Uttar Pradesh Rajarshi Tandon Open University

Bachelor of Science

SBSMM - 03

Elementary Analysis

Uttar Pradesh
RajarshiTandon
Open University

Bachelor Of Science

SBSMM - 03
Elementary Analysis

Block

1 Language of Mathematics, Relation & Mapping

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ELEMENTARY ANALYSIS

BLOCK-1

Language of Mathematics, Relation and Mapping

This First unit is most basic unit of this block as it introduces the concept of statements, statement variables. the five elementary operations and associated logical connectives. We introduce the well formed statement formulae, tautologies and equivalence of formulae. The law of duality is explained and established. It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly. Mathematics has a language of its own like most other sciences, which is very precise and communicates just what is required-neither more nor less. Language basically consists of words and their combinations called 'expression' or 'sentences'. However in Mathematics any expression or statement will not be called a 'sentence'.

In the Second unit of this block we introduce relations which have got a tremendous number of applications in almost every field, viz. sociology, economics, engineering, technology etc. Order relation has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of order relation is a major tool to learn to understand it more clearly.

IN the Third unit of this block we know the notion of a map which is one of the most fundamental concepts in mathematics and is used knowingly or unknowingly to our day to day life at every moment. Computer Science is an area where a number of applications of maps can be seen. We thought it would be a good idea to acquaint with some basic results about maps. Perhaps, we are already familiar with these results. But, a quick look through the pages will help us in refreshing our memory, and we will be ready to tackle the course. We will find a number of examples of bijective maps, direct and inverse image, Inverse map, composition of maps and various types of maps.

ELEMENTARY ANALYSIS

Block--01

Unit-01

Language of Mathematics

Structure

1.1. Introduction

This is most basic unit of this block as it introduces the concept of statements, statements, statement variables and the five elementary operations and associated logical connectives. We introduce the well formed statement formulae, tautologies and equivalence of formulae. The law of duality is explained and established. It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly. Mathematics has a language of its own like most other sciences, which is very precise and communicates just what is required-neither more nor less. Language basically consists of words and their combinations called 'expression' or 'sentences'. However in Mathematics any expression or statement will not be called a 'sentence'.

1.2. Objectives

After reading this unit we should be able to

1. Understand the concept of statement and statement variables

2. Use elementary operations like Conjunction, Disjunction, Negation, Implication, Double implication

3. Understand statement formulae, tautologies to equivalence of formulae

4. Use law of duality and functionally complete set of connectives

Logic is a field of study that deals with the method of reasoning Logic provides rules by which we can determine whether a given argument or reasoning is valid (correct) or not. Logical reasoning is used in Mathematics to prove theorems. In computer science logic is used to verify the correctness of programs.

1.3. Statements

Definitions: A statement (or proposition) is a sentence which is either true or false but not both.

Example1.1. Which of the following are statements?

(a) Indira Gandhi was one of the Prime Ministers of India.
(b) 8 is greater than 10.

(c) 2 + 4 = 6
(d) Blood is green.
(e) It is raining
(f) The sun will come out tomorrow.

Solution:

(a) is a statement because it is true.
(b) is a statement because it is false.
(c) is a statement because it is true.
(d) is a statement because it is false.
(e) is a statement because the sentence " it is raining" is either true of false but not both a given time.
(f) is a statement since it is either true or false but not both. Although, we would have to wait until tomorrow whether it is true or false.

If a sentence is a question (interrogative type) or a command or not free of ambiguity then the sentence cannot be answered as true or false and therefore such sentences are not statements.

Example1.2: The following are not statements.

(a) Is the number 6 a prime?
(b) 2 – x = 6
(c) What are you studying?
(d) Open the door.
(e) This statement is false.

Explanation:

(a) is not a statement because it is a question
(b) is not a statement because it is true or false depending on the value of *x*.
(c) is not a statement because it is a question.
(d) is a command and therefore it is not a statement.
(e) is not a statement because it is not possible to assign a definite true or false value to it. If we assume that sentence (e) is true then it says that statement (e) is false. On the other hand if we assume that sentence (e) is false then it implies that statement (e) is true. Hence it is not a statement.

1.4. Logical connectives:

There are some key words and phrases which are used to form new sentences from given sentences, as for example 'and' 'or', 'not', 'if…. then …., if and only if' etc. They are called sentential or logical connectives. **A** Sentence with some logical connective is called a 'Compound sentence' and a sentence without logical connective is called an ‘atomic sentence. As for example: A triangle is a plane figure. Water is cold, are atomic sentences. But the followings are the compound sentences.

(a) A triangle is a plane figure and is bounded by three straight lines.
(b) A real number is rational or irrational.
(c) 2013 is not a leap year.
(d) If a triangle is equilateral then it’s all angles are equal.
(e) If a triangle is isosceles then two of its angles are equal.

A part of a compound sentence that itself is a sentence is called a component of the sentence – thus the components of the sentence are also sentence.

1.5. Truth functional rules or truth tables:

The rules by which the truth or falsity of a compound sentence is determined from the truth or falsity of its components are called *truth functional rules*. The table giving the truth or falsity of the compound sentence depending upon the truth or falsity of its components is called its *truth table*. We shall say that T or F according as the sentence is true or false respectively.

1.6. Elementary Logical Operations:

The formation of compound sentence from given sentences by using the logical connectives are called *elementary logical operations* which are five in number in accordance with the five logical connectives used. They are: (**1) Conjunction (2) Disjunction (3) Negation (4) Implication (5) Double implication**.

Note: When we form compound sentence by using any of the five logical connectives, it is not necessary that the components of compound sentence should be related. As for example consider the compound sentence ‘Ram is a player and the earth revolves about the Sun. Here the components of the compound sentence are not related in the usual sense of conversation.

1.Conjunction: A sentence obtained by conjoining two sentences P, Q by using the connective ‘and’ is called the *conjunction* of the two sentences and will be denoted by $P \wedge Q$ (read as P and Q).

Example: Let *P*= U.S.A. sent Apollo 11 to the moon, *Q* = Russia sent Luna 15 to the moon. Then $P \wedge Q$= U.S.A. sent Apollo 11 and Russia sent Luna 15 to the moon.

***Truth functional rule for conjunction**:* $P \wedge Q$ is true if and only if both the sentences *P*, *Q* are true. How this truth functional rule is obtained is a matter of sophisticated logical reasoning and is beyond the purview of the present discussion.

Truth-Table for Conjunction: The following table gives the truth-values of $P \wedge Q$ for all possible truth values of P and *Q*:

P	*Q*	$P \wedge Q$
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*F*

2. Disjunction: A sentence obtained by joining two sentences *P*, *Q*, by the connective 'or' is called the *disjunction* of the two sentences and will be denoted by $P \vee Q$ (read as P or *Q*). For example: *P* = Ram is intelligent, *Q* = Ram is hard working, $P \vee Q$ = Ram is intelligent or hard working.

Truth functional rule for disjunction: $P \vee Q$ is true if at least one of *P*, *Q* is true, that is, $P \vee Q$ is false only when both *P* and *Q* are false. Truth Table for disjunction*:*

P	*Q*	$P \vee Q$
T	*T*	*T*
T	*F*	*T*
F	*T*	*T*
F	*F*	*F*

3. Negation: A sentence which has a truth value opposite to that of a sentence *P* is called the negation of *P* and is denoted by – *P* or ~ P. Negation of an atomic sentence is obtained by using the connective 'not' at proper place.

As for example: If *P*= The water is cold, Then – *P* =The water is not cold. Negation of $P \wedge Q$ is $(- P) \vee (- Q)$, that is, $- (P \wedge Q) \equiv (- P) \vee (- Q)$. Thus the negation of 'Ram is poor and honest' is 'Ram is not poor or not honest. This can be verified by the following Truth Table:

P	Q	$P \wedge Q$	$- P$	$-Q$	$\sim(P \vee Q)$	$-P \vee -Q$
T	*T*	*T*	*F*	*F*	*F*	*F*
T	*F*	*F*	*F*	*T*	*T*	*T*
F	*T*	*F*	*T*	*F*	*T*	*T*
F	*F*	*F*	*T*	*T*	*T*	*T*

The above table shows that the truth-values of $P \wedge Q$ (as given in the third column) are exactly opposite to those of $(– P) \vee (– Q)$ as given in the last column.

Note: that truth value of $\sim(P \vee Q)$ and $-P \vee -Q$ are same.

The negation of $P \vee Q$ is $(– P) \wedge (– Q)$, that is, $– (P \vee Q) \equiv (– P) \wedge (– Q)$

The negation of 'Mohan or Sohan has failed' is 'neither Mohan nor Sohan has failed' that is, 'Mohan has not failed and Sohan has not failed'.

4. Implication or a conditional sentence: A conditional sentence obtained by using the connective 'If ….then…' is called an implication. As for example: *P* = you read, *Q* = you will pass, By using the connective 'if ……… then' we get 'If you read then you will pass' which can be denoted by 'If P then *Q'*. It is also written as $P \Rightarrow Q$ (read as *P* implies *Q*). In the implication $P \Rightarrow Q$, P is called the *hypothesis* or antecedent and *Q* is called the *Summary or consequent.*

The Truth functional rule for implication:

$P \Rightarrow Q$ is false if P is true and Q is false;' otherwise it is true. The Negation of $P \Rightarrow Q$ is $P \wedge (-Q)$ that is,

$\sim(P \Rightarrow Q) \equiv P \wedge (-Q)$. This is proved by the following Truth Table:

P	Q	$P \Rightarrow Q$	$-Q$	$\sim(P \Rightarrow Q)$	$p \wedge -Q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	F	F

Truth values of $P \Rightarrow Q$ as given in third column are exactly opposite to those of

$P \wedge (-Q)$ as given in the last column or truth values of $\sim(P \Rightarrow Q)$and $p \wedge -Q$are same. Thus the Negation of the sentence 'If you read then you will pass' is 'You read and you will not pass. Note that 'If you do not read then you will not pass' is not the negation of the given sentence.

5. Double Implication:

A bi-conditional sentence obtained by using the connective 'If and only if'(briefly written as *iff*) between two sentences P, Q is called a double implication and is written a 'P *iff Q*'. It is also written as $P \Leftrightarrow Q$ (read as P implies and implied by Q). Thus we find that $P \Leftrightarrow Q$ is precisely the conjunction of $P \Rightarrow Q$, $Q \Rightarrow P$, that is $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The double implication $P \Leftrightarrow Q$ is true only when both P and Q are true or both are false. This is proved by the following table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$ i.e. $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note: If P= the Sun revolves about the earth, Q= The year consists of 400 days. Then 'P iff Q' or $P \Leftrightarrow Q$ = the Sun revolves about the earth iff the year consists of 400 days – which is true though P and Q are both false. The Negation of $P \Leftrightarrow Q$ is $(P \wedge - Q) \vee (Q \wedge - P)$. Thus the Negation of the sentence 'One is good teacher iff one is a good scholar' is 'One is a good teacher and a bad scholar or one is a good scholar and a bad teacher'.

Note: $-(P \Rightarrow Q) \equiv -(P \Rightarrow Q \wedge Q \Rightarrow P) \equiv (p \wedge -Q) \vee (Q \wedge -P)$

Example 1: Construct the truth table for ~p∨q. We must consider all possible

combination of truth values of p and q. All possible combinations of the truth values of the statements p and q are listed in the first two columns of the table. The truth values of $\sim p$ are entered in the third column and the truth values of $\sim p \vee q$ are entered in the fourth column.

P	***q***	***~p***	***~p∨q***
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth table for $\sim p \vee q$

Example 2: Construct the truth table for $p \wedge \sim p$.

Since the statement $p \wedge \sim p$ has only one distinct atomic statement. We have to consider 2 possible combinations of truth values. The truth table for $p \wedge \sim p$ is given below.

p	***~p***	***p∧~p***
T	F	F
F	T	F

Truth table for $p \wedge \sim p$

Example3: Construct the truth-table for $\sim(p \wedge \sim q)$.

In the first two columns, we list all the variable and the combinations of their truth values. In the third column, we write truth values for ~ q. The truth values of $p \wedge\sim q$ are listed in the next column. Finally we obtain the truth values of the proposition ~($p \wedge\sim q$). Thus we have the following truth table:

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Example4: Construct the truth-table for ($p \vee q$) ∧ ($p\vee r$).

Here, we have three atomic statements. Therefore we shall require eight rows to list all possible combinations of the truth values of statements p, q and r. Rest of the procedure will be the same as above. We shall proceed in steps and in the final column we will have the truth values of the given statements.

p	q	r	$p\vee q$	$p \vee r$	$(p\vee q)\wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Truth table for $(p\vee q)\wedge (p \vee r)$

Example 5: Prove that the truth values of the following pairs of sentences are the same.

(a) $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$
(b) $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$
(c) $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$
(d) $P \vee (Q \vee R)$ and $(P \vee Q) \vee R$

P	Q	R	$Q\vee R$	$P\wedge(Q\vee R)$	$P\wedge Q$	$P\wedge R$	$(P\wedge Q)\vee(P\wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

from columns fifth and eight we find that the truth values of $P\wedge (Q\vee R)$ ad $(P\wedge Q) \vee (P \wedge R)$ are the same in all cases. Solutions of other parts have been left as exercise.

Check your progress

1. Which of the following are statements?

(a) Is 3 a positive number?
(b) $x^2 – 5x + 6 = 0$
(c) There will be snow in December.
(d) Give me ten rupees.
(e) Ramesh is poor but honest
(f) No triangles are squares.

2. Let p be the proposition "Mathematics is easy" and let q be the proposition "five is greater than four." Write in English the proposition, which corresponds to each of the following: (a) $p \wedge q$ (b) $p \vee q$

(c) $\sim(p \wedge q)$ (d) $\sim p \wedge\sim q$

(e) $(p \wedge\sim q)\vee(\sim p \wedge q)$

3. Write the negation of each of the following statements:

(a) $2+7 \leq 1\ 3$

(b) 3 is an odd integer and 8 is an even integer.

(c) No nice people are dangerous.

4. Let p be the statement "Ravi is rich" and let q be the statement "Ravi is happy." Write the following statements in symbolic form:

(a) Ravi is poor but happy.
(b) Ravi is rich or unhappy.
(c) Ravi is neither rich nor unhappy.
(d) Ravi is poor or he is both rich and unhappy.

5. Construct the truth-table for the following functions:

(a) $(p'+q')'$ (b) $(p'q')'$

(c) $p(p+q)$ (d) $pqr + p'q'r'$

(e) $(p'+qr)'(pq+q'r)$

6. Given the truth values of p and q as true and those of r and s as false; find the truth values of the following:

(a) $p\vee (q\wedge r)$
(b) $(p \wedge (q \wedge r))\vee \sim((p \vee q)\wedge(r \vee s))$
(c)

Answers

1. (c),(e) and (f) are statements.
2. (a) Mathematics is easy and five is greater than four.
(b) Mathematics is easy or five is greater than four.

(c) Either Mathematics is not easy or five is not greater than four.

(d) Mathematics is not easy and five is not greater than four.

(e) Either Mathematics is easy and five is not greater than four or Mathematics is not easy and five is greater than four.

3. (a) It is false that $2 + 7 \leq 1\ 3$
   (b) Either 3 is not an odd integer or 8 is not an even integer.

   (c) Some nice people are dangerous.

4. (a) $\sim p \wedge q$ (b) $p \vee \sim q$
   (c) $\sim p \wedge q$ (d) $\sim p \vee (p \wedge \sim q)$

5. (a) True (b) True

Note: The symbols $\vee$, $\wedge$, $\sim$, $\rightarrow$ and $\leftrightarrow$ defined above are called **connectives.**

Converse, Inverse and Contra-positive of $p \rightarrow q$

Definition: Let $p \rightarrow q$ be any conditional statement. Then,

(a) the converse of $p \rightarrow q$ is statement $q \rightarrow p$.
(b) the inverse of $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$.
(c) the contra-positive of $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$.

Example1.15. Write the converse, inverse and contra-positive of the conditional statement "if 2 + 2 = 4 then I am not the Prime Minister of India."

Let p: 2+2 =4 and q: I am not the Prime Minister of India.

Then the given statement can be written as $p \rightarrow q$. Therefore, the converse is $q \rightarrow p$. That is, if I am not the Prime Minister of India then 2+2 = 4. The inverse of $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$. That is, if $2+2 \neq 4$ then I am Prime Minister of India.

The contra-positive of $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$. That is, contra-positive of the given statement is "if I am Prime Minister of India then $2+2 \neq 4$."

1.1.7 Tautology

Definition 1: A compound sentence is called a tautology if it is always true irrespective of the truth values of its component parts. i.e. A statement (or propositional function) which is true for all possible truth values of its propositional variables is called a tautology.

Definition 2: A statement which is always false is called a contradiction. A simple method to determine whether a given statement is a tautology is to construct its truth table. If the statement is tautology then the column corresponding to the statement in the truth table contains only T. Similarly a statement is contradiction if the column corresponding to the statement contains only F.

For example $P \vee - P$ is a tautology, since one of P and $-P$ must be true and so $P\vee -P$ is always true. Similarly $(- P \Rightarrow Q) \wedge -Q \Rightarrow P$ is a tautology as proved by the following table.

P	Q	$-P$	$-Q$	$-P\Rightarrow Q$	$(-P\Rightarrow Q)\wedge -Q$	$(-P\Rightarrow Q)\wedge -Q\Rightarrow P$
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	T	F	F	T

If $P \Rightarrow Q$ is a tautology then we also say $P\Rightarrow Q$ tautologically. Thus in the preceding example we can say that $(- P \Rightarrow Q) \wedge - Q \Rightarrow P$ tautologically.

Note: $P \Rightarrow Q$ cannot be a tautology if both P and Q are atomic sentence.

1.8. Tautological equivalence

Two sentence P and Q are said to be *tautologically equivalent* if $P\Rightarrow Q$ tautologically. And also $Q \Rightarrow P$ tautological equivalence if $P\Rightarrow Q$ tautologically, and also $Q \Rightarrow P$ tautologically. P and Q are tautologically equivalent may be written as $P\equiv Q$. It is clear that two compound sentence P and Q are tautologically equivalent if they have the same truth values in all the cases. i.e. Two statement p and q are said to be logically equivalent or equal if they have identical truth values.

One method to determine whether any two statements are equal is to construct a column for each statement in a truth table and compare these to see if they are identical.

For example $P \Rightarrow Q$ is tautologically equivalent to $- Q \Rightarrow - P$ as proved by the following table:

P	Q	$P\Rightarrow Q$	$-Q$	$-P$	$-Q\Rightarrow -P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T

F	*F*	*T*	*T*	*T*	*T*

We find that the truth values of $P \Rightarrow Q$ and $-Q \Rightarrow -P$ are the same in all the cases.

Hence $[P \Rightarrow Q] \Rightarrow [-Q \Rightarrow -P]$ and $[-Q \Rightarrow -P \Rightarrow [P \Rightarrow Q]$ are both tautologies.

The sentence –Q⇒ –P is called the contra-positive of the sentence $P \Rightarrow$Q. Hence very often to prove P⇒Q we prove –Q⇒–P.

Note: If $P \Rightarrow Q$ is a tautology, hen if P is true then Q must be true, since the implication is always true except when P is true and Q false.

Example1. Show that each of the following is a tautology

(a) $[p \wedge (p \rightarrow q)] \rightarrow q$
(b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
(a) We shall construct truth-table for the function $p \wedge (p \rightarrow q) \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*F*	*T*
F	*T*	*T*	*F*	*T*
F	*F*	*T*	*F*	*T*

Truth table for $p \wedge (p \rightarrow q)] \rightarrow q$

Since the column for $[p \wedge p \rightarrow q)] \rightarrow q$ contains only *T*, it is a tautology

(d) Here we construct the truth-table for $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow p \rightarrow r)]$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$	$[p \rightarrow r) \wedge [q \rightarrow r)] \rightarrow p \rightarrow r)$
T	*T*	*T*	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*T*	*F*	*F*	*F*	*T*
T	*F*	*T*	*F*	*T*	*T*	*F*	*T*

T	*F*	*F*	*F*	*T*	*F*	*F*	*T*
F	*T*	*T*	*T*	*T*	*T*	*T*	*T*
F	*T*	*F*	*T*	*F*	*T*	*F*	*T*
F	*F*	*T*	*T*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	*T*	*T*	*T*	*T*

Truth table for $[(p \to r) \wedge (q \to r)\ ] \to (p \to r)$

Since the last column corresponding to $[(p \to q) \wedge (q \to r)\ ] \to (p \to r)$ contains only *T*, it is a tautology.

Example2: Show that the statement $p\wedge{\sim}p$ is a contradiction. Consider the truth table for $p\wedge{\sim}p$.

P	~	$p\wedge{\sim}p$
T	*F*	*F*
F	*T*	*F*

Truth table for $p\wedge{\sim}p$

It follows from the table that $p\wedge{\sim}p$ is a contradiction.

Example3. Prove that $p \to q = {\sim}p \vee q$.

We shall construct truth table for statement $p \to q$ and ${\sim}p \vee q$.

p	*q*	$p \to q$	~*P*	${\sim}p \vee q$
T	*T*	*T*	*F*	*T*
T	*F*	*F*	*F*	*F*
F	*T*	*T*	*T*	*T*
F	*F*	*T*	*T*	*T*

Truth table for $p \to q$ and ${\sim}p \vee q$

We observe that the truth values in the columns for $p \to q$ and ${\sim}p \vee q$ are identical. Hence $p \to q = {\sim}p \vee q$.

Example4: Show that the statement $(p \wedge \sim p) \vee q$ and q are equal.

Consider the truth table for given statement.

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \vee q$
T	*T*	*F*	*F*	*T*
T	*F*	*F*	*F*	*F*
F	*T*	*T*	*F*	*T*
F	*F*	*T*	*F*	*F*

Truth table for $(p \wedge \sim p) \vee q$ and q

From the truth table we see that columns for $(p \wedge \sim p) \vee q$ and q are identical. Hence they are equal.

Example5: Show that two statements p and q are equivalent if bi-conditional statement $p \leftrightarrow q$ is a tautology. From the definition of bi-conditional statement we know that $p \leftrightarrow q$ is true whenever both p and q have the same truth values. Thus $p = q$ if $p \leftrightarrow q$ is a tautology.

Note. (1) Some authors have used the symbol '$\Leftrightarrow$' to denote equivalent or equal statements and symbol $\leftrightarrow$ is used for bi-conditional statement. From Example 1.20, we have $p \Leftrightarrow q$ if $p \leftrightarrow q$ is tautology.

(2) Two equivalent statements may contain different variables as is clear from Example 1.19 above.

Example6: Show that $p \rightarrow (q \rightarrow r) = (p \wedge q) \rightarrow r$

Consider the following truth table.

p	q	r	$q \rightarrow r$	$p \wedge q$	$p \rightarrow (q \rightarrow r)$	$(p \wedge q) \rightarrow r$
T	*T*	*T*	*T*	*T*	*T*	*T*
T	*T*	*F*	*F*	*T*	*F*	*F*
T	*F*	*T*	*T*	*F*	*T*	*T*
T	*F*	*F*	*T*	*F*	*T*	*T*

F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

Truth table for $p \to (q \to r) \& (p \wedge q) \to r$

We see that columns for $p \to (q \to r)$ and $(p \wedge q) \to r$ are identical hence given statement are equal.

Check your progress

(1) By constructing truth tables, show that the following are tautologies:

(a) $(P \wedge Q) \Rightarrow P$
(b) $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$
(c) $(P \Leftrightarrow Q) \wedge (Q \wedge R) \Rightarrow (P \Leftrightarrow R)$
(d) $(P \vee Q) \wedge - Q \Rightarrow P$
(e) $[P \Rightarrow Q] \Leftrightarrow [-P \vee Q]$

(2) Show that the following are tautological equivalences:

(a) $(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (- P \Rightarrow -Q)$
(b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
(c) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(d) $- (P \wedge Q) \equiv (- P) \vee (-Q)$
(e) $- (P \vee Q) \equiv (- P) \wedge (-Q)$
(f) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$
(g) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

The following theorem contains various laws satisfied by propositions. We shall use these laws for simplification of propositions.

Theorem 1: The following laws are satisfied by statements:

1. **Commutative laws:**

(a) $p \vee q = q \vee p$.   (b) $p \wedge q = q \wedge p$.

2. **Associative laws:**

(a) $p \vee (q \vee r) = (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

3. **Distributive laws:**

(a) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (b) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

4. **Idempotent laws:**

(a) $p \vee q = p$ (b) $p \wedge p = p$

5. **Laws of absorption:**

(a) $p \vee (p \wedge q) = p$ (b) $p \wedge (p \vee q) = p$

6. **Involution laws:**

(a) $\sim(\sim p) = p$

7. **Complement laws:**

(a) $p \vee \sim p = T$ (b) $p \wedge \sim p = F$

8. **De Morgan's laws:**

(a) $\sim(p \vee q) = \sim p \wedge \sim q$ (b) $\sim(p \wedge q) = \sim p \vee \sim q$

9. **Operation with *T***

(a) $p \vee T = T$ (b) $p \wedge T = p$

10. **Operation with *F***

(a) $F \vee p = p$ (b) $F \wedge p = F$

Here T and F denote statements, which are tautology and contradiction respectively.

Proof: We shall prove 3(a) and 8(a). The remaining laws can be proved exactly in the same way by constructing truth tables.

To prove 3(a), consider the following truth-table

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T

T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Since columns for $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical they are equal.

To prove 8(a), consider the following truth table:

P	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

It follows from the table that $\sim (p \vee q) = \sim p \wedge \sim q$.

Theorem 2: Show that (a) $p \rightarrow q = \sim p \vee q$, (b) $p \leftrightarrow q = (p \rightarrow q) \wedge = (q \rightarrow p)$

Proof: Using the definitions of $\rightarrow$ and $\leftrightarrow$ we have,

p	q	$\sim p$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$\sim p \vee q$	$(p \rightarrow q) \wedge = (q \rightarrow p)$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	F
F	F	T	T	T	T	T	T

Since the truth values in columns (4) and (7) are identical, we have (a), Similarly, since the truth values in columns (6) and (8) are identical, we have (b).

Theorem 3: Show that (a) $\sim(p \to q) = p \wedge \sim q$, (b) $\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$

Proof: Using the definitions of $\to$ and $\leftrightarrow$, we construct the truth table

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim(p \to q)$	$p \wedge \sim q$	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$p \leftrightarrow \sim q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	F	T	T	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	T	F	F	T	F	F

$\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$ have identical truth values, so $\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$

Check your progress

1. Prove that each of the following is a tautology:

(a) $p \to p$ (b) $p \wedge q \to p$

(c) $p \to (p \vee q)$ (d) $(p \wedge (p \to q)) \to q$

(e) $(p \to q) \to [(p \vee (q \wedge r)) \leftrightarrow q \wedge (p \vee r)]$

2. Write in words the converse, inverse, contra positive and negation of the implication "if she works then she will earn money."

3. Construct truth tables to determine whether each of the following is tautology or a contradiction:

(a) $p \wedge \sim p$ (b) $p \to (q \to p)$

(c) $p \to q \wedge p$ (d) $q \vee (\sim q \wedge p)$

4. Prove the following:

(a) $p \vee q = q \vee p$ (b) $p \wedge (q \wedge r) = (p \vee q) \wedge r$

(c) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r$ (d) $p \vee p = p$

(e) $\sim(p \wedge q) = \sim p \vee \sim q$ (f) $\sim(p \leftrightarrow q) = \sim p \leftrightarrow q$

5. Write in English the negation of each of the following:

(a) The weather is bad and I will not go to work.

(b) I grow fat only if I eat too much.

6. Show the following equivalences:

(a) $p \to (q \to q) \Leftrightarrow \sim p \to (p \to q)$

(b) $\sim (p \leftrightarrow q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q)$

7. We define $p \Rightarrow q$ if and only if $p \to q$ is tautology. Prove the following:

(a) $p \to q \Rightarrow p \to (p \wedge q)$

(b) $(p \to q) \to q \Rightarrow p \vee q$

7. Prove that $\sim (p \wedge q) \to (\sim p \vee (\sim p \vee q)) = \sim p \vee q$, without constructing truth table.

Answers

2. The converse of the statement is "if she earns money then she works." The inverse is "if she does not work then she will not earn money." The contra-positive is "if she does not earn money then she does not work" The negation of the statement is "she works and she will not earn money."

3. (a)Contradiction (b) Tautology

(c) Neither tautology nor contradiction

(d) Neither tautology nor contradiction

5.(a) The weather is bad but I will go to work.

(b) I grow fat and (although) I don't eat too much.

1.9. Law of Duality

In this section, we consider only those statements which contain the connectives $\wedge$, $\vee$ and ~ only.

Definition: Two statement p and p^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F and F by T.

It is obvious from the definition that dual of a statement is the statement itself. We now state (without proof) the principle of duality.

Principle of Duality

It states that if any two statements are equal then their duals are also equal.

Example.8: Prove the following:

(a) $\sim (p \wedge q) = \sim p \vee \sim q$ (b) $\sim (p \vee q) = \sim p \wedge \sim q$

Solution: We shall only prove (a). The result stated in (b) will follow by principle of duality.

To prove (a), consider the following table:

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, it follows that $\sim(p \wedge q) = \sim p \vee \sim q$

1.10. Sentential form

Consider the following expressions or statements: (1) x is mortal (2) x is a fraction

They are not sentences, since we do not know their truth values. If we take x to be a number in (1) and x to be a man in (2), then these two statements (1), (2) become meaningless and hence they are not sentences. But if we restrict x in (1) to men and in (2) to numbers, then these statements will be sentences either true or false. Here x will be called a variable. Such statements which contain variables like x which are not specified are called open sentences or sentential from. Similarly expressions containing pronouns, as for example ‘He is prime minister of India’, ‘It is a prime number’ are open sentences, since we do not know their truth value without additional information specifying the unknown pronouns which behave like variables. The open sentence ‘x is mortal’ will be denoted by $P(x)$.

1.11. Quantifiers

In the discussion of logic, some very important statements contain quantifiers. The following are examples of statements which contain quantifiers:

(1) Some people are honest.
(2) No woman is a player.
(3) All Americans are crazy.

The words *some*, *no* and all are known as quantifiers. From quantifiers, we know "how many" of a certain set of things is being considered.

1.11 (a) Universal Quantifier

Let p be a statement. We define the symbol $\forall xp$ to mean that for every value of x in the given set, the statement p is true. The symbol $\forall$ is called the universal quantifier. $\forall$ can also be read as 'for all' , ' for every' or 'for any.'

Illustration: The statement "for all natural numbers, $n + 4 > 3$" can be expressed as $\forall xp$, where x belongs to the set N of natural numbers and p is the statement '$n + 4 > 3$.'

1.11 (b) Existential Quantifier

Let p be a statement. We define the symbol $\exists xp$ to mean that for one or more elements x of a certain set, the proposition p is true. The symbol $\exists$ is called existential quantifier and is usually read as "these exists" or "for at least one "for some".

Illustration: (1) The statement "there exists a number x such that $x^2 - 4x = 16$" may be written as $\exists xp(x^2 - 4x = 16)$

(2) The statement $\exists_n (n + 4 < 7)$, where n is in the set of natural numbers is true since there exists a natural number, namely I, such that $n + 4 < 7$ is true.

1.12 Negation of Quantifiers

It is important to know how the negations of statements having quantifiers are formed. Consider the statement "All Americans are crazy"

The negation of this statement would be

"It is false that all Americans are crazy" or equivalently,

"There exists at least one American who is not crazy."

Example: The function f is said to approach the limit l near a if (1) $\forall \in > 0, \exists \delta > 0$ such that $\forall x, 0 < | x - a| < \delta \Rightarrow | f(x) - l| < \in$.

Putting $P(x)$: $0 < | x - a| < \delta \Rightarrow | f(x) - l | < \in$. It can be written as $\forall \in > 0 \; \exists \; \delta > 0$ such that $\forall(x) \; P(x)$. *Hence the negation of the above definition will be:*

The function f does not approach l at a if $\exists \in > 0$ Such that $\forall \delta > 0, \exists \, x \, (\sim P \, (x))$.

That is, if there exists some $\in > 0$, such that *for every* $\delta > 0$, there exists some x for which $0 < |x - a| < \delta$ and $| f(x) - l | \not< \in$.

Example: Write the negation of 'No teachers are wise'. Putting $P(x)$: x is wise (x is a teacher), the symbolic form of the above sentence is $\forall x$ (x is a teacher) x is not wise or $\forall x \, (\sim P(x))$ (x is a teacher). Hence its negation will be $\exists \, x \, P(x)$ that is, there exists a teacher x who is wise or' Some teachers are wise'.

Check your progress

(1). Given P is true, Q is false and R is true, find, find the truth values of:

(a) $(P \vee Q) \wedge (Q \vee R)$.
(b) $(P \Rightarrow Q) \Rightarrow (P \wedge - Q)$
(c) $[(P \wedge Q) \wedge - R] \Rightarrow (Q \Rightarrow P)$ [Ans. (a) T, (b) T, (c) T]

(2). Write the Negations of the following

(a) $(P \vee Q) \wedge R$,
(b) $P \wedge (Q \Rightarrow -R)$,
(c) $P \Rightarrow (Q \Rightarrow R)$.
(d) $P \wedge - Q \Leftrightarrow R$,
(e) $\forall x(x \neq 1, x \neq 2)$,
(f) $\exists x(x^2 < 0)$
(g) $\forall x(x \neq 0) \Rightarrow (x^2 > 0)$,
(h) $\exists x(x^2 = 1$ and $x^2 - 2x + 3 = 0)$
(i) Every Indian is honest. (j)If there is a will then there is a way.

(3). State if the following are sentence, giving reasons of your answer.

(a) Do you think you will pass in the examination?
(b) Mathematics is black .
(c) Walk right in.
(d) He is a President of India.

(e) 2/5 is a integer .
(f) If you pass in the examination, then the sun will revolve about the earth.
(g) Oh! How sand he is.

(4). By constructing Truth-tables shows that the following are tautologies:

(a) $(P \wedge Q) \Rightarrow P$
(b) $(P \vee Q) \wedge - Q \Rightarrow P$
(c) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$,
(d) $(- P \Rightarrow Q) \wedge - Q) \Rightarrow P$

(5). Prove the following tautological equivalences:

(a) $(P \Rightarrow Q) \vee (P \Rightarrow R) \equiv P \Rightarrow Q \vee R.$
(b) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv P \vee Q \Rightarrow R$
(c) $(P \Rightarrow Q) \wedge (P \Rightarrow R) \equiv P \Rightarrow Q \wedge R,$
(d) $(P \Rightarrow R) \vee (Q \Rightarrow R) \equiv \mathrm{P} \wedge Q \Rightarrow R.$

(6). Prove that the following are tautologies

(a) $[(P \Rightarrow Q) \wedge (R \Rightarrow S)] \Rightarrow (P \wedge R \Rightarrow Q \wedge S).$
(b) $[P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \vee R \Rightarrow Q \vee S).$

(7) Find the dual of the following:

(a). $(p \vee q) \wedge r$ (b). $(p \wedge q) \vee r$ (c). $\sim (p \vee q) \wedge (p \vee \sim (p \wedge s))$

(8) Form the negation of each of the following:

(a) "For all positive integers *x*, we have $x+2>8$"
(b) "All men are honest or some man is a thief."
(c) "There is at least one person who is happy all the time."
(d) "The sum of any two integers is an even integer."
(e) At least one student does not live in the dormitories.

Solution: (7) (a) Interchanging $\vee$ and $\wedge$, we have, dual as $(p \wedge q) \vee r$.

(b) Dual is $(p \vee q) \wedge r$

(8) Write the Negation of the following statement are:

(a) There exists a positive integer *x* such that $x+2>8$.
(b) There exists a man who is not honest and all men are not thief.
(c) No person is happy all the time.
(d) There exists two integers such that their sum is not an even integer.

(e) All students live in the dormitories.

1.13. Summary: After reading this unit we should be able to understand the concept of statement and statement variables, use elementary operations like Conjunction, Disjunction, Negation, Implication, Double implication, understand statement formulae, tautologies to equivalence of formulae, use law of duality and functionally complete set of connectives. We now that

Logic is a field of study that deals with the method of reasoning Logic provides rules by which we can determine whether a given argument or reasoning is valid (correct) or not. Logical reasoning is used in Mathematics to prove theorems. In computer science logic is used to verify the correctness of programs.

2.14. Terminal Questions

1. $By\ contructing\ truth\ tables, Show\ that\ the\ following\ are\ tautologics.$

   $(a)\ \ (P \wedge Q) \Rightarrow P$

   $(b)\ \ (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

   $(c)\ \ [p \Rightarrow Q] \Leftrightarrow [-PVQ]$

2. $Show\ that\ the\ following\ are\ tautological\ equi\ valences:$

   $(a)\ \ (P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (-P \Rightarrow -Q)$

   $(b)\ \ P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$

   $(c) - (P \wedge Q) \equiv (-P) \vee (-Q)$

3. $If\ P \Rightarrow Q\ and\ Q \Rightarrow R\ are\ tautologics, Show\ that\ P \Rightarrow$

   $R\ is\ also\ a\ tautology.$

4. $Given\ P\ is\ true, Q\ is\ False\ and\ R\ is\ True. Find\ the\ truth\ values\ of\ (PVQ) \wedge$

   $(QVR).$

5. $Write\ the\ Negations\ of\ the\ following:$

   $(a)\ \ (P \vee Q) \wedge R)$

   $(b)\ \ P \wedge (Q \Rightarrow R)$

6. $By\ contructing\ truth\ table, Show\ that\ the\ following\ are\ tautologics.$

   (a) $(P \wedge Q) \Rightarrow P$

   (b) $(P \vee Q) \wedge -Q \Rightarrow P$

7. $Write\ truth\ tables\ fo\ the\ sentence\ P \Rightarrow P\ and$

$$P \Rightarrow -P. Is\ the\ First\ sentence\ a\ tautology.$$

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Unit—II

Relation

Structure

2.1. Introduction

Relations have got a tremendous number of applications in almost every field, viz. sociology, economics, engineering, technology etc. In computer science the concept of a relation is a major tool to learn and understand it clearly. Order relation has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of order relation is a major tool to learn to understand it more clearly.

2.2. Objectives

After reading this unit we should be able to

- Recall the basic properties of relations
- Derive other properties with the help of the basic ones
- Identify various types of relations
- Understand the relationship between equivalence classes and partition.
- Recall the basic properties of order relations
- Derive other properties with the help of the basic ones
- Identify infimum and supremum
- Totally ordered set

2.3. Relation

Let X and Y be two sets, then a relation R from X to Y is defined to be a subset R of $X \times Y$, that is $R \subseteq X \times Y$.

If $(x, y) \in R$, we say that x does stand in relation R to y or briefly as xRy. In case $(x, y) \notin R$ we say (that is x is not R -related to y). Similarly we may define a relation R between two elements of the same set X or a relation R in X by $R \subseteq X \times X$.

If $(x_1, x_2) \in R$, then, $x_1 R x_2$.

Let X be the set of all women and Y the set of all men. Then the relation 'is wife of' between women (element of X) and men (element of Y) will give us a set of ordered pairs $R = \{(x, y): x \in X, y \in Y, \text{ and } x \text{ is wife of } y\}$.

The ordered pairs (Kamla Nehru, Jawahar Lal Nehru), (Kasturba Gandhi, Mahatma Gandhi) are elements of R. It is clear that $R \subseteq X \times Y$.

A relation is binary if it is between two elements. Thus 'is wife of' is a binary relation involving two persons, viz Kamla Nehru is the wife of Jawahar Lal Nehru). Conversely if we are given the set R of ordered pairs (x, y) which correspond to the relation 'is wife of' then even if we forget the meaning of 'is wife of' we can tell when a woman x is wife of a man y and when not, we are only to find if (x, y) does or does not belong to R. Hence we find that if we know the relation we know the set R and if we know the set R we know the relation.

Example: Let S be a set. Let R be a relation in P(S), P(S) is the power set of S, $R \subseteq$ P(S)×P(S) given by

$R = \{(A, B) : A, B \in P(S) \text{ and } A \subseteq B\}$, Now $(A, B) \in R \Leftrightarrow A \subseteq B$. Or $ARB \Leftrightarrow A \subseteq B$.

Example: Let X be a set and let Δ denote the *relation of equality or diagonal relation in X and we write* $x \Delta y$ iff $x = y$.

Example: If R = $X \times X - \Delta$. Then $(x, y) \in R \Rightarrow (x, y) \in X \times X$, $(x, y) \notin \Delta$ i.e. xRy iff $x \neq y$

R is called the relation of inequality in X. Thus we can say that the relation R of inequality in a set X is the complement of the diagonal relation Δ in $X \times X$.

Example: Let R be a relation in the set Z of integers given by $R = \{x, y): x < y, x, y \in Z\}$ where '<' has the usual meaning in Z. Since $3 < 4$, therefore $(3, 4) \in R$ or $3R4$. But $(4, 3) \notin R$, since $4 > 3$.

Example: Let A and B be two finite sets having m and n elements respectively. Find the number of distinct relations that can be defined from A to B.

The number of distinct relations from A to B is the total number of subsets of $A \times B$. Since $A \times B$ has mn elements so total number of subsets of $A \times B$ is 2^{mn} . Hence total number of possible distinct relations from A to B *is* $\mathbf{2^{mn}}$.

2. 4. Domain and Range of a Relation

The domain D of the relation R from set A to set B is defined as the set of elements of first element of the ordered pairs which belongs to R, i.e., $D = \{x \in A : (x, y) \in R$, for some $y \in B\}$.

The range E of the relation R is defined as the set of all elements of the second element of the ordered pairs which belong to R, i.e., $E = \{y \in B : (x, y) \in R$, for $x \in A\}$. Obviously, $D \subseteq A$ and $E \subseteq B$.

Example: Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Every subset of $A \times B$ is a relation from A to B. So, if $R = \{(2, a), (4, a), (4, c)\}$, then the domain of R is the set $\{2,4\}$ and the range of R is the set $\{a, c\}$

2.5. Types of Relation in a set

We consider some special types of relations in a set A.

Inverse Relation: Let R be a relation from the set A to the set B, then the inverse relation R^{-1} from the set B to the set A is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed, belong to R. Thus every relation R from the set A to the set B has an inverse relation R^{-1} from B to A.

Example: Let $A = \{1,2,3\}$, $B=\{a, b\}$ and $R=\{(1,a), (1, b), (3,a), (2, b)\}$ be a relation from A to B.The inverse relation of R is $R^{-1} = \{(a,1), (b, 1), (a, 3), (b,2)\}$

Example: Let $A= \{2,3,4\}$, $B=\{2,3,4\}$ and $R=\{x,y) : |x - y| = 1\}$ be a relation from A to B. That is, $R = \{3,2), (2,3), (4,3), (3, 4)\}$.

The inverse relation of R is $R^{-1}= \{(3,2), (2, 3), (4, 3), (3, 4)\}$. It may be noted that $R=R^{-1}$.

Note: every relation has an inverse relation. If R be a relation from A to B, then

R^{-1} is a relation from B to A and $(R^{-1})^{-1}=R$.

Theorem: If R be a relation from A to B, then the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

Proof: Let $y \in$ domain of R^{-1}. Then there exist $x \in A$ such that $(y, x) \in R^{-1}$. But $(y, x) \in R^{-1} \Rightarrow (x, y) \in R$. $\Rightarrow y \in$ range of R.

Therefore, $y \in$ domain $R^{-1} \Rightarrow y \in$ range of R. Hence domain of $R^{-1} \subseteq$ range of R. In a similar way we can prove that range of $R \subseteq$ domain of R^{-1}.

Therefore, domain of R^{-1} = range of R. In a similar manner it can be shown that domain of R =range of R^{-1}.

2. Universal Relation: A relation $A \times A$ in a set A is said to be the **universal relation** in A.

Example: Let $A = \{1, 2, 3\}$ then $R = A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ is a universal relation in A.

3. Void (empty) Relation: A relation R in a set A is said to be a void relation if R is a null set, i.e., if $R = \phi$.

Example: Let $A = \{2, 3, 7\}$ and let R be defined as 'aRb if and only if $2a = b$ ' then we observe that $R = \phi \subset A \times A$ is a void relation.

Example: Let $A = \{1, 2, 3\}$. We consider several relations on A.

(i) Let R_1 be the relation defined by $m < n$, that is, mR_1n if and only if $m < n$.

(ii) Let R_2 be the relation defined by mR_2n if and only if $|m - n| \leq 1$.

Define R_3 by $m \equiv n \pmod 3$, so that mR_3n if and only if $m \equiv n \pmod 3$, i.e.3 divides $m - n$.

(iii) Let E be the 'equality relation' on A, that is, mEn if and only if $m = n$.

2.6. Composition of Relations

Let R_1 be a relation from the set X to the set Y and R_2 *a* relation from the set Y to the set Z. That is $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$. *The composite of the two relations R_1 and R_2 denoted by $R_2 o R_1$ is a relation from the set X to Z,* that is $R_2 o R_1 \subseteq X \times Z$ defined by : $R_2 o R_1 = \{(x, z) \in X \times Z$: for some $y \in Y$, $(x, y) \in R_1$ and $(y, z) \in R_2\}$. I.e. $x(R_2 o R_1)z \Leftrightarrow$ for some $y \in Y$, xR_1y and yR_2z.

Example: Let X =Set of all women, Y=Set of all men, Z=Set of all human beings.

Let R_1 be a relation from X to Y given by $R_1 = \{(x, y) : x\in X, y\in Y$ and x is wife of $y\}$

And let R_2 be a relation from Y to Z given by $R_2= \{(y, z)$: $y\in Y$, $z\in Z$ and y is the father of $z\}$. Therefore

$R_2 o R_1= \{(x, z) \in X\times Z$: for some $y\in Y$ $(x, y)\in R_1$ and $(y, z)\in R_2\}$.

Here $(R_2 o R_1)$ is the relation 'is the mother of,' provided a man can have only one wife.

Example: if R_1 be a relation from the set X to the set Y, R_2 a relation from the set Y to the set Z and R_3 a relation from the set Z to the set W. Then $R_3 o (R_2 o R_1) = (R_3 o R_2) o R_1$, that is composition of relation is associative.

