velocity and distance softer time t.

8.7. Answers to Exercises

Test Your Progress 8.4

1) $V = \sqrt{\left(\frac{mg}{k}\right)} \tanh\left(\frac{9k}{m} t + c\right)$ **3)** $\frac{1}{k} \log_e 2$

Test Your Progress 8.5

1) $2\sqrt{v_0 / k}$ **2)** $v^2 = 2gx - \frac{\lambda}{m} x^2$

Block-III

The nth order linear differential equation with constant coefficients

U P RAJARSHI TANDON OPEN UNIVERSITY PRAYAGRAJ

UGMM-104 DIFFERENTIAL EQUATION

Unit –09 The n^{th} order linear differential equation with constant coefficients

Structure

9.1. Introduction

We have already studied the basics of differential equations, including separable first-order equations. In this unit, we go a little further and look at second-order, higher- order and n^{th}-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order and higher-order equations with constant coefficients.

Such equations have many practical applications. The operation of certain electrical circuits, known as resistor–inductor–capacitor (*RLC*) circuits, can be described by second-order differential equations with constant coefficients. These circuits are found in all kinds of modern electronic devices—from computers to smart phones to televisions. Such circuits can be used to select a range of frequencies from the entire radio wave spectrum, and are they commonly used for tuning AM/FM radios. We look at these circuits more closely later in illustrations.

Spring-mass systems, such as motorcycle shock absorbers, are a second common application of second-order differential equations. For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when landing could cost them the race. The movement of the shock absorber depends on the amount of damping in the system. In this chapter, we model forced and unforced spring-mass systems with varying amounts of damping.

When working with differential equations, usually the goal is to find a solution. In other words, we want to find a function (or functions) that satisfy the differential equation. The technique we use to find these solutions varies, depending on the form of the differential equation with which we are working. Second-order differential equations have several important characteristics that can help us determine which solution method to use. In this section, we examine some of these

Characteristics and the associated terminology.

9.2. Objectives

After reading this unit students should be able to:

- Identify the linear differential equations of second and higher order

- Recognize the homogeneous and non homogeneous linear differential equations of higher order
- Determine the characteristic equation of a homogeneous linear differential equation
- Determine the particular integrals by using different methods
- Find the general solution of the given homogeneous and non homogeneous linear differential equations
- Solve the initial value and boundary value problems involving linear differential equations.

9.3. The n^{th} order linear differential equation with constant coefficients

The linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the n^{th} order is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots + a_n y = F(x)$$

Where $a_0, a_1, a_2, \dots\dots\dots\dots a_n$ are real valued functions, and a_0 is not identically zero. F(x) is function of ‘x’ only.

If $a_0, a_1, a_2, ... a_n$ are all constants, then the equation is known as linear differential equation with constant coefficients.

If F(x) = 0 for every value of x, the equation is said to be a homogeneous linear equation.

If F(x) ≠ 0 for some value of x. Then, the equation is said to be a non homogeneous linear equation.

In particular,

A second order differential equation is linear if it can be written in the form

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = F(x)$$

Note:

1. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.
2. In linear differential equations, y and its derivatives can be raised only to the first power and they may not be multiplied by one another.
3. Terms involving y^2 or $\sqrt{y'}$ make the equation nonlinear. Functions of y and its derivatives, such as siny or $e^{y'}$, are similarly prohibited in

linear differential equations.

4. The equations may not always be given in standard form. It can be helpful to rewrite them in that form to decide whether they are linear, or whether a linear equation is homogeneous.

Example 9.3.1: Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or non homogeneous.

i. $y'' + 3x^4 y' + x^2 y^2 = x^3$

ii. $(\sin y) y'' + (\cos x) y' + 3y = 0$

iii. $4t^2 x'' + 3txx' + 4x = 0$

iv. $5y'' + y = 4x^5$

v. $(\cos x) y'' - \sin y' + (\sin x) y - \cos x = 0$

Solution: (i) We have, $y'' + 3x^4 y' + x^2 y^2 = x^3$**.** This equation is nonlinear because of the y^2 term.

(ii) We have, $(\sin y) y'' + (\cos x) y' + 3y = 0$**.** This equation is linear. There is no term involving a power or function of y, and the coefficients are all functions of x. The equation is already written in standard form, and F(x) is identically zero, so the equation is homogeneous.

(iii) We have, $4t^2x'' + 3txx' + 4x = 0$. This equation is nonlinear. Note that, in this case, x is the dependent variable and t is the independent variable. The second term involves the product of x and x', so the equation is nonlinear.

(iv) We have, $5y'' + y = 4x^5$ This equation is linear. Since $F(x) = 4x^5$, the equation is non homogeneous.

(v) We have, $(\cos x)y'' - \sin y' + (\sin x)y - \cos x = 0$. This equation is nonlinear, because of the $\sin y'$ term.

Test Your Progress 1
Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or non homogeneous. i. $8ty'' - 6t^2y' + 4ty - 3t^2 = 0$ ii. $\sin(x^2)y'' - (\cos x)y' + x^2y = y' - 3$ iii. $y'' + 5xy' - 3y = \cos y$

iv.	$\left(y''\right)^2 - y' + 8x^3 y = 0$
v.	$(\sin t)y'' + \cos t - 3ty' = 0$

9.4. General solution and Complimentary Function

We want to find a general solution (also known as complete solution) to a linear differential equation. Just as with first-order differential equations, a general solution (or family of solutions) gives the entire set of solutions to a differential equation. An important difference between first-order and second-order and higher order equations is that, with second-order and higher order equations, we typically need to find two different solutions or more solutions to the equation to find the general solution. If we find two or more solutions, then any linear combination of these solutions is also a solution. We state this fact as the following theorem.

Theorem 9.4.1: Superposition Principle

(1) If $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation of nth order, then the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants, is also a solution.

Consider, if $y_1(x)$ and $y_2(x)$ are only two solutions of the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + a_n y = 0 \quad \dots\dots\dots\dots\dots\dots \quad (1) \text{ then,}$$

$[say]\ u(x) = c_1 y_1(x) + c_2 y_2(x)$ is also its solution.

Since $y = y_1(x)$ and $y = y_2(x)$ are solutions of equation (1)

$$\therefore \quad a_0 \frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + a_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \dots\dots\dots + a_n y_1 = 0 \quad \dots\dots\dots\dots\dots\dots \ (2)$$

$$and \quad a_0 \frac{d^n y_2}{dx^n} + a_1 \frac{d^{n-1} y_2}{dx^{n-1}} + a_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \dots\dots\dots + a_n y_2 = 0 \quad \dots\dots\dots\dots\dots\dots \ (3)$$

(2) If c_1 and c_2 be two arbitrary constants, then we get

$$\text{LHS} = a_0 \frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + a_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \dots\dots\dots\dots + a_n (c_1 y_1 + c_2 y_2)$$

$$= c_1 \left[a_0 \frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots\dots\dots\dots + a_n y_1 \right] + c_2 \left[a_0 \frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots\dots\dots\dots + a_n y_1 \right]$$

$$= c_1(0) + c_2(0) = 0 = \text{RHS}$$

Thus, $a_0 \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + a_2 \frac{d^{n-2} u}{dx^{n-2}} + \dots\dots\dots + a_n u = 0 \quad \dots\dots\dots\dots\dots\dots$ (4). This proves the theorem.

Since the general solution of a differential equation of n[th] order contains 'n' arbitrary constants, it follows from above that if $y_1, y_2, y_3,$ ...., y_n are 'n' independent solutions of equation (1), then

$y = c_1y_1 + c_2y_2 + c_3y_3 +$ ....$+ c_n\, y_n$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$a_0 \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1}u}{dx^{n-1}} + a_2 \frac{d^{n-2}u}{dx^{n-2}} + \ldots\ldots\ldots + a_n u = X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$ then

$$a_0 \frac{d^n v}{dx^n} + a_1 \frac{d^{n-1}v}{dx^{n-1}} + a_2 \frac{d^{n-2}v}{dx^{n-2}} + \ldots\ldots\ldots + a_n v = X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Adding equations (4) and (6), we have

$$a_0 \frac{d^n (u+v)}{dx^n} + a_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + a_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + \ldots\ldots\ldots + a_n (u+v) = X$$

This shows that $y = u + v$ is the complete solution of equation (5).

The part u is called the complementary function (C. F) and the part v is called the particular integral (P.I) of equation (5).

Therefore, the complete solution (C. S) of equation (5) is $y = C.F + P.I$.

Thus in order to solve the equation (5), we have to first find the C. F., i.e., the complete solution of (1), and then the P.I., i.e., a particular solution of (5).

For Example: To solve the linear differential equation of second order

$a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$. Where a, b, and c are constants.

Solution: We have, $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$ .... (1)

Since all the coefficients are constants, the solutions are probably going to be functions with derivatives that are constant multiples of themselves. We need all the terms to cancel out, and if taking a derivative introduce a term that is not a constant multiple of the original function, it is difficult to see how that term cancels out. Exponential functions have derivatives that are constant multiples of the original function, so let's see what happens when we try a solution of the form $y(x)=e^{\lambda x}$, where λ is some constant.

$If\ \ y(x)=e^{\lambda x},\ then\ \ y^{/}(x)=\lambda e^{\lambda x}\ \ and\ \ y^{//}(x)=\lambda^2 e^{\lambda x}$. Substituting these expressions into equation (1), we get

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=ay^{//}+by^{/}+cy=a(\lambda^2e^{\lambda x})+b(\lambda e^{\lambda x})+ce^{\lambda x}=e^{\lambda x}(a\lambda^2+b\lambda+c)\,.$$

Since $e^{\lambda x}$ is never zero, this expression can be equal to zero for all x only if $a\lambda^2+b\lambda+c=0$. This is called as the characteristic equation of the differential equation.

Definition 9.4.1: The characteristic equation of the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \; Or \; ay^{//}+by^{/}+cy=0 .$$

The characteristic equation is very important in finding solution to differential equations of this form. We can solve the characteristic equation either by factorising or by using the quadratic formula $\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This gives four cases. The characteristic equation has

i. Distinct real roots

ii. A single, repeated real root

iii. Complex conjugate roots

iv. Complex conjugate repeated roots

Before considering each of these cases separately, let's have the idea of operator Din solving the linear differential equations of higher order.

Operator D: Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc, by D, D^2, D^3 etc, so that $\frac{dy}{dx}=Dy, \frac{d^2}{dx^2}=D^2y, \frac{d^3}{dx^3}=D^3y$ etc, the equation (5) above can be written in the symbolic form $(a_0D^n+a_1D^{n-1}+.............+a_n)y=X$, i.e., f(D)y = X, where $f(D)=a_0D^n+a_1D^{n-1}+.............+a_n$, i.e., a polynomial in D.

Thus, the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=(D^2+2D-3)y=(D+3)(D-1)y\ Or\ (D-1)(D+3)y$.

9.5. Methods of finding complimentary function

To solve the equation $a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_ny=0$

.... (1)

Where $a_0, a_1, a_2, ... a_n$ are constants.

The equation (1) in operator form is $(a_0D^n+a_1D^{n-1}+.............+a_n)y=0$

……….. (2)

Its symbolic coefficient equated to zero i.e., $a_0D^n+a_1D^{n-1}+.............+a_n=0$ is called the auxiliary equation (A.E). Let m_1, m_2, ……………., m_n be its roots.

<u>**Case I:**</u> If all the roots be real and different, then equation (2) is equivalent to

$(D-m_1)+(D-m_2)+.............+(D-m_n)y=0$ ……………… (3)

Now, equation (3) will be satisfied by the solution of $(D-m_n)y=0$, i.e.,

$\frac{dy}{dx}-m_n y=0$. This represents Leibnitz's linear and $I.F=e^{-m_n x}$.

Therefore, its solution is $ye^{-m_n x}=c_n$ i.e., $y=c_n e^{m_n x}$.

Similarly, since the factors in equation (3) can be taken in any order, it will be satisfied by the solutions of $(D-m_1)y=0, (D-m_2)y=0$ etc i.e., by $y=c_2 e^{m_2 x}$ etc.

Thus, the complete solution of equation (1) is

$$y=c_1 e^{m_1 x}+c_2 e^{m_2 x}+\ldots\ldots\ldots\ldots+c_n e^{m_n x} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Case II: A single, repeated real root. i.e., if two roots are equal (i.e., $m_1 = m_2$), then equation (4) becomes

$$y=(c_1+c_2)e^{m_1 x}+c_3 e^{m_3 x}\ldots\ldots\ldots\ldots+c_n e^{m_n x}$$
$$Or\ y=Ce^{m_1 x}+c_3 e^{m_3 x}\ldots\ldots\ldots\ldots+c_n e^{m_n x} \qquad [\because\ c_1+c_2=one\ arbitrary\ constant\ C$$

It has only n-1 arbitrary constants and is, therefore, not the complete solution of equation (1). In this case, we proceed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D-m_1)(D-m_1)y=0$.

Putting $(D-m_1)y=z$, *it becomes* $(D-m_1)z=0$ *Or* $\frac{dz}{dx}-m_1 z=0$

This is a Leibnitz's linear in z and $I.F = e^{-m_n x}$.

Therefore, its solution is $z e^{-m_n x} = c_1 \;\; Or \;\; z = c_1 e^{m_1 x}$.

Thus, $(D - m_1) y = z = c_1 e^{m_1 x} \;\; Or \;\; \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ …………………… (5)

Its I.F. being $e^{-m_n x}$, the solution of equation (5) is

$$y e^{-m_1 x} = \int c_1 e^{m_1 x} dx + c_2 = c_1 x + c_2 \;\; Or \;\; y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of equation (1) is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x} .$$

If, however, the A.E. has three equal roots (i.e., $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + ... + c_n e^{m_n x} .$$

Case III: Complex conjugate roots.

If one pair of roots be imaginary, i.e., $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$, then the complete solution is

$$\begin{aligned} y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x} \\ &= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + ... + c_n e^{m_n x} \\ &= e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + ... + c_n e^{m_n x} \end{aligned}$$

[$\because$ *by Euler's Theorem*, $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x] + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Where $C_1 = c_1 + c_2$ *and* $C_2 = i(c_1 - c_2)$.

Case IV: Complex conjugate repeated roots

If two points of imaginary roots be equal i.e.,

$m_1 = m_2 = \alpha + i\beta,\ m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos \beta x + (c_3 x + c_4)\sin \beta x] + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Example 9.5.1: Solve $\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 0$.

Solution: Given equation is $\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 0$ *Or* $y'' + 3y' - 4y = 0$

.... (1). This represents homogeneous linear ODE of second order with constant coefficients.

The symbolic form of equation (1) is $(D^2 + 3D - 4)y = 0$.

The corresponding auxiliary equation is $D^2 + 3D - 4 = 0$

$\Rightarrow D^2 + 4D - D - 4 = 0$ *Or* $(D+4)(D-1) = 0,\ \therefore\ D = 1,\ -4$, these are real and distinct roots.

Hence the general solution (Or complete solution) of equation (1) is

$y = c_1 e^{x} + c_2 e^{-4x}$.

Example 9.5.2: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$.

Solution: Given equation is $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$ *Or* $y'' + 6y' + 13y = 0$

.... (1). This represents homogeneous linear ODE of second order with constant coefficients.

The symbolic form of equation (1) is $(D^2 + 6D + 13)y = 0$.

The corresponding auxiliary equation is $D^2 + 6D + 13 = 0$

$\Rightarrow D = \frac{-6 \pm \sqrt{36 - 4(1)(13)}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$, these are real and

$\therefore\ D = -3 + 2i,\ -3 - 2i$

distinct roots.

Hence the general (Or complete) solution of equation (1) is

$y = e^{-3x}[c_1 \cos 2x + c_2 \sin 2x]$.

Example 9.5.3: Solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$.

Solution: Given equation is $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0$ Or $y^{//}+6y^{/}+9y=0$

.... (1). This represents homogeneous linear ODE of second order with constant coefficients.

The symbolic form of equation (1) is $(D^2+6D+9)y=0$.

The corresponding auxiliary equation is $D^2+6D+9=0$

$$\Rightarrow D=\frac{-6\pm\sqrt{36-4(1)(9)}}{2}=\frac{-6\pm 0}{2}=\frac{-6}{2}=-3\ (Twice)$$
$$\therefore\ D=-3,\ -3$$

These are real repeated roots.

Hence the general (Or complete) solution of equation (1) is $y=e^{-3t}[c_1t+c_2]$.

Example 9.5.4: Solve $\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=0$.

Solution: Given equation is $\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=0$ $Or\, y^{///}+y^{//}+4y^{/}+4y=0$

.... (1). This represents homogeneous linear ODE of third order with constant coefficients.

The symbolic form of equation (1) is $(D^3+D^2+4D+4)y=0$.

The corresponding auxiliary equation is $D^3 + D^2 + 4D + 4 = 0$

$$\Rightarrow D^2(D+1) + 4(D+1) \;\; Or \;\; (D^2+4)(D+1) = 0$$
$$\therefore \;\; (D^2+4) = 0 \Rightarrow D = \pm 2i \;\; and \;\; (D+1) = 0 \Rightarrow \;\; D = -1$$
$$\therefore \;\; D = -1, \; \pm 2i$$

Here, one is real root and another is complex conjugate case.

Hence the general (Or complete) solution of equation (1) is

$$y = c_1 e^{-t} + e^{0t}[c_2 \cos 2t + c_3 \sin 2t].$$

Or $y = c_1 e^{-t} + c_2 \cos 2t + c_3 \sin 2t$.

Example 9.5.5: Solve $\left[\dfrac{d^2y}{dt^2} + y\right]^3 = 0$.

Solution: Given equation is $\left[\dfrac{d^2y}{dt^2} + y\right]^3 = 0 \;\; Or \;\; [y^{//} + y]^3 = 0$ .... (1).

This represents homogeneous linear ODE of sixth order with constant coefficients.

The symbolic form of equation (1) is $(D^2+1)^3 y = 0$.

The corresponding auxiliary equation is $(D^2+1)^3 = 0$

$$\Rightarrow D^2 + 1 = 0 \;\; (Thrice)$$
$$\therefore \;\; D^2 = -1 \Rightarrow D = \pm i \;\; (Thrice)$$
$$\therefore \;\; D = \pm i, \pm i, \pm i$$

Here, it is a case of complex conjugate repeated roots.

Hence the general (Or complete) solution of equation (1) is

$$y = e^{0t}[(c_1 + c_2 t + c_3 t^2)\cos t + (c_4 + c_5 t + c_6 t^2)\sin t]$$
$$Or \;\; y = (c_1 + c_2 t + c_3 t^2)\cos t + (c_4 + c_5 t + c_6 t^2)\sin t$$

Example 9.5.6: Solve $\frac{d^4 y}{dt^4} + 4y = 0$.

Solution: Given equation is $\frac{d^4 y}{dt^4} + 4y = 0 \;\; Or\, y^{IV} + 4y = 0$ .... (1). This represents the homogeneous linear ODE of fourth order with constant coefficients.

The symbolic form of equation (1) is $(D^4 + 4)y = 0$.

The corresponding auxiliary equation is $D^4 + 4 = 0$

$$\Rightarrow (D^2)^2 + 4D^2 + 4 - 4D^2 = 0$$
$$\therefore \;\; (D^2 + 2)^2 - (2D)^2 = 0 \;\Rightarrow\; D^2 + 2D + 2 = 0,\; D^2 - 2D + 2 = 0$$
$$From,\; D^2 + 2D + 2 = 0,\;\; D = \frac{-2 \pm \sqrt{4-8}}{2} = 1 \pm i \;\; and$$
$$From,\; D^2 - 2D + 2 = 0,\;\; D = \frac{2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

Here, both the roots are complex conjugate with different real part.

Hence the general (Or complete) solution of equation (1) is

$$y = e^{t}[c_1 \cos t + c_2 \sin t] + e^{-t}[c_3 \cos t + c_4 \sin t].$$

Example 9.5.7: Solve the initial value problem,

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0 \ \ with \ \ y(0)=1, \ \ y^{/}(0)=0.$$

Solution: Given equation is $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0 \ \ Or \ \ y^{//}+2y^{/}+y=0$ ....

(1) This represents homogeneous linear ODE of second order with constant coefficients.

The symbolic form of equation (1) is $(D^2+2D+1)y=0$.

The corresponding auxiliary equation is $D^2+2D+1=0$

$\Rightarrow D^2+D+D+1=0 \ \ Or \ (D+1)^2=0, \ \therefore \ D=-1, \ -1$, these are real and repeated roots.

Hence the general solution (Or complete solution) equation (1) is

$y=(c_1x+c_2)e^{-x}$ .... (2)

Now, applying the given initial conditions: $y(0)=1$, equation (2) gives,

$1=c_2$

From equation (2), $y^{/}=-(c_1x+c_2)e^{-x}+c_1e^{-x}$, using the condition $y^{/}(0)=0$, we obtain

$0=-c_2+c_1 \ \ Or \ \ c_1=c_2 \ \Rightarrow c_1=1$.

Thus, the solution of the given initial value problem (known as particular solution) is $y=(x+1)e^{-x}$.

Example 9.5.8: Solve the initial value problem,

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0 \ \ with \ \ y(0)=0, \ \ y^{/}(0)=15.$$

Solution: Given equation is $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0 \ \ Or \ \ y^{//}+5y^{/}+6y=0$ .... (1) This represents homogeneous linear ODE of second order with constant coefficients.

The symbolic form of equation (1) is $(D^2+5D+6)y=0$.

The corresponding auxiliary equation is $D^2+5D+6=0$

$\Rightarrow D^2+3D+2D+6=0 \ \ Or \ (D+3)(D+2)=0, \ \therefore \ D=-2, \ -3$, these are real and repeated roots.

Hence the general solution (Or complete solution) equation (1) is

$y=c_1e^{-2x}+c_2e^{-3x}$ .... (2)

Now, applying the given initial conditions: $y(0)=0$, equation (2) gives,

$0=c_1+c_2$............... (3)

From equation (2), $y^{/} = -2c_1e^{-2x} - 3c_2e^{-3x}$, using the condition $y^{/}(0) = 15$, we obtain

$15 = -2c_1 - 3c_2 \;\; Or \;\; 2c_1 + 3c_2 = -15$ .... (4)

Solving equations (3) and (4), we get $c_1 = 15 \;\; and \;\; c_2 = -15$.

Thus, the solution of the given initial value problem (known as particular solution) is $y = 15e^{-2x} - 15e^{-3x} = 15(e^{-2x} - e^{-3x})$.

Test Your Progress 2

Solve the following differential equations.

1. $\frac{d^2y}{dt^2} - 16y = 0$
2. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$
3. $\frac{d^2y}{dt^2} + 16y = 0$
4. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$
5. $\frac{d^3y}{dt^3} + y = 0$
6. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$
7. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$
8. $(D^2 + 1)^2(D - 1)y = 0$
9. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0,\; y(0) = 0,\; y^{/}(0) = 2$.
10. $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 10y = 0,\; y(0) = 4,\; y^{/}(0) = 1$.

9.6. Methods of finding particular integrals

In this section, we examine how to solve non homogeneous differential equations. The terminology and methods are different from those we used for homogeneous equations, so let's start by defining some new terms.

Consider the non homogeneous linear differential equation

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_n y= f(x)$$

The associated homogeneous equation is

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_n y=0$$

is called the **auxiliary equation**. We have already seen that solving the auxiliary equation is an important step in solving a non homogeneous differential equation.

General Solution to a Non homogeneous Ordinary Linear Differential Equation:

Let $y_p(x)$ be any particular solution to the non homogeneous linear differential equation

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_n y= f(x)$$

Also, let $y_c = c_1 y_1(x)+c_2 y_2(x)+..........+c_n y_n(x)$ denote the general solution to the auxiliary equation. Then, the general solution of the non homogeneous linear differential equation is given by $y = y_c + y_p$. Where y_c is known as complementary function and y_p is called particular integral.

In the preceding section, we learned how to solve homogeneous equations with constant coefficients. Therefore, for non homogeneous equations of the form $a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_n y=F(x)$, we already know how to solve the auxiliary equation, and the problem boils down to finding a particular solution for the non homogeneous equation. We now examine two techniques for this: the method of undetermined coefficients and the method of variation of parameters.

