U. P. RajarshiTandon Open University, Prayagraj.

Bachelor of Science

DCEMM -113

Function of Complex Variables

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Block

1 Complex variables and Power series

Unit 1
Complex Variable

Unit- 2
Power Series

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Block-1

Complex variables and Power series

In the first Unit, we discussed function of complex variables which are useful in evaluating a large number of new definite integrals, the theory of differential equations, the study of electric fields, thermodynamics and fluid mechanics. The Functions of that depend only on the combination are called functions of a complex variable and functions of this kind that can be expanded in power series. They are vectors in this two-dimensional complex number space, each with a real and an imaginary part (or component). Since we can multiply z by itself and by any other complex number, we can form any polynomial in z and any power series also. Since all the operations that produce standard functions can be applied to complex functions, we can produce all the standard functions of a complex variable by the same steps as go to producing standard functions of real variables.

In the second unit we shall introduce the series representation of a complex valued function . We shall show that if f is analytic in some domain then it can be represented as a power series at any point in powers of (- which is the Taylor series of f about . If f fails to be analytic at a point , we cannot find Taylor series expansion of f about that point. However, it is often possible to expand f in an infinite series having both positive and negative powers of series. This series is called the Laurent series. In order to obtain and analyse Taylor and Laurent series, we need to develop some concepts related to series. We shall start the unit by discussing basic facts regarding the convergence of sequences and series of complex numbers in we have introduced the concept of radius of convergence of a power series and given the conditions for absolute and uniform convergence of the power series in relation to its radius of convergence.

UNIT-1: Complex Variable

Structure

1.1 Introduction

In this Unit, we discussed function of complex variables which are useful in evaluating a large number of new definite integrals, the theory of differential equations, the study of electric fields, thermodynamics and fluid mechanics. the Functions of $f(x, y)$ that depend only on the combination $(x + iy)$ are called **functions of a complex variable** and functions of this kind that can be expanded in power series in this variable are of particular interest. This combination $(x + iy)$ is generally called z, and we can define such functions as z^n, exp(z), sin z, and all the standard functions of z as well as of x. They are defined in exactly the same way the only difference being that they are actually complex valued functions, that is, they are vectors in this two-dimensional complex number space, each with a real and an imaginary part (or component). Since we can multiply z by itself and by any other complex number, we can form any polynomial in z and any power series as well. We define the exponential and sine functions of z by their power series expansions which converge everywhere in the complex plane. Since all the operations that produce standard functions can be applied to complex functions, we can produce all the standard functions of a complex variable by the same steps as go to producing standard functions of real variables.

1.2 Objectives

After reading this unit the learner should be able to understand about:

- Continuity of Complex Functions
- Uniformly Continuous
- Derivative
- Analytic Function
- Necessary and Sufficient Condition of function
- Milne's Thomson Method

1.3 Concept of a Complex Variable

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number. x is called the real part of $x + iy$ and is written as $R(x + iy)$ and y is called the imaginary part and is written $I(x + iy)$. It is represented by $z = x + iy$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other

If $z = x + iy$ then $\bar{z} = x - iy$.

1.4 Properties

1. The sum, difference, product and quotient of two complex numbers is a complex number.
2. If a complex number is equal to zero then its real and imaginary parts are separately equal to zero. Thus $x + iy = 0 \Rightarrow$ x = 0 and y = 0
3. If two complex numbers are equal, then their real and imaginary parts are separately equal. Thus $x + iy = a + ib \Rightarrow$ x = a and y = b
4. If two complex numbers are equal, then their conjugates are also equal

a. $a + ib = c + id \ \Rightarrow a - ib = c - id$

Note: Let $\mathbb{R}$ be the set of real number then an element $(x, y) \in \mathbb{R} \times \mathbb{R}$ is called a complex number if it satisfies

(i) $(x_1, y_1) = (x_2, y_2) \Rightarrow x_1 = x_2$ & $y_1 = y_2$

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2\ , y_1 + y_2)$

(iii) $(x_1, y_1).(x_2, y_2) = (x_1x_2 - \ y_1y_2, x_1y_2 + \ x_2y_1)$

If we take $z_1 = (x_1, y_1)$ & $z_2 = (x_2, y_2)$ then we get $z_1 = z_2;\ z_1 + z_2$ & $z_1.z_2$

5. Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ)

Put $x = rcos\theta$ ………………… (1)

$y = rsin\theta$ ……………………(2)

i.e. $z = x + iy = r(cos\theta + isin\theta)$

squaring and adding

$$r^2 = x^2 + y^2$$

Or $r = \sqrt{x^2 + y^2}$ (take +ve root only)

Dividing (2) by (1)

$$tan\theta = \frac{y}{x}$$

i.e. $\theta = tan^{-1}\frac{y}{x}$

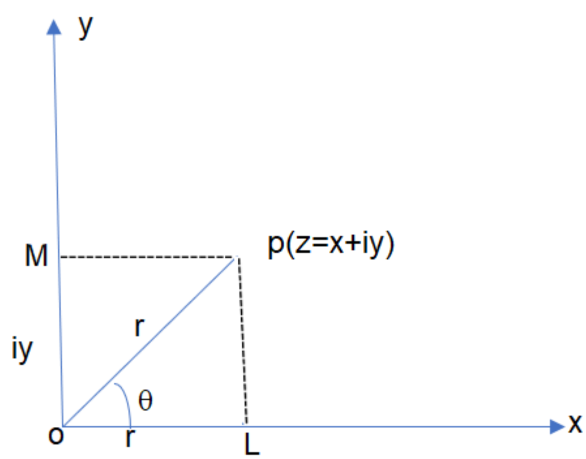
thus $x + iy = r(cos\theta + isin\theta)$ where $r = \sqrt{x^2 + y^2}$

and $$\theta = tan^{-1}\frac{y}{x}$$

the number $r = \sqrt{x^2 + y^2}$ is called the modulus of $x + iy$ and is written as mod$(x + iy)$ or $|x + iy|$

The angle θ is called the amplitude or argument of $x + iy$ and is written as amp $(x + iy)$ or org $(x + iy)$

Geometrical Representation of Complex Number:



Plane representing complex number as ordered pairs of real number (x, y) is called the complex plane or argand plane or gaussian plane.

(i) If z_1 and z_2 are two complex number, then addition of two complex number are $|z_1 + z_2| \leq |z_1| + |z_2|$.

(ii) Difference of two complex number are $|z_1 - z_2| \geq |z_1| - |z_2|$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (iv) $|z_1 . z_2| = |z_1| . |z_2|$

(v) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

De Moivre's Theorem:

If n is any integer +ve or -ve then $(cos\theta + isin\theta)^n = cosn\theta + isinn\theta$ and

If n is a fraction +ve and -ve, then one of the values of $(cos\theta + isin\theta)^n$ is $cosn\theta + isinn\theta$

Function of complex variable:

If for each value of the complex variable $z = x + iy$ in a given region R we have one or more value of

$$w = u + iv$$

Then w is said to be a complex functions of z and we write

$$w = u(x, y) + iv(x, y) = f(z)$$

Or $w = f(z)$

Where u and v are real functions of x and y.

For e.g. if $w = z^2$

$w = f(z) = u + iv$ and $z = x + iy$

Then $u + iv = (x + iy)^2$

$$= x^2 - y^2 + 2ixy$$

$\Rightarrow \qquad u = x^2 - y^2$

$$v = 2xy$$

Thus u and v are the real and imaginary part of w are functions of the real variables x and y.

Limit of f(z):- $\lim_{z \to 20} f(z). l$

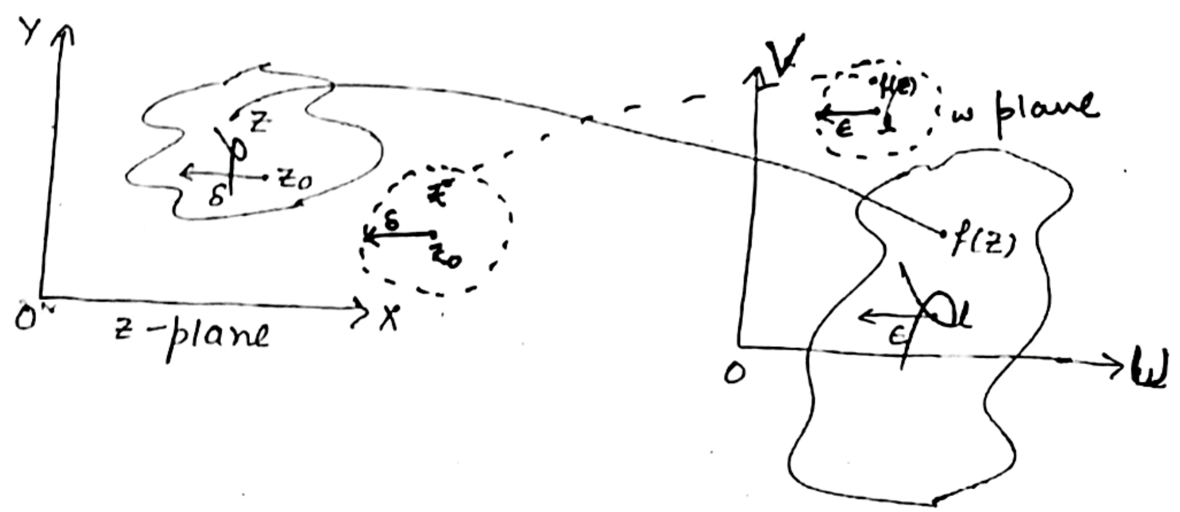
Definition:

A function $w = f(z)$ is said to tend to limit l as z approaches a point z_0 if for every real $\in$ we can find a +ve real δ s.t.

$|f(z) - l| < \varepsilon$ for $|z - z_0| < \delta$

i.e. for every $z \neq z_0$ in the δ-disc of z-plane f(z) has a value lying in the disc of w-plane.

We write $\lim_{z \to z_0} f(z) = l$



Note:- In real variable $x \to x_0$ implies that x approaches along the number line either from left or from right.

But in complex variables $z \to z_0$ implies that z – approaches z_0 along any path straight or curved since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

Means (i) along real axis (ii) image axis (iii) along the path y = mx.

Continuity of f(z):

A function $w = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$.

Further f(z) is said to be continuous in any region R of the z-plane, if f(z) is continuous at every point of that region.

1.5 Continuity of Complex Functions

In order to perform operations such as differentiation and integration of complex functions. We must be able to verify of the complex function is continuous. A complex function $f(z)$ is said to be continuous at a point z_0

If as z approaches z_0 (from any direction) then $f(z)$ can be made arbitrarily close to $f(z_0)$.

1.6 Uniformly Continuous

A function $f(z)$ is said to be uniformly continuous in a domain D if

Given $\varepsilon > 0, (however\ small), \exists\ \delta > 0\ (depending\ upon\ \varepsilon\ only)$.

s.t. $|f(z_1) - f(z_2)| < \varepsilon, \forall\ z_1, z_2 \in D$

whenever $0 < |z_1 - z_2| < \delta$

z_1, z_2 being any two points of the domain D.

Example: Show that $f(z) = \frac{1}{z^2}$ is uniformly continuous in the region $\frac{1}{2} \leq |z| \leq 1$.

Solution: Here $f(z) = \frac{1}{z^2}$

Suppose $f(z) = \frac{1}{z^2}$ is uniformly continuous in the region $\frac{1}{2} \leq |z| \leq 1$.

Then for a given $\varepsilon > 0$,we can choose $\delta > 0$

Such that $|f(z) - f(z_0)| = \left|\frac{1}{z^2} - \frac{1}{{z_0}^2}\right| <\in whenever\ |z - z_0| < \delta$

When δ depends only on ε and not on the particular choice of the point z_0, if z and z_0 are only point in $\frac{1}{2} \leq |z| \leq 1$

Then $\left|\frac{1}{z^2} - \frac{1}{{z_0}^2}\right| = \left|\frac{{z_0}^2 - z^2}{z^2 {z_0}^2}\right| = \frac{|z+z_0||z-z_0|}{|z|^2|z_0|^2}$

$$\leq \frac{|z+z_0||z-z_0|}{|z|^2|z_0|^2} \leq 2|z-z_0|[4 \times 4] = 32(z-z_0)$$

Since $|z| \geq \frac{1}{2}$ and $|z_0| \geq \frac{1}{2} \leftrightarrow \frac{1}{|z|} \leq 2$ and $\frac{1}{|z_0|} \leq 2$

Now if we choose $\delta = \frac{\varepsilon}{32}$, it follows that$\left|\frac{1}{z^2} - \frac{1}{{z_0}^2}\right| \leq \varepsilon$

This ensure that $f(z) = \frac{1}{z^2}$ is uniformly continuous in the region $\frac{1}{2} \leq |z| \leq 1$.

Bounded Function: A function f defined on some set X with real or complex values is called bounded if the set of its values is bounded. In other words, there exists a real number M such that for all x in X. A function that is not bounded is said to be unbounded.

Multivalued Function: If w takes two or more values for some values or all values of z in the region D,then w is said to be multivalued function of z.

Example: we have $w^2 = z = x + iy$

If we take $x = 5; y = 12$

Then $w^2 = z = 5 + 12i$

$$= (3 + 2i)^2$$

$$\Rightarrow w = \pm(3+2i)$$

Hence function is called **Multivalued.**

Branch: A branch of a multi-valued function is a single-valued analogue which is continuous on its domain.

Branch Cut: The set of points that have to be removed from the domain of a multivalued function to produce a branch of the function.

Branch Point: The point in the complex plane which lies in every branch cut of a complex function. It is often the origin.

1.7 Derivative

Let $w = f(z)$ be a single values function of the variable $z = x + iy$, then the derivative or differential coefficients of $w = f(z)$ is defined as

$f'(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z\to 0} \frac{f(z+\delta z)-f(z)}{\delta z}$$

Provided the limit exists and unique when $\delta z \to 0$ along different paths

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}$$

Or

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - \frac{i\partial u}{\partial y}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

Cauchy-Riemann Equations:

A necessary condition for $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in domain D is that u and v satisfy the **Cauchy-Riemann Equations**

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point D. ....………..(1)

If these partial derivatives (1) are also continuous, then **Cauchy-Riemann Equations** are sufficient condition for $f(z)$ to be analytic in D.

(i) Necessary Conditions for $f(z)$ to be analytic

We have $z = x + iy$

⇨$\Delta z = \Delta x + i\Delta y$.

Let $w = f(z) = u(x,y) + iv(x,y)$ be analytic function inside a region D.

It means differentiable of w exist at any point of this region i.e., $\frac{dw}{dz} = \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$ … … … … … . . (1)

Exist and unique along whatever path Δz may across to zero.

Now, we take $\Delta z \to 0$ along two paths $x - axis$ and $y - axis$.

Along $x - axis$, there is no change in $y\ i.e., \Delta y = 0$ therefore $\Delta z = \Delta x + i\Delta y = \Delta x$

Using equation (1), we have $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta x} \dots \dots \dots \dots .. (2)$

And when the motion is parallel to $y - axis$ then there is no change in x.

So $\Delta x = 0$, therefore $\Delta z = \Delta x + i\Delta y = i\Delta y$.

By equations (1) and (2) $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{i\Delta y} \dots \dots \dots \dots \dots \dots (3)$

Now by equations (2) and (3), $\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{i\partial y} \dots \dots \dots .. (4)$

We have $w = u + iv$

$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

And $\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial w}{i\partial y} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$

By equation (4) we have, $\frac{\partial w}{\partial x} = \frac{\partial w}{i\partial y}$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ or $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

These are the necessary condition for a function of analytic.

$u_x = v_y \qquad u_y = -v_x$.

(ii) Sufficient Condition: Sufficient condition of $f(z)$ to be analytic assume the existence and continuity of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

The Sufficient condition for the function $f(z)$ to be analytic required other the continuity four partial derivatives of $u\ and\ v$. Since u is a function of $x\ and\ y$.

So, $u = u(x, y) \Rightarrow u + \Delta u = u(x + \Delta x, y + \Delta y)$

$$\therefore \partial u = u(x + \partial x, y + \partial y) - u(x, y)$$

$$\Rightarrow u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y) + u(x + \Delta x, y) - u(x, y)$$

$$\Delta u = \Delta y \left[\frac{u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y)}{\Delta y}\right] + \Delta x[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}]$$

$\Rightarrow \Delta u = \Delta y. u_y(x + \Delta x, y + \theta_1 \Delta y) + \Delta x u_x(x + \theta_2 \Delta x, y)$ ....(1)

$$0 < \theta_1 < 1\ and\ 0 < \theta_2 < 1$$

$\Delta u = \Delta y(u_y + \varepsilon_1) + \Delta x(u_x + \varepsilon_2)$ ……(2)

And $\Delta v = \Delta y(v_y + \varepsilon_3) + \Delta x(v_x + \varepsilon_4)$ …….(3)

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0\ as\ \Delta z \to 0$$

Now by C-R equations

$\Delta u = \Delta y(-v_x + \varepsilon_1) + \Delta x(u_x + \varepsilon_2)$

And $\Delta v = \Delta y(u_x + \varepsilon_3) + \Delta x(v_x + \varepsilon_4)$

Or $\quad \Delta v = i\Delta y(u_x + \varepsilon_3) + i\Delta x(v_x + \varepsilon_4)$

On adding last two equation and simplify, we get

$\Delta u + i\Delta v = (\Delta x + i\Delta y)(u_x + iv_x) + \eta\Delta x + \eta'\Delta y$

$\eta = \quad \varepsilon_2 + i\,\varepsilon_4$

$$\eta' = \varepsilon_1 + i\varepsilon_3$$

Dividing by $\Delta x + i\Delta y$, we get

$\Rightarrow \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = (u_x + iv_x) + \eta\frac{\Delta x}{\Delta z} + \eta'\frac{\Delta y}{\Delta z}$

$\Rightarrow \frac{\Delta w}{\Delta z} = (u_x + iv_x) + \eta\frac{\Delta x}{\Delta z} + \eta'\frac{\Delta y}{\Delta z}$ $\qquad \because \Delta z = \Delta x + i\Delta y$

$\left|\eta\frac{\Delta x}{\Delta z} + \eta'\frac{\Delta y}{\Delta z}\right| \le |\eta| + |\eta'| \to \mathbf{0}$ $\qquad \begin{cases} \left|\frac{\Delta x}{\Delta z}\right| < 1 \\ \left|\frac{\Delta y}{\Delta z}\right| < 1 \end{cases}$

Therefore, in the $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence $\frac{dw}{dz}$ exist because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exists.

1.8 Analytic Functions

If a complex function f(z) is a single valued and differentiable at any point $z = z_0$ in the given region R then f(z) is called an analytic or regular or holomorphic function of z at the point $z = z_0$.

The point at which the function is not differentiable is called a singular point of the function.

A function f(z) which is single valued and possesses a derivative w.r.t. z at all parts of a region R, is called an analytic function.

Theorem: - The necessary and sufficient conditions for the derivatives of the function.

$w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are* continuous function of x and y in R.

(ii) (a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (b) $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

Or $u_x = v_y$ $u_y = v_x$

The relation (ii) are known as Cauchy Riemann equation or C-R equation.

Example. If $f(z) = u + iv$ is analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in the term of z.

Solution-we have $f(z) = u + iv$,then $iu - v = if(z)$

Adding we have $u + iv + iu - v = f(z) + if(z)$

$u-v+(u+v)=1+ifz=Fzsay$

$u - v = U \; and \; u + v = V \; then \; F(z) = U + iV$ is analytic function.

$$U = e^x(\cos y - \sin y)$$

Now $\frac{\partial U}{\partial x} = e^x(cosy - siny) or \; \frac{\partial U}{\partial y} = e^x(-siny - cosy)$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy$$

$dV = e^x(siny + cosy)dx + e^x(cosy - siny)dy$

Integrating, we have $V = e^x(siny + cosy) + c$

$$F(z) = U + iV = e^x(cosy - siny) + ie^x(siny + cosy) + ic$$

$$F(z) = e^z + ie^z + ic$$

$$(1 + i)f(z) = (1 + i)e^z + ic$$

$$f(z) = e^z + \frac{ic}{1 + i}$$

$$f(z) = e^z + c_1$$

Example. Using C-R equations show that $f(z) = |z|^2$ is not analytical at any point.

Solution- $we \; know \; that \; u = x^2 + y^2 \,, v = 0$

$$\frac{\partial u}{\partial x} = 2x\,, \frac{\partial u}{\partial y} = 2y\,, \frac{\partial v}{\partial x} = 0\,, \frac{\partial v}{\partial y} = 0.$$

By C-R equation

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$ $(C - R\ equation\ are\ satisfied\ only\ at\ origin\,)$

Since differentiable at origin but not at neighbourhood point. it is nowhere analytic.

1.9 The Necessary and Sufficient Condition for f(z) to be Analytic

The necessary condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

(a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (b) $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ provided

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists,

Sufficient condition for f(z) to be analytic: -

The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous function of x and y in region R.

C-R equation in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Remember:

1. If a function is analytic in a domain D, then $u\ ,v$ satisfy C-R conditions at all points in D.
2. C-R conditions are necessary but not sufficient for analytic function.
3. C-R conditions are sufficient if the partial derivative continuous.