Now $R_2 \, o \, R_1 \subseteq X \times Z$ and $R_2 \subseteq Z \times W$. Therefore $R_3 \, o \, (R_2 \, O \, R_1) \subseteq X\times W$, that is, a relation from X to W. Similarly $(R_3 o R_2) \, o \, R_1 \subseteq X\times W$; that is, a relation from X to W. Now $(x, w) \in R_3 o \, (R_2 o R_1)$

$\Leftrightarrow \exists z\in Z|(x,y)\in R_1$ and $(y, z)\in R_2$ & $(z, w)\in R_3$ for some $y\in Y$ *and* $z\in Z$

$\Leftrightarrow \exists z \in Z, y \in Y \, (x, y)\in R_1$ and $(y, z) \in R_2$ & $(z, w) \in R_3$

(Since $(P \wedge Q) \wedge R\equiv P\wedge (Q\wedge R)$)

$\Leftrightarrow\exists y \in Y \mid (x, y) \in R_1$ and $(y, w) \in R_3 o R_2$.

$\Leftrightarrow (x, w)\in(R_3 o R_2) o R_1$.

Therefore $R_3 o(R_2 o R_1)=(R_3 o R_2) o R_1$.

Check your progress

(3.1) Prove that $(R^{-1})^{-1}=R$.

(3.2) Prove that $(R_2 o R_1)^{-1}=R_1^{-1} o R_2^{-1}$.

Reversal Rule*: From the above we get the inverse of the composite of two relations is the composite of their inverses in the reverse order.*

2.7. Equivalence relation in a set

A relation R in a set S is called an equivalence relation if

(α) R is reflexive, that is $\forall \mathbf{x} \in S$, xRx or $(x, x) \in R$ that is, $\Delta \subseteq R$;

(β) R is symmetric, that is, $xRy \Rightarrow yRx$ or $(x, y) \in R \Leftrightarrow (y, x) \in R$ i.e. $R^{-1}=R$.

(γ) R is transitive, that is, $[xRy, yRz] \Rightarrow xRz$

Or $(x, y) \in R$, $(y, z) \in R \Rightarrow (x, z) \in R$., i.e. $RoR \subseteq R$.

Example: The diagonal or the equality relation Δ in a set S is an equivalence relation in S. For if $x, y \in S$ the $x\Delta y$ iff $x=y$. Thus

(α) $x\Delta x$ $\forall x \in S$ (reflexivity)

(β) $x\Delta y \Rightarrow x = y \Rightarrow y=x \Rightarrow y \Delta x$ (Symmetry)

(γ) for $x, y, z \in S$, $[x\Delta y, y\Delta z] \Rightarrow [x=y, y=z$ i.e. $x = z] \Rightarrow x\Delta z$.

Hence [$x\Delta y$, and $y\Delta z$] $\Rightarrow x\Delta z$ (transitivity).

Example: Let N be the set of natural numbers. Consider the relation R in $N \times N$ given by $(a, b)\ R(c, d)$ if $a+d=b+c$, where $a, b, c\ d \in N$ and + denotes addition of natural numbers, R is an equivalence relation in $N \times N$.

(α) $(a, b)R(a, b)$ since $a+b=b+a$ (Reflexivity) $\forall (a,b) \in N \times N$.

(β) $(a, b)R(c, d) \Rightarrow a+d=b+c$

$\Rightarrow c+b=d+a \Rightarrow (c,d)R(a, b)$ (Symmetry)

(γ) $[(a, b)R(c, d)$ and $(c, d)R(e, f)]$

$\Rightarrow [a+d=b+c$ and $c+f=d+e]$

$\Rightarrow (a+d+c+f=b+c+d+e)$

$\Rightarrow a+f=b+e$ (By cancellation laws in N)

$\Rightarrow (a, b)R(e, f)$ (transitivity)

Example: Let a relation R in the set N of natural numbers be defined by: If m, $n \in N$, then mRn if m and n are both odd.Then R is not reflexive, since 2 is not related to 2.Thus xRx *does not hold* $\forall x \in N$. But R is symmetric and transitive as can be verified.

Example: Let X be a set. Consider the relation R in P(X) given by: for $A, B \in P(X)$. ARB if $A \subseteq B$. Now R is reflexive, since $A \subseteq A$, $\forall A \in P(X)$ R is transitive, since $[A \subseteq B, B \subseteq C] \Rightarrow A \subseteq C$ where $A, B, C \in P(X)$. But R is not symmetric, since $A \subseteq B \not\Rightarrow B \subseteq A$.

Example: Let S be the set of all lines L in three dimensional space. Consider the relation R in S given by; for $L_1, L_2 \in S$, $L_1 R L_2$ *if* L_1 is coplanar with L_2. Now R is reflexive, since L_1 is coplanar with L_1, R is symmetric, since L_1 coplanar with $L_2 \Rightarrow L_2$ coplanar with L_1. But R is not transitive, since (L_1 coplanar with L_2 and L_2 coplanar with L_3) $\not\Rightarrow$ L_1 coplanar with L_3.

Example: (a) Let X $=\{x, x_2, x_3, x_4\}$. Define the following relations in X:

$R_1 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_2, x_3), (x_3, x_2)\}$

$R_2 = \{ x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4)\}$

$R_3 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3),(x_4, x_4), (x_2, x_3), (x_3, x_2), (x_3, x_4), (x_4, x_3)\}$

R_1 is symmetric, transitive but not reflexive since$(x_4, x_4) \notin R_1$

R_2 is reflexive, transitive but not symmetric since $x_3 R x_4$ but $(x_4, x_3) \notin R_2$

R_3 is reflexive, symmetric but not transitive since $x_2 R x_3$ and $x_3 R x_4$ but$(x_2, x_4) \notin R_3$.

Note: Examples prove that the three properties of an equivalence relation viz. reflexive, symmetric and transitive are independent of each other, i.e. no one of them can be deduced from the other two.

Example: Let A be the set of all people on the earth. Let us define a relation R in A, such that xRy if and only if 'x is father of y', Examine if R is (i) reflexive, (ii) symmetric, and (iii) transitive. We have

(i) For $x \in A$, xRx does not hold, because, x is not the father of x. That is R is not reflexive.

(ii) Let xRy, i.e., x is father of y, which does not imply that y is father of x. Thus yRx does not hold. Hence R is not symmetric.

(iii) Let xRy and yRz hold. i.e., x is father of y and y is father of z, but x is not father of z, i.e., xRz does not hold. Hence R is not transitive.

Example: Let A be the set of all people on the earth. A relation R is defined on the set A by aRb if and only if a loves b' for a, $b \in A$. Examine if R is (i) reflexive, (ii) symmetric, and (iii) transitive. Here,

(i) R is reflexive, because, every person loves himself. That is, aRa holds.

(ii) R is not symmetric, because, if a loves b then b not necessarily loves a, i.e., aRb does not always imply bRa. Thus, R is not symmetric.

(iii) R is not transitive, because, if a loves b and b loves c then a not necessarily loves c, i.e., if aRb and bRc but not necessarily aRc. Thus R is not transitive. Hence R is reflexive but not symmetric nor transitive.

Example: Let N be the set of all natural numbers. Define a relation R in N by 'xRy if and only if $x + y = 10$'. Examine R is (i) reflexive, (ii) symmetric, and (iii) transitive. Here,

(i) Since $3 + 3 \neq 10$ i.e., $3R3$ does not hold. Therefore R is not reflexive.

(ii) If $a + b = 10$ then $b + a = 10$, i.e., if aRb hold then bRa holds. Hence R is symmetric.

(iii) We have, 2+8=10 and 8+2=10 but 2+2≠10, i.e. $2R8$ and 8R2 holds but $2R2$ does not hold. Hence R is not transitive therefore R is not reflexive and transitive but symmetric.

Example: Let I be the set of all integers and R be a relation defined on I such that 'xRy if and only if $x > y$'. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Here,

(i) R is not reflexive, because, $x > x$ is not true, i.e., xRx is not true.

(ii) R is not symmetric also, because, if $x > y$ then $y \not> x$. i.e., R is not symmetric

(iii) *R is transitive because* if xRy and yRz holds then xRz hold. Therefore R is not reflexive and symmetric but transitive.

Example: A relation R is defined on the set $I*$ the set of all nonzero integers, by 'aRb if and only if $ab > 0$' for $a \neq 0$, $b \neq 0 \in I*$. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Here,

(i) Let $a \in I*$. Then $a.a.>0$ holds. Therefore aRa holds for all $a \in I*$. Thus R is reflexive.

(ii) Let $a, b \in I*$ and aRb holds. If $ab > 0$ then $ba > 0$. Therefore, $aRb \Rightarrow bRa$. Thus R is symmetric.

(iii) Let $a, b, c \in I*$ and aRb, bRc hold. Then $ab > 0$ and $bc > 0$. Therefore, $(ab)(bc)>0$. This implies $ac > 0$ since $b^2>0$. So aRb and $bRc \Rightarrow aRc$. Thus R is transitive. Hence R is reflexive, symmetric and transitive; hence R is an equivalence relation.

Example: Let R be a relation in a set S which is symmetric and transitive. Then consider the argument $aRb \Rightarrow bRa$ (by symmetry) [aRb and bRa] $\Rightarrow aRa$ (by Transitivity)

From this it may not be concluded that reflexivity follows from symmetry and transitivity. The fallacy involved in the above argument is:

for $a \in S$, to prove aRa, we have started with $aRb \Rightarrow bRa$.

Now it might happen that $\exists$ no element $b \in S$ such that aRb.

Example: Examine whether each of the following relations is an equivalence relation in the accompanying set –

(i) The geometric notion of similarity in the set of all triangles in the Euclidean plane.

Hint: It is an equivalence relation

(ii) The relation of divisibility of a positive integer by another, the relation being defined in the set of all positive integers as follows:

a is divisible by b if $\exists$ a positive integer c such that $a=bc$.

Hint: The relation is reflexive, transitive but not symmetric.

(iii) The relation R_1 in the set R of all real numbers defined as follows:

aR_1b if $\exists$ a non-negative number c such that $a+c=b$.

Hint: The relation is reflexive, since $a+0=a\ \forall a\in R$.

But the relation is not symmetric for $a\ R_1\ b \not\Rightarrow bR_1\ a$ (prove)

The relation is transitive for aR_1b and $bR_1c \Rightarrow a\ R_1c$ (Prove)

(iv) The relation < in the set of natural Numbers N is defined as follows:

(a). $a \nless a$ since $a + x = a$ has no solution in N. Hence < is non-reflexive

(b) . $a < b$ does not imply b <a. Hence < is not symmetric

(c). $a < b$ and $b < c \Rightarrow a < c$ Hence < is transitive.

Example: R is a relation in Z defined by: if $x, y, \in Z$, then xRy if $10+xy > 0$.

Prove that R is reflexive, symmetric but not transitive.

Hint: *-2R3* and *3R6* but *–2 not related 6*

2.8. Partition of a Set

Let X be a set. A collection C of disjoint non-empty subsets of X whose union is X is called a partition of X.

Example: Let X= $\{a, b, c, d, e, f\}$. Then a partition of X is $[\{a\}, \{b, c, d\}, \{e, f\}]$, since intersection of any two subsets of this collection is ϕ and their union is X. There may be other partitions of X. An equivalence relation in a set S is usually denoted by ~. Then '*x~a*' will be read as '*x* is equivalent to *a*'.

2.9. Equivalence Class

If ~ is an equivalence relation in a set S and $a\in S$, the set $\{x\in S: x \sim a\}$ is called an equivalence class of S determined by 'a' and will be denoted by $\bar{a}$. If the equivalence relation ~ is denoted by R, then the equivalence class of S determined by 'a' may be denoted by Ra.

Note: $\bar{a} = R_a = \{x \in S : xR_a\}$.

Theorem: If ~ is an equivalence relation in a set S, and a, $b \in S$, then

(i) $\bar{a}$, $\bar{b}$ are not empty.

(ii) $b \sim a$, if and only if $\bar{a} = \bar{b}$.

Proof: Since $a \sim a$ by reflexive property, $a \in \bar{a}$, hence $\bar{a}$ is a not empty. Similarly $\bar{b}$ is not empty.

(ii) Now $x \in \bar{a} \Rightarrow x \sim a$. $b \sim a \Rightarrow a \sim b$ (by symmetry). Hence we get $x \sim a$, and $a \sim b$. Therefore $x \sim b$ (by transitivity)

Consequently $x \in \bar{b}$ thus $x \in \bar{a} \Rightarrow x \in \bar{b}$. Therefore $\bar{a} \subseteq \bar{b}$. Similarly $\bar{b} \subseteq \bar{a}$. Hence $\bar{a} = \bar{b}$.

Conversely $\bar{a} = \bar{b} \Rightarrow b \in \bar{b}$ (since $b \sim b$), therefore $b \in \bar{a}$ ($\bar{a} = \bar{b}$). Therefore $b \sim a$

Theorem (3.2) Any equivalence relation in a set S partitiones S into equivalence classes. Conversely any partition of S into non empty subsets, induces an equivalence relation in S, for which these subsets are the equivalence classes.

(i) Given an equivalence relation ~ in S. We are to prove that the collection of equivalence classes is a partition of S. Let $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_i$, etc. be the equivalence classes where $x_i \in S$. We are to prove $\bigcup_{x \in S} \bar{x}_i = S$. We have $\bigcup_i \bar{x}_i \subseteq S$,

since $\bar{x}_i \subseteq S \; \forall \; x_i \in S$. Again $x_i \in S \Rightarrow x_i \in \bar{x}_i \Rightarrow x_i \in \bigcup_i \bar{x}_i$. Therefore $\bigcup_{x \in S} \bar{x}_i = S$.

Now we prove that any two equivalence class $\bar{x}$, $\bar{y}$ where x, y $\in$ S are disjoint or identical.

Let $\bar{x} \cap \bar{y} \neq \phi$ and $z \in \bar{x} \cap \bar{y}$, then $z \in \bar{x}$ and $z \in \bar{y}$.

Now $z \in \bar{x} \Rightarrow z \sim x \Rightarrow x \sim z$ (by symmetry) $z \in \bar{y}$

$\Rightarrow z \sim y$. Hence $z \in \bar{x} \cap \bar{y} \Rightarrow [x \sim z, z \sim y] \Rightarrow x \sim y$ (by transitivity) $\Rightarrow \bar{x} = \bar{y}$.

Thus $\bar{x} \cap \bar{y} \neq \phi \Rightarrow \bar{x} = \bar{y}$. Hence $\bar{x} \neq \bar{y} \Rightarrow \bar{x} \cap \bar{y} = \phi$.

This completes the proof of the first part of the theorem.

(ii) Let the collection $C = \{Ai\}$ be a partition of S.

Then $S = \cup A_i$ and A_i's are mutually disjoint non-empty subsets of S.

Now $x \in S \Rightarrow x \in A_i$ for exactly one i.

We define a relation R in S by: for $x, y \in S$. xRy if x and y are element of the same subset A_i. It can be proved that R is an equivalence relation is S and the subsets A_i are the corresponding equivalence classes.

2.10. Quotient set of a set S

The set of all equivalence classes obtained from an equivalence relation in a set S is called the quotient set of S which is denoted by $\bar{S}$ or by $S/\sim$, or by S/R when the equivalence relation is denoted by R.

Example: Let S be the set of all points in the x-y plane. We define a relation R in S by: For $a, b \in S$, aRb if the line through the point a parallel to the X-axis passes through the point b. It can easily be proved that R is an equivalence relation in S. Now the equivalence class $\bar{a}$ determined by the point 'a' is the line through the point 'a' parallel to the X- axis and the quotient set $\bar{S}$= set of all straight lines in the X-Y plane parallel to the X-axis.

Example: The diagonal relation or the relation of equality in a set S is an equivalence relation). If $a \in S$, then $\bar{a} = \{a\}$. I.e. each equivalence class is a singleton and $\bar{S}$= set of all singletons.

Example: If S is a set, then $R = S \times S$ is an equivalence relation in S and the only equivalence class is the set S. $\bar{S} = \{S\}$.

Example: If X be the set of points in a plane and R is a relation on X defined by A, $B \in X$, ARB if A and B are equidistant from the origin. prove that R is an equivalence relation. Describe the equivalence classes.

Hint: The equivalence class R_A=Set of points on the circle with centre as origin O and radius OA. Hence the quotient set X/R is the set of circles on the plane with centre as O.

2.11. Order relation:

Definition: A relation R in a set is called a partial order (or order) relation if

(1) R is reflexive i.e. $xRx\ \forall x \in S$
(2) R is anti symmetric i.e. xRy and $yRx \Rightarrow x$=y, where $x, y, \in S$
(3) R is transitive i.e. for $x, y, z \in S$. $[xRy, yRz] \Rightarrow xRz$. If in addition $\forall x, y \in S$, Either xRy or yRx, then R is called a linear order or total order relation. A set with a partial order relation is called a partially ordered set and a set with a total order relation is called a totally ordered set or a chain.

Note: Generally the partial order relation is denoted by the symbol $\leq$ and is read as 'less than or equal to'.

Example: In the set Z_+ of positive integers, the relation given by for $m, n \in Z_+$, $m \leq n$ if m divides n, is a partial order relation but not a total order relation.

For (1) $m \leq m\ \forall m \in Z$, since m divides m.

(2) $m \leq n$ and $n \leq m \Rightarrow m$ divides n and n divides m $\Rightarrow$ m = n.

(3) $[m \leq n, n \leq k] \Rightarrow m$ divides n, n divides k$\Rightarrow m$ divides k$\Rightarrow m \leq k$. Thus the relation is a partial order relation.

But it is not a total order relation, since for $m, n \in Z_+$ it may happen that neither m divides n nor n divides m i.e. neither $m \leq n$ nor $n \leq m$.

Example: In the set R of real numbers, the relation $\leq$ having its usual meaning in R is a total order relation. The proof is left as an exercise.

Example: If S be a set, then the relation in P(S) given by: for $A, B \in$ P(S). $A \leq B$

if A$\subseteq$ B is a partial order relation but not a total order relation. The proof is left as an exercise.

Definition: Let $(S, \leq)$ be a partially ordered set. If $x \leq y$ and $x \neq y$, then x is said to be strictly smaller than or a strict predecessor of y. We also say that y is strictly greater than or a strict successor of y, *then we* denote it by $x < y$.

An element $a \in S$ is said to be a least or first (respectively greatest or last) element of S if $a \leq x$ (respectively $x \leq a$) $\forall x \in S$.

An element $a \in S$ is called minimal (respectively maximal) element of S is $x \leq a$ (respectively $a \leq x$) implies $a = x$ where $x \in S$.

A least (greatest) element if it exists, unique and also the unique minimal (maximal) element in this case.

Example: $(N, \leq)$, (the relation $\leq$ having its usual meaning) is a partially ordered set.

2 is strictly smaller than 5 or $2 < 5$. 1 is the least or first element of N.

since, $1 \leq m \; \forall \; m \in N$, There is no greatest or last element of N. 1 is the only minimal element since if $x \in N$, Then $x \leq 1 \Rightarrow x = 1$.

Example: Consider the set S = {1, 2, 3, 4, 12}. Let $\leq$ be defined by $a \leq b$ if a divides b. Then 2 is strictly smaller than 4 or $2 < 4$. 12 is strictly greater than 4 or $4 < 12$. Since I divides each of the number 1,2,3,4,12 so $1 \leq x \; \forall x \in S$, hence *1* is the least element of S. Again since $x \leq 12 \forall x \in S$ i.e. each element of S divides 12, so 12 is the greatest or last element of S. Here also I is the only minimal element, since $x \in S$, then $x \leq 1$ i.e. x divides 1 implies $x = 1$.

Example: Let S be a set. Then ($\mathbf{p}$ (S), $\leq$) where $\leq$ is the set inclusion relation $\subseteq$, is a partially ordered set. Then ϕ is the least element, since $\phi \subseteq A \forall A \in P(S)$, and S is the greatest element since $A \subseteq S \; \forall \; A \in P(S)$, phi is the minimal element.

2.12. Infimum and Supremum

Let $(S, \leq)$ be a partially ordered set and A a subset of S. An element $a \in S$ is said to be a lower bound (respectively upper bound) of A if $a \leq x$ (respectively $x \leq a$) $\forall x \in A$.

In case A has a lower bound, we say that A is bounded below or bounded on the left. When A has an upper bound we say that A is bounded above or bounded on the right. Let $L(\neq\phi)$ be the set of all lower bounds of A, then greatest element of L, if it exists is called the greatest lower bound (*g l b*) or infimum of A. Similarly if $U(\neq\phi)$ be the set of all upper bounds of A, then the least element of U, if it exists, is called the least upper bounded (*l.u.b.*) or supremum of A

Example: Consider the partially ordered set $(N, \leq)$, where $m \leq n$ if m divides n. Consider the subset $A=\{12, 18\}$. 2 is a lower bound of A since 2 divides both 12 and 18. i.e. $2 \leq 12$ and $2 \leq 18$. The set of al lower bounds of A viz L= {1,2,3,6} and 6 is the greatest element of L. Hence (*g.l.b.* or infimum of A)=6. It is called the greatest common divisor (*g.c.d*) of A. Now 36, 72, 108 etc. are upper bounds of A since x divides 36, 72, 108 $\forall x \in A$ thus $x \leq 36, 72, 108\ \forall x \in A$. Now the set of upper bounds of A viz U = {36, 72, 108, …} has the least element 36. Hence the *l.u.b* or supremum of A= 36. It is also called the L.C.M. of 12 and 18.

Theorem: The least (respectively greatest) element of a partially set $(S, \leq)$, if it exists, is unique.

***Proof*:** If possible let l and l' be two least element of S. Since l is the least element, so $l \leq x\ \forall x \in S$ hence $l \leq l'$ since $l' \in S$. Similarly taking l' as least element

$l' \leq l$. Hence $l \leq l'$ and $l' \leq l$. Therefore by anti-symmetry $l = l'$.

A similar proof can be given for the greatest element.

Remark: In contrast to the above theorem, maximal and minimal elements of a partially ordered set X need not be unique. In example (1.3) we can show that every singleton is a minimal element. Sometimes minimal element can also be a maximal element. For example consider the partially ordered set $(X\ \Delta)$ where Δ is the diagonal relation. Every element of X is a minimal as well as a maximal element of X. For let $a \in X$. Then $x \Delta a \Rightarrow x = a,\ a \Delta x \Rightarrow x = a$.

2.13. Partially ordered set

A partially ordered set $(S, \leq)$ is said to be well ordered if every non empty subset of S has a least element.

Theorem: A well ordered set $(S, \leq)$ is always totally ordered or linearly ordered or a chain.

***Proof**:* Let x, y be any two element of S. Consider the subset $\{x, y\}$ of S, which is non empty and hence has a least element which is either x or y, then $x \leq y$ or $y \leq x$. Hence every two element of S are comparable and so S is totally ordered. We now state two important statements without proof.

Well ordering principle: Every set can be well ordered.

Zorn's Lemma: Let S be a non empty partially ordered set in which every chain i.e. every totally ordered subset has an upper bound, then S contains a maximal element.

2.14. Totally Ordered Sets

Two elements a and b are said to be not comparable

if $a \nleq b$ and $b \nleq a$, that is, if neither element precedes the other. A total order in a set A is a partial order in A with the additional property that $a < b$, $a=b$ or $b < a$

for any two elements a and b belonging to A. A set A together with a specific total order in A is called a totally ordered set.

Example: Let R be a relation in the set of natural numbers N defined by 'x is a multiple of y', then R is a partial order in N. 6 and 2, 15 and 3, 20 and 20 are all comparable but 3 and 5, 7 and 10 are not comparable. So N is not a totally ordered set, under this order relation.

Example: Let A and B be totally ordered sets. Then Cartesian product $A \times B$ can be totally ordered as follows: $(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b < b'$ This order is called the lexicographical order of A× B, since it is similar to the way words are arranged in a dictionary.

Theorem: Every subset of a well-ordered set is well-ordered.

Check your progress

(1) Let R be the relation in $A = \{1, 2, 3, 4, 5\}$ which is defined by 'x and y are relative prime'. Find the solution set of R and draw R on a coordinate diagram of $A \times A$.

(2) Let R be the relation in the natural numbers N defined by '$x - y$ is divisible 8'. Prove that R is an equivalence relation.

(3) Let L be the set of lines in the Euclidean plane and let

a. R be the relation in L defined by 'x is parallel to y'.

b. R' be the relation in L defined by 'x is perpendicular to y'. State whether or not R and R' are equivalence relations.

(4) For each of the following relations in the natural number N:

a. "$x > y$"

b. "x is a multiple of y"

c. "$x + 3y = 12$"

d. "$x \leq y$"

e. "$x^2 = y^2$"

find whether or not each of the relations are (i) reflexive (ii) symmetric (iii) anti-symmetric (iv) transitive.

(5) Let Z be the set of all integers. Define a relation R on Z in the following way. $R = \{(a, b) \in Z \times Z: (a - b) \text{ is divisible by } 7\}$.

Show that R is an equivalence relation.

Find all the distinct equivalence classes of the relation R.

(6) Show that if R and S be transitive relations on a set A,

then $R \cup S$ need not be transitive on A *in general*.

(7) Prove that a relation R on a set A is symmetric if and only if $R^{-1} = R$.

(8) Find the equivalence classes determined by the equivalence relation R on Z defined by 'aRb if and only if $a - b$ is divisible by 5' for $a, b \in Z$, the set of integer.

(9) Prove that an equivalence relation R on a set S determines partitions of S. Conversely, each partitions of S yields an equivalence relation on S.

(10) Find all of the partitions of $S = \{p, q, r, s\}$.

2.15. Summary:

After reading this unit we should be able to recall the basic properties of relations, to derive other properties with the help of the basic ones, identify various types of relations, to understand the relationship between equivalence classes and partition, recall the basic properties of order relations, derive other properties with the help of the basic ones, identify infimum and supremum and totally ordered set.

2.16. Terminal Questions

1. $Let\ A = \{a, b, c\},\ B = \{c, a, b\}, A = B$
2. $Prove\ that\ basic\ facts\ about\ union\ of\ sets\ \ A \subseteq B \Leftrightarrow A \cup B = B$
3. $Prove\ that\ A - B = A \cap B'$
4. $Prove\ that\ (R^{-1})^{-1} = R$
5. $The\ diagonal\ or\ the\ equality\ relation\ \&\ in\ a\ set\ S\ is\ an\ equivalence$

   $relation\ in\ S. For\ it\ x, y \in S\ the\ x\ y\ iff\ x = y.$
6. $Let\ a\ relation\ R\ in\ the\ set\ N\ of\ natural\ numbers\ be\ defined\ by:$

   $if\ m, n \in N. then\ mRn\ if\ m\ and\ n\ are\ both\ odd.$
7. $Let\ x\ be\ a\ set. Consider\ the\ relation\ R\ in\ \big(e(x)\big), given\ by:$

   $for\ A, B \in \big(e(n)\big) ARB\ if\ A \subseteq B.$

UNIT—III

Mapping

Structure

3. 1. INTRODUCTION

As we know the notion of a map is one of the most fundamental concepts in mathematics and is used knowingly or unknowingly to our day to day life at every moment. Computer Science is an area where a number of applications of maps can be seen. We thought it would be a good idea to acquaint with some basic results about maps. Perhaps, we are already familiar with these results. But, a quick look through the pages will help us in refreshing our memory, and we will be ready to tackle the course. We will find a number of examples of bijective maps, direct and inverse image, Inverse map, composition of maps and various types of maps.

3. 2. Objectives

After reading this unit you should be able to:

- Describe a map in its different forms
- Derive other properties with the help of the basic ones
- Define a map and examine whether a given map is one –one/onto

3.3. Mapping

Here we shall present some basic facts about functions which will help us refresh our knowledge. We shall look at various examples of maps and shall also define inverse maps.

Definition: If X and Y are two sets, a map f from X to Y, is a rule or a correspondence which connects every member of X to a unique member of Y. We write f: X $\rightarrow$ Y (read as "f is a map from X to Y) X is called the domain and Y is called the co-domain of f. We shall denote by f(x) that unique element of Y which is associated to X.

Note (1. 1). The map f from X to Y is written as f: X $\rightarrow$ Y

Note (1. 2). To find if f: $X \rightarrow Y$ is a map, we check that

(1) Every element of X must have an image in Y.
(2) If x_1, $x_2 \in X$, then $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$.

Note (1.3): Mapping of set X to a set Y, when X and Y are sets of numbers are also called functions.

Example: Let Z_+ be the set of positive integers and E the set of even positive integers. Let map f: $Z_+ \rightarrow E$ be defined by f (m) =2m $\forall$ $m \in Z_+$. Hence range

$f = f(Z_+) = E$.

Example: Let R be the set of the real numbers. Let function f: R$\rightarrow$ R be given by f(x) =e^x, x$\in R$, since $e^x > 0$ for $\forall x \in R$, therefore range $f = R_+$ (set of positive real numbers), since for every $x \in R_+$, $x = f(log_e\, x)$.

Example: Let X = set of all students of Allahabad University, Y = Set of ages in years. Since every student has some unique age, so we can define a map

$f: X \rightarrow Y$ by $f(x) = y$. Where x is student and y is his age in years.

Example: f: R→ R defined by $f(x) = \log x$, $x \in$R is not a map or function, since

$f(-3) = \log(-3)$ is not a real number. But f: $R_+ \rightarrow$R where R_+ is the set of positive real numbers defined by $f(x) = log\, x$ is a map.

Example: f: R_+ = R defined by $f(x) = \sqrt{x}$ is not a map, since $f(4) = \sqrt{4} = \pm 2$. Thus 4 has two f-images. But f(x) = + √x (positive value of the square root of x) will be a map or function from R_+ to R.

Example: The rule f(x) = x/2 does not define a function f: N → Z as odd natural numbers like 1, 3, 5 ……. From N cannot be connected to any member of Z.

Example: Every natural number can be written as a product of some prime numbers. Consider rule f(x) = a prime factor of x, which connects elements of N. Here since 6 = 2 × 3. f(6) has two values : f(6) = 2 and f(6) = 3. This rule does not associate a unique number with 6 and hence does not give a function from N to N.

Example: f: N → R, defined by f(x) = - x. is a map since the rule f(x) = - x associates a unique member (- x) of R to every member x of N. The domain here is N and the co-domain is R. To describe a map completely we have to specify completely we have to specify the following three things:

a. the domain
b. the co-domain, and
c. the rule which associates a unique member of the co-domain to each member of the domain.

The rule which defines a map need not always be in the form of a formula. But it should clearly specify (perhaps by actual listing) the correspondence between X and Y.

If f: X → Y, then y = f(x) is called the image of x under f or the f-image of x. The set of f-image of all members of X, i.e., {f(x): x ∈ X} is called the range of f and is denoted by f(X). We see that f(X) ⊆ Y.

Remark (a): We shall consider maps for each of which whose domain and co-domain are both subsets of R. Such maps are called real map or real-valued maps of a real variable. We shall, use the word 'map' to mean a real map.

(b) The variable x used in describing a map is often called a dummy variable because it can be replaced by any other letter. Thus, for example, the rule f(x) = -x, x ∈ N can as well be written in the form f(t) = -t, t ∈ N or as f(f) = -u, u∈N. The variable x (or t or u) is also called an independent variable, and f(x) is dependent on this independent variable.

Graph of a map We draw the graph of a map f: X → Y, we choose a system of coordinate axes in the plane. For each x ∈X, the ordered pair (x, f (x)) determines a point in the plane (see fig. 1). The set of all the points obtained by considering all possible values of x is the graph of the map f. Let us consider some more examples of maps and their graphs.

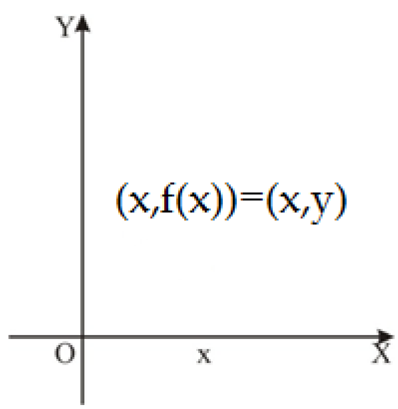


Fig.1

3.4. Type of maps:

Definition: A map f: $X \rightarrow Y$ is said to be injective or one-one if for $x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or equivalently $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

It is clear that if the mapping f is injective, then distinct elements of X have distinct images in Y.

Definition: A map $f: X \to Y$ is said to be **surjective** or onto if range $f = Y$ i.e. $f(X) = Y$. It is clear that if f is surjective, then $\forall\, y \in Y\ \exists\ x \in X$ such that $f(x) = y$.

Definition: A map $f: X \to Y$ is said to be **bijective** if it is both injective and surjective. X is said to be equipotent to Y, we write $X \simeq Y$. A bijective map f is also called a one-one correspondence.

Example: The map $f: Z_+ \to E$ given in example (6.1) is injective, because for m, $n \in Z_+$, $f(m)=f(n) \Rightarrow 2m = 2n \Rightarrow m = n$. f is surjective also, since for $\forall y \in E$,

$\exists\ y/2 \in Z_+$ such that $f(y/2) = y$. Thus f is a bijection or one-one correspondence from Z_+ to E.

Example: The map $f: R \to R$ given by $f(x) = e^x$, $x \in R$ is injective but not surjective for if x, $y \in R$ then $f(x) = f(y) \Rightarrow e^x = e^y \Rightarrow x = y$ therefore f is *injective*.

Again $e^x > 0\ \forall x \in R$, hence 0 or any negative real number is not the f-image of any real number of the domain set, and so f is not *surjective*.

Example: Let C be the set of complex numbers and R the set of real numbers. The map f: $C \to R$ given by $f(x + iy) = \sqrt{(x^2 + y^2)}$ is neither injective nor surjective for $f(x+iy) = f(x - iy) = \sqrt{(x^2+y^2)}$ and range (f) = $\{r \in R, r \geq 0\} \neq R$.but $x + iy \neq x - iy$

Example: The map f: $R \to R$ given by $f(x) = \sin x$ is neither injective nor surjective for $f(x) = f(\pi - x) = \sin x$ but $x \neq \pi - x$and there does not exist $x \in R$ such that $f(x) = \sin x = 2$.

Example: If A and B are two finite sets having the same number of elements, then $f: A \to B$ is injective (one-one) if and only if it is surjective (onto). For, let A and B both have n elements, if $f: A \to B$ is injective then the n elements of A will have n distinct images in B which will be the n elements of B and hence every element of B is the image of some element of A and so f is surjective.

Again if f is surjective, then each of the n element of B will be the image of at least one element of A, but any element of B cannot be the image of more than one element of A, for in that case A must have more than n elements. Hence each element of B is the image of exactly one element of A. So f is injective.

Example: Show that there exists a bijection between the set N of natural numbers and the set Z of integers. Define f: $N \to Z$ as follows:

$f(m) = m/2$ when m is an even natural number,

$f(m) = -(m-1)/2$ when m is an odd natural number.

This map is one-one correspondence (bijective).

Inclusion and Identity Maps: Let $X \subseteq Y$ **a**nd let f: $X \to Y$ be given by $f(x) = x$. $\forall x \in X$. Then f is called inclusion map of X into Y. An inclusion map is generally denoted by i_x in place of f. the inclusion map of X into X is called the **identity map** on X and is denoted by I_x, Thus I_x: $X \to$ X is given by $I_x(x) = x \; \forall \; x \in X$.

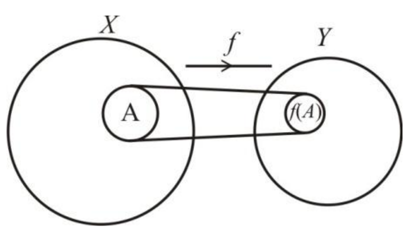
Equality of Mapping: Let f and g be two maps from X to Y, that is both f and g map the set X to the set Y. We define $f = g$ if $f(x) = g(x) \; \forall x \in X$.

If f; $R \to$ is defined as $f(x) = \frac{x^2 - 4}{x - 2}$ when $x \neq 2$ and $f(2) = 4$

and g: $R \to R$ is defined as $g(x) = x+2 \; \forall x \in R$. Then f = g.

3.5. Direct and Inverse image of sets

Let f: $X \to Y$ be a map and let $A \subseteq X$, $B \subseteq Y$, then the direct image of A under f denoted by $f(A)$ and is given by $f(A) = \{y \in \text{Y} \mid \exists \; x \in A$ with $f(x) = y\}$,

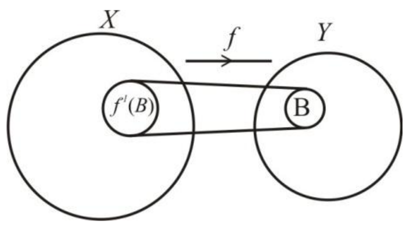


that is $f(A)$ is the set of images of all the elements of A. the above diagram illustrates it. Thus $x \in A \Rightarrow f(x) \in f(A)$ the reserve implication viz

$f(x) \in f(A) \Rightarrow x \in A$ is only true when f is injective. If $x \in X$, then $f(\{x\}) = \{f(x)\}$ and $f(X)$ = range f and f $(\phi) = \phi$.

The inverse image of B under f denoted by $f^{-1}(B)$ is given by $f^{-1}(B) = \{x \in X: f(x) \in B\}$ thus $x \in f^{-1}(B) \Rightarrow \exists\, x \in X$ *such that* $f(x) \in B$.

The reverse implication viz $f(x) \in B \Rightarrow x \in f^{-1}(B)$ is also true.



In case there is no element $x \in X$ such that $f(x) \in B$

(which may happen when f is not surjective), then $f^{-1}(B) = \phi$.

Example: Let f: R→R be given by $f(x) = x^2$, $x \in$ R.

Let $A = \{x \in \mathrm{R}: 1 \le x \le 2\} = [1, 2] \subset$ R.

Then $f(A) = \{y \in R: 1 \le y \le 4\} = [1, 4]$. [Since $1 \le x \le 2 \Rightarrow 1 \le x^2 \le 4$]

Let $B = \{y \in R: 4 \le y \le 9\} = [4, 9]$. Then $f^{-1}(B) = [-3, -2] \cup [2, 3]$.

If $C = [-4, -1]$, then $f^{-1}(C) = \phi$, since x∈R such that $f(x) = x^2 \in [-4, -1]$,

does not exist.

Example: (a) Let $A = \{n\pi: n$ is an integer$\}$ and R be the set of real numbers.

Let f: $A \to R$ be defined by $f(\alpha) = \cos\alpha\ \forall \alpha \in A$. Find $f(A)$ and $f^{-1}(\{0\})$.

Now $f(n\pi) = \cos n\pi = +1$ or -1, Hence $f(A) = \{-1, 1\}$.

If $f(\alpha) = 0$ or $\cos\alpha = 0$ or $\alpha = (2n + 1\} \frac{\pi}{2}$.

Hence $f^{-1}(\{0\}) = \{(2n+1)\frac{\pi}{2} \mid n \in \mathrm{Z}\}$

Now $(2n+1)\frac{\pi}{2} \notin \{n\pi\}$, So, $f^{-1}(0) = \phi$.

Example: Let $f: X \to Y$ be a map and let A and B be subsets of X, then

(i) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$

(ii) $f(A \cup B) = f(A) \cup f(B)$

(iii) $f(A \cap B) \subseteq f(A) \cap f(B)$. Equality holds when f is injective.

***Proof*: (i)** *If* $A \subseteq B$, then $x \in A \Rightarrow x \in B$. Now $y \in f(A)$

$\Rightarrow \exists\, x \in A$ s.t. $f(x) = y$.

$\Rightarrow \exists\, x \in B$ s.t. $y = f(x)$. $\Rightarrow y = f(x) \in f(B)$ since $x \in B$, $\Rightarrow f(x) \in f(B)$

Therefore $y \in f(A) \Rightarrow y \in f(B)$ hence $f(A) \subseteq f(B)$.

(ii) $y \in f(A \cup B) \Rightarrow \exists\, x \in (A \cup B)$ s.t. $y = f(x)$

$\Rightarrow \exists x \in A$ or $x \in B$ s.t. $y = f(x)$

$\Rightarrow y = f(x) \in f(A)$ or $y = f(x) \in f(B)$. (since $x \in A \Rightarrow f(x) \in f(A)$

and $x \in B \Rightarrow f(x) \in f(B)$).

Hence, $y \in f(A \cup B) \Rightarrow y \in f(A) \cup f(B)$.

Therefore $f(A \cup B) \subseteq f(A) \cup f(B)$.

Again $A \subseteq A \cup B$, $B \subseteq A \cup B$ therefore by (i) $f(A) \subseteq f(A \cup B)$,

$f(B) \subseteq f(A \cup B)$ therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$.

From the above we get $f(A \cup B) = f(A) \cup f(B)$.

(iii) $A \cap B \subseteq A$, $A \cap B \subseteq B$, therefore by (i) $f(A \cap B) \subseteq f(A)$,

$f(A \cap B) \subseteq f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Note: $f(A) \cap f(B) \subseteq f(A \cap B)$ is not true. Since $y \in f(A) \cap f(B)$

$\Rightarrow y \in f(A)$ and $y \in f(B) \Rightarrow \exists x_1 \in A \mid f(x_1) = y$ and

$\exists\, x_2 \in B$ *such that* $f(x_2) = y \not\Rightarrow \exists\, x \in A \cap B$ such that $f(x) = y$.

Since $x_1 \in A$ but x_1 may not be an element of B, similarly $x_2 \in$ B but x_2 may not be an element of A, so there may not exist a common element x of A and B such that $f(x)=y$.

But if f is injective, then $f(A) \cap f(B) \subseteq f(A \cap B)$ will be true and

Hence, in that case $f(A \cap B)=f(A) \cap f(B)$.

Example: When $f(A) \cap f(B) \nsubseteq f(A \cap B)$.

Consider map $f: R \rightarrow R$ given by $f(x) = x^2$, It is clear f is not injective.

Let A= {-1, - 2, -3, 4} and B= {1, 2, -3} be subsets of Dom f.

Then $A \cap B = \{-3\}$. So, $f(A \cap B) = \{(-3)^2\}$.

Now $f(A)=\{(-1)^2, (-2)^2, (-3)^2, (4)^2\}$, $f(B)=\{1^2, 2^2, (-3)^2\}$, ($x^2 = (-x)^2$)

So, $f(A) \cap f(B) = \{1^2, 2^2, (-3)^2\} \nsubseteq \{(-3)^2\}$.

So, $f(A) \cap f(B) \nsubseteq f(A \cap B)$.

Example: Let $f: X \rightarrow Y$ be a map and let A and B be subsets of Y.

Then (i) $A \subseteq B \Rightarrow f^{-1}(A) \Rightarrow f^{-1}(B)$

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(iii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

***Proof*:** (i) $x \in f^{-1}(A) \Rightarrow f(x) \in A \Rightarrow f(x) \in B$ (since $A \subseteq B$)

So, $x \in f^{-1}(B)$. Therefore, $f^{-1}(A) \subseteq f^{-1}(B)$.

(ii) $X \in f^{-1}(A \cup B) \Leftrightarrow f(x) \in A \cup B$

$\Leftrightarrow f(x) \in A$ or $f(x) \in B \Leftrightarrow x \in f^{-1}(A)$

or $x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$.

Therefore $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(iii) $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B$

$\Leftrightarrow f(x) \in A$ and $f(x) \in B \Leftrightarrow x \in f^{-1}(A)$

and $x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$.

Therefore $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Thus (ii) and (iii) show that union and intersection are preserved under inverse image.

Check your progress

(1.1) Prove that $f: X \to Y$ is injective iff $f^{-1}(\{y\}) = \{x\}$ $\forall y \in f(X)$, and some $x \in X$

(1.2) Prove that $f: X \to Y$ is sujrective iff $f^{-1}(B) \neq \phi$ $\forall$ $B \subseteq Y$ and $B \neq \phi$.

(1.3) Prove that $f: X \to Y$ *i*s bijective iff $\forall y \in Y, f^{-1}(\{y\}) = \{x\}, x \in X$.

(1.4) if $f: X \to Y$ and $A \subseteq X$, $B \subseteq Y$, prove that

(a). $f(f^{-1}(B)) \subseteq B$.

(b). $f^{-1}(f(A)) \supseteq A$.

(c). $f^{-1}(Y) = X$.

(d) let $f: X \to Y$ and let $A \subseteq Y$, then prove $f^{-1}(Y - A) = X - f^{-1}(A)$.

(1.5) Give examples when

(i) $f(f^{-1}(B))$ is a proper subset of B

(ii) A is a proper subset of $f^{-1}(f(A))$.

3.1.6. Answer/solution

1.5 (i) Consider map $f: R \to \mathrm{R}$ given by $f(x) = x^2$. So f is not surjective

Let $B = \{-1, -2, 3, 4\} \subseteq$ co-dom f. Then $f^{-1}(B) = \{\pm\sqrt{3}, \pm 2\}$,

hence $f(f^{-1}(B)) = \{(\pm\sqrt{3})^2, (\pm 2)^2\} = \{3, 4\}$.

Thus $f(f^{-1}(B))$ is a proper subset of B.

(ii) Consider the above map. Let $A = \{-1, -2, 3, 4\} \subset dom(f)$.

Then $f(A) = \{1^2, 2^2, 3^2, 4^2\}$.

Hence $f^{-1}\{f(A)\} = \{\pm 1, \pm 2, \pm 3, \pm 4\}$ (Prove).

Thus A is a proper subset of $f^{-1}((f(A))$.

3.6. Inverse map

Let $f: X \to \Upsilon$ be a map. Let us try to define a map

$\phi: Y \to X$ given by : if $y \in Y$, then $\phi(y) = x$ where $f(x) = y$.

if ϕ is to be map, then every $y \in Y$ must be the f image of some $x \in X$, that is f must be surjective. Further two different elements x_1 and x_2 of X must not have the same f-image $y \in Y$, for in that case $\phi(y) = x_1$ also x_2, so ϕ cannot be a map. Hence f must be injective. Thus when f is bijective we can define the above map ϕ which is called inverse of f and will be denoted by f^{-1}. Thus the inverse of a bijective map f is defined as: $f^{-1}: Y \to X$ given by $\forall y \in Y, f^{-1}(y) = x \in X$ such that $f(x) = y$.