9.7. Method of Undetermined Coefficients

To find the particular integral of

$[f(D)]y = [a_0 D^n + a_1 D^{n-1} + ...+a_n]y = F(x)$. We assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of F(x). Thus when

i) F(x) is an exponential: If $F(x) = 2e^{3x}$, trial solution $= ae^{3x}$

ii) F(x) is trigonometrical: If $F(x) = 3\sin 2x$, trial solution

$= a_1 \sin 2x + a_2 \cos 2x$

iii) F(x) is a polynomial: If $F(x) = 2x^3$, trial solution

$= a_1 x^3 + a_2 x^2 + a_3 x + a_4$

However when $F(x) = \tan x \ Or \ \sec x$, this method fails, since the number of terms obtained by differentiating $F(x) = \tan x \ Or \ \sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of $'x'$ which is large enough so that none of the terms which are then present, appear in the C.F.

Note: However, even if $F(x)$ included a sine term only or a cosine term only, both terms must be present in the trial solution. The method of undetermined coefficients also works with products of polynomials, exponentials, sines, and cosines.

Table 1. Some of the key forms of F(x) and the associated trial solutions for $y_p(x)$ are summarized in the following table.

Sl. No.	**F(x)**	**Trial solutions for $y_p(x)$**
1	k (a constant)	A (a constant)
2	$ax+b$	$Ax+B$ (Note: The trial must include both terms even if $b=0$)
3	ax^2+bx+c	Ax^2+Bx+c (Note: The trial must include all three terms even if b or c are zero)
4	Higher order polynomials	Polynomial of the same order as F(x)
5	$ae^{\lambda x}$	$Ae^{\lambda x}$
6	$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (Note: The trial must include all three terms even if b or c are zero)

7	$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
8	$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+c)e^{\lambda x}$
9	$(a_0x^2+a_1x+a_2)\cos\beta x+(b_0x^2+b_1x+b_2)\sin\beta x$	$(A_0x^2+A_1x+A_2)\cos\beta x+$ $(B_0x^2+B_1x+B_2)\sin\beta x$
10	$(a_0x^2+a_1x+a_2)e^{\alpha x}\cos\beta x+(b_0x^2+b_1x+b_2)e^{\alpha x}\sin$	$(A_0x^2+A_1x+A_2)e^{\alpha x}\cos\beta x+$ $(B_0x^2+B_1x+B_2)e^{\alpha x}\sin\beta x$

9.7.1 Working Method of Undetermined Coefficients

Step 1. Solve the auxiliary equation and write down the general solution.

Step 2: Based on the form of F(x), make an initial trial solution for $y_p(x)$.

Step 3: Check whether any term in the trial solution for $y_p(x)$ is a solution to the auxiliary equation. If so, multiply the trial solution by $'x'$. Repeat this step until there are no terms in $y_p(x)$ that solves the auxiliary equation.

Step 4: Substitute $y_p(x)$ into the differential equation and equate like terms to find values for the unknown coefficients in $y_p(x)$.

Step 5: Add the general solution to the auxiliary equation and the particular integral you just found to obtain the general solution to the non homogeneous equation.

Example 9.7.1: Find the general solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2e^{3x}$.

Solution: We have $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2e^{3x} \quad Or \quad y'' - y' - 2y = 2e^{3x}$ .... (1)

Here, we have $F(x) = 2e^{3x}$

The symbolic form of the equation (1) is $(D^2 - D - 2)y = 2e^{3x}$

The corresponding auxiliary equation is $D^2 - D - 2 = 0$

$\therefore\ D^2 - D - 2 = 0 \Rightarrow (D+1)(D-2) = 0 \Rightarrow D = -1, 2$.

The complementary function is given by $C.F = y_c(x) = c_1 e^{-x} + c_2 e^{2x}$.

Now, to find the particular integral $y_p(x)$, by the method of undetermined coefficients,

Since $F(x) = 2e^{3x}$, the particular solution might have the form $y_p(x) = Ae^{3x}$.

Then, we have $y_p'(x) = 3Ae^{3x}$ and $y_p''(x) = 9Ae^{3x}$. For $y_p(x)$ to be the part of the solution to the differential equation, we must find a value for A such that

$$y'' - y' - 2y = 2e^{3x}$$
$$\therefore\ 9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = 2e^{3x}$$
$$\Rightarrow\ 4Ae^{3x} = 2e^{3x}$$

$\therefore\ 4A = 2\ Or\ A = \frac{1}{2}$ Then, $y_p(x) = \left(\frac{1}{2}\right)e^{3x}$.

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{2}e^{3x}.$$

Example 9.7.2: Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$.

Solution: We have $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}\ \ Or\ \ y'' + 2y' + 4y = 2x^2 + 3e^{-x}$

.... (1)

Here, we have $F(x) = 2x^2 + 3e^{-x}$

The symbolic form of the equation (1) is $(D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$

The corresponding auxiliary equation is $D^2 + 2D + 4 = 0$

$$\therefore\ D^2 + 2D + 4 = 0 \Rightarrow D = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}.$$

The complementary function is given by

$$C.F = y_c(x) = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x).$$

Now, to find the particular integral $y_p(x)$, by the method of undetermined coefficients,

Since $F(x) = 2x^2 + 3e^{-x}$, the particular solution might have the form

$y_p(x) = a_0 x^2 + a_1 x + a_2 + a_3 e^{-x}$. Then, we have $y_p'(x) = 2a_0 x^2 + a_1 - a_3 e^{-x}$ and

$y_p^{//}(x) = 4a_0 x + a_3 e^{-x}$. For $y_p(x)$ to be solution to the differential equation.

Substituting these in the given equation, we get

$$y^{//} + 2y^{/} + 4y = 2x^2 + 3e^{-x}$$
$$4a_0 x + a_3 e^{-x} + 2(2a_0 x^2 + a_1 - a_3 e^{-x}) + 4(a_0 x^2 + a_1 x + a_2 + a_3 e^{-x}) = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$\therefore\ 4a_0 = 2,\quad 4a_0 + 4a_1 = 0,\ \ 2a_0 + 2a_1 + 4a_2 = 0,\ \ 3a_3 = 3$$

Then, $a_0 = \frac{1}{2},\ a_1 = -\frac{1}{2},\ a_2 = 0,\ a_3 = 1$.

Thus $P.I. = y_p = \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$

Thus the general (Or complete) solution is

$$y = C.F + P.I = y_c(x) + y_p(x) = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}.$$

Example 9.7.3: Find the general solution of $\frac{d^2y}{dx^2} + y = \sin x$.

Solution: We have $\frac{d^2y}{dx^2} + y = \sin x\ \ Or\ \ y^{//} + y = \sin x$ .... (1)

Here, we have $F(x) = \sin x$

The symbolic form of the equation (1) is $(D^2 + 1)y = \sin x$

The corresponding auxiliary equation is $D^2 + 1 = 0$

$\therefore\ D^2 + 1 = 0\ \ Or\ \ D^2 = -1\ \ \Rightarrow D = \pm i$.

The complementary function is given by $C.F = y_c(x) = (c_1 \cos x + c_2 \sin x)$.

Now, to find the particular integral $y_p(x)$, by the method of undetermined coefficients,

Since $F(x) = \sin x$, Let's assume the trial solution to have the particular solution, be of the form $y_p(x) = x(a_1 \cos x + a_2 \sin x)$ as these terms appear in the C.F, therefore we multiply it by x. Then, we have

$y_p^{/}(x) = (a_1 + a_2 x)\cos x + (a_2 - a_1 x)\sin x$ and

$y_p^{//}(x) = (2a_2 - a_1 x)\cos x - (2a_1 + a_2 x)\sin x$. For $y_p(x)$ to be solution to the differential equation. Substituting these in the given equation, we get

$$
\begin{aligned}
&y^{//} + y = \sin x \\
&(2a_2 - a_1 x)\cos x - (2a_1 + a_2 x)\sin x + x(a_1 \cos x + a_2 \sin x) = \sin x \\
&2a_2 \cos x - 2a_1 \sin x = \sin x
\end{aligned}
$$

Equating corresponding coefficients on both sides, we get

$$\therefore\ -2a_1 = 1 \ \Rightarrow\ a_1 = -\frac{1}{2},\ \ 2a_2 = 0 \ \Rightarrow\ a_2 = 0$$

Thus $P.I. = y_p = -\frac{1}{2} x \cos x$

Thus the general (Or complete) solution is

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 \cos x + c_2 \sin x) - \frac{1}{2} x \cos x.$$

Example 9.7.4: Find the general solution of $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 7\sin x - \cos x$.

Solution: We have

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=7\sin x-\cos x \;\; Or \;\; y^{//}-4y^{/}+4y=7\sin x-\cos x \;\;\; .... \;\; (1)$$

Here, we have $F(x)=7\sin x-\cos x$

The symbolic form of the equation (1) is $(D^2-4D+4)y=7\sin x-\cos x$

The corresponding auxiliary equation is $D^2-4D+4=0$

$\therefore \; (D-2)^2=0 \;\; Or \;\; D=2,\, 2$

The complementary function is given by $C.F=y_c(x)=(c_1x+c_2)e^{2x}$.

Now, to find the particular integral $y_p(x)$, by the method of undetermined coefficients,

Since $F(x)=7\sin x-\cos x$, Let's assume the trial solution to have the particular solution, be of the form $y_p(x)=(a_1\cos x+a_2\sin x)$. Then, we have $y_p^{/}(x)=-a_1\sin x+a_2\cos x$ and $y_p^{//}(x)=-a_1\cos x-a_2\sin x$. For $y_p(x)$ to be solution to the differential equation. Substituting these in the given equation, we get

$$y^{//}-4y^{/}+4y=7\sin x-\cos x$$
$$\therefore \; (-a_1\cos x-a_2\sin x)-4(-a_1\sin x+a_2\cos x)+4(a_1\cos x+a_2\sin x)=7\sin x-\cos x$$
$$Or \; (3a_1-4a_2)\cos x+(4a_1+3a_2)\sin x=7\sin x-\cos x$$

Equating corresponding coefficients on both sides, we get

$3a_1-4a_2=-1 \;\; and \;\; 4a_1+3a_2=7$, solving both the equations for $a_1 \;\; and \;\; a_2$, we obtain

$a_1 = 1 \ and \ a_2 = 1$.

Thus $P.I. = y_p = \cos x + \sin x$

Thus the general (Or complete) solution of eequation (1) is

$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 x + c_2)e^{2x} + \cos x + \sin x$.

Example 9.7.5: Find the general solution of $\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x$.

Solution: We have

$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x \ \ Or \ \ y'' - y = e^{3x}\cos 2x - e^{2x}\sin 3x$ .... (1)

Here, we have $F(x) = e^{3x}\cos 2x - e^{2x}\sin 3x$

The symbolic form of the equation (1) is $(D^2 - 1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$

The corresponding auxiliary equation is $D^2 - 1 = 0$

$\therefore \ D^2 - 1 = 0 \ \ Or \ \ D^2 = 1 \ \ \Rightarrow D = \pm 1$.

The complementary function is given by $C.F = y_c(x) = (c_1 e^{-x} + c_2 e^{x})$.

Now, to find the particular integral $y_p(x)$, by the method of undetermined coefficients,

Since $F(x) = e^{3x}\cos 2x - e^{2x}\sin 3x$, Let's assume the trial solution to have the particular solution, be of the form

$y_p(x) = e^{3x}(a_1\cos 2x + a_2\sin 2x) - e^{2x}(a_3\cos 3x + a_4\sin 3x)$ Then, we have

$y_p'(x) = e^{3x}[(3a_1 + 2a_2)\cos 2x + (3a_2 - 2a_1)\sin 2x] - e^{2x}[(2a_3 + 3a_4)\cos 3x + (2a_4 - 3a_3)\sin 3x]$

and

$$y_p^{//}(x) = e^{3x}[(5a_1 + 12a_2)\cos 2x + (5a_2 - 12a_1)\sin 2x] - e^{2x}[(12a_4 - 5a_3)\cos 3x - (5a_4 + 12a_3)\sin 3x]$$

For $y_p(x)$ to be solution to the differential equation. Substituting these in the given equation, we get

$$y^{//} - y = e^{3x}\cos 2x - e^{2x}\sin 3x$$
$$e^{3x}[(5a_1 + 12a_2)\cos 2x + (5a_2 - 12a_1)\sin 2x] - e^{2x}[(12a_4 - 5a_3)\cos 3x - (5a_4 + 12a_3)\sin 3x] -$$
$$e^{3x}(a_1\cos 2x + a_2\sin 2x) - e^{2x}(a_3\cos 3x + a_4\sin 3x) = e^{3x}\cos 2x - e^{2x}\sin 3x$$

Equating corresponding coefficients on both sides, we get

$$\therefore\ 4a_1 + 12a_2 = 1, \quad 4a_2 - 12a_1 = 0; \quad 12a_4 - 6a_3 = 0, \quad 6a_4 + 12c_3 = -1.$$

Solving for a_1, a_2, a_3 *and* a_4, we obtain $a_1 = \frac{1}{40}, a_2 = \frac{3}{40}, a_3 = -\frac{1}{15}$ *and* $a_4 = -\frac{1}{30}$

Thus $P.I = y_p = \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$

Thus the general (Or complete) solution of equation (1) is

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1e^{-x} + c_2e^{x}) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$$

Test Your Progress 3
Find the general solution of the following differential equations by finding the P.I.by Method of Undetermined Coefficients. 1. $y^{//} - 9y = -6\cos 3x$ 2. $y^{//} + 2y^{/} + y = 4e^{-x}$ 3. $y^{//} - 2y^{/} + 5y = 10x^2 - 3x - 3$

4. $y^{//} - 3y^{/} = -12t$	5. $(D^2 - 3D + 2)y = x^2 + e^x$	6. $\frac{d^2y}{dx^2} + y = 2\cos x$
7. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x$	8. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$	
9. $(D^2 - 2D + 3)y = x^3 + \cos x$	10. $(D^2 - 2D)y = e^x \sin x$	

9.8. Method of Variation of Parameters

Sometimes, $F(x)$ is not a combination of polynomials, exponentials, or sines and cosines. When this is the case, the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the **method of variation of parameters**.

To simplify our calculations a little, we are going to divide the differential equation through by a, so we have a leading coefficient of 1. Then the differential equation has the form

$y'' + py' + qy = F(x),$

where p and q are constants.

If the general solution to the complementary equation is given by $c_1y_1(x) + c_2y_2(x)$, we are going to look for a particular solution of the form

$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are assuming the coefficients are functions of *x*, rather than constants. We want to find functions *u*(*x*) and *v*(*x*) such that $y_p(x)$ satisfies the differential equation. We have

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \ (1)$$

$$y_p'(x) = u'(x)y_1(x) + u(x)y_1'(x) + v'(x)y_2(x) + v(x)y_2'(x)$$

$$y_p''(x) = (u'(x)y_1(x) + v'(x)y_2(x))' + (u'(x)y_1'(x) + u(x)y_1''(x) + v'(x)y_2'(x) + v(x)y_2''(x)$$

Substituting into the differential equation, we obtain

$$y'' + py' + qy =$$
$$y(u'(x)y_1(x) + v'(x)y_2(x))' + (u'(x)y_1'(x) + u(x)y_1''(x) + v'(x)y_2'(x) + v(x)y_2''(x)$$
$$+ p[u'(x)y_1(x) + u(x)y_1'(x) + v'(x)y_2(x) + v(x)y_2'(x)] + q[u(x)y_1(x) + v(x)y_2(x)$$

$$= u[y_1''(x) + py_1'(x) + qy_1(x)] + v(x)[y_2''(x) + py_2'(x) + qy_2(x)] + [u'(x)y_1(x) + v'(x)y_2(x)]' +$$
$$p[u'(x)y_1(x) + v'(x)y_2(x)] + [u'(x)y_1'(x) + v'(x)y_2'(x)]$$

Note that y_1 *and* y_2 are the solutions to the auxiliary equation, so the first two terms are zero. Thus, we have

$$[u'(x)y_1(x) + v'(x)y_2(x)]' + p[u'(x)y_1(x) + v'(x)y_2(x)] + [u'(x)y_1'(x) + v'(x)y_2'(x)] = F(x)$$

If we simplify this equation by imposing the additional condition $u'(x)y_1(x) + v'(x)y_2(x) = 0$, the first two terms are zero, and this reduces to

$u'(x)y_1(x)+v'(x)y_2(x)=F(x)$. So, with this additional condition, we have a system of two equations in two unknowns:

$$u'(x)y_1(x)+v'(x)y_2(x)=0$$

$$u'(x)y_1(x)+v'(x)y_2(x)=F(x)$$

Solving this system (by using Cramer's rule Or any suitable technique) gives us $u'=\frac{-y_2(x)\cdot F(x)}{W(x)}$ *and* $v'=\frac{y_1(x)\cdot F(x)}{W(x)}$, which we can integrate to find u *and* v. Where, $W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$

Substituting these in equation (1), we obtain the $y_p(x)$.

9.8.1. Working Method of Variation of Parameters

Step 1: Solve the auxiliary equation and write down complementary function: $y_c(x)=c_1y_1(x)+c_2y_2(x)$.

Step 2: Assume $y_p(x)=u(x)y_1(x)+v(x)y_2(x)$. Then, determine Wronskian $W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}=y_1(x)y_2'(x)-y_2(x)y_1'(x)$.

Step 3: Find $u \ and \ v$ using the formulae

$$u = -\int \frac{y_2(x)\cdot F(x)}{W(x)}\cdot dx \, and \ \ v = \int \frac{y_1(x)\cdot F(x)}{W(x)}\cdot dx \, .$$

Step 4: Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is the particular integral to the equation.

Step 5: Add the complementary function and the particular integral to obtain the general solution to the non homogeneous differential equation.

Example 9.8.1: Find the general solution of $\frac{d^2y}{dx^2} + y = \sec x$.

Solution: We have, $\frac{d^2y}{dx^2} + y = \sec x \quad Or \quad y'' + y = \sec x$ .... (1)

Here, we have $F(x) = \sec x$

The symbolic form of the equation (1) is $(D^2 + 1)y = \sec x$

The corresponding auxiliary equation is $D^2 + 1 = 0$

$\therefore \ D^2 + 1 = 0 \ \ Or \ \ D^2 = -1 \ \ \Rightarrow D = \pm i$.

The complementary function is given by $C.F = y_c(x) = (c_1 \cos x + c_2 \sin x)$.

Now, to find the particular integral $y_p(x)$, by the method of variation of parameters,

Let $y_p(x) = u(x)y_1(x) + v(x)y_2(x) = u(x)\cos x + v(x)\sin x$

$$\therefore W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x)\end{vmatrix} = y_1(x)y_2'(x)-y_2(x)y_1'(x)$$

$$=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x\end{vmatrix} = \cos^2 x+\sin^2 x=1$$

Now,

$$u=-\int \frac{y_2(x)\cdot F(x)}{W(x)}\cdot dx=-\int \frac{\sin x\cdot \sec x}{1}\cdot dx=-\log(\sec x)=\log(\cos x)$$

$$and \;\; v=\int \frac{y_1(x)\cdot F(x)}{W(x)}\cdot dx=\int \frac{\cos x\cdot \sec x}{1}\cdot dx=x$$

$$\therefore \;\; y_p(x)=\log(\cos x)\cos x+x\sin x$$

Thus the general (Or complete) solution of equation (1) is

$$y=C.F+P.I=y_c(x)+y_p(x)=(c_1\cos x+c_2\sin x)+\log(\cos x)\cos x+x\sin x$$

Example 9.8.2: Find the general solution of $\frac{d^2y}{dx^2}+4y=\tan 2x$.

Solution: We have, $\frac{d^2y}{dx^2}+4y=\tan 2x \;\; Or \;\; y''+y=\tan 2x$ .... (1)

Here, we have $F(x)=\tan 2x$

The symbolic form of the equation (1) is $(D^2+4)y=\tan 2x$

The corresponding auxiliary equation is $D^2+1=0$

$\therefore \; D^2+4=0 \;\; Or \;\; D^2=--4 \;\; \Rightarrow D=\pm 2i$.

The complementary function is given by $C.F=y_c(x)=(c_1\cos 2x+c_2\sin 2x)$.

Now, to find the particular integral $y_p(x)$, by the method of variation of parameters,

Let $y_p(x) = u(x)y_1(x) + v(x)y_2(x) = u(x)\cos 2x + v(x)\sin 2x$

$$\therefore W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1^{/}(x) & y_2^{/}(x) \end{vmatrix} = y_1(x)y_2^{/}(x) - y_2(x)y_1^{/}(x)$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin x & 2\cos x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

Now,

$$u = -\int \frac{y_2(x)\cdot F(x)}{W(x)} \cdot dx = -\int \frac{\sin 2x \cdot \tan 2x}{2} \cdot dx = -\frac{1}{2}\int (\sec 2x - \cos 2x)dx$$
$$= -\frac{1}{4}[\log(\sec 2x + \tan 2x) - \sin 2x]$$
$$and \quad v = \int \frac{y_1(x)\cdot F(x)}{W(x)} \cdot dx = \int \frac{\cos 2x \cdot \tan 2x}{2} \cdot dx = \frac{1}{2}\int \sin 2x \cdot dx$$
$$= -\frac{1}{4}\cos 2x$$

$$\therefore \quad y_p(x) = -\frac{1}{4}[\log(\sec 2x + \tan 2x) - \sin 2x]\cos 2x - \frac{1}{4}\cos 2x \sin 2x$$
$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x)]$$

Thus the general (Or complete) solution of equation (1) is

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1\cos 2x + c_2 \sin 2x) - \frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x)]$$

Example 9.8.3: Find the general solution of $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$.

Solution: We have, $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$ Or $y^{//}-6y^{/}+9y=\frac{e^{3x}}{x^2}$ …………….. (1)

Here, we have $F(x)=\frac{e^{3x}}{x^2}$

The symbolic form of the equation (1) is $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$

The corresponding auxiliary equation is $D^2-6D+9=0$

$\therefore\ D^2-6D+9=0\ \ Or\ \ (D-3)^2=0\ \ \Rightarrow D=3,3.$

The complementary function is given by $C.F=y_c(x)=(c_1+c_2x)e^{3x}$.

Now, to find the particular integral $y_p(x)$, by the method of variation of parameters,

Let $y_p(x)=u(x)y_1(x)+v(x)y_2(x)=u(x)e^{3x}+v(x)xe^{3x}$

$$\therefore W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1^{/}(x) & y_2^{/}(x) \end{vmatrix}=y_1(x)y_2^{/}(x)-y_2(x)y_1^{/}(x)$$

$$=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x}+e^{3x} \end{vmatrix}=3xe^{6x}+e^{6x}-3xe^{6x}=e^{6x}$$

Now,

$$u=-\int\frac{y_2(x)\cdot F(x)}{W(x)}\cdot dx=-\int\frac{xe^{3x}\cdot e^{3x}}{e^{6x}\cdot x^2}\cdot dx=-\int\frac{1}{x}\cdot dx=-\log x$$

$$and\ \ v=\int\frac{y_1(x)\cdot F(x)}{W(x)}\cdot dx=\int\frac{e^{3x}\cdot e^{3x}}{e^{6x}\cdot x^2}\cdot dx=\int\frac{1}{x^2}\cdot dx=-\frac{1}{x}$$

$\therefore\ y_p(x) = (-\log x)\cdot e^{3x} - \frac{1}{x}(xe^{3x}) = -e^{3x}(\log x + 1)$

Thus the general (Or complete) solution of equation (1) is

$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{3x} - e^{3x}(1 + \log x) = e^{3x}(c_1 + c_2 x - 1 - \log x)$

Example 9.8.4: Find the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$.

Solution: We have, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x\ \ Or\ \ y'' - 2y' + y = e^x \log x$ …………(1)

Here, we have $F(x) = e^x \log x$

The symbolic form of the equation (1) is $(D^2 - 2D + 1)y = e^x \log x$

The corresponding auxiliary equation is $D^2 - 2D + 1 = 0$

$\therefore\ D^2 - 2D + 1 = 0\ \ Or\ \ (D-1)^2 = 0\ \ \Rightarrow D = 1, 1$.

The complementary function is given by $C.F = y_c(x) = (c_1 + c_2 x)e^x$.