Example .1 If $w = logz$ find $\frac{dw}{dz}$ and determine where w is?

Solution:- We have $w = u + iv = \log\,(x + iy)$

$$u + iv = \frac{1}{2}log(x^2 + y^2) + itan^{-1}\frac{y}{x}$$

Equating real and imaginary parts.

$$u = \frac{1}{2}log(x^2 + y^2), \qquad v = tan^{-1}\frac{y}{x}$$

General value $logz = log|z| + i(\theta + 2n\pi)$

Preverbal value $logz = log|z| + i\theta$

$$\theta = tan^{-1}\frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{z}.\frac{1}{x^2 + y^2}.zx = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The Cauchy – Riemann equation are satisfied and the partial derivatives are continuous except at (0, 0).

Hence w is analytic everywhere except at z = 0.

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + \frac{-\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right)$$

$$= \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x - iy}{(x + iy)(x - iy)}$$

$$= \frac{1}{(x - iy)}$$

$$= \frac{1}{z}$$

Example .2 Show that the function $z|z|$ is not analytic anywhere.

Solution.:- Let $w = z|z|$. Here $w = u + iv$ and $z = x + iy$

$$u + iv = (x + iy)\sqrt{x^2 + y^2}$$

$$u = x\sqrt{x^2 + y^2}, \qquad v = y\sqrt{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = x.\frac{2x}{2\sqrt{x^2 + y^2}} + \sqrt{x^2 + y^2} = \frac{x^2 + x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$= \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial v}{\partial y} = y.\frac{2y}{2\sqrt{x^2 + y^2}} + \sqrt{x^2 + y^2} = \frac{y^2 + x^2 + y^2}{\sqrt{x^2 + y^2}}$$

$$= \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}$$

$\therefore \qquad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \qquad$ when x = y

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}}, \qquad \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

$\therefore \qquad \frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$

Hence C – R conditions are not satisfied at any point. The function $z|z|$ is not aualy til anywhere.

Example.3 Show that the function $f(z) = \sqrt{|xy|}$ equation is not regular at the origin although C – R equation are satisfied.

Solution:- Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$

Then $u(x, y) = \sqrt{|xy|}, \quad v(x, y) = 0$

At the origin (0, 0) we have

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{z\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$\therefore \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,\ y)-u(x,y)}{h}$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Clearly C – R equations are satisfied at the origin.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \ \&\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$

If $z \to 0$ along the line $y = mx$ we get

$$f'(0) = \lim_{y \to mx} \frac{\sqrt{|mx^2|}}{x(1+im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1+im}$$

Now this is not unique since it depends on m therefore $f'(0)$ does not exist.

Hence the function f(z) is not regular at the origin.

Example.2 Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}; z \neq 0$$

$f(0) = 0$ In the region including the origin.

Solution- we have $f(z) = u + iv = \frac{x^2 y^5 (x+iy)}{x^4+y^{10}}; z \neq 0$

Equating real and imaginary parts , we get

$u = \frac{x^3 y^5}{x^4+y^{10}}$, $v = \frac{x^2 y^6}{x^4+y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{x \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{x\to 0} \frac{v(0,y) - v(0,0)}{x} = 0$$

From the above results that , it is clear that

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence , C-R equations are satisfied at the origin.

$$f'^{(0)} = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x\to 0 \\ y\to 0}} \left[\frac{x^2y^5(x+iy)}{x^4+y^{10}} - 0\right] \frac{1}{x+iy}$$

$$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2y^5}{x^4+y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$,then

$$f'^{(0)} = \lim_{x\to 0} \frac{m^5x^7}{x^4+m^{10}x^{10}} = \lim_{x\to 0} \frac{x^3m^5}{1+m^{10}x^6}$$

$$= \frac{0}{1} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$,then

$$f'^{(0)} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

Which show that $f'^{(0)}$does not exist. here $f(z)$is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Example.6 If f(z) is an analytic function with constant modulus show that f(z) is constant.

Solution: Let an analytic function be $f(z) = u + iv$

Taking modulus

$|f(z)| = \sqrt{u^2 + v^2}$ squaring both sides

$|f(z)|^2 = u^2 + v^2$

Given $|f(z)|$ = constant = c (say) and c ≠ 0,

$c^2 = u^2 + v^2$ ………………. (1) diff. w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ …………………(ii)

Diff. (i) partially w.r.y. y

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ ………………. (iii)

Squaring and adding (ii) and (iii)

$$u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=0$$

$$u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=0$$

$$\therefore \qquad \Rightarrow (u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow c^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow \left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0 \qquad \therefore \quad u^2+v^2=c^2\neq 0$$

$$\Rightarrow |f'(z)|^2=0 \qquad \left(\frac{\partial u}{\partial x}\right)^2=i\left(\frac{\partial v}{\partial x}\right)^2=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0$$

$$\left[\because f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]$$

$$\Rightarrow \quad f'(z)=0$$

$\Rightarrow$ f(z) = constant.

1.10 Milne's Thomson Method

This method is used to find $f(z)=u(x,y)+iv(x,y)$ in terms of z when $u(x,y)$ and $v(x,y)$ are given.

We have $z=x+iy$ and $\bar{z}=x-iy$

$\therefore z+\bar{z}=2x \; or \; x=\frac{z+\bar{z}}{2}$ and $z-\bar{z}=2iy \; or \; y=\frac{1}{2i}(z-\bar{z})$

$$f(z) = u(x,y) + iv(x,y) = u\left\{\frac{z+\bar{z}}{2}, \frac{1}{2i}(z-\bar{z})\right\} + iv\left\{\frac{z+\bar{z}}{2}, \frac{1}{2i}(z-\bar{z})\right\}$$

This being an identity in two independent variables $and\ \bar{z}$.

(i) When u(x, y) is real part is given then f(z) is obtained as follows.

$$f(z) = \int [\emptyset_1(z,0) - i\emptyset_2(z,0)]dz + c$$

Where $\emptyset_1(x,y) = \frac{\partial u}{\partial x}$ and $\emptyset_2(x,y) = \frac{\partial u}{\partial y}$

(ii) When v(x, y) is imaginary part is given then f(z) is obtained as follows.

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]dz + c$$

Where $\psi_1(x,y) = \frac{\partial v}{\partial y}$, $\quad \psi_2(x,y) = \frac{\partial v}{\partial x}$

Example 1 Determine the analytic function where real part is $log\sqrt{x^2+y^2}$

Solution: Let $u(x,y) = log\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$\therefore \quad f(z) = \int [\emptyset_1(z,0) - i\emptyset_2(z,0)]dz + c$

$$= \int \frac{z}{z^2} dz + c$$

$$= logz + c$$

Example.2 Find the regular function whose imaginary part is $e^x siny$.

Solution: Let $v(x, y) = e^x siny$

$$\frac{\partial v}{\partial x} = e^x siny \quad , \quad \frac{\partial v}{\partial y} = e^x cosy$$

$$\therefore \qquad f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]dz + c$$

$$= \int e^z dz + c$$

$$= e^z + c$$

Example.3 Find the analytic function $f(z) = u + iv$ if $2u + v = e^x(cosy - siny)$.

Solution: Let $f(z) = u + iv$

Or $\quad 2f(z) = 2u + i2v$ ……………………(1)

$if(z) = iu - v$ ……………………… (2)

Subtract (2) from (1)

$$(2 - i)f(z) = (2u + v) + i(2v - u)$$

Let $(2 - i)f(z) = F(z)$

$$2u + v = u$$

And $\quad 2v - u = v$ then

$$f(z) = u + iv$$

$\therefore$ $f(z)$ is analytic because $f(z)$ is analytic and u be the real part

$$\frac{\partial u}{\partial x} = e^x(cosy - siny) = \phi_1(x, y)$$

$$\frac{\partial u}{\partial y} = e^x(-siny - cosy) = \phi_2(x, y)$$

$$F(z) = \int[e^z cos\theta - ie^z(-cos0)]dz + c$$

$$= \int[e^z cos\theta + ie^z]dz + c$$

$$= \int e^z(1 + i)dz + c$$

$$= (1 + i)e^z + c$$

$\Rightarrow$ $(2 - i)f(z) = (1 + i)e^z + c$

$\Rightarrow$ $f(z) = \frac{(1+i)e^z}{2-i} + \frac{c}{2-i}$

$$= \frac{(1+i)(2+i)}{(2-i)(2+i)}e^z + c_1$$

$$= \frac{2+i+2i-1}{4+1}e^z + c_1$$

$$= \frac{3i+1}{5}e^z + c_1 \qquad \text{Ans.}$$

1.11 Harmonic Function: -

Any function which satisfies the Laplace equation is known as a Harmonic functions. A function $f(z) = u + iv$ is said to Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ etc.

Theorem: If $f(z) = u + iv$ is an analytic function then u and v both are harmonic function.

Proof: Let $f(z) = u + iv$ be an analytic function then

We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ …………….. (1)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ……………….(2)

Diff partially equation (1) w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{………………..} (3)$$

Diff. partially equation (2) w.r.t. y

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial x \partial y} \qquad \text{………………..} (4)$$

Adding (3) and (4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{…………….} (5)$$

Similarly by diff. partially (1) w.r.t.y and (2) w.r.t. x and subtracting we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Thus both the functions u and v satisfy the Laplace equation in two variables. For this reason they are known as harmonic functions and their theory is called potential theory. Such functions u and v are called conjugate harmonic function as $u + iv$ is also analytic function.

Example.1 If $u(x, y) = x^2 - y^2$, prove that the u satisfies Laplace equations.

Solution: we have $u(x, y) = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \ and \ \frac{\partial u}{\partial y} = 2y$$

Then $\frac{\partial^2 u}{\partial x^2} = 2 \ and \ \frac{\partial^2 u}{\partial y^2} = 2$

Hence Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

Example.2 Prove that $u = x^2 - y^2$ and $v = y/x^2 + y^2$ are harmonic function of (x, y) but are not harmonic conjugates.

Solution: $u = x^2 - y^2, \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Now $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

∴ u(x, y) satisfy Laplace equation, hence u(x, y) is harmonic function

$$v = y/x^2 + y^2$$

$$\frac{\partial v}{\partial x} = \frac{-y.2x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) - (-2xy)2(x^2+y^2)(2x)}{[(x^2+y^2)^2]^2}$$

$$= \frac{(x^2+y^2)[-2x^2y - 2y^3 + 8x^2y]}{[(x^2+y^2)^2]^2}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)2(x^2+y^2)(2y)}{[(x^2+y^2)^2]^2}$$

$$= \frac{(x^2+y^2)[-2x^2y - 2y^3 - 4x^2y + 4y^3]}{(x^2+y^2)^4}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

Hence, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

∴ v(x, y) also satisfy Laplace equation.

Hence v(x, y) is also harmonic function.

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Therefore u and v are not harmonic conjugates.

Example.3 If f(z) is a regular function of z prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)| = 4|f'(z)|^2$.

Solution: Let $f(z) = u(x,y) + iv(x,y)$ so that

$$|f(z)| = \sqrt{u^2 + v^2}$$

and $\quad |f(z)|^2 = u^2 + v^2 = \phi(x,y)$ say,

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[4\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly $\frac{\partial^2 \phi}{\partial y^2} = 2\left[4\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$

Adding

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Since u and v have to satisfy C-R equation and the Laplace equation.

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4|f'(z)|^2$$

Example.4 Define a harmonic function. Show that the function

$$u(x, y) = x^4 - 6x^2y^2 + y^4$$

is harmonic. Also find the analytic function

$$f(z) = u(x, y) + iv(x, y).$$

Solution- A function $u(x, y)$ of x , y which processes continuous partial derivatives of the first and second orders and satisfies

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, Laplace's equation is called a harmonic function.

We have

$$u(x, y) = x^4 - 6x^2y^2 + y^4$$

$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$ putting $x \to z \; and \; y \to 0$

$$then\ \emptyset_1(z, 0) = 4z^3$$

$\frac{\partial u}{\partial y} = -12x^2y + 4y^3$ putting $x \to z \; and \; y \to 0$

then $\emptyset_2(z,0) = 0$

$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$ (1)

$\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$ (2)

Adding equation (1) and (2) we have

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ hence, u is a harmonic function.

$f(z) = \int [\emptyset_1(z,0) - i \;\; \emptyset_2(z,0)]dz + c$ (3)

Putting value $\emptyset_1(z,0) \; and \;\; \emptyset_2(z,0)$ then

$f(z) = \int 4z^3 + c$

1.12 SUMMARY

We shall formally define the definition of the limit of a complex function to a point and use this definition to define the concept of continuity in the onctext of a complex function of a complex variable. A variable that can take on the value of a complex number. In basic algebra, the variables x and y generally stand for values of real numbers. The algebra of complex numbers (complex analysis) uses the complex variable z to represent a number of the form a + bi.

We conclude with summarizing what we have covered in this unit.

- Analytic function
- Cauchy-Riemann equations
- Milne Thompson Method.

1.13 Terminal Questions

1. Prove that $f(z) = z^2$ is uniformly continuous in the region $|z| \leq 1$.

2. The function $f(z) = (z-3)^{1/2}$has a branch point at $z = 3$.

3. Construct a function $f(z)$ which has a real function $u(x, y) = e^x(x\cos y - y\sin y)$ for its real part, satisfying Laplace's equation.

4. Prove that $w = |z|^2$ is continuous everywhere but nowhere differentiable except at $z = 0$.

5. Prove that the function $f(z) = xy + iy$ is everywhere continuous but not analytic.

6. Prove that analytic function with constant real part is constant.

7. If $u = \frac{1}{2}\log(x^2 + y^2)$, find v such that $f(z) = u + iv$ is analytic. Determine $f(z)$ in terms of z.

8. Show that the following functions are not analytic (i) $|z|^2$ (ii) $\bar{z}$

9. Show that the following functions are not analytic (i) z^3(ii) e^z

10. Continuity of functions of a complex variable .

(i) Let $g(z) = \frac{\bar{z}-1}{z-1}$ for $z \neq 1$ and $g(1) = 1$ if g continuous at 1? Is g continuous at 0?

(ii) Let $h(z) = \frac{3}{z^5}$ for $z \neq 0$, can h be defined at 0 so that the new function is continuous at 0.

(iii) It the function $f(z) = z - \bar{z}$ continuous at every point where it is defined?

UNIT- 2: POWER SERIES

Structure

2.1 Introduction

In this unit we shall introduce you to the series representation of a complex valued function . We shall show that if f is analytic in some domain D , then it can be represented as a power series at any point $z_0 \epsilon D$ in powers of z-z_0 which is the Taylor series of f about z_0. If f fails to be analytic at a point z_0, we cannot find Taylor series expansion of f about that point. However, it is often possible to expand f in an infinite series having both positive and negative powers of z-z_0. This series is called the Laurent series. In order to obtain and analyse Taylor and Laurent series, we need to develop some concepts related to series. We shall start the unit by discussing basic facts regarding the convergence of sequences and series of complex numbers in we have introduced the concept of radius of convergence of a power series and given the conditions for absolute and uniform convergence of the power series in relation to its radius of convergence.

2.2 Objectives

After studying this unit, you should be able to:

- discuss the convergence of sequence of complex numbers;
- use the properties of convergence and absolute convergence of infinite series of complex numbers in order to check the convergence of any given series;

- obtain the Taylor series representation of a complex-valued function about a point at which the function is analytic;
- Obtain a series representation of a complex-valued function about a point at which the function is not analytic in terms of Laurent series;
- Obtain the radius of convergence of a power series.

2.3 Power Series

A series of the types $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, whose terms are variable, is called a power series about z_0, where z is a complex variable, a_n, z_0 are complex constants and a_n is independent of z.

By substituting $z - \zeta + z_0$, the above power series becomes $\sum_{n=0}^{\infty} a_n \zeta^n$, where ζ is the new complex variable. Since the first form of the power series can be reduced to the second form merely by changing the origin, it is sufficient to consider the series of the form $\sum_{n=0}^{\infty} a_n z^n$.

2.4 The circle of convergence of power series

The circle $|z| = R$ such that the power series $\sum a_n z^n$ is convergent for every z within it is called the circle of convergence of the series.

2.5 Power series and analytic function

Sum function of a power series- If $f(z) = \sum a_n z^n$, then f(z) is called the sum function of the power series $\sum a_n z^n$.

In the following theorem we show that the derivative of a power series has the same radius of convergence as the original series.

Theorem 1: The power series $\sum na_n z^{n-1}$ obtained by term by term differentiation of the power series $\sum a_n z^n$ has the same radius of convergence as the original series $\sum a_n z^n$.

Power series as an analytic function, the following important theorem says that every power series can be treated as an analytic function by means of its sum functions.

Theorem 2: The sum function f(z) of the power series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside its circle of convergence.

Further, every power series processes derivative of all orders within its circle of convergence and these derivatives are obtained through term by term differentiation of the series.

2.6 Radius of convergence of power series

Consider the power series $\sum a_n z^n$. Here $U_n(z) = a_n z^n$. By nth root test, this series is convergent if $\lim_{n \to o} |U_n(z)|^{1/n} < 1$

i.e. $\lim_{n\to\infty} |a_n . z^n| < 1$ i.e. $\lim\limits_{n\to\infty} |a_n|^{1/n} . |z| < 1$

i.e. $|z| \lim\limits_{n\to\infty} |a_n|^{1/n} . < 1,$ i.e. $|z| < R,$

where R is given by

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The relation (1) is known as Hadamard formula for radius of convergence.

Thus $\sum a_n z^n$ is convergent or divergent according as

$$|z| < R, \quad \text{or } |z| > R$$

The above discussion leads to the following result.

Radius of Convergence: The number R such that the power series $\sum a_n z^n$ is convergence of the series.

Thus, of the radius of the circle of convergence is the radius of convergence of the series.

There are three possibilities for R:

i. R=0 In this case, the series is convergent only at z = 0
ii. R is finite and positive. In this case, the series is convergent at every point within the circle $|z| < R$.
iii. R is infinite. In this case, the series is convergent for all values of z.

Note:- If the given power series $\sum a_n (z-a)^n$ and $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$, then the circle of convergence is $|z-a| < R$.

Example1- Find the radius of convergence for each of the following power series:

(i) $\sum \frac{z^n}{n^n}$

(ii) $\sum(logn)^n z^n$

Solution: (i) Comparing the given $\sum \frac{z^n}{n^n}$ with the standard form $\sum a_n z^n$, we have

$$a_n = \frac{1}{n^n}$$

So, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$

This gives R = ∞

Hence the radius of convergence of the given series is ∞.

(iii) Here $a_n = (logn)^n$. We have

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |(logn)^n|^{1/n}$$

$$= \lim_{n\to\infty} |logn| = \lim_{n\to\infty} logn = \infty$$

Thus $\frac{1}{R} = \infty$ So that R = 0

Example 2: Find the radius of convergence of the power series $\sum b_n^n z^n$, where for each n:

$$b_n = 1 + 1 + \frac{1}{2!} + \cdots \ldots \ldots + \frac{1}{n!}$$

Solution:- Here $a_n = b_n^n = \left(1 + 1 + \frac{1}{2!} + \cdots \ldots \ldots + \frac{1}{n!}\right)^n$. Therefore

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left(1 + 1 + \frac{1}{2!} + \cdots \dots \dots + \frac{1}{n!}\right) = e$$

Hence R (radius of convergence) = 1/e.

Example 3. Find the radius of convergence for

$$\sum (4 + 3i)^n z^n$$

Solution:- Here $a_n = (4 + 3i)^n$. Therefore

$$|a_n| = |(4 + 3i)^n| = \left\{\sqrt{(4^2 + 3^2)}\right\}^n = 5^n$$

Now $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}(5^n)^{1/n} = \lim_{n\to\infty} 5 = 5$

Example.4 Find the radius of convergence of the power series

$$\sum \frac{2^{-n} z^n}{1 + in^2}$$

Solution:- Here $a_n = \frac{2^{-n}}{1+in^2}$. We have

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{2^{-n}}{|1+in^2|}\right)^{1/n} = \lim_{n\to\infty}\left\{\frac{2^{-n}}{(1+n^4)^{1/2}}\right\}^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{2(1+n^4)^{1/2n}} = \frac{1}{2}\lim_{n\to\infty} \frac{1}{(n^4)^{1/2n}(1+1/n^4)^{1/2n}}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{1}{(n^2)^{1/n}(1+1/n^4)^{-1/2n}}$$

$$= \frac{1}{2}\lim_{n\to\infty} \left\{\frac{1}{\left(n^{1/n}\right)^2}\left(1 - \frac{1}{2n^5} + \cdots \dots \dots\right)\right\}$$

$$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2} \quad \text{Ans.}$$

Example 5.:- Find the radius of convergence of the power series:

$$\sum \left(\frac{n\sqrt{2}+i}{1+2in}\right) z^n.$$

Solution:- Here $a_n = \frac{n\sqrt{2}+i}{1+2in}$. We have

$$|a_n| = \left|\frac{n\sqrt{2}+i}{1+2in}\right| = \left(\frac{2n^2+1}{4n^2+1}\right)^{1/2} = \left(\frac{2n^2}{4n^2}\right)^{1/2} \left(\frac{1+1/2n^2}{1+1/4n^2}\right)^{1/2}$$

So, $\qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{2^{1/2n}} \frac{\left(1+1/2n^2\right)^{1/2n}}{(1+1/4n^2)^{1/2n}} = \frac{1}{2^0} \cdot \frac{1}{1} = 1.$

Thus R = 1. Here the radius of convergence of the given power series is 1.