As we will notice, the map g is also one-one and onto and therefore it will also have an inverse. You must have already guessed that the inverse of g is the map f. From this discussion we have the following:

If f is one-one and onto map from X to Y, then there exists a unique map

$g: Y \to X$ such that for each $y \in Y$, $g(y) = x \Leftrightarrow y = f(x)$. The function g so defined is called the inverse of f. Further, if g is the inverse of f, then f is the inverse of g, and the two map f and g are said to be the inverse of each other. The inverse of a function f is usually denoted by f^{-1}.

Solve the equation $f(x) = y$ for x. The resulting expression for x (in terms of y) defined the inverse map. Thus, if $f(x) = \frac{x^5}{5} + 2$, we solve $\frac{x^5}{5} + 2 = y$ for x. This gives us $x = \{5(y-2)\}^5$. Hence f^{-1} is the map defined by $f^{-1}(y) = \{5(y-2)\}^{1/5}$.

Graphs of Inverse map

Let f; $X \to Y$ be a one-one and onto map, and let $g: Y \to X$ be the inverse of f. A point (p, q) lies on the graph of $f \Leftrightarrow q. = f(p) \Leftrightarrow p = g(q) \Leftrightarrow (q, p)$ lies on the graph of g. Now the points (p, q) and (q, p) are reflections of each of the with respect to (w.r.t.) the line y = x. Therefore, we can say that the graphs of f and g are reflections of each other w.r.t. the line y = x.

Therefore, it follow that, if the graph of one of the map f and g is given, that of the other can be obtained y reflecting it w.r.t. the line y = x. As an illustration, the graphs of the maps $y = x^3$ and $y = x^{1/3}$ are given in Fig. 8. Do you agree that these two maps are inverse of each other? If the sheet of paper on which the graphs have been drawn is folded along the line y = x, the two graphs will exactly coincide.

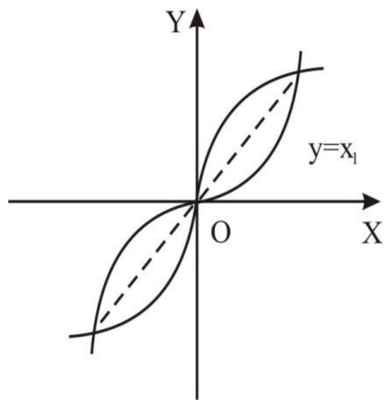


Fig. 8

Since we know that sin (x + 2π) = sin x, obviously this map is not one-one on R. But if we restrict it to the interval [- π + 2π], we find that it is one-one. Thus, if f(x)=sin x ∀ x ∈ [x/2], π/2], then we can define $f^{-1}(x) = \sin^{-1}(x) = y$ if sin y = x.

Similarly, we can define $\cos^{-1}$ and $\tan^{-1}$ maps as inverse of cosine and tangent maps if we restrict the co-domain to [0, π] and]- π/2, π/2[, respectively.

Remarks: Inverse map of f should not be confused with the inverse image of a subset under f, denoted by the same symbol viz f^{-1}.

Note1: Inverse of the map $f: X \rightarrow Y$ only exists when f is bijective that is the inverse map $f^{-1}: X \rightarrow Y$ only exists when f is bijective and the inverse map $f^{-1}: Y \rightarrow X$ is such that $f^{-1}((y) = x \Leftrightarrow f(x) = y$.

Note2: Let $X = [- \pi/2, \pi/2]$, $Y = [- 1, 1]$. Let $f: X \rightarrow Y$ be given by $f(x) = \sin x$, $x \in X$. It can be easily proved that f is a bijection. So $f^{-1}: Y \rightarrow X$ given by

$f^{-1}(y) = \sin^{-1} y = x \in X$. such that $\sin x = y$. Thus $\sin^{-1} y = x . \Leftrightarrow \sin x = y$.

Note3: If f: $X \rightarrow Y$ is a bijection, then the inverse map f^{-1}: $Y \rightarrow X$ is also a bijection. For let $f^{-1}(y_1) = x_1$, where $y_1 \in Y$ and $x_1 \in X$. Then $f(x_1) = y_1$, and $f^{-1}(y_2) = x_2$, $y_2 \in Y$ and $x_2 \in X$. Then $f(x_2) = y_2$. Now $f^{-1}(y_1)= f^{-1}(y_2) \Rightarrow x_1= x_2 \Rightarrow f(x_1) = f(x_2)$

[since f is map] $\Rightarrow y_1= y_2$. Therefore f^{-1} is injective.

Again since f is bijective, every element $y \in Y$ is the f-image of a unique element $x \in X$. Hence every $x \in X$ is the f^{-1} image of an element $y \in Y$. Therefore, f^{-1} is surjective.

3.7. Composition of Maps

Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be two maps. Their composite denoted by (gof) is the map gof: $X \rightarrow Z$ given by $(gof)(x)$ = z such that for some $y \in Y$, $f(x) = y$ and $g(y) = z$. Thus we get $(gof)(x) = z = g(y) = g(f(x))$.

Let f: X →Y and g: Y → Z be two maps. We define a function h: X → Z by setting h(x) = g(f(x)). To obtain h(x), we first take the f-image, f(x), of an element x of X. This f(x) ∈ Y, which is the domain of g. We then take the g image of f(x), that is, g(f(x)), which is an element of Z. This scheme has been shown in Fig. 11 .

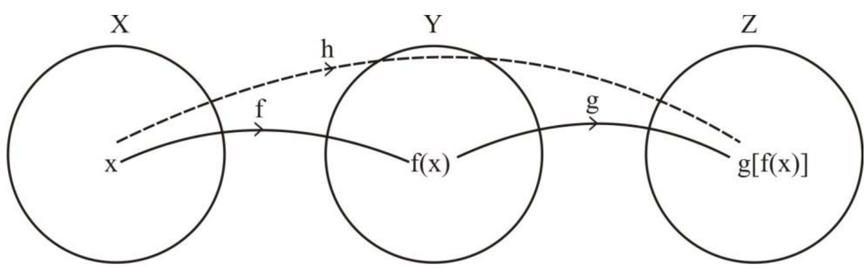


Fig. 11

The function h, defined above, is called the composite of and g and is written as gof. Note the order. We first find the f-image and then is g-image. Try to distinguish it form (fog), which will be defined only when Z is a subset of X. Also, in that case, fog is a function from Y to Y.

Example: Consider the function $f(x) = x^2 \ \forall x \in R$ and

$g(x) = 8x + 1 \ \forall x \in R$.

(fog) is a function from R to itself defined by $(fog)(x) = f\big(g(x)\big)$

$= f(8x + 1) = (8x + 1)^2$. Thus (gof) are both define, but are different from each other.

The concept of composite map is used not only to combine map, but also to look upon a given map as made up of two simpler maps.

For example, consider the function. $h(x) = sin\,(3x + 7)$. We can think of it as the composite (gof) of the function $f(x) = 3x + 7\ \forall x \in R$ and $g(u) = sin\,u\ \forall u \in R$.Now let us try to find the composites fog and gof of the functions: $f(x) = 2x + 3\ \forall x \in R$, and $g(x) = (1/2)\,x - 3/2\ \forall x \in R$. Note that f and g are inverse of each other now $gof\ (x) = g(f(x)) = g(2x+3) = \frac{1}{2}(2x + 3) - \frac{3}{2} = x$. Similarly $fog(x) = f(g(x)) = f(x/2 - 3/2) = 2(x/2 - 3/2) + 3 = x$. Thus, we see that $gof(x) = x$ and $fog(x) = x$ for all $x \in R$. In other words, each of gof and fog is the identity map on R. We have observed here is true for any two maps f and g which are inverse of each other. Thus, if $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are inverses of each other, then gof and fog are identity functions. Since the domain of gof is X and of fog is Y, we can write this as: $gof = i_X, fog = i_Y$.

Note: In general $gof \neq fog$.

Note: The maps being particular types of relations, composite of maps has been defined exactly in the same way as composite of relations.

Example: let $f: R \rightarrow R$ be given by $f(x) = e^x, x \in R$ and $g: R \rightarrow R$ be given $g\,(y) = sin\,y, y \in R$. Then $(gof): R \rightarrow R$ given by

$$(gof)\,(x) = g\,(f\,(x)) = g\,(e^x) = sin\,(e^x).$$

Here Range f = f (R) = R^+ (the set of positive real numbers) $\subseteq R$.

Here(fog) is also defined, viz. $(fog): R \rightarrow R$ given by

$(fog)\,(y) = f(g(y)) = f(sin\,y) = e^{Siny}, \forall\, y \in R$.

Hence $(fog)\,(x) = f(g(x)) = e^{Sinx}$.

Thus $(gof) \neq (fog)$.

(Since $(gof)(0) = sin1 \neq 1 = (fog)(0)$)

Example: Let f: R $\rightarrow$ R be given by $f(x) = x^3 + 3$

and $g: R \rightarrow R$ be given by $g(x) = x^2 - 7$.

Then (gof): R $\rightarrow$ R given by $(gof)(x) = g(f(x)) = g(x^3+3) = (x^3+3)^2 - 7$.

Now (fog): $R \rightarrow R$ given by $(fog)(x) = f(g(x))$

$= f(x^2 - 7) = (x^2 - 7)^3 + 3$. Thus $(gof) \neq (fog)$.

Remarks: If (gof) is defied, then (fog) need not be defined.

Theorem: Let $f: X \rightarrow Y, g: Y \rightarrow Z, h: Z \rightarrow W$ be three maps.

Then $(ho(gof)) = ((hog)\ of)$, that is composition of maps is associative just like the composition of relations, as shown below.

Both $(ho\ (gof))$ and $((hog)\ of)$ maps from $X \rightarrow W$.

Now $(ho\ (gof))\ (x) = h\ ((gof)(x)) = h(g\ (f\ (x))$ and

$((hog)\ of)\ x = (hog)\ (f\ (x)) = h\ (g\ (f\ (x)))\ \forall x \in X$.

Hence $(ho\ (gof)) = ((hog)\ of)$.

Theorem: Let $f: X \rightarrow Y$ be a bijection, prove that f^{-1} o $f = I_X$ and $fof^{-1} = I_Y$.

Solution: Let $f(x) = y, x \in X$. Then $x = f^{-1}(y)$. Now f^{-1} o $f : X \rightarrow X$, given by

$(f^{-1}of)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_X(x)$.

Therefore $(f^{-1}\ of)(x) = I_X(x)\ \forall x \in X$.

Hence $f^{-1}\ of = I_X$. The other part can similarly be proved.

Example: Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be both bijection.

Prove that $go\ f$ is bijection and $(go\ f)^{-1} = f^{-1}\ og^{-1}$ (reversal rule)

Solution: If $x_1, x_2 \in X$, then $(go\ f)(x_1) = (go\ f)(x_2)$

$\Rightarrow g(f(x_1) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ (since g is injective) $\Rightarrow x_1 = x_2$ (since f is injective).

Hence $go\,f$ is injective.

Now we prove $go\,f$ is surjective.

Since gof: $X \rightarrow Z$. and $(gof)(X) = g(f(X)) = g(Y)$

(since f $(X) = Y$, f being surjective) $= Z$ (since $g(Y) = Z$, g being surjective)

Therefore, gof is sujrective. Hence gof is bijective.

Now both $(gof)^{-1}$ and $f^{-1} og^{-1}$ map from $Z \rightarrow X$.

Let $(gof)^{-1}(z) = x$ where $z \in Z$, $x \in z$. Then $(gof)(x) = z$.

Let $f(x) = y$. Then $g(y) = z$. Now $(f^{-1} og^{-1})(z) = f^{-1}(g^{-1}(z))$

$= f^{-1}(y)$[since $g(y) = z \Rightarrow g^{-1}(z) = y$]$= x$. [since $f(x) = y \Rightarrow f^{-1}(y) = x$].

Hence $(gof)^{-1}(z) = (f^{-1} og^{-1})(z)\ \forall z \in Z$. Consequently, $(gof)^{-1} = f^{-1} og^{-1}$.

Check your progress

(2.1) Let f: $X \rightarrow Y$ and g: $Y \rightarrow X$ be both bijection such that $gof = I_X$

and $fog = I_Y$. Prove that $g = f^{-1}$ and $f = g^{-1}$. Also prove that

(i) gof injective $\Rightarrow f$ is injective and

(ii) gof surjective $\Rightarrow g$ is surjective.

(2.2) Let Z be the set of integers. Define f: $Z \rightarrow Z \times Z$ by

$f(m) = (m - 1, 1)$, $m \in Z$.

g: $Z \times Z \rightarrow Z$ by $g(m, n) = m + n$, $m, n \in Z$.

Prove that $gof = I_Z$. Discuss the mapping fog.

3.8. Different useful maps

(1) **A constant map:** The simplest example of a map is a constant map. A constant map sends all the elements of the domain to just one element of the co-domain. For example, lef f: R $\rightarrow$ R be defined by f(x) = 1. $\forall$ x $\in$R. The graph of is as shows in fig. it is the y = 1

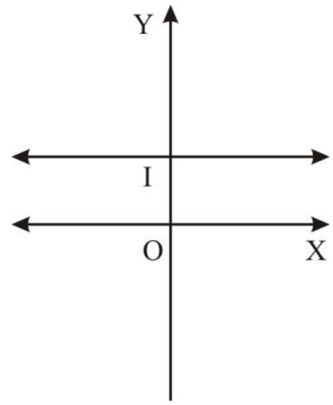


Fig.2

In general, the graph of a constant map f: R $\rightarrow$ R be defined by f(x) = c. $\forall$ x $\in$R is straight line which is parallel to the x-axis at a distance of |c| units from it.

(2) **Absolute Value function**: Another interesting function is the absolute value map (or modulus function) which can be defined by using the concept of the absolute value of a real number as : $f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

The graph of this map is shown in fig. 3. It consists of two rays, both starting at the origin and making angles $\pi/4$ and $3\pi/4$, respectively, with the positive direction of the x-axis.

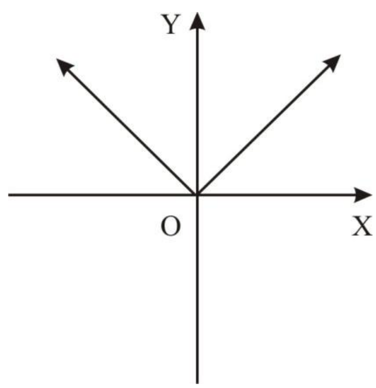


Fig. 3

(3) **The identity Map:** Another important example of a map is a map which sends every element of the domain to itself. Let X be any non empty set, and let f be the map on X defined by setting $f(x) = x \;\; \forall\, x \in X$. this map is known as the identity map on X and is denoted by $\mathbf{i_x}$. The graph of i_R is the line y = x.

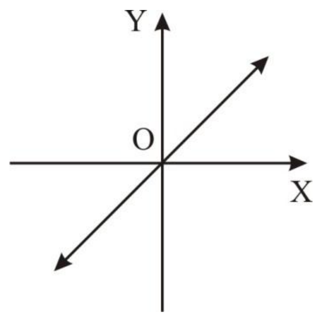


Fig. 4

(4) T**he Exponential Map:** If a is a positive real number other than 1, we can define a map f as f: $R \to R$, $f(x) = a^x$ $\{a > 0, a \neq\}$. This map is known as the exponential map. A special case of this map, where a = e, is often found useful. Fig. 5 shows the graph of the map f: $R \to R$ such that $f(x) = e^x$. This map is also called the natural exponential map. Its range is the set R^+ of positive real numbers.

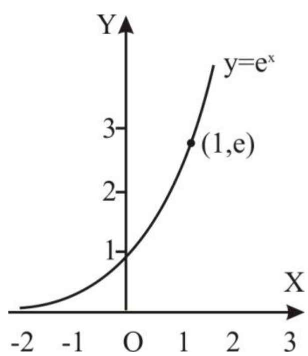


Fig.5

(5) **The natural logarithmic Map:** This map is defined on the set R^+ of positive real numbers, with f: $R^+ \to R$ such that f(x) = 1n (x). The range of this map is R. Its graph is shown in Fig. 6.

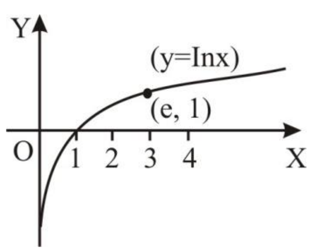


Fig. 6

(6) **The Greatest Integer Map:** Take a real number x. Either it is an integer, say n (so that x = n) or it is not an integer. If it is not an integer, we can find (by the Archimedean property of real numbers) an integer n, such that $n < x < n + 1$. Therefore, for each real number x we can find an integer n, such that $n \leq x < n + 1$. Further, for a given real number x, we can find only one such integer n. We say that n is the greatest integer not exceeding x, and denote it by [x].

For example, [3] = 3 and [3.5] = 3, [- 3.5] = -4. Let us consider the function the map defined on R by setting f(x) = [x]. This map is called the greatest integer map. The graph of the map is a shown in Fig. 7. Notice that the graph consists of infinitely many line segments of unit length, all parallel to the x-axis.

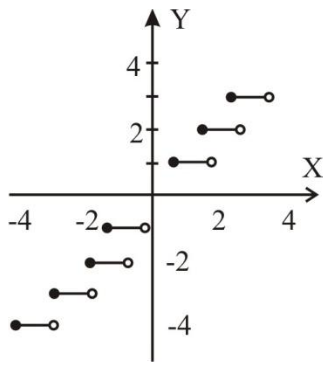


Fig.7

(7) **Other Maps:** The following are some important classes of functions.

a. Polynomial Maps: $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots a_n$ where $a_0, a_1, \ldots\ldots, a_0$ are given real numbers (constants) and n is a positive integer.

b. Rational Map: f(x) = g (x) / k(x), where g (x) and k(x) are polynomial function of degree n and m. this is defined for all real, for which $k(x) \neq 0$.

c. Trigonometric or Circular Map: f(x) = sin x, f(x) = cos x, f(x) = tan x, f() = cos x, f(x) = sec x, f(x) = cosec x.

d. Hyperbolic Map: $f(x) = \cosh x = \frac{(e^x + e^{-x})}{2}$, $f(x) = \sinh x = \frac{(e^x - e^{-x})}{2}$

Check your progress

(6) given below are the graphs of four map depending on the notion of absolute value. The map s are $x \to -|x|$, $x \to |x| + 1$, $x \to |x + 1|$, $x \to |x - 1|$, though not necessarily in this order. (The domain in each case is R). Can you identify them?

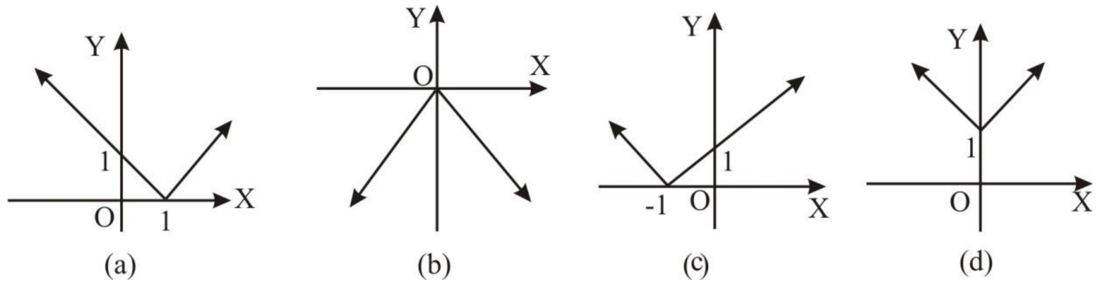


Fig.9

(7) Compare the graphs of 1n x and e^x given in Fig. 6 and 7 and verify that they are inverses of each other. If a given map is not one-one on its domain, we can choose a subset of the domain on which it is one-one, and then define its inverse map. For example, consider the map f(x) = x.

(8) Which of the following maps are one-one?

(a) f: R $\rightarrow$ R defined by f(x) = |x|
(b) f: R $\rightarrow$ R defined by f(x) = 3x – 1
(c) f: R $\rightarrow$ R defined by f(x) =x^2
(d) f: R $\rightarrow$ R defined by f(x) = 1

(9) Which of the following maps are onto?

(a) f : R $\rightarrow$ R defined by f(x) = 3x + 7
(b) f : R $\rightarrow$ R defined by f(x) = $\sqrt{x}$
(c) f : R $\rightarrow$ R defined by f(x) = x^2 + 1
(d) f : R $\rightarrow$ R defined by f(x) = 1/x

(10) Show that the map f: X $\rightarrow$ X such that f (x) = $\frac{x+1}{x-1}$, where X is the set of al real numbers expect 1, is one-one and onto. Find its inverse.

(11) Give one example of each of the following:

(a) a one-one map which is not onto.
(b) Onto map which is not one-one
(c) A map which is neither one-one nor onto.

3.9. Operation on Maps

1. Scalar Multiple of a Map

Consider the map f : x $\rightarrow$ $3x^2$+ 1 $\forall$ x $\in$ R. the map g : x $\rightarrow$ 2 ($3x^2$ + 1) $\forall$ x x$\in$ R is such that g (x) = 2f(x) $\forall$ x $\in$ R. We say that g = 2f, and that g is a scalar multiple of by 2. In the above example there is nothing special about the number 2. We could have taken any real number to construct a new map from f. Also, there is nothing special about the particular map that we have considered. We could as well have taken any other map. This suggests the following definition:

Let f be a map with domain D and let k be any real number. The scalar multiple of f by k is a map with domain D. It is denoted by kf and is defined by setting (kf) (x) = kf(x). Two special cases of the above definition are important.

(i) Given any map f, if k = 0, the map kf turns out to be the zero map. That is,0.0=0.

(ii) If k = - 1, the map kf is called the negative of f and is denoted simply by –f instead of the clumsy – f.

2. Absolute Value Map(or modulus map) of a given map

Let f be a map with domain D. The absolute value map of f, denoted by |f| and read as mod f is defined by setting (| f |) (x) = | f(x) |, for all x $\in$ D.

Since | f(x) | = f(x), if f(x) $\geq$ 0, find f and |f| have the same graph for those value of x for which f (x) $\geq$ 0.Now let us consider those values of x for which f(x) < 0.

Here | f(x) | = - f(x). Therefore, the graphs of f and |f| are reflections of each other w.r.t. the x-axis for those value of x for which f(x) < 0.

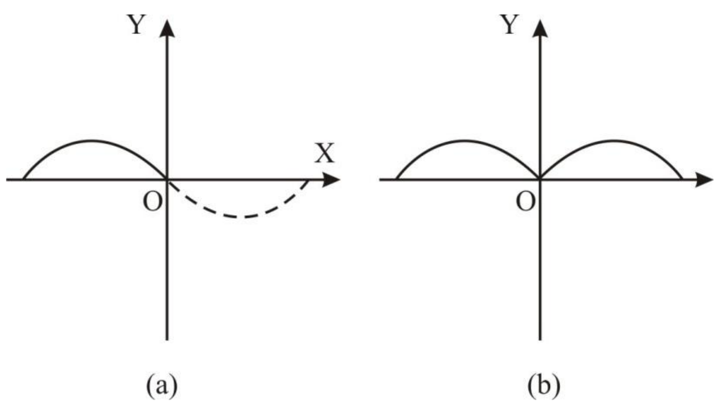


Fig 10

As an example, consider the graph in Fig. 10(a). The portion of the graph below the x-axis (that is the portion for which f(x) < 0) has been shown by a dotted line.

To draw the graph of |f| we retain the undotted portion in Fig. 10 (a) as it is and replace the dotted portion by its reflection w.r.t the x-axis (see fig. 10b)

3. Sum, difference, Product and Quotient of two Maps

If we are given two map s with a common domain, we can form several, new maps by applying the four fundamental operations of addition, substraction, multiplication and division on them.

(i) Define a map s on D by setting $s(x) = f(x) + g(x)$.
The maps is called the sum of the maps f and g, and is denoted by $f + g$. Thus, $(f + g)(x) = f(x) + g(x)$

(ii) Define a map d on D by setting $d(x) = f(x) - g(x)$.
The map d is the map obtained by subtracting g from f, and it denoted by $f - g$. Thus, for all $x \in D$. $(f - g)(x) = f(x) - g(x)$.

(iii) Define a map p on D by setting $p(x) = f(x)\, g(x)$.
The function p, called the product of the function f and g, is denoted by fg. Thus, for all $x \in D$. $(fg)(x) = f(x)\, g(x)$

(iv) Defined a function q on D by setting $q(x) = f(x)/g(x)$, provided $g(x) \neq 0$ for $x \in D$. The function q is called the quotient of f by g and is denoted by f/g. thus,$(f/g)(x) = f(x)/g(x)$ $(g(x) \neq 0$ for any $x \in D)$.

Remark: In case $g(x) = 0$ for some $x \in D$. We can consider the set, say D, of all those values of x for which $g(x) \neq 0$, and define f/g on D by setting $(f/g)(x) = f(x)/g(x)\ \forall x \in D$.

Example: Consider the function $f(x) = x^2$ and $g(x) = x^3$. Then the function f + g, f – g, fg are defined as $(f + g)(x) = x^2 + x^3$,

$(f - g)(x) = x^2 - x^3$ and $(fg)(x) = x^5$

Now, $g(x) = 0 \Leftrightarrow x^3 = 0 \Leftrightarrow x = 0$. Therefore, in order to define the function f/g, we shall consider only non-zero values of x, If $x \neq 0$, $f(x)/g(x) = x^2/x^3=1/x$. Therefore f/g is the function. $f/g(x) = 1/x$ whenever $x \neq 0$.

All the operations defined on functions till now, were similar to the corresponding operatios on real numbers. In the next subsection we are going to introduce an operation which has no parallel in R. Composite functions play a very important role in calculus.

(a) Even Function: A function f, defined on R is even, if for each $x \in R$, $f(-x) = f(x)$. **A** function f defined on R by $f(x) = x^2 \; \forall \; x \in R$. Notice that $f(-x) = (-x)^2 = x^2 = f(x) \; \forall \; x \in R$. This is an example of an even function. We find that the graph (a parabola) is symmetrical about the y-axis. If we fold the paper along the y-axis, we shall see that the parts of the graph on both sides of the y-axis completely coincide with each other. Such functions are called even functions.

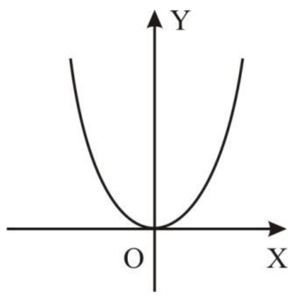


Fig. 12

(b). Odd Map: A map f defined on R is said to be an odd map if

$f(–x) = -f(x) \; \forall \; x \in R$. Now let us consider the map f defined by $f(x) = x^3 \; \forall \; x \in R$. We observe that $f(–x) = (– x) = (–x)^3 = -x^3 = -f(x) \; \forall \; x \in R$. If we consider another map g given by g(x) = sinx we shall be able to note again the g(– x) = sin x = – g(x). Such maps are called odd maps. The graph of an odd map is symmetric with respect to the origin. In other words, if we turn the graph of an odd map through 180^o about the origin we will find that we get the original graph again. Conversely, if the graph of a map is symmetric with respect to the origin, the map must be an odd map. The above facts are often useful while handling odd maps.

Check your progress

(12) Given below are two examples of even maps, along with their graphs. Try to convince yourself, by calculations as well as by looking at the graphs, that both the maps are, indeed, even maps.

The absolute value map on R f (x)= |x|.The graph is shown alongside.

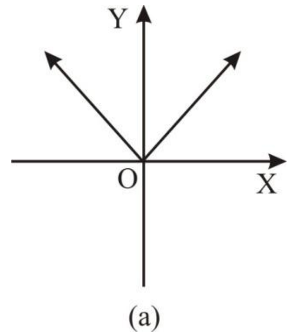
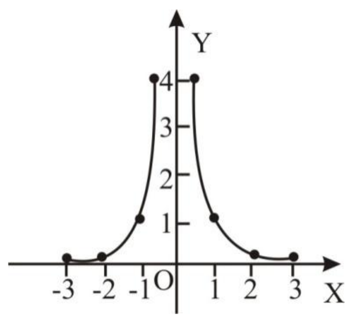


Fig.13(a)

The map g defined on the set on non-zero real numbers by setting $g(x) = 1/x^2$, $x \neq 0$.The graph of g is shown alongside



(Fig. 13b)

(13) There are two maps along with their graphs. By calculation as well as by looking at the graphs, find out for each whether it is even or odd.

(a) The identity map on R ; $f(x) = x$

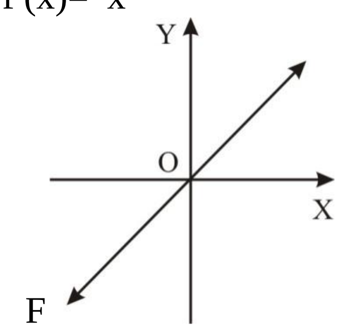


fig.14 (a)

(b) The map g defined on the set of non-zero real numbers of setting $g(x) = 1/x$, $x \neq 0$

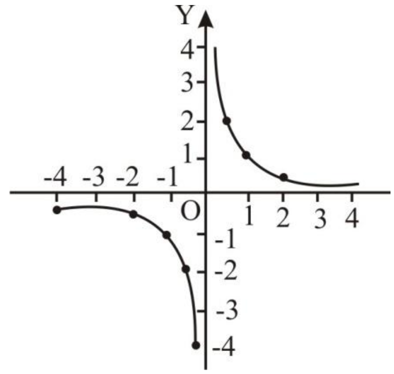


Fig. 14(b)

(14) Which of the following maps are even, which are odd, and which are neither even nor odd?

(a) $x \to x^2 + 1, \forall x \in R$
(b) $x \to x^3 + 1, \forall x \in R$
(c) $x \to \cos x + 1, \forall x \in R$
(d) $x \to x|x|, + 1, \forall x \in R$

$$(e)\, f(x) = \begin{cases} 0 \ if \ \ x \ is \ rational \\ 1, \ if \ \ x \ is \ irrational \end{cases}$$

Note: There are many map which are neither even nor odd.

Consider, for example, the map f (x) = $(x + 1)^2$. Here $f(-x) = (-x + 1)^2$. Is $f(x) = f(-x) \ \forall x \in R$? Here f is not an even map. Is $f(x) = -f(-x) \ \forall x \in R$? f is not an odd map.

3.10. Monotonic Maps

We consider two types of maps: (i) Increasing and (ii) Decreasing

Any map which conforms to any one of these types is called a monotone map. Does the profit of a company increase with production? Does the volume of gas decrease with increase in pressure? Problems like these require the use of increasing or decreasing maps. Now let us see what we mean by an increasing map. Consider the map g defined by $g(x) = \begin{cases} -x, \ if \ \ x \leq 0 \\ 1, \ if \ \ x > 0 \end{cases}$

Note that whenever $x_2 > x_1$, implies $g(x_2) > g(x_1)$.

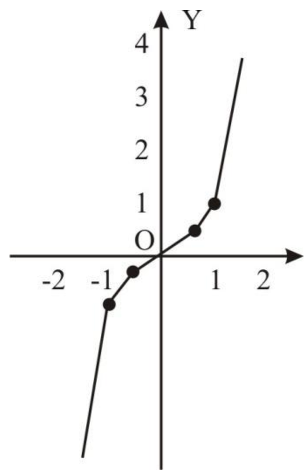


Fig. 15

In other words, as x increases, g(x) also increases. In this case we see that if $x_2 > x_1$, Equivalently, we can say that g(x) increase (or does not decreases) as x increases. Map like g is called increasing or non-decreasing map.

Thus, a map f defined on a domain D is said to be **increasing (or non-decreasing**) if, for every pair of elements $x_1, x_2 \in D$, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$. Further, we say that f is strictly increasing if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).

Clearly, the map g (x) = x^3 discussed above, is a strictly increasing map We shall now study another concept which is, in some sense, complementary to that of an increasing map. Consider the map f_1 defined on R by setting.

$$f_1(x) = \begin{cases} 1, if\ x \leq -1 \\ -x, if\ -1 < x < 1 \\ -1, if\ x \geq 1 \end{cases}$$ The graph of f_1 is as shown in

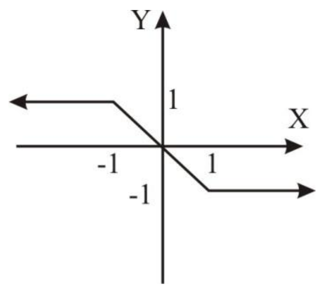


Fig.(16)

From the graph we can easily **s**ee that as x increases f_1 does not increase.

That is, $x_2 > x_1 \Rightarrow f_1(x_2) \leq f_1(x_1)$. Now consider the map $f_2(x) = -x^3$ $(x \in R)$

The graph of f_2 is shown in

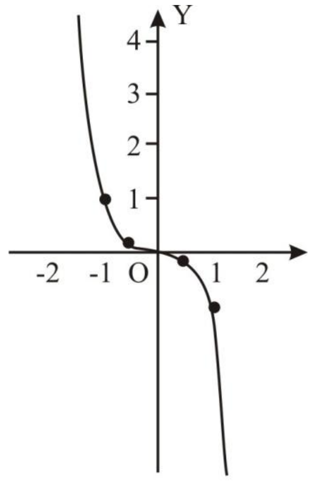


fig. (17).

Since $x_2 > x_1 \Rightarrow -x_2^3 > x_2^3 \Rightarrow f_2(x_2) < f_2(x_1)$, we find that as x increases, $f_2(x)$ decreases. Maps like f_1 and f_2 are called decreasing or non-increasing maps. A map f defined on a domain D is said to be **decreasing (or non-increasing)** if for every pair of elements x_1, x_2, $x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$. Further, f is said to be strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

We have seen that, f_2 is strictly decreasing, while f_1 is not strictly decreasing.

A map f defined on a domain D is said to be a **monotone** map if it is either increasing or decreasing on D. The maps (g, f_1, f_2) discussed above are monotone maps. The word 'monotonically increasing' and 'monotonically decreasing' are used for 'increasing' and 'decreasing', respectively.

There are many other maps which are not monotonic. $f(x) = x^2$ ($x \in R$).

This map is neither increasing nor decreasing in R. If we find that a given map is not monotonic, we can still determine some subsets of the domain on which the map is increasing or decreasing. The map $f(x) = x^2$ is strictly decreasing on $]-\infty, 0]$ and is strictly increasing in $[0, \infty[$.

3.11. Periodic Maps

A map f defined on a domain D is said to be a **periodic** map if there exists a positive real number p such that $f(x+p) = f(x)$ for all $x \in D$. The smallest positive integer p with the property described above is called the period of f.

Periodic maps occur very frequently in application of mathematics to various branches of science. Many phenomena in nature such as propagation of water waves, sound waves, light waves, electromagnetic waves etc. are periodic and we need periodic maps to describe them. Similarly, weather conditions and prices can also be described in the term of periodic maps. We must have a similar situation occurs in the graphs of periodic maps. Look at the graphs in Fig. 18.

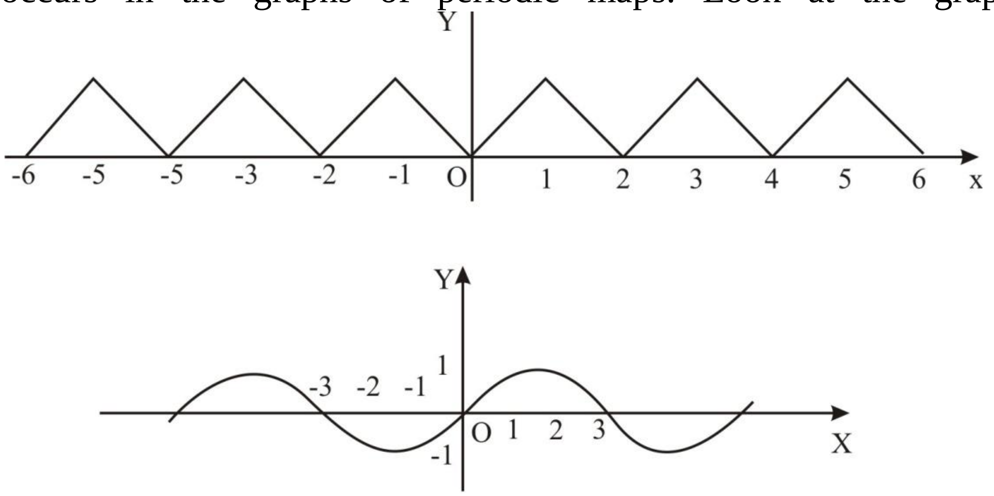


Fig. 18

In each of figures shown above the graph consists of a certain pattern repeated infinitely many times. Both these graphs represent periodic maps. Consider the graph in Fig. 18(a). Te partition of the graph lying between x = -1 and x = 1 is the graph of the map $x \to |x|$ on the domain $-1 \leq x \leq 1$.

This portion is being repeated both to the left as well as to the right, by translating (pushing) the graph through two units along the x-axis. That is to say, if x is any point of [-1, 1], then the ordinates at x, $x \pm 2$, $x \pm 4$, $x \pm 6$, ……. are all equal. The map f defined by $f(x) = |x|$, if $-1 \leq x \leq 1$

The graph in Fig. 18(b) is the graph of the since map, $x \to \sin x$, $\forall\, x \in R$. You will notice that the portion of the graph between 0 and 2π is repeated both to the right and to the left. You now already that $\sin(x + 2\pi) = \sin x$, $\forall\, x \in R$.

We know, $\tan(x + n\pi) = \tan x\ \forall\, n \in N$. This means that $n\pi$, $n \in N$ are all periods of the tangent map. The smallest of $n\pi$, is the period of the tangent map.

Remark: Monotoncity and periodicity are two properties of maps which cannot exist together, a monotonic map can never be periodic, and a periodic map can never be monotonic.

In general, it may not be easy to decide whether a given map is periodic or not. But sometimes it can be done in a straight forward manner. Suppose we want to find whether the map f: $x \to x^2\ \forall\ x \in R$ is periodic or not. We start by assuming that it is periodic with period p.Then we must have p > 0 and $f(x + p) = f(x)\ \forall x \Rightarrow (x + p)^2 = x^2\ \forall x \Rightarrow 2xp + p^2 = 0\ \forall x \Rightarrow p(2x + p) = 0\ \forall x$

Considering $x \neq -p/2$ we find that $2x + p \neq 0$. Thus, p = 0. This is a contradiction.

Therefore, there does not exist a positive number p such that $f(x + p) = f(x), \forall x \in R$ and, consequently, f is not periodic. As another example of a periodic map, consider the map f defined on R by setting f(x) = x – [x]. Let us recall that [x] stands for the greatest integer not exceeding x. The graph of this map is as shown in Fig. 19. From the graph we can easily see that $f(x+n) = f(x)\ \forall\ x \in R$. and for each positive integer n.

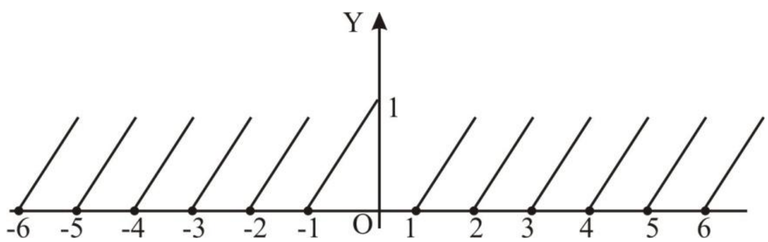


Fig.19

The given map is therefore periodic, the numbers 1, 2, 3 4 being all periods. The smallest of these, namely 1, is the period. Thus the given map is periodic and has the period 1.

Check your progress

(15) Given below are the graphs of some maps. Classify them as non-decreasing, strictly decreasing, neither increasing nor decreasing.

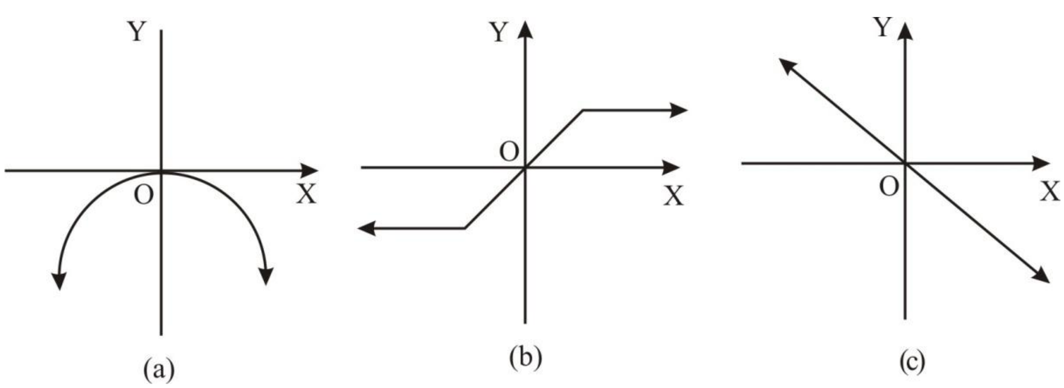


Fig. 20

(16) (a) What is period each of the maps given in Fig. 17(a) and (b)?

(b) Can you give one other period of each of these maps?

(17) Examine whether the following maps are periodic or not. Write the periods of the periodic maps.

(a) $x \to \cos x$ (b) $x \to x + 2$

(c) $x \to \sin 2x$ (d) $x \to \tan 3x$

(e) $x \to \cos (2x + 5)$ (f) $x \to \sin x + \sin 2x$

(18) The graphs of three map are given below: classify the function as periodic and non-periodic. Is sum of two periodic maps, a periodic map?

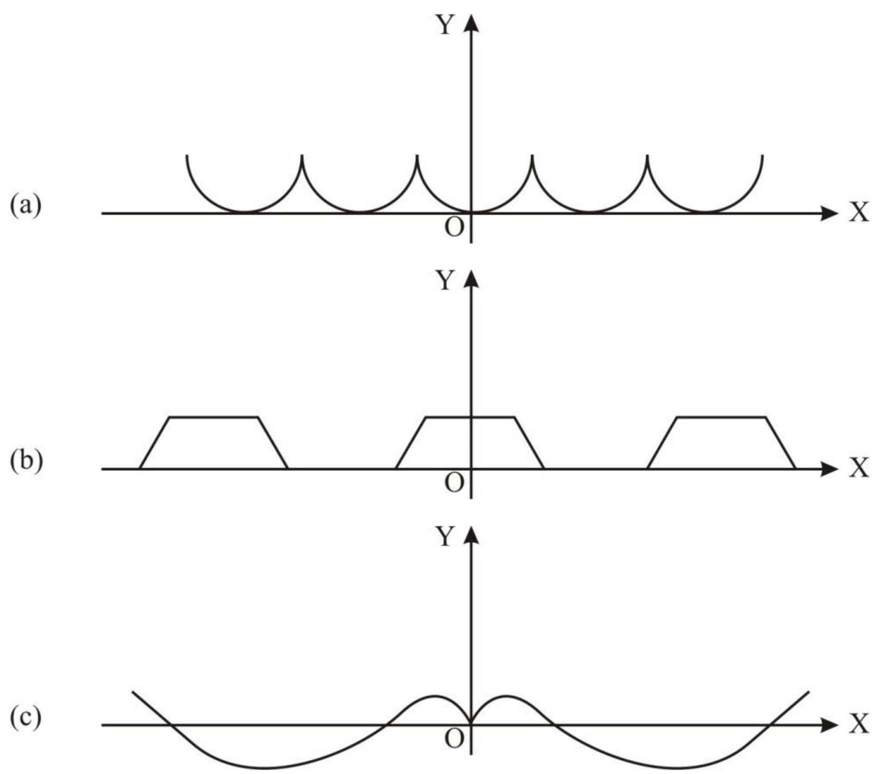


Fig. (21).

3.12. Summary:

After reading this unit we should be able to describe a map in its different forms, derive other properties with the help of the basic ones, define a map and examine whether a given map is one –one/onto and its related concepts, monotonic and periodic maps.

3.13. Terminal Questions

1. $Let\ A = \{n\pi : n\ is\ an\ integer\} and\ R\ be\ the\ set\ of\ real\ numbers.$
2. $Let\ f: X \to Y\ be\ a\ map\ and\ let\ A\ and\ B\ subsets\ of\ X, then\ A \subseteq B \Rightarrow f(A) \subseteq f(B)$
3. $When\ f(A) \cap f(B) \not\subset f(A \cap B) Considermap\ f: R \to R\ given\ by\ f(x) = x^2, It\ is\ clear\ f\ is\ not\ injective.$
4. $Let\ f: X \to Y\ be\ a\ map\ and\ let\ A\ and\ B\ be\ subsets\ of\ Y. Then\ A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$
5. $Prove\ that\ f: X \to Y\ is\ injective\ iff \quad f^{-1}(\{y\}) = (\{x\}) \forall y \in f(X), x \in X$
6. $Prove\ that\ f: X \to Y\ is\ surjective\ iff \quad f^{-1}(B) \neq \emptyset\ where\ B \subseteq Y\ and\ B \neq \emptyset$
7. $Let\ X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y = [-1,1]$

   $Let\ f: X \to Y\ given\ by\ f(x) = sinx, x \in X.$

Uttar Pradesh

RajarshiTandon
Open University

Bachelor Of Science

SBSMM - 03

Elementary Analysis

Block

2 Real Number System & Division in z

Unit -4
REAL NUMBER SYSTEM

Unit -5
Division in Integers

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BLOCK-2

Real number system and Division in Integers

First unit of this block is the basic part of calculus. We thought it would be a good idea to acquaint with some basic results about the real number system and related concepts, before start our study of calculus, perhaps we are already familiar with these results. In this section of this unit, we shall present some results about the real number system.

In the second unit of this block we thought it would be a good idea to acquaint the reader with some basic results about the set of integers. A quick look through the pages will help us in refreshing our memory. We shall illustrate with a number of examples also.

Unit –IV

REAL NUMBER SYSTEM

Structure

4.1. INTRODUCTION

This is the first unit of the course on calculus. We thought it would be a good idea to acquaint with some basic results about the real number system and related concepts, before start our study of Calculus, perhaps we are already familiar with these results. In this section of this unit, we shall present some results about the real number system.