Now, to find the particular integral $y_p(x)$, by the method of variation of parameters,

Let $y_p(x) = u(x)y_1(x) + v(x)y_2(x) = u(x)e^x + v(x)xe^x$

$$\therefore W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

$$= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

Now,

$$u=-\int \frac{y_2(x)\cdot F(x)}{W(x)}\cdot dx=-\int \frac{xe^x\cdot e^x\log x}{e^{2x}}\cdot dx=-\int x\log x\cdot dx=-\frac{x^2}{2}\left(\log x-\frac{1}{2}\right)$$

$$and\ \ v=\int \frac{y_1(x)\cdot F(x)}{W(x)}\cdot dx=\int \frac{e^x\cdot e^x\log x}{e^{2x}}\cdot dx=\int \log x\cdot dx=x(\log x-1)$$

$$\therefore\ \ y_p(x)=-\frac{x^2}{2}\left(\log x-\frac{1}{2}\right)e^x-x(\log x-1)xe^x=-e^x\left[\frac{x^2}{2}\left(\log x-\frac{1}{2}\right)+x(\log x-1)x\right]$$

Thus the general (Or complete) solution of equation (1) is

$$y=C.F+P.I=y_c(x)+y_p(x)=(c_1+c_2x)e^x--e^x\left[\frac{x^2}{2}\left(\log x-\frac{1}{2}\right)+x(\log x-1)x\right]$$

Example 9.8.5: Find the general solution of $\frac{d^2y}{dx^2}-y=\frac{2}{(1+e^x)}$.

Solution: We have, $\frac{d^2y}{dx^2}-y=\frac{2}{(1+e^x)}$ *Or* $y^{//}-y=\frac{2}{(1+e^x)}$ ....

(1)

Here, we have $F(x)=e^x\log x$

The symbolic form of the equation (1) is $(D^2-1)y=\frac{2}{(1+e^x)}$

The corresponding auxiliary equation is $D^2-1=0$

$\therefore\ D^2-1=0 \quad \Rightarrow D=1,-1$.

The complementary function is given by $C.F=y_c(x)=(c_1e^x+c_2e^{-x})$.

Now, to find the particular integral $y_p(x)$, by the method of variation of parameters,

Let $y_p(x) = u(x)y_1(x) + v(x)y_2(x) = u(x)e^x + v(x)e^{-x}$

$$\therefore W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$
$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

Now,

$$u = -\int \frac{y_2(x) \cdot F(x)}{W(x)} \cdot dx = -\int \frac{e^{-x} \cdot 2}{(-2)(1+e^x)} \cdot dx = \int \frac{e^{-x}}{(1+e^x)} \cdot dx = \int \frac{1}{e^x(1+e^x)} \cdot dx$$
$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) \cdot dx = e^{-x} - \int \frac{e^{-x}}{e^{-x}+1} \cdot dx = -e^{-x} + \log(1+e^{-x})$$
$$and \;\; v = \int \frac{y_1(x) \cdot F(x)}{W(x)} \cdot dx = \int \frac{e^x \cdot 2}{(-2)(1+e^x)} \cdot dx = -\int \frac{e^x}{(1+e^x)} \cdot dx = -\log(1+e^x)$$

$$\therefore \;\; y_p(x) = e^x[-e^{-x} + \log(1+e^{-x})] - e^{-x}\log(1+e^x) = -1 + e^x \log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Thus the general (Or complete) solution of equation (1) is

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1e^x + c_2e^{-x}) - 1 + e^x \log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Test Your Progress 4
Find the general solution of the following differential equations by finding the P.I.by Method of Variation of parameters.

1. $y'' - 2y' + y = \frac{e^t}{t^2}$	2. $y'' + y = 3\sin^2 x$	3. $\frac{d^2y}{dx^2} + a^2 y = \cos ecax$
4. $\frac{d^2y}{dx^2} + y = \tan x$	5. $\frac{d^2y}{dx^2} + y = x\sin x$	6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$
7. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}$	8. $y'' - 2y' + 2y = e^x \tan x$	9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$
10. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$		

9.9 Summary

Method for finding the complementry fuction C.F. and methods for finding P.I. in certain standerd cases is given. Method of variation of parameters and method of undetermined coefficent has been discussed to find P.I. for non stunderd cases.

9.10. Terminal Questions

1. Solve

   $(D^2 - 4D + 3)y = x^3 . e^{2x}$

2. Solve

   $(D^2 - 2D + 1)y = xe^{-x} cosx$

3. Solve by method of variation of parameters

   i. $(D^2 - 2D + 2)y = e^x tanx$

   ii. $(D^2 + 1)y = cosecx$

4. Solve

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

Ans.

1. $y = c_1 e^x + c_2 e^{3x} - e^{2x}(x^3 + 6x)$
2. $y = (c_1 x + c_2)e^{-x} + e^{-x}(-xcosx + 2sinx)$
3. (i.) $y = e^x(c_1 cosx + c_2 sinx) - e^x cosx \log(secx + tanx)$

   (ii) $y = c_1 cosx + c_2 sinx - xcosx + sinx \log(sinx)$
4. $y = (c_1 x + c_2)e^{3x} - e^{3x}.(1 + logx)$

9.11. Answers to Check Your Progress

Test Your Progress 1

(i) We have, $8ty'' - 6t^2 y' + 4ty - 3t^2 = 0$. This equation is linear. Rewriting it in standard form gives $8ty'' - 6t^2 y' + 4ty = 3t^2$. With the equation in standard form, we can see that $F(t) = 3t^2$. So the equation is non homogeneous.

(ii) We have, $\sin(x^2)y'' - (\cos x)y' + x^2 y = y' - 3$. This equation looks like its linear, but we should rewrite it in standard form to be sure. We get $\sin(x^2)y'' - (\cos x + 1)y' + x^2 y = -3$. This equation is, indeed, linear. With $F(x) = -3$, it is non homogeneous.

(iii) We have, $y'' + 5xy' - 3y = \cos y$. This equation is nonlinear because of the cosy term.

(iv) We have, $\left(y''\right)^2 - y' + 8x^3 y = 0$. This equation is nonlinear because of $(y'')^2$ term.

(v) We have, $(\sin t)y'' + \cos t - 3ty' = 0$ *Or* $(\sin t)y'' - 3ty' = -\cos t$. This is linear. With $F(x) = -\cos t$, it is non homogeneous.

Test Your Progress 2

1. $y = c_1 e^{-4t} + c_2 e^{4t}$

2. $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$

3. $y = c_1 \cos 4t + c_2 \sin 4t$

4. $y = (c_1 + c_2 x)e^{-3x}$

5. $y = c_1 e^{-t} + e^{t/2}(c_2 \cos \frac{\sqrt{3}}{2} t + c_3 \sin 2x \frac{\sqrt{3}}{2} t)$

6. $y = (c_1 + c_2 x + c_3 x^2)e^x$

7. $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

8. $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + c_5 e^x$

9. $y = \frac{2}{3} e^{2x} \sin 3x$

10. $y = e^x (4\cos 3x - \sin 3x)$

Test Your Progress 3

1. $y = c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{3}\cos 3x$

2. $y = (c_1 + c_2 x)e^{-x} + 2t^2 e^{-x}$

3. $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 2x^2 + x - 1$

4. $y = c_1 e^{tx} + c_2 + 2t^2 + \frac{4}{3}t$

5. $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}(x^2 + 3x + 3.5 - 2xe^x)$

6. $y = c_1 \cos x + c_2 \sin x - x\sin x$

7. $y = c_1 e^{2x} + c_2 e^{3x} + xe^{3x} + \frac{1}{10}(\sin x + 3\cos x)$

8. $y = c_1 e^x + c_2 e^{-2x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

9. $y = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \frac{1}{4}(\cos x - \sin x)$

10. $y = c_1 + c_2 e^{2x} - \frac{1}{2}e^x \sin x$

Test Your Progress 4

1. $y = c_1 e^t + c_2 te^t - e^t \log t$

2. $y = c_1 \cos x + c_2 \sin x + 1 + \cos^2 x$

3. $y = (c_1 - x/a)\cos ax + [c_2 + (1/a^2)\log\sin ax]\sin ax$

4. $y = c_1 \cos x + c_2 \sin ax - \cos x \log(\sec x + \tan x)$

5. $y = c_1 \cos x + c_2 \sin x + \frac{x}{2}\sin x - \frac{x^2}{4}\cos x$

6. $y = (c_1 + c_2 x)e^x + xe^x \log x$

7. $y=(e^{x}+e^{2x})\log(1+e^{x})+(c_1-1-x)e^{x}+(c_2-x)e^{2x}$

8. $y=e^{x}(c_1\cos x+c_2\sin x)-e^{x}\cos x\log(\sec x+\tan x)$

9. $y=c_1+c_2e^{2x}-\frac{1}{2}e^{x}\sin x$

10. $y=c_1\cos x+c_2\sin x+\sin x\log(1+\sin x)-x\cos x-1$

Unit 10: Methods of finding particular integrals by inverse operator method

Structure

10.1. Introduction

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general differential equation of the nth order is of the form

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots + k_n y = F(x)$$

Where $k_0, k_1, k_2, \dots\dots\dots\dots\dots\dots\dots k_n$ are real valued functions, and k_0 is not identically zero. F(x) is function of 'x' only.

If $k_0, k_1, k_2, \dots\dots\dots\dots\dots\dots\dots k_n$ are all constants, then the equation is known as linear differential equation with constant coefficients.

Operator D: Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc, by D, D^2, D^3 etc, so that

$\frac{dy}{dx} = Dy, \frac{d^2}{dx^2} = D^2 y, \frac{d^3}{dx^3} = D^3 y$ etc, the equation (5) above can be written in the symbolic form $(k_0 D^n + k_1 D^{n-1} + \dots\dots\dots\dots + k_n) y = F(x)$, i.e., f(D)y = F(x), where $f(D) = k_0 D^n + k_1 D^{n-1} + \dots\dots\dots\dots + k_n$, i.e., a polynomial in D.

Thus, the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order.

Let $y_c = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$ denote the general solution to the auxiliary equation. Then, the general solution of the non homogeneous linear differential equation is given by $y = y_c + y_p$. Where y_c is known as complementary function and y_p is called particular integral.

In the preceding unit, we learnt how to solve non homogeneous equations with constant coefficients. Therefore, for non homogeneous equations of the form $k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + k_n y = F(x)$, we already know how to solve the auxiliary equation and to find a particular solution for the non homogeneous equation. We have seen two methods of finding particular integral. Now examine one more technique for this: the inverse operator method.

Inverse Operator

Definition 10.1.1: $\frac{1}{f(D)}F(x)$ is that function of x, not containing the arbitrary constants when operated upon by $f(D)$ gives $F(x)$.

$i.e.,\ f(D)\left\{\frac{1}{f(D)}F(x)\right\} = F(x)$.

Thus $\frac{1}{f(D)}F(x)$ satisfies the equation $f(D)y = F(x)$ and is, therefore, its particular integral.

Obviously, $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Definition 10.1.2: $\frac{1}{D}F(x) = \int F(x)\cdot dx$

Let $\frac{1}{D}F(x) = y$ . ... (1)

Operating by D,

$$D\frac{1}{D}F(x) = Dy \ \ i.e.,\ F(x) = \frac{dy}{dx}$$

Integrating both sides w. r. t. x, $y = \int F(x)\cdot dx$, no constant being added as equation (1) does not contain any constant.

Thus $\frac{1}{D}F(x) = \int F(x)\cdot dx$

Definition 10.1.3: $\frac{1}{D-a}F(x) = e^{ax}\int F(x)\cdot e^{-ax}dx$

Let $\frac{1}{D-a}F(x) = y$. ………….. (2)

Operating by $(D-a)$, $(D-a)\cdot\frac{1}{(D-a)}\cdot F(x) = (D-a)y$

Or $F(x) = \frac{dy}{dx} - ay$ *i.e.*, $\frac{dy}{dx} - ay = F(x)$. This represents a Leibnitz's linear equation.

$\therefore\ I.F = e^{\int -a\,dx} = e^{-a\,x}$, its general solution is given by

$y\cdot e^{\int -a\,dx} = \int F(x)\cdot e^{-a\,x}\cdot dx$, no constant being added as equation (2) does not contain any constant.

Thus $\frac{1}{(D-a)}\cdot F(x) = y = e^{ax}\int F(x)\cdot e^{-ax}\cdot dx$

Rules for Finding the Particular Integral

$$k_0\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + ... + k_n y = F(x)$$

It's symbolic form is $(k_0D^n + k_1D^{n-1} + ... + k_n)y = F(x)$. Then,

$$P.I = \frac{1}{(k_0 D^n + k_1 D^{n-1} + \ldots\ldots\ldots\ldots + k_n)} F(x)$$

10.2. Objectives

After reading this unit students should be able to:

- Identify the method of particular integral to apply
- Recognize the method of particular integral to solve the non homogeneous linear differential equations of higher order
- Determine the particular integrals by using inverse operator method

10.3. Case of Exponential Function

When $F(x) = e^{ax}$

Since $De^{ax} = a e^{ax}$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$D^3 e^{ax} = a^3 e^{kax}$$

....

....

$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore \quad (k_0 D^n + k_1 D^{n-1} + ...+ k_n) e^{ax} = (k_0 a^n + k_1 a^{n-1} + ...+ a_n) e^{ax}$$

$$i.e., \quad f(D) e^{ax} = f(a) e^{ax}$$

Operating on both sides by

$$\frac{1}{f(D)}, \ \frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad Or \quad e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$\therefore$ *Dividing by* $f(a)$,

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad .... \ (1), \quad \text{provided } f(a) \neq 0.$$

If $f(a) = 0$, the above rule fails and we proceed further.

Since 'a' is a root of auxiliary equation

$$f(D) = k_0 D^n + k_1 D^{n-1} + ...+ k_n = 0$$

$\therefore (D-a)$ is a factor of $f(D)$. Suppose $f(D) = (D-a)\cdot\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(a)}e^{ax}$$

$$= \frac{1}{\phi(a)}\cdot\frac{1}{D-a}\cdot e^{ax} = \frac{1}{\phi(a)}\cdot e^{ax}\int e^{ax}\cdot e^{-ax}dx$$

$$= \frac{1}{\phi(a)}\cdot e^{ax}\int dx = x\cdot\frac{1}{\phi(a)}e^{ax}$$

$$\therefore\ \frac{1}{f(D)}e^{ax} = x\frac{1}{f^{/}(a)}e^{ax} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\begin{bmatrix} \because\ f^{/}(D) = (D-a)\phi^{/}(D) + 1\phi(D) \\ \therefore\ f^{/}(a) = 0\times\phi^{/}(a)\cdot\phi(a) \end{bmatrix}$$

If $f^{/}(a) = 0$, then applying equation (2) again, we obtain,

$$\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f^{/}(a)}e^{ax},\ \ provided\ \ f^{/}(a) \neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3) and so on.}$$

Note 10.3.1:

Definition 10.3.1: If $F(x) = k$, where 'k' is arbitrary constant. Then,

$$y_p(x) = \frac{1}{f(D)}\cdot k = k\frac{1}{f(D)}e^{0x} = k\frac{1}{f(0)},\ \ provided\ \ f(0) \neq 0$$

$If\ \ f(0) = 0.\ Then,$

$$y_p(x) = k\cdot\frac{x}{f^{/}(D)}e^{0x} = k\cdot\frac{x}{f^{/}(0)},\ \ provided\ \ f^{/}(0) \neq 0$$

$If\ \ f^{/}(0) = 0.\ Then,$

$$y_p(x) = k\cdot\frac{x^2}{f^{//}(D)}e^{0x} = k\cdot\frac{x^2}{f^{//}(0)},\ \ provided\ \ f^{//}(0) \neq 0\ and\ so\ on$$

Definition 10.3.2: If $F(x) = a^x$, where 'a' is arbitrary constant. Then,

$$y_p(x) = \frac{1}{f(D)} \cdot a^x = \frac{1}{f(D)} e^{\log a^x} = \frac{1}{f(D)} e^{(\log a)x}$$

$$= \frac{1}{f(\log a)}, \ \text{provided} \ f(\log a) \neq 0$$

If $f(\log a) = 0$. *Then,*

$$y_p(x) = \cdot \frac{x}{f^{/}(D)} e^{(\log a)x} = \frac{x}{f^{/}(\log a)}, \ \text{provided} \ f^{/}(\log a) \neq 0$$

If $f^{/}(\log a) = 0$. *Then,*

$$y_p(x) = \frac{x^2}{f^{//}(D)} e^{(\log a)x} = \frac{x^2}{f^{//}(\log a)}, \ \text{provided} \ f^{//}(\log a) \neq 0 \ \text{and so on}$$

Example10.3.1: Find the general solution of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

Solution: We have

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2 \quad Or \quad y^{//} - 6y^{/} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

.... (1)

Here, we have $F(x) = 6e^{3x} + 7e^{-2x} - \log 2$

The symbolic form of the equation (1) is

$$(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$$

The corresponding auxiliary equation is $D^2 - 6D + 9 = 0$

$$\therefore \ D^2 - 6D + 9 = 0 \Rightarrow (D-3)^2 = 0 \Rightarrow D = 3, 3.$$

The complementary function is given by $C.F = y_c(x) = (c_1 + c_2 x)e^{3x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$y_p(x) = \frac{1}{(D^2 - 6D + 9)} \cdot (6e^{3x} + 7e^{-2x} - \log 2)$$
$$= 6 \cdot \frac{1}{(D^2 - 6D + 9)} e^{3x} + 7 \cdot \frac{1}{(D^2 - 6D + 9)} e^{-2x} - \log 2 \frac{1}{(D^2 - 6D + 9)} e^{0x}$$
$$= 6 \cdot \frac{x}{(2D - 6)} e^{3x} + 7 \cdot \frac{1}{[-2^2 - 6(-2) + 9]} e^{-2x} - \log 2 \frac{1}{[0^2 - 6(0) + 9]} e^{0x}$$
$$= 6 \cdot \frac{x^2}{2} e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$
$$\therefore \quad y_p(x) = 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9} .$$

Example10.3.2: Find the general solution of $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$

Solution: We have

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2 = 1 - 2e^x + e^{2x} \quad Or \quad y'' + y' + y = (1 - e^x)^2 = 1 - 2e^x + e^{2x}$$
.... (1)

Here, we have $F(x) = 1 - 2e^x + e^{2x}$

The symbolic form of the equation (1) is $(D^2 + D + 1)y = 1 - 2e^x + e^{2x}$

The corresponding auxiliary equation is $D^2+D+1=0$

$$\therefore\ D^2+D+1=0 \Rightarrow D=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2} \Rightarrow D=\frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2}.$$

The complementary function is given by

$$C.F = y_c(x) = e^{-\frac{1}{2}x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x).$$

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$\begin{aligned}
y_p(x) &= \frac{1}{(D^2+D+1)}\cdot(1-2e^x+e^{2x}) \\
&= \frac{1}{(D^2+D+1)}e^{0x} - 2\cdot\frac{1}{(D^2+D+1)}e^x + \frac{1}{(D^2+D+1)}e^{2x} \\
&= 1-2\cdot\frac{e^x}{[1^2+(1)+1]} + \frac{e^{2x}}{[2^2+(2)+1]} \\
\therefore\quad y_p(x) &= 1-\frac{2e^x}{3}+\frac{e^{2x}}{7}
\end{aligned}$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = e^{-\frac{1}{2}x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) + 1 - \frac{2e^x}{3} + \frac{e^{2x}}{7}$$

Example10.3.3: Find the general solution of $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{-x}$

Solution: We have , $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{-x} \quad Or \quad y^{///}+2y^{//}+y^{/}=e^{-x}$ ....

(1)

Here, we have $F(x)=e^{-x}$

The symbolic form of the equation (1) is $(D^3+2D^2+D)y=e^{-x}$

The corresponding auxiliary equation is $D^3+2D^2+D=0$

$\therefore\ D^3+2D^2+D=0\ Or\ D(D^2+2D+1) \Rightarrow D=0,\ (D+1)^2=0. Thus,\ D=0, -1, -1$

.

The complementary function is given by $C.F=y_c(x)=c_1+(c_2+c_3x)e^{-x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$\begin{aligned} y_p(x) &= \frac{1}{(D^3+2D^2+D)}\cdot e^{-x} \\ &= x\cdot\frac{1}{3D^2+4D+1}\cdot e^{-x} \\ &= x^2\cdot\frac{1}{6D+4}\cdot e^{-x}=\frac{x^2e^{-x}}{-2} \\ \therefore\ y_p(x) &= -\frac{x^2e^{-x}}{2} \end{aligned}$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y=C.F+P.I=y_c(x)+y_p(x)=c_1+(c_2+c_3x)e^{-x}-\frac{x^2e^{-x}}{2}$$

Example10.3.4: Find the general solution of $\frac{d^2y}{dx^2} - y = 5e^x + 3^{-x}$

Solution: We have $\frac{d^2y}{dx^2} - y = 5e^x + 3^{-x}$ *Or* $y'' - y = 5e^x + 3^{-x}$ .... (1)

Here, we have $F(x) = 5e^x + 3^{-x}$

The symbolic form of the equation (1) is $(D^2 - 1)y = 5e^x + 3^{-x}$

The corresponding auxiliary equation is $D^2 - 1 = 0$

$\therefore D^2 - 1 = 0 \Rightarrow D = \pm 1$ *Or* $D = 1, -1.$

The complementary function is given by $C.F = y_c(x) = (c_1 e^x + c_2 e^{-x})$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{(D^2 - 1)} \cdot (5e^x + 3^{-x})$$

$$= 5 \cdot \frac{1}{(D^2 - 1)} \cdot e^x + \frac{1}{(D^2 - 1)} \cdot 3^x$$

$$= 5 \cdot \frac{x}{[2D]} \cdot e^x + \frac{1}{[D^2 - 1]} \cdot e^{x(\log 3)}$$

$$\therefore \quad y_p(x) = \frac{5xe^x}{2} + \frac{3^x}{[(\log 3)^2 - 1]}$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 e^x + c_2 e^{-x}) + \frac{5xe^x}{2} + \frac{3^x}{[(\log 3)^2 - 1]}$$

Test Your Progress 1
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method.

1. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=3e^{2x}+5^x$
2. $\frac{d^3y}{dx^3}-5\frac{d^2y}{dx^2}+7\frac{dy}{dx}-3y=10e^{2x}$
3. $(D^3-D)y=e^{2x}+15$
4. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+25y=100e^{2x}$
5. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=55e^{-2x}$
6. $(D^4+D^2+1)y=e^{-x/2}$

10.4. Case of Hyperbolic Functions

When $F(x)=\sinh(ax+b)$ *Or* $\cosh(ax+b)$

Since $D[\sinh(ax+b)]=a\cosh(ax+b)$

$$D^2[\sinh(ax+b)]=a^2\sinh(ax+b)$$

$$D^3[\sinh(ax+b)]=a^3\cosh(ax+b)$$

$$D^4[\sinh(ax+b)]=a^4\sinh(ax+b)$$

i.e., $D^2[\sinh(ax+b)]=(a^2)\sinh(ax+b)$

$$(D^2)^2[\sinh(ax+b)] = (a^2)^2 \sinh(ax+b)$$

In general, $(D^2)^r[\sinh(ax+b)] = (a^2)^r \sinh(ax+b)$

$\therefore \quad f(D^2)[\sinh(ax+b)] = f(a^2)\sinh(ax+b)$

Operating $\dfrac{1}{f(D^2)}$ on both sides,

$$\frac{1}{f(D^2)} f(D^2)\sinh(ax+b) = \frac{1}{f(D^2)} f(a^2)\sinh(ax+b)$$

$$Or \quad \sinh(ax+b) = f(a^2)\frac{1}{f(D^2)}\sinh(ax+b)$$

Dividing by $f(a^2)$, we obtain

$$\frac{1}{f(D^2)}\sinh(ax+b) = \frac{1}{f(a^2)}\sinh(ax+b) \ \ provided \ \ f(a^2) \neq 0.$$

If $f(a^2) = 0$, the above rule fails and we proceed further as follows;

$$\frac{1}{f(D^2)}\sinh(ax+b) = x \cdot \frac{1}{f'(a^2)}\sinh(ax+b) \ \ provided \ \ f'(a^2) \neq 0.$$

If $f'(a^2) = 0$, $\dfrac{1}{f(D^2)}\sinh(ax+b) = x^2 \cdot \dfrac{1}{f''(a^2)}\sinh(ax+b) \ \ provided \ \ f''(a^2) \neq 0$

If $f^{//}(a^2)=0$, $\frac{1}{f(D^2)}\sinh(ax+b)=x^3\cdot\frac{1}{f^{///}(a^2)}\sinh(ax+b)$ *provided* $f^{///}(a^2)\neq 0$

, and so on

Similarly,

$$\frac{1}{f(D^2)}\cosh(ax+b)=\frac{1}{f(a^2)}\cosh(ax+b) \quad provided \ f(a^2)\neq 0.$$

If $f(a^2)=0$, the above rule fails and we proceed further as follows;

$$\frac{1}{f(D^2)}\cosh(ax+b)=x\cdot\frac{1}{f^{/}(a^2)}\cosh(ax+b) \quad provided \ f^{/}(a^2)\neq 0.$$

If $f^{/}(a^2)=0$,

$$\frac{1}{f(D^2)}\cosh(ax+b)=x^2\cdot\frac{1}{f^{//}(a^2)}\cosh h(ax+b) \quad provided \ f^{//}(a^2)\neq 0$$

If $f^{//}(a^2)=0$,

$\frac{1}{f(D^2)}\cosh(ax+b)=x^3\cdot\frac{1}{f^{///}(a^2)}\cosh(ax+b)$ *provided* $f^{///}(a^2)\neq 0$, and so on

Note 10.4.1:

We can make use of the definitions of

$\sinh x=\frac{e^x-e^{-x}}{2}$ *and* $\cosh x=\frac{e^x+e^{-x}}{2}$ and then one can opt for the case of exponential to find the particular integral.