Example 6: Find the radius of convergence of the power series.

$$\sum \frac{(-1)^n (z-2i)^n}{n}$$

Solution: Comparing the given series with the standard form $\sum a_n (z-a)^n$, we find that a = 2i, which is the centre of the circle of convergence. Also,

$$a_n = \frac{(-1)^n}{n}$$

So, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$

Thus R = 1. Hence the radius of convergence is 1 i.e., the given power series is convergent in the circle $|z - 2i| < 1$.

Theorem: $\lim \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

Example 7.: Find the radius of convergence of the series:

$$\frac{1}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+\cdots.$$

Solution:- The coefficient of z^n in the given series will be:

$$a_n = \frac{1.3.5\ldots\ldots\ldots(2n-1)}{2.5.8\ldots\ldots\ldots(3n-1)}$$

So, $$a_{n+1} = \frac{1.3.5\ldots\ldots\ldots(2n-1)(2n+1)}{2.5.8\ldots\ldots\ldots(3n-1)(3n+2)}$$

Whence $$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2n+1}{3n+2} = \frac{2}{3}$$

Therefore the required radius of convergence is $\frac{3}{2}$.

Example 8: Find the radius of convergence for each of the following power series

(i) $\sum\left(\frac{z^n}{2^n+1}\right)$, (ii) $\sum\left(1+\frac{1}{n}\right)^{n^2} z^2$

Solution: (i) Here $a_n = \frac{1}{2^n+1}$ and $a_{n+1} = \frac{1}{2^{n+1}+1}$. So

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{2^{n+1}+1}{2^n+1}\right) = \lim_{n\to\infty}\left(\frac{2+1/2^n}{1+1/2^n}\right) = \frac{2+0}{1+0} = 2$$

Hence the radius of convergence of the given series is 2.

(ii) Here $a_n = \left(1+\frac{1}{n}\right)^{n^2}$. So,

$$\frac{1}{R} = \lim_{n\to\infty} |a|^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$$

Thus R = 1/e. Hence the radius of convergence of the given series is 1/e.

2.7 Exponential Function of a complex variable

When x is real, we are already familiar with the exponential function

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots \ldots \ldots \ldots \ldots . + \frac{x^n}{\underline{|n}} + \cdots \ldots \ldots . \infty$$

Similarly, we define the exponential function of the complex variable z = x + iy as

$$e^z = \exp(z) = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \cdots \ldots \ldots \ldots \ldots . + \frac{z^n}{\underline{|n}} + \cdots \ldots \ldots . \infty$$

……(1)

Putting x = 0 in (1) we get

$$e^{x+iy} = e^{iy} = 1 + iy + \frac{(iy)^2}{\underline{|2}} + \frac{(iy)^3}{\underline{|3}} + \cdots \ldots \ldots \ldots \ldots . + \frac{(iy)^n}{\underline{|n}} + \cdots \ldots \ldots . \infty$$

$$= \left[1 - \frac{y^2}{\underline{|2}} + \frac{y^4}{\underline{|4}} + \cdots\right] + i\left[y + \frac{y^3}{\underline{|3}} + \frac{y^5}{\underline{|5}} + \cdots\right]$$

$$= cosy + isiny$$

Thus $e^z = e^x . e^{iy} = e^x(cosy + isiny) = re^{i\theta}$

Also $x + iy = r(cos\theta + isin\theta) = re^{i\theta}$

Changing i to $-i$, $e^{-iy} = cosy - isiny$

Example.1 Split up into real and imaginary parts $e^{5+i\frac{\pi}{2}}$.

Solution:- $e^{5+i\frac{\pi}{2}} = e^5.e^{i\frac{\pi}{2}} = e^5\left[cos\frac{\pi}{2} + isin\frac{\pi}{2}\right] = ie^5$

$\therefore$ Real part of $e^{5+i\frac{\pi}{2}} = 0$

Imaginary part of $e^{5+i\frac{\pi}{2}} = e^5$

Circular function of a complex variable

Since $e^{iy} = cosy + isiny$ and $e^{-iy} = cosy - isiny$

The circular functions of real angles can be written as

$$siny = \frac{e^{iy}-e^{-iy}}{2i}, \qquad cosy = \frac{e^{iy}+e^{-iy}}{2} \text{ and so on.}$$

It is therefore natural to define the circular function of the complex variable z by the equation.

$$sinz = \frac{e^{iz}-e^{-iz}}{2i}, \qquad cosz = \frac{e^{iz}+e^{-iz}}{2}$$

$$tanz = \frac{sinz}{cosz} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$$

Example.1 Prove that $[\sin(\alpha - \theta) + e^{-i\alpha}sin\theta]^n = sin^n\alpha e^{-in\theta}$

Solution.:- L.H.S. = $[\sin(\alpha - \theta) + e^{-i\alpha}sin\theta]^n$

$$= [sin\alpha cos\theta - cos\alpha sin\theta + (cos\alpha - isin\alpha)sin\theta]^n$$

$$= [sin\alpha cos\theta - cos\alpha sin\theta + cos\alpha sin\theta - isin\alpha sin\theta]^n$$

$$= sin^n\alpha[cos\theta + isin\theta]^n$$

$$= sin^n\alpha(e^{-i\theta})^n$$

$$= sin^n\alpha e^{-in\theta}$$

= R.H.S.

2.8 Hyperbolic Functions

If x be a real or complex

(1)$\frac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and written as sinhx

(2)$\frac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x and written coshx.

$$sinh0 = 0, \qquad cosh0 = 1, \qquad tanh0 = 0$$

2.9 Trigonometric Functions

The definitions of sine and cosine are unacceptably vague because they involve measuring of an angle without giving a precise algorithm for doing so. We are now in a position to remedy this defect. Namely, we take the Taylor expansions

$$\cos x = \sum_0^\infty \frac{(-1)^k x^{2k}}{(2k)!}, \sin x = \sum_0^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Relation between hyperbolic and trigonometry circular function

Since for all values of θ, $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$

Putting $\theta = ix$ we have

$$sinix = \frac{e^{i.ix}-e^{-i.ix}}{2i} = \frac{e^{-x}-e^{x}}{2i}$$

$$= \frac{-(e^{x}-e^{-x})}{2i}$$

$$= \frac{i^2(e^{x}-e^{-x})}{2i}$$

$$= \frac{i(e^{x}-e^{-x})}{2}$$

$$= isinhx$$

So, $sinix = isinhx$

And $cosix = \frac{e^{-x}+e^{x}}{2} = coshx$

$cosix = coshx$

$tanix = itanhx$

Cor.: $sinhix = isinx$

$coshix = cosx$

$$tanh ix = i tan x$$

Fundamental formula

(i) $cosh^2x - sinh^2x = 1$

(ii) $sech^2x + tanh^2x = 1$

(iii) $coth^2x - cosech^2x = 1$

Proof:- for all values of θ

$cos^2\theta + sin^2\theta = 1$ putting $\theta - ix$ we get

$$cos^2 ix + sin^2 ix = 1$$

$$(cosh x)^2 + i^2(sinh x)^2 = 1$$

$$cosh^2x - sinh^2x = 1$$

(ii) Since we know that $cosh^2x - sinh^2x = 1$ dividing by $cosh^2x$

$$\Rightarrow 1 - tanh^2x = sech^2x$$

$$\Rightarrow sech^2x + tanh^2x = 1$$

(iii) We know $cosh^2x - sinh^2x = 1$ dividing by $sinh^2x$

$$\Rightarrow coth^2x - 1 = cosech^2x$$

$$\Rightarrow coth^2x - cosech^2x = 1$$

Inverse Hyperbolic function

If $sinhu = z$, then u is called the hyperbolic sine inverse of z and written as $u = sinh^{-1}z$

Similarly we define $cosh^{-1}z$ and $tanh^{-1}z$.

The inverse hyperbolic function like other inverse functions are many values, but we shall consider only their principle values.

Example.1 Prove that $tan^{-1}z = \frac{i}{2}log\left(\frac{i+z}{i-z}\right)$

Solution:- Let $tan^{-1}z = \theta$

$\Rightarrow tan\theta = z$ divide by I, we get

$$\frac{tan\theta}{i} = \frac{z}{i}$$

$$\Rightarrow \frac{i}{tan\theta} = \frac{i}{z}$$

By componendo and dividendo

$$\Rightarrow \frac{i+tan\theta}{i-tan\theta} = \frac{i+z}{i-z}$$

$$\Rightarrow \frac{i+\frac{sin\theta}{cos\theta}}{i-\frac{sin\theta}{cos\theta}} = \frac{i+z}{i-z}$$

$$\Rightarrow \frac{icos\theta+sin\theta}{icos\theta-sin\theta} = \frac{i+z}{i-z}$$

$$\Rightarrow \frac{i(cos\theta-isin\theta)}{i(cos\theta+isin\theta)} = \frac{i+z}{i-z}$$

$$\Rightarrow \frac{e^{-i\theta}}{e^{i\theta}} = \frac{i+z}{i-z}$$

$$\Rightarrow e^{-2i\theta} = \frac{i+z}{i-z}$$

$$\Rightarrow -2i\theta = log_e\left[\frac{i+z}{i-z}\right]$$

$$\Rightarrow \theta = \frac{-1}{2i} log_e\left[\frac{i+z}{i-z}\right]$$

$$\Rightarrow tan^{-1}z = \frac{i}{2} log_e\left[\frac{i+z}{i-z}\right]$$ hence proved.

Example.2 To show that $sinh^{-1}z = log_e\left[z + \sqrt{z^2+1}\right]$.

Solution:- Let $sinh^{-1}z = u$

$$\Rightarrow sinhu = z$$

$$\Rightarrow \frac{e^u - e^{-u}}{2} = z$$

$$\Rightarrow 2z = e^u - e^{-u}$$

$$\Rightarrow 2z = e^u - \frac{1}{e^u}$$

$$\Rightarrow 2ze^u = e^{2u} - 1$$

$$\Rightarrow e^{2u} - 2ze^u - 1 = 0$$

$$e^u = \frac{2z \mp \sqrt{4z^2+4}}{2} = z \mp \sqrt{z^2+1}$$

We take only +ve sign then

$$e^u = z + \sqrt{z^2+1}$$

$$\Rightarrow u = log_e\left[z + \sqrt{z^2+1}\right]$$

$$\Rightarrow sinh^{-1}z = log_e\left[z + \sqrt{z^2+1}\right]$$ hence proved

Example.3 If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$. Prove that

(i) $tanh\frac{u}{2} = tan\frac{\theta}{2}$ (ii) $coshu = sec\theta$

Solution:- Since, we have $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$\Rightarrow e^u = tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\Rightarrow \frac{e^{u/2}}{e^{-u/2}} = \frac{tan\frac{\pi}{4} + tan\frac{\theta}{2}}{1 - tan\frac{\pi}{4}.tan\frac{\theta}{2}} = \frac{1 + tan\frac{\theta}{2}}{1 - tan\frac{\theta}{2}}$$

By componendo and dividend

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{1 + tan\frac{\theta}{2} - 1 + tan\frac{\theta}{2}}{1 - tan\frac{\theta}{2} + 1 + tan\frac{\theta}{2}}$$

$$\Rightarrow tanh\frac{u}{2} = \frac{2tan\frac{\theta}{2}}{2}$$

$$\Rightarrow tanh\frac{u}{2} = tan\frac{\theta}{2}$$ hence proved

Example.4: If $y = logtanx$, prove that $\sin h\, xy = \frac{1}{2}[tan^n x - cot^n x]$.

Solution: We have $y = logtanx$

$\Rightarrow e^{y} = tanx$

$\Rightarrow e^{ny} = tan^{n}x$ and $e^{-ny} = cot^{n}x$

$\Rightarrow e^{ny} - e^{-ny} = tan^{n}x - cot^{n}x$

$\Rightarrow \frac{e^{ny}-e^{-ny}}{2} = \frac{1}{2}[tan^{n}x - cot^{n}x]$ hence proved.

Example.5 Prove that $(cosh\theta + sinh\theta)^{n} = coshn\theta + sinhn\theta$.

Solution: $(cosh\theta + sinh\theta)^{n} = \left(cosi\theta + \frac{1}{i}sini\theta\right)^{n}$

$= \left(cosi\theta + \frac{i}{i^{2}}sini\theta\right)^{n}$

$= (cosi\theta - isini\theta)^{n}$

$= cosni\theta - isin\, ni\theta$

$= coshn\theta - i(isin\, hn\theta)$

$= coshn\theta + sin\, hn\theta$ hence proved.

Example.6 Find $tanhx$ if $5sinhx - coshx = 5$

Solution: Give $5sinhx - coshx = 5$

$\Rightarrow 5\left[\frac{e^{x}-e^{-x}}{2}\right] - \left[\frac{e^{x}+e^{-x}}{2}\right] = 5$

$\Rightarrow 5e^{x} - 5e^{-x}$-$e^{x} - e^{-x} = 10$

$\Rightarrow 4e^{x} - 6e^{-x} - 10 = 0$

$$\Rightarrow 4e^{2x} - 10e^{-x} - 6 = 0$$

$$e^x = \frac{10 \mp \sqrt{100-4(4)(-6)}}{2\times 4}$$

$$= \frac{10 \mp \sqrt{100+96}}{8}$$

$$= \frac{10 \mp \sqrt{196}}{8}$$

$$= \frac{10+14}{8} \quad \text{and} \quad e^x = \frac{10-14}{8}$$

$$= \frac{24}{8} \qquad\qquad = \frac{-4}{8}$$

$$= 3 \qquad\qquad = \frac{-1}{2}$$

If $e^x = 3$

Then $tanhx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3-1/3}{3+1/3} = \frac{9-1}{9+1} = \frac{8}{10} = \frac{4}{5}$

If $e^x = \frac{-1}{2}$

$$tanhx = \frac{-\frac{1}{2}+2}{-\frac{1}{2}-2} = \frac{-1+4}{-1-4} = \frac{3}{-5} = \frac{-3}{5}$$

Real and imaginary parts of circular and hyperbolic function

(i) $\sin(x+iy) = sinxcosiy + cosxsiniy$

$$= sinxcoshy + icosxsinhy$$

$\therefore \quad cosiy = coshy$

$$siniy = isinhy$$

(ii) $\cos(x+iy) = cosx.cosiy - sinx.siniy$

$$= cosx.coshy - isinx.sinhy$$

Example.1 If $\sin(A+iB) = x+iy$. Prove that

$$\frac{x^2}{cosh^2B} + \frac{y^2}{sinh^2B} = 1,\ \frac{x^2}{sin^2A} + \frac{y^2}{cos^2A} = 1$$

Solution: It is given that $\sin(A+iB) = x+iy$

$$\Rightarrow sinA.cosiB + cosA.siniB = x+iy$$

$$\Rightarrow sinAcoshB + icosAsinhB = x+iy$$

Equating real and imaginary parts

$x = sinAcoshB$; $\quad y = cosAsinhB$

$\Rightarrow \frac{x}{coshB} = sinA$; $\quad \frac{y}{sinhB} = cosA$

$$\Rightarrow \frac{x^2}{cosh^2B} + \frac{y^2}{sinh^2B} = sin^2A + cos^2A = 1$$

Now, $\frac{x}{sinA} = coshB$; $\quad \frac{y}{cosA} = sinhB$

$$\frac{x^2}{sin^2A} + \frac{y^2}{cos^2A} = cosh^2B + sinh^2B = 1$$

Example.2 If $\tan(A+iB) = x+iy$, prove that

(i) $x^2 + y^2 + 2xcot2A = 1$

(ii) $x^2 + y^2 - 2ycoth2B + 1 = 0$

Solution:- Given $\tan(A + iB) = x + iy$

Now changing i to $-i$

$x - iy = \tan\,(A - iB)$

Now, $tan2A = \tan\,[(A + iB) + (A - iB)]$

$$= \frac{\tan(A+iB)+\tan\,(A-iB)}{1-Tan(A+iB)\tan\,(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\frac{1}{cot2A} = \frac{2x}{1-(x^2+y^2)}$$

$\Rightarrow 1 - x^2 - y^2 = 2xcot2A$

$\Rightarrow x^2 + y^2 + 2xcotA = 1$

(iii) Now, $\tan(2iB) = \tan\,[(A + iB) - (A - iB)]$

$$= \frac{\tan(A+iB)-\tan\,(A-iB)}{1+\tan\,(A+iB)\tan\,(A-iB)}$$

$$= \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$$

$$\Rightarrow \frac{1}{cot2iB} = \frac{x+iy-x+iy}{1+x^2+y^2}$$

$\tan(2iB) = itanh2B$
$\cot(2iB) = \frac{1}{i}coth2B$

$\Rightarrow x^2 + y^2 + 1 = 2iycot2iB$

$\Rightarrow x^2 + y^2 - 2iycot2iB + 1 = 0$

$\Rightarrow x^2 + y^2 - 2iy.\frac{1}{i}coth2B + 1 = 0$ hence proved

Example.3 If $\tan(\theta + i\phi) = tan\alpha + isec\alpha$. Prove that

$$e^{2\phi} = \mp cot\frac{\alpha}{2} \quad \text{and} \quad 2\theta = \left(n + \frac{1}{2}\right)\pi + \alpha.$$

Solution: We have $\tan(\theta + i\phi) = tan\alpha + isec\alpha$

Replacing i as $-i$

$$\tan(\theta - i\phi) = tan\alpha - isec\alpha$$

$$= \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{tan\alpha+isec\alpha-tan\alpha+isec\alpha}{1+(tan\alpha+isec\alpha)(tan\alpha-isec\alpha)}$$

$$= \frac{2isec\alpha}{1+tan^2\alpha-i^2sec^2\alpha}$$

$$= \frac{2isec\alpha}{sec^2\alpha+sec^2\alpha}$$

$$itanh2\phi = \frac{2isec\alpha}{2sec^2\alpha}$$

$$= icos\alpha$$

$$\Rightarrow \frac{e^{2i\phi}-e^{-2i\phi}}{e^{2i\phi}+e^{-2i\phi}} = \frac{cos\alpha}{1} = \frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}}$$

Apply componendo and dividendo

$$tan2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{tan(\theta+i\phi)+tan(\theta-i\phi)}{1-tan(\theta+i\phi).tan(\theta-i\phi)}$$

$$= \frac{tan\alpha+isec\alpha+tan\alpha-isec\alpha}{1-(tan\alpha+isec\alpha)(tan\alpha-isec\alpha)}$$

$$= \frac{2tan\alpha}{1-tan^2\alpha-sec^2\alpha}$$

$$= \frac{2tan\alpha}{1-tan^2\alpha-1-tan^2\alpha}$$

$$= \frac{2tan\alpha}{2tan^2\alpha}$$

$$= -cot\alpha$$

$\Rightarrow\frac{e^{2i\phi}+e^{-2i\phi}+e^{2i\phi}-e^{-2i\phi}}{e^{2i\phi}+e^{-2i\phi}-e^{2i\phi}+e^{-2i\phi}} = \frac{cos\alpha+1}{-cos\alpha+1}$

$\Rightarrow\frac{2e^{2i\phi}}{2e^{-2i\phi}} = \frac{2cos^2\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}}$

$\Rightarrow e^{4i\phi} = cot^2\frac{\alpha}{2}$

$\Rightarrow e^{2i\phi} = \mp cot\frac{\alpha}{2}$ hence proved

$\therefore\ tan2\theta = tan\left(\frac{\pi}{2}+\alpha\right)$

$2\theta = n\pi + \frac{\pi}{2} + \alpha = \left(n+\frac{1}{2}\right)\pi + \alpha$

Example.4 If $\cos(\alpha + i\beta) = r(cos\theta + isin\theta)$. Prove that

$$e^{2\beta} = \frac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$$

Solution: Give $\cos(\alpha + i\beta) = r(cos\theta + isin\theta)$.

$$\Rightarrow cos\alpha.cosi\beta - sin\alpha sini\beta = rcos\theta + irsin\theta$$

$$\Rightarrow cos\alpha cosh\beta - isin\alpha sinh\beta = rcos\theta + irsin\theta$$

Equating real and imaginary parts

$$cos\alpha cosh\beta = rcos\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots..\ (1)$$

$$-sin\alpha sinh\beta = rsin\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Dividing (2) by (1)

$$\frac{-sin\alpha sinh\beta}{cos\alpha cosh\beta} = \frac{sin\theta}{cos\theta}$$

$$\Rightarrow -tan\alpha tanh\beta = tan\theta$$

$$\Rightarrow tanh\beta = \frac{tan\theta}{tan\alpha}$$

$$\Rightarrow \frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}} = \frac{-sin\theta}{cos\theta} \cdot \frac{cos\alpha}{sin\alpha}$$