4.2. Objectives

After reading this unit you should be able to:

- recall the basic properties of real numbers
- derive other properties with the help of the basic ones
- Derive Archimedian property
- Derive Rational density theorem
- Define absolute value of a number
- identify various types of bounded and unbounded intervals

4.3. Basic Properties of R

The real number system is the foundation on which a large part of mathematics, including calculus, rests. Thus, before we actually start learning calculus, it is necessary to understand the structure of the real number system.

We are already familiar with the operations of addition, subtraction, multiplication and division of real numbers, and also with inequalities. Here we shall quickly recall some of their properties. We start with the operation of addition.

A1: **R is closed under addition:** If x and y are real number, then x + y is unique real number.

A2: **Addition is associative:** x + (y + z) = (x + y) + z holds for all x, y, z in R.

A3: **Zero exists:** There is real number 0 such that x + 0 = 0 + x = x for all x in R.

A4: **Negatives exist:** For each real number x, there exists a real number y (called a negative o an additive inverse of x, and denoted by – x) such that

x + y + x = 0.

A5: **Addition Commutative:** x + y = y + x holds for all x, y in R. Similar to these properties of addition, we can also list some properties of the operation of multiplication.

M1: **R is closed under multiplication:** If x and y are real numbers, then x, y is a unique real number.

M2: **Multiplication is associative:** x (y, z) = (x, y), z holds for all x, y, z in R.

M 3: **Unit element exists:** There exists a real number 1 such that x.1 = 1.x = x for every x in R.

M4: **Inverse exists:** For each real number x other than 0, there exists a real number y (called a multiplicative inverse of x and denoted by x^{-1} or by 1/x) such that x.y=y.x=1

M 5: **Multiplication is commutative:** x.y = y.x holds for all x,y in R. The next property involves addition as well as multiplication.

D: **Multiplication is distributive over addition:** x.(y + z) = x.y + x.z holds for all x, y, z in R.

Remark(a): The fact that the above eleven properties are satisfied is often expressed by say that the real numbers form a field with respect to the usual addition and multiplication operations.

Remark (b): Usually the operator '.' Is dropped in expression, i.e $x.y$ may be denoted as xy. In addition to the above mentioned properties, we can also define an order relation on R with the help of which we can compare any two real numbers. We write x > y to mean that x is greater than y. The order relation '>' has the following properties:

Order axiom:

01 **Law of Trichotomy holds:** For any two real numbers a, b, one and only one of the following holds: a **> b, or a = b, or b > a.**

02 **Transitivity**: '>' is **transitive:** If a > b and b > c, then a > c, ∀ a, b, c ∈ R

03 **Addition is monotone:** If a, b, c in R are such that a > b, then a + c > b + c.

04 **Multiplication is monotone in the following sense:** If a, b, c in R are such that a > b and c > 0, then ac > bc.

Caution : a > b and c < 0 ⇒ac < bc.

Example: (i). $(\mathbb{R}, +, .)$ is an ordered field.

(ii). $(\mathbb{Q}, +, .)$ is an ordered field.

(iii). $(\mathbb{C}, +, .)$ is not an ordered field.

Note: That if $z_1 \& z_2$ are complex number then$z_1 > z_2$ or $z_1 < z_2$ is not defined.

4.4. Archimedian Property(Principle): If x and y are any two real numbers with $x > 0$, then there exists a positive integer n such that $nx >$ y .

Proof: We prove this by contradiction. If it possible then suppose that $nx \leq y$ for all $n \in N$. We define a set $S = \{nx : n \in N\}$.

The set is non empty and bounded above. Therefore, by completeness axiom the set S must have its supremum. Let sup(S)= α, then $nx \leq \alpha \, \forall \, n \in N$, or, $(n + 1)x \leq \alpha \, \forall \, n \in N$, or $nx \leq \alpha - x \, \forall \, n \in N$.

This shows that $(\alpha - x)$ is less than α. This contradicts the assumption that α is supremum of S. Hence, $nx \nless y$. So, $nx >$ y.

Corollary1: For every real number y, there exists a natural number n such that $n > y$.

Proof: By Archimedian Property we have $nx >$ y Put $x = 1$ then we get $n.1 >$ y or, $n >$ y

Corollary2: For every real number $y \neq 0$, there exists a natural number n such that $1|n < y$.

Proof: Since $y \neq 0$ be a real number then $1|y$ is also a real number. So, $n > 1|y$ Multiplying both sides by $n^{-1}y$, we have

$n(n^{-1}y) > (n^{-1}y)1|y,$

Or, $(nn^{-1})y > n^{-1}(y.1|y),$ or, $y > n^{-1}$ i.e. $1|n < y$.

Corollary3: Let $x\epsilon\mathbb{R}$, then there exists an integer K such that $(x - 1) \leq K < x$.

Proof: By Archimedian Property we have $n > (-1)$x $= -x$ or, $-n < x$. Put $-n = m$ then $m < x$, So, $m < x < n$.

Let $K = max\{m, m + 1, m + 2, \ldots\ldots, n\}$ such that $K \leq x$ then $x - 1 \leq K$.

Hence by combining these above statements we have $(x - 1) \leq K < x$.

Corollary4: For any real number x, there exists one and only one integer n such that $n \leq x < n + 1$.

Proof: From the corollary(3) we have two integers m and n such that $m < x < n$.

Let be two integers such that $m_1 < x < m_2$.

Let $n = max\{m_1, m_1 + 1, m_1 + 2, \ldots\ldots, m_2\}$ such that $n \leq x$ so, $n + 1 > x$

Hence, there exists an unique integer n such that $n \leq x < n + 1$.

4.5. Rational Density theorem: Between any two different real numbers, there exists at least one rational number.

Proof: Let a and b be two real numbers with $a < b$, $b - a > 0$.

By corollary (2), $1|n < y$ so that $1|n < b - a,$ put $y = b - a > 0$

Or, $a < b - 1|n$., again there exists an integer K such that $x - 1 \leq K < x$

Put $nb = x,$ then $nb - 1 \le K < nb,$ or $b - 1|n \le K|n < b,$ or $a < K|n < b,$ since, ($a = b - 1|n$) or, $a < r < b; r = K|n$

Remark: Any field together with a relation > satisfying order axioms 01 to 04 is called an ordered field. Thus R with the usual > is an example of an ordered field.

Notations: We write x < y (and read x is less than y) to mean y > x. We write x ≤ (and read x less than or equal to y) to mean either x < y or x = y. We write x ≥ y (and read x is greater than or equal to y) if either x > y or x = y.

A number x is said to be positive or negative according as x > 0 or x < 0. If x ≥ 0, we say that x non-negative.

Now, we know that given any number x ∈ R. We can always find a number y ∈ R such that y ≥ x. (in fact, there are infinitely many such real numbers y). Let us see what happens when we take any sub set of R instead of a single real number x. Do you think that, given a set S ⊆ R, it is possible to find u ∈ R such that u ≥ x for all x ∈ S? Discuss the special case when S is empty.

Definition: Let S be a subst of R. An element u in R is said to be an upper bound of S if u ≥ x holds for every x in S. We say that S is bounded above, if there is an upper bound of S. Now we can reword our earlier questions as follows : Is it possible to find an upper bound for a given set?

Let us consider the set Z^{-} ={ –1, –2, –3, –4, …..}

Now, each x ∈ Z^{-} is negative. Or, in other words, x < 0 for all x ∈ Z. It is easily seen that, in this case, we are able to find an upper bound, namely zero, for our set Z^{-}.

On the other hand, if we consider the set of natural numbers, N = {1, 2, 3…..}, obviously we will not be to find an upper bound. Thus N is not bounded above.

We will, of course, realize that if u is an upper bound for a set S then u + 1, u + 2, u + 3…., (in fact, u + r, where r is any positive number) are all upper bounds of S. For example, we have seen that 0 is an upper bound for Z^{-}. Check that 1,2,3,8,…. Are all upper bounds of Z. From among all the upper bounds of a set, which is bounded above, we can choose an upper bound u such that u is less than or equal to every upper bound of S. It is easily seen that, if such a u exists, then it is unique. We call this u the least upper bound or the supremum of S. For example, consider the set T = { x ∈ R : x^2 ≤ 4 } = {x∈ R : –2 ≤ 2 }

Now 2, 3, 4, 5, 4, 4 + π are all upper bounds for this set. But we will see that 2 is less than any other upper bound. Hence 2 is the supremum or the least upper bound of T. We will agree that -1 is the l.u.b (least upper bound) of Z^-.

Note: Both the sets T and Z^-, the l.u.b. belongs to the set. This may not be true in general. Consider the set of all negative real number $R^- = \{x : x < 0\}$. The l.u.b. of this set 0. But $0 \notin R^-$. Working on similar lines we can also define a lower bound for a given set S to be a real number v such that $v \leq x$ for all $x \in S$. We shall say that a set is bounded below, if we can find a lower bound for it. Further, form among all the lower bounds of a set S, which is bounded below, we can choose a lower bound v such that v is greater than or equal to every lower bound of S. It is easily seen that, if such a v exists, then it is unique. We call this v the greatest lower bound or the infimum of S.

Note: As in the case of l.u.b, remember that the g.l.b of a set may or may not belong to the set. We shall say that a set $S \subset R$ is bounded if it has both and upper bound and a lower bound. Based on this discussion you will be able to solve the following exercise.

4.6. Absolute Value

Definition: If x is a real number, its absolute value, denoted by |x| (read as modulus of x, or mod x), is defined by the following rules.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

For example, we get $|5| = 5$, $|-5| = 5$, $|1.7| = 1.7$, $|-2| = 2$, $|0| = 0$

It is obvious that |x| is defined for all $x \in R$. The following theorem gives some of the important properties of |x|.

Theorem: If x and y be any real numbes, then

a. $|x| = \max \{ -x, x\}$
b. $|x| = |-x|$
c. $|x|^2 = x^2 = |-x|^2$
d. $|x+y| \leq |x| + |y|$ (the triangle inequality)
e. $|x - y| \geq | |x| - |y||$

Proof.

a. By the law of trichotomy (01) applied to the real numbes x and 0, exactly one of the following holds : (i) $x > 0$, (ii) $x = 0$ or (iii) $x < 0$

Let us consider these one by one.

(i) If $x > 0$, then $|x| = x$ and $x > -x$, so that Max $\{ -x, x\} = x$ and hence $|x| = \max \{-x, x\}$

(ii) If $x = 0$, then $x = 0 = -x$, and therefore, Max $\{ -, x \} = 0$, Also $|x| = 0$, so that $|x| = \max \{-x, x\}$.

(iii) If $x < 0$, then $|x| = -x$, and $-x > x$, so that Max $\{-, x\} = -x$. Thus, again $|x| = \max \{-x, x|$ From this it follows that $x \leq |x|$

b. $|-x| = \max \{ -(-x), = \max \{x, -x\} = \max \{-x, x\} = |x|$.

c. If $x \geq 0$, then $|x| = x$, so that $|x|^2 = x^2$. If $x < 0$, then $|x| = -x$, so that $|x|^2 = (-x)^2 = x^2$ .Therefore, for all $x \in R$, $|x|^2 = x^2$. Also $|-x|^2 = |x|^2$, because $|-x| = |x|$ by (b). Thus, we have $|x|^2 = x^2$

d. We shall consider two different cases according as (i) $x + y \geq 0$ or (ii) $x + y < 0$. Let $x + y \geq 0$. Then $|x + y| = x + y$. Now $x \leq |x|$ and $y \leq |y|$ by (a).

Therefore $|x + y| = x + y \leq |x| + |y| = |x + y| = x + y \leq |x| + |y|$

Let $x + y < 0$. Then $-(x + y) > 0$, that is, $(-x) + (-) > 0$ and we can use the result of (i) for $-x$ and $-y$. Now $|x+y| = |-(x + y)|$ by (b)

Thus $|x + y| = (-x) + (-y) + (-y) | \leq| -x | + -y|$, by (i) $= |x| + |y|$, by (b) Therefore, we get $|x + y| \leq |x| + |y|$. Thus we find that for all $x, y \in R$, $|x + y| \leq x | + | y |$.

e. By writing $x = (x - y) + y$ and applying the triangle inequality to the numbers $x - y$ and y, we have $|x| = | (x - y) + y | \leq | x - y| + | y |$,

so that $|x| - |y| \leq x -y|$. since (10 holds for all x and y in R,

Therefore, by interchanging x and y in (1) we have

$|y| - |x| \leq |y - x| = | - (x - y) |= | x - y|$. So that $-(|x| - | y |) \leq | x - y|$.

From (1) and (2) we find that $|x| - | y |$ and its negative $-(|x| - |y|)$ are bothe less than or at the most equal to $| x - y|$. Therefore, $\max \{|x| - |y|, - |y|)\} \leq |x - y|$. But the left hand side of the above inequality is simply $||x| - |y||$. Therefore, we have $||x| - | y || \leq x - y|$. That is, $|x - y| \geq ||x | - |y ||$ for all $x, y \in R$.

Now you should be able to prove some easy consequence of this theorem. The following exercise will also give you some practice in manipulating absolute values. This practice will come in handy when you study unit 2.

Check your progress

(1). Give example to illustrate the following:

a. A set of real numbers having a lower bound.
b. A set of real numbers without any lower bound,
c. A set of real numbers whose g.l.b. does not belongs to it

(2). prove the following:

a. $x = 0 \Leftrightarrow |x| = c$
b. $|xy| = |x| . |y|$
c. $| 1/x | = 1/|x|$, if $x \neq 0$
d. $|x - y| \leq |x| + |y|$
e. $|x + y + z| \leq | x | + |y| + | z|$
f. $|xyz| = |x| . |y| . |z|$

(e) and (f) can be extended to any number of reals. Now if $a \in R$ and $\delta > 0$, then $|x - a| < \delta \Rightarrow x - a < \delta$, and $-(x - a) < \delta$. $x - a < \delta$, this means that $x < a + \delta$

$-(x - a) < \delta$, this means that $a - \delta$ Thus, we get that $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$.

This means that the difference between x and a is not more than δ.

4.7. Intervals on the Real line

The real numbers in the set R can be put into one-to-one correspondence with the points on a straight line L. That is we shall associate a unique point on L to each real number and vice versa. Consider a straight line L. Mark a point O on it. We shall use the part to the left of O for representing negative real number and the part to the right of O for representing positive real numbers. We choose a point A on L which is to the right of O. OA can now serve as a unit. To each positive real number x we can associate exactly one point plying to the right of O on l, so that OP = |x| units. A negative real number y will be represented by a point Q lying to the left of O on the straight line L, so that OQ = |y| = - y units. We find that to each real number we can associate a point on the line. Also, each point S on the line represent unique real number z, such that |z| = OS. Further, z is positive if S is to the right of O, and z is negative if S is to the left of O.

This representation of real number by points on a straight line is often very useful. Because of this one-to-one correspondence between real number and the points of a straight line, we often call a real number “a point of R”. Similarly L is called a “number line”. Note that the absolute value or the modulus of any number x is nothing but is distance from the point O or the number line. In the same way,

| x – y| denotes the distance between the two numbers x and y (see. Fig. 1(b).

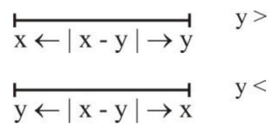


Figure 1 : (a) Number line (b) Distance between x and y is | x – y |

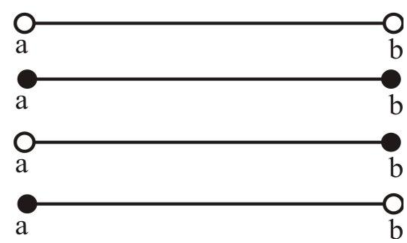
Now let us consider the set of the real numbers which is between two given real numbers a and b, where a ≤ b. Actually, there will be four different sets satisfying this loose condition. These are:

$(i)]a,b[=\{x: a < x < b\}$

$(ii)[a,b] = \{x: a \le x \le b\}$

$(iii)]a,b] = \{x: a < x \le b\}$

$(iv)[a,b[= \{x: a \le x < b\}$



Note: we also write$]a,b[= (a,b),]a,b[\&[a,b[= [a,b)$

The representation of each of these sets is given alongside. Each of these sets is called an interval, and a and b are called the end points of the interval. The interval [a, b[, in which the end points are not included, is called an open interval.

Note: In this case we have drawn a hollow circle around a and b to indicate that they are not included in the graph. The set [a, b] contains both its end points and is called a closed interval. In the representation of this closed interval, we have put thick black dots at a and b to indicate that they are included in the set.

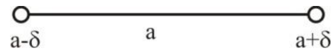
The sets [a, b [and] a, b] are called **half-open (or half – closed) intervals or semi-open (or semi closed) intervals,** as they contains only one end points. This fact is also indicated in their geometrical representation.

If a = b,]a, a [=] a, a] = [a, a[= ϕ and [a, a] =a.

Each of these intervals is bounded above by b and bounded below by a.

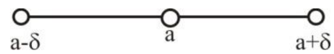
Can we represent the set I = {x: | x – a | < δ } on the number line? We know that | x – a| can be thought of as the distance between x and a. this means I is the set of all numbers x, whose distance from a is less than δ. Thus,

1 = { x : | x – a | < δ }



Is the open interval]a - δ, a + δ [. Similarly, I_1= {x: |x – a | ≤ δ } is the closed interval [a - δ, a + δ]. Sometimes we also come across sets like I_2= {x : 0 < | x – a | < δ}. The means if x ∈ I_2, then the distance between x and a is less than δ, but is not zero. We can also say that the distance between x and a is less than δ, but x ≠ a. Thus,

I_2 =]a - δ, a + δ [\ {a}



=] a - δ, a [∪] a, a + δ [.

Apart from the four types of intervals listed above, there are few more types. These are :

]a, ∞ [= { x : a < x} (open right ray)

[a, ∞ [= {x : a ≤ x} (closed right ray)

] - ∞, b [= {x : x < b} (open left ray)

] - ∞, b] = {x : x ≤ b} (closed left ray)

] - ∞, ∞ [= R (open interval)

As we can see easily, none of these sets are bounded. For instance,]a, ∞ [is bounded below, but is not bounded above,] - ∞, b] is bounded above, but is not bounded below. Note that ∞ and - ∞ does not denote a real number, it merely indicates that in interval extend without limits.

We note further that if S is any interval (bounded or umbounded) and if c and d are two elements of S, then all numbers lying between c and d are also elements of S.

4.8. Summary:

After reading this unit you should be able to recall the basic properties of real numbers, derive other properties with the help of the basic ones, derive Archimedian property, derive Rational density theorem. define absolute value of a number and to identify various types of bounded and unbounded intervals.

4.9. Terminal Questions

1. Prove That the following:

g. $|xy| = |x|.\ |y|$, if $x \neq 0$

h. $|x + y + z| \leq |x| + |y| + |z|$

2. $(\mathbb{R}, +, .)$ is an ordered field.

(ii). $(\mathbb{Q}, +, .)$ is an ordered field.

(iii). $(\mathbb{C}, +, .)$ is not an ordered field.

3. Show that the let $x \epsilon \mathbb{R}$, then there exists an integer K such that $(x - 1) \leq K < x$.

4. Give example to illustrate the following:

d. A set of real numbers having a lower bound.

e. A set of real numbers without any lower bound,

5. Show that

(a) $|n| \geq 0$ for every $n \in \mathbb{Z}$

(b) $|n| \geq n$ for every $n \in \mathbb{Z}$

Unit – V

Division in Integers

Structure

5.1. INTRODUCTION

We thought it would be a good idea to acquaint the reader with some basic results about the set of integers. A quick look through the pages will help us in refreshing our memory. We shall illustrate with a number of examples also.

5.2.Objectives

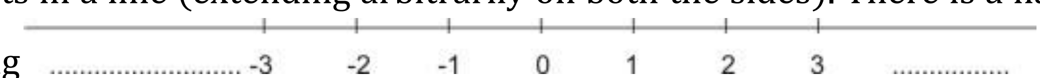
After reading this unit you should be able to:

- Recall the basic properties of set of integers
- Derive other properties with the help of the basic ones
- Describe a division algorithm and the Euclidean algorithm for the set of all integers.
- Define a Prime number and the concepts related to it
- Define the greatest common divisor and least common multiple and calculate them.
- State Fundamental Theorem of Arithmetic
- Define the least common multiple of two integers

5.3. Basic Properties of *Z*

1. Order and Inequalities: We start with the set of integers

$\mathbb{Z} = \{- - - - - - - - - - 2, -1, 0, 1, 2\ 3\ - - - - -\}$ as being given as a subset of the set $\mathbb{R}$ of all real numbers which may be identified with the set of all points in a line (extending arbitrarily on both the sides). There is a natural ordering



on the real numbers given by a < b (a is less than b) if a lies to the left of b.

$$\mathbb{N} = \{1,2,3\ - - - - -\} \subseteq \mathbb{Z} \subseteq \mathbb{Q} = \left\{\frac{m}{n} \mid m, n, \in \mathbb{Z}, n \neq 0\right\}$$

We know that sum and product of any two elements in $\mathbb{N}$ are again in $\mathbb{N}$, sum, difference and product of any two elements of $\mathbb{Z}$ are again in $\mathbb{Z}$ and a similar statement holds in $\mathbb{Q}$ and $\mathbb{R}$ also. In $\mathbb{Q}$ and $\mathbb{R}$, quotient of any element by any nonzero element is also there. This property does not always hold in $\mathbb{Z}$ (and $\mathbb{N}$) as may be easily seen (take 5 ad 3 for example). When it does, (say for 6 and 3) it is a special situation called divisibility.

The order relation $<$ and its inverse relation $>$ in $\mathbb{R}$ (consequently in $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$ too) satisfies:

(1) $0 < 1$.

(2) Given any a, b in $\mathbb{R}$, exactly one of $a < b, a = b, b < a$ holds.
(3) $a < b \Rightarrow a + c < b + c$ for every $c \in R$
(4) $a < b, o < c \Rightarrow a . c < b . c$
(5) $a < b, b < c \Rightarrow a < c.$

As remarked above, these properties hold in $\mathbb{Z}$ also.

Proposition:

(a) $a < b, c < o \Rightarrow b c < ac.$ in $\mathbb{Z}$
(b) for every $m \neq 0$ in $\mathbb{Z}$, $m^2 > 0$.
(c) $a + b = a + c \Rightarrow b = c$.
(d) $ab = ac, a \neq 0 \Rightarrow b = c$.
(e) $ab = 0 \Rightarrow a = 0$ or $b = 0$.

5.4. Absolute Value

With every integer n (respectively rational number, real number x) we associate an integer $|n|$ (respectively rational, or real $|x|$) as follows:

$$|n| = \begin{cases} n \text{ if } n \geq 0 (\text{i.e. if } n \text{ is nonegative}) \\ -n \text{ if } n < 0 \ (\text{i.e. if } n \text{ is negative}) \end{cases}$$

Properties:

(1) $|n| \geq 0$ for every $n \in \mathbb{Z}$
(2) $|n| \geq n$ for every $n \in \mathbb{Z}$
(3) $|n| = |-n|$ for every $n \in \mathbb{Z}$
(4) $|m + n| \leq |m| + |n|$ for every $m, n, \in \mathbb{Z}$.

Here $a \leq b$ means $a < b$ or $a = b$.

Example: for $m = 2, n = -3$.

$|m + n| = |-1| = 1 < 5 = 2 + 3 = |m| + |n|$.

But for $m = -6, n = -9,$

$$|m + n| = |-15| = -(-15) = 15 = 6 + 9 = |m| + |n|$$

$\therefore$ both possibilities $|m + n| < m| + |n| \; or \; |m + n| = |m| + |n|$ actually may occur for different sets of values for m and n.

Check your progress

1. Prove that $|m + n| = |m| + |n|$ occurs if and only if m and n have same sign (positive or negative) or one of them at least is zero and that $|m + n| < |m| + |n|$ if and only if they are of opposite signs.

2. Prove that $||a| - |b|| \leq |a - b|$ for any $a, b \in \mathbb{Z}, \mathbb{Q} \; or \; \mathbb{R}$.

5.5. Mathematical Induction

The principle of Mathematical induction is of great help in proving results involving a natural member for every n or for every $n \geq$ some positive integer m.

Principle of Mathematical Induction: If $P(n)$ is a statement involving a positive integer n for which

(1). $P(m)$ is true for some integer m.
(2). Truth of $P(l) \Rightarrow$ Truth of $P(l+1) \;\; \forall l \geq m$.

Then $P(n)$ is true for every $n \geq m$. The particular case of this result for $m = 1$ is usually referred to as the principle of mathematical induction and in fact the general version stated above can be obtained from this particular case. The above principle is popularly stated as if a statement holds for $n = 1$ and whenever it is true for $n = k$, it holds for $n = k + 1$, then it holds for all natural numbers n.

Example: $2^n > n^2$ for all $n \geq 5$. Clearly the statement does not hold for $n = 2, 3, 4$.

$2^5 = 32 > 25 = 5^2$ & so if holds for $n = 5$.

Take any $l \geq 5$ and assume that $2^l > l^2$.

Then $2^{l+1} = 2.2^l = 2^l + 2^l. > l^2 + l^2$ (by hypothesis)

$\geq l^2 + 5l \qquad (\because \; l \geq 5)$

$= l^2 + 2l + 3l \geq l^2 + 2l + 3\times5$ ($\because\ l \geq 5$)

$> l^2 + 2l + 1$ ($\because\ 15 > 1$)

$= (l + 1)^2$ i.e. $2^{l+1} > (l + 1)^2\ \ \forall l \geq 5$.

$\therefore$ Assumption of truth of the statement for $l \geq 5$ implies its truth for $l + 1$. $\therefore$ the above statement is true for every $l \geq 5$ by the above result.

Sometimes the statements $P(n)$ do not imply $P(n+1)$ and in this case the above principle cannot be used. For such situations we have the stronger.

Second Principle of Induction: If $P(n)$ is a statement involving a natural number n and Truth of $P(l)\ \forall l < m \Rightarrow$ Truth of $P(m)$, then the statement is true for all natural numbers n. We shall illustrate its uses later in this unit.The second principle of induction is a consequence of the well ordering property of the set $\mathbb{N}$ of natural number or of $\mathbb{N} \cup \{0\}$.

5.6. Well ordering property

Every non empty subset A of $\mathbb{N}$ (or of $\mathbb{N} \cup \{0\}$ has a least element i.e. there is an element $l \in A$ for which $l \leq a$ for every $a \in A$.

We are omitting the proof but the reader may satisfy himself by considering various subsets of $\mathbb{N}$ and obtain least elements of them.

This result does not hold for $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

Check your progress

1. Prove that $3^n > 2^n + 1$ for all $n \geq 2$.

2. Prove that $1^3+2^3+\ldots\ n^3= \dfrac{n^2(n+1)^2}{4}\ \forall\ n \geq 1$.

3. Prove that for any real number $x > -1$, $(1 + x)^n \geq (1 + nx)\ \forall n \geq 1$.

4. Prove that $n! > 2^n\ \forall n \geq 4$.

5. Prove that $n! > 4^n\ \forall n \geq 9$.

6. Let $a_1 = 1$ and $a_n = \sqrt{3a_{n-1}+1}\ \forall n \geq 2$. Prove that $a_n < \frac{7}{2}\ \forall$ integer $n \geq 1$.

7. Prove that $2^n > n^3$ for all $n \geq 10$

5.7. Division Algorithm in $\mathbb{Z}$

Recall that we call by the same name, 'division' two different kind of notions: real or rational division of one real or rational number, by another non zero such number to get again such a number: For $r, s \in \mathbb{R}$ or $\mathbb{Q}$ with $s \neq o, r \div s$ $(= r/s) \in \mathbb{R}$ or $\mathbb{Q}$ respectively. Similarly for $m, n \in \mathbb{Z}, n \neq 0$, we have $m/n \in \mathbb{Q}$. Also we have the notion of 'integer division' where in upon division of an integer 'a' by a non zero integer 'b' we get an integer q(quotient) and a remainder r which is non negative and numerically smaller than the dividend'b'.

Theorem: If a, b are integers with $b \neq 0$, we can find unique integers 'q' and 'r' such that $a = b\,q + r$, with $0 \leq r < |b|$

Proof: Consider the set

$S = \{a - b\,q \mid q \in \mathbb{Z}\} = \{\dots\dots\dots.., a - 2\,b\,, a - b\,, a, a + b, a + 2b, \dots\dots\dots\}$.

$Or, \{\dots\dots\dots\ a + 2b, a + b, a, a - b, a - 2b, \dots\dots\dots.\}$

This set must have some nonnegative elements because $b \neq 0$ and so by adding or subtracting sufficiently many multiples of b according as $b > 0$ or $b < 0$, we may prove that $S \cap (\mathbb{N} \cup \{0\}) \neq \Phi$. As it is a nonempty subset of $\mathbb{N} \cup \{0\}$, by the well ordering property of $(\mathbb{N} \cup \{0\})$, we can get a least element in $S \cap (\mathbb{N} \cup \{0\})$ say r, then $0 \leq r$ and $r = a - b\,q$ for some $q \in \mathbb{Z}$.

$\therefore\ a = b\,q + r$, Also $r < |b|$ for if $b < 0$ then $r \geq |b| \Rightarrow r \geq -b$. then $r_1 = r \quad + b < r$ and $r_1 = r + b \geq 0\ \therefore\ r_1 = r + b = a - b\,q + b = a - b(q - 1) \in S \cap (\mathbb{N} \cup \{0\})$ and $r_1 < r$ contradicting the fact that r is the least element of $S \cap (\mathbb{N} \cup \{0\})$.

Similarly if $b > 0$ and $r \geq |b| = b$ then $r_2 = r - b$ contradicts minimality of r in $(S \cap (\mathbb{N} \cup \{0\})$.

Exercise: Show that if $a = bq_1 + r_1 = bq_2 + r_2$ with $0 \leq r_1, r_2, < |b|$

then $q_1 = q_2$ and $r_1 = r_2$.

Hint: Use properties of absolute values.

Example:

1. For $a = -73, b = -10$
   $-73 = (-10)\,8 + 7$ $i.e.\ q = 8$ and $r = 7$.
2. For $a = 60, b = -8$
   $60 = (-8)(-7) + 4$ $\quad i.e. q = -7, r = 4$
3. For $a = -81, b = 7$
   $-81 = 7(-12) + 3$ $i.e. q = -12, r = 3$
4. For Observe that the above procedure is same what is taught in schools for positive values of a and b. We term q as the quotient and r is the remainder, also note that

$a = bq + r$ in case $b > 0 \Rightarrow \frac{a}{b} = q + r/b$ where $0 \le \frac{r}{b} < 1$

$\therefore\ q = \left[\frac{a}{b}\right]$ where $[x]$ means the integral part of x i.e. the greatest integer $\le x$

$\therefore$ the integer quotient q is the integral part of $\frac{a}{b}$ in case $b > 0$.

Check your progress

10. Find q and r if

1. $a = 100, b = -13$
2. $a = 20, b = -5.$
3. $a = -121, b = 12.$

Divisibility in $\mathbb{Z}$: Given any integers a and b, we say that 'b' divides a' or that 'a is a multiple of b' or that a is divisible by b if $a = bc$, holds for some $c \in \mathbb{Z}$. Here b may be 0 also. In case $b \neq 0$, divisibility of a by b means that the remainder (when 'a' is divided by 'b', using division algorithm) is 0. We write $b|a$ to denote 'b divides a'.

Example:

1. 11 divides 132 as 132 = 11 × 12.
2. 5 does not 127 as 127 = 5 × 25 + 2.

Properties of divisibility: For any integers a, b, c etc.

1. $(\pm a) \mid a \ \forall a \in \mathbb{Z}$
2. $\pm 1|a \ \forall a \in \mathbb{Z}$
3. $a \,|0 \ \forall a \in \mathbb{Z}$
4. $a|b, a \mid c \Rightarrow a \mid (mb + nc) \ \forall m, n \in \mathbb{Z}$. In particular $a \mid (b \pm c)$
5. $a|b \Rightarrow a \mid mb \ \forall m \in \mathbb{Z}$
6. $a \mid b, b \mid a \Rightarrow b = \pm a$.
7. $a \mid b, b \mid c \Rightarrow a \mid c$.
8. $a \mid b, a > 0, b > 0 \Rightarrow a \leq b$.
9. If $m \neq 0$ $then\ a \mid b \Leftrightarrow ma|mb$
10. $a|b, c|d \Rightarrow ac|bd$.

Remark: $a|b$ is a statement to be read as 'a divides b' and is not to be confused with the fraction a/b.

Proof:

1. $a = a.1 = (-a)(-1) \ \therefore \ \pm a|a$
2. $a = 1.a = (-1)(-a) \ \therefore \ \pm 1|a$
3. $0 = a.0 \ \therefore \ a|0$ as $0 \in \mathbb{Z}$
4. $b = aa_1$ & $c = aa_2 \Rightarrow mb + nc = a(ma_1 + na_2)$, *for some* $a_1, a_2 \in \mathbb{Z}$
   $\therefore \ a|(mb + nc) \ \forall m, n \in \mathbb{Z}$.
   Taking $m = 1, n = 1$, we get $a|(b + c)$
   and taking $m = 1, n = -1$, we get $a|(b - c)$
5. $b = ac \Rightarrow mb = a(mc)$
   $\therefore \ a|mb$ as $mc \in \mathbb{Z}$.
6. $a|b, b|a \Rightarrow b = aa_1$ and $a = bb_1$, *for some* $a_1, b_1 \in \mathbb{Z}$
   $\therefore a = bb_1 = b = a(a_1 b_1)$,
   If $a \neq 0$ then $a.1 = a(a_1 b_1)$.

$\Rightarrow a_1 b_1 = 1$ *if* $a \neq 0$

$\therefore a_1 = 1 = b_1$ or $a_1 = -1 = b_1$.

Accordingly $a = b$ or $a = -b$.

If $a = 0$, again, $a = b\ (= 0)$ holds.

$\therefore a = \pm b$.

7. $a \mid b, b \mid c \Rightarrow b = aa_1$ and $c = bb_1$

$\therefore c = bb_1 = (aa_1)b_1 = a\,(a_1 b_1) \therefore a|c$.

8. $a|b \Rightarrow b = aa_1$ for some $a_1 \in \mathbb{Z}$. As $a > 0, b > 0$

$\therefore a_1 > 0$ also $\therefore a_1 \geq 1$ as $a_1 \in \mathbb{Z}$

$\therefore$ b = $aa_1 \geq a\,.1$ as $a_1 \geq 1$ $\therefore a \leq b$

9. If $a|b$ and $m \neq 0$ is in $\mathbb{Z}$ then $b = ac$

$\Rightarrow mb = (ma)\,c \quad \therefore ma \mid mb$.

If $ma \mid mb$ *then* $mb = (m\,a)c$, for some $c \in \mathbb{Z}$

$\therefore b = ac$ as $m \neq 0 \therefore a|b$

$\therefore a|b$ iff $ma|mb$.

10. If $a|b$ *and* $c|d$ then $b = aa_1$ and $d = cc_1$ for some $a_1, c_1, \in \mathbb{Z}$

$\therefore bd = (a\,c)\,(a1, c1) \therefore ac|\,bd$.

Whenever 'a' divides b, $(-a)$ clearly also divides b. In such cases '$-a$' is also said to be a divisor of 'b. For any integer n, the set of all positive divisors of n will be denoted by $D(n)$.Thus $D\,(n) = \{d \in \mathbb{Z} \mid d|n \text{ and } d \geq 1\}$

Clearly $D\,(0) = \mathbb{N}, D(1) = \{1\}, D(5) = \{1, 5\}. D\,(12) = \{1, 2, 3, 4, 6, 12\}$.

5.8. Greatest common Divisor

Given any two non zero integers a and b, we say that an integer 'g' is a greatest common divisor (g.c.d) of a and b if

1. $g|a, g|b$ (i.e. g is a common divisor of a and b)
2. $d|a, d|b \Rightarrow d \mid g$ (i.e. g is a multiple of every common divisor of a and b).

It is clear that if g is a g.c.d. of a and b, then so also is $-g$.

Exercise: If g is a g.c.d of a and b, the only other g. c. d of a and b is $-g$.

Solution: $g|a, g|b \Rightarrow a = g\,a_1$, and $b = g\,b_1$, for some $a_1, b_1 \in \mathbb{Z}$

$\therefore a = (-g)\,(-a_1)$ and $b = (-g)\,(-b_1)$

$\therefore\ -g\ |a$ and $-g\ |\ b$

Further if $d|a$ and $d|b$ then $d|g$ i.e. $g\ =\ dg_1$ for some $g_1, \in\mathbb{Z}$

$\therefore\ (-g)\ =\ d\ (-g1)$

$\therefore\ d|(-g)\ \therefore\ (-g\ )$ is also a greatest common divisor.

If $g'$is any other g.c.d of a and b, then $g\nmid a, g\nmid b$ as $g'$is a common divisor

$\therefore\ g\nmid g$ as g is a g.c.d. Similarly $g|\ g'$.

$\therefore g|g'$and $g\nmid g$

$\therefore\ g'\ =\ \pm g$ as proved earlier in the properties of divisibility. The following result shows the existence of a g.c.d for any pair of integers a and b.

Theorem: Given any non zero integers a and b, the set $\{ma\ +\ nb\ |\ m, n\ \in\mathbb{Z}\}$ has some (actually infinitely many) positive integers. The least positive member g of this set is a g.c.d of a and b and in fact g is a divisor of every element of this set.

The symbol (a, b) will denote the unique positive g.c.d of a and b.

Example:

1. $(24, 36)\ =\ (-\ 24, 36)\ =\ 12$
2. $(5, 11)\ =\ 1$
3. $(28, 105)\ =\ 7$

5.9. Properties of greatest common divisor

1. $(1, a)\ =\ 1\ \forall 0\neq a\ \in\mathbb{Z}$
2. $(a, a)\ =\ (-a, a)\ =\ (a, -a)\ =\ |a\ |\ \forall 0\neq a\ \in\mathbb{Z}$.
3. $(a, b)\ =\ (-a, b)\ =\ (b, a)$. $\forall 0\neq a, b\ \in\mathbb{Z}$
4. $(a, b)\ =\ (a, b+ma)\ \forall m\ \in\mathbb{Z}$, if $b+ma\neq 0$
5. $(a, b)\ =\ 1$ if and only if $ma\ +\ nb\ =\ 1$ for some $m, n\ \in\mathbb{Z}$.
6. If $m\ (>\ 0)\in\mathbb{Z}$ then $(ma, mb)\ =\ m\ (a, b)$.
7. if $(a, m)\ =\ (b, m)\ =\ 1$ then $(ab, m)\ =\ 1\ \forall 0\neq m\in\mathbb{Z}$.

8. $c \mid ab, (b,c) = 1 \Rightarrow c|a$.
9. If $a|b$ then $(a,b) = |a|$

Proof: Most of the proofs are straight forward e.g.

(4). Put $(a,b) = g$ and $(a, b + ma) = g'$

then $g|a,\ g|b \Rightarrow g|m.a + 1.b\ i.e.\ g|(b+ma)$

$\therefore\ g|g'as\ g' = (a, b+ma)$ & $g|a, g|b+ma$.

Similarly $g \nmid a. g'|(b+ma) \Rightarrow g \nmid [(-m)\ a + 1.(b+ma)]$

$i.e.\ g \nmid b\ \therefore\ g'|g.\ \therefore g' = \pm g$.

$\therefore\ g' = g$ as both are positive $\therefore\ (a, b+ma) = (a,b)$

(8) As $(b,c) = 1\ \therefore\ mb + nc = 1$ for some $m, n \in \mathbb{Z}$

$\therefore\ mab + nac = a$.

Now, $c|ab$ *(given) and* $c|ac$

$\Rightarrow\ c|\ m(ab) + n\,(a\,c).\ i.e.\ c|a$

Proofs of the other parts are left as exercise.

The following result gives an equivalent characterization for prime numbers.

5.10. **Prime Integers:** We have seen that for any integer n, ± 1 and $\pm n$ are always divisors of n. It may or may not have other divisors.

Definition: An integer $p\ (\neq 0, \neq \pm 1)$ is said to be a prime if the only divisors of p are ± 1 and $\pm p$. It may be checked that whenever p is a prime, so also is $-\ p$. It may be checked that
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$ are all the positive primes between 1 and 100. This list continues indefinitely and there are infinitely many positive primes. We shall derive other properties of primes only after studying some other concepts.

Theorem: An integer $p \neq 0, \pm 1$, is a prime if and only if $p \mid ab \Rightarrow p|a \text{ or } p|b$ (i.e. whenever p is a divisor of a product of two integers, it divides at least one of them).

Remark: $6 \mid 3 \times 4$ but $6 \nmid 3, 6 \nmid 4$. (because 6 is not a prime).

Proof: Suppose that p is a prime and $p \mid ab$. If $p \mid a$, the proof of this part is finished.

$\therefore$ Suppose that p does not divide a.

But g.c.d $(p, a) \mid p$ As p is a prime, and $(p, a) > 0$.

$\therefore (p, a) = 1 \text{ or } |p|$. But $(p, a) \mid a$ also. If $(p, a) = |p| = \pm p$

we would get $p|a$, which is not true by supposition.

$\therefore (p, a) = 1 \therefore 1 = mp + na$ for some $m, n \in \mathbb{Z}$

$\therefore b = mpb + n\,a\,b$. Since $p \mid pb$ & $p \mid ab$ we get $p \mid b$.

Conversely let $p \mid ab \Rightarrow p \mid a \text{ or } p \mid b$.

Let $d \mid p \therefore p = dq$ for some $q \in \mathbb{Z} \therefore p \mid dq$

$\therefore$ by hypothesis $p \mid d$ or $p \mid q$. If $d \mid p$, from $d \mid p$ and $p \mid d$ we get $d = \pm p$

If $p|q$, we get $p|q$ and $q|p$ (since p = dq)

$$\Rightarrow q = \pm p \therefore p = d(\pm p) \Rightarrow d = \pm 1 \text{ as } p \neq 0.$$

$\therefore d = \pm 1 \text{ or } \pm p$ for every divisor of p i.e. p is a prime.

The following result gives a method to calculate the g.c.d. (a, b) and also shows how to write it as $ma + nb$ for some $m, n \in \mathbb{Z}$.

Theorem: (**Euclid**) The set of positive primes is infinite.

Proof. If possible, let $p_1, p_2, \ldots\ldots. p_m$ be the set of all positive distinct primes.

Put n = p_1p_2……….. p_m+1. Then any prime factor of n can not be any of the above p_i, for then p_i with divide n – p_1, p_2.......p_r i.e. p_i| 1 ∴p_i =1 or p_i = -1 (contradiction) ∴ there are primes other than p_1 p_2 …….. p_m.

∴ The set of primes is infinite.

5.10.Euclidean Algorithm: Any two integers a and b with $a \neq 0$ have a greatest common divisor.

Proof: Let S be the set of integers of the form $ax + by$. Let d be the least positive integer such that $d = am + bn$ in the set S. Now by division algorithm

$a = dq + r,$ with $0 \leq r < |d|$ $so,$ $r = a - dq = a - (am + bn)q = a(1 - mq) + b(-nq).$ $Hence,$ $r \in S.$ $Since,$ $r < d$ which contradicts the assumption that d is the least positive integer in the set S. Hence, $r = 0$ and so $a = dq$ $i.e. d|a$.

Similarly, we can show that $d|b$. Suppose that $c|a$ and $c|b$then $c|(am + bn)$ or, $c|d$. Thus d is the greatest common divisor of a and b.

If d' be an another greatest common divisor of a and b then $d|d'$ and $d'|d$, then $d' = \pm d$.

Remark: This is the good old 'repeated division' method for finding g.c.d′s. working back upwards starting from the obtained g.c.d., expresses it in the form $ma + nb$ where $m, n \in \mathbb{Z}$.

Example: Find the g.c.d of 4235 and 854 and express it as $4235\,m + 854n$.

Solution: We have $4235 = 4 \times 854 + 819$

$$854 = 1 \times 819 + 35$$

$$819 = 23 \times 35 + 14$$

$$35 = 2 \times 14 + 7$$

$$14 = 2 \times 7.$$

$$\therefore (4235, 854) = 7 = 35 - 2 \times 14$$

$$= 35 - 2\,(819 - 23 \times 35\,)$$

$$= 47 \times 35 - 2 \times 819$$

$$= 47\,(854 - 819) - 2 \times 819$$

$$= 47 \times 854 - 49 \times 819$$

$$= 47 \times 854 - 49 \times 819$$

$$= 47 \times 854 - 49 \times (4235 - 4 \times 854)$$

$$= 4235\,(-49\,) + 854 \times 243$$

This is not the only way to write gcd 7 $as\ 4235\, m + 854n$.

$$In\ fact\ 7 = 4235\,(-49\,) + 854 \times 243$$

$$= 4235\,(-49) + 4235 \times 854 - 854 \times 4235 + 854 \times 243$$

$$= 4235 \times 805 + 854\,(-3992)\ etc.$$

Check your progress

11. Find the set of all positive divisors of $3, 16, 36, 105, 107, 121$ and 141.

12. Find the gcd of a and b and express it as $ma + nb$ in more than one ways:

(i) $a = 196, b = 192$

(ii) $a = 1024, b = 384$

(iii) If $(a, 4) = 2 = (b, 4)$ prove $(a + b, 4) = 4$.

(iv) If n is odd, prove that $8|(n^2 - 1\,)$

5.11.Unique factorization theorem for integers: Every integer $a(|a| > 1)$ can be expressed as a unit times a product of positive primes.

Proof: If a is prime then it can be expressed as a unit times a product of positive primes. But if a is not a prime means it is a composite number then $|a| = |m||n|$ where $|m|$ and $|n|$ are positive integers less than $|a|$.