Example10.4.1: Find the general solution of $\frac{d^2y}{dx^2} - 4y = \sinh 2x$

Solution: We have $\frac{d^2y}{dx^2} - 4y = \sinh 2x \quad Or \quad y^{//} - 4y = \sinh 2x$ .... (1)

Here, we have $F(x) = \sinh 2x$

The symbolic form of the equation (1) is $(D^2 - 4)y = \sinh 2x$

The corresponding auxiliary equation is $D^2 - 6D + 9 = 0$

$\therefore\ D^2 - 4 = 0 \Rightarrow D^2 = 4 \Rightarrow D = \pm 2$.

The complementary function is given by $C.F = y_c(x) = (c_1 e^{2x} + c_2 e^{-2x})$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{(D^2 - 4)} \cdot \sinh 2x$$
$$= \frac{x}{2D}\sinh 2x = \frac{x}{2}\int \sinh 2x \cdot dx = \frac{x}{2} \cdot \frac{\cosh 2x}{2} = \frac{x}{4}\cosh 2x$$
$$\therefore\ y_p(x) = \frac{x}{4}\cosh 2x$$

Or, alternatively; use $\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$

$$\therefore \quad y_p(x) = \frac{1}{(D^2-4)} \cdot \frac{e^{2x}-e^{-2x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-4)}(e^{2x}) - \frac{1}{(D^2-4)}(e^{-2x})\right] = \frac{1}{2}\left[\frac{x}{2D}e^{2x} - \frac{x}{2D}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{xe^{2x}}{4} + \frac{xe^{-2x}}{4}\right] = \frac{x}{4}\left[\frac{e^{2x}+e^{-2x}}{2}\right] = \frac{x}{4}\cosh 2x$$

$$\therefore \quad y_p(x) = \frac{x}{4}\cosh 2x$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 e^{-2x} + c_2 e^{2x}) + \frac{x}{4}\cosh 2x \, .$$

Example10.4.2: Find the general solution of $\frac{d^4 y}{dx^4} - y = \cosh x$

Solution: We have $\frac{d^4 y}{dx^4} - y = \cosh x \quad Or \quad y^{/v} - y = \cosh x$ .... (1)

Here, we have $F(x) = \cosh x$

The symbolic form of the equation (1) is $(D^4 - 1)y = \cosh x$

The corresponding auxiliary equation is $D^4 - 1 = 0$

$\therefore (D^2)^2 - 1 = 0 \Rightarrow D^2 - 1 = 0,\ D^2 + 1 = 0 \Rightarrow D = \pm 1,\ \pm i$.

The complementary function is given by

$C.F = y_c(x) = (c_1 e^{x} + c_2 e^{-x}) + (c_3 \cos x + c_4 \sin x)$.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$y_p(x) = \frac{1}{(D^4-1)} \cdot \cosh x$$

$$= \frac{x}{4D^3}\cosh x = \frac{x}{4}\iiint \cosh x \cdot dx$$

$$= \frac{x}{4}\iint \sinh x \cdot dx = \frac{x}{4}\int \cosh x \cdot dx = \frac{x}{4}\sinh x$$

$$\therefore \quad y_p(x) = \frac{x}{4}\sinh x$$

$$Or \; y_p(x) = \frac{x}{4D^3}\cosh x == \frac{x}{4D^2 D}\cosh x = \frac{x}{4(1^2)}\int \cosh x \cdot dx = \frac{x}{4}\sinh x$$

Or, alternatively; using $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\therefore \quad y_p(x) = \frac{1}{(D^4-1)} \cdot \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D^4-1)}(e^x) + \frac{1}{(D^4-1)}(e^{-x})\right] = \frac{1}{2}\left[\frac{x}{4D^3}e^x + \frac{x}{4D^3}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{xe^x}{4} - \frac{xe^{-x}}{4}\right] = \frac{x}{4}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{x}{4}\sinh x$$

$$\therefore \quad y_p(x) = \frac{x}{4}\sinh x$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 e^x + c_2 e^{-x}) + (c_3 \cos x + c_4 \sin x) + \frac{x}{4}\sinh x\,.$$

Example10.4.3: Find the general solution of

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 8\cosh 5x - 7\sinh 4x$$

Solution: We have ,

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 8\cosh 5x - 7\sinh 4x \qquad \text{..............} \quad (1)$$

$$Or \quad y''' - 5y'' + 5y' - 3y = 8\cosh 5x - 7\sinh 4x$$

Here, we have $F(x) = 8\cosh 5x - 7\sinh 4x$

The symbolic form of the equation (1) is

$(D^3 - 5D^2 + 7D - 3)y = 8\cosh 5x - 7\sinh 4x$

The corresponding auxiliary equation is $D^3 - 5D^2 + 7D - 3 = 0$

$\therefore\ (D-1)^2(D-3) = 0 \Rightarrow D = 1, 1, 3$.

The complementary function is given by $C.F = y_c(x) = (c_1 + c_2x)e^x + c_3e^{3x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$\begin{aligned} y_p(x) &= \frac{1}{D^3 - 5D^2 + 7D - 3}\cdot(8\cosh 5x - 7\sinh 4x) \\ &= 8\frac{1}{D^3 - 5D^2 + 7D - 3}\cosh 5x - 7\frac{1}{D^3 - 5D^2 + 7D - 3}\sinh 4x; \ \ Put \ \ D^2 = 5^2 \ \ and \ \ D^2 = 4^2 \\ &= 8\cdot\frac{1}{25D - 125 + 7D - 3}\cosh 5x - 7\cdot\frac{1}{16D - 80 + 7D - 3}\sinh 4x \end{aligned}$$

$$= 8 \cdot \frac{1}{32D - 128} \cosh 5x - 7 \cdot \frac{1}{23D - 83} \sinh 4x$$

$$= 8 \cdot \frac{32D + 128}{1024D^2 - 16384} \cosh 5x - 7 \cdot \frac{23D + 83}{529D^2 - 6889} \sinh 4x$$

$$= 8 \cdot \frac{32(5\sinh 5x) + 128\cosh 5x}{1024(5^2) - 16384} - 7 \cdot \frac{23(4\cosh 4x) + 83\sinh 4x}{529(4^2) - 6889}$$

$$= 8 \cdot \frac{160\sinh 5x + 128\cosh 5x}{9216} - 7 \cdot \frac{92\cosh 4x + 83\sinh 4x}{1575}$$

$$\therefore \quad y_p(x) = \frac{160\sinh 5x + 128\cosh 5x}{1152} - \frac{92\cosh 4x + 83\sinh 4x}{225}$$

$$Or \quad y_p(x) = \frac{5\sinh 5x + 8\cosh 5x}{72} - \frac{92\cosh 4x + 83\sinh 4x}{225}$$

Or, alternatively; use $\sinh 4x = \dfrac{e^{4x} - e^{-4x}}{2}$ *and* $\cosh 5x = \dfrac{e^{5x} - e^{-5x}}{2}$

$$\therefore \quad y_p(x) = 8 \cdot \frac{1}{D^3 - 5D^2 + 7D - 3} \cdot \frac{e^{5x} + e^{-5x}}{2} - 7 \cdot \frac{1}{D^3 - 5D^2 + 7D - 3} \cdot \frac{e^{4x} - e^{-4x}}{2}$$

$$= 4 \cdot \left[\frac{1}{D^3 - 5D^2 + 7D - 3} \cdot e^{5x} + \frac{1}{D^3 - 5D^2 + 7D - 3} \cdot e^{-5x} \right] - \frac{7}{2}\left[\frac{1}{D^3 - 5D^2 + 7D - 3} \cdot e^{4x} + \frac{1}{D^3 - 5D^2 + 7D - 3} \cdot e^{-4x} \right]$$

$$\therefore \quad y_p(x) = 4 \cdot \left[\frac{e^{5x}}{32} + \frac{e^{-5x}}{288} \right] - \frac{7}{2}\left[\frac{e^{4x}}{9} - \frac{e^{-4x}}{175} \right] = \frac{e^{5x}}{8} + \frac{e^{-5x}}{72} - \frac{7}{2}\left[\frac{e^{4x}}{9} - \frac{e^{-4x}}{175} \right]$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{x} + c_3 e^{3x} + \frac{e^{5x}}{8} + \frac{e^{-5x}}{72} - \frac{7}{2}\left[\frac{e^{4x}}{9} - \frac{e^{-4x}}{175} \right].$$

Example10.4.4: Find the general solution of $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x$.

Also find y when $y=0,\ \frac{dy}{dx}=1\ at\ \ x=0$.

Solution: We have $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x\ \ Or\ \ y^{//}+4y^{/}+5y=-2\cosh x$

.... (1)

Here, we have $F(x)=-2\cosh 2x$

The symbolic form of the equation (1) is $(D^2+4D+5)y=-2\cosh x$

The corresponding auxiliary equation is $D^2+4D+5=0$

$\therefore\ D^2+4D+5=0 \Rightarrow D=\frac{-4\pm\sqrt{16-20}}{2} \Rightarrow D=-2\pm i$.

The complementary function is given by $C.F=y_c(x)=e^{-2x}(c_1\cos x+c_2\sin x)$

.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x)=\frac{1}{(D^2+4D+5)}\cdot(-2\cosh x)$$
$$=-2\cdot\frac{1}{4D+6}\cosh x=-\frac{1}{2D+3}\cosh x=-\frac{2D-3}{4D^2-9}\cdot\cosh x=\frac{2\sinh x-3\cosh x}{5}$$
$$\therefore\ \ y_p(x)=\frac{2\sinh x-3\cosh x}{5}$$

Or, alternatively; use $\cosh x = \dfrac{e^x - e^{-x}}{2}$

$$\therefore \quad y_p(x) = \frac{-2}{2} \cdot \left[\frac{1}{(D^2 + 4D + 5)} \cdot e^x - \frac{1}{(D^2 + 4D + 5)} \cdot e^{-x} \right]$$
$$= \left[\frac{1}{10}(e^x) - \frac{1}{2}(e^{-x}) \right]$$
$$= \frac{1}{2}\left[\frac{e^x}{5} - e^{-x} \right]$$
$$\therefore \quad y_p(x) = \frac{1}{2}\left[\frac{e^x}{5} - e^{-x} \right]$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{2\sinh x - 3\cosh x}{5}$$

………….. (2)

Now, using the given conditions; $y = 0, \ \dfrac{dy}{dx} = 1 \ at \ \ x = 0$. From equation (2),

$$0 = c_1 - \frac{3}{5} \ \ Or \ \ c_1 = \frac{3}{5}.$$

Again, from equation (2);

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{2\sinh x - 3\cosh x}{5}$$
$$\Rightarrow y' = -2e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}(-c_1 \sin x + c_2 \cos x)$$
$$\therefore U\sin g \ \ at \ \ x = 0, \ \ \frac{dy}{dx} = 1, \ \ we \ \ obtain,$$
$$1 = -2(c_1 + 0) + (0 + c_2) \ \ Or \ \ -2c_1 + c_2 = 1, \ \ put \ \ c_1 = \frac{3}{5} \Rightarrow \ c_2 = \frac{11}{5}$$

Thus, the particular solution of equation 1) is given by

$$y = e^{-2x}\left(\frac{3}{5}\cos x + \frac{11}{5}\sin x\right) + \frac{2\sinh x - 3\cosh x}{5}$$

$$Or\ \ y = \frac{e^{-3x}}{5}\,(3\cos x + 11\sin x) + 2\sinh x - 3\cosh x$$

Test Your Progress 2
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method. 1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 10\sinh x$ 2. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = \cosh x$ 3. $(D^3 - D)y = 2\sinh x - \cosh 2x$ 4. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 50\cosh 3x$ 5. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \sinh 2x$

10.5. Case of Trignometric Functions

When $F(x) = \sin(ax+b)\ \ Or\ \ \cos(ax+b)$

Since $D[\sin(ax+b)] = a\cos(ax+b)$

$$D^2[\sin(ax+b)] = -a^2 \sin(ax+b)$$

$$D^3[\sin(ax+b)] = -a^3 \cos(ax+b)$$

$$D^4[\sin(ax+b)] = a^4 \sin(ax+b)$$

i.e., $D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$

$$(D^2)^2[\sin(ax+b)] = (-a^2)^2 \sin(ax+b)$$

In general, $(D^2)^r[\sin(ax+b)] = (-a^2)^r \sin(ax+b)$

$\therefore \quad f(D^2)[\sin(ax+b)] = f(-a^2)\sin(ax+b)$

Operating $\dfrac{1}{f(D^2)}$ on both sides,

$$\frac{1}{f(D^2)} f(D^2)\sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

Or $\sin(ax+b) = f(-a^2)\dfrac{1}{f(D^2)}\sin(ax+b)$

Dividing by $f(a^2)$, we obtain

$\dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)$ *provided* $f(-a^2) \neq 0$.

If $f(-a^2) = 0$, the above rule fails and we proceed further as follows;

$$\frac{1}{f(D^2)}\sin(ax+b) = x\cdot\frac{1}{f^{/}(-a^2)}\sin(ax+b) \;\; \textit{provided} \;\; f^{/}(-a^2)\neq 0 .$$

If $f^{/}(-a^2)=0$,

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\cdot\frac{1}{f^{//}(-a^2)}\sin(ax+b) \;\; \textit{provided} \;\; f^{//}(-a^2)\neq 0$$

If $f^{//}(-a^2)=0$,

$$\frac{1}{f(D^2)}\sin(ax+b) = x^3\cdot\frac{1}{f^{///}(-a^2)}\sin(ax+b) \;\; \textit{provided} \;\; f^{///}(-a^2)\neq 0$$, and so on

Similarly,

$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b) \;\; \textit{provided} \;\; f(-a^2)\neq 0 .$$

If $f(-a^2)=0$, the above rule fails and we proceed further as follows;

$$\frac{1}{f(D^2)}\cos(ax+b) = x\cdot\frac{1}{f^{/}(-a^2)}\cos(ax+b) \;\; \textit{provided} \;\; f^{/}(-a^2)\neq 0 .$$

If $f^{/}(-a^2)=0$, $\frac{1}{f(D^2)}\cos(ax+b) = x^2\cdot\frac{1}{f^{//}(a^2)}\cos(ax+b) \;\; \textit{provided} \;\; f^{//}(-a^2)\neq 0$

If $f^{//}(-a^2)=0$,

$$\frac{1}{f(D^2)}\cos(ax+b) = x^3\cdot\frac{1}{f^{///}(-a^2)}\cos(ax+b) \;\; \textit{provided} \;\; f^{///}(-a^2)\neq 0$$, and so on.

Example10.5.1: Find the general solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{2x}-\cos^2 x$

Solution: We have $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{2x}-\cos^2 x$ $\quad Or \quad$ $y^{//}+2y^{/}+y=e^{2x}-\cos^2 x$

.... (1)

Here, we have $F(x)=e^{2x}-\cos^2 x$

The symbolic form of the equation (1) is $(D^2+2D+1)y=e^{2x}-\cos^2 x$

The corresponding auxiliary equation is $D^2+2D+1=0$

$\therefore\ D^2+2D+1=0 \Rightarrow (D+1)^2=0 \Rightarrow D=-1,\ -1$.

The complementary function is given by $C.F=y_c(x)=(c_1+c_2x)e^{-x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{D^2+2D+1}\cdot(e^{2x} - \cos^2 x)$$

$$= \frac{1}{D^2+2D+1}e^{2x} - \frac{1}{D^2+2D+1}\cos^2 x$$

$$= \frac{1}{9}e^{2x} - \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right) = \frac{e^{2x}}{9} - \frac{1}{2}\left[\frac{1}{D^2+2D+1}\cdot e^{0x} + \frac{1}{D^2+2D+1}\cdot\cos 2x\right]$$

$$\therefore \quad y_p(x) = \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{1}{-2^2+2D+1}\cdot\cos 2x\right] = \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{1}{2D-3}\cdot\cos 2x\right]$$

$$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{2D+3}{4D^2-9}\cdot\cos 2x\right] = \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{2D+3}{4(-2^2)-9}\cdot\cos 2x\right]$$

$$= \frac{e^{2x}}{9} - \frac{1}{2}\left[1 + \frac{-4\sin 2x + 3\cos 2x}{7}\right] = \frac{e^{2x}}{9} - \frac{1}{2}\left[\frac{7 - 4\sin 2x + 3\cos 2x}{7}\right]$$

$$Or \;\; y_p(x) = \frac{e^{2x}}{9} + \frac{4\sin 2x - 3\cos 2x - 7}{14}$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{-x} + \frac{e^{2x}}{9} + \frac{4\sin 2x - 3\cos 2x - 7}{14}.$$

Example10.5.2: Find the general solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x\cos 2x$

Solution: We have

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x\cos 2x \quad Or \quad y'' - 4y' + 3y = \sin 3x\cos 2x \quad \text{.........} \quad (1)$$

Here, we have $F(x) = \sin 3x\cos 2x$

The symbolic form of the equation (1) is $(D^2 - 4D + 3)y = \sin 3x\cos 2x$

The corresponding auxiliary equation is $D^2 - 4D + 3 = 0$

$\therefore\ D^2-4D+3=0 \Rightarrow D^2-3D-D+3=0\ Or\ (D-3)(D-1)=0 \Rightarrow D=1,\ 3$.

The complementary function is given by $C.F=y_c(x)=c_1e^{x}+c_2e^{3x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x)=\frac{1}{D^2-4D+3}\cdot\sin 3x\cos 2x$$

$$U\sin g\ \sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$=\frac{1}{2}\cdot\left[\frac{1}{D^2-4D+3}\cdot\sin 5x+\frac{1}{D^2-4D+3}\cdot\cos x\right]$$

$$=\frac{1}{2}\cdot\left[\frac{1}{25-4D+3}\cdot\sin 5x+\frac{1}{1-4D+3}\cdot\cos x\right]\qquad Here\ -a^2=-5^2\ and\ -a^2=-1^2$$

$$=\frac{1}{2}\cdot\left[\frac{1}{28-4D}\cdot\sin 5x+\frac{1}{4-4D}\cdot\cos x\right]$$

$$\therefore\quad y_p(x)=\frac{1}{8}\cdot\left[\frac{7+D}{49-D^2}\cdot\sin 5x+\frac{1+D}{1-D^2}\cdot\cos x\right]=\frac{1}{8}\cdot\left[\frac{7\sin 5x+5\cos 5x}{49+25}+\frac{\cos x-\sin x}{2}\right]$$

$$Or\ y_p(x)=\frac{1}{16}\cdot\left[\frac{7\sin 5x+5\cos 5x}{37}+\cos x-\sin x\right]$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y=C.F+P.I=y_c(x)+y_p(x)=c_1e^{x}+c_2e^{3x}+\frac{1}{16}\cdot\left[\frac{7\sin 5x+5\cos 5x}{37}+\cos x-\sin x\right].$$

Example10.5.3: Find the general solution of $\frac{d^2x}{dt^2}+n^2x=k\cos(nt+\alpha)$

Solution: We have $\dfrac{d^2x}{dt^2}+n^2x=k\cos(nt+\alpha)$ *Or* $x^{//}+n^2x=k\cos(nt+\alpha)$

.... (1)

Here, we have $F(x)=k\cos(nt+\alpha)$

The symbolic form of the equation (1) is $(D^2+n^2)x=k\cos(nt+\alpha)$

The corresponding auxiliary equation is $D^2+n^2=0$

$\therefore\ D^2+n^2=0 \Rightarrow D=\pm in$.

The complementary function is given by $C.F=x_c(t)=c_1\cos nt+c_2\sin nt$.

Now, to find the particular integral $x_p(t)$, by the inverse operator method

,

Consider,

$$x_p(t)=\frac{1}{D^2+n^2}\cdot k\cos(nt+\alpha)$$

$$U\sin g\ \cos(nt+\alpha)=\cos nt\cos\alpha-\sin nt\sin\alpha$$

$$=\left[\cos\alpha\cdot\frac{1}{D^2+n^2}\cdot\cos nt-\sin\alpha\cdot\frac{1}{D^2+n^2}\cdot\sin nt\right]$$

$$=\left[\cos\alpha\cdot\frac{1}{D^2+n^2}\cdot\cos nt-\sin\alpha\cdot\frac{1}{D^2+n^2}\cdot\sin nt\right] \qquad Here\ \ -a^2=-n^2\ \ in\ both\ cases$$

$$=\left[\cos\alpha\cdot\frac{t}{2D}\cdot\cos nt-\sin\alpha\cdot\frac{t}{2D}\cdot\sin nt\right]=\frac{t}{2n}\left[\cos\alpha\cdot\sin nt+\sin\alpha\cdot\cos nt\right]$$

$$\therefore\ \ x_p(t)=\frac{t}{2n}\sin(nt+\alpha)$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$x(t) = C.F + P.I = x_c(t) + x_p(t) = c_1 \cos nt + c_2 \sin nt + \frac{t}{2n}\sin(nt+\alpha).$$

Example10.5.4: Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3\sin x + 4\cos x,\ y(0)=1,\ \ y^{/}(0)=0.$$

Solution: We have

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3\sin x + 4\cos x \ \ Or \ \ y^{//} + 4y^{/} + 4y = 3\sin x + 4\cos x \ \ \dots \ \ (1)$$

Here, we have $F(x) = 3\sin x + 4\cos x$

The symbolic form of the equation (1) is $(D^2 + 4D + 4)y = 3\sin x + 4\cos x$

The corresponding auxiliary equation is $D^2 + 4D + 4 = 0$

$\therefore\ D^2 + 4D + 4 = 0 \Rightarrow (D+2)^2 = 0 \ \Rightarrow D = -2, -2.$

The complementary function is given by $C.F = y_c(x) = (c_1 + c_2 x)e^{-2x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{(D^2+4D+4)} \cdot (3\sin x + 4\cos x)$$

$$= 3\cdot\frac{1}{D^2+4D+4}\sin x + 4\cdot\frac{1}{D^2+4D+4}\cos x, \ \ \textit{here} \ \ D^2 = -a^2 = -1^2 = -1 \ \textit{in both cases}$$

$$= 3\cdot\frac{1}{4D+3}\sin x + 4\cdot\frac{1}{4D+3}\cos x = 3\cdot\frac{4D-3}{16D^2-9}\sin x + 4\cdot\frac{4D-3}{16D^2-9}\cos x$$

$$= \frac{3(3\sin x - 4\cos x)}{25} + \frac{4(4\sin x + 3\cos x)}{25} = \frac{25\sin x}{25}$$

$$\therefore \ \ y_p(x) = \sin x$$

Thus the general (Or complete) solution of equation (1) is given by,

$$y = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{-2x} + \sin x \quad \ldots\ldots\ldots\ldots.. \ (2)$$

Now, using the first given condition; $y(0) = 1$. From equation (2),

$$1 = c_1$$

Again, from equation (2);

$$y(x) = (c_1 + c_2 x)e^{-2x} + \sin x$$
$$y'(x) = -2e^{-2x}(c_1 + c_2 x) + c_2 e^{-2x} + \cos x$$

Using the second condition $y'(0) = 0$; $0 = -2c_1 + c_2 + 1$ *Or* $2c_1 - c_2 = 1$, put

$1 = c_1$ then $1 = c_2$

Thus, the particular solution of equation (1) is given by

$$y(x) = y_c(x) + y_p(x) = (1 + x)e^{-2x} + \sin x$$

Test Your Progress 3
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method. 1. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=\sin t$ 2. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4\cos^2 x$ 3. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+25y=\sin x$ 4. $(D^3-D)y=4\cos x$ 5. $(D^2+1)^2 y=2\sin x\cos x$

10.6. Case of a polynomial

.