Now, Apply componendo and dividend

$$\frac{e^{\beta}-e^{-\beta}+e^{\beta}+e^{-\beta}}{e^{\beta}-e^{-\beta}-e^{\beta}-e^{-\beta}} = \frac{-sin\theta.cos\alpha+cos\theta.sin\alpha}{-sin\theta.cos\alpha-cos\theta.sin\alpha}$$

$$\Rightarrow \frac{2e^{\beta}}{-2e^{-\beta}} = \frac{\sin(\alpha-\theta)}{-\sin(\alpha+\theta)}$$

$$\Rightarrow e^{2\beta} = \frac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$$ Hence proved

Example.5 If $\tan(x+iy) = \sin(u+iv)$. Prove that

$$\frac{sin2x}{sinh2y} = \frac{tanu}{tanhv}$$

Solution: We have $\tan(x+iy) = \sin(u+iv)$ ……….. (1)

Changing i as $-i$ in (1)

$$\tan(x-iy) = \sin(u-iv) \text{ ……….. (2)}$$

Dividing (1) by (2)

$$\frac{\sin(u+iv)}{\sin(u-iv)} = \frac{\tan(x+iy)}{\tan(x-iy)}$$

$$\Rightarrow \frac{\sin(u+iv)}{\sin(u-iv)} = \frac{\sin(x+iy).\cos\,(x-iy)}{\cos(x+iy).\sin\,(x-iy)}$$

Apply componendo and dividend

$$\Rightarrow \frac{\sin(u+iv)+\sin(u-iv)}{\sin(u+iv)-\sin(u-iv)} = \frac{\sin(x+iy).\cos(x-iy)+\cos(x+iy).\sin\,(x-iy)}{\sin(x+iy).\cos(x-iy)-\cos(x+iy).\sin\,(x-iy)}$$

$$sinA + sinB = 2sin\left(\frac{A+B}{2}\right) cos\left(\frac{A-B}{2}\right)$$

$$sinA - sinB = 2cos\left(\frac{A+B}{2}\right) sin\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \frac{2sin\left(\frac{u+iv+u-iv}{2}\right).cos\left(\frac{u+iv-u+iv}{2}\right)}{2cos\left(\frac{u+iv+u-iv}{2}\right).sin\left(\frac{u+iv-u+iv}{2}\right)} = \frac{\sin\,(x+iy+x-iy)}{\sin\,(x+iy-x+iy)}$$

$$\Rightarrow \frac{sinu.cosiv}{cosu.siniv} = \frac{sin2x}{sin2iy}$$

$$\Rightarrow \frac{sinu.coshv}{icosu.sinhv} = \frac{sin2x}{isinh2y}$$

$$\Rightarrow \frac{tanu}{tanhv} = \frac{sin2x}{sinh2y}$$ hence proved

Example.6 If $sin^{-1}(u + iv) = \alpha + i\beta$, Prove that $sin^2\alpha$ and $cosh^2\beta$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0$$

Solution: We have $sin^{-1}(u + iv) = \alpha + i\beta$

$$\Rightarrow u + iv = \sin\,(\alpha + i\beta)$$

$$\Rightarrow u + iv = sin\alpha cosi\beta + cos\alpha.\,sini\beta$$

$$= sin\alpha cosh\beta + icos\alpha sinh\beta$$

Equating real and imaginary parts

$$u = sin\alpha cosh\beta$$

$$v = cos\alpha sinh\beta$$

Now,

$$1 + u^2 + v^2 = 1 + sin^2\alpha cosh^2\beta + cos^2\alpha sinh^2\beta$$

$$= 1 + sin^2\alpha cosh^2\beta + (1 - sin^2\alpha)(cosh^2\beta - 1)$$

$$= 1 + sin^2\alpha cosh^2\beta + cosh^2\beta - 1 - sin^2\alpha cosh^2\beta + sin^2\alpha$$

$$= sin^2\alpha + cosh^2\beta$$

= sum of roots.

Hence required equation is

$x^2 - x$(sum of roots) + (product of root) = 0

$\Rightarrow x^2 - x(1 + u^2 + v^2) + u^2 = 0$ $\quad\quad\quad$ $[\therefore sin^2\alpha cosh^2\beta = u^2]$

Hence proved

2.10 Logarithmic functions of a complex variable

If $z = x + iy$ and $w = u + iv$ be so related that $e^w = z$ then w is said to a logarithmic of z to the base e and written as $w = log_e z$ ……….. (1)

Also $\quad e^{w+2in\pi} = e^{w}.e^{2in\pi}$

$$= z.1 \qquad \because e^{2in\pi} = cos2n\pi + isin2n\pi = 1$$

$$= z$$

$logz = w + 2in\pi$ …………………… (ii)

i.e. the logarithm of a complex number has an infinite number of values and is therefore a multivalued function. The general values of the logarithm of z is written as logz so as to distinguish it form its principle value which is written as logz. This principal value is obtained by taking n = 0 in logz.

$\log(x + iy) = 2in\pi + \log\,(x + iy)$ [put $x = rcos\theta$, $y = rsin\theta$]

$$= 2in\pi + \log\,[r(cos\theta + isin\theta)]$$

$$= 2in\pi + log_e(re^{i\theta})$$

$$= 2in\pi + log_e r + log_e e^{i\theta}$$

$$= 2in\pi + log_e r + i\theta$$

$$= 2in\pi + log\sqrt{x^2 + y^2} + itan^{-1}\frac{y}{x}$$

$$= log\sqrt{x^2 + y^2} + i\left[2n\pi + tan^{-1}\frac{y}{x}\right]$$

$$= log|z| + i(2n\pi + \theta)$$

$$logz = log|z| + i\theta = log\sqrt{x^2 + y^2} + itan^{-1}\frac{y}{x}$$

Example.1 Prove that $log\left(\frac{a+ib}{a-ib}\right) = 2itan^{-1}\frac{b}{a}$. Hence evaluate $cos\left[ilog\left(\frac{a+ib}{a-ib}\right)\right]$

Solution: Put $a = rcos\theta, \quad b = rsin\theta$ so that $\theta = tan^{-1}\frac{b}{a}$

Now, $\quad log\left(\frac{a+ib}{a-ib}\right) = log\frac{r(cos\theta+isin\theta)}{e(cos\theta-isin\theta)} = log\frac{e^{i\theta}}{e^{-i\theta}}$

$$= loge^{2i\theta}$$

$$= 2i\theta$$

$$= 2itan^{-1}\frac{b}{a}$$

Thus, $cos\left[ilog\left(\frac{a+ib}{a-ib}\right)\right] = cos[i(2i\theta)] = cos2\theta$

$$= \frac{1-tan^2\theta}{1+tan^2\theta}$$

$$= \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}$$

$$= \frac{a^2-b^2}{a^2+b^2}$$

Example.2 Prove that $tan\left[ilog\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Solution: Let $a + ib = r(cos\theta + isin\theta) = re^{i\theta}$

Then equating real and imaginary parts

$$a = rcos\theta$$

$$b = rsin\theta$$

Also $a - ib = r(cos\theta - isin\theta) = re^{-i\theta}$

L.H.S. $= tan\left[ilog\left(\frac{a-ib}{a+ib}\right)\right]$

$$= tan\left[ilog\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right]$$

$$= tan\left[iloge^{-2i\theta}\right]$$

$$= \tan\,[i(-2i\theta)]$$

$$= tan2\theta$$

$$= \frac{2tan\theta}{1-tan^2\theta}$$

$$= \frac{2b/a}{1-b^2/a^2}$$

$$= \frac{2b/a}{\frac{a^2-b^2}{a^2}}$$

$$= \frac{2ab}{a^2-b^2}$$

= L.H.S. Hence proved

Example.3 If $i^{\alpha+i\beta} = \alpha + i\beta$. Prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

Solution: We have

$\alpha + i\beta = i^{\alpha+i\beta} = e^{logi^{\alpha+i\beta}} = e^{(\alpha+i\beta)logi}$

$= e^{\alpha+i\beta}.e^{logi}$

Let $z = i$ $\theta = tan^{-1}y/x = tan^{-1}1/0 = tan^{-1}\infty$

$|z| = 1$ $= tan^{-1}tan\frac{\pi}{2}$

$logi = log|i| + i(\theta + 2n\pi)$ $= \frac{\pi}{2}$

$= logi + i\left(\frac{\pi}{2} + 2n\pi\right)$

$= 0 + i\pi\left(\frac{4n+1}{2}\right)$

$= i\pi\left(\frac{4n+1}{2}\right)$

$\therefore$ $\alpha + i\beta = e^{\alpha+i\beta}.e^{i\pi\left(\frac{4n+1}{2}\right)}$

$= e^{\pi/2.(\alpha+i\beta)(4in+1)}$

$= e^{\frac{i\pi}{2}.(4n+1)\alpha-\beta(4n+1)\frac{\pi}{2}}$

$= e^{-\beta(4n+1)\frac{\pi}{2}}..e^{\frac{i\pi}{2}.(4n+1)\alpha}$

$= e^{-\beta(4n+1)\frac{\pi}{2}}\left[\cos(4n+1)\frac{\pi}{2}\alpha + isin(4n+1)\frac{\pi}{2}\alpha\right]$

$= e^{-\beta(4n+1)\frac{\pi}{2}}.\cos(4n+1)\frac{\pi}{2}\alpha + ie^{-\beta(4n+1)\frac{\pi}{2}}sin(4n+1)\frac{\pi}{2}\alpha$

Equating real and imaginary parts

$$\alpha = e^{-\beta(4n+1)\frac{\pi}{2}}.\cos(4n+1)\frac{\pi}{2}\alpha$$

$$\beta = e^{-\beta(4n+1)\frac{\pi}{2}} sin(4n+1)\frac{\pi}{2}\alpha$$

Squaring and adding

$$\alpha^2 + \beta^2 = e^{-2\beta(4n+1)\frac{\pi}{2}}\left[\cos^2(4n+1)\frac{\pi}{2}\alpha + isin^2(4n+1)\frac{\pi}{2}\alpha\right]$$

$$= e^{-\beta(4n+1)\pi}[1]$$

$$= e^{-\beta(4n+1)\pi}$$

Example.4 If $(a+ib)^p = m^{x+iy}$ then prove that

$$\frac{y}{x} = \frac{2tan^{-1}(b/a)}{log(a^2+b^2)}$$

Solution: We have

$$(a+ib)^p = m^{x+iy}$$

Taking log both sides

$$\Rightarrow plog(a+ib) = (x+iy)logm$$

$$\Rightarrow p\left[\frac{1}{2}\log(a^2+b^2) + itan^{-1}(b/a)\right] = xlogm + iylogm$$

Equating real and imaginary parts

$$xlogm = \frac{p}{2} log(a^2 + b^2) \quad \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$ylogm = ptan^{-1}(b/a) \quad \ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Dividing (2) by (1)

$$\frac{y}{x} = \frac{2tan^{-1}(b/a)}{log(a^2+b^2)}$$

Example.5 If $tanlog(x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $tanlog(x^2 + y^2) = \frac{2a}{1-a^2-b^2}$

Solution: Since $tanlog(x + iy) = a + ib$ …… (1)

$$tanlog(x - iy) = a - ib \quad \ldots\ldots\ldots \text{(2)}$$

Changing i as $-i$

Now, $tanlog(x^2 + y^2) = \tan\,[\log(x + iy)\,(x - iy)]$

$$= \tan\,[\log(x + iy) + log\,(x - iy)]$$

$$= \frac{tanlog(x+it)+tanlog(x-iy)}{1-tanlog(x+it).tanlog(x-iy)}$$

$$= \frac{a+ib+a-ib}{1-(a+ib)(a-ib)}$$

$$= \frac{2a}{1-(a^2+b^2)}$$

$$= \frac{2a}{1-a^2-b^2}$$

Example.6 If $sin^{-1}(x + iy) = \log(A + iB)$, show that

$$\frac{x^2}{sin^2u} - \frac{y^2}{cos^2u} = 1, \text{ where } A^2 + B^2 = e^{2u}$$

Solution: We have

$$sin^{-1}(x + iy) = \log(A + iB)$$

Let $\log(A + iB) = u + iv$

Then given equation becomes

$$sin^{-1}(x + iy) = u + iv$$

$$\Rightarrow x + iy = \sin\,(u + iv)$$

$$= sinu.cosiv + cosu.siniv$$

$$= sinu.coshv + icosu.sinhv$$

$\therefore$ $x = sinu\; coshv$ and

$$y = cosu.sinhv$$

$$\frac{x}{sinu} = coshv \text{ and } \frac{y}{cosu} = sinhv \text{ ……….. (2)}$$

We know that

$$cosh^2v - sinh^2v = 1$$

$$= \frac{x^2}{sin^2u} - \frac{y^2}{cos^2u} = 1 \text{ ……….. (3)}$$

Now, $\log(A + iB) = u + iv$

$$\frac{1}{2}log(A^2 + B^2) + itan^{-1}\frac{B}{A} = u + iv$$

$$\therefore \qquad u = \frac{1}{2}log(A^2 + B^2)$$

$$\Rightarrow 2u = log(A^2 + B^2)$$

$$\Rightarrow e^{2u} = A^2 + B^2 \qquad ....(4)$$

Hence the equation (3) holds only when equation (4) is true.

Example.7 if $i^{i^{i^{\cdot^{\infty}}}} = A + iB$ prove that $tan\frac{\pi A}{2} = \frac{B}{A}$ and $A^2 + B^2 = e^{-\pi B}$

Solution: We have $i^{i^{i^{\cdot^{\infty}}}} = A + iB$

$$\Rightarrow i^{A+iB} = A + iB$$

$$\Rightarrow (A + iB)logi = \log\,(A + iB)$$

$$\Rightarrow A + iB = e^{(A+iB).logi}$$

$$= e^{(A+iB).i\frac{\pi}{2}}$$

$$= e^{i\pi A/2}.e^{-\pi B/2}$$

$$\Rightarrow A + iB = e^{-\pi B/2}.\left[cos\frac{\pi A}{2} + isin\frac{\pi A}{2}\right]$$

$$A = e^{-\pi B/2}cos\frac{\pi A}{2}; \qquad B = e^{-\pi B/2}.sin\frac{\pi A}{2}$$

$\therefore \ \frac{B}{A} = tan\frac{\pi A}{2}$ hence proved

And $A^2 + B^2 = e^{-\pi B}.\left[cos^2\frac{\pi A}{2} + sin^2\frac{\pi A}{2}\right] = e^{-\pi B}$

Example.8 Expand $\frac{1}{(z+1)(z+3)}$ in the regions $|z| < 1$.

Solution- we have $f(z) = \frac{1}{(z+1)(z+3)}$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

If the regions $|z| < 1$

$f(z) = \frac{1}{2}[(1+z)^{-1} - (3+z)^{-1}]$,$then\ \ |z| < 1\ and\ \frac{|z|}{3} < 1$

$$f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}[1+\frac{z}{3}]^{-1}$$

$$f(z) = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots)$$

$$f(z) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(-\frac{z}{2} + \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) + \left(-\frac{z^3}{2} - \frac{z^3}{162}\right)$$

$$f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 \ldots..$$

Example.9 Expand $f(z) = \frac{1}{(z-1)(z-2)}\ for\ 1 < |z| < 2$

Solution-$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

In first bracket $|z| < 2$,we take out 2 as common and from second bracket z is taken out common as $1 < |z|$.

$$f(z) = -\frac{1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] - \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$f(z) = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} \ldots . - \frac{1}{z} - \frac{1}{z^3} - \frac{1}{z^4} \ldots$$

Example.10 Expand the function $Sin^{-1}z$ in powers of z.

Solution- Let $w = Sin^{-1}z$

$$\frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}}$$

$$= (1-z^2)^{-\frac{1}{2}} \qquad (1)$$

On expanding the R.H.S. of binomial theorem, we have

$$\frac{dw}{dz} = 1 - \frac{1}{2}(-z^2) + \frac{\frac{-1}{2}\left(-\frac{3}{2}\right)}{2!}(-z^2)^2 + \cdots$$

$$\frac{dw}{dz} = 1 + \frac{z^2}{2} + \frac{3}{8}z^4 + \cdots$$

On integrating, we have $w = z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots + c$

Putting $z = 0$ $then$ $w = sin^{-1}z = 0$

$i.e. \quad c = 0$

We have $sin^{-1}z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots$

2.11 SUMMARY

In this unit, we have covered the following:

1. If $Z_n = x_n + iy_n$ for $n = 1,2, \ldots$ and $z = x + iy$ $then$

$$\lim_{n\to\infty} Z_n = z \leftrightarrow \lim_{n\to\infty} x_n = x \ and \ \lim_{n\to\infty} y_n = y$$

2. If $Z_n = x_n + iy_n$ for $n = 1,2$ ...and $s = x + iy$ then $\sum_{n=1}^{\infty} z^n = s \leftrightarrow \sum_{n=1}^{\infty} x^n = X$ and $\sum_{n=1}^{\infty} y^n = Y$
3. If f analytic in a domain D, Then it can be represented as a power series at any point $z_0 \in D$ in powers of $z - z_0$ which is the Taylor series of f about z_0. If f fails to be analytic at a point z_0. It is possible to expand f in an infinite series having both positive and negative powers of $z - z_0$ using the Laurent series.
4. If f is an analytic function in the disk $|z - z_0| < \rho$. Then f has the power series representation $f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n, |z - z_0| < \rho$.
5. To every power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ there corresponds $0 \leq R \leq \infty$, for the series

(i) Converges absolutely in $|z - z_0| < R$,if $0 \le R \le \infty$.

(ii) Converges absolutely in $|z - z_0| < r < R$,if $0 \le R \le \infty$.

(iii) Converges absolutely in $|z - z_0| > R$,if $0 \le R \le \infty$.

The number R is called the radius of convergence of the power series and the circle $|z - z_0| = R$ is called its circle of convergence.

6. A power series always converges inside and diverges outside the circle of convergence $|z - z_0| = R$. But a power series may converge at all, none, or some of the points on the circle of convergence.

2.12 Terminal Questions

1. Find the radius of convergence of the series $\sum_{n=0}^{\infty}(n + 2^n)z^n$.

2. Find the radius of convergence R of the following power series:

(i) $\sum_{n=0}^{\infty} z^n$ (ii) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ (iii) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

Discuss the behaviour of each series at the points $z = R$ and $z = -R$

3. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \left(\frac{2n+3}{4n+1}\right)^n (z-2)^n$$

4. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = 2^n$ if n is even and $a_n = -3^n$ if n is odd.

5. Find the radius of convergence of the following power series

(i) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ (ii) $\sum_{n=1}^{\infty} n!\, z^n$

6. Show that if $\sum_{n=1}^{\infty} z_n = s$, then $\sum_{n=1}^{\infty} \bar{z}_n = \bar{s}$

7. Does the series $\sum_{n=1}^{\infty} z^n$ converges if $|z| \geq 1$? Justify your answer.

8. Determine whether there the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges.

9. To prove $cosh^{-1}z = log_e\left[z + \sqrt{z^2 - 1}\right]$. Try yourself.

10. To prove $tanh^{-1}z = \frac{1}{2} log \frac{1+x}{1-x}$. Try yourself.

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Bachelor of Science
DCEMM -113

Function of Complex Variables

Block

2 Complex Integration and Expansion of series

Unit 3
Complex Integration

Unit 4
Expansion in Series and Singularities

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Block – 2

Complex Integration and Expansion of series

In the first unit we will introduce complex integration. We begin by discussing the integration of complex valued function of a real variable. We also discuss some basic results regarding differentiation of such functions. we introduce contour integration which is a powerful tool in Complex Analysis, discuss the existence of anti-derivatives, Cauchy-Goursat theorem which is an important result in complex analysis.

In this unit, we shall study a result called Cauchy integral formula (CIF) and use it to evaluate certain integrals along simple closed contours. In this section, we shall also discuss Morera's theorem, a converse of Cauchy's theorem, Liouville's theorem. We shall also prove here the fundamental theorem of algebra as a consequence to Liouville's theorem. Finally, we shall show that moduli of analytic functions behave strangely if we talk about their maximum or minimum.

In the second unit we shall introduce the series representation of a complex valued function f. We shall show that if f is analytic in some domain D., then it can be represented as a power series at any point ∈D in powers of which is the Taylor series of f about . If f fails to be analytic at a point , we cannot find Taylor series expansion of f about that point. However, it is often possible to expand f in an infinite series having both positive and negative powers of . This series is called the Laurent series.

This leads up to study of Morera's theorem, Cauchy's inequality, Liouville's theorem, maximum Modulus theorem. The study of complex integration will be incomplete without the study of Taylor's series and Laurent's series, which we also kept up in this unit.

Unit-3: Complex Integration

Structure

3.1 INTRODUCTION

In the earlier units you have studied differentiation of complex analytic functions. In this unit we will introduce you to complex integration. We begin by discussing the integration of complex valued function of a real variable. We also discuss some basic results regarding differentiation of such functions. In we introduce you to contour integration, a powerful tool in Complex Analysis, discuss the existence of antiderivatives. In we prove Cauchy-Goursat theorem an important result in complex analysis.