Let $|m| = p_1p_2\, p_3 \dots \dots \dots .\, p_r$ and $|n| = q_1q_2\, q_3 \dots \dots \dots .\, q_s$ where p_i s and q_j s are positive primes. So,

$|a| = |m||n| = p_1p_2\, p_3 \dots \dots \dots .\, p_r q_1 q_2\, q_3 \dots \dots \dots .\, q_s$, or,

$|a| = k.p_1 p_2\, p_3 \ldots \ldots \ldots . p_r q_1 q_2\, q_3 \ldots \ldots \ldots . q_s$, where $k = 1\ or - 1$

If one of the p_i s or q_j s are not positive primes then we repeat this process again and again tills each factors are not prime. So. $|a|$ can be expressed as a unit times a product of positive primes.

Uniqueness: If it possible then suppose that there are two such factorization of $|a|$. Suppose another factorization of $|a|$ is $|a| = w_1 w_2\, w_3 \ldots \ldots \ldots . w_k$

Then, $|a| = w_1 w_2\, w_3 \ldots \ldots \ldots . w_k = p_1 p_2\, p_3 \ldots \ldots \ldots . p_r q_1 q_2\, q_3 \ldots \ldots \ldots . q_s$.

This shows that w_i must divide one of the primes of another side say p_j.

This shows that $w_i = p_j$., because both p_j and w_i are primes, so we replace these from the expression. Hence, we have $w_1 w_2\, w_3 \ldots\, w_{i-1} w_{i+1} \ldots \ldots . w_k$

$$= p_1 p_2\, p_3 \ldots p_{j-1} p_{j+1} \ldots \ldots . p_r q_1 q_2\, q_3 \ldots \ldots \ldots . q_s$$

We proceed this process in the same way to these equal products a finite number of times, we get $k = r + s$.

So, the factorization of $|a|$ is unique except for the order of primes.

5.12. Fundamental Theorem of Arithmetic: This theorem states that the primes are the building blocks of the integers in the sense that almost (i.e. except $0, 1, -\ 1$) every integer is a product of primes.

Example: $357 = 3^1 \times 7^1 \times 17^1$, $196 = +\ 2^2 \times\ 7^2$, $47 = 47^1$

Theorem: If $a\ (\neq\ 0, \pm 1)$ is any integer then it can be uniquely expressed as $a = {p_1}^{m_1} {p_2}^{m_2}\, {p_3}^{m_3} \ldots \ldots \ldots \ldots {p_r}^{m_r}$ where p_1, p_2, ….. p_r are distinct positive primes and m_1, …… m_r are integers > 1. (This representation is unique except for the order of appearance of p_1, ….. p_r)

Proof: Suppose that $a > 1$ and the result holds for all integers $1 < b < a$. If 'a' is itself a prime, the proof is finished as it is then a singleton product of primes with

$r = 1$ in the statement. If 'a' s not a prime, $a = bc$ with $1 < b, c < a$.

By the induction hypothesis b and c can be written as a product of primes (not necessarily distinct)

$\therefore\ a = b.c$ is also a product of primes contained in both the lists along with repetitions if any.

The uniqueness part of the proof can be proved using the property of primes : $p \mid ab \Rightarrow p \mid a \text{ or } p \mid b$ and is being omitted.

The above prime factorization gives alternative formulas for g.c.d's and l.c.m's and throws light on their relationship.

Note that a least common multiple (or l.c.m) of nonzero integers in defined as

Definition: A least common multiple 'l' of non zero integers 'a' and 'b' is defined by the properties:m_1

(i) $a \mid l, b \mid l$ (i.e. l is a common multiple)

(ii) $a \mid m , b \mid m \Rightarrow l \mid m$ (i.e. l is a divisor of every common multiple)

The set M of all common multiples of a and b is nonempty as $ab \in M$ and $M \cap \mathbb{N} \neq \phi$ as negative of an element of M is again an element of M. The least positive element of M (which exists by well ordering property of ℕ) may be seen to satisfy the above defining properties. As in the case of g.c.d'.s it is unique upto a ' – ' sign and $[a, b]$ denotes the positive l.c.m. of a and b.

Theorem: let $a = p_1^{m1} \ldots\ldots\ldots p_r^{mr}, b = p_1^{m_1^1} \ldots\ldots\ldots p_r^{m_r^1} (0 \le m_1, m_1^1)$ and let $s_i = \min(m_i, m_i')$, $t_i = \max(m_i, m_i')$. Then $(a,b) = p_1^{s1} \ldots\ldots p_r^{sr}$, $[a,b] = p_1^{t_1} \ldots\ldots p_r^{t_r}$ and $(a, b) . [a, b] = a.b$, where a, b are positive, and p_1 p_2……….. p_r are distinct positive primes. Here the powers are allowed to be zero in order to able to write unrelated integers a, b as a product of primes from the same set $\{p_1, \ldots\ldots\ldots, p_r\}$.

Check your progress

1. If $a = \pm\, p_1^{m1} \ldots\ldots p_r^{mr}$ and d|a, show that $d = \pm\, p_1^{n_1} \ldots\ldots p_r^{n_r}$ with $0 \le n_i \le m_i$ for each i = 1, …… r. Hence derive the total number of positive divisors of 'a'. (Ans : $(m_1 + 1)(m_2 + 1) \ldots\ldots (m_r + 1)$)
2. Prove by induction that $11^{n+2} + 12^{2n+1}$ is divisible by 133.
3. Prove by induction that $6^{n+2} + 7^{2n+1}$ is always divisible by 43.
4. Prove that induction that $15 \mid (3n^5 + 5n^3 + 7n) \; \forall \; n \ge 1$.
5. Prove by induction that $n(n^2 - 1)(3n - 2)$ is a multiple of 24.

5.13. Congruence Relation

Let n be a fixed integer ≥ 1. We define a relation ≡ in the set ℤ of all integers by

a ≡ b(mod n) if (b – a) is a multiple of n.

Example: $78 \equiv 104 \pmod{13}$ ∵ 104 – 78 = 26 is a multiple of 13.

Similarly $196 \equiv 42 \pmod{14}$

∵ 42 – 196 = – 154 and 14 |(- 154).

But 196 ≡ 40 (mod 14) ∵ 14 ∤(- 156)

The statement a ≡ b (mod n) is read as a is congruent to b module n'.

Theorem: For any integer n ≥ 1

1. a≡ a (mod n) ∀ a ∈ℤ
2. a ≡b (mod n) ⇒ b ≡ a (mod n)
3. a ≡ b (mod n), b ≡ c (mod n) ⇒ a ≡ c (mod n)
4. a ≡ 0 (mod n), if and only if n|a
5. a ≡ b (mod n), c ≡ d (mod n) ⇒ (ax + cy) ≡ (bx + dy) (mod n) in particular (a + c) ≡ (b +d) (mod n) and (a – c) ≡ (b – d) (mod n)
6. a ≡ b (mod n), c ≡ d (mod n) ⇒ ac ≡ bd (mod n)
7. a ≡ b (mod n) ⇒ a + c ≡ b + c (mod n) ∀ c∈ℤ
8. a ≡ b (mod n) ⇒ ac ≡ bc (mod n) ∀c∈ℤ
9. a ≡ b (mod n) ⇒ a ≡ b (mod d) ∀ d | n.
10. ax ≡ ay (mod n) ⇒ x ≡ y $\left(\bmod \frac{n}{(a,n)}\right)$. In particular if (a, n) = 1 then x ≡ y(mod n) if and only if ax ≡ ay (mod n)
11. if x ≡ y (mod n) then (x, n) = (y, n).

Remark: It follows from (1), (2) and (3) above that the relation a≡b (mod n) is reflexive, symmetric and transitive and hence partition ℤ into equivalence classes. For example if a is divided by n using division algorithm, : a = nq + r . Then r∈{0, 1, ……,n-1} as a - r = nq (a multiple of n) ∴ a≡r (mod n) for a unique r determined by a. The class of r = [r] = {a∈ℤ| r ≡ a(mod n)} = {r +tn | n ∈ℤ}

= { ……….r – 2n, r – n, r, r + n, r + 2n ………..}

ℤ = [0] ∪ [1] ∪ …….∪ [n – 1] (mutually disjoint blocks)

The set of all classes is denoted by

$\mathbb{Z}_n$= {[0], [1], ……. [n – 1]}.

Just as the equation ax = b for a, b ∈ℤ may or may not have an integer solution for the unknown x, we may consider the congruence equation ax ≡ b (mod n) for any given integer a and b and integer n ≥ 1. (called a linear congruence equation).

5.14. Summary:

We are able to understand the basic properties of set of integers. And to derive other properties with the help of the basic ones, describe a division algorithm and the Euclidean algorithm for the set of all integers, define a Prime number and the concepts related to it, define the greatest common divisor and least common multiple and calculate them, the Fundamental Theorem of Arithmetic and to define the least common multiple of two integers.

5.15. Terminal Questions

1. Prove that $1^3+2^3+\ldots\ldots n^3 = \frac{n^2(n+1)^2}{4}$ $\forall$ n ≥ 1.
2. Prove that $2^n > n^3$ for all $n \geq 10$
3. Find the set of all positive divisors of 36, 105, 121 and 141.
4. Prove by induction that $11^{n+2}+12^{2n+1}$ is divisible by 133.
5. Find the gcd of a and b and express it as $ma + nb$ in more than one ways:

$$a = 196, b = 192$$

$$a = 1024, b = 384$$

Uttar Pradesh

RajarshiTandon
Open University

Bachelor Of Science

SBSMM - 03

Elementary Analysis

Block

3 Sequence & Series

Unit -6
SEQUENCE OF REAL NUMBER

Unit -7
Infinite Series

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BLOCK – 03

Sequence, and Infinite Series

First unit of this block is most basic unit of this block as it introduces the concept of sequence of real numbers, convergent and divergent sequence, subsequence, Cauchy sequence, their applications and interrelationship between convergence of a sequence and a Cauchy sequence. Sequence has an important role in the field of Analysis. It has many important applications in analysis like as application in almost every field, social, economy, engineering, technology etc.

In the second unit of this block we introduce the concept of Partial sum of series, convergent and divergent series of non- negative terms and use of different tests for convergence of series of non- negative terms. use of different tests for convergence of series of non- negative terms. The theory of Absolutely convergence and conditionally convergence.

Series of non negative terms has an important role in the field of Analysis·

Unit:6 SEQUENCE OF REAL NUMBER

Structure

6. 1. Introduction

This is most basic unit of this block as it introduces the concept of sequence of real numbers, convergent and divergent sequence , subsequence, Cauchy sequence, their applications and interrelationship between convergence of a sequence and a Cauchy sequence.

Sequence has an important role in the field of Analysis·

It has many important applications in analysis like as application in almost every field, social, economy, engineering, technology etc.

6. 2. Objectives

After reading this unit we should be able to

1. Understand the concept of Sequence of real numbers.

2. Concept of convergence of a real sequence.

3. Understand the Concept of divergence and Oscillatory sequences of a real numbers

4. Understand the concept of subsequences, Cauchy sequence and its uses.

Sequences has an important role in the field of Analysis·

It has many important applications in Analysis.

6.3. Sequence of real numbers

Definition: A map $a: \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural numbers and real numbers respectively, is called a **sequence in** $\mathbb{R}$**.**

Since we shall be dealing with real sequences only, we shall use the term **sequence**to denote a **real sequence.**

Notation: As we know a map $\boldsymbol{a}: \mathbb{N} \to \mathbb{R}$is of the form,

$a = \{(n, a(n)| \ n \in \mathbb{N}\}$

$or, \quad a = \{(1, (1)),(2, a(2)), (3, a(3)), \dots \dots, (n, a(n)), \dots \dots \}$

$or, ingeneralitcanbeexpressedasthe\boldsymbol{orderedset},$

$a = \{a(1), a(2), a(3), \dots a(n), \dots\}$or,$a = \{a_1, a_2, a_3, \dots, a_n, \dots\}$; where a_n denotes $a(n)$, when the domain is $\mathbb{N}$**.**

Since the domain of a sequence $\boldsymbol{a}: \mathbb{N} \to \mathbb{R}$in always $\mathbb{N}$, a sequence is specified by the values , $n \in \mathbb{N}$. Thus a sequence may be denoted as,

$$\{a_n\}_{n=1}^{\infty} \quad or \quad \{a_n\}_{n\in\mathbb{N}} \quad or \quad \{a_1, a_2, a_3, \dots, a_n, \dots\}.$$

Remarks:

1. The values a_1, a_2 ,$_3$ … … are called the first, second, third,…… terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

2. The m^{th} and n^{th} terms a_m and a_n for $m \neq n$are treated as distinct terms of the sequence even if $a_m = a_n$.

3. A sequence is an **ordered set**. *i.e.* the terms of a sequence are arranged in a definite order as first, second, third,… terms and the terms occurring at different positions are treated as distinct terms even if they have the same value.

4. The number of terms in a sequence $\{a_n\}_{n=1}^{\infty}$ is always infinite.

Examples:

1 $The\ sequence\ \{a_n\}_{n=1}^{\infty}\ given\ by,\ a_n = \frac{1}{n}\ ;\forall\ n \in \mathbb{N}.$

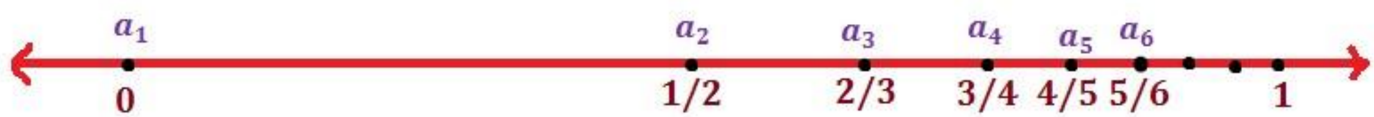
Here, $a_1 = 1\ , a_2 = \frac{1}{2}\ , a_3 = \frac{1}{3}\ , \dots \dots$. *The graph of the sequence is as follows:*

Now, observe that the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ approaches to the real number 0 as n approaches to infinity.

2 $The\ sequence\ \{a_n\}_{n=1}^{\infty}\ given\ by,\ a_n = \frac{n-1}{n}\ ;\forall\ n \in \mathbb{N}.$

$Here\ , a_1 = 0\ , a_2 = \frac{1}{2}\ , a_3 = \frac{2}{3}\ , a_4 = \frac{3}{4}, a_5 = \frac{4}{5}, \dots \dots \dots$

The graph of the sequence is as follows:

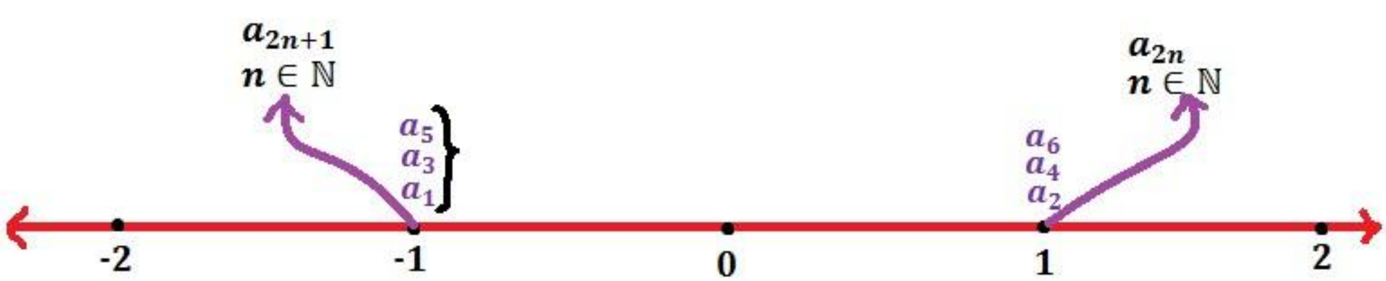


Observe that the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ approaches to the real number 1 as n approaches to infinity.

3 $The\ sequence\ \{a_n\}_{n=1}^{\infty}\ given\ by,\ a_n = (-1)^n\ ;\forall\ n \in \mathbb{N}.$

$Here\ , a_1 = -1\ , a_2 = 1\ , a_3 = -1\ , a_4 = 1, a_5 = -1, \dots \dots \dots$

The graph of the sequence is as follows:

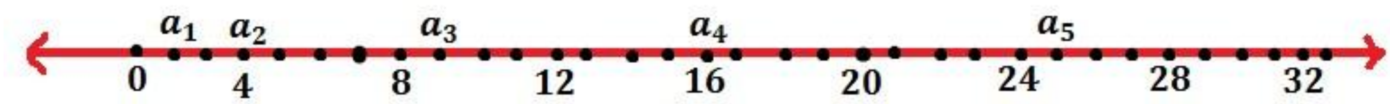


Observe that the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ doesn't approaches to the any one real number as n approaches to infinity.

4 $The\ sequence\ \{a_n\}_{n=1}^{\infty}\ given\ by,\ a_n = n^2\ ;\forall\ n \in \mathbb{N}.$

$Here\ , a_1 = 1\ , a_2 = 4\ , a_3 = 9\ , a_4 = 16, a_5 = 25, \dots \dots \dots$

The graph of the sequence is as follows:



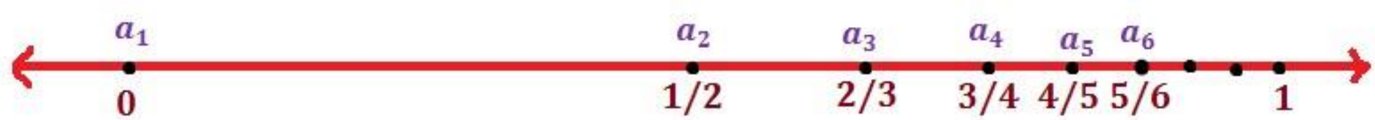
Observe that the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ approaches to infinity (∞) as n approaches to infinity.

6.4 The convergence of a real sequence:

Now, as we see in examples 3.3.1 and 3.3.2, the sequence $\{a_n\}_{n=1}^{\infty}$ approaches to a real number ($say l$), as n approaches to infinity. "What does it actually mean?" It means, whatever positive real number (ε) we choose, however small, the distance between the terms of the sequence and the real number l must be less than ε, after some finite number of terms (N).

Let us consider an example to understand this concept:

Consider the sequence, then $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ we observe that the terms of the sequence $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ approaches to the real number 1 as n approaches to infinity.



So, choose $\varepsilon = \frac{1}{10}\ (> 0)$, we will show that after finite number of terms *(say N)*, the distance between the terms of the sequence and the real number l is less than ε.

i.e. in Mathematical notation, for $\varepsilon = \frac{1}{10}\ (> 0),\ \exists\ N \in \mathbb{N}$ *such that*

$$n \geq N \Longrightarrow |a_n - l| < \varepsilon$$

Here, $|a_n - l| < \varepsilon \quad if \ \left|\frac{n-1}{n} - 1\right| < \frac{1}{10}$

$or\ if\ \left|\frac{n-1-n}{n}\right| < \frac{1}{10}$

$or\ if\ \left|\frac{-1}{n}\right| < \frac{1}{10}$

$or\ if\ \frac{1}{n} < \frac{1}{10}$

$or if n > 10.$

i.e. if we choose $N = 11$, then $n \geq N \Longrightarrow |a_n - l| < \varepsilon$.

$\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$

Does such N exists for any $\varepsilon > 0$ for this sequence ?

Yes. Let us take $\varepsilon = 0.002$ (>0), then we will try to find $N \in \mathbb{N}$, such that

$$n \geq N \Longrightarrow |a_n - l| < \varepsilon$$

$$|a_n - l| < \varepsilon \quad if \ \left|\frac{n-1}{n} - 1\right| < 0.002$$

For,

$$or \ if \ \left|\frac{n-1-n}{n}\right| < 0.002$$
$$or \ if \ \left|\frac{-1}{n}\right| < 0.002$$
$$or \ if \ \frac{1}{n} < 0.002$$
$$or if n > 500.$$

i.e. if we choose $N = 501$, then $n \geq N \Longrightarrow |a_n - l| < \varepsilon$.

$$\lim_{n\to\infty} a_n = l$$

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be convergent and converges to a real number l if and only if $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$, *depending on* ε, *such that* $n \geq N \Longrightarrow |a_n - l| < \varepsilon$.

Note: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number l, then we write,

.

Illustrative Examples :

Prove that the limit of the sequence $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ is 1.

Solution: Here, $a_n = \frac{n-1}{n}$ and $l = 1$. Choose any $\varepsilon > 0$, We have to find a natural number N, depending on ε, such that $n > N \Longrightarrow |a_n - l| < \varepsilon$.

Now, $|a_n - l| < \varepsilon \quad if \ \left|\frac{n-1}{n} - 1\right| < \varepsilon$

$$or \ if \ \left|\frac{n-1-n}{n}\right| < \varepsilon$$
$$or \ if \ \left|\frac{-1}{n}\right| < \varepsilon$$
$$or \ if \ \frac{1}{n} < \varepsilon$$

$or\ if\ \ n > \frac{1}{\varepsilon}.$

If we choose the first natural number N just after $\frac{1}{\varepsilon}$, i.e. choose $N = \max\left\{1, \left[\frac{1}{\varepsilon}\right] + 1\right\}$,

then for, $n \geq N > \frac{1}{\varepsilon}$, $|a_n - l| < \varepsilon$.

Showing that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to l.

i.e. $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = 1$. #

Prove that the limit of the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ is zero.

Solution: Here, $a_n = \frac{(-1)^n}{n}$ and $l = 0$.

Choose any $\varepsilon > 0$, We have to find a natural number N, depending on ε, such that $n > N \Longrightarrow |a_n - l| < \varepsilon$.

Now, $|a_n - l| < \varepsilon \quad if \ \left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$

$or\ if\ \ \left|\frac{(-1)^n}{n}\right| < \varepsilon$

$or\ if\ \ \frac{1}{n} < \varepsilon$

$or\ if\ \ n > \frac{1}{\varepsilon}.$

If we choose the first natural number N just after, $\frac{1}{\varepsilon}$

i.e. choose $N = \max\left\{1, \left[\frac{1}{\varepsilon}\right] + 1\right\}$, then for, $n > N > \frac{1}{\varepsilon}$, $|a_n - l| < \varepsilon$.

Showing that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to l. *i.e.* $\lim_{n\to\infty}\left(\frac{(-1)^n}{n}\right) = 0$. #

Prove that, $\lim_{n\to\infty}\left(\frac{3n+1}{2n+4}\right) = \frac{3}{2}$.

$a_n = \frac{3n+1}{2n+4}$ and $l = \frac{3}{2}$

Solution: Here, Choose any $\varepsilon > 0$, We have to find a natural number N, depending on ε, such that $n > N \Longrightarrow |a_n - l| < \varepsilon$.

Now, $|a_n - l| < \varepsilon \quad if \ \left|\frac{3n+1}{2n+4} - \frac{3}{2}\right| < \varepsilon$

$or\ if \ \left|\frac{(6n+2)-(6n+12)}{2(2n+4)}\right| < \varepsilon$

$or\ if \ \left|\frac{-10}{2(2n+4)}\right| < \varepsilon$

$or\ if \ \frac{5}{2n+4} < \varepsilon$

$or\ if \ \frac{2n+4}{5} > \frac{1}{\varepsilon}$

$or\ if \ n > \frac{1}{2}\left(\frac{5}{\varepsilon} - 4\right)$

If we choose the first natural number N just after $\frac{1}{2}\left(\frac{5}{\varepsilon} - 4\right)$,

i.e. choose $N = \max\left\{1, \left[\frac{1}{2}\left(\frac{5}{\varepsilon} - 4\right)\right]\right\}$, then for, $n \geq N > \frac{1}{\varepsilon}$, $|a_n - l| < \varepsilon$.

Showing that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to l.

i.e. $\lim_{n\to\infty} \left(\frac{3n+1}{2n+4}\right) = \frac{3}{2}$. #

Check your progress

1. Define convergence of a real sequence $\{a_n\}_{n=1}^{\infty}$. Prove that:
   - (i) The sequence $\left\{\frac{1}{n^2}\right\}$ converges to zero.
   - (ii) The sequence $\left\{\frac{1}{n^p}\right\}$ converges to zero, where p is a fixed positive real number.
   - (iii) $\lim_{n\to\infty} \left(\frac{6n-5}{5n+5}\right) = \frac{6}{5}$.
   - (iv) The sequence $\left\{\frac{3}{2}, \frac{7}{5}, \frac{11}{8}, \frac{15}{11}, \frac{19}{14}, \ldots\right\}$ is a convergent sequence and converges to the limit $\frac{4}{3}$.
2. Find a natural number N, such that
$$n \geq N \Rightarrow \left|\frac{8n-3}{4n+9} - 2\right| < \frac{21}{1000}.$$
3. Prove that $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$.

Proposition: Let $x \in \mathbb{R}$. Then,

(i) $x < \varepsilon \ ; \forall \varepsilon > 0 \Longrightarrow x \leq 0.$

(ii) $|x| < \varepsilon \ ; \forall \varepsilon > 0 \Longrightarrow x = 0.$

Proof: (i) Let $x \in \mathbb{R}$ and $x <$; $\forall \varepsilon > 0$.

If possible suppose, $x > 0$ then choose $\varepsilon = \frac{x}{2} (> 0)$, then

$$x > \frac{x}{2} \Rightarrow x > \varepsilon \quad \left(as \ \ \varepsilon = \frac{x}{2}\right)$$

which contradicts the fact that $x < \varepsilon$; $\forall \varepsilon > 0$.

Hence, $x \ngtr 0$ or $x \leq 0$ (by Trichotomy Principle). #

(ii) Let $x \in \mathbb{R}$ and $|x| <$; $\forall \varepsilon > 0$. Choose $|x| = y$, then $y \geq 0$

Also, $y <$; $\forall \varepsilon > 0$

So, by (i), $y \leq 0$

Hence, $y = 0$ (as $y \geq 0$ and $y \leq 0$)

$\Longrightarrow |x| = 0$

$\Longrightarrow x = 0$. #

Theorem: A sequence $\{a_n\}$ cannot converge to more than one point.

Proof: If possible, suppose $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = l'$

Then, for any given $\varepsilon > 0$, $\exists$ $N_1, N_2 \in \mathbb{N}$, depending on ε, such that

$$n > N_1 \Longrightarrow |a_n - l| < \varepsilon/2 \qquad \text{......(1)}$$

$$n > N_2 \Longrightarrow |a_n - l'| < \varepsilon/2 \qquad \text{......(2)}$$

Choose, $= Max. (N_1, N_2)$, then (1) and (2) reduces to,

$$n > N \Longrightarrow |a_n - l| < \varepsilon/2 \qquad \text{......(3)}$$

$$n > N \Longrightarrow |a_n - l'| < \varepsilon/2 \qquad \text{......(4)}$$

Now, $n \geq N \Longrightarrow |l - l'| = |l - a_n + a_n - l'|$

$\leq |l - a_n| + |a_n - l'|$ (by Triangle inequality)

$\leq |a_n - l| + |a_n - l'|$ $(as\ |-x| = |x|\ \forall x \in \mathbb{R})$

$< \varepsilon/2 + \varepsilon/2$ (From (3) and (4))

$= \varepsilon$.

Thus, $n \geq N \Longrightarrow |l - l'| <$; for any given $\varepsilon > 0$.

Hence, $l - l'$
$= 0$ or, $l = l'$.

6.5. Divergent and Oscillatory Sequence

A sequence $\{a_n\}$ is said to be **divergent and diverges to** $+\infty$

$(in\ symbols,\ \lim_{n\to\infty} a_n = +\infty)$ if for any positive real number k, however large, there exists a natural number N, such that

$n \geq N \Longrightarrow a_n > k$

i.e. $\lim_{n\to\infty} a_n = +\infty$ *if* $\forall\, k \in \mathbb{R}^+$*, however large,* $\exists N \in \mathbb{N}$*, such that* $n \geq N \Longrightarrow a_n > k$.

Example: The sequences $\{n\}$, $\{n^2\}$ and $\{-3n^2\}$are divergent sequences and diverges to $+\infty$.

A sequence $\{a_n\}$ is said to be **divergent and diverges to** $-\infty$ $\left(\text{in symbols, } \lim_{n\to\infty} a_n = -\infty\right)$ if for any positive real number k, however large, there exists a natural number N, such that

$n \geq N \Longrightarrow a_n < -k$

i.e. $\lim_{n\to\infty} a_n = -\infty$ *if* $\forall\, k \in \mathbb{R}^+$*, however large,* $\exists N \in \mathbb{N}$*, such that* $n \geq N \Longrightarrow a_n < -k$.

Example: The sequences $\{-n\}$, $\{-n^2\}$ and $\{-3n^{-2}\}$ are divergent sequences and diverges to $-\infty$.

A real sequence $\{a_n\}$ is said to be <u>oscillatory sequence</u> if it is neither convergent nor divergent.

Example: The sequences $\{(-1)\ \}$, $\{(-2)^n\}$ are oscillatory sequence. [

Hint: The sequence $\{(-1)^n\}$ oscillates finitely between -1 and 1.

The sequence $\{(-2)^n\}$ oscillates infinitely between $-\infty$ and $+\infty$.]

Theorems on Limits:

Let $\{a_n\}$ and $\{b_n\}$are two sequences such that $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$, then:

(i) $\lim_{n\to\infty}(a_n + b_n) = l + m$ and $\lim_{n\to\infty}(a_n - b_n) = l - m$

(ii)Let $(k \neq 0) \in \mathbb{R}$, then $\lim_{n\to\infty} ka_n = kl$

(iii) $\lim_{n\to\infty}(a_n . b_n) = l.m$

(iv) If $b_n \neq 0$; $\forall n \in \mathbb{N}$ and $m \neq 0$, then $\lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{m}$.

(v) If $b_n \neq 0$; $\forall n \in \mathbb{N}$ and $m \neq 0$, then $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{l}{m}$.

Proof: (i) Given that, $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$.

Let $\varepsilon' > 0$ be arbitrarily chosen. Then $\exists\, N_1 \in \mathbb{N}$and $N_2 \in \mathbb{N}$ such that

$n \geq N_1 \Longrightarrow |a_n - l| < \varepsilon'$… (1)

$n \geq N_2 \Longrightarrow |b_n - m| < \varepsilon'$… (2)

Now, choose $N = max. \{N_1, N_2\}$, then above equations can be expressed as,

$$n \geq N \Longrightarrow |a_n - l| < \varepsilon' \ldots (3)$$

$$n \geq N \Longrightarrow |b_n - m| < \varepsilon' \ldots (4)$$

To show: $\lim_{n\to\infty}(a_n + b_n) = l + m$.

If $n \geq N$, consider, $|(a_n + b_n) - (l + m)| = |(a_n - l) + (b_n - l)|$

$\leq |a_n - l| + |b_n - l|$ (by Triangle inequality)

$< \varepsilon' + \varepsilon'$ (From (3) and (4))

$= 2\varepsilon'$

Hence, $n \geq N \Longrightarrow |(a_n + b_n) - (l + m)| < 2\varepsilon'$

So, if we choose $\varepsilon' = \varepsilon/2$, then $\varepsilon > 0$ is arbitrarily chosen and then $\exists N \in \mathbb{N}$ such that

$n \geq N \Longrightarrow |(a_n + b_n) - (l + m)| < \varepsilon$

Hence, $\lim_{n\to\infty}(a_n + b_n) = l + m$. #

Note: $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(ii) Given that, $\lim_{n\to\infty} a_n = l$ and $(\neq 0) \in \mathbb{R}$.

Let $\varepsilon' > 0$ be arbitrarily chosen. Then $\exists\ N \in \mathbb{N}$ such that

$n \geq N_1 \Longrightarrow |a_n - l| < \varepsilon' \ldots (1)$

To show: $\lim_{n\to\infty} ka_n = kl$

If $n \geq N$, consider, $|(ka_n) - (kl)| = |k(a_n - l)|$

$= |k|.\ |a_n - l|$

$< |k|.\ \varepsilon'$ (From (1))

Hence, $n \geq N \Longrightarrow |(ka_n) - (kl)| < |k|.\ \varepsilon'$

So, if we choose $\varepsilon' = \varepsilon/|k|$, then $\varepsilon > 0$ is arbitrarily chosen and then $\exists N \in \mathbb{N}$ such that

$n \geq N \Longrightarrow |(ka_n) - (kl)| < \varepsilon$

Hence, $\lim_{n\to\infty} ka_n = kl$. #

Note: $\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty} a_n$

(iii) Given that, $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$.

First we will show that if $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$ then $\lim_{n\to\infty} \alpha_n . \beta_n = 0$

Let $\varepsilon' > 0$ be arbitrarily chosen. Then $\exists\ N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

$n \geq N_1 \Longrightarrow |\alpha_n| < \varepsilon' \ldots (1)$

$n \geq N_2 \Longrightarrow |\beta_n| < \varepsilon' \ldots (2)$

Now, choose $N = max.\{N_1, N_2\}$, then above equations can be expressed as,

$n \geq N \Longrightarrow |\alpha_n| < \varepsilon'$ ... (3)

$n \geq N \Longrightarrow |\beta_n| < \varepsilon'$ ... (4)

To show: $\lim_{n\to\infty} \alpha_n.\beta_n = 0$.

If $n \geq N$, consider, $|\alpha_n.\ \beta_n| = |\alpha_n|.\ |\beta_n|$

$< \varepsilon'.\ \varepsilon'$ (From (3) and (4))

$= (\varepsilon')^2$

Hence, $n \geq N \Longrightarrow |\alpha_n.\ \beta_n| < (\varepsilon')^2$

So, if we choose $\varepsilon' = \sqrt{\varepsilon}$, then $\varepsilon > 0$ is arbitrarily chosen and then $\exists N \in \mathbb{N}$ such that

$n \geq N \Longrightarrow |\alpha_n.\ \beta_n| < \varepsilon$

Hence, $\lim_{n\to\infty} \alpha_n.\beta_n = 0$.

To show: $\lim_{n\to\infty} a_n.b_n = lm$.

Let $\alpha_n = a_n - l$ and $\beta_n = b_n - m$, then $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}(a_n - l) = l - l = 0$ and $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty}(b_n - m) = m - m = 0$.

Hence, from above,

$\lim_{n\to\infty} \alpha_n.\beta_n = 0$

$\Longrightarrow \lim_{n\to\infty}(a_n - l).(b_n - m) = 0$

$\Longrightarrow \lim_{n\to\infty} a_n.b_n - \lim_{n\to\infty} l.b_n - \lim_{n\to\infty} m.a_n + \lim_{n\to\infty} l.m = 0$

$\Longrightarrow \lim_{n\to\infty} a_n.b_n - l.\lim_{n\to\infty} b_n - m.\lim_{n\to\infty} a_n + \lim_{n\to\infty} l.m = 0$

$\Longrightarrow \lim_{n\to\infty} a_n.b_n - l.m - m.l + l.m = 0$

$\Longrightarrow \lim_{n\to\infty} a_n.b_n = lm$. #

Note: $\lim_{n\to\infty} a_n.b_n = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$

(iv) First we will prove the following Lemma:

If $\lim_{n\to\infty} b_n = m$ and $m \neq 0$, then $\exists N \in \mathbb{N}$, such that $n \geq N \Rightarrow |b_n| < \frac{|m|}{2}$

Proof: Since, $\lim_{n\to\infty} b_n = m$, then for $\varepsilon = \frac{|m|}{2} > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |b_n - m| < \frac{|m|}{2} \quad \text{... (1)}$$

$$|m| = |m - b_n + b_n| \leq |m - b_n| + |b_n| < \frac{|m|}{2} + |b_n|.$$

Thus, $n \geq N \Rightarrow |m| < \frac{|m|}{2} + |b_n|$

$\Rightarrow |b_n| > \frac{|m|}{2}$. #

Now, we will show, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{m}$.

If $n \geq N$, consider,

$$\left|\frac{1}{b_n} - \frac{1}{m}\right| = \frac{|b_n - m|}{|b_n|.|m|}$$

$$< \frac{\varepsilon'}{\frac{|m|}{2}.|m|} \quad (from\ above\ lemma\ \&\ equation\ (1))$$

$$= \frac{2}{|m|^2}.\varepsilon'$$

Hence,

$n \geq N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{m}\right| < \frac{2}{|m|^2}.\varepsilon'$

So, if we choose $\varepsilon' = \frac{1}{2}|m|^2\varepsilon$, then $\varepsilon > 0$ is arbitrarily chosen and then $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{m}\right| < \varepsilon$

Hence, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{m}$.

(V). Easy to prove and is left for the exercise.

6.6. Subsequences: Let $\{u_n\}$ be a given sequence. If $\{n_k\}$ is a strictly increasing sequence of natural numbers i.e.$(n_1 < n_2 < n_3 < \cdots)$, then$\{u_{n_k}\}$ is called a subsequence of $\{u_n\}$.

Example: 1. The sequences $\{1, 3, 5, \ldots, 2n-1\}$, $\{2, 4, 6, \ldots .2n \ldots\}$ and $\{1, 4, 9, \ldots n^2, \ldots\}$ are all subsequences of the sequence $\{n\}$.

Remark: The terms of a sequence occur in the same order in which they occur in the original sequence.

Theorem: If a sequence $\{u_n\}$ converges to l, then every subsequence of $\{u_n\}$ converge to l.

Proof: Let $\{u_{n_k}\}$ be a subsequence of the sequence $\{u_n\}$ and this sequence $\{u_n\}$ converges to l.

That is for choosen any $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n

$|u_n - l| < \varepsilon \ \ \forall\, n > N$

$\Rightarrow |u_{n_k} - l| < \varepsilon \ \ \forall\, n_k > N$

$\Rightarrow$ the subsequence $\{u_{n_k}\}$ converges to l.

Remark:1. All subsequences of a convergent sequence converges to the same limit point.

2.The converse of the above theorem need not be true.

Example: The subsequence $\{1,1,1,\dots.\}$ *and* $\{-1,-1,-1,\dots..\}$ of the sequence $\{(-1)^n\}$ are convergent where as the sequence $\{(-1)^n\}$ is not convergent.

3.In the order to prove that a sequence is not convergent, it is sufficient to show that any two of its subsequence converge to different limits.

Example: The subsequence $\{1,1,1,\dots.\}$and $\{-1,-1,-1,\dots..\}$ of the sequence $\{(-1)^n\}$ are convergent and converge to 1 *and* -1 respectively so the limits of two subsequence of this sequence are different so the sequence $\{(-1)^n\}$ is not convergent.

Theorem: If the subsequences $\{u_{2n-1}\}$and $\{u_{2n}\}$ of a sequence $\{u_n\}$ converges to the same limit l, then sequence $\{u_n\}$ converges to l.

Proof: Since the subsequences $\{u_{2n-1}\}$and $\{u_{2n}\}$ converge to l, so for each $\varepsilon > 0, \exists$a positive integer N_1 and N_2 depending on ε, such that

$|u_{2n-1} - l| < \varepsilon \ \ \forall\, n > N_1$ and

$|u_{2n} - l| < \varepsilon \ \ \forall\, n > N_2$

Let $N = \max\{N_1, N_2\}$ then both of them reduces to

$|u_{2n-1} - l| < \varepsilon \ \ \forall\, n > N$ and

$|u_{2n} - l| < \varepsilon \ \ \forall\, n > N$

Hence,$|u_n - l| < \varepsilon \ \ \forall\, n > N$

So, the sequence $\{u_n\}$ converges to l.

Note: 1.If a sequence $\{u_n\}$ diverges to ∞, then every subsequence of sequence $\{u_n\}$ also diverges to ∞.

2. If a sequence $\{u_n\}$ diverges to $-\infty$, then every subsequence of sequence $\{u_n\}$ also diverges to $-\infty$.

6.7. Cauchy Sequence: A sequence $\{u_n\}$ is said to be a Cauchy Sequence if for each $\varepsilon > 0, \exists$a positive integer N depending on ε, such that for all positive integers m, n

$$m, n > N \Rightarrow |u_n - u_m| < \varepsilon$$

In particular taking $m = N + 1 > N$,

We have

$$n > N \Rightarrow |u_n - u_{N+1}| < \varepsilon$$

Or, $p > 0, n > N \Rightarrow |u_n - u_{n+P}| < \varepsilon$

Example: show that the sequence $\{u_n\}$ is a Cauchy sequence where

$u_n = \frac{n}{n+1}$.

Proof: Since $u_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$

Choosen any $\varepsilon > 0$. Let m, n be any positive integers such that $m > n$. Now,

$$m, n > N \Rightarrow |u_n - u_m| = \left|\frac{1}{n+1} - \frac{1}{m+1}\right| = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1}$$

Now, $\frac{1}{n+1} < \varepsilon \; if \; n > \frac{1}{\varepsilon} - 1$

Hence, the given sequence $u_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$ is a Cauchy sequence.

Example: show that the sequence $\{u_n\}$ is not a Cauchy sequence where $u_n = 1 + 1/3 + 1/5 + \cdots \ldots . + \frac{1}{2n-1}$.

Proof: We have $u_n = 1 + 1/3 + 1/5 + \cdots \ldots . + \frac{1}{2n-1}$ *and*

$$u_{2n} = 1 + 1/3 + 1/5 + \cdots \ldots . + \frac{1}{2n-1} + \frac{1}{2n+1} + \ldots + \frac{1}{4n-1}$$

Now, $u_{2n} - u_n = \left[1 + 1/3 + 1/5 + \cdots \ldots . + \frac{1}{2n-1} + \frac{1}{2n+1} + \ldots + \frac{1}{4n-1}\right] - \left[1 + 1/3 + 1/5 + \cdots \ldots . + \frac{1}{2n-1}\right] = \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n-1}$

$$\Rightarrow u_{2n} - u_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{4n-1} > n\, 1/4n = 1/4$$

Since, $\frac{1}{2n+1} > 1/4n$ and so on also there are n terms.

$\Rightarrow u_{2n} - u_n > 1/4 \; \forall \, n \in N$

$\Rightarrow$ $\exists$ a positive integer k such that $|u_n - u_k| > 1/4$ whenever $n \geq k$

$\Rightarrow$This sequence is not a Cauchy sequence

Theorem: Every Cauchy sequence is bounded.

Proof: Let $\{u_n\}$ be a Cauchy sequence, choosen any $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n.

$n > N \Rightarrow |u_n - u_{N+1}| < \varepsilon$

Or, $n > N \Rightarrow u_{N+1} - \varepsilon < u_n < u_{N+1} + \varepsilon$

Thus $u_{N+1}, N+2, N+3, \ldots.$ that is all the terms after u_N are less than $u_{N+1} + \varepsilon$. Hence, $K = Max\{u_1, u_2, u_3, \ldots. \; u_N, u_{N+1} + \varepsilon\}$

Then $u_N \leq K$ for all positive integers n. Hence, the given sequence is bounded above.
Again we see that each term of the sequence after u_N is greater than $u_{N+1} - \varepsilon$, so if $H = Min\{u_1, u_2,\ u_3, \ldots.\ u_N, u_{N+1} - \varepsilon\}$, then $H \leq u_N$ for all n.
Hence the given sequence is bounded below.
Hence the given sequence is bounded.
Remark: Converse of this theorem need not be true.
Example: The sequence $\{(-1)^n\}$ is bounded but it is not a Cauchy sequence. That is every bounded sequence need not be a Cauchy sequence.
6.8. Theorem (Cauchy's criterion for convergence): A sequence converges if and only if it is a Cauchy sequence.
Proof: Suppose that the sequence $\{u_n\}$ be a convergent sequence and $\lim_{n\to\infty} u_n = l$. Now we want to show it is a Cauchy sequence. For each $\varepsilon > 0, \exists$a positive integer N depending on ε, such that for all positive integer n we have
$n > N \Rightarrow |u_n - l| < \varepsilon/_2$............(1)
In particular for $n = m$, equation (1) reduces to
$m > N \Rightarrow |u_m - l| < \varepsilon/_2$............(2)
Now, $m \geq n > N \Rightarrow |u_n - u_m| = |(u_n - l) - (u_m - l)| \leq |(u_n - l)| + |u_m - l| \leq \varepsilon/_2 + \varepsilon/_2$
Thus $m \geq n > N \Rightarrow |u_n - u_m| \leq \varepsilon$.
Hence, the sequence $\{u_n\}$ is a Cauchy sequence.
Conversely, Suppose that the sequence $\{u_n\}$ be a Cauchy sequence. Now we want to show it is a convergent sequence. Since every Cauchy sequence is bounded. Also we know that every bounded sequence has a limit point, so, $\{u_n\}$ has a limit point say$l$ be the limit point of $\{u_n\}$. Now we want to show that $\lim_{n\to\infty} u_n = l$. The sequence $\{u_n\}$ is a Cauchy sequence then for each $\varepsilon > 0, \exists$a positive integer N depending on ε, such that for all positive integers m, n
$m, n > N \Rightarrow |u_n - u_m| < \varepsilon/_3$......(1)
Since l be the limit point of $\{u_n\}$, every nbd of l contains infinitely many terms of $\{u_n\} \Rightarrow u_n \in (l - \varepsilon/_3, l + \varepsilon/_3)$ for infinitely many values of n so, we find a positive integer $k > m$ such that $u_k \in (l - \varepsilon/_3, l + \varepsilon/_3)$
or, $k > m \Rightarrow |u_k - l| < \varepsilon/_3$............(2)
Also from equation (1) we have $m, k > N \Rightarrow |u_k - u_m| < \varepsilon/_3$......(3)
Now, $|u_n - l| = |(u_n - u_m) + (u_m - u_k) + (u_k - l)|$

$\leq |(u_n - u_m)| + |(u_m - u_k)| + |(u_k - l)| < {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3} + {}^{\varepsilon}/_{3}\ \forall n \geq m.$

Thus we have for each $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n we have

$$n > N \implies |u_n - l| < \varepsilon$$

Hence, the sequence $\{u_n\}$ is a convergent sequence.