When $F(x)=x^m$

In this case, $P.I=\frac{1}{f(D)}\cdot x^m=[f(D)]^{-1}\cdot x^{m.}$. We expand $[f(D)]^{-1}$ in ascending powers of D as for as the term in D^m are zero, we need not consider terms beyond D^m.

Equivalently,

When $F(x) = x^m$, 'm' being a positive integer.

$P.I = \frac{1}{f(D)} \cdot x^m = [f(D)]^{-1} \cdot x^{m.}$. To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by binomial theorem as for as D^m and operate on x^m term by term.

Example10.6.1: Find the general solution of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$$

Solution: We have $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$ .... (1)

$$Or \quad y^{//} - 6y^{/} + 25y = e^{2x} + \sin x + x$$

Here, we have $F(x) = e^{2x} + \sin x + x$

The symbolic form of the equation (1) is $(D^2 - 6D + 25)y = e^{2x} + \sin x + x$

The corresponding auxiliary equation is $D^2 - 6D + 25 = 0$

$$\therefore \; D^2 - 6D + 25 = 0 \;\Rightarrow\; D = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm 4i \quad .$$

The complementary function is given by

$C.F = y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$y_p(x)=\frac{1}{D^2-6D+25}\cdot(e^{2x}+\sin x+x)$$

$$=\left[\frac{1}{D^2-6D+25}\cdot e^{2x}+\frac{1}{D^2-6D+25}\cdot\sin x+\frac{1}{D^2-6D+25}\cdot x\right]$$

$$=\left[\frac{e^{2x}}{4-8+25}+\frac{1}{-1^2-6D+25}\cdot\sin x+\frac{1}{25}\left(1-\frac{(6D-D^2)}{25}\right)^{-1}\cdot x\right]$$

$$=\left[\frac{e^{2x}}{21}+\frac{1}{24-6D}\cdot\sin x+\frac{1}{25}\left(1+\frac{6D-D^2}{25}+\left(\frac{6D-D^2}{25}\right)^2+..........\right)\cdot x\right]$$

$$\therefore\quad y_p(x)=\left[\frac{e^{2x}}{21}+\frac{24+6D}{576-36D^2}\cdot\sin x+\frac{1}{25}\left(x+\frac{6}{25}\right)\right]=\frac{e^{2x}}{21}+\frac{24\sin x+6\cos x}{612}+\frac{x}{25}+\frac{6}{625}$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y=C.F+P.I=y_c(x)+y_p(x)=e^{3x}(c_1\cos 4x+c_2\sin 4x)+\frac{e^{2x}}{21}+\frac{24\sin x+6\cos x}{612}+\frac{x}{25}+\frac{6}{625}$$

.

Example10.6.2: Find the general solution of

$$\left(\frac{d^2y}{dx^2}+1\right)^2 y=x^4+2\sin x\cos 3x$$

Solution: We have $\left(\frac{d^2y}{dx^2}+1\right)^2 y=x^4+2\sin x\cos 3x$ .... (1)

$$Or\ \left(y^{//}+y\right)^2=x^4+2\sin x\cos 3x$$

Here, we have $F(x)=x^4+2\sin x\cos 3x$

The symbolic form of the equation (1) is $(D^2+1)^2 y=x^4+2\sin x\cos 3x$

The corresponding auxiliary equation is $(D^2+1)^2=0$

$\therefore\ (D^2+1)^2=0 \Rightarrow D^2=-1\ \ (Twice) \Rightarrow D=\pm i\ \ (Twice)$.

The complementary function is given by

$C.F = y_c(x) = (c_1+c_2x)\cos x + (c_3+c_4x)\sin x)$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{(D^2+1)^2}\cdot(x^4+2\sin x\cos 3x)$$

$$= \left[\frac{1}{(D^2+1)^2}\cdot x^4 + \frac{1}{(D^2+1)^2}\cdot 2\cos 3x\sin x\right]$$

$Using\ \ 2\cos A\cdot\sin B = \sin(A+B)-\sin(A-B)$

$$= \left[(1+D^2)^{-2}\cdot x^4 + \frac{1}{(D^2+1)^2}\cdot\sin 4x - \frac{1}{(D^2+1)^2}\cdot\sin 2x\right]\ ;\ D^2=-4^2\ \ and\ \ \ D^2=-2^2$$

$Using\ the\ expansion\ (1+x)^{-2} = 1-2x+3x^2-4x^3+................$

$$= (1-2D^2+3D^4-4D^6+................)\cdot x^4 + \frac{1}{(-15)^2}\cdot\sin 4x - \frac{1}{(-3)^2}\cdot\sin 2x$$

$$\therefore\ \ y_p(x) = [x^4-2(12x^2)+3(24)] + \frac{\sin 4x}{225} - \frac{\sin 2x}{9} = (x^4-24x^2+72) + \frac{\sin 4x}{225} - \frac{\sin 2x}{9}$$

Thus, the general (Or complete) solution of equation (1) is given by,

$y = C.F + P.I = y_c(x) + y_p(x) =$

$$= (c_1+c_2x)\cos x + (c_3+c_4x)\sin x + (x^4-24x^2+72) + \frac{\sin 4x}{225} - \frac{\sin 2x}{9}\ .$$

Example10.6.3: Find the general solution of

$$\left(\frac{dy}{dx}-2y\right)^2=8(e^{2x}+\sin 2x+x^2)$$

Solution: We have $$\left(\frac{dy}{dx}-2y\right)^2=8(e^{2x}+\sin 2x+x^2) \quad \text{.........} \quad (1)$$

$$Or \ (y^{/}-2y)^2=8(e^{2x}+\sin 2x+x^2)$$

Here, we have $F(x)=8(e^{2x}+\sin 2x+x^2)$

The symbolic form of the equation (1) is $(D-2)^2 y=8(e^{2x}+\sin 2x+x^2)$

The corresponding auxiliary equation is $(D-2)^2=0$

$\therefore\ (D-2)^2=0 \Rightarrow D=2,\ 2$.

The complementary function is given by $C.F=y_c(x)=(c_1+c_2x)e^{2x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method,

Consider,

$$y_p(x) = 8 \cdot \frac{1}{(D-2)^2} \cdot (e^{2x} + \sin 2x + x^2)$$

$$= 8\left[\frac{1}{(D-2)^2} \cdot e^{2x} + \frac{1}{(D-2)^2} \cdot \sin 2x + \frac{1}{(D-2)^2} \cdot x^2\right]$$

$$= 8\left[\frac{x}{2(D-2)} \cdot e^{2x} + \frac{1}{D^2 - 4D + 4} \cdot \sin 2x + \frac{1}{D^2 - 4D + 4} \cdot x^2\right]$$

$$= 8\left[\frac{x^2}{2} \cdot e^{2x} - \frac{1}{4D} \cdot \sin 2x + \frac{1}{4}\left(1 - \left(\frac{4D - D^2}{4}\right)\right)^{-1} \cdot x^2\right]$$

$$\therefore \quad y_p(x) = 8\left[\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} \cdot + \frac{1}{4}\left(1 + \frac{4D - D^2}{4} + \left(\frac{4D - D^2}{4}\right)^2 + \ldots\ldots\ldots\ldots\right) \cdot x^2\right]$$

$$= 8\left[\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{1}{4}\left(x^2 + 2x - \frac{1}{4}(2) + 2\right)\right] = 8\left(\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{1}{4}(x^2 + 2x)\right)$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{2x} + 8\left(\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{1}{4}(x^2 + 2x)\right).$$

Test Your Progress 4
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method. 1. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = 5t^2 - \sin t$ 2. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x + 4\cos^2 x$ 3. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 3x + 5e^x$ 4. $(D^3 - D)y = 2x^3$ 5. $(D^2 + 1)^2 y = e^{2x} - 4x^2$

10.7. Case of combination of $e^{ax}V$

When $F(x)=e^{ax}\cdot V$**, where V is a function of** $'x'$**.**

In this case, $P.I=\frac{1}{f(D)}e^{ax}\cdot V=\frac{1}{f(D+a)}\cdot V$ and then evaluate $\frac{1}{f(D+a)}\cdot V$

as in case of 10.3, 10.4, 10.5 and 10.6.

Example10.7.1: Find the general solution of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=x+e^x\cos x$

Solution: We have

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=x+e^x\cos x \quad Or \quad y^{//}-2y^{/}+2y=x+e^x\cos x \quad .... \quad (1)$$

Here, we have $F(x)=x+e^x\cos x$

The symbolic form of the equation (1) is $(D^2-2D+2)y=x+e^x\cos x$

The corresponding auxiliary equation is $D^2-2D+2=0$

$\therefore\ D^2-2D+2=0 \Rightarrow D=\frac{2\pm\sqrt{4-8}}{2}=1\pm i \ \Rightarrow\ D=1+i,\ 1-i$.

The complementary function is given by $C.F=y_c(x)=e^{-x}(c_1\cos x+c_2\sin x)$

.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2-2D+2}\cdot(x+e^x\cos x)\\
&= \frac{1}{D^2-2D+2}\cdot x+\frac{1}{D^2-2D+2}\cdot e^x\cos x\\
&= \frac{1}{2}\left[1-\left(\frac{2D-D^2}{2}\right)\right]^{-1}\cdot x+e^x\frac{1}{(D+1)^2-2(D+1)+2}\cos x\\
&= \frac{1}{2}\left[1+\frac{2D-D^2}{2}+\left(\frac{2D-D^2}{2}\right)^2+....\right]\cdot x+e^x\frac{1}{D^2+1}\cos x
\end{aligned}$$

$$\therefore\quad y_p(x)=\frac{1}{2}(x+1)+\frac{xe^x}{2D}\cdot\cos x=\frac{1}{2}\left[(x+1)+xe^x\sin x\right]$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y=C.F+P.I=y_c(x)+y_p(x)=e^{-x}(c_1\cos x+c_2\sin x)+\frac{1}{2}\left[(x+1)+xe^x\sin x\right].$$

Example10.7.2: Find the general solution of

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=e^{-x}\sin x+xe^{3x}$$

Solution: We have

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=e^{-x}\sin x+xe^{3x}\quad Or\quad y''+4y'+3y=e^{-x}\sin x+xe^{3x}\quad ...\ (1)$$

Here, we have $F(x)=e^{-x}\sin x+xe^{3x}$

The symbolic form of the equation (1) is $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

The corresponding auxiliary equation is $D^2+4D+3=0$

$\therefore\ D^2+4D+3=0 \Rightarrow D=(D+1)(D+3) \Rightarrow D=-1,\ -3$.

The complementary function is given by $C.F = y_c(x) = (c_1e^{-x}+c_2e^{-3x})$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2+4D+3}\cdot(e^{-x}\sin x + xe^{3x}) \\
&= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\cdot\sin x + e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}\cdot x \\
&= e^{-x}\cdot\frac{1}{D^2+2D}\cdot\sin x + e^{3x}\frac{1}{D^2+10D+24}\cdot x \\
&= e^{-x}\cdot\frac{2D-1}{4D^2-1}\cdot\sin x + \frac{e^{3x}}{24}\left[1+\left(\frac{10D+D^2}{24}\right)\right]^{-1}\cdot x
\end{aligned}$$

$$\begin{aligned}
\therefore\quad y_p(x) &= \frac{e^{-x}}{5}(\sin x - 2\cos x) + \frac{e^{3x}}{24}\left[1+\frac{10D+D^2}{24}+\left(\frac{10D+D^2}{24}\right)^2+\ldots\ldots\ldots\right]\cdot x \\
&= \frac{e^{-x}}{5}(\sin x - 2\cos x) + \frac{e^{3x}}{24}\left(x+\frac{5}{12}\right)
\end{aligned}$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1e^{-x}+c_2e^{-3x}) + \frac{e^{-x}}{5}(\sin x - 2\cos x) + \frac{e^{3x}}{24}\left(x+\frac{5}{12}\right)\ .$$

Example10.7.3: Find the general solution of $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Solution: We have $\frac{d^2y}{dx^2} - 4y = x\sinh x$ Or $y^{//} - 4y = x\sinh x$ ... (1)

Here, we have $F(x) = x\sinh x$

The symbolic form of the equation (1) is $(D^2 - 4)y = x\sinh x$

The corresponding auxiliary equation is $D^2 - 4 = 0$

$\therefore\ D^2 - 4 = 0 \Rightarrow\ D = 2,\ -2$.

The complementary function is given by $C.F = y_c(x) = (c_1 e^{2x} + c_2 e^{-2x})$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{D^2-4} \cdot x \sinh x$$

$$Using \ \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{D^2-4} \cdot x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2} \cdot \left[\frac{1}{D^2-4} xe^x - \frac{1}{D^2-4} xe^{-x}\right]$$

$$= \frac{1}{2} \cdot \left[e^x \frac{1}{(D+1)^2-4} x - e^{-x} \frac{1}{(D-1)^2-4} x\right]$$

$$= \frac{1}{2} \cdot \left[e^x \frac{1}{D^2+2D-3} x - e^{-x} \frac{1}{D^2-2D-3} x\right]$$

$$\therefore \ y_p(x) = \frac{1}{2} \cdot \left[e^x \cdot \frac{1}{-3}\left[1 - \left(\frac{2D+D^2}{3}\right)\right]^{-1} x - e^{-x} \cdot \frac{1}{-3}\left[1 + \left(\frac{2D-D^2}{3}\right)\right]^{-1} x\right]$$

$$= \frac{1}{-6} \cdot \left[e^x \cdot \left[1 + \left(\frac{2D+D^2}{3}\right) + \left(\frac{2D+D^2}{3}\right)^2 + \ldots..\right] x - e^{-x}\left[1 - \left(\frac{2D-D^2}{3}\right) + \left(\frac{2D-D^2}{3}\right)^2\right] x\right]$$

$$= \frac{1}{-6}\left[e^x\left(x + \frac{2}{3}\right) - e^{-x}\left(x - \frac{2}{3}\right)\right] = -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 e^{2x} + c_2 e^{-2x}) - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Test Your Progress 5	
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method.	
1. $(D^4 + D^2 + 1)y = e^{-x/2} \cos \frac{\sqrt{3}}{2} x$	2. $\frac{d^2 y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$

3. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=xe^{3x}+\sin 2x$ 4. $(D^3+2D^2+D)y=x^2e^{2x}+\sin^2 x$

10.8. Case of combination of $x^m \cdot V$

When $F(x)=x^m \cdot V$**, where V is cosax or sinax.**

Then, $P.I=\frac{1}{f(D)}x^m\cos ax \;\; Or \;\; x^m \sin ax$

$$\Rightarrow \frac{1}{f(D)}x^m[\cos ax+i\sin ax]=\frac{1}{f(D)}x^m e^{iax}$$

$$=e^{iax}\frac{1}{f(D+ia)}x^m$$

And $\frac{1}{f(D+ia)}x^m$, can be evaluated by the method mentioned in 10.6, the case of polynomial then equating the real and imaginary parts, we get the required results.

Remark 10.8.1: When $F(x)=x\cdot V$**,** 'V' being any function of 'x'. Then we use the formula,

$$\frac{1}{f(D)}x\cdot V=\left[x-\frac{f'(D)}{f(D)}\right]\cdot\frac{1}{f(D)}\cdot V.$$

This rule is applicable if

i) Power of 'x' is one

ii) $\frac{1}{f(D)} \cdot V$ is not a case of failure

iii) If the power of 'x' is one and $\frac{1}{f(D)} \cdot V$ is a case of failure then do not apply xV rule. In this case we apply rule given by case 10.7.

Remark 10.8.2: When $F(x) = f(x)$**,** being any function of 'x'. Then

$P.I = \frac{1}{f(D)} \cdot f(x)$. If $f(D) = (D-m_1)(D-m_2)\ ....\ (D-m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ .... \ \frac{A_n}{D-m_n}$$

$$\therefore \quad P.I = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ .... \ \frac{A_n}{D-m_n} \right] \cdot f(x)$$

$$= A_1 \frac{1}{D-m_1} \cdot f(x) + A_2 \cdot \frac{1}{D-m_2} \cdot f(x) + \ .... + A_n \cdot \frac{1}{D-m_n} \cdot f(x)$$

$$= A_1 \cdot e^{m_1 x} \int e^{-m_1 x} \cdot f(x) \cdot dx + A_2 \cdot e^{m_2 x} \int e^{-m_2 x} \cdot f(x) \cdot dx + ... + A_n \cdot e^{m_n x} \int e^{-m_{n1} x} \cdot f(x) \cdot dx$$

Equivalently,

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on f(x) remembering that

$$\frac{1}{D-a}\cdot f(x)=e^{ax}\int f(x)e^{-ax}dx$$

Note: This method is general one and, can therefore, be employed to obtain a particular integral in any given case.

Example10.8.1: Find the general solution of $\frac{d^2y}{dx^2}-y=x\sin 3x+\cos x$

Solution: We have $\frac{d^2y}{dx^2}-y=x\sin 3x+\cos x$ Or $y^{//}-y=x\sin 3x+\cos x$ ... (1)

Here, we have $F(x)=x\sin 3x+\cos x$

The symbolic form of the equation (1) is $(D^2-1)y=x\sin 3x+\cos x$

The corresponding auxiliary equation is $D^2-1=0$

$\therefore$ $D^2-1=0 \Rightarrow D=1,\ -1.$

The complementary function is given by $C.F=y_c(x)=(c_1e^x+c_2e^{-x})$.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$y_p(x) = \frac{1}{D^2-1}\cdot x\sin 3x + \cos x$$

$$= \frac{1}{D^2-1}\cdot x\sin 3x + \frac{1}{D^2-1}\cdot \cos x = I.P\left(\frac{1}{D^2-1}\cdot xe^{3ix}\right) - \frac{\cos x}{2} \quad as \;\; a^2 = -1^2 = -1$$

$$= I.P\left(e^{3ix}\frac{1}{(D+3i)^2-1}\cdot x\right) - \frac{\cos x}{2} = I.P\left(e^{3ix}\frac{1}{D^2+6iD-10}\cdot x\right) - \frac{\cos x}{2}$$

$$\therefore \;\; y_p(x) = I.P\left(\frac{e^{3ix}}{-10}\left[1-\frac{(6iD+D^2)}{10}\right]^{-1}\cdot x\right) - \frac{\cos x}{2}$$

$$= I.P\left(\frac{e^{3ix}}{-10}\left[1+\left(\frac{6iD+D^2}{10}\right)+\left(\frac{6iD+D^2}{10}\right)^2+.........\right]^{-1}\cdot x\right) - \frac{\cos x}{2}$$

$$= I.P\left(\frac{e^{3ix}}{-10}\left[1+\frac{6iD}{10}+\frac{D^2}{10}+\frac{36i^2D^2}{100}+\frac{6iD^3}{50}+\frac{D^4}{100}+.........\right]\cdot x\right) - \frac{\cos x}{2}$$

$$= I.P\left(\frac{e^{3ix}}{-10}\left[x+\frac{3i}{5}\right]\right) - \frac{\cos x}{2} = I.P\left(\frac{\cos 3x + i\sin 3x}{-10}\left[x+\frac{3i}{5}\right]\right) - \frac{\cos x}{2}$$

$$\therefore \;\; y_p(x) = I.P\left(\frac{x\cos 3x + ix\sin 3x}{-10} + \frac{3i\cos 3x - 3\sin 3x}{-50}\right) - \frac{\cos x}{2} = \frac{(5x\sin 3x + 3\cos 3x - 25\cos x)}{-50}$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1e^x + c_2e^{-x}) - \frac{(5x\sin 3x + 3\cos 3x + 25\cos x)}{50}$$

Example10.8.2: Find the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Solution: We have $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x \;\; Or \;\; y'' - 2y' + y = xe^x \sin x$

.... (1)

Here, we have $F(x) = xe^x \sin x$

The symbolic form of the equation (1) is $(D^2-2D+1)y = xe^x \sin x$

The corresponding auxiliary equation is $D^2-2D+1=0$

$\therefore\ (D-1)^2 = 0 \Rightarrow\ D=1,\ 1.$

The complementary function is given by $C.F = y_c(x) = (c_1+c_2x)e^x$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$
\begin{aligned}
y_p(x) &= \frac{1}{(D-1)^2}\cdot xe^x \sin x \\
&= e^x \frac{1}{(D+1-1)^2}\cdot x\sin x = e^x \frac{1}{D^2}\cdot x\sin x = e^x \iint x\sin x \cdot dx\cdot dx \\
&= e^x \int\left[-x\cos x + \int \cos x\cdot dx\right]\cdot dx = e^x \int(-x\cos x+\sin x)\cdot dx \\
\therefore\quad y_p(x) &= e^x\left[-\int x\cos x\,dx + \int \sin x\,dx\right] = e^x\left[-x\sin x + \int \sin x\,dx - \cos x\right] \\
&= e^x\left[-x\sin x - \cos x - \cos x\right] = -e^x[-x\sin x - 2\cos x]
\end{aligned}
$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1+c_2x)e^x - e^x[-x\sin x - 2\cos x]$$

Example10.8.3: Find the general solution of $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2\cos x$

Solution: We have $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2\cos x\ \ Or\ \ y^{/v} + 2y^{//} + y = x^2\cos x$ ...

(1)

Here, we have $F(x) = x^2 \cos x$

The symbolic form of the equation (1) is $(D^4 + 2D^2 + 1)y = x^2 \cos x$

The corresponding auxiliary equation is $D^4 + 2D^2 + 1 = 0$

$\therefore\ (D^2 + 1)^2 = 0 \Rightarrow\ D^2 = \pm i\ (Twice)$.

The complementary function is given by

$C.F = y_c(x) = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$.

Now, to find the particular integral $y_p(x)$, by the inverse operator method ,

Consider,

$$y_p(x) = \frac{1}{D^4 + 2D^2 + 1} \cdot x^2 \cos x$$
$$= \text{Re}\left(\frac{1}{(D^2+1)^2} \cdot x^2 e^{ix} \right) = \text{Re}\left(e^{ix} \frac{1}{[(D+i)^2 + 1]^2} \cdot x^2 \right)$$
$$= \text{Re}\left(\frac{1}{(D^2 + 2iD)^2} \cdot x^2 \right) = \text{Re}\left(e^{ix} \left\{ \frac{1}{-4D^2} \left(1 - \frac{i}{2} D \right)^{-2} \right\} \cdot x^2 \right)$$
$$\therefore\ y_p(x) = \text{Re}\left(\frac{e^{ix}}{-4} \left\{ \frac{1}{D^2} \left(1 + 2\frac{iD}{2} + 3\left[\frac{iD}{2} \right]^2 + \dots\dots\dots \right) \right\} \cdot x^2 \right)$$
$$= \text{Re}\left(\frac{e^{ix}}{-4} \left\{ \frac{1}{D^2} \left(x^2 + 2ix - \frac{3}{2} \right) \right\} \right) = \text{Re}\left(\frac{e^{ix}}{-4} \left\{ \frac{1}{D} \left(\frac{x^3}{3} + ix^2 - \frac{3}{2} x \right) \right\} \right)$$
$$= -\frac{1}{4} \text{Re}\left[e^{ix} \left(\frac{x^4}{12} + \frac{i\,x^3}{3} - \frac{3x^2}{4} \right) \right] = -\frac{1}{48} \text{Re}\left[(\cos x + i \sin x)\left(x^4 + 4ix^3 - 9x^2 \right) \right]$$
$$\therefore\ y_p(x) = -\frac{1}{48} \left[\cos x \left(x^4 - 9x^2 \right) - 4x^3 \sin x \right]$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - \frac{1}{48}\left[\cos x\left(x^4 - 9x^2\right) - 4x^3 \sin x\right]$$

Example10.8.4: Find the general solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2 e^{2x} \sin 2x$

Solution: We have

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2 e^{2x} \sin 2x \quad Or \quad y^{//} - 4y^{/} + 4y = 8x^2 e^{2x} \sin 2x \quad \ldots\ldots.. \; (1)$$

Here, we have $F(x) = 8x^2 e^{2x} \sin 2x$

The symbolic form of the equation (1) is $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

The corresponding auxiliary equation is $D^2 - 4D + 4 = 0$

$\therefore \; (D-2)^2 = 0 \Rightarrow \; D = 2, \; 2$.