In this unit, we shall study a result called Cauchy integral formula (CIF) and use it to evaluate certain integrals along simple closed contours. we shall start with proving the formula and discuss its applications in the remaining sections of this unit. It has been shown in that an analytic function is infinitely differentiable and all its derivatives are again analytic. In this section, we shall also discuss Morera's theorem, a converse of Cauchy's theorem, which you studied in Unit 4. As an application to Cauchy integral formula, we shall prove, Liouville's theorem and show that an entire bounded function has to be a constant. We shall also prove here the fundamental theorem of algebra as a consequence to Liouville's theorem. Finally, we shall show that moduli of analytic functions behave strangely if we talk about their maximum or minimum.

3.2 Objectives

After studying this unit, you should be able to:

• find the derivatives of complex valued functions of a real variable;

• state and apply the chain rule for differentiation of complex valued functions;

• define the concepts of arc, contour, rectifiable arc and the arc length of a rectifiable arc;

• define, state and apply the properties of complex valued functions of a real variable;

• define the integral of a function over a contour and state its basic properties; and

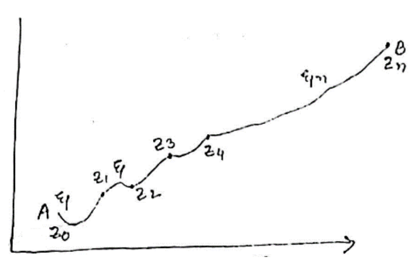
• state the Cauchy-Goursat theorem and apply it to evaluate contour integrals whenever possible.

3.3 Complex line Integral:

Consider a continuous function f(z) of the complex variable z = x + iy defined at all points on a curve c having end points A & B. Divide c into n parts at the points z_0, z_1, z_2, z_3, …….z_n.

Let A = z_0 and B = z_n

We choose a point ξ/k on each are joining z_{k-1} to z_k



From the sum $\delta n = \sum_{r=1}^{\infty} f(\xi/r)(z_r - z_{r-1})$

Suppose maximum value of $(z_r - z_{r-1}) \to 0$ as $n \to \infty$.

Then the sum δn bends to fixed limit which does not depend upon the made of sub division and denote this limit by

$$\int_a^b f(z)dz \quad or \int_c f(z)dz$$

Which is called the complex line integral or line integral of f(z) along c. an evaluation of integral by such method is also called ab-initio method. In case of real variable the path of integration of $\int_a^b f(z)dz$ is always along the x-axis from x = a to x = b. but in case of complex function f(z) the path of the definite integral $\int_a^b f(z)dz$ can be along the curve from z = a to z = b. its value depends upon the path of integration. But the value of integral from a to b remains the same if the different curves from a to b are regular curves.

Note: By the symbol $\int_c f(z)dz$ we mean the integral of f(z) along a boundary c in the +ve sense. In case of closed path the +ve direction is anticlockwise. The integral also c is often called contour integral.

If $f(z) = u(x,y) + iv(x,y)$ then since $z = x + iy,\ dz = dx + idy$

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy)$$

$$= \int_c (udx - vdy) + i\int_c (udx - vdy)$$

Which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integral of real functions.

3.4 Jordan Arc.:

A continuous are without multiple points is called a Jordan Arc.

Thus, for a point z on a Jordan are, z as expressed in equation (A), is one valued and $\phi(+), \psi(+)$ are also continuous. In addition, if $\phi'(+), \psi'(+)$ are also continuous in the range $a \leq + \leq B$, then the arc is called a regular of a Jordan curve.

A continuous Jordan Curve consists of a chain of finite number of continuous arcs.

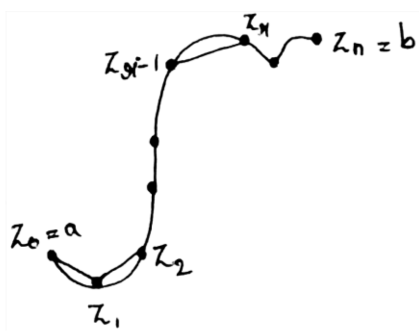
3.5 Rectifiable Arc:

Let $z = z(+) = x(+) + iy(t)$ be any given curve and let t take up any value between a and b, i.e., $a \leq t \leq b$.

Let $P = \{t_0, t_1, t_2, \dots \dots .. t_n\}$ be a partition of [a, b]. if $P_0, P_1, P_2, \dots \dots,$ P_n be the points on the curve corresponding to the points $t_0, t_1, t_2, \dots \dots .. t_n$, then the lines $P_0, P_1, P_2, P_3 \dots \dots, P_{n-1}\, P_n$.

Let $Z_0, Z_1, Z_2, \dots\dots, Z_n$ be the points on the curve corresponding to the values $t_0, t_1, t_2, \dots \dots .. t_n$, i.e. $Z(t_r) = Z_r$ then the length of the polygonal line

$$= \sum_{r=1}^{n} |Z_r - Z_{r-1}|$$



The value of this sum depends upon the made of the sub division and is called the length of an inscribed polygon.

If the curve is such that this sum (the length of the inscribed polygon) have a finite upper bound l, for all modes of the subdivision, the curve is said to be rectifiable and l is called the length of the curve.

3.6 Contour:

By contour, we mean a continuous chain of a finite no. of regular arcs. If the contour is closed and doesn't intersect itself then it is called a closed contour.

Example: Boundaries of triangle and rectangle.

By Contour we mean a Jordan Curve consisting of continuous chain of a finite number of regular arcs.

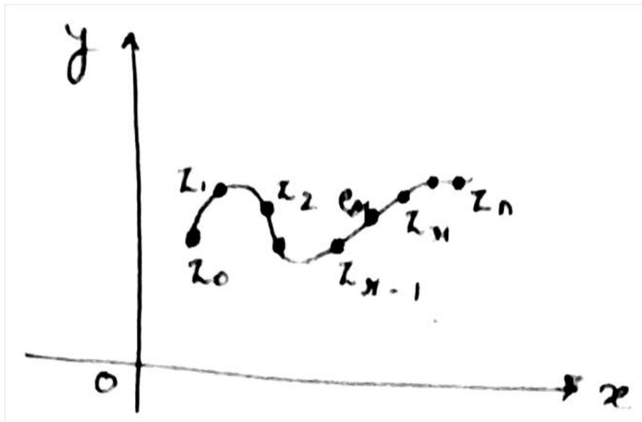
If A be the starting point of the first arc and B the end point of the last arc, then integral along such a curve is written as $\int_{AB} f(z)dz$.

If the starting point A of the arc coincides with the end point B of the last arc, then the contour AB is said to be closed.

The integral along such closed contour is written as $\int_c \ f(z)dz$, and is read as integral f(z) taken over the closed contour C. Although $\int_c \ f(z)dz$ does not indicate the direction along the curve, but it is conventional to take the direction positive which is anticlockwise, unless indicated otherwise.

3.7 Complex Integration:

Let $z = z(t) = x(t) + iy(t),\ a \le t \le b$, be a given curve C joining a and b and let f(z) be a function of a complex variable z defined and continuous on C.



Consider the partition $P = \{a = t_0, t_1, t_2, \dots \dots \dots t_n = b\}$ of the interval [a, b]. Let $z_0, z_1, z_2, \dots . z_n$ be the points on the curve corresponding to the values to $t_0, t_1, t_2, \dots \dots \dots t_n$, i.e., $z(t_r)$. On each are joining z_{r-1} to z_r, choose a point $e_r = z_r$. Where r = 1, 2, ….., n, on the arc joining z_{r-1} and z_r. Form the following sum S_p for the partition P;

$$S_p = f(e_1)(z_1 - z_0) + f(e_2)(z_2 - z_1) + \cdots \dots . . + f(e_r)(z_r - z_{r-1}) + \cdots + f(e_n)(z_n - z_{n-1})$$

i.e., $$S_p = \sum_{r=1}^{n} f(e_r)\,(z_r - z_{r-1})$$

$$S_p = \sum_{r=1}^{n} f(e_r)\,\Delta z_r\ ,$$

Where $\Delta z_r = z_r - z_{n-1}$

As $n \to \infty$, i.e., the largest of the chord lengths $|\Delta z_r|$ approaches to zero and if for every partition P and for every choice of points e_r, the sum S_p

tends to a uniue limit, then the function f(z) is said to be integrable from a to b along C. this limit is denoted by

$$\int_a^b f(z)dz \quad or \int_c f(z)dz$$

And is called the complex line integral or briefly the line integral of f(z) along the curve C.

It is also known as the definite integral of f(z) from a to b along the curve C. Thus

$$\int_c f(z)dz = \lim_{n\to\infty}\sum_{r=1}^{n} f(e_r)\,(z_r - z_{r-1})$$

It C happens to be a closed contour, then the line integral $\int_c f(z)dz$ is usually denoted by $\int_c f(z)dz$.

Example 1: Evaluate $\int_c dz$.

Solution: By the definition of complex integral, we have

$$\int_c f(z)dz = \lim_{n\to\infty}\sum_{r=1}^{n} f(e_r)\,(z_r - z_{r-1}) \quad \ldots\ldots\ldots\ldots.. (1)$$

Here f(z) = 1, so that $f(e_r) = 1$. Now from (1), we have

$$\int_c f(z)dz = \lim_{n\to\infty}\sum_{r=1}^{n} 1\,(z_r - z_{r-1})$$

$$= \lim_{n\to\infty}[(z_1 - z_0) + (z_2 - z_1) + \cdots \ldots\ldots + (z_n - z_{n-1})]$$

$$= \lim_{n\to\infty}(z_n - z_0) = b - a, \text{ Since } z_0 = a \text{ and } z_n = b.$$

Note: If C is a closed curve, then the points a and b coincide, i.e., b = a.

Hence $\int_c \; dz = 0$, (for closed curve).

Example 2.: Evaluate $\int_c \; |z|dz$, where C is the upper half of circle $|z| = 1$.

Solution: By the definition of complex integral, we have

$$\int_c \; f(z)dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(e_r)\,(z_r - z_{r-1}),$$

Where e_r is any point on the arc joining the points z_{r-1} and z_r.

Here $f(z) = |z|$ so that $f(e_r) = |e_r| = 1$

Since e_r lies on the unit circle

Hence $\int_c \; |z|dz = \lim_{n\to\infty} \sum_{r=1}^{n} 1\,(z_r - z_{r-1})$

$= \lim_{n\to\infty}[(z_1 - z_0) + (z_2 - z_1) + \cdots \ldots\ldots + (z_n - z_{n-1})]$

$= \lim\limits_{n\to\infty}(z_n - z_0) = b - a$, Since $z_0 = -1$ and $z_n = 1$.

$= \lim\limits_{n\to\infty}\{1 - (-1)\}$

= 2

3.8 Some elementary properties of complex line integrals:

Property 1.: $\int_c \; \{f(z) + g(z)\}dz = \int_c \; f(z)dz + \int_c \; g(z)dz.$

This property can be generalized for any finite number of functions.

Property 2.: $\int_c f(z)dz = -\int_{-c} f(z)dz$

Where $-c$ is the opposite arc of C.

Property 3.: $\int_{c_1+c_2} f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz.$

Where the end point of C, coincides with the initial point of C_2.

This property can be extended for a finite number of arcs provided the end point of the preceding arc coincides with the initial point of the arc which follows it.

Property. 4.: $\int_c Rf(z)dz = R\int_c f(z)dz,$

Where R is any complex constant.

Property 5.: $\int_c \{R_1 f_1(z) + R_2 f_2(z) + \cdots .. + R_n f_n(z)\}dz$

$$= R_1 \int_c f_1(z)dz + R_2 \int_c f_2(z)dz + \cdots . + R_n \int_c f_n(z)dz$$

Where R_1, R_2, ….R_n are complex constants.

This property is a direct consequence of properties 1 and 4.

Property.6: $\left|\int_c f(z)dz\right| \leq \int_c |f(z)||dz|$

Example.1 Evaluate $\int_c \frac{dz}{z-a}$, where c represents the circle $|z-a| = r$

Solution: Parametric equation of the circle $|z-a| = r$ is

$z = a + re^{i\theta}$, where $0 \leq \theta \leq \pi$

Therefore, $d_z = ire^{i\theta}d\theta$. Hence

$$\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta$$

$$= i[\theta]_0^{2\pi} = 2\pi i$$

3.9 Cauchy's Theorem:

If f(z) is analytic in a simply connected domain D, and C is any closed continuous rectifiable curve in D, then

$$\int_c f(z)dz$$

Proof: First we shall prove the following lemma known as Goursat's lemma.

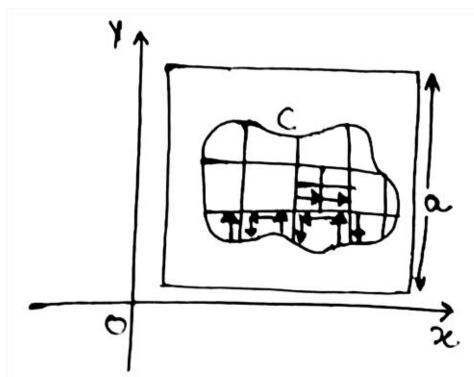
Lemma, let f(z) be analytic within and on a closed contour C. Then for every $\varepsilon > 0$, it is possible to divide the region within C into a finite number of squares and partial squares whose boundaries are denoted by S_i(i = 1, 2, ……., n) such that there exists a point z_i within each S_i such that

$$\left|\frac{f(z)-f(z_i)}{z-z_i} - f'(z_i)\right| < \varepsilon \qquad \text{.................. (1)}$$

For each point $z(\neq z_i)$ within or on S_i(i = 1, 2, ……., n)

Proof of the Lemma. Suppose the lemma is false. It means the lemma does not hold at least in one mesh, i.e., there exists on $\varepsilon > 0$ such that in however small meshes (squares and partial squares) we subdivide the region within C, there will be at least one mesh (square are a partial square) where the inequality (1) does not hold good.

Let R denote the region within and on the closed contour C. Cover the region R by a network of finite number of meshes (squares and partial squares) by drawing lines parallel to the co-ordinate axes. Then as per assumption there is at least one mesh for which (1) does not hold. Let us denote it by σ_0. It may be a square or a partial square. Then at least one of these squares contains the points of R for which (1) is not true. Suppose it is σ_1. Quadrisect σ_1 and repeat the above process. If this process comes to an end after a finite number of steps we arrive at a contradiction and the lemma is proved.

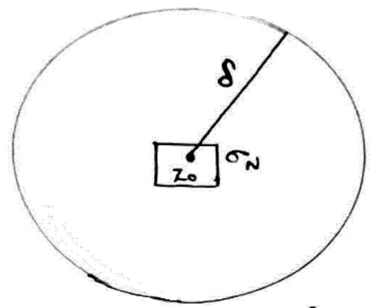


On the other hand if the process is continued indefinitely, we obtain a nested sequence of squares $\sigma_0, \sigma_1, \sigma_2, \dots \dots, \sigma_n, \dots.$ each contained in the previous one, for which lemma is not true. Consequently there exists a point z_0 which is the limit point of the set of points in R. Also $z_0 \in R$ because R is closed. Since f(z) is analytic at every point which lies within and on the closed contour C, f(z) is differentiable at z_0. So for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{f(z)-f(z_i)}{z-z_i} - f'(z_i)\right| < \varepsilon \qquad \dots\dots\dots\dots\dots\dots (2)$$

For all z for which $|z - z_0| < \delta$.

We can choose a positive integer N so large that the diagonal of the square σ_n is less than δ. Then all the squares $\sigma_n (n \geq N)$ are contained in the neighbourhood.



$|z - z_0| < \delta$ of z_0.

Also $z_0 \in \sigma_n$

Thus there exists a point z_i (here z_i is z_0) within each S_i for which inequality (1) is satisfied which contradicts the hypothesis. Thus the lemma is true

Proof of the theorem: The inequality (1) can be written as

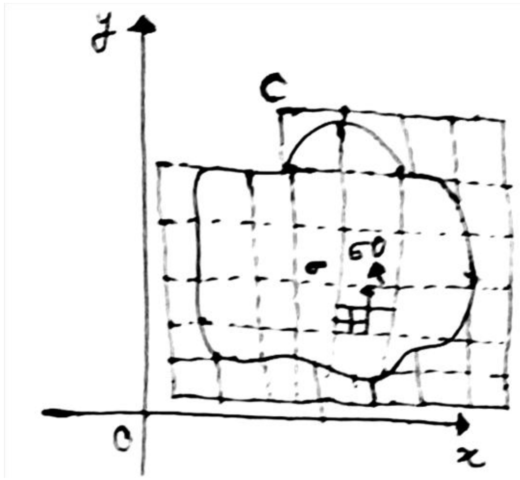
$$f(z) = f(z_i) - z_i f'(z_i) + z f'(z_i) + (z - z_i) n_i(z) \qquad \ldots\ldots\ldots (3)$$

Since (3) also gives the value of f(z) at any point on the boundary of S_i, integrating (3) around S_i, we get

$$\int_{S_i} f(z)dz = \{f(z_i) - z_i f'(z_i)\} \int_{S_i} dz + f'(z_i) \int_{S_i} zdz + \int_{S_i} (z - z_i) n_i(z)dz$$

$$= \int_{S_i} (z - z_i) n_i(z)dz, \quad \text{since } \int_{S_i} dz = 0 = \int_{S_i} zdz \qquad \ldots\ldots(4)$$

It is clear from the adjoining diagram that the integral around the closed curve C is equal to the sum of the integrals around all the Si's because the line integrals along the common boundaries of every pair of adjacent meshes cancel each other. We are left only with the integrals along the arcs which from parts of C.



Hence $\int_c f(z)dz = \sum_{i=1}^{n} \int_{s_i} f(z)dz$

From (4) and (5) we have

$$\int_c f(z)dz = \sum_{i=1}^{n} \int_{s_i} (z - z_i)n_i(z)dz,$$

i.e., $\left|\int_c f(z)dz\right| = \left|\sum_{i=1}^{n} \int_{s_i} (z - z_i)n_i(z)dz\right|$

$$\leq \sum_{i=1}^{n} \left|\int_{s_i} (z - z_i)n_i(z)dz\right|$$

$$\leq \sum_{i=1}^{n} \int_{s_i} |z - z_i||n_i(z)|dz$$

$$\leq \varepsilon \sum_{i=1}^{n} \int_{s_i} |z - z_i||dz| \quad \text{................(6)}$$

The boundary S_i of a mesh either completely or partially coincides with the boundary of a square. Let a_i be the length of a side of that square. The point z lies on S_i and z_i lies either on the boundary of S_i or inside δ_i. Therefore the distance between the points z and z_i cannot be greater than the length $a_i\sqrt{2}$ of the diagonal of that square i.e.,

$$|z - z_i| \leq a_i\sqrt{2}$$

So, $\int_{S_i} |z - z_i|\,|dz| \leq a_i\sqrt{2} \int_{S_i} |dz|$ …………… (7)

We know that $\int_{c} |dz| =$ length of Si

$= 4a_i$, if S_i is a complete square; and $\int_{S_i} |dz| \leq 4a_i + l_i$, if Si is a partial square, l_i denotes the length of arc of C which forms a part of S_i.

Substituting these values in (7), we get

$\int_{S_i} |z - z_i|\,|dz| \leq a_i\sqrt{2} = 4a_i^2\sqrt{2}$ ……………… (8)

If S_i is a square

and $\int_{S_i} |z - z_i|\,|dz| \leq a_i\sqrt{2}(4a_i + l_i) \leq 4a_i^2\sqrt{2} + al_i.\sqrt{2}$ ………………. (9)

If S_i is a partial square, where a denotes the length of the side of the square which encloses the curve C together with the squares which are used in covering C. obviously the sum of the areas a_i^2 of these squares cannot exceed a^2.

If l denotes the arc length of C, we have from (6), (8) and (9) that

$$\left|\int_{c} f(z)dz\right| < \varepsilon \sum_{i=1}^{n}\left(4\sqrt{2}a_i^2 + \sqrt{2}al_i\right)$$

$$< \varepsilon\left(4\sqrt{2}a^2 + \sqrt{2}al\right)$$

$$= \varepsilon \text{ (a constant)}$$

Hence $\qquad \int_c f(z)dz = 0$

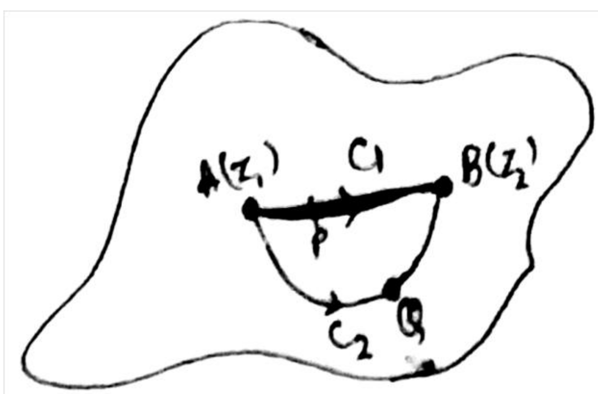
3.10 Extension of Cauchy's theorem on contours:

The following result is regarded as an extension of Cauchy's theorem.