For each $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n we have

$n > N \implies |u_n - l| < {}^{\varepsilon}/_{3}$............(1)

In particular for $n = m$, equation (1) reduces to

$m > N \implies |u_m - l| < {}^{\varepsilon}/_{3}$............(2)

Also, the sequence $\{u_n\}$ is a Cauchy sequence then for each $\varepsilon > 0, \exists$a positive integer N depending on ε, such that for all positive integers m, n

$m, n > N \implies |u_n - u_m| < \varepsilon$......(3)

Theorem (Cauchy's general principle of convergence): A necessary and sufficient condition for a sequence $\{u_n\}$ to be convergent is that to each $\varepsilon > 0$, there corresponds a positive integer N such that

$\forall p \geq 1, \forall\, n > N \implies |u_n - u_{n+p}| < \varepsilon$

Proof: Let us suppose that the sequence $\{u_n\}$ be convergent and this sequence converges to l, then for a given $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n we have

$n > N \implies |u_n - l| < {}^{\varepsilon}/_{2}$

Since, $\forall p \geq 1$, $n + p \geq n \geq N$. Then

$\forall p \geq 1, n + p \geq n > N \implies |u_{n+p} - l| < {}^{\varepsilon}/_{2}$

Now we consider, $|u_{n+p} - u_n| = |u_{n+p} - l + l - u_n| \leq |u_{n+p} - l| + |l - u_n|$

$$< {}^{\varepsilon}/_{2} + {}^{\varepsilon}/_{2} = \varepsilon$$

$\implies$ The sequence $\{u_n\}$ be a Cauchy sequence.

Conversely, suppose the sequence $\{u_n\}$ be a Cauchy sequence

$\implies$ The sequence $\{u_n\}$ be a bounded sequence.

Suppose m *and* M be the lower and upper bounds of u_{n+p} respectively, then $m \leq u_{n+p} \leq M \implies |M - m| = |(u_{n+p} + \varepsilon) - (u_{n+p} + \varepsilon)| < 2\varepsilon$

Hence, $M - m = 0 \implies M = m$

That is $M - \varepsilon < u_{n+p} < M + \varepsilon$ i.e. $|u_{n+p} - M| < \varepsilon$

Thus, the sequence $\{u_n\}$ be a convergent sequence.

Theorem(Cauchy's first theorem on limits): If $\lim_{n\to\infty} u_n = l$, then

$$\lim_{n\to\infty} \frac{u_1 + u_2 + \cdots . + u_n}{n} = l$$

Proof: Suppose that $u_n = v_n + l$ then, $u_1 = v_1 + l$, $u_2 = v_2 + l$ so on

We have $u_n - l = v_n$

$$\Rightarrow \lim_{n\to\infty} v_n = \lim_{n\to\infty} (u_n - l) = \lim_{n\to\infty} u_n - l = l - l = 0$$

Now,

$$\lim_{n\to\infty} \frac{u_1 + u_2 + \cdots . + u_n}{n} = \lim_{n\to\infty} \frac{v_1 + l + v_2 + l + \cdots . + v_n + l}{n}$$

$$\lim_{n\to\infty} \frac{v_1 + v_2 + \cdots . + v_n + nl}{n} = \lim_{n\to\infty} \frac{v_1 + v_2 + \cdots . + v_n}{n} + l$$

Now we want to show that $\lim_{n\to\infty} \frac{v_1 + v_2 + \cdots . + v_n}{n} = 0$

Since the sequence $\{u_n\}$ be a convergent sequence so, the sequence $\{v_n\}$ be a convergent sequence, hence, it is bounded. There exists a positive integer k such that $|v_n| \le k \forall n \in N$. Also, $\{v_n\}$ is convergent and converges to 0 then for a given $\varepsilon > 0, \exists$ a positive integer N depending on ε, such that for all positive integer n we have

$$n > N \Rightarrow |v_n - 0| < \varepsilon/2$$

Now consider, $\left[\frac{v_1 + v_2 + \cdots . + v_n}{n}\right] = \left[\frac{v_1 + v_2 + \cdots . + v_m + v_{m+1} + \cdots .. + v_n}{n}\right]$

$$\le \frac{|v_1| + |v_2| + \cdots . + |v_m|}{n} + \frac{(v_{m+1} + \cdots .. + v_n)}{n} < \frac{mk}{n} + \frac{(n-m)}{2}\varepsilon, \forall n \ge m$$

$< \frac{mk}{n} + \frac{\varepsilon}{2}$ if $n > \frac{2mk}{\varepsilon}$

Again, let α be any positive integer, $\alpha > \frac{2mk}{\varepsilon}$ $i.e. n \ge \alpha$

So, we have $\frac{mk}{n} \le \frac{\varepsilon}{2}$

Let $\beta = \max\{m, \alpha\}$, then for every $n \ge \beta$, we have

$$\left[\frac{v_1 + v_2 + \cdots . + v_n}{n}\right] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{v_1+v_2+\cdots.+v_n}{n}\right]=0$$

Hence, $\lim_{n\to\infty}\frac{u_n}{n}=l$

Remark: Converse of this theorem need not be true.

Example: Let $u_n=\{(-1)^n\}, then$

$\frac{u_1+u_2+\cdots.+u_n}{n}=0,$ if n is even and $\frac{u_1+u_2+\cdots.+u_n}{n}=-\frac{1}{n}$ if n is odd

Therefore, $\lim_{n\to\infty}\frac{u_1+u_2+\cdots.+u_n}{n}=0$.While the sequence $u_n=\{(-1)^n\}$ is not convergent.

Theorem (Cauchy's Second theorem on limits): If $\{u_n\}$ is a sequence of positive terms and $\lim_{n\to\infty}u_n=l$, then $\lim_{n\to\infty}(u_1.u_2.u_3\ldots u_n.)^{1/n}=l$.

Proof: Let $\{v_n\}$ is a sequence defined by $v_n=\ log u_n,\ \forall n\in N$

Therefore, $\lim_{n\to\infty}u_n=l\Rightarrow\lim_{n\to\infty}v_n=\lim_{n\to\infty}(log u_n)=log l$

Now, by Cauchy's first theorem on limit $\lim_{n\to\infty}\left[\frac{v_1+v_2+\cdots.+v_n}{n}\right]=\lim_{n\to\infty}v_n=log l$

$$\lim_{n\to\infty}\left[\frac{log u_1+log u_2+\cdots.+log u_n}{n}\right]=\lim_{n\to\infty}\frac{1}{n}(log u_1+log u_2+\cdots.+log u_n)=log l$$

$$\lim_{n\to\infty}\frac{1}{n}(log u_1u_2\ldots u_n)=log l=\lim_{n\to\infty}(log u_1u_2\ldots u_n)^{1/n}=log l$$

Or, $\lim_{n\to\infty}(u_1.u_2.u_3\ldots u_n.)^{1/n}=l$.

Remark: Converse of this theorem need not be true

Check your progress

1.Show that the sequence $\{\frac{(-1)^n}{n}\}$ is convergent.

2**.** Show that the sequence $\{\frac{1}{3^n}\}$ is convergent.

3. Show that the sequence $\{\sqrt{n+1}-\sqrt{n}\}\ \forall n\in N$ is convergent.

4. Show that the sequence $\left\{\frac{1}{p^n}\right\}, p>0$ is convergent.

5. Show that the sequence $\{\frac{1}{2^n-1}\}$ is convergent.

6. Show that the sequence $\{log\frac{1}{n}\}$ diverges to $-\infty$.

7. show that $\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ ...+\frac{1}{n}\right)=0.$

8. show that $\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\ ...+\frac{1}{2n-1}\right)=0.$

9. show that $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ ...+\frac{1}{(2n)^2}\right)=0.$

10. Show that $\lim_{n\to\infty}\{\left(\frac{(3n)!}{(n!)^3}\right)\}^{1/n}=27.$

11. show that for any number x, $\lim_{n\to\infty}\frac{(x)^n}{n!}=0.$

12. If $|x|>0 and\ k>0$ show that $\lim_{n\to\infty}\frac{(n)^k}{(x)^n}=0.$

13. Using Cauchy's general principle of convergence, show that the sequence $\{u_n\}$,

Where $u_n=1+\frac{1}{2}+\frac{1}{3}+\ ....+\frac{1}{n}$ is not convergent.

14. Show that the sequence $\{u_n\}$, Where $u_n=1+\frac{1}{3}+\frac{1}{5}+\ ....+\frac{1}{(2n-1)}$ is not a Cauchy sequence. Is it convergent?

15. Using Cauchy's general principle of convergence, to show that the sequence $\{u_n\}$, where $u_n=1+\frac{1}{4}+\frac{1}{7}+\ ....+\frac{1}{3n-2}$ is not convergent.

16. Show that the sequence $\{u_n\}$, Where $u_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ ....+\frac{(-1)^{n-1}}{(n)}$ is a Cauchy sequence.

18. If $\{u_n\}$ is a sequence such that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l$ where $|l|<1$ then $\lim_{n\to\infty}u_n=0.$

19. If $\{u_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l$ then $(u_n)^{1/n}=l.$

20. Prove that $\lim_{n\to\infty}(n)^{1/n}=1.$

21. Prove that $[1 + \left(2\right)^{\frac{1}{2}} + (3)^{1/3} + \cdots . + (n)^{1/n}\Big] = 1/n$.

22. Show that $\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = e^2$.

23. Show that the sequence $\{n\}$ is not a Cauchy sequence.

24. Show that the sequence $\{n^2\}$ is not a Cauchy sequence.

25. If If $\{u_n\}$ is a sequence of positive real numbers and$\{(u_n)^2\}$ is convergent. Find whether $\{u_n\}$ is convergent or not.

6.9. Monotonic sequence: A sequence $\{u_n\}$ is said to be monotonically increasing if $u_n \leq u_{n+1}\ \forall\, n \in N$.

A sequence $\{u_n\}$ is said to be monotonically decreasing if $u_n \geq u_{n+1}\ \forall\, n \in N$.

A sequence $\{u_n\}$ is said to be strictly increasing if $u_n < u_{n+1}\ \forall\, n \in N$

A sequence $\{u_n\}$ is said to be strictly decreasing if $u_n > u_{n+1}\ \forall\, n \in N$

Remark: 1. A sequence $\{u_n\}$ is a monotonic sequence if it is either monotonic increasing or monotonic decreasing sequence.

2.A strictly monotonic sequence may be monotonic after a certain number of $(m-1)\ terms$.

3. A monotonic sequence can not oscillate, as it is bounded below or bounded above according as it is increasing or decreasing.

Example: 1. Sequence $\{2, 2, 2, 4, 4, 6, \ldots\}$ is monotonically increasing.

2.Sequence $\{1, 1/2, 1/4, 1/6 \ldots\}$ is monotonically decreasing.

3. Sequence $\{2, 3, 4, 5, 6, 7, \ldots\}$ is strictly increasing.

4. Sequence $\{1/n\}$ is monotonically decreasing.

5. Sequence $\{0, 2, 0, 2, 0, 2, \ldots\}$ is not monotonically sequence

Theorem: A monotonic increasing sequence is convergent (converges to least upper bound) if and only if it is bounded.

Proof: Suppose that a sequence $\{u_n\}$be a monotonic increasing sequence. Let $S = \{u_n : n \in N\}$ which is a non empty set also it is bounded above. Therefore, there exists a number l which is least upper bound of S.

Claim: l is the limit point of the sequence $\{u_n\}$. Let $\varepsilon > 0$, since $l - \varepsilon < l$.

So $l - \varepsilon$ is not an upper bound of S, so, there exists a positive integer m such that $u_m > l - \varepsilon$. Since, the sequence $\{u_n\}$be a monotonic increasing sequence, therefore, $u_n \geq u_m > l - \varepsilon \; \forall \, n \geq m$.

Since, $lub(S) = l \implies u_n < l < l + \varepsilon \; \forall n \in N$

From these two results we conclude that $l - \varepsilon < u_n < l + \varepsilon \; \forall \, n \geq m$

$\implies |u_n - l| < \varepsilon \; \forall \, n \geq m$

$\implies \lim_{n\to\infty} u_n = l$. Hence the sequence $\{u_n\}$ is convergent and converges to l.

Theorem: A monotonic decreasing sequence is convergent (converges to greatest lower bound) if and only if it is bounded.

Proof: Suppose that a sequence $\{u_n\}$be a monotonic decreasing sequence.

Since every convergent sequence is bounded, therefore, A monotonic decreasing convergent sequence will be bounded.

Let $S = \{v_n = -u_n : n \in N\}$ which is a non empty set also it is bounded monotonic increasing sequence, so, it must be convergent. Therefore, there exists a number l which is greatest lower bound of S.

Claim: l is the limit point of the sequence $\{v_n\}$.

Let $\lim_{n\to\infty} u_n = l = \lim_{n\to\infty} -v_n$

$\implies \lim_{n\to\infty} v_n = -l$ which is a finite quantity. Hence the sequence $\{v_n\}$ is convergent and converges to $-l$. So, the sequence $\{u_n\}$ is convergent and converges to l.

Remark: 1. A monotonic sequence is convergent if and only if it is bounded.

2. A monotonic decreasing sequence, which is not bounded below, diverges to $-\infty$.

3. A monotonic increasing sequence, which is not bounded above, diverges to ∞.

4. Every monotonic decreasing sequence, is either convergent or divergent and diverges to $-\infty$.

5. Every monotonic increasing sequence, is either convergent or divergent and diverges to ∞.

Example: Show that the sequence $\{\frac{1}{2n}\}$ is monotonically decreasing.

Proof: Let $u_n = \frac{1}{2n}$ and $u_{n+1} = \frac{1}{2n+2}$

Now, $u_n - u_{n+1} = \frac{1}{2n} - \frac{1}{2n+2} = \frac{1}{n(n+1)} > 0 \; \forall \, n \in N$

$\Rightarrow u_n - u_{n+1} > 0 \; \forall \, n \in N \Rightarrow u_n > u_{n+1} \forall \, n \in N.$

Hence, the sequence $\{\frac{1}{2n}\}$ is monotonically decreasing.

Example: Show that the sequence $\{u_n\}$ defined by $u_n = \frac{(2n-7)}{(3n+2)}$ is convergent. Find its limit.

Proof: Since, $u_n = \frac{(2n-7)}{(3n+2)}$ and $u_{n+1} = \frac{(2n-5)}{(3n+5)}$

Now, $u_n - u_{n+1} = \frac{(2n-7)}{(3n+2)} - \frac{(2n-5)}{(3n+5)} = \frac{25}{(3n+2)(3n+5)} < 0 \; \forall \, n \in N$

$\Rightarrow u_n - u_{n+1} < 0 \; \forall \, n \in N \Rightarrow u_n < u_{n+1} \forall \, n \in N.$

Hence, the sequence $\{\frac{(2n-7)}{(3n+2)}\}$ is monotonically increasing sequence.

Since, $u_n = \frac{(2n-7)}{(3n+2)} \geq -1 \; \forall \, n \in N.$

Also, $u_n = \frac{(2n-7)}{(3n+2)} \; 1 - \frac{(n+9)}{(3n+2)} < 1 \; \forall \, n \in N.$

Thus, $-1 \leq u_n = \frac{(2n-7)}{(3n+2)} \leq 1 \; \forall \, n \in N.$

Hence, the sequence $\{ \frac{(2n-7)}{(3n+2)}\}$ is bounded so, it is convergent.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(2n-7)}{(3n+2)} = \lim_{n\to\infty} \frac{(2 - {}^{7}/_{n})}{(3 + {}^{2}/_{n})} = {}^{2}/_{3}$$

Check your progress

1.Prove that the sequence $\left\{\left(1+{}^{1}/_{n}\right)^{n}\right\} \forall\, n \in N$ is convergent.

2. Prove that the sequence $\{u_n\}$ defined by $u_1 = 1$ and $u_{n+1} = \sqrt{3u_n}$ is monotonic increasing and converges to 3.

3. Prove that the sequence $\{u_n\}$ defined by $u_1 = 1$ and$u_{n+1} = \sqrt{\alpha + u_n}$ is converges to the positive root of the equation $x^2 - x - \alpha = 0$.

4. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ is converges to the positive root of the equation $x^2 - x - 7 = 0$.

5. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ is monotonic increasing and converges to 2.

6. Show that the sequence $\{{}^{1}/_{n!}\}$ is convergent.

7. Show that the sequence $\{n^2\}$ is a monotonically increasing sequence.

8. Show that the sequence $\{2^n\}$ is a monotonically increasing sequence.

9. Show that the sequence $\{-n^2\}$ is a monotonically decreasing sequence.

10. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2 + u_n}$ is converges to the positive root of the equation $x^2 - x - 2 = 0$.

6.10. Summary:

We are able to understand the concept of Sequence of real numbers, concept of convergence of a real sequence, understand the Concept of divergence and Oscillatory sequences of a real numbers, and understand the concept of subsequences, Cauchy sequence and its uses.

6.11. Terminal Questions

1. Show that the limit of the sequence $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ is one.
2. Define convergence of a real sequence $\{a_n\}_{n=1}^{\infty}$. Prove that:

   The sequence $\left\{\frac{1}{n^2}\right\}$converges to zero.
3. Find a natural number N, such that

   $$n \geq N \Rightarrow \left|\frac{8n-3}{4n+9} - 2\right| < \frac{21}{1000}$$

4. show that the sequence $\{u_n\}$ is a Cauchy sequence where

$$u_n = \frac{n}{n+1}\ .$$

5. Show that the sequence $\{\frac{1}{2^n-1}\}$ is convergent.

UNIT-7

Infinite Series

Structure

7. 1 Introduction

This is most basic unit of this block as it introduces the concept of Partial sum of series, convergent and divergent series of non- negative terms and use of different tests for convergence of series of non- negative terms.We introduce use of different tests for convergence of series of non- negative terms. The theory of Absolutely convergence and conditionally convergence.

Series of non negative terms has an important role in the field of Analysis·

It has many important applications in analysis like as application in almost every field, social, economy, engineering, technology etc.

7. 2 Objectives

After reading this unit we should be able to

1. Understand the concept of Partial sum of series, convergent and divergent series of non- negative terms.

2. Use of different tests for convergence of series of non- negative terms.

3. Understand Use of different tests for convergence of series of non- negative terms.

4. Understand absolutely convergence and conditionally convergence.

Series of non negative terms has an important role in the field of Analysis·

It has many important applications in Analysis.

7.3 Partial sum of series

Let $\{u_n\}$ be a given series then the form $u_1+u_2+u_3+\ ...+u_n+\cdots=\sum_{n=1}^{\infty}u_n$ or $\sum u_n$ is called an infinite series. We define an another sequence $\{S_n\}$ as

$$S_n=u_1+u_2+u_3+\ ...+u_n.$$

Thus S_n denotes the sum of the first n terms of the infinite series $\sum_{n=1}^{\infty}u_n$. This sequence $\{S_n\}$ is said to be sequence of partial sums of the series $\sum_{n=1}^{\infty}u_n$.

7.4 Convergence and divergence of non negative series

Definition: 1. The infinite series $\sum_{n=1}^{\infty}u_n$ is said to be **convergent** if the sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty}u_n$ is convergent. If $\lim_{n\to\infty}S_n=S$, then S is called the sum of the series $\sum_{n=1}^{\infty}u_n$ and written as $\sum_{n=1}^{\infty}u_n=S$.

2. The infinite series $\sum_{n=1}^{\infty}u_n$ is said to be **divergent** if the sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty}u_n$ is divergent.

3. The infinite series $\sum_{n=1}^{\infty} u_n$ is said to be **Oscillate** if the sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$ is oscillate.

Note: 1.The replacement, addition or omission of finite number of terms of a series $\sum_{n=1}^{\infty} u_n$ has no effect on its convergence.

2. The convergence of a series remains unchanged if each of its term is multiplied by a non-zero constant.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$.

Proof: $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = 1 - \frac{1}{(n+1)}$$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 - \frac{1}{(n+1)}) = 1$

$\Longrightarrow$ The sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$ is convergent and converges to 1.

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Example: Consider the series $\sum_{n=1}^{\infty} n^2$

Proof: $u_n = n^2$ and $S_n = u_1 + u_2 + u_3 + \ldots + u_n = 1 + 2^2 + 3^2 + \cdots + n^2$.

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$

$\Longrightarrow$ The sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$ is divergent and diverges to ∞.

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Consider the series $\sum_{n=1}^{\infty}(-1)^n$

We have $S_1 = 1,\ S_2 = 1 - 1 = 0, S_3 = 1 - 1 + 1 = 1, \ldots$

Therefore, $\{S_n\} = \{1, 0, 1, 0, \ldots\}$ which oscillate between 0 and 1

$\Rightarrow$ the series $\sum_{n=1}^{\infty} u_n$ is oscillatory.

7.5 Theorem (Necessary condition for the convergence of an infinite series): If the infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$.

Proof: Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$ and let $\lim_{n\to\infty} S_n = S$ so that $\lim_{n\to\infty} S_{n-1} = S$. We have $u_n = S_n - S_{n-1}$

$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$.
Hence, $\lim_{n\to\infty} u_n = 0$.
Note: The converse of this theorem need not be true.
Example: Consider the series $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots. + 1/\sqrt{n} + ..$

Here, $u_n = 1/\sqrt{n}$ so, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$

But the above series is not convergent. Since,

$S_n = u_1 + u_2 + u_3 + \ldots + u_n =$

$1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots. + 1/\sqrt{n} > 1/\sqrt{n} + 1/\sqrt{n} + \ldots + 1/\sqrt{n} = n/\sqrt{n} = \sqrt{n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$

$\Rightarrow$ The sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$ is divergent and diverges to ∞.

$\Rightarrow$ the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Note: If $\lim_{n\to\infty} u_n \neq 0$, then the series cannot converge.

Example: Consider the series $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \ldots. + \sqrt{\frac{n}{2(n+1)}} + \cdots$ doesnot converge.

Proof: Since $u_n = \sqrt{\frac{n}{2(n+1)}}$, So, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2}} \neq 0$

$\Rightarrow$ the series $\sum_{n=1}^{\infty} u_n$ is not convergent.

Check your progress

1.Show that the series $\sum_{n=1}^{\infty} \cos\left(1/n^2\right)$ is not convergent.

2. Show that the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ ...+\ \frac{n}{n+1}+\cdots$is not convergent.

3. Show that the series $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\ ....+\sqrt{\frac{n}{(n+1)}}+\cdots$ is not convergent.

4. Show that the series $\sum_{n=1}^{\infty}\cos\left({}^{1}/_{n}\right)$ is not convergent.

5. Show that the series $\sum_{n=1}^{\infty}\left({}^{1}/_{n}\right)^{1/n}$ is not convergent.

6. Show that the series $\sum_{n=1}^{\infty}\frac{n}{(2^{-n}+1)}$ is not convergent.

7. Show that the series $\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$ is not convergent.

7.6. Cauchy's General principle of Convergence for series: A necessary and sufficient condition for a series $\sum_{n=1}^{\infty}u_n$ to be convergent is that to each $\varepsilon>0$, there exists a positive integer m such that $|u_{m+1}+u_{m+2}+\cdots+\cdots.+u_n|<\varepsilon\ \forall\ n>m$.

Proof: Suppose that series $\sum_{n=1}^{\infty}u_n$ is convergent. Suppose that $\{S_n\}$ be a sequence of partial sums of the series $\sum_{n=1}^{\infty}u_n$.

The series $\sum_{n=1}^{\infty}u_n$ is convergent$\Leftrightarrow\{S_n\}$ to be convergent.

$\Leftrightarrow$ each $\varepsilon>0$, there exists a positive integer m such that

$|S_n-S_m|<\varepsilon\ \forall\ n>m$. By Cauchy's General principle of Convergence for sequences

$\Leftrightarrow|(u_1+u_2+u_3+\ ...+u_m+u_{m+1}+u_{m+2}+\cdots+\cdots.+u_n)-(u_1+u_2+u_3+\ ...+u_m)|<\varepsilon\ \forall\ n>m$ ....(1)

$\Leftrightarrow|u_{m+1}+u_{m+2}+\cdots+\cdots.+u_n|<\varepsilon\ \forall\ n>m$.

Example: Show that the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ ...+\ \frac{1}{n}+\cdots$is not convergent.

Proof: If it possible suppose the given series is convergent. Suppose $\varepsilon=\frac{1}{4}$.

By Cauchy's General principle of Convergence, we can find a positive integer m such that $|u_{m+1}+u_{m+2}+\cdots+\cdots.+u_n|<\varepsilon\ \forall\ n>m$.

$\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\ ...+\ \frac{1}{n}<\frac{1}{4}\ \ \forall\ n>m$......(1)

Taking $n=2m$, we have $\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\ ...+\ \frac{1}{n}=\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\ ...+\frac{1}{2m}>\frac{1}{2m}+\frac{1}{2m}+\frac{1}{2m}+\ .....+\frac{1}{2m}=m\frac{1}{2m}=\ \frac{1}{2}$.

$\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+ \ ...+\ \frac{1}{n}>\frac{1}{2}. \forall\, n\ > m$

This contradicts the result (1). Hence, the given series is not convergent.

7.7. Convergence of Positive term series: A necessary and sufficient condition for a positive term series $\sum_{n=1}^{\infty} u_n$ to be convergent is that the sequence $\{S_n\}$of partial sums of the series $\sum_{n=1}^{\infty} u_n$ defined by $S_n = u_1+u_2+u_3+ \ ...+u_n$ is bounded above.

Proof: Since $u_n \geq \ \forall\, n\ \in N$, The sequence $\{S_n\}$ is monotonically increasing.

A necessary and sufficient condition for a series $\sum_{n=1}^{\infty} u_n$ to be convergent is that to the sequence $\{S_n\}$ of its partial sums is convergent. Again a necessary and sufficient condition for a sequence $\{S_n\}$ which is monotonic increasing sequence is to be convergent (converges to least upper bound) is that if it is bounded above.

Hence the result.

Note:1 A positive term series is divergent if and only if the sequence of its parsial sums is not bounded above.

2.If a positive term series is divergent then it diverges to ∞. In fact the sequence of partial sums of positive term series being monotonically increasing, it either tend to a finite limit or to plus infinity.

3. A positive term series $\sum_{n=1}^{\infty} u_n$ to be convergent if and only if there exists a number k such that $S_n = u_1+u_2+u_3+ \ ...+u_n < k\ \forall\, n\ \in N$.

4. A positive term series $\sum_{n=1}^{\infty} u_n$ to be divergent if each term after a fixed stage is greater than some fixed positive number.

5. A series $\sum_{n=1}^{\infty} u_n$ of positive terms such that $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ diverges.

6. If each term of a series $\sum_{n=1}^{\infty} u_n$ of positive terms does not exceed the corresponding term of a convergent series $\sum_{n=1}^{\infty} v_n$ of positive terms, then $\sum_{n=1}^{\infty} u_n$ is convergent.

7. If each term of a series $\sum_{n=1}^{\infty} u_n$ of positive terms exceed or equals to the corresponding term of a divergent series $\sum_{n=1}^{\infty} v_n$ of positive terms, then $\sum_{n=1}^{\infty} u_n$ is divergent.

8. If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two convergent series, then $\sum_{n=1}^{\infty} u_n + v_n$ and$\sum_{n=1}^{\infty} u_n - v_n$ are also convergent.

9. If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two divergent series, then $\sum_{n=1}^{\infty} u_n + v_n$ is also divergent.

10. A series $\sum_{n=1}^{\infty} u_n$ whose terms are not necessarily positive may fail to be convergent even if the sequence $\{S_n\}$ is bounded above.
Example: $u_n = (-1)^n$ then $S_n = 0, if\ n$ is even and $S_n = -1, if\ n$ is odd. Here, the sequence $\{S_n\}$ is bounded but not convergent because the sequence $\{S_n\}$ has two limits points Viz. $0\ and - 1$.

Example: Show that the series $\sum_{n=1}^{\infty} (1/n)^{1/n}$ is not convergent.

Solution: Here, $u_n = (1/n)^{1/n} \Rightarrow log u_n = 1/n\, log\, 1/n = -(logn)/n$

$\lim_{n\to\infty} log\, u_n = -\lim_{n\to\infty} (logn)/n$, which is of $\frac{\infty}{\infty}\ form$, then by L Hospital's rule $= -\lim_{n\to\infty} \frac{1/n}{1}$. Thus $\lim_{n\to\infty} log\, u_n = 0 \Rightarrow \log(\lim_{n\to\infty} u_n) = 0$
$\Rightarrow (\lim_{n\to\infty} u_n) = e^0 = 1 \neq 0$.
Hence the given series is divergent.

Theorem (Geometric series): The positive term infinite geometric series $1 + r + r^2 + r^3 + r^4 + \ldots. + r^n + \cdots$ is convergent if and only if $0 \le r < 1$.
Proof: We have $S_n = \frac{1-r^n}{1-r}\ if\ r \neq 1$, and equal to $n\ if\ r = 1$.
Case1: Let $0 \le r < 1$, then $S_n = \frac{1-r^n}{1-r} = S_n = \frac{1}{1-r} - \frac{r^n}{1-r} \le \frac{1}{1-r}\ \forall\, n \in N$
So the sequence $\{S_n\}$ is bounded above and as such the given series is convergent.
Since, $0 \le r < 1 \Rightarrow \lim_{n\to\infty}(r^n) = 0$, we see that in this case the sum of the infinite geometrical series is $\frac{1}{1-r}$.
Case2: Let $r = 1$ then $S_n = n$, so that the sequence $\{S_n\}$ is not bounded and so the such series is not convergent.
Case3: Let $r > 1$ then $S_n = r^n > 1, \forall\, n \in N$, so that the sequence $\{S_n\}$ is not bounded and so the such series is not convergent.

7.8. Comparison test of the first type:1. Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that (i). $\sum_{n=1}^{\infty} v_n$ is convergent and (ii). There exists $m \in N$ such that $u_n \le v_n\ \forall\, n \ge m$, then the series $\sum_{n=1}^{\infty} u_n$ is convergent.
2. Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that (i). $\sum_{n=1}^{\infty} v_n$ is divergent and (ii). There exists $m \in N$ such that $u_n \ge v_n\ \forall\, n \ge m$, then the series $\sum_{n=1}^{\infty} u_n$ is divergent.
Proof: We have $S_n = u_1 + u_2 + u_3 + \ldots + u_n$ and $T_n = v_1 + v_2 + v_3 + \ldots + v_n$
Suppose that $n \ge m$

We write $u_1+u_2+u_3+\ ...+u_m=a$ and$v_1+v_2+v_3+\ ...+v_m=b$ We see that $\forall n\geq m,\ S_n-a\leq T_n-b\ \Rightarrow S_n\leq T_n+a-b$ ……(1)
Since, $\sum_{n=1}^{\infty}v_n$ is convergent then the sequence $\{T_n\}$ of it's partial sums is convergent and therefore bounded so there exists a number k such that
$T_n\leq k\ \forall n\in N$……..(2)
From (1) and (2) we see that $S_n\leq k+a-b\ \forall n\geq m$
We see that the sequence $\{S_n\}$ of partial sums of the infinite series $\sum_{n=1}^{\infty}u_n$ is bounded and as such the series $\sum_{n=1}^{\infty}u_n$ is convergent.
Note: Proof of the second part is for Exercise.
Remark: Let $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ be two positive term series such that $\lim_{n\to\infty}\frac{u_n}{v_n}=l\neq 0$, then the two series $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ have identical behaviours in relation to convergence.
Example: Examine the convergence of the infinite series
(1). $\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$, (2). $\sum_{n=1}^{\infty}\frac{bn-a}{bn^2+a^2}$ (3). $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ (4). $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n^4+2}}$
Solution: We have $u_n=\frac{1}{n^2+a^2}$ take $v_n=\frac{1}{n^2}$
Now, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{n^2+a^2}.n^2=\lim_{n\to\infty}\frac{1}{1+{a^2}/{n^2}}.=1$
Also the series $\sum_{n=1}^{\infty}v_n$ is convergent. Thus the series $\sum_{n=1}^{\infty}u_n$ is also convergent.
(2). We have $u_n=\frac{bn-a}{bn^2+a^2}$ take $v_n=\frac{1}{n}$
Now, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{bn-a}{bn^2+a^2}.n=\lim_{n\to\infty}\frac{1}{1+{a^2}/{n}}.=1$
Also the series $\sum_{n=1}^{\infty}v_n$ is convergent. Thus the series $\sum_{n=1}^{\infty}u_n$ is also convergent.
(3). We have $u_n=\frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ take $v_n=\frac{1}{2\sqrt{n}}$
Now, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{(n+1)}}.2\sqrt{n}=2$
Also the series $\sum_{n=1}^{\infty}v_n$ is divergent. Thus the series $\sum_{n=1}^{\infty}u_n$ is also divergent.
(4). We have $u_n=\sqrt{\frac{n}{n^4+2}}$ take $v_n=\sqrt{\frac{n}{n^4}}=\sqrt{\frac{1}{n^3}}$
Now, $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\sqrt{\frac{n}{n^4+2}}.1\Big/\sqrt{\frac{1}{n^3}}=1$

Also the series $\sum_{n=1}^{\infty}v_n$ is convergent. Thus the series $\sum_{n=1}^{\infty}u_n$ is also convergent.

Example: Examine the convergence of the infinite series

(1). $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots.$

(2). $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n!}}$ (3). $\sum_{n=3}^{\infty}\frac{1}{n^2 logn}$ (4). $\sum_{n=1}^{\infty}\sqrt{n^4+1}-\sqrt{(n^4-1)}$

Solution: We have $u_n = \frac{1}{n^n}$.

Since, $n^n > 2^n$ for $n > 2$ so, $\frac{1}{n^n} < \frac{1}{2^n}$ for $n > 2$

Also, the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with common ratio $\frac{1}{2} < 1$

So the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is convergent. Hence, by comparison test $\sum_{n=1}^{\infty}\frac{1}{n^n}$ is convergent.

(2). We have $u_n = \frac{1}{\sqrt{n!}}$.

Since, $\frac{1}{\sqrt{n!}} < \frac{1}{2^{(n-1)/2}}$ for $n \geq 2$

Since the series $\sum_{n=1}^{\infty}\frac{1}{2^{(n-1)/2}}$ being a geometric series with common ratio $\frac{1}{\sqrt{2}} < 1$ is convergent. Hence the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}$ is convergent.

(3). We have $u_n = \frac{1}{n^2 logn}$.

Since, $\frac{1}{n^2 logn} < \frac{1}{n^2}$ for $n \geq 3$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. Hence, by comparison test $\sum_{n=1}^{\infty}\frac{1}{n^2 logn}$ is convergent.

We have $u_n = \sqrt{n^4+1}-\sqrt{(n^4-1)} = \frac{2}{\sqrt{n^4+1}+\sqrt{(n^4-1)}}$

We take $v_n = \frac{2}{\sqrt{n^4}+\sqrt{(n^4)}} = \frac{1}{n^2}$

Now, $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2n^2}{\sqrt{n^4+1}+\sqrt{(n^4-1)}}. = \lim_{n\to\infty}\frac{2}{\sqrt{1+1/n^4}+\sqrt{(1-1/n^4}} = 1,$ which is finite and non-zero.

Also, the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. Hence, by comparison test $\sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{(n^4-1)})$ is convergent.

Check your progress

1.Examine the convergence of the infinite series$\sum_{n=1}^{\infty}\{(n^3+1)^{1/3}-n\}$.

2. Examine the convergence of the infinite series $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ \ldots+\ \ldots.$

3. Examine the convergence of the infinite series $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \ ....+ \sqrt{\frac{n}{(n+1)}} + \cdots$.

4. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{\text{n}} \sin\left({}^{1}/_{n}\right)$.

5. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{\text{n}^2} \sin\left({}^{1}/_{n}\right)$.

6. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n}{(n^2+3)}$.

7. Examine the convergence of the infinite series $\sum_{n=1}^{\infty}(\sqrt{n^3+1} - \sqrt{(n^3-1)})$.

8. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n+1}{(n^p)}$.

9. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n^5+1)}$.

10. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n}{(n^2+3)}$

7.9. Comparison test of the second type: Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that (i). $\sum_{n=1}^{\infty} v_n$ is convergent and (ii). There exists $m \in N$ such that $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$ $\forall\, n \geq m$, then the series $\sum_{n=1}^{\infty} u_n$ is convergent.

2. Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that (i). $\sum_{n=1}^{\infty} v_n$ is divergent and (ii). There exists $m \in N$ such that $\frac{u_{n+1}}{u_n} \geq \frac{v_{n+1}}{v_n}$ $\forall\, n \geq m$,, then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Proof: We have $S_n = u_1 + u_2 + u_3 + \ ... + u_n$ and $T_n = v_1 + v_2 + v_3 + \ ... + v_n$

Suppose that $n \geq m$,

Let $\frac{u_2}{u_1} < \frac{v_2}{v_1}, \frac{u_3}{u_2} \leq \frac{v_3}{v_2}$ etc.

Thus $S_n = u_1 + u_2 + u_3 + \ ... + u_n = u_1(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \cdots$

Or $S_n \leq u_1\left[1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}.\frac{v_2}{v_1}\ ....\right]$

$$S_n \leq \frac{u_1}{v_1}(v_1 + v_2 \ldots \ldots + v_n)$$

Or $S_n \leq (\frac{u_1}{v_1})T_n$.

Since, $\sum_{n=1}^{\infty} v_n$ is convergent then the sequence $\{T_n\}$ of it's partial sums is bounded. Thus, the sequence $\{S_n\}$ is also bounded and hence, convergent.

Note: Proof of the second part is for Exercise.

7.10. D' Alembert's ratio test: Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$, then the series is (i), convergent if $l > 1$,(ii). Divergent if $l < 1$ and if $l = 1$ then series may converge or diverge.

Example: 1. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1/_{n+1}}{1/_n} = 1.$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1/_{(n+1)^2}}{1/_{n^2}} = 1.$

3. Test the convergence of the following series

(1). $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ ... + \frac{(n+1)!}{3^n} + \cdots.$

(2). $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n+1}$

(3). $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(4). $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots.$

(5). $\frac{1^2 2^2}{1!} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \ ....$

(6). $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \ ....$

Solution:1. Here, $u_n = \frac{(n+1)!}{3^n}$ and $u_{n+1} = \frac{(n+2)!}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{3^n} . \frac{1}{\frac{(n+2)!}{3^{n+1}}} = \lim_{n\to\infty} \frac{3}{n+2} = 0 < 1$$

Hence, by Ratio test, the given series is divergent.

2. Here, $u_n = \frac{2^{n-1}}{3^n+1}$ and $u_{n+1} = \frac{2^n}{3^{n+1}+1}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{n-1}}{3^n+1} . \frac{1}{\frac{2^n}{3^{n+1}+1}} = \lim_{n\to\infty} \frac{1}{2} \frac{3^{n+1}+1}{3^{n+1}} = \frac{3}{2} > 1$$

Hence, by Ratio test, the given series is convergent.

3. Here, $u_n = \frac{n!}{n^n}$ and $u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{n^n} . \frac{1}{\frac{(n+1)!}{(n+1)^{(n+1)}}} = \lim_{n\to\infty} \frac{1}{n+1} \left(\frac{n+1}{n}\right)^n . (n+1)$$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1.$ Hence, by Ratio test, the given series is convergent.

4. Here, $u_n = \frac{n^p}{n!}$ and $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^p}{n!} \cdot \frac{1}{\frac{(n+1)^p}{(n+1)!}} = \lim_{n\to\infty} \left(\frac{n+1}{(1+\frac{1}{n})^p} \right) = \infty$$

Hence, by Ratio test, the given series is convergent.

5. Here, $u_n = \frac{n^2(n+1)^2}{n!}$ and $u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{n!} \cdot \frac{1}{\frac{(n+1)^2(n+2)^2}{(n+1)!}} = \lim_{n\to\infty} (n+1) \frac{1}{(1+\frac{2}{n})^2} = \infty > 1$$

Hence, by Ratio test, the given series is convergent.

6. Here, $u_n = \frac{1.2.3 .....n}{3.5.7.9....(2n+1)}$ and $u_{n+1} = \frac{1.2.3 .....n(n+1)}{3.5.7.9....(2n+1)(2n+3)}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1.2.3 .....n}{3.5.7.9 ....(2n+1)} \cdot \frac{1}{\frac{1.2.3 .....n(n+1)}{3.5.7.9....(2n+1)(2n+3)}}$$

$$= \lim_{n\to\infty} \frac{(2n+3)}{(n+1)} = 2 > 1$$

Hence, by Ratio test, the given series is convergent.

Check your progress

1.Examine the convergence of the infinite series$\sum_{n=1}^{\infty}\{\frac{(n^3+a)}{(2^n+a)}\}$.

2. Examine the convergence of the infinite series $\frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \ ... + \ ....$

3. Examine the convergence of the infinite series$\frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5}$ ....

4. Examine the convergence of the infinite series$\sum_{n=1}^{\infty} \frac{r^n}{n!}$, $r > 0$.

5. Examine the convergence of the infinite series$\sum_{n=1}^{\infty} \frac{r^n}{n^n}$. $r > 0$

6. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n^n x^n}{(n!)}$.

7. Examine the convergence of the infinite series$\sum_{n=1}^{\infty}(\sqrt{n^2+1} - n)x^{2n}$.

8. Examine the convergence of the infinite series$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

9. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{\sqrt{n}\, x^n}{\sqrt{(n^2+1)}}$ $x > 0$.

10. Examine the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{(x^n+x^{-n})}$.

7.11. Cauchy's n^th^ root test: Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty} u_n{}^{1/n} = l$, then the series is (i), convergent if $l < 1$,(ii). Divergent if $l > 1$ and if $l = 1$ then series may converge or diverge (test fails).

Proof: **Case1**: suppose that $l < 1$. Let ρ be a number such that $l < \rho < 1$.

Then there exists m such that $\forall n \geq m,\ u_n{}^{1/n} < \rho\ \Rightarrow u_n < \rho^n$.

Now $\rho < 1$, the geometric series $\sum_{n=1}^{\infty} \rho^n$ is convergent. Thus, by the comparison test of the first type, it follows that the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Case2: Suppose that $l > 1$Let ρ be a number such that $1 < \rho < l$.

Then there exists m such that $\forall n \geq m,\ u_n{}^{1/n} > \rho\ \Rightarrow u_n > \rho^n$.

Now $\rho > 1$, the geometric series $\sum_{n=1}^{\infty} \rho^n$ is divergent. Thus, by the comparison test of the first type, it follows that the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Case3: Suppose $l = 1$.consider the two series (i) $\sum_{n=1}^{\infty} 1/n$ and (ii). $\sum_{n=1}^{\infty} 1/n^2$

The series (i) is divergent and the series (ii) is convergent. We have $\lim_{n\to\infty} (1/n)^{1/n} = 1$ and $\lim_{n\to\infty} (1/n^2)^{1/n} = 1$. In these cases the two series have the same limit $l = 1$ but while one series is convergent and other is divergent.

7.12. Raabe's test: Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty} n[\frac{u_n}{u_{n+1}} - 1] = l$, then the series is (i), convergent if $l > 1$,(ii). Divergent if $l < 1$ and if $l = 1$ then series may converge or diverge (test fails).

Proof: **Case1**: suppose that $l > 1$. Let ρ be a number such that $1 < \rho < l$.

Then there exists m such that $\forall n \geq m,\ \left[n\frac{u_n}{u_{n+1}} - 1\right] > \rho$

$\Rightarrow nu_n - nu_{n+1} > \rho u_{n+1}$.

$\Rightarrow nu_n - (n+1)u_{n+1} > (\rho - 1)u_{n+1}$

Replacing $n\ by\ m, m+1, m+2, \dots, n$ and adding, we get

$mu_m - (n+1)u_{n+1} > (\rho - 1)[u_{m+1} + \cdots . + u_{n+1}]$

$\Rightarrow (\rho - 1)[u_{m+1} + \cdots . + u_{n+1}] <\ mu_m, \Rightarrow [u_{m+1} + \cdots . + u_{n+1}] < {mu_m}/{(\rho - 1)}$

Thus $\forall\ n \in N$, we have $S_n \leq [u_1 + \cdots . + u_{m-1}] +\ mu_m/(\rho - 1)$.

So the sequence $\{S_n\}$ is bounded and as series is convergent.

Case2: suppose that $l < 1$. Let ρ be a number such that $l < \rho < 1$.

Then there exists m such that $\forall n \geq m,\ \left[n\frac{u_n}{u_{n+1}} - 1\right] < 1$

$\Rightarrow nu_n - nu_{n+1} < u_{n+1}. \Rightarrow \frac{u_n}{u_{n+1}} > 1 + {}^{1}/_{n} = {}^{(n+1)}/_{n}$

Replacing n by $m, m+1, m+2, \dots, n-1$ and multiplying them together, we get $\frac{u_n}{u_m} > m/n,\ \forall n \geq m \Rightarrow u_n > \frac{k}{n} where\ k = mu_m$

Thus $\forall\ n \in N$, we have the series $\frac{1}{n}$ is divergent. Hence, $\sum_{n=1}^{\infty} u_n$ is divergent.

7.13. Logarithmic Test: Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty}[nlog\frac{u_n}{u_{n+1}}] = l$, then the series is (i), convergent if $l > 1$,(ii). Divergent if $l < 1$ and if $l = 1$ then series may converge or diverge (test fails).

Proof: Let $l > 1$ and let us choose $\varepsilon > 0$ such that $l - \varepsilon = \gamma > 1$

Now, $\lim_{n\to\infty}[nlog\frac{u_n}{u_{n+1}}] = l \Rightarrow$ there exists a positive integer m such that

$l - \varepsilon < nlog\frac{u_n}{u_{n+1}} < l + \varepsilon\ \forall n \geq m$

$\Rightarrow nlog\frac{u_n}{u_{n+1}} > \gamma\ \forall n \geq m, \Rightarrow \frac{u_n}{u_{n+1}} > e^{\gamma/n}\ \forall n \geq m$…..(1)

We know that the sequence $\{(1+1/n)^n\}$ converges to e and hence,

$e \geq (1+1/n)^n\ \ \forall n \in N$

$\Rightarrow e^{\gamma/n} \geq (1+1/n)^n\ \forall n \in N$ ……(2)

From equation (1) and (2), we have

$\frac{u_n}{u_{n+1}} > (1+1/n)^{\gamma} = ((n+1)/n)^{\gamma} = \frac{v_n}{v_{n+1}}, \forall n \geq m$ …..(3)

Where $v_n = 1/n^{\gamma}$

Since, $\gamma > 1$, so $\sum_{n=1}^{\infty} v_n$converges, then using comparison test of second type it follows that the given series $\sum_{n=1}^{\infty} u_n$ also convergent.

Note: second part has been proved as above.