The complementary function is given by $C.F = y_c(x) = (c_1 + c_2 x)e^{2x}$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{(D-2)^2} \cdot 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} \cdot x^2 \sin 2x = 8e^{2x} \frac{1}{D^2} \cdot (x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$\therefore \quad y_p(x) \quad = 8e^{2x} \cdot \frac{1}{D} \int \left[x^2 \left(-\frac{\cos 2x}{2} \right) - \int \left(-\frac{\cos 2x}{2} \right) \cdot 2x \, dx \right]$$

$$= 8e^{2x} \cdot \frac{1}{D} \int \left[-\frac{x^2}{2} \cos 2x + x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx \right]$$

$$\therefore \quad y_p(x) = 8e^{2x} \cdot \int \left[-\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{\cos 2x}{4} \right] dx$$

$$= 8e^{2x} \cdot \int \left[\left(-\frac{x^2}{2} \frac{\sin 2x}{2} - \int (-x) \frac{\sin 2x}{2} dx \right) + \left(\int \frac{x}{2} \sin 2x \, dx \right) + \frac{\sin 2x}{8} \right]$$

$$= 8e^{2x} \left[\left(\frac{-x^2}{4} + \frac{1}{8} \right) \sin 2x + \int x \sin 2x \, dx \right]$$

$$= 8e^{2x} \left[\left(\frac{1}{8} - \frac{x^2}{4} \right) \sin 2x + x \left(\frac{-\cos 2x}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2x}{2} \right) dx \right]$$

$$= 8e^{2x} \left[\left(\frac{1}{8} - \frac{x^2}{4} \right) \sin 2x - \frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \right]$$

$$\therefore \quad y_p(x) = e^{2x} \left[(3 - 2x^2) \sin 2x - 4x \cos 2x \right]$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = (c_1 + c_2 x)e^{2x} + e^{2x} \left[(3 - 2x^2) \sin 2x - 4x \cos 2x \right]$$

Example10.8.5: Find the general solution of $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

Solution: We have $\frac{d^2 y}{dx^2} + a^2 y = \sec ax \;\; Or \;\; y'' + a^2 y = \sec ax$ ... (1)

Here, we have $F(x) = \sec ax$

The symbolic form of the equation (1) is $(D^2 + a^2)y = \sec ax$

The corresponding auxiliary equation is $D^2 + a^2 = 0$

$\therefore D^2 = -a^2 \Rightarrow \; D = \pm ai$.

The complementary function is given by $C.F = y_c(x) = c_1 \cos ax + c_2 \sin ax$.

Now, to find the particular integral $y_p(x)$, by the inverse operator

method ,

Consider,

$$y_p(x) = \frac{1}{D^2 + a^2} \cdot \sec ax = \frac{1}{(D+ai)(D-ai)} \cdot \sec ax$$

Resolving into partial fractions, we obtain

$$\therefore \quad y_p(x) = \frac{1}{2a}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$
$$= \frac{1}{2a}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right]$$

Now, consider,

$$\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax \cdot e^{-iax}\, dx$$
$$\because \frac{1}{D-a} f(x) = e^{ax}\int e^{-ax} f(x)\, dx$$
$$\frac{1}{D-ia}\sec ax = e^{iax}\int \frac{\cos ax - i \sin ax}{\cos ax}\, dx = e^{iax}\int (1 - i \tan ax)\, dx$$
$$= e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to –i, we have

$$\frac{1}{D+ia}\sec ax = e^{-iax}\int \frac{\cos ax + i\sin ax}{\cos ax}dx = e^{-iax}\int (1+i\tan ax)\,dx$$
$$= e^{-iax}\left(x - \frac{i}{a}\log\cos ax\right)$$

Thus

$$y_p(x) = \frac{1}{2ai}\left[e^{iax}\left(x+\frac{i}{a}\log\cos ax\right) - e^{-iax}\left(x-\frac{i}{a}\log\cos ax\right)\right]$$
$$= \frac{x}{a}\cdot\frac{e^{iax}-e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax\cdot\frac{e^{iax}+e^{-iax}}{2}$$
$$\therefore\ y_p(x) = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax\cdot\cos ax$$

Thus, the general (Or complete) solution of equation (1) is given by,

$$y = C.F + P.I = y_c(x) + y_p(x) = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax\cdot\cos ax$$

Test Your Progress 6
Find the general solution of the following differential equations by finding the P.I.by Inverse Operator Method.
1. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = x\cos x$ 2. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = x\sin 2x$
3. $\frac{d^2y}{dx^2}+y = 24x\cos x$ 4. $\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y\right)^2 = xe^{-x}\cos 2x$

10.9 Summary

It is again a method to find P.I. when methods for standerd cases are not aplicable to solve $f(D)y = f(x)$. In this case use of integration is involved.

10.10 Answers to Check Your progress

Test Your Progress 1

1. $y = (c_1 + c_2 x)e^{-x} + \frac{1}{3}e^{2x} + \frac{5^x}{(\log 5)^2 + 2(\log 5) + 1}$

2. $y = (c_1 e^x + c_2 e^{2x}) - 10e^{2x}$

3. $y = (c_1 + c_2 e^x + c_3 e^{-x}) + \frac{e^{2x}}{6} - 15x$

4. $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{100e^{2x}}{17}$

5. $y = (c_1 e^{-2x} + c_2 e^{-3x}) + 55xe^{-2x}$

6. $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{21}{16}e^{-x/2}$

Test Your Progress 2

1. $y = (c_1 + c_2 x)e^x - \frac{5}{2}(\cosh x + \sinh x)$

2. $y = c_1 + e^{\frac{5}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{8\sinh x + 5\cosh x}{39}$

3. $y = (c_1 + c_2 e^{x} + c_3 e^{-x}) + x\sinh x - \frac{1}{6}\sinh 2x$

4. $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{25}{218}[17\cosh 3x + 9\sinh 3x]$

5. $y = (c_1 e^{-2x} + c_2 e^{-3x}) - \frac{1}{5}\left(\frac{x\sinh 2x}{4} - \cosh 2x\right)$

Test Your Progress 3

1. $x(t) = e^{-t}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t) + \frac{\sin t - \cos t}{4}$

2. $y(x) = (c_1 e^{-x} + c_2 e^{-2x}) + 1 - \frac{3}{8}\sin 2x + 4\cos 2x$

3. $y(x) = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{6}\left(\frac{4\sin x + \cos x}{17}\right)$

4. $y(x) = (c_1 + c_2 e^{x} + c_3 e^{-x}) + 2\sin x$

5. $y(x) = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{1}{9}\sin 2x$

Test Your Progress 4

1. $y = e^{-t}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t) + \frac{5}{3}\left(t^2 - \frac{4}{5}t + \frac{2}{3}\right) + \frac{1}{4}(\cos t - \sin t)$

2. $y(x) = (c_1 e^{-x} + c_2 e^{-2x}) + \frac{1}{2}(x + \frac{3}{2}) + \cos 2x + \frac{1}{10}(3\sin 2x - \cos 2x)$

3. $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{3}{25}\left(x + \frac{6}{25}\right) + \frac{x}{4}$

4. $y(x) = (c_1 + c_2 e^{x} + c_3 e^{-x}) - \frac{x^4}{2} - 6x^2$

5. $y(x) = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{e^{2x}}{25} - 4(x^2 - 4)$

Test Your Progress 5

1. $$y = e^{-x/2}\left[(c_1 + x/4)\cos(x\sqrt{3/2}) + (c_2 + x/4\sqrt{3})\sin(x\sqrt{3/2})\right] + e^{x/2}\left[c_3 \cos\sqrt{3x/2} + c_4 \sin\sqrt{3x/2}\right]$$

2. $y = (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^{x}}{17}(4\sin 2x - \cos 2x)$

3. $y(x) = (c_1 e^{x} + c_2 e^{2x}) + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$

4. $y(x) = (c_1 + c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7x}{8} + \frac{11}{6}\right) + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

Test Your Progress 6

1. $y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}(x\sin x + \cos x - \sin x)$

2. $y = (c_1 e^{-2x} + c_2 e^{-x}) + \left(\frac{7 - 30x}{200}\right)\cos 2x + \left(\frac{12 - 5x}{100}\right)\sin 2x$

3. $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - x^3 \cos x + 3x^2 \sin x$

4. $y = [e^{-x}(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x \frac{1}{2}(x\sin x + \cos x - \sin x)]$

$-\frac{e^{-x}}{32}\left[(x^3 - x^2)\cos 2x - \frac{2}{3}x^3 \sin 2x\right]$

10.11 Terminal Questions

Solve

1) $(D^2 + 3D + 2)y = e^{e^x}$

2) $(D^2 + D)y = \frac{1}{1+e^x}$

3) $(D^2 + 3D + 2)y = e^{e^x} + cos\ e^x$

4) $(D^2 + 1)y = \ cosce\ x$

By Method of inverse operator.

Ans-

1) $\gamma = C_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

2) $\gamma = c_1 + c_2 e^{-x} + x - log(1 + e^2) - e^{-x} log(1 + e^x)$

3) $\gamma = C_1 e^{-2x} + c_2 e^{-x} + e^{-2x}\left(e^{e^x} - cose^x\right)$

4) $\gamma = c_1\ cosx + c_2 \sin x + sinx\ log\ \ sinx - x\ cosx$

Unit-11 : Equation Reducible to linear with Constant Coefficients

Structure:

11.1 Introduction

In this chapter we will study Cauchy homogeneous linear different equation with variable coefficients this types of differential equation consisting of three main terms first homogeneous second linear and third differential equation with variable coefficients. So now we explain all these terminology.

1- Homogeneous means power of x in the coefficients are equal to the orders of the derivatives associated with them i.e.

   $\underline{x}.\frac{dy}{d\underline{x}}$, $\underline{x}^2\frac{d^2y}{d\underline{x}^2}$, $\underline{x}^3\frac{d^3y}{d\underline{x}^3}$ and so on.

2- Linear means dependent variable y and its derivatives appear in the first degree and there is no more terms not multiplied together.

3- Different equation means dependent variable and its derivatives occurs in the equation.

   If all these properties comes together in a differential equation such types of equation is called homogeneous linear differential equation with variable coefficient such equations can be solve by reduction procedure of suitable substitution and transform the give linear ordinary differential equations with variable coefficients to linear ordinary differential equations with constant coefficient after then we dicused Legendre’s linear differential equation and then simultaneous linear D.E.W. constant coefficient.

11.2 Objectives:-

After reading this unit you should be able to

- Solve Chauchy’s linear differential equations
- From the given equations we will be able to identify the given equation is Cauchy equation or not.
- Solve simultaneous linear differential equation
- Solve as illustrated in the problems of oscillation and electric circuits among other.

11.3 . Cauchy Linear Differential Equations

A differential equations is said to be Cauchy homogeneous linear differential equation with variable coefficient which is defend as

$$\sum_{i=0}^{n} a_i\, x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} = \quad Q(x)$$

It can be expanded as

$$a_0\, x^n \frac{d^n y}{dx^n} + a_1\, x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2\, x^{n-1} \frac{d^{n-2}y}{dx^{n-2}} + \cdots ..+a_{n-1}\, x \frac{dy}{dx} + a_n \frac{d^0 y}{dx^0} = Q(x)$$

Where $a_0 = 1$ and $\frac{d^0 y}{dx^0}$ =y, so above equation becomes.

$$x^n \frac{d^n y}{dx^n} + a_1\, x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2\, x^{n-1} \frac{d^{n-2}y}{dx^{n-2}} + \cdots ..+a_{n-1}\, x \frac{dy}{dx} + a_n \frac{d^0 y}{dx^0} = Q(x)\ldots\ldots(1)$$

Where a_i's are constants, x are variable and Q is the function of x and power of x in the coefficients are equals to the order of the derivative associated with them so it is called Homogeneous linear differential equation this types of equation was first used by Euler's and Chauchy hence, because of their name it is called Eular Chauchy homogeneous linear differential equation with variable coefficients.

Such types of equation can be reduced to linear differential equation with constant coefficients by the substitution.

Reduction Process

$$Let\ x = e^t\ or\ t = logx\ \ and\ \frac{d}{dt} = D$$

$$Now\ \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy}{dt}.\frac{1}{x}, or\ x\frac{dy}{dx} = \frac{dy}{dt} = Dy$$

Origin differentiate, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}, \frac{1}{x}\right)$$

$$= \frac{1}{x}.\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(\frac{d}{dx}.\frac{1}{x}\right)$$

$$= \frac{1}{x}.\frac{d}{dt}.\frac{dy}{dt}.\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2}.\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = D^2y - Dy = (D^2 - D)y = D(D-1)y$$

Similarly

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2}\right)\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dt}\right)$$

Hence

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dt}$$

$$= D^3y - 3D^2y + 2Dy$$

$$= \ (D^3 - 3D^2 + 2D)y$$

$$= D(D^2 - 3D + 2)y$$

$$= D\ (D-1)(D-2)y$$

Therefore, we can write

$$x^r\frac{d^ry}{dx^r} = D\ (D-1)(D-2)\ (D-3)\ \ldots\ldots(D-r+1)y\ and\ so\ on.$$

Substituting these values in equation (1) we get a linear differential equation with constant coefficients which can be solved by the methods already discussed.

Example 1 :- solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^3$

Solution:- Given equation is a Cauchy's homogeneous linear different equation. So Put

$$x = e^t \text{ or } t = \log x \text{ and } \frac{dt}{dx} = \frac{1}{x} \text{ and } x\frac{dy}{dx} = Dy \text{ where } D$$

$$= \frac{d}{dt} \quad , \qquad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Substituting these values in the given equation it reduces to

$$D(D-1)y - Dy + y = (e^t)^3$$

$$= (D^2 - D - D + 1) = e^{3t}$$

$$= (D^2 - 2D - D + 1) = e^{3t}$$

which is a linear equation will constant coefficients. Its A.E. is D^2 - 2D+1=0 i.e.

$(D-1)^2 = 0$ or D = 1,1

So C.F is y= $(c_1+c_2t)\, e^t$, where c_1 and c_2 are arbitrary constants.

Now calculate PI = $\frac{1}{(D-1)^2}\ e^{3t}$

$$= \frac{1}{(3-1)^2} = e^{3t}$$

$$= \frac{1}{4} e^{3t}$$

Therefore the complete solution is y= CF+PI== $(c_1+c_2t)\, e^t + \frac{1}{4} e^{3t}$

Now putting t= log x Hence y=$(c_1 + c \log x) e^{logx} + \frac{1}{4} x^3$

Example.2:- solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 4y = x^3 =$ cos (log) x + x sin (log x)

This equation in a cauchy`s homogenous liner differential equation we put

$x = e^t\ or\ t = logx$ then we have

$$x \frac{dy}{dx} = Dy\ and\ x^2 \frac{d^2y}{dx^2} = D\,(D-1)y\ where\ D = \frac{d}{dt}$$

the given equation is transformed into

$$D\,(D-1)y - Dy + 4y = cos\,(t) + e^t \sin t$$

$or\ (D^2 - D - D + 4)y = cos\ t + e^t \sin t$

$or\ (D^2 - 2D + 4)y = cos\ t + e^t \sin t$

which in a liner differential equation with constant coefficient. Now the A.E. is

$D^2 - 2D + 4 = 0\ \Rightarrow D = 1\ \pm i\sqrt{3}$

Hence C.F. in Given by

$y = e^t\{c_1\ \cos\sqrt{3}\ .t + c_2 \sin\sqrt{3}\,.t\}$

$$and\ P.I = \frac{1}{D^2 - 2D + 4} Cost + \frac{1}{D^2 - 2D + 4} e^t.Sint$$

$$= \frac{1}{-1 - 2D + 4} Cost + e^t \frac{1}{(D+1)^2 - (D+1) + 4} Sint$$

$$= \frac{1}{3 - 2D} Cost + e^t \frac{1}{D^2 + 3} Sint$$

$$= \frac{2D + 3}{4D^2 - 9} Cost + e^t \frac{1}{-1 + 3} Sint$$

$$= \frac{(2D + 3)cost}{-4 - 9} + e^t \frac{1}{2} Sint$$

$$\frac{1}{13}(-2\ sint + 3\ cost) + \frac{1}{2}e^t sint$$

Hence the general solution in

$$y = e^t\left[c_1 \cos\left(\sqrt{3}.t\right) + c_2 sin\left(\sqrt{3}.t\right)\right] + \frac{1}{13}[3cost - 2sint] + \frac{1}{2}e^t sint$$

$putting\ t = logx\ we\ get$

$$y = x\left[c_1 \cos\sqrt{3}\ (logx)\ + c_2 sin\sqrt{3}\ (logx)\right]$$

$$+\frac{1}{13}[3\cos\ (logx) - 2sin\ (logx)] + \frac{1}{2}x\ \sin\ (logx)$$

Example. 3: – solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 . \sin\ (logx)$

It is a cauchy`s homogenous linear diffintial equation. we put

$x = e^t$ or $t = \log x$

then $x\frac{dy}{dx} = Dy$ & $x^2\frac{d^2y}{dx^2} = D\ (D-1)y$

therefore the given equation is changed in the form :

$$[D(D-1) - 3D + 5)y = e^{2t} sint$$

$or\ (D^2 - 4D + 5)y = \ e^{2t}\sin t$

$The\ A.E.is\ D^2 - 4D + 5 = 0\ \&\ there\ fore\ D$

$= 2 \pm \mathrm{i}\ \&\ C.F\ is\ given\ by$

$(putting\ t = logx)$

$y = e^{2t}[c_1\text{cost} + c_2 sint] = x^2[c_1\cos\text{(logx)} + c_2\sin(logx)$

$$and\ P.I = \frac{1}{D^2 - 4D + 5}e^{2t}sint$$

$$= e^{2t}\frac{1}{(D+2)^2 - 4(D+2) + 5}sint$$

$$= e^{2t}\frac{1}{D^2 + 1}sint$$

$$= -e^{2t}\frac{t}{2}cost = -\frac{1}{2}e^{2t}.t\ cost = -{}^{1}\!/_{2}\,x^2(logx)\cos\,(logx)$$

Therefore the general solution in given by -

$y = C.F. + P.I$

$$= x^2[c_1\cos(logx) + c_2\sin(logx)] - \frac{1}{2}x^2(logx)\cos\,(logx)$$

11.3.1.Test your knowledge

Solve the following differential equation -

1. $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$
2. $x^3 \frac{d^3y}{dx^3} - x^2 . \frac{d^2y}{dx^2} - 2y = x^2 + x^{-3}$
3. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin\,(logx^2)$
4. $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2logx$
5. $(x^2D^2 - xD + 1)y = x. logx$
6. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0\ (mult. by\ x$

Ans:-

1. $y = c_2x^2 + c_2x^3 + \frac{x^5}{6}$
2. $y = c_1x^2 + c_2\cos(logs) + c_3\sin(logx) + \frac{x^5}{5}logx - \frac{1}{50}x^{-3}$
3. $y = c_1\cos(logx) + c_2 sin\,(logx) - \frac{1}{3}sine(logx^2)$
4. $y = c_1x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{1}{2}logx + {}^3/_8$
5. $y = x[c_1logx + c_2] + \frac{x}{6}(logx)^3$

6. $y = c_1 + c_2 logx - \frac{x^2}{4}$

11.4 Legendre's Linear Differential Equations

$$\sum_{k=0}^{n} P_k (a+bx)^{n-k} \ \frac{d^{n-k}y}{dx^{n+k}} = \quad Q(x) \ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Or

$$P_0(a+bx)^{n-k} \ \frac{d^{n-k}y}{dx^{n+k}} + P_1(a+bx)^{n-1} \ \frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$+P_{n-1}\frac{dy}{dx} + P_n(a+bx)^{n-n} \ \frac{d^{n-1}y}{dx^{n+1}} = Q(x)$$

$$\Rightarrow P_0(a+bx)^{n-k} \ \frac{d^{n-k}y}{dx^{n+k}} + P_1(a+bx)^{n-1} \ \frac{d^{n-1}y}{dx^{n-1}} +..P_{n-1}\frac{dy}{dx}$$

$$+ P_n y = \ Q(x)$$

Where P_0 = 1 and $\frac{d^0y}{dx^0}$ = y

So above equation can be written as

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots . + P_{n-1}\frac{dy}{dx} + P_n y$$

$$= Q(x) - - - - - - (2)$$

Equation (1) and (2) both are same. Equation (2) is also a Legendre's linear different equation.

Reduction Process

Let a+ bx = e^t

Taking log on both sides

Log (a+ bx) = t log e

Now consider

$$\frac{dy}{dx} = \frac{d}{dx}\log(a+bx) = \frac{d}{dx}(t \log e) \qquad loge^e = 1$$

$$= \frac{1}{a+bx}. b = \frac{dt}{dx} = \frac{dt}{dy}.\frac{dy}{dx}$$

$$= \frac{b}{a+bx} = \frac{dy}{dx}.\frac{dt}{dy}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dt} = b\ Dy \qquad where\ D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \left(\frac{b}{a+bx}\right)\frac{dy}{dt}$$

Again differentiate, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\right)\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \left(\frac{b}{a+bx}\right)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\,.\frac{(a+bx)x0 - b.b}{(a+bx)^2}$$

$$= \frac{b}{a+bx}\left(\frac{dt}{dx}\right) - \frac{b^2}{(a+bx)^2}\frac{dy}{dt}$$

$$= \frac{b}{a+bx}\frac{d^2y}{dx^2}.\frac{dt}{dx} = -\frac{b^2}{(a+bx)^2}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{b}{a+bx}.\frac{d^2y}{dt^2}\left(\frac{b}{a+bx}\right) - \frac{b^2}{(a+bx)^2}\frac{dy}{dt}$$

$$= \frac{b^2}{(a+bx)^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$(a+bx)^2\frac{d^2y}{dt^2} = b^2(D^2y - Dy)$$

$$(a+bx)^2\frac{d^2y}{dt^2} = b^2\,D(D-1)y$$

Similarly

$$(a+bx)^3\frac{d^3y}{dx^3} = b^3\ D(D-1)(D-2)y$$

Substituting these values in equation 2, we get a linear different equation with constant coefficients, which can be solved by the methods already discussed.

Example 1 : Solve

$$(5+2x)^2\frac{d^2y}{dx^2} + 6(5+2x)\frac{dy}{dx} + 8y = 0$$

Given equation is a Legendre's linear D.E. put (5+2x) = e^t

$$\log(5+2x) = t.loge = t \quad so\, \frac{2}{5+2x} = \frac{dt}{dx}$$

Now consider

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{2}{5+2x}\right)$$

$$(5+2x)\frac{dy}{dx} = 2\frac{dy}{dt} = 2Dy\ \ where\ D = \frac{d}{dt}$$

Again diff, we get $(5+2x)^2\frac{d^2y}{dx^2} = 2^2D(D-1)y$

Putting all these values in given equation, it reduces

4D(D-1)y+6.2Dy+8y=0

$$(D^2 - D + 3D + 2)y = 0$$

$$(D^2 + 2D + 2)y = 0$$

Which is a linear equation with constant coefficients it's A. E. is $D^2+2D+2=0$

$$D = \frac{-2 \pm \sqrt{4 - 4 \times 2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

So C.F. = $e^{-t}[C_1 Cos\ t + i\ C_2 \sin t]$ $PI = 0$

Hence the C.S. is y=$e^{-t}[C_1 Cos\ t + i\ C_2 \sin t]$

Putting t= log(5+2x), So, we have

$y = e^{-\log(5+2x)}[C_1 Cos\ t + i\ C_2 \sin \log(5 + 2x)]$

$y = \frac{1}{5 + 2x}[C_1 Cos\ t + i\ C_2 \sin \log(5 + 2x)]$

Example 2 : Solve

$$(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}+y=4\,Cos\log(x+1)$$

Solution: Given differential equation is Legendre's L.D.E. put x+1 = e^t i.e.

t = log(x+1)

$$So \quad (x+1)\frac{dy}{dx}=Dy \;\; and \;(x+1)^2\frac{d^2y}{dx^2}=D(D-1)y$$

Substituting these values in the given equation it reduces to

D(D-1)y + Dy +y = 4 Cos t

(d^2-D+D+1) y= 4Cos t

Here A.E. is $D^2 + 1 = 0$ D =$\pm$ i

C.F. is C_1 Cos t + c_2 Sin t

So PI is $y=\frac{1}{D^2+1}4\,Cos\,t$

$$=\frac{4t}{2D}\;Cos\;t=\frac{2t}{D}Cost=2t\frac{1}{D}\;\{Cost\;dt\}=2t\;Sin\;t$$

Therefore, the complete solution is

CS = CF+PI

$y = C_1 \cos t + C_2\, Sin\, t + 2t \sin t$

$putting\ t = \log(x+1) so, we\ have$

$$y = C_1\, Cos\, (\log(x+1) + C_2\, Sin \log \quad (x+1)$$
$$+ 2\log(x+1). Sin \log(x+1)$$

11.5 Simultaneous Linear differential equation with Constant Coefficient

Since we know that equations in which there is one independent variable and two or more than two dependent variables, such equations are called simultaneous linear equations. Just like that if two linear ordinary differential equation with two or more dependent variables and a single independent variable then this types of equation is known as simultaneous linear ordinary differential equations. For example

Let x and y are two dependent variable and t is the independent variable. Consider the simultaneous linear differential equation with constant coefficients.