Corollary 1: If f(z) is analytic in a simply connected domain D, then the integral along any rectifiable curve in D joining any two given points of D is the same, i.e. it does not depend upon the curve joining the two points.

Proof: suppose the two points A(z_1) and B(z_2) of the simply connected domain D are joined by the curves C and C_2 as shown in the figure given below. Then by Cauchy's theorem, we have

$$\int_{APBQA} f(z)dz = 0$$



i.e., $\qquad \int_{APB} f(z)dz + \int_{BQA} f(z) = dz = 0$

i.e., $\qquad \int_{c_1} f(z)dz - \int_{c_2} f(z)dz = 0$

(Using property 2 of section 5.3 in the second term)

i.e., $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$

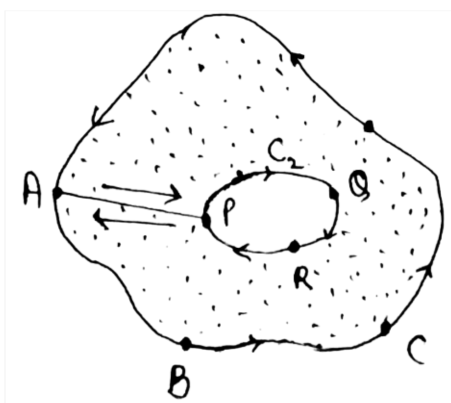
3.11 Defining multiply connected regions:

The following corollary may be called Cauchy's theorem for multiply connected domain.

Corollary 2. If a closed contour C_1 contains another closed contour C_2 and f(z) is analytic at every point lying in the ring-shaped domain bounded by C_1 and C_2, then

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

Proof: First we connect the outer contour C_1 to the inner contour C_2 by making a narrow cross cut joining a point A on C_1 to a point P on C_2.



Now ABCDAPQRPA is a simply connected domain. Clearly f(z) is analytic in this domain and is continuous on its boundary. Hence by Cauchy's theorem, we have

$$\int_{ABCDAPQRPA} f(z)dz = 0$$

i.e., $\int_{ABCDA} f(z)dz + \int_{AP} f(z)dz + \int_{PQRP} f(z)dz + \int_{PA} f(z)dz = 0$

$\int_{ABCDA} f(z)dz + \int_{PQRP} f(z)dz = 0$

Since $\int_{PA} f(z)dz = -\int_{PA} f(z)dz,$

So, the second and fourth integrals cancel each other

i.e., $\int_{c_1} f(z)dz + \int_{-c_2} f(z)dz = 0$

i.e. $\int_{c_1} f(z)dz - \int_{c_2} f(z)dz = 0$

i.e., $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$

Deduction: If the contour C contains non-intersecting contours C_1, C_2, ……C_n, then

$$\int_{c} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots . + \int_{C_n} f(z)dz$$

Example1: Evaluate: $\int_{C} \frac{dz}{z-5}$, where C is $|z| = 3$

Solution: Since the point z = 5 is outside the simple closed curve C : $|z| = 3$, the function $\frac{1}{z-5}$ is analytic inside and on C. Hence by Cauchy's theorem, we have

$$\int_{c} \frac{dz}{z-5} = 0$$

Example2.: Evaluate $\int_c \frac{dz}{z-1}$, where C is the circle $|z| < 1$.

Let $|z| = r$ be the circle C, where r < 1, then the function $\frac{1}{z-1}$ is analytic on and inside C. Hence by Cauchy's theorem, we have

$$\int_c \frac{dz}{z-1} = 0$$

Remark: We know that the complex integration is defined along a curve. The inequality $|z| < 1$ taken in the statement of the question represents, in fact, an open unit disc and not a circle, as stated. So we have taken $|z| = r$ as the equation of the circle.

Example 3.: Evaluate the integral $\int_0^{1+i} z^2\, dz$.

Solution: Refer the figure of example 3 of the preceding section.

Let B be the point of affix $1+i$ in the z-plane join OB. Here the given function $f(z) = z^2$ is analytic for all finite values of z. therefore, its integral between two fixed points will be the same irrespective of the path joining the two fixed points.

We choose the straight-line OB as the path for integration.

On OB: y = x so that $dy = dx$ …………… (1)

So, $\int_0^{1+i} z^2\, dz = \int_{OB}(x+iy)^2 d(x+iy)$

$$= \int_{OB}(x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_0^1 2ix^2\,(1+i)dx, \text{ using (1)}$$

$$= 2i(1+i)\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}i(1+i) = \frac{2}{3}(-1+i)$$

3.12 Cauchy's integral formula:

In this section we prove Cauchy's integral formula and several other related theorems. These results are found to be of great help in solving various problems of complex integration.

Theorem:

Let f(z) be an analytic function in a simply connected domain D enclosed by a rectifiable Jordan curve C and let f(z) be continuous on C. Then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_0} dz$$

Where z_0 is any point of D.

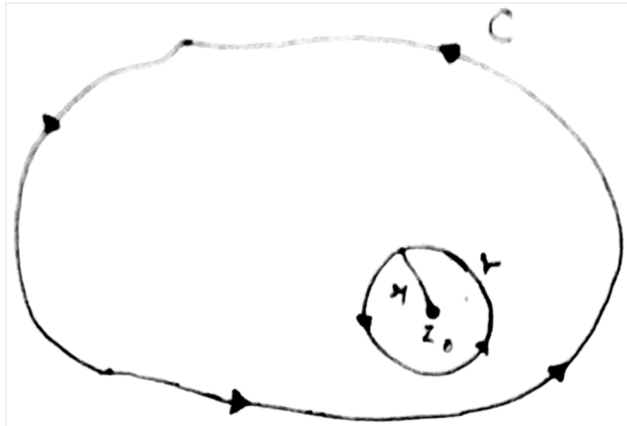
Proof: With z_0 as centre draw a circle of radius r lying entirely within C. Equation of the circle r is

$$|z - z_0| = r \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Consider a function $\phi(z)$ defined by

$$\phi(z) = \frac{f(z)}{z - z_0}$$

Then $\phi(z)$ is analytic in the doubly connected region bounded by C and r. By Cauchy's theorem for multiply connected regions, we have



$$\int_c \phi(z)\,dz = \int_r \phi(z)\,dz$$

When c and r are both traversed in anticlockwise directions.

i.e. $\int_c \frac{f(z)}{z-z_0} dz = \int_r \frac{f(z)}{z-z_2} dz$

i.e., $\int_c \frac{f(z)}{z-z_0} dz - \int_r \frac{f(z)}{z-z_2} dz = \int_r \frac{f(z)-f(z_0)}{z-z_0} dz$ ………. (2)

Putting $z - z_0 = re^{i\theta}$, $dz - ie^{i\theta}$, we have

$$\int_r \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

Substituting this value in (2), we have

$\int_c \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) = \int_r \frac{f(z)-f(z_0)}{z-z_0} dz$ …………….. (2)

The function f(z) is continuous at z_0. Therefore for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(z) - f(z_0)| < \varepsilon$ …………… (4)

For all z satisfying $(z - z_0) < \delta$

Since r is arbitrary, we can choose $r < \delta$ so that (4) is satisfied for all points on r.

Taking modulus of both sides of (2), we have

$$\left|\int_c \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0)\right| = \left|\int_r \frac{f(z)-f(z_0)}{z-z_0}dz\right|$$

$$\leq \int_r \frac{|f(z)-f(z_0)|}{|z-z_0|}|dz|$$

$< \int_r \frac{\varepsilon}{r}|dz|$, from (1) and (4)

$< \frac{\varepsilon}{r}\int_r |dz| = \frac{\varepsilon}{r}2\pi r = 2\pi\varepsilon$

Since ε is arbitrarily small and positive, we have

$\int_c \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0,$

i.e., $\quad f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz.$

3.13 Derivative of an analytic function:

Theorem 4. If a function f(z) is analytic within and on a closed contour C and ‘a’ is any point within C, then derivatives of all orders are analytic and are given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^{n+1}}$$

Proof: We known Cauchy’s integral formula

$$f'(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^2}$$

This shows that the required result is true for n = 1 suppose that this result is true for n = k so that

$$f^{(k)}(a) = \frac{k!}{1\pi i}\int_c \frac{f(z)dz}{(z-a)^{k+1}}$$

Let a + h be a point in the neighbourhood of a. then

$$\frac{f^{(k)}(a+h) - f^{(k)}(a)}{h} = \frac{k!}{1\pi ih}\int_c f(z)\left|\frac{1}{(z-a-h)^{k+1}} - \frac{1}{(z-a)^{k+1}}\right| dz$$

$$= \frac{k!}{1\pi ih}\int_c \frac{f(z)dz}{(z-a)^{k+1}}\left\{\left(1 - \frac{h}{z-a}\right)^{-(k+1)} - 1\right\} dz$$

$$= \frac{k!}{1\pi ih}\int_c \frac{f(z)}{(z-a)^{k+1}}\left\{\frac{(k+1)h}{z-a} + \frac{(k+1)(k+2)h^2}{2!}\frac{1}{(z-a)^2} + \cdots\right\} dz$$

$$= \frac{k!}{1\pi i}\int_c \frac{f(z)}{(z-a)^{k+1}}\left\{\frac{k+1}{z-a} + \frac{(k+1)(k+2)h}{2!}\frac{1}{(z-a)^2} + \cdots\right\} dz$$

Taking limit as h → 0, this gives

$$\lim_{h\to 0}\frac{f^{(k)}(a+h) - f^{(k)}(a)}{h} = \frac{k!}{1\pi i}\int_c \frac{f(z)}{(z-a)^{k+1}}\left\{\frac{k+1}{z-a} + 0 + 0 + \cdots\right\} dz$$

i.e. $\quad f^{(k+1)}(a) = \frac{ki(k+1)}{2\pi i}\int_c \frac{f(z)}{(z-a)^{k+1}.(z-a)} dz,$

i.e., $\quad f^{(k+1)}(a) = \frac{i(k+1)}{2\pi i}\int_c \frac{f(z)}{(z-a)^{k+2}} dz,$

Thus the required result is true for n = k + 1 if it is true for n = k, Hence by the principle of mathematical induction, it is true for all positive integral values of n.

Problem. Prove that $f'''(a) = \frac{3!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^4}$,

Where c is any contour containing z = a

This result follows by taking n = 3 in the above theorem.

Example 1: Evaluate $\int_c \frac{f(z)}{z-1}$, where C is $|z| = 2$

Solution: Cauchy's integral formula is:

$$\int_c \frac{f(z)}{z-a}dz = 2\pi i f(a),$$

Where z = a is a point inside contour C and f(z) is analytic within and on C.

Here C is $|z| = 2$, which represents a circle centred at z = 0 and having radius 2. Also, a = 1, which lies inside C.

Taking f(z) – 1, it follows that

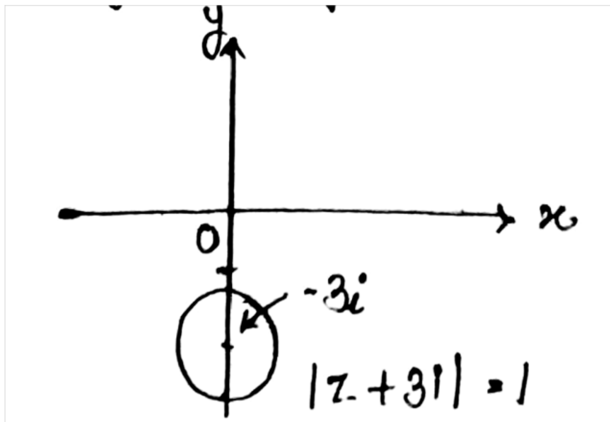
$$\int_c \frac{dz}{z-1} = 2\pi i f(1).$$

i.e., $\int_c \frac{dz}{z-1} = 2\pi i,$ since f(1) = 1

Example 2: Using Cauchy's integral formula, calculate the following integrals:

$\int_c \frac{dz}{z(z+\pi i)}$ where C is $(z + \pi i) = 1$

Solution: By Cauchy's integral formula, we have



$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a). \qquad \ldots\ldots.. (1)$$

Where z = a is a point inside the contour C and f(z) is analytic within and on C.

Let $I = \int_c \frac{dz}{z(z+\pi i)}$ take $f(z) = \frac{1}{z}$. Then

$$I = \int_c \frac{dz}{z-(-\pi i)} = 2\pi i f(-\pi i), \text{ by (1)}$$

$$= 2\pi i \left(\frac{1}{-\pi i}\right) = -2$$

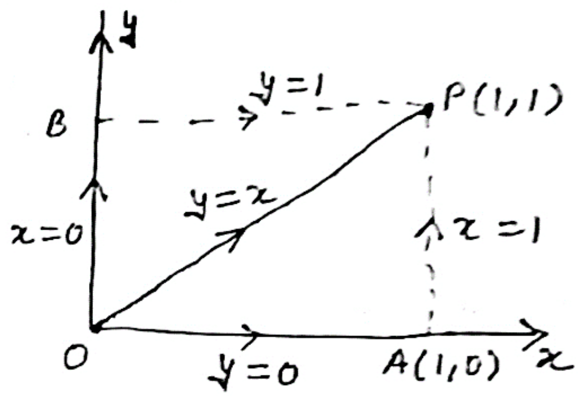
Here $z = -\pi i$ lies inside C and f(z) is analytic within C.

Example 3: Evaluate $\int_0^{1+i} (x - y + ix^2) dz$

1. Along the straight line from z = 0 to z = 1 + i
2. Along the real axis from z = 0 to z = 1 and then along a line parallel to imaginary axis from z = 1 to z = 1 + i.
3. Along the imaginary axis from z = 0 to z = I and then along a line parallel to real axis from z = i to z = 1 + i

Solution: Along the straight line OP joining O(z = 0) and P(z = 1 + i) y = k so that

dy = dx and x varies from 0 to 1.



$\int_0^{1+i}(x - y + ix^2)dz$

$= \int_0^{1+i}(x - y + ix^2)(dx + idy)$

$= \int_0^1 (x - x + ix^2)(dx + idx)$

$= \int_0^1 i(1 + i)x^2 dx$

$= (i - 1)\left[\frac{x^3}{3}\right]_0^1$

$= \frac{i-1}{3}$

(ii) Along the path OAP where A is z = 1

$$\int_0^{1+i}(x - y + ix^2)dz = \int_{OA}(x - y + ix^2)dz + \int_{AP}(x - y + ix^2)dz$$

Now along OA, y = 0,   along AP z = 1

$$\Rightarrow dy = 0 \qquad\qquad \Rightarrow dz = 0$$

$$\int_0^1 (x + ix^2)(dx + idy) + \int_0^1 (x - y + ix^2)(dx + idy)$$

$$= \int_0^1 (x + ix^2)dx + \int_0^1 (1 + i - y)(idy)$$

$$= \left(\frac{x^2}{2} + \frac{ix^3}{3}\right)_0^1 + (1 + i)y - \frac{y^2}{2}\int_0^1 (1 + i - y)(dy)$$

$$= \frac{1}{2} + \frac{i}{3} + \left[(1 + i)y - \frac{iy^2}{2}\right]_0^1$$

$$= \frac{1}{2} + \frac{i}{3} + i - 1 - \frac{i}{2}$$

$$= \frac{3 + 2i + 6i - 6 - 3i}{6}$$

$$= \frac{5i - 3}{6}$$

(iii) Along the path OBP where B is z = 1

$$\int_0^{1+i} (x - y + ix^2)dz = \int_{OB} (x - y + ix^2)dz + \int_{BP} (x - y + ix^2)dz$$

Now along OB,   x = 0,   along BP y = 1

$$\Rightarrow dy = 0 \qquad\qquad \Rightarrow dy = 0$$

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 -yidy + \int_0^1 (x - y + ix^2)dz$$

$$= \left[\frac{-iy^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x + \frac{ix^3}{2}\right]_0^1$$

$$= \frac{-i}{2} + \frac{1}{2} - i + \frac{i}{3}$$

$$= \frac{-3i+3-6+2i}{6}$$

$$= \frac{-i-3}{6}$$

$$= \frac{-(i+1)}{6} \quad \text{Ans.}$$

Example: Find the value of the integral $\int_c \ (x+y)dx + x^2ydy$

(a) Along $y = x^2$ having (0, 0), (3, 9) end points.

(b) Along $y = 3x$ between the same points.

Solution: Along $y = x^2$

$\Rightarrow \quad dy = 2xdx$ and x varies from 0 to 3

$$\int_c \ (x+y)dx + x^2ydy = \int_0^3 (x+x^2)dx + x^2.x^2(2x)dx$$

$$= \int_0^3 (x+x^2+2x^5)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^6}{6}\right]_0^3$$

$$= \left[\frac{3^2}{2} + \frac{3^3}{3} + \frac{23^6}{6}\right]$$

$$= 256\frac{1}{2} \quad \text{Ans.}$$

(b) Along $y = 3x$

$\Rightarrow dy = 3dx$ and x varies from 0 to 3

$$\int_c (x+y)dx + x^2y\,dy = \int_0^3 (x+3x)dx + x^2(3x)3dx$$

$$= \int_0^3 (4x + 9x^3)dx$$

$$= \left[\frac{4x^2}{2} + \frac{9x^4}{4}\right]_0^3$$

$$= 4.\frac{3^2}{2} + 9\frac{3^4}{4}$$

$$= 200\frac{1}{4} \qquad \text{Ans.}$$

Example: Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along

(a) Real axis to 2 and then vertically to $2+i$

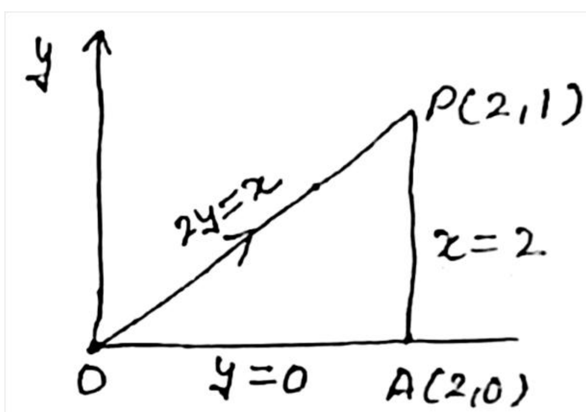
(b) Along the line $2y = x$

Solution: Let $z = x + iy$ then

$\bar{z} = x + iy$

$\Rightarrow (\bar{z})^2 = (x+iy)^2 = x^2 - y^2 - i2xy$

Along the path OAP where A(2, 0) and P(2, 1)



$$\int_0^{2+i} (x^2 - y^2 - i2xy)dz$$

$$= \int_{OA} (x^2 - y^2 - i2xy)dz + \int_{AP} (x^2 - y^2 - i2xy)dz$$

Along OA; y = 0; Along AP; x = 0

$dy = 0$ $dx = 0$

$$= \int_0^2 x^2 dx + \int_0^1 (4 - y^2 - 4iy)idy$$

$$= \left(\frac{x^3}{3}\right)_0^2 + \int_0^1 (4i - iy^2 - 4y)dy$$

$$= \frac{2^3}{3} + 4i[y]_0^1 - i\left[\frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{8}{3} + 4i - \frac{i}{3} + \frac{4}{2}$$

$$= \frac{16+24i-2i+12}{6}$$

$$= \frac{11i+14}{3}$$ Ans.

(b) Along the OP, 2y = x or dx = 2dy

And y varies from 0 to 1

$$\int_0^{2+i} (x^2 - y^2 - i2xy)dz = \int_0^1 [4y^2 - y^2 - 2i(2y)y][2dy + idy]$$

$$= (i+2)\int_0^1 (3y^2 - 4y^2 i)dy$$

$$= (i+2)\left(y^3 - 4i\frac{y^3}{3}\right)_0^1$$

$$= (i+2)\left(1 - \frac{i4}{3}\right)$$

$$= \frac{3i+6+4-8i}{3}$$

$$= \frac{10-5i}{3} \text{ Ans.}$$

Example 4: Use the Cauchy's integral formula to calculate

(i) $\int_c \frac{3z^2+7z+1}{z+1} dz$ where c is $|z| = \frac{1}{2}$

(ii) $\int_c \frac{2z+1}{z^2+z} dz$ where c is $|z| = \frac{1}{2}$

(iii) $\int_c \frac{z}{z^2-3z+2} dz$ where c is $|z-2| = \frac{1}{2}$

Solution: $\int_c \frac{3z^2+7z+1}{z+1} dz$

Since a = -1 which lies outside the circle c is $|z| = \frac{1}{2}$. Hence

$\int_c f(z)\, dx = 0$

$\Rightarrow \int_c \frac{3z^2+7z+1}{z+1} dz = 0$

(ii) $\int_c \frac{2z+1}{z^2+z} dz$ where c is $|z| = \frac{1}{2}$

Since z = 0 and z = -1 but z = -1 lies outside the circle and z = 0 is only point lie inside the circle.

$$\int_c \frac{2z+1}{z^2+z} dz = \int_c (2z+1)\left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

$$= \int_c \frac{2z+1}{z} dz - \int \frac{2z+1}{z+1} dz \text{ ....(1)}$$

$f(z) = 2z + 1$ at a = 0

$f(0) = 0$

$1 = \frac{1}{2\pi i}\int_c \frac{2z+1}{z}dz$

$\Rightarrow \int_c \frac{2z+1}{z}dz = 2\pi i$

At z = -1 , $\int \frac{2z+1}{z+1} = 0$

$\therefore$ from (1)

Hence $\int_c \frac{2z+1}{z(z+1)}dz = 2\pi i$ Ans.