2.The above logarithmic test is alternative to Raabe's test and should be used when D'Alembert's ratio test fails and when either e occurs in $\frac{u_n}{u_{n+1}}$ or n occurs as an exponent in$\frac{u_n}{u_{n+1}}$.

Example: Test the convergence of the series:

$(i). \sum_{n=1}^{\infty}(1+1/n)^{-n^2}$, (ii). $\sum_{n=1}^{\infty} {n^{n^2}}\Big/{(1+n)^{n^2}}$

$(iii). (\frac{2^2}{1^2} - \frac{2}{1})^{-1} + (\frac{3^2}{2^2} - \frac{3}{2})^{-2} + (\frac{4^2}{3^2} - \frac{4}{3})^{-3} + \dots$

(iv). $\sum_{n=1}^{\infty}(1+nx)^n/n^n$

Solution: (1). $u_n = \{(1+1/n)^{-n}\}^n$

Therefore, $\lim_{n\to\infty}[u_n{}^{1/n}] = \lim_{n\to\infty}[\{(1+1/n)^{-n}]$

$= \lim_{n\to\infty}[\{1/(1+1/n)^n] = 1/e < 1.$

Hence, by Cauchy's root test, the given series is convergent.

(ii). $u_n = n^{n^2}\Big/(1+n)^{n^2} = \{(\frac{n}{1+n})^n\}^n$

Therefore, $\lim_{n\to\infty}[u_n{}^{1/n}] = \lim_{n\to\infty}[\{(\frac{n}{1+n})^n\}] = \lim_{n\to\infty}[1/\{(1+1/n)\}^n] = 1/e < 1.$

Hence, by Cauchy's root test, the given series is convergent.

(iii).The n^{th} term $of\ u_n$ of the given series is given by

$u_n = \{(\frac{1+n}{n})^{n+1} - (\frac{1+n}{n})\}^{-n} \Rightarrow u_n{}^{1/n} = \{(\frac{1+n}{n})^{n+1} - (\frac{1+n}{n})\}^{-1}$

Therefore, $\lim_{n\to\infty}[u_n{}^{1/n}] = \lim_{n\to\infty}[\{(\frac{1+n}{n})^{-1}\}\{(\frac{1+n}{n})^n - 1\}^{-1}]$

$= \lim_{n\to\infty}[\{(1+\frac{1}{n})^{-1}\}\{(1+\frac{1}{n})^n - 1\}^{-1}] = 1\times(e-1)^{-1} = \frac{1}{(e-1)} < 1.$

Hence, by Cauchy's root test, the given series is convergent.

(iv). $u_n = (1+nx)^n/n^n$

Therefore, $\lim_{n\to\infty}[u_n{}^{1/n}] = \lim_{n\to\infty}[\{\{(1+nx)/n\}^n\}^{1/n}]$

$= \lim_{n\to\infty}[(1+nx)/n] = \lim_{n\to\infty}[(x+1/n)] = x.$

Hence, by Cauchy's root test, the given series is convergent if $x < 1$ and divergent if $x > 1$.When $x = 1$, the Cauchy's root test fails. In this case

$u_n = (1+n)^n/n^n = \{(1+n)/n\}^n.$

Therefore, $\lim_{n\to\infty}[u_n] = \lim_{n\to\infty}\{\{(1+n)/n\}^n\} = \lim_{n\to\infty}[(1+1/n)^n] = e > 1.$

So, the given series is convergent.

Hence, the given series is convergent if $x \le 1$ and divergent $if\ x > 1$.

Example: Examine the following infinite series for convergence:

$$\sum_{n=1}^{\infty}\frac{(1.3.5\ldots\ldots(2n-1)}{(2.4.6\ldots.2n)}.\frac{x}{2n}^{2n}$$

Solution: Here $u_n = \frac{(1.3.5\ldots..(2n-1)}{(2.4.6\ldots.2n)}.\frac{x}{2n}^{2n}$ and $u_{n+1} = \frac{(1.3.5\ldots..(2n+1)}{(2.4.6\ldots.(2n+2))}.\frac{x}{(2n+2)}^{2n+2}$

$$\frac{u_n}{u_{n+1}} = \frac{(1.3.5 \ldots\ldots(2n-1)}{(2.4.6 \ldots .2n)} \cdot \frac{x}{2n}^{2n} \times 1/\frac{(1.3.5 \ldots\ldots(2n+1)}{(2.4.6 \ldots .(2n+2))} \cdot \frac{x}{(2n+2)}^{2n+2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n+1)} \frac{(2n+2)}{(2n)} . 1/x^2$$

$$\lim_{n\to\infty} [\frac{u_n}{u_{n+1}}] = \frac{(1+1/n)^2}{(1+1/2n)} \times \frac{1}{x^2} = 1/x^2$$

And as such by D' Alembert's ratio test the series converges if

$\frac{1}{x^2} > 1 \Leftrightarrow x^2 < 1 \Leftrightarrow x < 1$; x being non negative

And diverges if $\frac{1}{x^2} < 1 \Leftrightarrow x^2 > 1 \Leftrightarrow x > 1$; x being non negative

Now we put $x^2 = 1$ In this case we have, $\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n+1)} \frac{(2n+2)}{(2n)}$

$$\Rightarrow \lim_{n\to\infty} \text{n} [\frac{u_n}{u_{n+1}} - 1] = \lim_{n\to\infty} [\frac{n(6n+4)}{2n(2n+1)}] = 3/2 > 1$$

So, the series converges if $x \leq 1$, and diverges if $x > 1$.

Example: Example: Examine the following infinite series for convergence:

$$\sum_{n=1}^{\infty} \frac{1.3.5 \ldots\ldots(4n-5)(4n-3)}{2.4.6 \ldots .(4n-4)(4n-2)} \cdot \frac{x}{4n}^{2n}$$

Solution: Here $u_n = \frac{1.3.5 \ldots\ldots(4n-5)(4n-3)}{2.4.6 \ldots .(4n-4)(4n-2)} \cdot \frac{x}{4n}^{2n}$ and

$$u_{n+1} = \frac{(1.3.5 \ldots .(4n-3).(4n+1)}{(2.4.6 \ldots .(4n-2)(4n+2)} \cdot \frac{x^{2n+2}}{(4n+4)}$$

$$\frac{u_n}{u_{n+1}} = \frac{1.3.5 \ldots\ldots(4n-5)(4n-3)}{2.4.6 \ldots .(4n-4)(4n-2)} \cdot \frac{x}{4n}^{2n} \times \frac{1}{\frac{(1.3.5 \ldots .(4n-3).(4n+1)}{(2.4.6 \ldots .(4n-2)(4n+2)} \cdot \frac{x^{2n+2}}{(4n+4)}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(4n+2)}{(4n+1)} \frac{(4n+4)}{(4n)} . 1/x^2$$

$$\lim_{n\to\infty} [\frac{u_n}{u_{n+1}}] = \frac{\left(4+\frac{2}{n}\right)\left(4+\frac{4}{n}\right)}{4\left(4+\frac{1}{n}\right)} \times \frac{1}{x^2} = \frac{1}{x^2}$$

And as such by D' Alembert's ratio test the series converges if

$\frac{1}{x^2} > 1 \Leftrightarrow x^2 < 1 \Leftrightarrow x < 1$; x being non negative

And diverges if $\frac{1}{x^2} < 1 \Leftrightarrow x^2 > 1 \Leftrightarrow x > 1$; x being non negative

Now we put $x^2 = 1$ In this case we have

$$\frac{u_n}{u_{n+1}} = \frac{(4n+2)}{(4n+1)}\frac{(4n+4)}{(4n)}$$

$$\Rightarrow \lim_{n\to\infty} \text{n} [\frac{u_n}{u_{n+1}} - 1] = \lim_{n\to\infty} \text{n}\{\frac{(4n+2)}{(4n+1)}\frac{(4n+4)}{(4n)} - 1\} = \frac{20n+8}{4(4n+1)} = \frac{20}{16} = \frac{5}{4} > 1$$

So, by Raabe's test, the series converges if $x \leq 1$, and diverges if $x > 1$.

Example: Examine the following infinite series for convergence:

$$\sum_{n=1}^{\infty} \frac{2.4.6 \ldots .2n}{1.3.5.7 \ldots \ldots (2n+1)}$$

Solution: Here $u_n = \frac{2.4.6 \ldots .2n}{1.3.5.7 \ldots \ldots (2n+1)}$ and $u_{n+1} = \frac{2.4.6 \ldots .2n(2n+2)}{1.3.5.7 \ldots \ldots (2n+1)(2n+3)}$

$$\frac{u_n}{u_{n+1}} = \frac{2.4.6 \ldots .2n}{1.3.5.7 \ldots \ldots (2n+1)} \times \frac{1}{\frac{2.4.6 \ldots .2n(2n+2)}{1.3.5.7 \ldots \ldots (2n+1)(2n+3)}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(2n+2)} \Rightarrow \lim_{n\to\infty} [\frac{u_n}{u_{n+1}}] = \frac{\left(2+\frac{3}{n}\right)}{(2+\frac{2}{n})} = 1$$

And as such by D' Alembert's ratio test fails and we apply Raabe's test

$$\Rightarrow \lim_{n\to\infty} \text{n} [\frac{u_n}{u_{n+1}} - 1] = \lim_{n\to\infty} \text{n}\{\frac{(2n+3)}{(2n+2)} - 1\} = \frac{n}{(2n+2)} = \frac{1}{2} < 1$$

So, by Raabe's test, the series diverges.

Example: Examine the following infinite series for convergence:

$$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots .. \text{for x} > 0.$$

Solution: Here $u_n = \frac{n^n x^n}{n!}$ and $u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^n x}{n!} \times \frac{1}{\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}},$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{(n)!}\frac{(n^n)}{(n+1)^{n+1}}.\frac{1}{x} = \left(\frac{n}{n+1}\right)^n.\frac{1}{x}$$

$$\lim_{n\to\infty}[\frac{u_n}{u_{n+1}}] = \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{x} = \frac{1}{ex}$$

And as such by D' Alembert's ratio test the series converges if

$\frac{1}{ex} > 1 \Leftrightarrow x < \frac{1}{e}$; x being non negative

And diverges $\frac{1}{ex} < 1 \Leftrightarrow x > \frac{1}{e}$; x being non negative

Now we put $x = \frac{1}{e}$ In this case we shall apply logarithmic test

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n e \Rightarrow \lim_{n\to\infty} \text{n} \log(\frac{u_n}{u_{n+1}}) = \lim_{n\to\infty} \text{nlog}\{\left(\frac{n}{n+1}\right)^n e\}$$

$$= \lim_{n\to\infty} \text{nlog}\left\{\text{e} \times \left(\frac{n+1}{n}\right)^{-n}\right\} = \lim_{n\to\infty} \text{n}\{\text{loge} - \text{nlog}\left(1+\frac{1}{n}\right)\}$$

$$= \lim_{n\to\infty} n\left\{1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ \ldots\right)\right\} = \frac{1}{2} < 1$$

So, by logarithmic test the series$\sum_{n=1}^{\infty}(u_n)$ diverges. Hence the series$\sum_{n=1}^{\infty}(u_n)$ converges if $x < \frac{1}{e}$, and diverges if $x \geq \frac{1}{e}$.

Example: Examine the following infinite series for convergence:

$$\frac{(a+x)}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \frac{(\text{a}+4x)^4}{4!} + \cdots..\text{for x} > 0.$$

Solution: Here $u_n = \frac{(\text{a}+\text{n}x)^n}{\text{n}!}$ and $u_{n+1} = \frac{(\text{a}+(\text{n}+1)x)^{n+1}}{(\text{n}+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{(\text{a}+\text{n}x)^n}{\text{n}!} \times \frac{1}{\frac{(\text{a}+(\text{n}+1)x)^{n+1}}{(\text{n}+1)!}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{(n)!}\frac{(a+nx)^n}{(a+(n+1)x)^{n+1}} = \frac{(n+1)(a+nx)^n}{(a+(n+1)x)^{n+1}}$$

$$= \frac{n\left(\frac{1}{n}+1\right)\left(\frac{a}{nx}+1\right)^n n^n x^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}(n+1)^{n+1}x^{n+1}} = \frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^n}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}.\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}[\frac{u_n}{u_{n+1}}] = \frac{e^{\frac{a}{x}}}{e} \times \frac{1}{e^{\frac{a}{x}}} = \frac{1}{ex}$$

And as such by D' Alembert's ratio test the series $\sum_{n=1}^{\infty}(u_n)$ converges if $\frac{1}{ex} > 1 \Leftrightarrow x < \frac{1}{e}$; x being non negative

And diverges $\frac{1}{ex} < 1 \Leftrightarrow x > \frac{1}{e}$; x being non negative

Now we put $= \frac{1}{e}$, the test fails. In this case we shall apply logarithmic test

$\frac{u_n}{u_{n+1}} = e.\frac{\left(\frac{ae}{n}+1\right)^n}{\left(\frac{ae}{(n+1)}+1\right)^{n+1}}.\frac{1}{\left(1+\frac{1}{n}\right)^n} \Rightarrow \lim_{n\to\infty} \log(\frac{u_n}{u_{n+1}})$

$= \log(e) + n\log\left\{\frac{ae}{n}+1\right\} - nlog\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ea}{n+1}\right) = 1 + n\left(\frac{ea}{n} - \frac{e^2a^2}{n^2} + \frac{e^3a^3}{n^3} - \cdots.\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - (n+1)\left[\frac{ea}{n+1} - \frac{e^2a^2}{2(n+1)^2} + \frac{e^3a^3}{3(n+1)^3} - \cdots.\right] = \frac{1}{n}\left[\left(\frac{1}{2} - \frac{e^2a^2}{2}\right) + \frac{e^2a^2}{2(n+1)} + \frac{1}{n^2}\left(\frac{e^3a^3}{3} - \frac{1}{3}\right) + \ ....\right] u_{n+1} \leq u_n \Rightarrow$

$\lim_{n\to\infty} \text{nlog}(\frac{u_n}{u_{n+1}})\Big) = \lim_{n\to\infty}\left[\left(\frac{1}{2} - \frac{e^2a^2}{2}\right) + \frac{ne^2a^2}{2(n+1)} + \frac{1}{n}\left(\frac{e^3a^3}{3} - \frac{1}{3}\right) + \ ...\right]$

$= \frac{1}{2} - \frac{e^2a^2}{2} + \frac{e^2a^2}{2} = \frac{1}{2} < 1.$

So, by logarithmic test the series $\sum_{n=1}^{\infty}(u_n)$ diverges. Hence the series $\sum_{n=1}^{\infty}(u_n)$ converges if $x < \frac{1}{e}$, and diverges if $x \geq \frac{1}{e}$

7.14. Cauchy's condensation test: If $f(n)$ is a monotonically decreasing function of $n \in N$, then two infinite series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ are converge or diverge together, a being a positive integer greater than unity.

Proof: $\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \ ....+ f(a)$

$+f(a+1) + f(a+2) + f(a+3) + \ .....+ f(a^2)$

$+f(a^2+1) + f(a^2+2) + f(a^2+3) + \ .....+ f(a^3)$

$+ \cdots ......+ \cdots ..$

$f(a^k+1) + f(a^k+2) + f(a^k+3) + \ .....+ f(a^{k+1})$-----(1)

The term in the k^{th}group are

$f(a^k+1) + f(a^k+2) + f(a^k+3) + \ .....+ f(a^{k+1})$-----(2)

The number of terms in this group $= a^{k+1} - a^k = a^k(a-1)$

Since, $\{f(n)\}$ is a decreasing sequence, it follows that $f(a^{k+1})$ is the smaller term in the k^{th}group. Therefore,

$f(a^k+1) + f(a^k+2) + f(a^k+3) + \ .....+ f(a^{k+1}) > f(a^k+1) + f(a^k+1) + \cdots .+ f(a^k+1)$

i.e. $f(a^k + 1) + f(a^k + 2) + f(a^k + 3) + \dots\dots + f(a^{k+1})$

$> a^k(a-1)f(a^k + 1) = \frac{a^{k+1}(a-1)}{a} f(a^k + 1)$ …..(3)

Putting $k = 0, 1, 2, 3, \dots\dots$ in equation (3)

$$f(2) + f(3) + \quad \dots . + f(a) > \frac{a-1}{a}\{af(a)\}$$

$$f(a+1) + f(a+2) + \cdots \dots . f(a^2) > \frac{a-1}{a}\{a^2 f(a^2)\}$$

$$f(a^2+1) + f(a^2+2) + \cdots \dots . f(a^3) > \frac{a-1}{a}\{a^3 f(a^3)\}$$

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We adding the above inequalities, we get

$\sum_{n=1}^{\infty} f(n) - f(1) > \frac{a-1}{a} \sum_{n=1}^{\infty} a^n f(a^n)$ -----(4)

Since, $\sum_{n=1}^{\infty} a^n f(a^n)$ is divergent, then by comparison test (4) shows that $\sum_{n=1}^{\infty} f(n)$ is also divergent sequence. It follows that $f(a^k)$ is the greater than each term in the k^{th}group (2)

$$f(a^k + 1) + f(a^k + 2) + f(a^k + 3) + \dots\dots + f(a^{k+1})$$
$$< f(a^k) + f(a^k) + \cdots . + f(a^k)$$

i.e.

$$f(a^k + 1) + f(a^k + 2) + f(a^k + 3) + \dots\dots + f(a^{k+1}) < (a-1)\{a^k f(a^k)\}$$

Putting $k = 0, 1, 2, 3, \dots\dots$ in equation (3)

$$f(2) + f(3) + \quad \dots . + f(a) < (a-1)\{f(1)\}$$

$$f(a+1) + f(a+2) + \cdots \dots . f(a^2) < (a-1)\{af(a)\}$$

$$f(a^2+1) + f(a^2+2) + \cdots \dots . f(a^3) < (a-1)\{a^2 f(a^2)\}$$

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We adding the above inequalities, we get

$$\sum_{n=1}^{\infty} f(n) - f(1) < (a-1)\sum_{n=1}^{\infty} a^n f(a^n) + (a-1)f(1)$$

Or,

$\sum_{n=1}^{\infty} f(n) < a\, f(1) + (a-1)\sum_{n=1}^{\infty} a^n f(a^n)$-----(5)

If $\sum_{n=1}^{\infty} a^n f(a^n)$ is convergent, then by comparison test (5) shows that $\sum_{n=1}^{\infty} f(n)$ is also convergent sequence. Thus $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ both converges or diverges together.

Remark: The given series $\sum_{n=2}^{\infty} 1/n(logn\,)^p$ converges if $p > 1$ and diverges if $p \leq 1$.

Case(1). Let $p > 0$. then $f(n)$is a positive monotonically decreasing function of n for all $n \geq 2$. Hence by Cauchy's condensation test we have

$$a^n f(a^n) = a^n \frac{1}{a^n(loga^n\,)^p} = \frac{1}{(nloga)^p} = \frac{1}{n^p(loga)^p}$$

So, $\sum_{n=2}^{\infty} a^n f(a^n) = \frac{1}{(loga)^p}\sum_{n=2}^{\infty}\frac{1}{n^p}$, where $\frac{1}{(loga)^p}$ is a constant.

We know that $\sum_{n=2}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. So, it follows that $\sum_{n=1}^{\infty} a^n f(a^n)$ converges if $p > 1$ and diverges if $0 < p \leq 1$. Again by Cauchy's condensation test, the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ both converges or diverges together. Hence, the series $\sum_{n=2}^{\infty} 1/n(logn\,)^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Case(2): Let $p \leq 0$. Then $\frac{1}{n(logn\,)^p} \geq \frac{1}{n}\ \forall n > 1$

But the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Hence, by comparison test the series $\sum_{n=2}^{\infty} 1/n(logn\,)^p$ diverges if $p \leq 0$.

Hence, The given series $\sum_{n=2}^{\infty} 1/n(logn\,)^p$ converges if $p > 1$ and diverges if $p \leq 1$.

Check your progress

Test the convergence of the given series whose n^{th} term is

(i).$(1 + 1/\sqrt{n})^{-n^{3/2}}$

(ii). $(n - logn)^n/(2n)^n$

(iii). $\{logn/\log(n+1)\}^{n^2 logn}$

2. Test the convergence of the series $\sum_{n=1}^{\infty}(n^{1/n} + x)^n$ for all positive values of x.

3.Test the convergence of the given series

(i). $\sum_{n=1}^{\infty}(1/n^n)$,

(ii). $\sum_{n=2}^{\infty}(1/(logn)^n)$

(iii). $\sum_{n=1}^{\infty}(n + 1/n + 2)^n x^n)\,, x > 0$

(iv). $\sum_{n=1}^{\infty} {(1 + 1/n)^{2n}}/{e^n}$

(v). $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)x^2+\ \left(\frac{4}{5}\right)x^3+\cdots.,if\ x>0$

(vi). $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\frac{4^3}{3^4}x^3+\cdots.+\ \frac{(n+1)^n}{n^{n+1}}x^n+\cdots.,if\ x>0$

4.Examine the following infinite series for convergence:

(i). $1+\frac{x}{1}+\frac{1}{2}.\frac{x^3}{3}+\frac{1}{2}.\frac{3}{4}\frac{x^5}{5}+\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7}+\cdots..$

(ii). $x+\frac{1}{2}.\frac{x^2}{4}+\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\ \frac{x^4}{8}+\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{7}{8}.\frac{9}{10}\ \frac{x^6}{12}+\cdots..$

5.Examine the following infinite series for convergence:

(i). $\sum_{n=1}^{\infty}\frac{logn}{n}$

(ii). $\sum_{n=2}^{\infty}\frac{1}{(logn)^p}$.

(iii). $\sum_{n=1}^{\infty}\frac{1}{n}$

(iv). $\sum_{n=2}^{\infty}\frac{1}{(nlogn)}$

(v). $\sum_{n=2}^{\infty}\frac{1}{(nlogn)^p}$

6.Examine the following infinite series for convergence:

(i). $1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\ \frac{5^4x^4}{5!}+\cdots..$

(ii). $1+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\ \frac{5^5x^5}{5!}+\cdots..$

7.Examine the following infinite series for convergence:

$$\frac{(log2)^2}{2^2}+\frac{(log3)^2}{3^2}+\frac{(log4)^2}{4^2}+\ \ldots.+\frac{(logn)^2}{n^2}+\cdots$$

7.15. Alternating series: A series whose terms are alternatively positive and negative is referred as an alternating series. i.e. A series of the form $u_1-u_2+u_3-\ u_4+\ \ldots=\ \sum_{n=1}^{\infty}(-1)^{n-1}u_n$ where $u_n>0\ \forall\ n\in N$.

Example:$1-$ ½ $+\frac{1}{3}-\frac{1}{4}+$ ….. is an alternating series.

Leibnitz's theorem: Let $\{u_n\}$ be a sequence such that $\forall\ n\in N,(a).u_n\geq 0$, $(b).u_{n+1}\leq u_n\ \ (c).\lim_{n\to\infty}u_n=0$.Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

Proof: We write $S_n=u_1-u_2+u_3-\ u_4+\ \ldots+(-1)^{n-1}u_n$So that $\{S_n\}$ is the sequence of partial sums of the given series. Firstly we want to show that the sequence $\{S_n\}$ converges. Since,

$S_{2n}=u_1-u_2+u_3-\ u_4+\ \ldots+u_{2n-1}-\ u_{2n}$

$S_{2n+2} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$
$S_{2n+2} - S_{2n} = (u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}) -$
$(u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n}) = u_{2n+1} - u_{2n+2} \geq 0,$
$since, u_{n+1} \leq u_n \; \forall \, n \in N \Rightarrow S_{2n+2} \geq S_{2n}$
$\Rightarrow \{S_{2n}\}$ is a monotonically increasing sequence. Also, we have
$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n})$
Each term of this expansion being positive, so, S_{2n} is positive. Also,
$S_{2n} = u_1 - (u_2 - u_3) - u_4 + \ldots - (u_{2n-1} - u_{2n-1}) - u_{2n}$
$= u_1 - [(u_2 - u_3) + u_4 + \ldots + (u_{2n-1} - u_{2n-1}) + u_{2n}]$
Therefore, $S_{2n} \leq u_1. since, u_{n+1} \leq u_n \; \forall \, n \in N.$
$\Rightarrow \{S_{2n}\}$ is bounded above. Thus, $\{S_{2n}\}$ is a bounded monotonically increasing sequence and is as such convergent.
Let $\lim_{n\to\infty} S_{2n} = l \ldots \ldots (1)$
Claim: To show that $\lim_{n\to\infty} S_{2n+1} = l$
Now, $S_{2n+1} = u_1 - u_2 + u_3 - u_4 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1}$
$\Rightarrow S_{2n+1} = S_{2n} + u_{2n+1} = l + 0$. Since, $. \lim_{n\to\infty} u_n = 0, \forall \, n \in N$
$\Rightarrow \lim_{n\to\infty} S_{2n+1} = l \ldots \ldots (2)$
This shows that the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit l.
Now we want to show $\lim_{n\to\infty} S_n = l$
Let $\varepsilon > 0$ be given. Now $\lim_{n\to\infty} S_{2n+1} = l$
$\Rightarrow$ there exists a positive integer $m"$ such that
$|S_{2n+1} - l| < \varepsilon \;\; \forall \, n \geq m" \;\; \ldots \ldots (3)$
Again, $\lim_{n\to\infty} S_{2n} = l \Rightarrow$ there exists a positive integer m' such that
$|S_{2n} - l| < \varepsilon \;\; \forall \, n \geq m' \;\; \ldots \ldots (4)$
From (3) and (4) If we take $m = \max\{m", m'\}$, we see that $|S_n - l| < \varepsilon \;\; \forall \, n \geq m$
So, we see that the sequence $\{S_n\}$ converges to l, thus the given series is convergent.
Note: Let $\{u_n\}$ be a sequence such that $\forall \, n \in N, (a). u_n \geq 0,$ either $(b). u_{n+1} \nleq u_n$ $(c). \lim_{n\to\infty} u_n \neq 0$.Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is not convergent.

7.16. Absolute convergence and conditional convergence: A series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is said to be **absolutely convergent** if the positive term series $\sum_{n=1}^{\infty} |(-1)^{n-1} u_n| = \sum_{n=1}^{\infty} |u_n|$ is convergent.

Example: $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = 1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+$ ... ... is absolutely convergent, because, $\sum_{n=1}^{\infty}|(-1)^{n-1}u_n| = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+$ ... ... is an infinite geometric series of positive terms with common ratio $r=\frac{1}{2}<1$ and it is convergent.

A series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is said to be **conditionally convergent**, if it is convergent without being absolutely convergent. i.e. $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is said to be conditionally convergent if the positive term series $\sum_{n=1}^{\infty}|(-1)^{n-1}u_n| = \sum_{n=1}^{\infty}|u_n|$ is divergent.

Example: $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ ... ... is conditionally convergent, because, $\sum_{n=1}^{\infty}|(-1)^{n-1}u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ ... ... is a series of positive terms which is divergent, but the series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ ... ... is convergent by Leibnitz's test.

Note: Every absolutely convergent series is convergent.

Theorem: Every absolutely convergent series is convergent.

Proof: Let the series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ be any absolutely convergent series. We associate with the series two positive term series $\sum_{n=1}^{\infty}v_n$ and $\sum_{n=1}^{\infty}w_n$ defined as follows: $v_n = u_n, if\ u_n \geq 0, and\ = 0\ if\ u_n < 0$

Also, $w_n = -u_n, if\ u_n \leq 0, and\ = 0\ if\ u_n > 0$

From this we see that $|u_n| = v_n + w_n$ and $u_n = v_n - w_n$

Now for $\forall\ n \in N$, we have $v_n \leq |u_n|,\ and\ w_n \leq |u_n|$

The series $\sum_{n=1}^{\infty}|(-1)^{n-1}u_n| = \sum_{n=1}^{\infty}|u_n|$ being to be convergent, by comparison test, it follows that series $\sum_{n=1}^{\infty}v_n$ and $\sum_{n=1}^{\infty}w_n$ are both convergent. Hence, $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

Note: 1. $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent without being absolutely convergent, i.e. if $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is conditionally convergent, then each of the positive term series $\sum_{n=1}^{\infty}v_n$ and $\sum_{n=1}^{\infty}w_n$ are both divergent and diverge $to\ \infty$.

Since $v_n = \frac{1}{2}\{u_n + |u_n|\}$and $w_n = \frac{1}{2}\{|u_n| - u_n\}$.

Theorem: Every absolutely convergent series is convergent. The converse need not be true.

Proof: We have to show that $\sum_{n=1}^{\infty}|u_n|$ is convergent$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

$Let\ \sum_{n=1}^{\infty}(-1)^{n-1}u_n$ be an absolutely convergent series. i.e. $\sum_{n=1}^{\infty}|u_n\,|$ is convergent. For given any $\varepsilon > 0$, we have to show that $\exists\ m \in N$ such that

$|u_{n+1} + u_{n+2} + u_{n+3} + \cdots .. + u_{n+p}\,| < \varepsilon, \forall n \geq m, \forall p \geq 1$

Since, $\sum_{n=1}^{\infty}|u_n\,|$ is convergent, then by Cauchy's principle of convergence, there exists $m \in N$ such that

$|u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \cdots .. + |u_{n+p}\,| < \varepsilon, \forall n \geq m, \forall p \geq 1$...... (1)

Also, we know, $\forall n \in N, and\ \forall p \geq 0$

$|u_{n+1} + u_{n+2} + u_{n+3} + \cdots .. + u_{n+p}\,| < |u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \cdots .. + |u_{n+p}\,|$

From these we have, $|u_{n+1} + u_{n+2} + u_{n+3} + \cdots .. + u_{n+p}\,| < \varepsilon, \forall n \geq m, \forall p \geq 1$

Hence, by Cauchy's general principle of convergence the given series$\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent. The converse of this theorem need not be true.

Example: We consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ .....$

Here, $u_n = \frac{1}{n}\ and\ \lim_{n\to\infty}\frac{1}{n} = 0, also, \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n, \forall\ n \in\ N.$

By Leibnitz's test $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ .....$ is convergent. But the series $\sum_{n=1}^{\infty}|(-1)^{n-1}\left(\frac{1}{n}\right)| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ ..... = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Hence, a convergent series need not be absolutely convergent.

Theorem: In an absolutely convergent series, the series of its positive terms and the series of negative terms are both convergent.

Proof: Suppose that S_n and T_n be the n^{th} partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ and $\sum_{n=1}^{\infty}|u_n|$ respectively. Therefore,

$S_n = u_1 + u_2 + u_3 + \cdots .. + u_n$ and $T_n = |u_1| + |u_2| + |u_3| + \cdots .. + |u_n|$

Suppose that P_n and Q_n denote the sum of positive and negative terms in S_n. Then we have $S_n = P_n - Q_n$ and $T_n = P_n + Q_n$.......(1)

From this we have, $P_n = \frac{T_n + S_n}{2},\ Q_n = \frac{T_n - S_n}{2}$ ......(2)

Now, it is given that $\sum_{n=1}^{\infty}u_n$ is absolutely convergent.

$\Rightarrow \sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}|u_n|$ are both convergent.

$\Rightarrow \{S_n\}$ and $\{T_n\}$ are both convergent sequences.

Let $\lim_{n\to\infty}S_n = S\ and\ \lim_{n\to\infty}T_n = T$ ....(3)

Now, $\lim_{n\to\infty}P_n = \lim_{n\to\infty} = \frac{T_n + S_n}{2} = \frac{1}{2}(T + S)$ and $\lim_{n\to\infty}Q_n = \lim_{n\to\infty} = \frac{T_n - S_n}{2} = \frac{1}{2}(T - S)$

From these we see that $\{S_n\}$ and $\{T_n\}$ are both convergent sequences. Hence, the series of positive terms and the series of negative terms are separately convergent.

Theorem: In a conditionally convergent series, the series of its positive terms and the series of negative terms are both divergent.

Proof: Suppose that $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent.

$\Rightarrow \{S_n\}$ is convergent and $\{T_n\}$ is divergent sequence.

Let $\lim_{n\to\infty} S_n = S \; and \; \lim_{n\to\infty} T_n = T$ …….(3)

Now, $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} = \frac{T_n+S_n}{2} = \frac{1}{2}(T+S) = \infty \; (\because T \to \infty)$

and $\lim\limits_{n\to\infty} Q_n = \lim\limits_{n\to\infty} = \frac{T_n-S_n}{2} = \frac{1}{2}(T-S) = \infty$

From these we see that $\{S_n\}$ and $\{T_n\}$ are both divergent sequences. Hence, the series of positive terms and the series of negative terms are separately divergent.

Example: Test the convergence, absolute convergence and conditionally convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}1}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} - \; \ldots\ldots for \; p > 0$

Proof: Since, $p > 0$ and $(n+1)^p > n^p$, therefore, $\frac{1}{(n+1)^p} < \frac{1}{n^p}$

$\Rightarrow u_{n+1} < u_n, \forall n \in N$

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0, since, p > 0$

Therefore, by Leibnitz test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}1}{n^p}$ is convergent.

Now, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}1}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent, if $p > 1$, and divergent if $p \leq 1$.

Hence, the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 \leq p \leq 1$.

Example: Test for absolute convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{(n+1)!}$

Proof: We have $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{\frac{n^2}{(n+1)!}}{\frac{(n+1)^2}{(n+2)!}} = \lim\limits_{n\to\infty} \frac{n^2}{(n+1)!} \times \frac{(n+2)!}{(n+1)^2}$

$= \lim\limits_{n\to\infty} \frac{(n+2)}{(1+\frac{1}{n})^2} = \infty.$

So, by ratio test we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{(n+1)!}$ is convergent so, the given series is absolutely convergent.

Example: Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}logn}{(n)^2}=\frac{log2}{2^2}-\frac{log3}{3^2}+\frac{log4}{4^2}-\ \dots\dots=\sum_{n=1}^{\infty}\frac{(-1)^{n}logn}{n^2}$.

Proof: We have, $u_n=\frac{logn}{n^2}\to 0\ as\ n\to\infty. Thus\ \lim_{n\to\infty}u_n=0$

Now we want to show that $u_{n+1}<u_n, \forall n\in N$

Let $u(x)=\frac{logx}{x^2}, so\ that\ u'(x)=\frac{1-2logx}{x^3}<0\ \ \forall\ x>e^{1/2}$

(since, $x>e^{1/2}\Leftrightarrow logx>\frac{1}{2}\ \Leftrightarrow 1-2logx<0$)

$\Rightarrow u(x)$ is a decreasing function $\forall\ x>e^{1/2}$

$\Rightarrow u_{n+2}\le u_{n+1}\ \forall\ n\in N, (since, n+2>n+1>e^{\frac{1}{2}}\ \ \forall\ n\in N)$

$$\Rightarrow \frac{\log{(n+2)}}{(n+2)^2}\le\frac{\log{(n+1)}}{(n+1)^2}\ \forall\ n\in N$$

So, that $u_{n+1}\le u_n\ \forall\ n\in N.$

Hence, the both conditions of Leibnitz's test are satisfied and so the given series is convergent.

Example: Show that the series $\sum_{n=1}^{\infty}\frac{(x)^n}{n!}=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots..$ converges absolutely for all values of x.

Solution: We have $u_n=\frac{(x)^n}{n!}\ and\ u_{n+1}=\frac{(x)^{n+1}}{(n+1)!}$

$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|}=\lim_{n\to\infty}\frac{\frac{(x)^n}{n!}}{\frac{(x)^{n+1}}{(n+1)!}}=\lim_{n\to\infty}\frac{(n+1)!}{n!|x|}\ =\lim_{n\to\infty}\frac{(n+1)}{|x|}\ =\infty.$ Provided $x\neq 0$. By ratio test, $\sum_{n=1}^{\infty}|u_n|$ is convergent for all values of x. Hence, the given series converges absolutely for all values of x.

Check your progress

Test the convergence of the given series whose n^{th} term is

(i).$\frac{(-1)^{n+1}1}{3n+2}$

(ii). $\frac{(-1)^{n+1}1}{n^2}$

(iii). $\frac{(-1)^{n+1}1}{2n(2n+1)}$

2. Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n}1}{n^p}$ is absolutely convergent for all values of $p>0$ and conditionally convergent for all values of $p<0$

3.Test the convergence of the given series. Are these series absolutely or conditionally convergent ?

(i). $\sum_{n=1}^{\infty}(-1)^{n+1}(1/\sqrt{n})$,

(ii). $\sum_{n=2}^{\infty}(-1)^{n+1}(1/\log(n+1))$

(iii). $\sum_{n=1}^{\infty}(-1)^{n+1}(1/\log(n+1)$,

(iv). $\sum_{n=1}^{\infty}(-1)^{n+1}(1/\sqrt{n+1})$

(v). $\frac{1}{3.2^2}+\frac{1}{5.3^2}+\frac{1}{7.4^2}+\frac{1}{9.5^2}+\cdots.,$

(vi). $\frac{1}{3.4}+\frac{1}{5.4^2}+\frac{1}{7.4^3}+\cdots.+\ \ldots.,$

4.Examine the following infinite series for convergence where a being positive:

(i). $\sum_{n=1}^{\infty}(-1)^{n}(1/\sqrt{(n+a)})$,

(ii). $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{1}{\sqrt{(n}+\sqrt{a}}\right)$

(iii). $\sum_{n=1}^{\infty}(-1)^{n}(1/(n+a))$,

(iv). $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{1}{(\sqrt{(n}+\sqrt{a})^{2}}\right)$

5.Examine the following infinite series for convergence:

(i). $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\sqrt{(n+1)}-\sqrt{(n-1)}\right)$

(ii). $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\sqrt{(n+1)}-\sqrt{n}\right)$.

(iii). $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{a}{n}\right), a>0$

(iv). $\sum_{n=1}^{\infty}(-1)^{n}n\left(1-\cos\left(\frac{a}{n}\right)\right), a>0$

(v). $\sum_{n=2}^{\infty}\frac{1}{n(logn)^p}$

6.Examine the following infinite series for convergence:

(i). $\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{7}}+\ldots\ldots..$

(ii). $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}+\frac{1}{9.10}+\cdots..$

(iii). $(\frac{1}{2})^2-(\frac{1.3}{2.4})^2+(\frac{1.3.5}{2.4.6})^2-\ \ldots\ldots$

7.Examine the infinite series for convergence $\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{1}{2n+3})$.

8.Examine the infinite series for convergence $\sum_{n=1}^{\infty}[(n^3+1)^{\frac{1}{3}}-n]\frac{1}{logn}$.

9.Show that $1-\frac{1}{4.3}+\frac{1}{4^2.5}-\frac{1}{4^3.7}+\ \ldots\ldots$is convergent.

10.Show that the series $1 + x + x^2 + x^3 + \cdots ... + x^{n-1} +$ ... is convergent if $x < 1$ and divergent if $x > 1$.

7.17. Summary: .

We are able to understand the concept of Partial sum of series, convergent and divergent series of non- negative terms, use of different tests for convergence of series of non- negative terms, understand the use of different tests for convergence of series of non- negative terms and to understand absolutely convergence and conditionally convergence of an alternating series .

7.18. Terminal Questions

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$.
2. Show that the series $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \ldots . + \sqrt{\frac{n}{(n+1)}} + \cdots$ is not convergent.
3. Examine the convergence of the infinite series $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \ldots . + \sqrt{\frac{n}{(n+1)}} + \cdots$.
4. Test the convergence of the following series
$$\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots + \frac{(n+1)!}{3^n} + \cdots.$$
5. Examine the following infinite series for convergence:
   (i). $\sum_{n=2}^{\infty} \frac{1}{(logn)^p}$.
   (ii). $\sum_{n=2}^{\infty} \frac{1}{(nlogn)}$
   (iii). $\sum_{n=2}^{\infty} \frac{1}{(nlogn)^p}$

Bachelor Of Science

SBSMM - 01

Elementary Analysis

Uttar Pradesh
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Block

4 Multiple Integral and Its Applications

Unit -8
Double and Triple Integral

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BLOCK-4

Multiple Integral and Its Applications

This unit of this block is most useful unit of this block as it introduces the concept of double and triple integral and its applications for area and volume for a given curve. It has many important applications in analysis like as application in almost every field, social, economy, engineering, technology etc. It has also many applications in applied mathematics and physics.

UNIT-8
Double and Triple Integral

Structure

6. 1. Introduction

This is most useful unit of this block as it introduces the concept of double and triple integral and its applications for area and volume for a given curve.

It has many important applications in analysis like as application in almost every field, social, economy, engineering, technology etc. It has also many applications in applied mathematics and physics.

6. 2. Objectives

After reading this unit we should be able to

1. Understand the concept of double and triple integral.

2. Concept of applications for area and volume for a given curve.

3. Understand the concept of Gamma function and Beta function.

4. Understand the concept of relation between multiple integral and these special functions.

8.3. Evaluation of Double Integrals: The use-fullness of double integrals would be limited if it were necessary to take limit of sums in order to evaluate them. Fortunately there is an alternative way of evaluating double integrals by successive single integrations. We show that the two methods are equivalent.

If A is a region bounded by the curves$y = f_1(x), y = f_2(x) x = a \text{ and } x = b$, then

$\int\int f(x, y) dA = \int_a^b \{\int_{f_1(x)}^{f_2(x)} f(x, y) dy\} dx \ldots . (1)$

where, the integration with respect to y is performed first treating x as a constant.

To prove this, we divide the region A by a network of lines parallel to the axes of coordinates into rectangular elements of width h and height k.Then $\iint \mathrm{f(x, y)dA} = \lim_{h\to 0, k\to 0} \sum f(x_r, y_r)\mathrm{hk}$ .....(2), where (x_r, y_r) denotes a point inside the r^{th} rectangle, and the summa tion extends over all the complete elementary rectangles. In evaluating the limit, the order in which the elements are summed up is immaterial. We can first sum up over all the elementary rectangles which lié one above another in a vertical column. This gives the sum of $f(x, y)hk$ over a particular vertical strip. We can then take the sum over A strip by strip, from the first to the last strip. Thus, we can write the sum (2), as

$\lim_{h\to 0, k\to 0} \sum_{r=1}^{p} \{\sum_{s=1}^{m} f(x_r, y_s)\, k\} h$ ....(3).

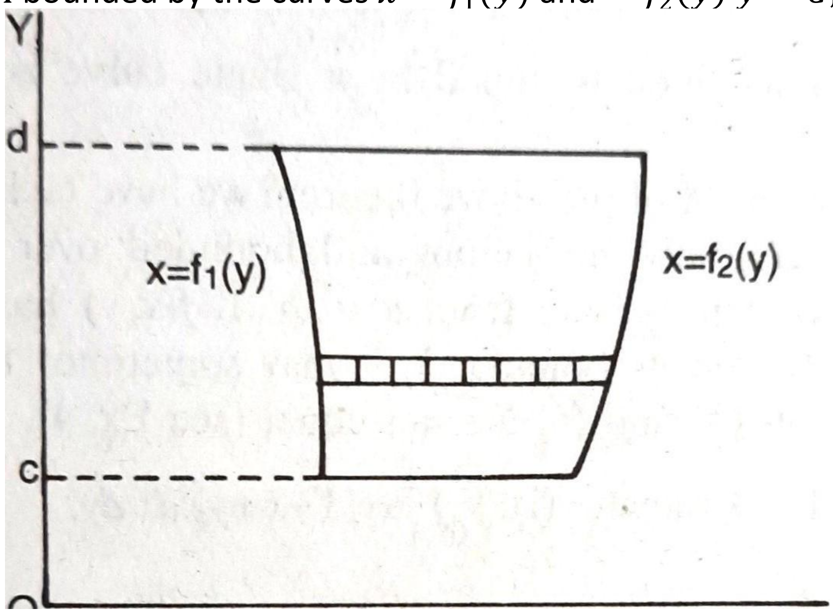
where (x_r, y_s) is a point inside the s^{th} rectangle in the r^{th} vertical strip, p is the total number of strips and m the number of rectangles in the r^{th} strip. The summation inside the brackets is performed first.During this summation x_r can be kept as constant.Taking the limits of the two sums in (3) successively, the terms inside the brackets. $\lim_{k\to 0} \sum_{s=1}^{m} f(x_r, y_s)\, k\} = \int_{Y_1}^{Y_2} f(x_r, y)\, dy$.Where, Y_1 and Y_2 are the extreme values of y in the r^{th} strip. Since the region A is bounded below and above by the curves $y = f_1(x_r)$ and $y = f_2(x_r)$.We can take $y_1 = f_1(x_r)$ and $y_2 = f_2(x_r)$. Thus the limit of the terms inside the brackets in (3) is $\int_{f_1(x_r)}^{f_2(x_r)} f(x_r, y)\, dy$ $=F(x_r)$, say, after integration, (3) becomes

$\lim_{h \to 0} \sum_{r=1}^{p} F(x_r)\, h = \int_a^b F(x)\, dx = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x\,, y)\, dy \right\} dx$. We, therefore, see that $\iint \mathrm{f(x,y)dA} = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x\,, y)\, dy \right\} dx$ .....(4)

Generally the brackets are omitted from the integral on the right hand side of (4) and it is written either as

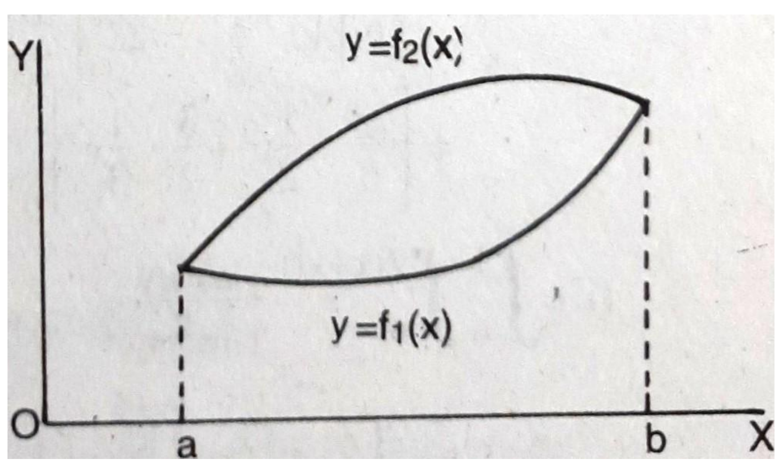
$\int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x\,, y)\, dy \right\} dx = \int_a^b dx \int_{f_1(x)}^{f_2(x)} f(x\,, y)\, dy$.