$(i)\ \frac{dx}{dt} + \frac{dy}{dt} + \ 3x + y = 0\ and \frac{dy}{dt} + 5x + 3y = 0$

$(ii) \frac{d^2x}{dx^2} - 3x - 4y = 0\ and\ \frac{d^2y}{dt^2} + x + y = 0$

$(iii)\ \frac{dx}{dt} + \frac{dy}{dt} + y = 1\ and \frac{dx}{dt} - \frac{dz}{dt} + 2x + z$

$= 1\ \ and \frac{dy}{dt} + \frac{dz}{dt} + y + 2z = 0$

where x, y and z are dependent variables and t is the independent variable.

Working Process

Step 1- Firstly we will convert the given equation into operator from.

Step 2- Then solving both these equations with the help of elimination method. We get the value of x or y.

Step 3- If the value of x (or 'y') is obtained then we will get the value of y by substitution method in any one of the given original equations.

Example 1 :- Solve $\frac{dx}{dt} = 7x - y$

$$\frac{dy}{dt} = 2x + 5y$$

Here x, y are independent variable and t is independent variable So it is simultaneous liner differential equation because it has two dependent variable and one independent variable and both differential equation of degree one so it is linear and its coefficient are constant so it is simultaneous linear differential equation wish constant coefficients.

Solution:- Given equation can be rewritten as in operator from

$$Dx = 7x - y \quad where\ D = \frac{d}{dt}$$

$$Dy = 2x + 5y \quad t\ is\ independent\ variable.$$

Or

$$Dx - 7x + y = 0 \quad or\ (D - 7)x + y = 0 \qquad (1)$$

or

$$-2x + Dy - 5y = 0 \quad or\ -2x + (D - 5)y = 0 \qquad (2)$$

Now we will eliminate t, for this multiply in equation (1) by (D-5) and then subtraction equation (1) from equation (2)

$$(D-5)(D-7)x+(D-5)y=0$$

$$-2x+\qquad\qquad(D-5)y=0$$

$$(D-5)(D-7)x+2x=0$$

$$(D^2-12D+37)x=0$$

which is a linear equation with constant coefficients. It's A.E. is

$$D^2-12D+37=0$$

$$D=\frac{12\pm\sqrt{12^2-4\times 37}}{2\times 1}=\frac{12\pm 2i}{2}=6\pm i$$

So C.F = e^{bt} [C_1 Cost + C_2 Sint] and P.I= 0

So C.S = C.F + P.I

x= e^{bt} [C_1 Cos t C_2 Sin t]

Now we will obtained the value of y with the help of x. For this substituting the value of x and $\frac{dx}{dt}$ in equation (1) we have.

x= e^{6t} [C_1 Cost + C_2 Sint]

$$\frac{dx}{dt}=e^{6t}[-C_1 sint+C_2 cost]+6e^{6t}[C_1 cost+C_2 Sin\ t]$$

$and\ equation\ (1) is\ \frac{dx}{dt} = 7x - y\ so\ y = 7x - \frac{dx}{dt}$

$= 7e^{6t}[\mathrm{C_1\ Cos\ t + C_2\ Sin\ t}] - e^{bt}[6C_1 + C_2] Cos\ (6C_2 + C_1)\ Sin\ t$

$y = e^{6t}[(7c_1 - 6C_2 - C_2)\ Cos\ t + (7C_2 - 6C_2 + C_1)\ Sin\ t]$

$y = e^{6t}[(C_1 - C_2)\ Cos\ t + (C_1 + C_2)\ Sin\ t]$

Ex.2: Solve the Simultaneous linear differential equations.

$$\frac{dx}{dt} - 3x - 6y = t^2 \qquad -------- (1)$$

$$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t \qquad -------- (2)$$

$taking\ D = \frac{d}{dt}\ equations\ (1) and\ (2) can\ be\ written\ as$

$$(D-3)x - 6y = t^2 \qquad -------- (3)$$

$$Dx + (D-3)y = t^2 \qquad -------- (4)$$

$To\ eliminate\ x\ from\ (3) \&\ (4) we\ operate\ (3) by\ D\ and\ (4) by (D - 3) we\ get.$

$$D(D-3)x - 6\ Dy = D(t^2) = 2t \qquad -------- (5)$$

$$or\ D(D-3)x+(D-3\ )^2y=(D-3)e^t=e^t-3e^t-2e^t \quad --(6)$$

$now\ subtracting(5)for\ (6)we\ get.$

$$(D^2+9)y=-2e^t-2t$$

$Whose\ general\ solution\ in\ (get\ C.F.\&\ P.I.)$

$$y=c_1\cos 3t+c_2 sing3t-\frac{e^t}{5}-\frac{2t}{9}$$

$Now\ to\ eliminate\ y\ fram\ (3)\&\ (4)we\ oprate\ (3)by\ (D-3)$

$and\ multiply\ (4)by\ (6)and\ then\ subtracting\ we\ get.$

$$(D^2+9)x=6e^t-3t^2+2t$$

Whose general solution is

$$x=c_3cos3t+c_4sint+\frac{3e^t}{5}-\frac{t^2}{3}+\frac{2t}{9}+\frac{2}{7}$$

11.4.1.Test your Knowledge

Solve the following Simultaneous equations:-

1. $\frac{dx}{dt}+y=e^t$

$$\frac{dy}{dt} + x = e^t$$

2. $\frac{dx}{at} + 5x - 2y = t$

$$\frac{dy}{dt} + 2x + y = 0$$

$$given\ that\ x = y = 0\ where = 0$$

3. $4\frac{dx}{dt} + 9\frac{dy}{dt} + 44x + 49y = t$

$$3\frac{dx}{dt} + 7\frac{dy}{dt} + 34x + 38y = e^t$$

Ans:-

1. $x = c_1 cost + c_2 sint + \frac{1}{2}(e^t - e^t)$

$$y = c_1 sint - c_2 cost + \frac{1}{2}(e^t - e^t)$$

2. $x = -\frac{1}{27}(1 + 6t)e^{3t} + \frac{1}{27}(1 + 3t)$

$$y = -\frac{2}{27}(2 + 3t)e^{3t} + \frac{2}{27}(2 - 3t)$$

3. $x = c_1 + c_2 e^t + \frac{19}{3}t - \frac{56}{9} - \frac{29}{7}e^t$

$$y = c_1 e^t + 4c_2 e^{6t} - \frac{17}{3}t + \frac{55}{9} + \frac{24}{7}e^t$$

11.6.Summary

The idea of chauchy`s linear differential equation is introduced and method of solving it is givin by examples. then legendre`s Differential equation is introduced with examples. After that simultaneous liner equation with coefficient is introduced and method of solving it, is given by examples. In the end terminal questions are giving of each type for practice of the students.

11.7.Terminal Question

Solve the following differential equation .

1. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x.\log x.$
2. $x^2 \frac{d^{2y}}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1+x^{2.}}$
3. $(x^2D^2 - Dx + 1)y = x.\log x$
4. $(1 + x)^2.\frac{d^2y}{dx^2}+(1 + x)\frac{dy}{dx} + y = 4\, cos[log(1 + x)]$
5. $(4x + 1)^2.\frac{d^2y}{dx^2}+2(4x + 1)\frac{dy}{dx} + y = 2x + 1.$

6. $\frac{dx}{dt} + x - y = t.e^{t}$

$2y - \frac{dx}{dt} + \frac{dy}{dt} = e^{t}$

7. $(D-1)x + Dy = t$

$3x + (D+4)y = t^{2}$

Ans:-

1.

2. $y = c_1 x + \frac{c_2}{x} + \frac{x}{4} log(1+x^2) - \frac{x}{4} + \frac{1}{4x} log(x^2+1).$

3. $y = x[c_1 logx + c_2] + \frac{x}{6}\ (logx)^3$

4. $y = c_1[log(x+1)] + c_2 sin[log(1+x)]$

5. $y = c_1 + c_2\ log(4x+1)(4x+1)^{1/4} + \frac{1}{18}(4x+1) + \frac{1}{2}$

6.

7. $x = -2c_1 e^{2t} - \frac{2}{3} c_2 e^{-2t} - \frac{1}{4} - \frac{1}{2} t$

$y = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4} t + \frac{1}{4} t^2 + \frac{1}{8}.$

Unit – 12 Linear differential equation of Second Order

Structure:

12.1. Introduction

An equation of the type

$\frac{d^2y}{dx^2} + P\frac{dy}{dx}$ +Qy = R Where P, Q, R

are functions of x alone is called linear differential

equation of second order, here coefficient of $\frac{dy}{dx}$ *is* 1

Note - If the coeffient P & Q are constant then it becomes linear differential equation with constant coeffient wich has been discussed earliar.

12.2. Objective

By reading the matter the student able be identify the type of differential equation & by the Method givin in each section the student can solve the given differential equation.

12.3 . Method of reduction in equation where one part of the C.F. is known.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \text{ ....(1)}$$

Let y = u be the known part of the complementary function

Then $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$ ....(2)

Now putting $y = \upsilon u$ we get $\frac{dy}{dx} = \upsilon\frac{du}{dx} + u\frac{d\upsilon}{dx}$

and $\frac{d^2y}{dx^2} = v.\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}$

Putting the values of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

in equation (1)

We get

$$\left(\upsilon\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{d\upsilon}{dx} + u\frac{d^2\upsilon}{dx^2}\right) + p\left(\upsilon\frac{du}{dx} + u\frac{d\upsilon}{dx}\right) + Q.\upsilon u = R$$

Or $\upsilon\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2\upsilon}{dx^2} + P\frac{d\upsilon}{dx}\right) + 2\frac{du}{dx}.\frac{d\upsilon}{dx} = R$

With the help of equation (2) and dividing by u we get

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + \frac{2}{u}.\frac{du}{dx}.\frac{dv}{dx} = \frac{R}{u}$$

Or $$\frac{d^2\upsilon}{dx^2} + \left(P + \frac{2}{u}.\frac{du}{dx}\right).\frac{d\upsilon}{dx} = \frac{R}{u} \text{..................}(3)$$

We put $\frac{d\upsilon}{dx} = Z$ *then* $\frac{d^2\upsilon}{dx^2} = \frac{dz}{dx}$

$\therefore$ from (3) we get

$$\frac{dz}{dx} + \left(P + \frac{2}{u}.\frac{du}{dx}\right).Z = R/u \text{......................}(4)$$

which is a linear differential equation of order one, its

I.F.
$$= e^{\int}\left(P + \frac{2}{u}\frac{du}{dx}du\right)$$
$$= \int_e \left(P.dx + \frac{2}{u}du\right)$$
$$= u^2.\int_e P.dx$$

& so the solution of equation (4) is

$$Z.u^2 \int_e Pdx = \int\left(\frac{R}{u}.u^2 \int_e Pdx\right)dx + C \text{ ....(5)}$$

From which we get

$$Z = \frac{d\upsilon}{dx}$$

And integrating we get the value of υ & thus we get the

Solution y = υu.

This is the complete solution of equation (1)

Ex.1 : Solve

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)y = 0. \textit{It is given} \text{..............(1)}$$

That $y = e^{x2}$ is one solution in the complementary function.

Solution: Here $u = e^{x2}$ therefore putting $y = v.e^{x2}$ in (1)

We get

$$\frac{d^2\upsilon}{dx^2} + \left[P + \frac{2}{u}.\frac{du}{dx}\right]\frac{d\upsilon}{dx} = 0$$

$(\because R = 0)$

From equation (4) where

$$Z = \frac{d\upsilon}{dx}$$

Here $P = -4x,\ Q = 4x^2 - 2, \qquad and \qquad R = 0$

$$\therefore \frac{d^2v}{dx^2} + \left[-4x + \frac{2}{e^{x2}}.(2xe^{x2})\right]\frac{dv}{dx} = 0$$

Or $\dfrac{d^2u}{dx^2} + [-4x + 4x]\dfrac{d\upsilon}{dx} = 0 \ or \ \dfrac{d^2\upsilon}{dx^2} = 0$

Or $\dfrac{d\upsilon}{dx} = c_1 \qquad or \quad \upsilon = c_1 x = c_2$

$\therefore$ *The solution is* $y = u\upsilon = e^{x^2}.\ (c_1 x + c_2 x)$

Ex. 2: Solve $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\ y = 0$

When $y = e^x$ is one solution in the C.F.

Solution : Dividing the equation by x we get

$$\frac{d^2y}{dx^2} - \frac{(2x-1)}{x}\frac{dy}{dx} + \left(\frac{x-1}{x}\right)y = 0 \text{ ....(1)}$$

We put $y = \upsilon e^x$, then we get from (1)

$$\frac{d^2\upsilon}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{d\upsilon}{dx} = 0$$

Putting P = $-\frac{(2x-1)}{x}$ & $u = e^x$ and $\frac{du}{dx} = z$

We get $\frac{dz}{dx} + \left[\frac{-(2x-1)}{x} + \frac{2}{e^x}e^x\right].z = 0$

Or $-\left[\frac{dz}{dx} + \frac{-2x+1+2x}{x}.z\right] = 0$

Or $\frac{dz}{dx} + \frac{z}{x} = 0 \quad or \quad \frac{dz}{x} = -\frac{dx}{x}$

Or Integrating log z = log x + log c

Or z = $\frac{c}{x}$ or $\frac{d\upsilon}{dx} = \frac{c}{x}$ or $\upsilon = c \log x + c_1$

$\therefore$ The complete solution is

$$y = u.\upsilon = e^x\,(c \log x + c_1)$$

Ex. 3: Solve $x^2.\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$ It is given that

$x+\frac{1}{x}$ is one solution .

Solution : dividing the equation by x^2

We get $\frac{d^2y}{dx^2}+\frac{1}{x}.\frac{dy}{dx}-\frac{y}{x}=0$

Putting y = $\upsilon.\left(x+\frac{1}{x}\right)$ *the* equation becomes

$$\frac{d^2\upsilon}{dx^2}+\left(P+\frac{2}{u}.\frac{du}{dx}\right).\frac{d\upsilon}{dx}=0 \text{ Where } u=x+\frac{1}{x}$$

Or $\frac{d^2\upsilon}{dx^2}+\left\{\frac{1}{x}+\frac{2(1-\frac{1}{x^2}}{x+1/x}\right\}.\frac{d\upsilon}{dx}=0$

Or $\frac{d^2\upsilon}{dx^2}+\frac{3x^2-1}{x(x^2+1)}.\frac{d\upsilon}{dx}=0$

Or putting $z=\frac{d\upsilon}{dx}$ we get $\frac{dz}{dx}+\frac{3x^2-1}{x(x^2+1)}.z=0$

Or $\frac{dz}{z}+\frac{3x^2-1}{x(x^2+1)}dx=0$

Or $\frac{dz}{z}+\left(-\frac{1}{x}+\frac{4x}{x^2+1}\right)dx=0$

& so integrating log z – log x + 2 log $(x^2 + 1)$ = log c_1

Or $$Z = \frac{C_1 x}{(x^2+1)} \quad or \quad \frac{d\upsilon}{dx} = \frac{C_1 x}{(x^2+1)^2}$$

Or integrating $$\upsilon = -\frac{C_1}{2(x^2+1)} + C_2$$

$\therefore$ The complete solution is y = υu

Or $$y = \upsilon.\left(\frac{x^2+1}{x}\right)$$

Or $$y = \frac{A}{x} + C_2\left(x + \frac{1}{x}\right) = \frac{A_1}{x} + B_1.x$$

Check your Progress 1

Exercise- Solve the following differential equations

1. $Sin^2 x.\frac{d^2y}{dx^2} = 2y$ When $y = \cot x$ is a solution.

2. $\left((x\,Sin\,x + Cos\,x)\frac{d^2y}{dx^2}\right) - x\cos x.\frac{dy}{dx} + y\cos x = 0$

When y = x is a solution.

3. $x\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right) + (1-x)y = x^2c^{-x}$ *when* $y = e^x$ is a solution.

4. $\frac{d^2y}{dx^2} - x^2.\frac{dy}{dx} + xy = x$ When y = x is a solution.

Ans. (1) cy = 1 + (C_1 – x) cot x (2) y = C_2x – C_1 cos x

(3) $y = C_2e^x + C_1(2x+1)e^{-x} - \frac{1}{4}\left(2x^2 + 2x+1)e^{-x}\right)$

(4) $y = 1C_1x\int\frac{1}{x^2}\frac{x^3}{e^3}.dx + C_1x$

12.4 Method to find out part of the complementary function of

$$\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = 0 \text{ ....(1)}$$

Rule 1: $y = e^{mx}$ *is a solution if* $m^2 + Pm + Q = 0$

If $y = e^{mx}$ *then* $\frac{dy}{dx} = me^{ms}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore$ If $y = e^{mx}$ in a solution then from (1)

$m^2e^{ms} + Pme^{mx} + e^{mx} = 0$

Or $m^2 + Pm + Q = 0.$

In Particular if $y = e^x$ is a solution there of 1 + P + Q = 0 & if

m = -1 then e^{-x} will be a solution if 1 – P + Q = 0

Rule 2 : $y = x^m$ is a solution

Then $\frac{dy}{dx} = m x^{m-1}$ & $\frac{d^2 y}{dx^2} = m(m-1)\, x^{m-2}$

Therefore from (1) we have

$m\,(m-1)\, x^{m-1} + P\, m x^{m-1} + Q\, x^m = 0$

Or $m(m-1) + P\, mx + Qx^2 = 0$

In particular if m = 1 then y = x will be a solution if P + Qx = 0

& Putting m = 2, y = x^2 will be a solution

If 2 + P.x + Q x^2 = 0.

12.5 Procedure for solving the differential equation.

(i) We put the equation in the standard form

$$\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy \quad = \; R$$

in which the coefficient of $\frac{d^2y}{dx^2}$ is one.

(ii) if 1 + P + Q = 0 then $y=e^x$ in solution

If 1 – P + Q = 0 then $y=e^{-x}$ is a solution

(iii) if P + xQ = 0 then y = x is a solution.

(iv) if $2 + 2Px + Qx^2 = 0$ then $y = x^2$ is a solution

(v) Put y = υu there reduced equation will be

$$\frac{d^2y}{dx^2} + \left(P + \frac{2}{u}.\frac{du}{dx}\right)\frac{d\upsilon}{dx} = \frac{R}{u}$$

(vi) Put $z=\frac{d\upsilon}{dx}$ & solve the linear equation in z & x.

Ex.: 1. Solve $\frac{d^2y}{dx^2} - x^2.\frac{dy}{dx} + xy = x$

Here $P = -x^2$, Q = x & so P + xQ = 0.

Therefore y = x in a part of the C.F.

& so u = x. Now putting y = $\upsilon u = \upsilon x$

$$\therefore \frac{dy}{dx} = x.\frac{d\upsilon}{dx} + \upsilon$$

$$\& \frac{d^2y}{dx^2} = x\frac{d^2\upsilon}{dx^2} + 2.\frac{d\upsilon}{dx}$$

then Equation becomes

$$\left(x\frac{d^2\upsilon}{dx^2} + 2.\frac{d\upsilon}{dx}\right) - x^2\left(x\frac{d\upsilon}{dx} + \upsilon\right) + x.\,\upsilon x = x$$

Or $\quad x\dfrac{d^2\upsilon}{dx^2} + (2 - x^3)\dfrac{d\upsilon}{dx} = x.\quad Now\ putting\ \dfrac{d\upsilon}{dx} = z$

We get $\quad x.\dfrac{dz}{dx} + (2 - x^3)\,z = x$

Or $\quad \dfrac{dz}{dx} + \left(\dfrac{2}{x} - x^2\right).z = 1$............................(1)

This is linear and the I.F. $= e^{\int\left(\frac{2}{x} - x^2\right)dx}$

$$= x^2 e^{-\frac{1}{3}x^3}$$

Hence the solution of (1) in

$$Z.\, x^2 \, e^{-\frac{1}{3}x^3} = \int x^2 \, e^{-\frac{1}{3}x^2} \, . \, dx + C_1$$

$$= -\, e^{-\frac{1}{3}x^3} + C_1 \qquad \text{(by putting } \frac{1}{3}x^3 = t \text{)}$$

So we get $Z = \dfrac{d\upsilon}{dx} = -\dfrac{1}{x^2} + \dfrac{C_1 e^{\frac{1}{3}x^3}}{x^2}$

Or $\quad \upsilon = \dfrac{1}{x} + \int C_1 e^{\frac{1}{3}x^3} . x^{-2} dx + C_2$

$\therefore$ The complete solution is

$$y = \upsilon u = 1 + x C_1 \int e^{\frac{1}{3}x^3} x^{-2} \, dx + C_2 x.$$

Ex. Solve $\; x^2 . \dfrac{d^2 y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)\, y = x^3$

Solution: Dividing by x^2 the equation in the standard form is

$$\frac{d^2 y}{dx^2} - \frac{2(1+x)}{x} . \frac{dy}{dx} + \frac{2(1+x)}{x^2} \, y = x$$

Hence $\; P = \dfrac{-2(1+x)}{x}, \; Q. = \dfrac{2(1+x)}{x^2}$

and P + x. Q = 0 Therefore y = x is a part of C.F.