(iii) $\int_c \frac{z}{z^2-3z+2}dz$ where c is $[z-2] = \frac{1}{2}$

$= \int_c \frac{z}{z^2-2z-z+2}dz$

$= \int_c \frac{z}{(z-1)(z-2)}dz$ here a = +1, a = +2

Now $[z-2] = \frac{1}{2}$

$\Rightarrow |x+iy-2| = \frac{1}{2}$

$\Rightarrow (x-2)^2 + y^2 = \left(\frac{1}{2}\right)^2$

Which is a circle having centre (2, 0) and radius $\frac{1}{2}$. $(x-h)^2 + (y-k)^2 = r^2$ centre is (-h, k) and radius is r.

$= \int_c \frac{z}{(z-1)(z-2)}dz = \int_c z\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$

$$= \int_c \frac{z}{z-2} dz - \int_c \frac{z}{z-1} dz$$

At a = 2 f(2) = 2

$$\Rightarrow 2 = \frac{1}{2\pi i} \int \frac{z}{z-2} dz$$

$$\Rightarrow \int \frac{z}{z-2} dz = 4\pi i$$

At a = 1 which lies outside the circle.

$$\int_c \frac{z}{z-1} dz = 0$$

Hence $\int_c \frac{z}{(z-1)(z-2)} dz = 4\pi i$ Ans.

Example 5: (i) Prove that $\int_c \frac{dz}{z-a} = 2\pi i$

(ii) $\int_c (z-a)^n dz = 0$ (n, any integer $\neq -1$)

Where c is the circle $|z-a| = r$

Solution: The parametric equation of c is $z - a = re^{i\theta}$ where θ varies from 0 to 2π as z describes c once in the anticlockwise direction.

$$dz = ire^{i\theta} d\theta \quad \& \quad |z-a| = r$$

(i) $\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = 1[\theta]_0^{2\pi} = 2\pi i$

(ii) $\int_c (z-a)^n dz = \int_0^{2\pi} \left(re^{i\theta}\right)^n ie^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)i\theta} d\theta$$

$$= ir^{n+1} \left[\frac{e^{(n+1)i\theta}}{(n+1)i}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{(n+1)i2\pi} - 1\right]$$

$$= \frac{r^{n+1}}{n+1}[1 - 1]$$

$$= 0$$

Example 6: $\int_c \frac{sin^2 zdz}{\left(z-\frac{\pi}{6}\right)^3}$ where c is the circle $|z| = 1$

Solution: $f(z) = sin^2 z$ is analytic inside the circle c : $|z| = 1$ and the point $a = \frac{\pi}{6} = \frac{3.14}{6} = 0.5app.$ lies within c.

By Cauchy's integral formula

$$f''(a) = \frac{2}{2\pi i}\int_c \frac{f(z)}{(z-a)^{2+1}}$$

Now $\Rightarrow f(z) = sin^2 z$

$\Rightarrow f'(z) = 2sinzcosz = sin2z$

$\Rightarrow f''(z) = 2cos2z$

$\therefore \ f''\left(\frac{\pi}{6}\right) = 2cos2.\frac{\pi}{6} = 2.\frac{1}{2} = 1$

$\therefore \quad 1 = \frac{2}{2\pi i}\int \frac{sin^2 zdz}{\left(z-\frac{\pi}{6}\right)^3}$

$\Rightarrow \int \frac{sin^2 zdz}{\left(z-\frac{\pi}{6}\right)^3} = \pi i$ Ans.

Example 7: Evaluate using Cauchy's integral formula $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where c is a circle $|z| = 3$

Solution: we have $f(z) = e^{2z}$ is analytic within the circle $|z| = 3$ and two singular point a = 1 and a = 2 lie inside c.

$$\int_c \frac{e^{2z}}{(z-1)(z-2)} dz = \int_c e^{2z} \left(\frac{1}{(z-2)} - \frac{1}{(z-1)} \right) dz$$

$$= \int_c e^{2z} \left(\frac{1}{(z-2)} \right) dz - \int_c e^{2z} \left(\frac{1}{(z-1)} \right) dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i (e^4 - e^2) \quad \text{Ans.}$$

Example 8: Evaluate $\int_c \frac{dz}{z^2-1}$ where c is the circle $x^2 + y^2 = 4$

Solution: Here $z = \mp 1$

$$\int_c \frac{dz}{(z-1)(z+1)} = \frac{1}{2} \int_c \frac{dz}{z-1} - \frac{1}{2} \int_c \frac{dz}{z+1}$$

When a = 1; when a = -1

$\int_c \frac{dz}{z-1} = 2\pi i$ $\int_c \frac{dz}{z+1} = +2\pi i$

Hence $\int_c \frac{dz}{(z-1)(z+1)} = \frac{1}{2} [2\pi i - 2\pi i] = 0$ Ans.

3.14 Morera’s Theorem:

The significance of the following theorem is that it is a sprt pf cpmverse pf the celebrated Cauchy’s theorem.

Theorem (Morera's theorem): If f(z) be continuous in a simply connected domain D and

$$\int_{\tau} f(z)dz = 0$$

Where τ is any rectifiable closed Jordan curve in D, then f(z) is analytic in D.

Proof.: Suppose z is any variable point and z_0 is a fixed point in the region D. also suppose τ_1 and τ_2 are ay two continuous rectifiable curves in D joining z_0 to z and τ is the closed continuous rectifiable curve consisting of τ_1 and $-\tau_2$. Then we have

$$\int_{\tau} f(z)\, dz = \int_{\tau_1} f(z)dz + \int_{-\tau_2} f(z)dz$$

And $\int_{\tau} f(z)dz = 0$ (given)

So $\int_{\tau_1} f(z)dz = -\int_{-\tau_2} f(z)dz = \int_{\tau_2} f(z)dz$

i.e., the integral along every rectifiable curve in D joining z_0 to z is the same.

Now consider a function f(z) defined by

$$f(z) = \int_{z_0}^{z} f(t)dt$$

As we know that the integral (1) depends only on the end points z_0 to z. we have

$$f(z+h) = \int_{z_0}^{z+h} f(t)dt \qquad \text{..............} \ (2)$$

From (1) and (2) we have

$$f(z+h) - f(z) = \int_{z_0}^{z+h} f(t)dt - \int_{z_0}^{z} f(t)dt$$

$$= \int_{z_0}^{z+h} f(t)dt + \int_{z}^{z_0} f(t)dt$$

$$= \int_{z}^{z+h} f(t)dt \qquad \text{......}(3)$$

Since the integral on the right hand side of (3) is path independent, it may be taken along the straight line joining z to z+h, so that

$$\frac{f(z+h) - f(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(t)dt - \frac{f(z)}{h}h$$

$$= \frac{1}{h}\left[\int_{z}^{z+h} f(t)dt - f(z)\int_{z}^{z+h} dt\right]$$

$$= \frac{1}{h}\int_{z}^{z+h}[f(t) - f(z)]dt \ . \ \text{.........}(4)$$

The function f(t) is given to be continuous at z. therefore for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(t) - f(z)| < \varepsilon,$

Where $|t - z| < \delta$

Since h is arbitrary, choose $|h| < \delta$ so that every point t lying on the line joining z to z+h satisfies (5) from (4) and (5), we have

$$\left|\frac{f(z+h) - f(z)}{h} - f(z)\right| \leq \frac{1}{|h|}\int_{z}^{z+h} |f(t) - f(z)|\,|dt|$$

$$< \frac{1}{|h|} \varepsilon \int_z^{z+h} |dt|, \text{ from } (5)$$

$$< \frac{1}{|h|} \varepsilon |h| = \varepsilon$$

Since ε is small and positive, we have

$$\lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} - f(z) \right| = 0$$

i.e. $\lim_{h \to 0} \left| \frac{f(z+h)-f(z)}{h} - f(z) \right| = f(z)$

Hence f'(z) = f(z), i.e. f(z) is differentiable for all values of z in D. consequently f(z) is analytic in D. Since the derivative of an analytic function is analytic, it follows that f(z) is analytic in D.

3.15 SUMMARY

In this unit we have discussed:

• Integration and differentiation of complex valued function of a real variable;

• The concept of an arc and the concept of a contour;

• The concept of a rectifiable arc, some conditions for an arc to be rectifiable and the concept of arc length of a rectifiable arc;

• Integration of a continuous function defined on a domain of the complex plane along a contour;

• The concept of antiderivative of a function and conditions for the existence of a function in a domain;

The results of this unit may be summarised as follows:

• An oriented curve is an ordered aggregate of points; the order being induced by the manner in which the parameter of the curve varies.

• A continuous curve in the complex plane C is a continuous mapping from a closed interval into C.

• If a continuous mapping is one-to-one, it is called a Jordan arc.

• A Jordan curve is a Jordan arc z (t) such that z (a) = z (b), when t ∈ (a, b).

• Jordan curve z divides the complex plane into two parts: the interior and exterior of z.

• A domain bounded by a Jordan curve is called a Jordan Domain.

• If two end points of a curve meet but it does not intersect at any other point, the curve is called a Closed Curve.

• A curve z (t) = x (t) + i y (t) is called a Smooth Curve if x (t) and y (t) have continuous derivatives at all points of its interval and the derivatives do not vanish simultaneously in the interval.

• For a curve z (t) = x (t) + i y (t), if x (t) and y (t) have continuous derivatives at all points except at finite number of points, then the curve is called Sectionally or Piece-wise Smooth Curve.

• Jordan arcs with continuously turning tangent are called Regular arcs.

• A contour is a continuous curve consisting of finite number of regular arcs.

• A region in which every closed curve can be contracted to a point without passing out of the region is called a Simply-connected Region, otherwise the space is Multiply-connected.

3.16 Terminal Questions

1. Evaluate $\int_{c} \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z| = 3$

2. Using Cauchy integral formula, calculate the following integrals.

(i) $\int_{c} \frac{z}{(9-z^2)(z+i)} dz$, where C is the circle $|z| = 2$ described in positive sense.

(ii) $\int_{c} \frac{\cos(\pi z)}{z(z^2+1)} dz$, where C is the circle $|z| = 2$

3. Evaluate $\int_{c} \frac{dz}{z(z+\pi i)}$ where C is $|z + 3i| = 1$

4. Evaluate $\int_{0}^{3+i} z^2 dz$ along the line joining the points (0, 0) and (3, 1).

5. Evaluate $\int_{c} \frac{dz}{(z-a)}$, where C is a closed curve and $z = a$ is (i) outside c and (ii) inside c.

6. Evaluate $\int_{\boldsymbol{c}} \frac{\boldsymbol{dz}}{\boldsymbol{z-2}}$ $for\ n = 2,3,4$ ...where $z = a$ is a point inside the simple closed curve c.

7. Evaluate $\int_{c} \frac{dz}{z-2}$ around (i) the circle $|z - 2| = 4$ and (ii) the circle $|z - 1| = 5$.

8. Show that $\mathrm{f(z)} = \int_{0}^{1} \mathrm{e}^{-\mathrm{z}^2\mathrm{t}^2} \mathrm{dt}$ is entire, and compute $\mathrm{f}'(\mathrm{z})$.

9. Evaluate $\int_{c} \frac{dz}{z-2}$ around the square vertices at $2 \pm 2i, -2 \pm 2i$.

10. Evaluate $\int_{c} \frac{e^{z}dz}{z^{2}+1}$ over the circular path $|z| = 2$

Unit – 4: Expansion in Series and Singularities

Structure

4.1 Introduction:

In this unit we shall introduce you to the series representation of a complex valued function f . We shall show that if f is analytic in some domain D., then it can be represented as a power series at any point $Z_0 \in$ D in powers of $Z - Z_0$ which is the Taylor series of f about Z_0. If f fails to be analytic at a point Z_0 , we cannot find Taylor series expansion of f about that point. However, it is often possible to expand f in an infinite series having both positive and negative powers of $Z - Z_0$. This series is called the Laurent series.

This leads up to study of Morera's theorem, Cauchy's inequality, Liouville's theorem, Maximum Modulus theorem. The study of complex integration will be incomplete without the study of Taylor's series and Laurent's series, which we also kept up in this unit.

You have already studied that the sum of a power series with non-zero radius of convergence is an analytic function, regular within the circle of convergence. We now prove the converse theorem, known as Taylor's theorem concerning analytic function of a complex variable.

4.2 Objectives:

After studying this unit, you should be able to

• obtain the Taylor series representation of a complex-valued function about a point at which the function is analytic;

• obtain a series representation of a complex-valued function about a point at which the function is not analytic in terms of Laurent series;

• obtain the radius of convergence of a power series.

• Learn the results of Morera's theorem, Cauchy's inequality, Maximum as well as Minimum Modulus Theorem and Lioville's theorem, and

• study the use of Taylor's series and Laurent's series for development of series of complex functions.

4.3 Taylors series:

You may recall from your knowledge of real analysis that certain real-valued functions can be approximated by polynomials using Taylor theorem. Under certain conditions, an infinitely differentiable function in a neighbourhood of a point $x_0 \in R$ has a Taylor series expansion about that point. The Taylor series about zero is referred as Maclaurin series. Some of the well-known Maclaurin series expansions are:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, x \in R$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots, x \in R$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots, x \in R,$

We shall now extend these series expansions for functions of complex variables.

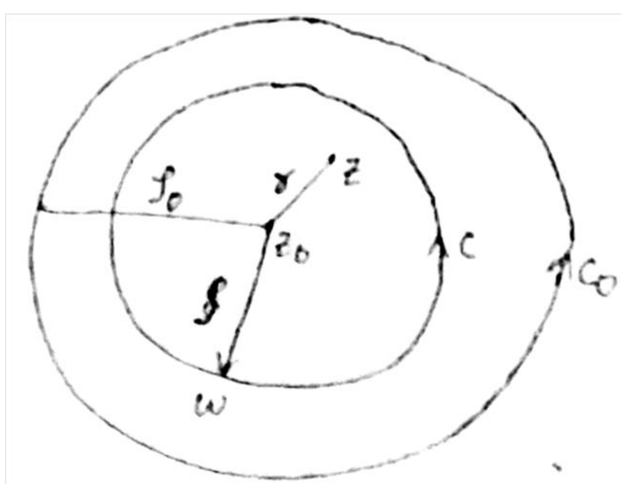
If a function f(z) is analytic within a circle c_o with centre z_0 and radius to then for every point z within c_0

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{\underline{|2}} f''(z_0) + \cdots \ldots \ldots$$
$$+ \frac{(z - z_0)^n}{\underline{|n}} f^n(z_0) + \cdots$$

Or

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where}$$

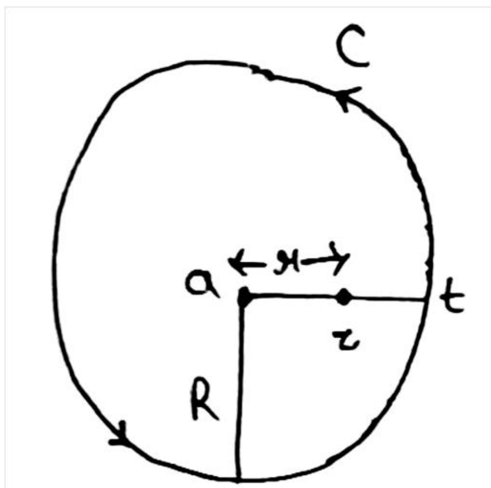
$$a_n = \frac{f^n(z_0)}{\underline{|n}}, \ r < \delta, \qquad |z - z_0| = r$$



Taylor's Theorem:- If a function f(z) is analytic within a circle c with its centre z = a and radius R, then at every point z inside c.

$f(z) = \sum_{n=0}^{\infty} f^n(a) \frac{(z-a)^n}{n!}$ i.e., $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

Where $a_n = \frac{f^n(a)}{n!}$



[The series on the right-hand side is known as Taylor's series of f(x).]

Proof: - Let f(z) be analytic within a circle c whose equation is $|t-a| = R$. Let z be any point within c such that $|z-a| = r < R$.

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-a)-(z-a)}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(t)}{t-a}\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1} dt$$

$$= \frac{1}{2\pi i}\int_c \frac{f(t)}{t-a}.\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right) \quad +\cdots \ldots \ldots dt\right]$$

$$= etc = \sum f^n(a).\frac{(z-a)^n}{n!}$$

4.4 Cauchy's Inequality:

Statement: If f (z) is analytic within and on a circle $C\colon |z-z_0| = R < \rho$ and if $|f(z)| \leq M$ on C,Then $|f^n(z_0)| \leq \frac{n!m}{R^n}$

Proof: From the n^th derivative of an analytic function, we have

$$|f^n(z_0)| \leq \frac{n!\,m}{R^n}\int_c \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$\leq \frac{n!\,m}{2\pi}\int_c \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}\int_c |dz|$$

$$= \frac{n!}{2\pi}.\frac{M}{r^{n+1}}.2\pi R = \frac{n!.M}{R^n}$$

Which prove Cauchy's Inequality.

For the special case, $n = 0$, Cauchy's Inequality becomes $|f(z)| \leq M$.

which shows that on every circle around z_0, no matter how small, | f (z) | has a maximum value M which is at least as great as f (z_0). This result is usually referred to as the Maximum Modulus Theorem, which may stated as

The absolute value of non-constant function f (z) cannot have a maximum at any point where the function is analytic. If f (z) is analytic at all points of a closed region R, bounded by a simple closed curve C, then the real function | f (z) | must have a maximum at some point of R. By the Maximum Modulus Theorem, the maximum cannot occur in the interior of Ri hence it must occur on the boundary C. This gives the following result:

Corollary of the Maximum Modulus Theorem If f (z) is a non-constant function which is analytic over a closed region R bounded by a simple closed occur C, then the maximum value of | f (z) | over R occurs on the boundary C. A similar result is true for the minimum value of | f (z) | over R, provided f (z) ≠ 0 in R. This result is known as Minimum Modulus Theorem, which states If f (z) is analytic inside and on a simple closed curve C, and f (z) ≠ 0 inside C, then | f (z) | must assume the minimum value on C. This result can be proved by applying the maximum modulus theorem to $\frac{1}{f(z)}$.

Example: If f (z) be analytic within and on the boundary of a bounded domain D, show that Re f (z) and Im f (z) attain maximum values on the boundary of D.

Solution: Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function within and on the boundary of D. Then g (z) = exp [f (z)] is analytic in D.

Hence, by maximum modulus theorem,

$|g(z)| = |e^{f(z)}| = |e^{u+iv}| = e^u$ is attained on the boundary of D.

Since real exponential function is an increasing function, therefore $Re\ f(z) = u = u(x, y)$ also attains its maximum exactly at the same points at e^u , i.e. on the boundary of D.

Similarly, by considering g (z) = − i f (z), we can show that Im f (z) = v (x, y) also attains its maximum on the boundary of D.

Another theorem, which can be readily deduced from Cauchy's Inequality is called Lioville's Theorem, which we take up next.

4.5 Liouville's Theorem: -

Bounded function: - A function f(z) analytic in a domain D is said to be bounded if there exists a number M > 0 such that

$$|f(z)| \leq M, \qquad \forall z \in D$$

Theorem: - If a function f(z) is analytic for all finite values of z and is bounded, then it is a constant.

Or

If f(z) be an integral function satisfying the inequality $|f(z)| \leq M$ for all finite values of z, where M is a positive constant, then f(z) is constant.

Proof: - Let z_1, z_2 be any two points of the z-plane take the contour C to be a large circle, with its centre origin and radius R, enclosing the points z1 and z2, so that $R > |z_1|$ and also $R > |z_2|$. By Cauchy's integral formula, we have

$$f(z_1) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z - z_1}$$

$$f(z_2) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z - z_2}$$

So, $\quad f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z-z_1} - \frac{1}{2\pi i} \int_c \frac{f(z)dz}{z-z_2}$

$$= \frac{1}{2\pi i} \int_c \frac{(z_1-z_2)f(z)dz}{(z-z_1)(z-z_2)}$$

Whence

$$| f(z_1) - f(z_2)| = \left| \frac{1}{2\pi i} \int_c \frac{(z_1 - z_2)f(z)dz}{(z - z_1)(z - z_2)} \right|$$

$$\leq \left| \frac{1}{2\pi i} \right| \int_c \frac{|z_1 - z_2||f(z)||dz|}{|z - z_1||z - z_2|}$$

$$\leq \frac{1}{2\pi} |z - z_1| M \int_c \frac{|dz|}{(|z| - |z_1|)(|z| - |z_2|)}$$

$$= \frac{1}{2\pi} \frac{|z_1 - z_2|M}{(R - |z_1|)(R - |z_2|)} \int_c |dz|$$

$$= \frac{1}{2\pi} \frac{|z_1 - z_2|M}{(R - |z_1|)(R - |z_2|)} \int_0^{2\pi} R d\theta$$

Since $z = Re^{i\theta}$, $\quad |dz| = Rd\theta$

$$= \frac{|z_1 - z_2|M2\pi R}{2\pi(R - |z_1|)(R - |z_2|)} = \frac{|z_1 - z_2|M}{R\left(1 - \frac{|z_1|}{R}\right)\left(1 - \frac{|z_2|}{R}\right)}$$

4.6 Laurent's Theorem

Suppose a function $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R'($R' < R$).