Similarly by summing along a horizontal strip first, we can show that for a region A bounded by the curves $x = f_1(y)$ and $= f_2(y)$ $y = C, y = d$,



we have $\iint \mathrm{f(x,y)dA} = \int_c^d \left\{ \int_{f_1(y)}^{f_2(y)} f(x\,, y)\, dx \right\} dy$.......(5)

We have thus reduced the double integration to a process of successive single integrations. The student should note that in (4), the first integration is along a vertical strip. Only y is treated as a variable in this integra.tion, x being treated as a constant. The limits for this integration are the values of y for the lowest and the highest points of the strip. Naturally, these are given by the



equations of the curves bounding the strip below and above. The second integration with respect to x performs a strip-wise summation. The limits of integration are the values of x for the points of region A at the extreme left and the extreme right. Similar remarks apply to the integral (5). These points will help the student in determining the limits for successive integrations the region of integration A is given. It is seen from the above that even when the region of integration is bounded by two curves. we can consider it bounded by the four lines. $y = f_1(x),\ y = f_2(x), x = a, x = b$.

In the proof of the above theorem we have tacitly assumed that $f(x, y)$ is uniformly continuous and bounded over A, and $F(x)$ is bounded and integrable from a to b. If $f(x, y)$ has discontinuities within A (or on its boundary, it may sometimes happen that the two integrals (4) and (5) are not equal.

Example1. Evaluate (i) $\int_0^3 \int_1^2 xy(1 + x + y)dxdy$.

$$(ii).\ \int_0^1 \int_1^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$$

Solution: (i) $\int_0^3 \int_1^2 xy(1 + x + y)dxdy = \int_0^3 \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right] dx, for\ y = 1\ to\ y = 2$.

$= \int_0^3 \left[\frac{23x}{6} + \frac{3x^2}{2}\right] dx = \left[\frac{23x^2}{12} + \frac{x^3}{2}\right] at\ x = 0\ to\ x = 3 = \frac{23.9}{12} + \frac{27}{2} = \frac{123}{4}.$

$$= \int_0^3 \left[\frac{1}{2}x(4-1)+\frac{1}{2}x^2(4-1)+\frac{1}{3}x(8-1)\right] dx$$

$$= \int_0^3 \left[\left(\frac{3}{2}+\frac{7}{3}\right)x+\frac{3}{2}x^2\right] dx.$$

$$= \left[\frac{23}{6}\cdot\frac{1}{2}x^2+\frac{3}{2}\cdot\frac{1}{3}x^3\right]_0^3 = \frac{23}{6}\cdot\frac{9}{2}+\frac{27}{2}=30\frac{3}{4}.$$

(ii) $\int_0^1\int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}$

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(1+x^2)}}\right]_0^{\sqrt{(1+x^2)}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}[\tan^{-1}1-\tan^{-1}0]dx=\frac{1}{4}\pi\int_0^1\frac{dx}{\sqrt{(1+x^2)}}$$

$= \frac{\pi}{4}\left[log\left\{x+\sqrt{(1+x^2)}\right\}\right] atx = 0\ to\ x = 1, = \frac{\pi}{4}\ log\ (1+\sqrt{2})\,.$

Example2. Evaluate $\iint xy\,dxdy$ over the region in the positive quadrant for which $x + y \le 1$.

Solution: The region of the integration is the area A bounded by the two axes and the straight line $x + y = 1$.We can consider it as the area bounded by the lines $y = 0, y = 1 - x, x = 0\ and\ x = 1$.

Therefore, $\iint xy\,dxdy = \int_0^1\int_0^{1-x} xydxdy = \int_0^1 \frac{1}{2}x(1-x)^2\,dx = \frac{1}{2}\int_0^1 (x - 2x^2 + x^3)dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right] at\ x = 0\ to\ x = 1;\ = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}$

Example3. Evaluate $\iint (x+y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.For the ellipse $\frac{y}{b} = \pm\sqrt{(1 - x^2/a^2)}$.

So the region of integration can be considered as bounded by the lines $y = -b\sqrt{(1 - x^2/a^2)}, y = b\sqrt{(1 - x^2/a^2)}\ and\ x = -a\ to\ x = a$.Therefore the given double integral

$\int_{-a}^{a}\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}(x^2\ +2xy+y^2)\ dxdy = 2\int_{-a}^{a}\int_{0}^{b\sqrt{(1-x^2/a^2)}}(x^2\ +y^2)\ dxdy =$ $4\int_{0}^{a}\int_{0}^{b\sqrt{(1-x^2/a^2)}}(x^2\ +y^2)\ dxdy = 4\int_{0}^{a}[x^2y+\frac{1}{3}y^3]\ at\ y=0\ to\ y=$ $b\sqrt{(1-x^2/a^2)}$ = $4\int_{0}^{a}\{x^2b\sqrt{(1-x^2/a^2)}+\frac{1}{3}b^3(1-x^2/a^2)^{3/2}\}dx =$ $4b\int_{0}^{\pi/2}\{a^2sin^2\theta cos\theta+\frac{1}{3}b^2cos^3\theta\}acos\theta d\theta$, Putting $x=asin\theta, dx=$ $acos\theta d\theta, So, 4ab\int_{0}^{\pi/2}\{a^2sin^2\theta cos^2\theta+\frac{1}{3}b^2cos^4\theta\}d\theta = 4ab\left\{a^2\frac{\frac{\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{2}}{2.2.1}+\right.$ $\left.\frac{1}{3}b^2\frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}\ .\sqrt{\pi}}{2.2.1}\right\} = 4ab\left\{\frac{1}{16}\pi a^2+\frac{1}{16}\pi b^2\right\} = \frac{1}{4}\pi ab(a^2+b^2).$

NOTE: In step (1) we have halved the range of integration of y. The terms x^2+y^2, which are even function of y, get multiplied by 2. The term xy, which is an odd function of y, gives a zero term. Similarly, in step (2) we hålve the range of integration of x and multiply by a factor 2. This is possible, since the integrand and also the limit $b\sqrt{(1-x^2/a^2)}$ which enters in the second integration, are even functions of x.

Example4. Show that $\int_{0}^{1}dx\int_{0}^{1}\frac{(x-y)}{(x+y)^3}dy \neq \int_{0}^{1}dy\int_{0}^{1}\frac{(x-y)}{(x+y)^3}dx$. Find the values of the two integrals.

Solution: The integral on the left

$$= \int_0^1 dx\int_0^1 \frac{2x-(x+y)}{(x+y)^3}dy = \int_0^1 dx\int_0^1\left\{\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\right\}dy$$

$$= \int_0^1\left[\frac{-x}{(x+y)^2}+\frac{1}{x+y}\right]_0^1 dx = \int_0^1\left[-\frac{x}{(1+x)^2}+\frac{1}{x}+\frac{1}{1+x}-\frac{1}{x}\right]dx$$

$$= \int_0^1\frac{dx}{(1+x)^2} = \left[-\frac{1}{1+x}\right]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}.$$

The integral on the right

$$= \int_0^1 dy\int_0^1\left\{\frac{1}{(x+y)^2}-\frac{2y}{(x+y)^3}\right\}dx = \int_0^1\left[-\frac{1}{x+y}+\frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y}\right] dy = -\int_0^1 \frac{dy}{(1+y)^2} = \frac{1}{2} - 1 = -\frac{1}{2}.$$

We see that the two integrals arc not cqual.

Check your Progress

Evaluate the following double integrals.

1. $\int_0^a \int_0^b (x^2 + y^2) dxdy$
2. $\int_0^1 \int_0^2 (x + 2) dxdy$
3. $\int_1^a \int_1^b (\frac{1}{xy}) dxdy$
4. $\int_1^2 \int_0^x \frac{1}{x^2+y^2} dxdy$
5. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2 y) dxdy$
6. $\int_0^{\pi/2} \int_{\pi/2}^{\pi} cos(x + y) dydx$
7. $\int_0^a \int_0^{\sqrt{a^2-y^2}} (\sqrt{a^2-x^2 - y^2}) dydx$
8. $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x) dxdy$

9. Evaluate $\iint x^2y^2 \, dxdy$ over the region $x^2 + y^2 \leq 1$.10.Evaluate $\iint (x^2 + y^2) \, dxdy$ over the region in the positive quadrant for which $x + y \leq 1$.

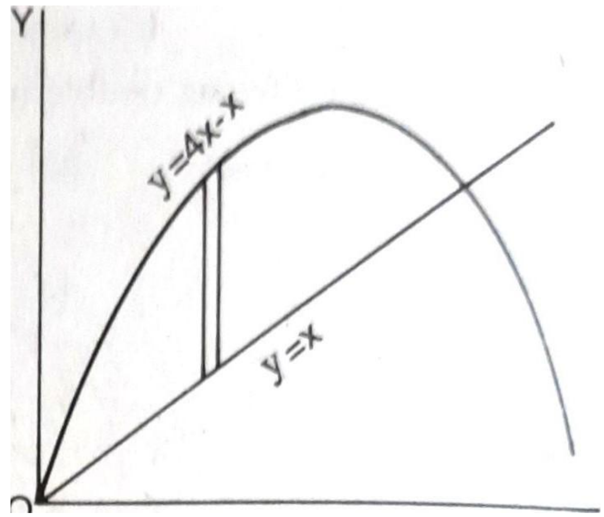
11. Evaluate $\iint xy/\sqrt{1 - y^2} \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

12. Evaluate $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$.
13. Find the mass of a plate in the form of a quadrant of an ellipse $x^2/a^2 + y^2/b^2 = 1$, whose density per unit area is given by $\rho = kxy$.
18. Find the mass of the area between $y = x^3$ and $x = y^2$, if $\rho = k(x^2 + y^2)$.

8.4. Area by double integration: Putting $f(x, y) = 1$ in the definition of the double integral, we see that the area A lying between the curves $y = f_1(x), y = f_2(x), x = a, and\ x = b$ is $\iint dA = \int_a^b \int_{f_1(x)}^{f_2(x)} dxdy$.

Example: Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution: The two curves intersect at points whose abscissae are given by $4x - x^2 = x, i.e. x = 0\ or\ x = 3$. The area can be considered as lying between the curves by $4x - x^2 = y, y = x, x = 0\ and\ x = 3$.So So, integrating along a vertical strip first, we see that the required area



$\int_0^3 \int_x^{4x-x^2} dxdy = \int_0^3 (4x - x^2 - x)dx = \int_0^3 (3x - x^2)dx = \frac{27}{2}$.

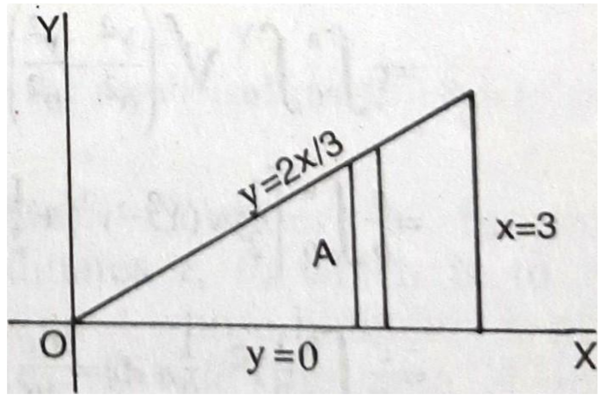
8.5. Volume under a surface: Let A be a region in the plane XOY enclosed by the curve $\varphi(x, y) = 0$. In three dimensions $\varphi(x, y) = 0$ represents a cylinder based on this curve and with generators parallel to z −axis. It is required to find the volume inside the cylinder, enclosed by the surface $z = f(x, y)$ and the planeXOY. Divide the region A by a network of lines parallel to OX and OY into a number of small rectangles of area hk, and consider one of these rectangles $PQRS$. Construct a vertical prism on $PQRS$ as base, bounded at the top by the portion $P'Q'R'S'$ of the surface $z = f(x, y)$. If z_1 and z_2 are the minimum and maximum ordinates of the surface $P'Q'R'S'$ the volume of the prism lies between $hkz_1$and hkz_2.So the volume is hkz, where$z_1 < z < z_2$. Evidently z is the ordinate of some point on $P'Q'R'S'$..

The volume between the surface $z = f(x, y)$ and the region A is composed of similar prisms constructed over all the elementary rectangles in A. Therefore, the required volume, $\lim_{h\to 0,k\to 0} \sum zhk = \iint zdA$. If the region A may be considered as enclosed by the curves$y = f_1(x), y = f_2(x), x = a\ and\ x = b$, we can write the volume as$\int_a^b \int_{f_1(x)}^{f_2(x)} dxdy$.

NOTE: When writing the integral for the volume, the student should keep in mind that the integrand $f(x, y)$ is taken from the surface $z = f(x, y)$ which covers the top of the volume while the limits a, b, f_1, f_2 are taken from the base area A in the xy −plane.

Example: Find the volume under the plane $x + y + z = 6$ and above the triangle in the xy −plane bounded by $2x = 3y, y = 0, x = 3$.

Solution: The required volume $V = \iint zdA = \iint (6 - x - y)dA$. Where A is the region shown in the figure. Integrating along a vertical strip first, we have



$$V = \int_0^3 \int_0^{2x/3} (6 - x - y)dxdy$$

$$= \int_0^3 [4x - \frac{2}{3}x^2 - \frac{2}{9}x^2]dx$$

$$= \int_0^3 \left[4x - \frac{8}{9}x^2\right] dx = 18 - 8 = 10.$$

Example: Find the volume in the positive octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution: The required volume lies between the ellipsoid, $z = c\sqrt{(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})}$

And the plane XOY, and and is bounded on the sides by the planes $x = 0, y = 0$. The given ellipsoid cuts XOYplane in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$. Therefore, the region A above which the required volume lies, is bounded by curves $y = 0, y = b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$,

$x = 0 \ and \ x = a$. Therefore, the required volume is

$$= \iint zdA = \int_0^a \int_0^{b\sqrt{(1-\frac{x^2}{a^2})}} c\sqrt{(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})}dxdy$$

$= c \int_0^a \int_0^Y \sqrt{(\frac{Y^2}{b^2} - \frac{y^2}{b^2})}dxdy$, putting $\sqrt{(1 - \frac{x^2}{a^2})} = \frac{Y}{b}$

$$\frac{c}{b}\int_0^a \frac{1}{2}Y^2 . \frac{\pi}{2}dx = \frac{\pi c}{4b}\int_0^b b^2\left(1 - \frac{x^2}{a^2}\right) dx = \frac{1}{6}\pi abc.$$

Check your Progress

Find by double integration the area of the region enclosed by the following curves:

1.$x^2 + y^2 = a^2$ and $x + y = a$ (in the first quadrant).

2.$y^2 = x^3$ and $y = x$.

3.$9xy = 4$ and $2x + y = 2$.

4.$(x^2 + 4a^2)y = 8a^3, 2y = x, and\ x = 0$.

5.Show by doublc integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

6.Find the volume of the cylinder $x^2 + y^2 - ax = 0$ bounded by the planes $z = 0$ and $z = x$.

7.Find the volume under the plane $x + z = 2$, above $z = 0$ and within the cylinder $x^2 + y^2 = 4$.

8.Find the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.

9.Find the volume bounded by the coordinate planes and the plane $x/a + y/b + z/c = 1$.

10.Find the volume bounded by $4z = 16 - 4x^2 - y^2$ and the plane $z = 0$.

11.Find the volume enclosed by the cylinders $y^2 = z$ and $x^2 + y^2 = a^2$, and the plane $z = 0$.

12.Find the volume in the first octant bounded by the parabolic cylinders $z = 9 - x^2$, $x = 3 - y^2$.

13.Find the volume in the first octant bounded by $z = x^2 + y^2$ and $y = 1 - x^2$.

14.Find the volume inside the paraboloid $x^2 + 4z^2 + 8y = 16$ and on the positive side of xz −plane.

15. $\int_0^{\pi} \int_0^{asin\theta} rd\theta dr$

16. $\int_0^{\pi/2} \int_0^{acos\theta} rsin\theta d\theta dr$

17. $\int_0^{\pi}\int_0^{a(1+cos\theta)} r^2 cos\theta d\theta dr$

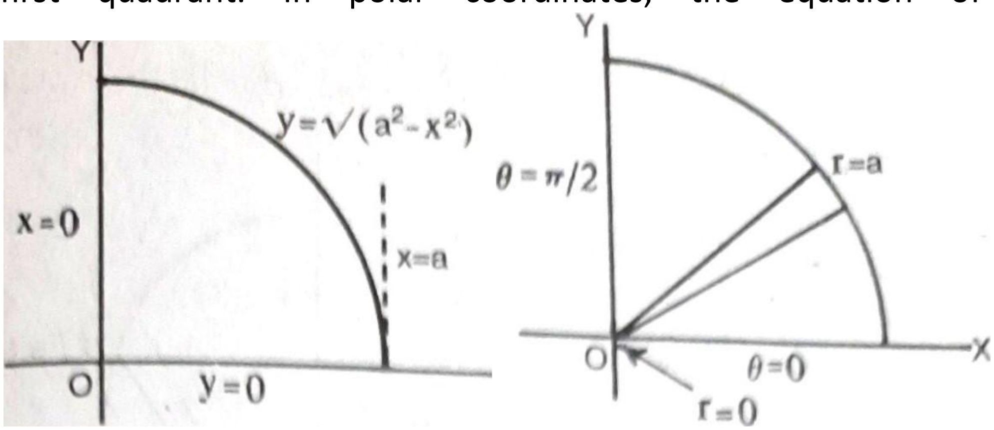
18. $\int_0^{a}\int_y^{a} \frac{x}{x^2+y^2} dydx$

19. $\int_0^{2}\int_y^{\sqrt{(2x-x^2)}} \frac{x}{\sqrt{x^2+y^2}} dxdy$. 20. $\int_0^{1}\int_0^{x} \frac{x^3}{x^2+y^2} dxdy$

Example: Transform the integral $\int_0^{a}\int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{x^2+y^2}dxdy$ by changing to polar coordinates, and hence evaluate it.

Solution: The given limits of integration show that the region of integration lies between the curves $y = 0, y = \sqrt{(a^2-x^2)}\; x = 0, x = a.$

Thus the region of integration is the part of the circle $x^2 + y^2 = a^2$ in the first quadrant. In polar coordinates, the equation of the circle is



$$r^2cos^2\theta + r^2sin^2\theta = a^2, i.e. r = a$$

Hence, in polar coordinates, the region of integration is bounded by the curves:

$r = 0, r = a, \theta = 0, \theta = \frac{\pi}{2}.$

Therefore, $\int_0^{a}\int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{x^2+y^2}dxdy = \int_0^{\pi/2}\int_0^{a} r^2sin^2\theta . r. rd\theta dr$

$$= \frac{1}{5}a^5\int_0^{\pi/2} sin^2\theta\, d\theta = \frac{1}{20}\pi a^5$$

8.6. Change of order of Integration: We have seen that in evaluation of a double integration by successive integrations, we may integrate it with respect to y first and then x, or we may integrate in the reverse order. Given the region of the integration$A$, we determine the limits of integration in the former case by taking a strip parallel to the $y-$ axis, and in the latter case by taking one parallel to the $x-$ axis. When it is required to change the order of integration in an integral for which the limits are given, we first of all ascertain from the given limits the region A of integration. Knowing the region of integration, we can then put in the limits for integration in the reverse order.

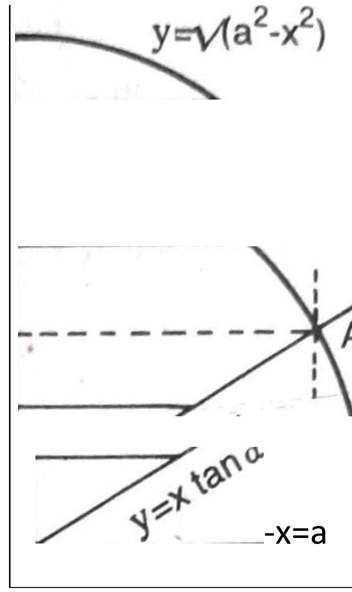
Example: Change the order of integration in the integral $\int_0^{acos\alpha}\int_{xtan\alpha}^{\sqrt{a^2-x^2}} f(x,y)dxdy.$

Solution: The given limits show that the region of integration is bounded by the curves $y = xtan\alpha, y = \sqrt{a^2-x^2}, x = 0\ and\ x = acos\alpha$.

The first is a line through the origin and the circle $x^2 + y^2 = a^2$. These intersect at the point $(acos\alpha, asin\alpha)$.Therefore,the region of integration is OAB in the figure. When we integrate with respect to x first along a horizontal strip, the strip starts from $x = 0$. But some of the strips end on $)A$ while the others end on AB. The line of demarcation is the line CA, $y = asin\alpha$, so the region OAB must be subdivided in the subregions OAC and CAB. These are respectively bounded by the curves $x = 0, x = ycot\alpha, y = 0, y = asin\alpha$,

And $x = 0, x = \sqrt{a^2 - y^2}$, $y = asin\alpha, y = a$. Hence on changing of the order of the integration, the double integral becomes

$\int_0^{asin\alpha}\int_0^{acot\alpha} f(x,y)dydx + \int_{asin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)dydx.$



$cos\alpha$

Check your Progress

Change the order of integration in the following integrals

1. $\int_0^a \int_0^x f(x,y)\, dxdy$

2. $\int_0^4 \int_x^{2\sqrt{x}} f(x,y)\, dxdy$

3. $\int_0^a \int_x^{\frac{a^2}{x}} f(x,y)\, dxdy$

4. $\int_0^{\frac{a}{2}} \int_{\frac{x^2}{a}}^{x-\frac{x^2}{a}} f(x,y)\, dxdy$

5. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\, dxdy$

6. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x,y)\, dydx$

7. show that $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} f(x,y)\,dxdy = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} f(x,y)\,dydx$. Indicate the region of integration and evaluate the integral. What does it represent ?

9. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dxdy$

10. $\int_0^1 \int_{\sqrt{ax}}^{\sqrt{2-x^2}} \frac{x}{\sqrt{y^2+x^2}}\,dxdy$

11. $\int_0^1 \int_{e^x}^{e} \frac{1}{logy}\,dxdy$

12. $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}}\,dxdy.$

8.7. Triple Integrals: Let V be a region of the three dimensional space, and let $f(x,y,z)$ be a function of the independent variables x, y, z defined at every point in V. Divide the region V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$, and let (x_r, y_r, z_r) be any point inside the r^{th} subdivision δV_r. Form the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$ ..... (1)

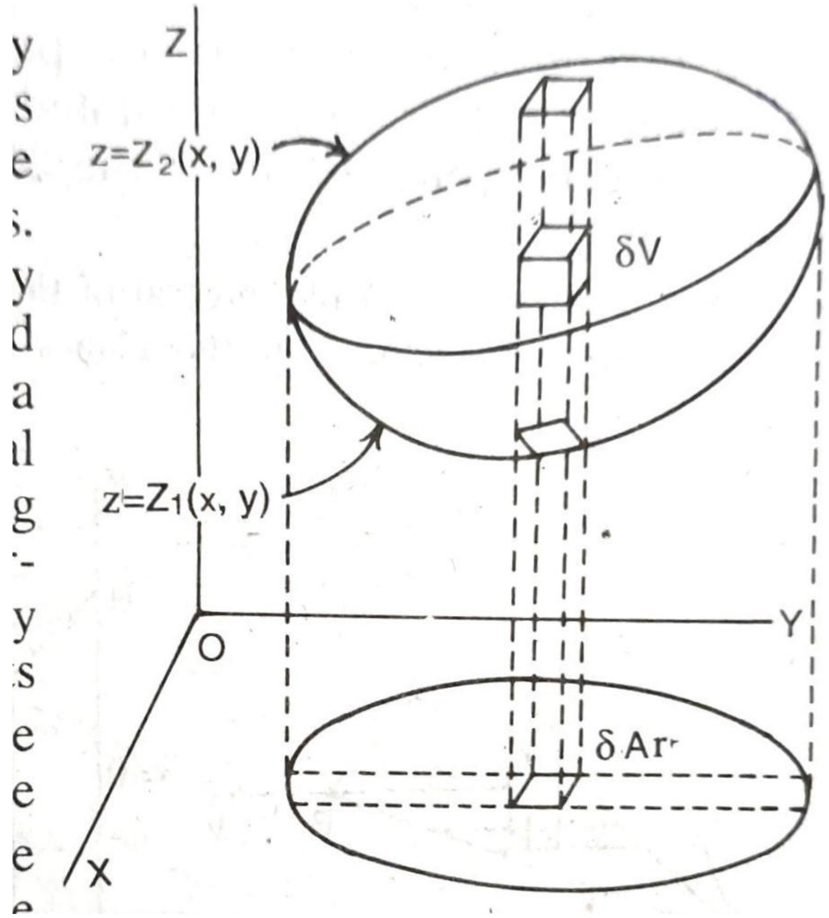
Then the limit of this sum, as n tends to infinity and the dimensions of each subdivision tend to zero, is called the triple integral of $f(x,y,z)$ over the region V, and is denoted by $\iiint f(x,y,z)dV$ .... (2)

This definition is similar to that of a double integral. For the purpose of evaluation the triple integral also can be expressed as a repeated integral. The triple integral can be used to evaluate a number of physical quantities. For example, if we put $f(x,y,z) = l$ in (2), we find that the volume $v = \iiint f(x,y,z)dV = \iiint 1dV$ .... (3)

Similarly, the mass of a body of density $\rho = f(x,y,z)$, occupying a volume V is

$$M = \iiint f(x,y,z)dV = \iiint \rho dV \text{ ....(4)}$$

To express the triple integral as a repeated integral, divide the region V into elementary cuboids by planes parallel to the three coordinate planes. The volume V may then be considered as the sum of a number of vertical columns extending from the lower surface of V, say $z = Z_1(x, y)$, to its upper surface $z = Z_2(x, y)$ The bases of these columns are the elementary areas δA_r which cover a certain region A in the xy −plane when all the columns in V are taken. (Only one of these columns is shown in the figure.)



Therefore, if we sum up over the elementary cuboids in the same vertical column first, and then take the sum for all the columns in V, we can write (1) as, $\sum_{r=1}^{n}\{\sum_{k=1}^{n} f(x_r, y_r, z_k)\delta z\}\delta A_r$

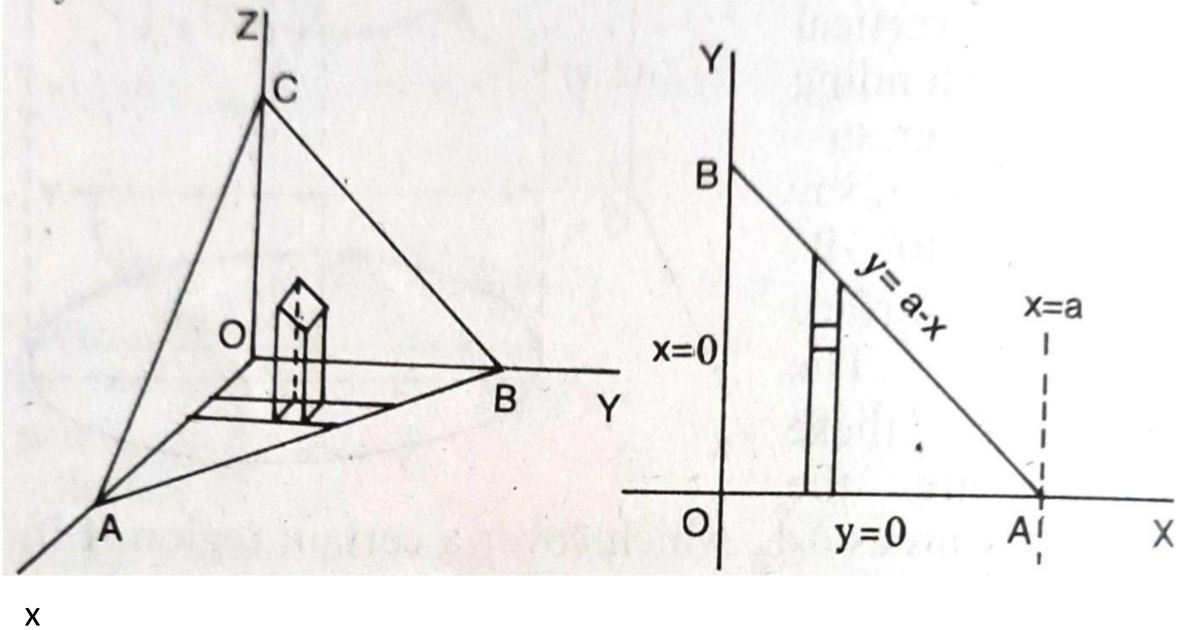
where, (x_r, y_r, z_k) is a point in the k^{th} cuboid above the r^{th} element. Taking the limit when the dimensions of δA_r, and δz tend to zero, this becomes equal to $\iiint_{Z_1(x,y)}^{Z_2(x,y)} f(x, y, z)\, dz\}dA$.....(5).

The integration with respect to z is performed first, keeping x and y constant. The remaining integration is performed as for the double integral. Therefore if A is bounded by the curves $y = Y_1(x), y = Y_2(x), x = a, x = b$, the triple integral (5) may be written as $\int_a^b \int_{Y_1(x)}^{Y_2(x)} \int_{Z_1(x,y)}^{Z_2(x,y)} f(x, y, z) dxdydz$,

where the three integrations are performed in order from right to left. It should be noted that

the region A is the projection on xy −plane of the bounding surface of the volume V.

Example: Evaluate the triple integral of the function $f(x, y, z) = x^2$ over the region V enclosed by the planes $x = 0, y = 0, z = 0$ and $x+y+z=a.$



Solution: Here a vertical column is bounded by the planes $z = 0, z = a—x—y$. The latter plane cuts the xy −plane in the line $a—x—y = 0$.

So the area A above which the volume stands is the region in xy −plane bounded by the lines $y = 0, y = a— x, x = 0, x = a$.

Hence the triple integral $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dxdydz = \int_0^a \int_0^{a-x} x^2(a - x -$ *y)dxdy =120ax2(a−x)2dx=12(13−12+15)a5=160a5.*

Check your Progress

Evaluate the following integrals:

1. $\int_0^a \int_0^b \frac{xy\,dx\,dy}{\sqrt{(1-x^2-y^2)}}$. 2. $\int_2^3 \int_0^{y-1} \frac{dy\,dx}{y}$.

3. $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} (x+y)\,dx\,dy$. 4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}$.

5. $\int_0^{\pi} \int_0^{a(1+COS\theta)} r d\theta dr$ 6. $\int_0^1 \int_0^{2\sqrt{z}} \int_0^{\sqrt{(4z-x^2)}} dzdxdy$

7.Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dxdydz$, and state precisely what is the region of integration.

8.Evaluate $\iiint zdxdydz$ over the volume enclosed between the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = 1$ on the positive side of xy −plane.

9.Find by triple integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$.

10.Find the volume in the first octant bounded by the cylinder $x = 4 — y^2$, and the planes $z = y, x = 0, z = 0$.

11.Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$, if the density at any point is kyz.

12.Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

13.Find the value of $\iiint x^2\, dxdy\, dz$ over the volume bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Hint. Put $x = aX, y = bY, z = cZ$, integrate with respect to Z, and then transform to polars.]

14.Find by double integration the area between $y = \frac{3x}{x^2+2}$ and $4y = x^2$.

8.8. Gamma function: We define the gamma function, $\Gamma(x)$, for $x > 0$, by the relation $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

We have already seen that the integral defining $\Gamma(x)$ is convergent for $x > 0$.

The student should note that the integrand on the right is a function of both x and t. But on integration and substitution of the limits, t gets removed. The resulting function of x is denoted by $\Gamma(x)$. In actual practice, the integration is possible only for special values of x. For other values of x, recourse must be had to numerical methods for evaluating the integral. However, a number of properties of the function can be derived from the definition itself.

An important property: On integrating by parts, we see that $\int_0^\infty t^x e^{-t}\,dt = [-t^x e^{-t}\,] + x\int_0^\infty t^{x-1}e^{-t}\,dt.$ The integrated part vanishes at the lower limit, since $x > 0$. At the upper limit, we have by L' Hospital's $\lim_{t\to\infty} t^x e^{-t} = \lim_{t\to\infty}\frac{t^x}{e^t} = \lim_{t\to\infty}\frac{xt^{x-1}}{e^t} = \lim_{t\to\infty}\frac{x(x-1)t^{x-2}}{e^t} = \cdots$

By differentiating the numerator and denominator again and again till we get a zero or negative exponent n the numerator, we see that the limit is zero. Hence

$$\int_0^\infty t^x e^{-t}\,dt = x\int_0^\infty t^{x-1}e^{-t}\,dt \quad i.e.\ \Gamma(x+1) = x\Gamma(x) \ldots\ldots (1)$$

This fundamental property of gamma function helps us in its evaluation.

For example, a repeated application of (1) gives

$\Gamma(5) = 4\Gamma(4) = 4.3\Gamma(3) = 4.3.2\Gamma(2) = 4.3.2.1\Gamma(1) = 5.4.3.2.1 = 4!$

Hence , $\Gamma(5) = 4!$.We can similarly show that $\Gamma(\mathrm{n}+1) = \mathrm{n}\Gamma(\mathrm{n}) = \cdots = \mathrm{n}!$

when n is a positive integer. We can proceed in a similar fashion for non integral values e.g.$\Gamma(5.3) = 4.3\Gamma(4.3) = 4.3 \times 3.3\Gamma(3.3) = 4.3 \times 3.3 \times 2.3\Gamma(2.3) = 4.3 \times 3.3 \times 2.3 \times 1.3\Gamma(1.3)$.

We generally stop at this stage, and substitute the value of $\Gamma(1.3)$ from the table of gamma functions. The relation (1) can also be used for defining the gamma function for negative values of x. Thus the complete definition is

$$\Gamma(\mathrm{x}) = \int_0^\infty t^x e^{-t}\,dt \ (x > 0);\ \frac{\Gamma(x+1)}{x} = \Gamma(x)\ (\mathrm{x} < 0).$$

We notice that the gamma function is undefined for $x = 0$ or a negative integer,

A graph showing the values of $\Gamma(\mathrm{x})$is given in the margin.

Example: Evaluate in terms of gamma function the integral $\int_0^\infty e^{-x^4}\,dt$.

Solution: Putting $x^4 = t, x = t^{\frac{1}{4}}; dx = \frac{1}{4}t^{-\frac{3}{4}}dt$, we have, $\int_0^\infty e^{-x^4}\,dt = \int_0^\infty e^{-t}\,\frac{1}{4}t^{-\frac{3}{4}}dt = \frac{1}{4}\,\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{5}{4}\right)$.

8.9. Product of two single integrals: We shall use in the next article the following theorem : $\int_a^b f(x)dx \times \int_c^d g(y)dy = \int_a^b\int_c^d f(x)g(y)\,dx\,dy$

To prove it, suppose that $\int g(y)dy = G(y)$, Then $\int_a^b f(x)dx \times \int_c^d g(y)dy = \int_a^b f(x)[G(y)]dx = \int_a^b f(x)[G(d) - G(c)]dx$

$$= \int_a^b f(x)dx \times [G(d) - G(c)] = \int_a^b f(x)dx \times \int_c^d g(y)dy$$

The theorem holds for improper integrals also, provided each of the integrals is convergent.

Value of $\Gamma(1/2)$. To obtain the value of $\Gamma(\frac{1}{2})$, we notice that

$$\boldsymbol{\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt = 2\int_0^\infty e^{-x^2}\, dx} \text{ ----(1)}$$

on putting $t = x^2$ and $dt = 2x\, dx$.

Rewriting this result with y instead of x, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy ---- \qquad (2)$$

Multiplying (1) and (2), we get

$$[\Gamma\left(\frac{1}{2}\right)]^2 = 4\int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy =$$

$$4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, rd\theta dr$$

on transforming to polars, $= 2\int_0^{\frac{\pi}{2}} [-e^{r^2}]\, d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

8.10. Integral of $\boldsymbol{sin^{2m-1}x\, cos^{2n-1}x}$: For this we put $t = x^2$ in the definition of $\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt$ we obtain $\Gamma(m) = \int_0^\infty x^{2m-2} e^{-x^2} 2xdx = 2\int_0^\infty x^{2m-1} e^{-x^2} dx$------(1)

Similarly, $\Gamma(n) = \int_0^\infty y^{2n-2} e^{-x^2} 2ydy = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$------(1)

We multiplying (1) and (2),

$$\Gamma(m)\Gamma(n) = 2\int_0^\infty x^{2m-1} e^{-x^2} dx \times 2\int_0^\infty y^{2n-1} e^{-y^2} dy$$

$$= 4\int_0^\infty \int_0^\infty x^{2m-1}\, y^{2n-1} e^{-(x^2+y^2)dxdy}$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty (r\cos\theta)^{2m-1}\, (r\sin\theta)^{2n-1} e^{-(r^2)rd\theta dr}$$

$$= 4\int_0^{\frac{\pi}{2}} cos^{2m-1}\theta sin^{2n-1}\theta d\theta \times \int_0^\infty (r)^{2m+2n-1}\, e^{-(r^2)dr}$$

$$= 2\int_0^{\frac{\pi}{2}} cos^{2m-1}\theta sin^{2n-1}\theta d\theta . \Gamma(m+n)$$

Hence, $\int_0^{\frac{\pi}{2}} cos^{2m-1}\theta sin^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

This result is true for all positive values of m and n, integral as well as fractional. An alternative form of the above result is

$$\int_0^{\frac{\pi}{2}} cos^m\theta sin^n\theta d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \; ; m > -1, n > -1$$

Example: Evaluate $\int_0^1 \sqrt{1-x^4}\ dx$

Solution: Put $x = \sqrt{sin\theta}\,, dx = \frac{1}{2}(sin\theta)^{-\frac{1}{2}} cos\,\theta\, d\theta$; then

$$\int_0^1 \sqrt{1-x^4}\ dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(sin\theta)^{-\frac{1}{2}} cos^2\,\theta\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{6}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

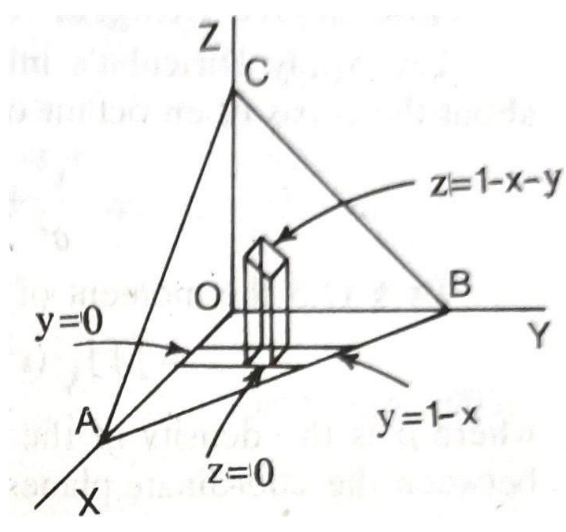
8.11. Beta function: We define the beta function $B(m,n), for\ m > 0, n > 0$, by the relation $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$.

Putting $x = sin^2\theta$, in the above integral, we find that

$$B(m,n) = \int_0^{\frac{\pi}{2}}(sin\theta)^{2m-2}(cos\theta)^{2n-2} 2sin\theta cos\theta\, d\theta.$$

$$= 2\int_0^{\frac{\pi}{2}}(sin\theta)^{2m-1}(cos\theta)^{2n-1}d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

8.12. Dirichet's Integral: We shall now show how to evaluate the triple integral $\iiint x^{p-1}y^{m-1}\, z^{n-1}dxdydz$ over the volume enclosed by the three coordinate planes and the plane $x+y+z = 1$.



The triple integral may be written as

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{p-1}y^{m-1}z^{n-1}dxdydz$$

$= \int_0^1 \int_0^{1-x} x^{p-1} y^{m-1} (1-x-y)^n dxdy.$

In the first integration here, which respect to y, put $y = (l—x)\, t, dy = (l—x)\, dt$, then the required integral

$$= \frac{1}{n} \int_0^1 \int_0^1 x^{p-1} [(1-x)t]^{m-1} [1-x-(1-x)t]^n (1-x) dx\, dt$$

$$= \frac{1}{n} \int_0^1 \int_0^1 x^{p-1} (1-x)^{m+n}\, t^{m-1} (1-t)^n\, dx\, dt$$

$$= \frac{1}{n} \int_0^1 x^{p-1} (1-x)^{m+n}\, dx \times \int_0^1 t^{m-1} (1-t)^n\, dt$$

$$\frac{1}{n} B(p, m+n+1).B(m, n+1)$$

$$= \frac{1}{n} \cdot \frac{\Gamma(p)\Gamma(m+n+1)}{\Gamma(p+m+n+1)} \cdot \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$= \frac{\Gamma(p)\Gamma(m)\Gamma(n)}{\Gamma(p+m+n+1)}$, since, $\Gamma(n+1) = n\Gamma(n)$.

This gives the important result, $\iiint x^{p-1} y^{m-1} z^{n-1} dxdydz = \frac{\Gamma(p)\Gamma(m)\Gamma(n)}{\Gamma(p+m+n+1)}$

where $V$is the legion given by $x \geq 0, y \geq 0, z \geq 0,\ x+y+z \leq 1$. The above integral is known as Dirichlet's Integral.

Example: Apply Dirichlet's integral to find the moment of inertia about the z —axis of an octant of the ellipsoid,$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.---(1)

(1)

Solution: The moment of inertia $\iiint (x^2 + y^2)\, \rho dxdydz$ ----(2)

where ρ is the density of the ellipsoid and V the volume enclosed between the coordinate planes and the surface (1). Put

$x = a\sqrt{X}, y = b\sqrt{Y}, z = c\sqrt{Z},$

$and\ dx = \frac{1}{2} aX^{-\frac{1}{2}} dX, dy = \frac{1}{2} aY^{-\frac{1}{2}} dY, dz = \frac{1}{2} aZ^{-\frac{1}{2}} dZ.$

Then the surface (1) reduces to the surface $X + Y + Z = 1$-----(3)

and the integral (2) becomes

$\rho \iiint (a^2 X + b^2 Y) \frac{1}{8} abc (X^{-\frac{1}{2}} Y^{-\frac{1}{2}} Z^{-\frac{1}{2}}) dXdYdZ$,----- (4)

where now V denotes the volume enclosed between (he coordinate planes and the surface (3).

The integral (4) is the sum of two Dirichlet's integrals $\frac{1}{8}\rho abc \iiint (a^2 X^{\frac{1}{2}} Y^{-\frac{1}{2}} Z^{-\frac{1}{2}} + b^2 (X^{\frac{-1}{2}} Y^{\frac{1}{2}} Z^{-\frac{1}{2}})\, dX dY dZ$. Hence, its value is

$$= \frac{1}{8}\rho abc\{a^2 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} + b^2 \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})}\}$$

$$= \frac{1}{8}\rho abc(a^2 + b^2) \frac{\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}.\sqrt{\pi}}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} = \frac{1}{30}\pi\rho abc(a^2 + b^2).$$

Since, the mass M of the Octant $= \frac{1}{8}(\frac{4}{3}\pi\rho abc)$.

we can also write the moment of inertia as $\frac{1}{5}M(a^2 + b^2)$.

Check your progress

1. When $s, x > 0$, prove that $\int_0^\infty e^{-st} t^{x-1}\, dt = s^{-x}\Gamma(\mathrm{x})$.
2. Show that $\Gamma(\mathrm{n}) = \int_0^1 (\log\frac{1}{y})^{n-1}\, dy$.
3. Show that if $c > 1$, $\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(\mathrm{c}+1)}{(\log c)^{c+1}}$.
4. Evaluate the following integrals:
   (i). $\int_0^\infty x^6 e^{-2x}\, dx$; (ii). $\int_0^\infty \sqrt{x} e^{-x^3}\, dx$; (iii). $\int_0^\infty 4x^4 e^{-x^4}\, dx$.

(iv). Prove that $\int_0^1 x^m (\log\, x)^{\mathrm{n}}\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

5. Show that if $n > -1$ then $\int_0^1 x^n e^{-k^2 x^2}\, dx = \frac{1}{2(k)^{n+1}}\Gamma(\frac{\mathrm{n}+1}{2})$

   Hence, or otherwise, evaluate $\int_{-\infty}^{\infty} e^{-k^2 x^2}\, dx$.
6. Show that $\int_0^1 \frac{1}{\sqrt{(1-x^n)}}\, dx = \sqrt{\pi}\Gamma(\frac{1}{\mathrm{n}})/(\mathrm{n}\Gamma\left(\frac{1}{\mathrm{n}} + \frac{1}{2}\right))$
7. Evaluate the integral: (i). $\int_0^1 (1 - x^3)^{-\frac{1}{2}}\, dx$; (ii). $\int_0^1 (1 - x^3)^{\frac{1}{2}}\, dx$;
   (iii). $\int_0^1 x^2/(1 - x^3)^{\frac{1}{2}}\, dx$; (iv). $\int_0^1 1/(1 - x^4)^{\frac{1}{2}}\, dx$.

7.13. Summary: We are able to understand the concept of double and triple integral, concept of applications for area and volume for a given curve, understand the concept of Gamma function and Beta function and understand the concept of relation between multiple integral and these special functions.

8.14. Terminal Questions

1. Evaluate $\int_0^3 \int_1^2 xy(1+x+y)dxdy$
2. Evaluate $\iint xy\, dxdy$ over the region in the positive quadrant for which $x+y \le 1$.
3. Evaluate the following double integrals.
   (a) $\int_0^a \int_0^b (x^2+y^2)dxdy$
   (b) (b) $\int_1^a \int_1^b (\frac{1}{xy})dxdy$
4. Evaluate $\int_1^2 \int_0^x \frac{1}{x^2+y^2} dxdy$
5. Evaluate $\iint xy/\sqrt{1-y^2}\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=1$.
6. Find the volume inside the paraboloid $x^2+4z^2+8y=16$ and on the positive side of xz −plane.
7. Evaluate $\int_0^1 \int_0^{2\sqrt{z}} \int_0^{\sqrt{(4z-x^2)}} dzdxdy$