& so u = x. Therefore putting y = ux the equation becomes

$$\left(x\frac{d^2\upsilon}{dx^2}+2\frac{d\upsilon}{dx}\right)-\frac{2(1+x)}{x}\left(x\frac{d\upsilon}{dx}+\upsilon\right)+\frac{2(1+x)}{x^2}.\upsilon x=x$$

Or $x\frac{d^2\upsilon}{dx^2}-2x\frac{d\upsilon}{dx}=x.$ *Now Putting* $\frac{d\upsilon}{dx}=Z$

We get $x\frac{dz}{dx}-2xz=x$

Or $\frac{dz}{dx}-2z=1$

Which is linear & so the I.F. = e^{-2x}

$\therefore$ the solution is

$$Z.e^{-2x}=\int e^{-2x}.\,dx+C_1=-\frac{1}{2}e^{-2x}+C_1$$

Or $Z=-\frac{1}{2}+C_1e^{2x}$ *or* $\frac{d\upsilon}{dx}=C_1e^{2x}-\frac{1}{2}$

Integrating $\upsilon=\frac{C_1}{2}e^{2x}-\frac{1}{2}x+C_2$

Hence the complete solution is

$$y=\upsilon x=\frac{C_1\,x\,e^{2x}}{2}-\frac{1}{2}x^2+C_2x.$$

Ex. : Solve $\frac{d^2y}{dx^2} - \cot x.\frac{dy}{dx} - (1-\cot x)\ y = e^x.\sin x.$

Hence P = - cot x , Q = - (1 – cot x)

& So 1+P + Q = 0 therefore y = e^x is a part of

The C.F. Putting y = $\upsilon u = \upsilon.e^x$

$$\therefore \quad \frac{dy}{dx} = e^x.\left(\frac{d\upsilon}{dx} + \upsilon\right), \quad and \quad \frac{d^2y}{dx^2} = e^x\left(\frac{d^2\upsilon}{dx^2} + 2\frac{d\upsilon}{dx} + \upsilon\right)$$

& So the equation becomes

$$e^x\left[\left(\frac{d^2\upsilon}{dx^2} + 2\frac{d\upsilon}{dx} + \upsilon\right) - \cot x\left(\frac{d\upsilon}{dx} + \upsilon\right) - (1-\cot - x)\upsilon\right]$$

Or $$\frac{d^2\upsilon}{dx^2} + (2-\cot x).\frac{d\upsilon}{dx} = \sin x.$$

Putting $\frac{d\upsilon}{dx} = Z$ we get

$\frac{dz}{dx} + (2-\cot x).z = \sin x$ which is linear

$$\therefore \quad I.F. = (e^{\int 2-\cot x}) = e^{2x-\log\sin x} = \frac{e^{+2x}}{\sin x}$$

$\therefore$ Solutiion is

$$Z.\frac{e^{-2x}}{\sin x} = \int \sin x.\,\frac{e^{-2x}}{\sin x}\,dx + C_1 = \frac{1}{2}e^{2x} + C_1$$

Or $Z = \frac{1}{2} \sin x + C_1 \sin x\, e^{-2x}$

Or $\frac{d\upsilon}{dx} = \frac{1}{2} \sin x + C_1 \sin x e^{-2x}$. Integrating

$$\upsilon = -\frac{1}{2} \cos x - \frac{C_1}{2^2+1} e^{-2x} (\cos x + 2 \sin x) + C_2$$

$$\left(\because \int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right)$$

$\therefore$ The complete solution is y = υe^{x}

Or $y = -\frac{1}{2}e^{x} \cos x - \frac{C_1}{5} e^{-x} (\cos x + 2 \sin x) + C_2 e^{x}$

Check Your Progress 2

Exercise - Solve the following differential equations :

1. $(x \sin + \cos x)\frac{d^2 y}{dx^2} - x \cos x \frac{dy}{dx} + y \cos = 0$

Of which y = x is a solution.

2. $$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = \sin x$$

(Hint 1 – P + Q = 0)

Ans. (1) y = - $C_1 \cos x + C_2 x$

(2) $y = C_1 e^{-x} + C_1(\sin x - \cos x) - \frac{1}{10}(\sin 2x - 2\cos 2x).$

12.6 Complete solution of the second order differential equation by changing the dependent variable

$\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy = R$ by changing the dependent . Variable (or equations which do not contain x directly).

The equation which do not contain x directly are of the form

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx,}, y\right) = 0 ....(1)$$

To solve such equations we put

$\frac{dy}{dx} = p$ and from (1) we get

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dP}{dy}.\frac{dy}{dx} = \frac{dp}{dy}\ p$$

$\therefore$ Putting in (1) we get

$$f\,(p.\frac{dP}{dy},\ p,\ y) = 0 \qquad \text{.......................(2)}$$

Now equation (2) can be solved for P

Let $p = f_1(y)$ & $so\ \frac{dy}{dx} = f_1\ (y)$

Or $\frac{dy}{f_1(y)} = dx \quad or \quad \int \frac{dy}{f_1(y)} = x + c.$

Ex.: Solve $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2$ ....(1)

Solution : Put $\frac{dy}{dx} = p$ & $so\ \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}.\frac{dy}{dx} = p.\frac{dp}{dy}$

$\therefore$ from (1) we get $y.p\frac{dp}{dx} + p^2 = y^2$ ....(2)

We get $2\text{p}\frac{dp}{dy} = \frac{dz}{dy}$ $\therefore$ from (2)

$$\frac{1}{2}\ \frac{dz}{dy} + \frac{z}{y} = y \qquad or \qquad \frac{dz}{dy} + \frac{2z}{y} = 2y$$

Which is linear and so

I.F. $= e^{\int \frac{2}{y} dy} = e^{2\log y} = e^{\log y^2} = y^2$

$\therefore$ The solution in

$$zy^2 = \int 2y(y^2)\, dy + c_1$$

Or $p^2y^2 = \dfrac{y^4}{2} + C_1 \quad or \quad 2\,p^2\,y^2 = y^4 + 2C_1$

Or $\left(\sqrt{2}\right) y\,p = \sqrt{y^4 + 2C_1}$

Or $\sqrt{z.y}\,\dfrac{dy}{dx} = \sqrt{y^4 + 2C_1} \quad or \quad \sqrt{2}\,\dfrac{y\,dy}{\sqrt{y^4 + k}} = dx \quad (K = 2C_1)$

Or Putting $y^2 = t$ $\therefore$ $2ydy = dt$

Or $\dfrac{1}{\sqrt{2}}\,\dfrac{dt}{\sqrt{t^2 + K}} = dx \; or \; \dfrac{1}{\sqrt{2}}\sinh^{-1}\left(\dfrac{t}{\sqrt{k}}\right) = x + C_2$

Or $\sin h^{-1}\left(\dfrac{y^2}{\sqrt{x}}\right) = \left(\sqrt{2}\right)x + C_2 \quad or \quad y^2 = \sqrt{k}.\sin h\left((\sqrt{2})\,x + C_2\right)$

Ex. Solve $y(1 - \log y)\dfrac{d^2y}{dx^2} + (1 + \log y)\left(\dfrac{dy}{dx}\right)^2 = 0$ ....(1)

Put $\frac{dy}{dx} = p$ & so $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \left(\frac{dp}{dy}\right) . p$

$\therefore$ *from* (1) $y(1 - \log y)\frac{dp}{dy} . p + (1 + \log y) p^2 = 0$

Or $y(1 - \log y) \frac{dp}{dy} + (1 + \log y)\, p = 0$

OR $y(1 - \log y) \frac{dp}{dy} = -(1 + \log y)\, p$

or $\frac{dp}{p} = -\frac{(1 + \log y)}{y(1 - \log y)} . dy$ ....(2)

Putting log y = t & so $\frac{dy}{y} = dt$ $\therefore$ from (2)

$\frac{dp}{p} = \frac{-(1+t)}{1-t} dt = \left(1 + \frac{2}{t-1}\right) dt$

Or Integrating $\log p = t + 2\log(t-1) + \log C$

Or $\log p = t + \log C.(t-1)^2$

Or $p = ce^t (t-1)^2 = Cy(\log y - 1)^2$ $\therefore y = e^t$

Or $\frac{dy}{dx} = Cy\,(\log y - 1)^2$ *or* $\frac{dy}{y(\log y - 1)^2} = c_1 dx$

Or integrating we get

$$-\frac{1}{(\log y-1)}\ Cx + C_1$$

Or (1- log y) = $\frac{1}{Cx+C_1}$

Check Your Progress 3

Ex. Solve the following differential equations :

1. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1$ $\qquad$ $Ans.\ \ y^2=x^2+ax+b$

2. $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}=0$ $\qquad$ $Ans.\ Cy+2=de^{Cx}$

3. $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\log y$ $\qquad$ $Ans.\ \log=be^{x}+a\,e^{-x}$

12.7 Complete Solution of the differential equation by changing the independent variable

$\frac{d^2y}{dx^2}+P.\frac{dy}{dx}+Qy=R$ *by changing the independent*

Variable. ……………………..(1)

We change the independent variable from x to z with the help of Z = f(x).

Then

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} \, and \; \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}.\frac{d^2z}{dx^2}$$

Putting these values in (1) we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}.\frac{d^2z}{dx^2} + P.\left[\frac{dy}{dZ}.\frac{dZ}{dx}\right] + Qy = R$$

Or $$\frac{d^2y}{dZ^2}\left(\frac{dz}{dx}\right)^2 + \left(P\frac{dZ}{dx} + \frac{d^2Z}{dx^2}\right)\frac{dy}{dz} + Qy = R$$

Or $$\frac{d^2y}{dz^2} + \frac{\left(p\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2}.\frac{dy}{dz} + \frac{Qy}{\left(\frac{dz}{dx}\right)^2} = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad \text{............(2)}$$

Now putting $P_1 = \dfrac{P.\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$.

$\therefore$ from (2) we have

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad ....(3)$$

Hence P_1, Q_1, R_1 are functions of x can be expressed a function of Z with the help of z = f (x).

Case 1: $when\ P_1 = 0\ if\ we\ choose\ z\ such\ that\ p_1 = 0\ ie\ \frac{d^2z}{dx^2} + p\frac{dz}{dx} = 0$

$or\ \frac{d}{dx}\left(\frac{dz}{dx}\right) + p\frac{dz}{dx} = 0\ which\ in\ linear\ in\ \frac{dz}{dx}$

$\therefore$ The solution is

$$\frac{dz}{dx} = e^{-\int pdx} \quad or \quad z = \int \left[e^{-\int pdx}\right] dx$$

equation (3) becomes $\frac{d^2y}{dz^2} + Q_1 y = R_1$ $\quad which\ can\ be\ solved\ if$

$(i)\ \ Q_1\ \ is\ constant\ then\ it\ is\ a\ linear\ equation\ with\ constant\ coefficent$

$(ii)\ \ Q_1\ \ is\ of\ the\ form\ \frac{k}{z^2}\ \ then\ it\ is\ \ linear\ homogenous\ equation$

$with\ variable\ coefficent.$

case - 2. We choose Z such that $Q_1 = a^2$ = costout

Or $\frac{Q}{\left(\frac{dz}{dx}\right)^2} a^2 \Rightarrow a\left(\frac{dZ}{dx}\right) = \sqrt{Q}$

Or $dZ = \frac{\sqrt{Q}}{a} dx \;\; or \;\; Z = \int \frac{\sqrt{Q}}{a} dx$

Note: Hence Q is taken in such a way that Q remain the whole Square of a

function without surd and its negative sign is ignored we choose Z such that

$\left(\frac{dz}{dx}\right)^2 = Q.\; these \;\; find \;\; \frac{dz}{dx} \;\; and \;\; find \; Z \;\&\; \frac{d^2z}{dx^2}$ then find P_1, Q_1, R_1 etc.

Ex.: Solve $\frac{d^2y}{dx^2} + \cot x\left(\frac{dy}{dx}\right) + 4y\operatorname{cosec}^2 x = 0$ …………..(1)

OR $\sin^2 x.\frac{d^2y}{dx^2} + \sin x.\cos x.\frac{dy}{dx} + 4y = 0$ by changing the independent variable

Solution:

From (1) $P = \cot x,\; Q = 4\operatorname{cosec}^2 x,\; R = 0$

We choose Z such that

$$\left(\frac{dZ}{dx}\right)^2 = Q. \quad or \quad \left(\frac{dz}{dx}\right)^2 = 4\cos ec^2 x$$

Or $\quad \frac{dz}{dx} = 2\cos ec\, x \qquad or \quad Z = 2\log\tan {x}/{2}$

$$\therefore \quad P_1 = \frac{P.\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cot x.\cos ec\, x - 2\cos ec\, x\cot x}{4\cos ec^2 x} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

Hence the transformed equation is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$\frac{d^2y}{dz^2} + y = 0 \quad or \quad (D^2+1)y = 0 \quad or \quad D = \pm i$$

$\therefore$ *The solution is* $y = C_1\cos(z + C_2)$

Or $\; y = C_1 \cos \; (2 \log \tan x/2 + C_2)$

Ex: Solve $x^6.\frac{d^2y}{dx^2} + 3x^5.\frac{dy}{dx} + a^2y = \frac{1}{x^2}$

Solution : Dividing by x^6 the equation is

$$\frac{d^2y}{dx^2}+\frac{3}{x}.\frac{dy}{dx}+\frac{a^2}{x^6}.y=\frac{1}{x^8}$$

$$\therefore\ P=\frac{3}{x},\ Q=\frac{a^2}{x^6},\quad R=\frac{1}{x^8}$$

We choose Z such that $\left(\frac{dz}{dx}\right)^2=Q=\frac{a^2}{x^6}$

Or $\left(\frac{dz}{dx}\right)^2=\frac{1}{x^6}$ (taking a = 1)

Or $\frac{dz}{dx}=\frac{1}{x^3}\quad or\quad Z=-\frac{1}{2x^2}\ \&\ \frac{d^2z}{dx^2}=\frac{-3}{x^4}$

$$\therefore\quad P_1=\frac{\frac{d^2z}{dx^2}+P.\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2};\ Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}\ \&\ R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\therefore\quad P_1=\frac{\left[\frac{-3}{x^4}+\frac{3}{x}\left(\frac{1}{x^3}\right)\right]}{\left(\frac{1}{x^3}\right)^2}=0,\qquad Q_1=\frac{\left(\frac{a^2}{x^6}\right)}{\frac{1}{x^3}}=a^2$$

$$\&\qquad R_1=\frac{\frac{1}{x^8}}{\frac{1}{x^6}}=\frac{1}{x^2}=2z$$

$$\left(\because \quad z = -\frac{1}{2x^2}\right)$$

Therefore the transformed equation is

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Or $\quad \dfrac{d^2y}{dz^2} + a^2y = -2z \quad or \quad \left(D^2 + a^2\right)y = -2z$

$\therefore$ The C.F. = $C_1 \cos (az + C_2)$

& P.I. = $\dfrac{-2z}{(D^2+a^2)} = (.a^2 + D^2)^{-1}.(-2z)$

$$= \frac{-2z}{a^2}$$

$\therefore$ The solution is

$\therefore$ y = C.F. + P.I. = $C_1 \cos\left(C_2 - \dfrac{a}{2x^2}\right) + \dfrac{1}{a^2x^2}$

(on putting z = - $\dfrac{1}{2x^2}$

Ex.: Solve $\dfrac{d^2y}{dx^2} + (3\sin x - \cot x)\dfrac{dy}{dx} + 2y\sin^2 x$

$$= e^{cosx}, \sin^2 x$$

Solution: Hence P = $\left(3\sin x - \cot x\right),\ Q = 2\sin^2 x$

And R = $e^{-cosx}\sin^2 x$

We choose z such that $\left(\dfrac{dz}{dx}\right)^2 = Q.$

Or We take $\left(\dfrac{dz}{dx}\right)^2 = \sin^2 x$ *therefore* $\dfrac{dz}{dx} = \sin x$

Or z= -cos x & $\dfrac{d^2z}{dx^2} = \cos x$

$$\therefore \quad P_1 = \frac{\dfrac{d^2z}{dx^2} + P.\dfrac{dz}{dx}}{\left(\dfrac{dz^2}{dx}\right)^2} = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x} = 3$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{2\sin^2 x}{\sin^2 x} = 2$$

and $$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{e^{-cosx}.\sin^2 x}{\sin^2 x} = e^{-cosx} = e^{z}$$

$(\because\ z = -\cos x 0$

& so the transformed equation is

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$$

Or $$\frac{d^2y}{dz^2}+3\frac{dy}{dz}+2y=e^z$$

Or $$(D^2+3D+2)y=e^z$$

$\therefore$ $A.E. is\ D^2+3D+2=0 \quad or \quad (D+2)(D+1)=0$

Or D = -2, -1

& so the C.F. = $C_,e^{-2}+C_2e^{-2z}$

And P.I. = $\frac{e^z}{D^2+3D+2}=\frac{1}{6}e^z$

Therefore the complete solution is

$$-y=C_1e^{-z}+C_2e^{-2z}+\frac{1}{6}e^z$$

Or $$y=C_1e^{cosx}+C_2e^{cosx}+\frac{1}{6}e^{cosx}$$

12.8. Removable of the First degree term

(Reduction to Normal form)

When we fail in obtion a part of the C.F

then the differential equations $\frac{d^2y}{dx^2} + p\frac{dy}{dx}Qy = R \quad -(1)$

may be solved by Removing the first derivative. We do it as follos :

We put $y = vy_1$

then $\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1.\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = v.\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}.\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}$$

and then equation (1) becomes

$$\left[v.\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}.\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}\right] + P\left[\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qvy_1 = R\right]$$

or $y_1\frac{d^2v}{dx^2} + y_1\left[P + \frac{2}{y_1}.\frac{dy_1}{dx}\right]\frac{dv}{dx} + v\left[\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = R\right]$

$$\left[Here\ \frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = o\right]$$

$because\ y_1\ in\ not\ a\ part\ of\ the$ Solution

We choose y_1 such that the first derivative is removed

$$ie\ \ P + \frac{2}{y_1}.\frac{dy_1}{dx} = o$$

$$or \quad \frac{dy_1}{y_1} = -\frac{1}{2}P.dx \qquad or \quad integrating$$

$$logy_1 = -\frac{1}{2}\int Pdx \qquad or \quad y_1 = e^{-\frac{1}{2}\int Pdx} \qquad ---(2)$$

then the above equation becomes

$$\frac{d^2v}{dx^2} + \frac{v}{y_1}\left[P.\frac{d^2y_1}{dx^2} + P.\frac{dy_1}{dx} + Qy_1\right] = \frac{R}{y_1} \qquad ----(3)$$

$$from\ (2)\ \ \frac{dy_1}{dx} = \left(e^{-\frac{1}{2}\int Pdx}\right)\left(-\frac{1}{2}P\right) = -\frac{1}{2}Py_1$$

$$or \quad \frac{d^2y_1}{dx^2} = -\frac{1}{2}P.\frac{dy_1}{dx} - \frac{1}{2}\frac{dp}{dx}.y_1$$

$$= -\frac{1}{2}P\left(-\frac{1}{2}Py_1\right) - \frac{1}{2} - \frac{dp}{dx}.y_1$$

$$= \frac{1}{4}p^2y_1 - \frac{1}{2}y_1\frac{dp}{dx}$$

Putting these values in (3) we get

$$\frac{d^2v}{dx^2} + v\left[\frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx} - P.\frac{1}{2}P + Q\right] = \frac{R}{e^{-\frac{1}{2}\int Pdx}}$$

$$or \quad \frac{d^2v}{dx^2} + v\left[Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx}\right] = R.e^{\frac{1}{2}\int Pdx}$$

$$or \quad \frac{d^2v}{dx^2} + Q.v = R_1 \qquad ----------(4)$$

$$Where \;\; Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx} \;\; and \;\; R_1 = R.e^{\frac{1}{2}\int Pdx}$$

The reduced equation (4) is called a differential equation in normal form.

Ex.1. Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$

$Here \;\; P = -4x, Q = 4x^2 - 3, R = e^{x^2}$

to remove the first derivative we choose

$$y_1 = e^{\frac{1}{2}\int Pdx} = e^{\int 2xdx} = e^{x^2}$$

Putting $y = vy_1$ the equation after removing the first derivative becomes

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \qquad --------(1)$$

$$Where\ \ Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx} = (4x^2 - 3) - 4x^2 + 2 = -1$$

$$R_1 = \frac{R}{y_1} = \frac{e^{x^2}}{e^{x^2}} = 1$$

We get for (1) $\frac{d^2v}{dx^2} - v = 1 \ \ or \ \ (Q^2 - 1)v = 1$

$A.E. is\ D^2 - 1 = 0 \quad or \quad D = \pm 1 \quad and$

$C.F. = C_1 e^x + \ C_2 e^{-x}$

$$and\ \ P.I. = \frac{1}{D^2 - 1} = -1$$

$Hence \quad v = C_1 e^x + \ C_2 e^{-x} - 1$

$\therefore \ \ The\ complete$ solution $is\ y = vy_1$

$or\ \ y = e^{x^2}(C_1 e^x + \ C_2 e^{-x} - 1)$

Ex.2. Solve $\frac{d^2y}{dx^2} - 2tanx\frac{dy}{dx} + 5y = seex.e^x$

To remove the first derivative we choose

$$y_1 = e^{-\frac{1}{2}\int pdx} = e^{\int tanx\, dx} = e^{logseex} = seex$$

$Now\ putting\ y = v.y_1 \quad the\ above\ quation\ becomes$

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \qquad ---------(1)$$

$$where\ \ Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx} = 5 - tan^2x + see^2x = 6$$

$$\& \ R_1 = \frac{R}{y_1} = \frac{seex.e^x}{seex} = e^x$$

$$Hence\ form\ (1) We\ get \quad \frac{d^2v}{dx^2} + 6v = e^x$$

$$A.E. is\ \ D^2 + 6 = 0 \quad or \quad Q = \pm i\sqrt{6}$$

$$\therefore \quad C.F. = \ c_1 cos\left(\sqrt{6}.x + c_2\right)$$

$$and\ \ P.I. = \ \frac{e^x}{D^2 + 6} = \frac{e^x}{7}\ therefore$$

$$v = c_1 cos\left(\sqrt{6}.x + c_2\right) + \frac{1}{7}e^x$$

and so the complete solution *is*

$$y = c_1 seex\ cos\left(\sqrt{6}.x + c_2\right) + \frac{1}{7}seex.e^x$$

Ex.3. solve $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2+5)y = xe^{-\frac{1}{2}x^2}$

$$Hence\ P = 2x, Q = x^2 + 5, R = xe^{-\frac{1}{2}x^2}$$

To remove the first derivative we choose

$$y_1 = e^{-\frac{1}{2}\int pdx} = e^{-\frac{1}{2}x^2} now\ putting\ \ y = v.y_1$$

$the\ above\ quation\ becomes$

$$\frac{d^2v}{dx^2} + Q_1.v = R_1 \qquad ---------(1)$$

$$where\ \ Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx} = (x^2+5) - x^2 - 1 = 4$$

$$\&\quad R_1 = \frac{R}{y_1} = x$$

$$form\ (1) We\ get \quad \frac{d^2v}{dx^2} + 4v = x \qquad or \quad (D^2+4)v = x$$

$$A.E.is\ \ D^2 + 4 = 0 \Rightarrow D = \pm 2i$$

$$\therefore\ \ C.F. = \ c_1 cos(2x + c_2)$$

$$and\ \ P.I. = \frac{x}{D^2+4} = \frac{1}{2}\left(1 - \frac{D^2}{4}\right)x = \frac{1}{4}x$$

$$v = c_1 cos(2x + c_2) + \frac{1}{4}x$$

$complete$ solution is $\quad y = v.y_1$

$$y = e^{-\frac{1}{2}x^{2},}\left[c_1 cos(2x + c_2) + \frac{1}{4}x\right]$$

Check Your Progress 4

1. Solve $\frac{dy^2}{dx^2} + 4x\frac{dy}{dx} + 4x^2y = 0 \qquad by\ removing$

   the first derivative

2. Solve $x^2.\frac{d^2y}{dx^2} - 2(x^2 + x)\frac{dy}{dx} + (x^2 + 2x + 2 + 2)y = 0$

   by removing the first derivative. (Hint divide x^2)

3. $\frac{d^2y}{dx^2} + \frac{2}{x}.\frac{dy}{dx} + y = \frac{\sin 2x}{x}$

4. Solve $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0$

5. $x^2\frac{d^2y}{dx^2} + (x - 4x^2)\frac{dy}{dx} + (1 - 2x + 4x^2)y = 0$

12.9. Summary

First the method of reduction then been discussed when one part of the C.F. is known. Then complete solution of $\frac{d^2y}{dx^2} + P.\frac{dy}{dx} + Qy =$

R has been discussed by chaing the dependent variable and by chaenging the independet variable. In the end the mehead of solving the above diffrential equaction by romoving the first derivative (or norml form hes bing discussed.

12.10. Terminal Question

Solve the the following different equation.

1) $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2 \quad ans. (c_1 + c_1 log)x^2 + x^2(log)^2$

2) $x^2 \frac{d^2x}{dx^2} - x \frac{dy}{dx} + 4y = cos(log\ x) + x\ sin(\log x). ans\ y =$

$x\left(A\ cos\sqrt{3}(\log x) + B\ Sin\sqrt{3}(\log x)\right) + \frac{1}{13}\Big[3\ cos(\log x) -$

$2sin(\log x) + \frac{1}{2} x\ sin(\log x)\Big]$

3) Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} 36y = 3x^2 + 4x + 1.$

4) $x^2 - \frac{d^2y}{dx^2} - 2x^2(1+x)\frac{dy}{dx} + 2(1+x)y = x^{2.}$

5) $Solve\ (x\ sin + cos) \frac{d^2y}{dx^2} - x \cos x\ \frac{dy}{dx} + y \cos x =$

$0\ of\ which\ y = x$

6) Solv $x^5. \frac{d^2y}{dx^2} + 3x^4. \frac{dy}{dx} + a^2y = \frac{1}{x^2}.$