If z is any point of the annulus. Then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$

Where $a_n = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-a)^{n+1}}, b_n = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-a)^{-n+1}}$.

Proof: Let $f(z)$ be analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R'($R' < R$).

Then if z is any point within the ring space, then

$$R' < |z-a| = r < R$$

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-z)} - \frac{1}{2\pi i}\int_{c'} \frac{f(t)dt}{(t-z)}$$

$$f(z) = \frac{1}{2\pi i}\int_{c} \frac{f(t)dt}{(t-a)-(z-a)} + \frac{1}{2\pi i}\int_{c'} \frac{f(t)dt}{(z-a)(t-a)}$$

$$f(z) = \frac{1}{2\pi i}\int_{c} \frac{f(t)}{(t-a)}\left[1-\frac{z-a}{t-a}\right]^{-1} dt + \frac{1}{2\pi i}\int_{c'} \frac{f(t)}{(z-a)}\left[1-\frac{t-a}{z-a}\right]^{-1} dt$$

$f(z) = \frac{1}{2\pi i}\int_{c} \frac{f(t)}{t-a}\left[1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots\ldots.+\left(\frac{z-a}{t-a}\right)^n+\left(\frac{z-a}{t-a}\right)^{n+1}\left[1-\frac{z-a}{t-a}\right]^{-1}\right]+$

$\frac{1}{2\pi i}\int_{c} \frac{f(t)}{t-a}\left[1+\left(\frac{t-a}{z-a}\right)+\left(\frac{t-a}{z-a}\right)^2+\cdots\ldots.+\left(\frac{t-a}{z-a}\right)^n+\left(\frac{t-a}{z-a}\right)^{n+1}\left[1-\frac{t-a}{z-a}\right]^{-1}\right] dt$……………….(1)

We let $a_n = \frac{1}{2\pi i}\int_{c} \frac{f(t)dt}{(t-a)^{n+1}}, b_n = \frac{1}{2\pi i}\int_{c} \frac{f(t)}{(t-a)^{-n+1}}|dt = a_{-n}$

Then by equation (1), we have

$f(z) = [a_n + (z-a)a_1 + \cdots + a_n(z-a)^n + U_{n+1}] + \left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_n}{(z-a)^n}+V_{n+1}\right]$…………(2)

Where $U_{n+1} = \frac{1}{2\pi i}\int_{c} \frac{f(t)}{(t-a)}\left[\frac{z-a}{t-a}\right]^{n+1}\left[1-\frac{z-a}{t-a}\right]^{-1} dt \left[\left[1-\frac{z-a}{t-a}\right]^{-1} = \left(\frac{t-z}{z-a}\right)^{-1} = \left(\frac{t-a}{t-z}\right)\right]$

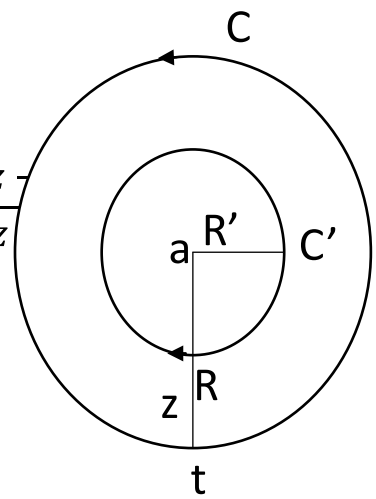
$$= \frac{1}{2\pi i}\int_{c} \frac{f(t)(z-a)^{n+1}}{(t-a)(t-a)^{n+1}}\left(\frac{t-a}{z-a}\right)$$

$$=\frac{1}{2\pi i}\int_c \frac{f(t)(z-a)^{n+1}}{(t-z)(t-a)^{n+1}}\,dt \ldots\ldots\ldots\ldots\text{(3)}$$

And

$$V_{n+1}=\frac{1}{2\pi i}\int_c \frac{f(t)}{(Z-a)}\left[\frac{t-a}{z-a}\right]^{n+1}\left[1-\frac{t-a}{z-a}\right]^{-1} dt$$

$$=\frac{1}{2\pi i}\int_c \frac{f(t)(t-a)^{n+1}}{(z-a)(z-a)^{n+1}}\left(\frac{z-}{z}\right.$$



$$=\frac{1}{2\pi i}\int_c \frac{f(t)(t-a)^{n+1}}{(z-t)(z-a)^{n+1}}\,dt \ldots\ldots\ldots\ldots\text{(3)}$$

Let $M = max.value\ of\ |f(t)|\ on\ C$

$$M' = max.value\ of\ |f(t)|\ on\ C'.$$

From equation (3),

$$U_{n+1}=\frac{1}{2\pi i}\int_c \frac{f(t)}{(t-a)}\left[\frac{z-a}{t-a}\right]^{n+1} dt$$

$$|U_{n+1}| = \frac{1}{2\pi}\int_c \frac{|f(t)||z-a|^{n+1}}{|t-z||t-a|^{n+1}}|dt| \quad \left\{\begin{array}{c} |z-a| = r \\ |t-a| = R \\ |t-z| = |(t-a)-(z-a)| = R-r \\ |\int_c |dt|| = 2\pi R \end{array}\right.$$

$$=\frac{M}{2\pi}\int_C \frac{r^{n+1}}{(R-r)R^{n+1}}|dt|$$

$$= \frac{M}{2\pi}\frac{r^{n+1}}{R^{n+1}}\frac{1}{R-r}.2\pi R = M\left(\frac{r}{R}\right)^{n+1}.\frac{1}{1-\frac{r}{R}}$$

$\lim_{n\to\infty}|U_{n+1}| = \lim_{n\to\infty} M\left(\frac{r}{R}\right)^{n+1}.\frac{1}{1-\frac{r}{R}} \to 0\ as\ \frac{r}{R} < 1$…………….(5)

Now from equation (4), $|V_{n+1}| \leq \frac{1}{2\pi}\int_{C'} \frac{|f(t)|}{|z-t|}\frac{|t-a|^{n+1}}{|z-a|^{n+1}}|dt|$ $\{|z-t| = |(z-a)-(t-a)| = r - R'$

Solving above

$\lim_{n\to\infty}|V_{n+1}| \leq \lim_{n\to\infty} M'\left(\frac{R'}{r}\right)^{n+1}.\frac{1}{\frac{r}{R'}-1} \to 0\ as\ \frac{R'}{r} < 1\ or\ \frac{r}{R'} > 1$……………..(6)

Now by equation (2), using equations (5) and (6), we have

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$……………….(7)

Where $a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}$

$b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(t)dt}{(t-a)^{-n+1}}$

4.7 Laurent's Series: -

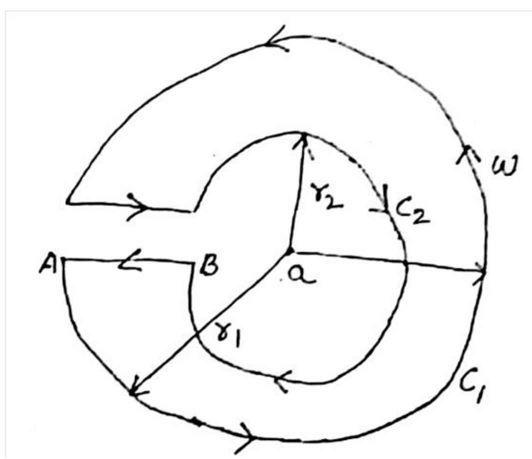
If f(z) is analytic in the ring-shaped region (annulus region) R bounded by two concentric circles c_1 and c_2 with centre a and radius r_1 and r_2 ($r_1 > r_2$) then all z in R.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Where $$a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)dw}{(w-a)^{n+1}}$$

$$b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(w)dw}{(w-a)^{-n+1}}$$

z be any point in the annulus region s.t. $|z-a| = r$, $r_1 > r_2$ and w be any point on the circle c_1.



Remark 1 It should be noted that the coefficients of the positive powers of (z – a) in Laurent's expression, although identical in form with the integrals of Taylor's Theorem, cannot be replaced by the derivative expressions $\frac{f^{(n)}(a)}{n!}$,

since f (z) is not analytic throughout the entire interior of C_2 (or C) and hence Cauchy's generalised integral formula cannot be applied. Specifically, f (z) many have many points of non-analyticity within C_1 and therefore within C_2 (or C).

Remark 2 The Laurent expansion of a function over a given annulus, if it exists, is unique. **Remark 3** As in the case of Taylor's series, the Laurent expansion of a given function in a given annulus is usually not found through the use of Laurent's Theorem but rather by algebraic manipulations suggested by the nature of the function. Such procedures are correct because Laurent expression, if exist, is unique. Thus if an expansion of the Laurent form is found by any process, it must be the Laurent expansion.

Remark 4 The real importance of Laurent's theorem rests in the fact that it is an existence theorem. It shows that an analytic function can be expanded, under certain circumstances, as a series of a given type, but it does not necessarily provide the simplest method of calculating the coefficients.

Remark 5 It should be observed that Laurent's theorem will not provide an expansion of the logarithm of z as a series of positive and negative powers of z – for log z is a many valued functions, whose principal value, log z, is discontinuous along the negative half of the real axis and so is not regular in any annulus with centre at the origin.

Example.1 Find the first four terms of the Taylor's series expansion of the complex variable function $f(z) = \frac{z+1}{(z-3)(z-4)}$ about z = 2

Solution. We have

$$f(z) = \frac{z+1}{(z-3)(z-4)} \qquad \text{at } z = 2, \quad f(2) = 3/2$$

To make the differentiation easier let us

$$\frac{z+1}{(z-3)(z-4)} = \frac{-4}{z-3} + \frac{5}{z-4}$$

$$f'(z) = \frac{4}{(z-3)^2} - \frac{5}{(z-4)^2}$$

$$f'(z) = 4 - \frac{5}{4} = \frac{11}{4}$$

$$f^{"}(z) = \frac{-8}{(z-3)^3} - \frac{5}{(z-4)^3}$$

$$f^{"}(z) = 8 - \frac{10}{8} = \frac{27}{4}$$

$$f'"(z) = \frac{24}{(z-3)^4} - \frac{30}{(z-4)^4}$$

$$f"'(z) = 24 - \frac{30}{16} = \frac{177}{8}$$

Then Taylor's Series is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{\underline{|2}} f"(a) + \cdots$$

$$=\frac{3}{2}+(z-2).\frac{11}{4}+\frac{(z-2)^2}{2}.\frac{27}{4}+(z-2)^3\frac{59}{16}$$

Example.2 expand $f(z)=\frac{1}{(z-1)(z-2)}$ in the region

(a) $|z|<1$ (b) $1<|z|<2$

Solution: by Partial fraction

$\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$ .... (1)

$\because$ $\frac{|z|}{1}<1=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{-1(1-z)}$

$$=\frac{-1}{2}\left\{1-\frac{z}{2}\right\}^{-1}+(1-z)^{-1}$$

$$=\frac{-1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}++\cdots\ldots\ldots.\right)+(1+z+z^2+\cdots..)$$

$$=\frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\cdots\ldots\ldots$$

Which is a Taylor's series.

(b) $1<|z|<2$ i.e. $\frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$

$$=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots \ldots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \ldots\right)$$

$$= - - - - - - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - - - -$$

Which is Laurent's series.

Example.3 Obtain the Taylor's & Laurent's series which represent the function $\frac{z^2-1}{(z+2)(z+3)}$

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

Solution: - if $|z| < 2$

Or $\frac{|z|}{2} < 1$

$$f(z) = 1 - \frac{5z+7}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$f(z) = 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots \ldots.\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots \ldots.\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{2^n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^n}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n$$

Which is Taylor's series valid in $|z| < 2$

(ii) when $2 < |z| < 3$, $\frac{2}{|z|} < 1$ & $\frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots..\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots..\right)$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots..\right) - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n \left[\frac{3.2^n}{z^{n+1}} - \frac{8.z^n}{3^{n+1}}\right]$$

Which is Laurent's series in the annulus region.

(iii) $|z| > 3$ or $3 < |z|$ or $\frac{3}{|z|} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots.\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots ...\right)$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n \frac{2^n}{z^n} - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n \frac{3^n}{z^n}$$

$$= 1 + 3\sum_{n=0}^{\infty}(-1)^n \frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n \frac{3^n}{z^{n+1}}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+1}}[3.2^n - 8.3^n] \quad \text{Ans.}$$

Example.4 Find the Laurent's series expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < |z+1| < 3$.

Solution: The given function is $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

Let $z + 1 = u$ then

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

By partial function

$$f(z) = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

Since $1 < |u| < 3$

$$\Rightarrow \quad \frac{1}{|u|} < 1, \qquad \frac{|u|}{3} < 1$$

$$\therefore \quad f(z) = \frac{-3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)}$$

$$= \frac{-3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

$$= \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots.\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \cdots..\right)$$

$$= \left(\frac{-2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \frac{1}{u^4} + \cdots\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \cdots..\right)$$

Hence

$$f(z) = \left[\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots \ldots \ldots\right]$$

$$- \frac{2}{3}\left[1 + \frac{(z+1)}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \cdots \ldots ..\right]$$

Example.5 Find the Laurent's series expansion of $f(z) = \frac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in region

$$3 < |z+2| < 5$$

Solution: The given function is $f(z) = \frac{z^2-6z-1}{(z-1)(z+2)(z-3)}$

By partial fraction

$$f(z) = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

Since $3 < |z+2| < 5$ writing $z + 2 = t$

$\therefore \quad 3 < |t| < 5 \quad \frac{3}{|t|} < 1; \quad \frac{|t|}{5} < 1$

$$\therefore \quad f(z) = \frac{1}{t-2-1} - \frac{1}{t-2-3} + \frac{1}{t}$$

$$= \frac{1}{t-3} - \frac{1}{t-5} + \frac{1}{t}$$

$$= \frac{1}{t\left(1-\frac{3}{t}\right)} - \frac{1}{5\left(1-\frac{t}{5}\right)} + \frac{1}{t}$$

$$= \frac{1}{t}\left(1 - \frac{3}{t}\right)^{-1} + \frac{1}{5}\left(1 - \frac{t}{5}\right)^{-1} + \frac{1}{t}$$

$$= \frac{1}{t}\left(1 + \frac{3}{t} + \frac{9}{t^2} + \frac{27}{t^3} + \cdots..\infty\right) + \frac{1}{5}\left(1 + \frac{t}{5} + \frac{t^2}{25} + \frac{t^3}{125} + \cdots.\right) + \frac{1}{t}$$

$$= \left(\frac{2}{t} + \frac{3}{t^2} + \frac{9}{t^3} + \cdots..\infty\right) + \frac{1}{5}\left(1 + \frac{t}{5} + \frac{t^2}{25} + \frac{t^3}{125} + \cdots.\right) + \frac{1}{t}$$

$$= \left(\frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{9}{(z+2)^3} + \cdots..\infty\right) + \frac{1}{5}\left[1 + \frac{z+2}{5} + \frac{(z+2)^2}{25} + \frac{(z+2)^3}{125} + \cdots..\infty\right]$$

Which is required Laurent's series.

Example.6 Expand $f(z) = \frac{1}{(z-1)(z-2)}\ for\ 1 < |z| < 2$

Solution-$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

In first bracket $|z| < 2$,we take out 2 as common and from second bracket z is taken out common as $1 < |z|$.

$$f(z) = -\frac{1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] - \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right] = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$f(z) = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} \ldots. - \frac{1}{z} - \frac{1}{z^3} - \frac{1}{z^4} \cdots$$

Example.7 Expand $\frac{1}{(z+1)(z+3)}$ in the regions $|z| < 1$.

Solution- we have $f(z) = \frac{1}{(z+1)(z+3)}$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

If the regions $|z| < 1$

$f(z) = \frac{1}{2}[(1+z)^{-1} - (3+z)^{-1}]$ $, then\ \ |z| < 1\ and\ \frac{|z|}{3} < 1$

$$f(z) = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}[1+\frac{z}{3}]^{-1}$$

$$f(z) = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots)$$

$$f(z) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(-\frac{z}{2} + \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) + \left(-\frac{z^3}{2} - \frac{z^3}{162}\right)$$

$$f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 \ldots..$$

Example.8 Expand the function $Sin^{-1}z$ in powers of z.

Solution- Let $w = Sin^{-1}z$

$$\frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}}$$

$= (1-z^2)^{-\frac{1}{2}}$ (1)

On expanding the R.H.S. of binomial theorem, we have

$$\frac{dw}{dz} = 1 - \frac{1}{2}(-z^2) + \frac{\frac{-1}{2}\left(-\frac{3}{2}\right)}{2!}(-z^2)^2 + \cdots$$

$$\frac{dw}{dz} = 1 + \frac{z^2}{2} + \frac{3}{8}z^4 + \cdots$$

On integrating, we have $w = z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots + c$

Putting $z = 0\ then\ w = sin^{-1}z = 0$

$i.e.\quad c = 0$

We have $sin^{-1}z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots$

4.8 Singular point:

A point at which a function f(z) is not analytic is known as a singular point or singularity of the function.

For e.g. $f(z) = \frac{1}{z-2}$ has a singular point at z – 2 = 0 or z = 2

(1)Isolated singularity: -

If z = a is a singularity of f(z) such that f(z) is analytic at each point in its neighbourhood (i.e. there exists a circle with centre ‘a’ which does not contain other singularity), then z = a is called an isolated singularity.

In such a case, f(z) can be expanded in a Laurent’s series around z = a, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots .. + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots ..$$

…….(1)

e.g. of non isolated singularity:-

$f(z) = \frac{1}{sin\frac{\pi}{z}}$ is not analytic at the points where $sin\frac{\pi}{z} = 0$ i.e. at the points $\frac{\pi}{z} = n\pi$ i.e., the points $z = \frac{1}{n}(n = \mp 1, \mp 2, \mp 3, \dots \dots)$, thus $z = \mp 1,\ \mp\frac{1}{2},\ \mp\frac{1}{3}, \dots \dots$ are all isolated singularities or there is no other singularity in their neighbourhood.

But when n is large, z = 0 is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus z = 0 is the non-isolated singularity of f(z).

(ii) Removable singularity: -

If all the negative powers of (z – a) in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$. There the singularity can be removed by defining f(z) at z = a in such a way that it becomes analytic at z = a such a singularity is called a removable singularity.

Thus if $\lim_{z=a} f(z)$ exists finitely, then z = a is a removable singularity.

(iii) Poles:

If all the negative powers of (z – a) in (1) after the nth are missing, then the singularity at z = a is called a pole of order n.

A pole of first order is called a simple pole.

(iv) Essential singularity: -

If the number of negative powers of (z – a) in (1) is infinite, then z = a is called an essential singularity. In this case $\lim_{z=a} f(z)$ does not exist.

Example.1 Find the nature and location of singularities of the following functions.

(i) $\frac{z-sinz}{z^2}$ (ii) $(z+1)sin\left(\frac{1}{z-2}\right)$ (iii) $\frac{1}{cosz-sinz}$

Solution: (i) Here z = 0 is a singularity.

Also $\frac{z-sinz}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots.\right)\right\}$

$$= \frac{z}{3!} + \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots.$$

Since there are no negative powers of z in the expantion z = 0 is a removable singularity.

(ii) $(z+1)sin\left(\frac{1}{z-2}\right)$, writing z – 2 = t

$\therefore$ $= (t+2+1)sin\frac{1}{t}$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} \ldots\ldots\right\}$$

$$= \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} \ldots\ldots\right) + \left\{\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} \ldots\ldots\right\}$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \cdots \dots \dots \dots \dots \dots$$

$$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \frac{1}{120(z-2)^4} - \cdots \dots \dots \dots \dots \dots$$

Since there are infinite number of terms in the negative powers of (z – 2), z = 2 is an essential singularity.

(iii) Poles of $f(z) = \frac{1}{cosz - sinz}$ are given by equating denominator to zero, i.e. by

$cosz - sinz = 0$ or $tanz = 1$ or $z = \frac{\pi}{4}$ clearly $z = \frac{\pi}{4}$ is a simple pole of f(z).

4.9 Zeros of an analytic function: -

The zeros of analytic function: - The value of z for function f(z) becomes zero is said to be the zero of f(z).

If f(z) is analytic in a domain D and z_0 is any point of D, then we can expand f(z) as Taylar's series about $z = z_0$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

If $a_0 = a_1 = a_2 = \dots\dots = a_{n-1} = 0$ and $a_m \neq 0$, f(z) is said to have a zero of order m at $z = z_0$

In this case Taylor's expansion of f(z) reduces to

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n = \sum_{n=m}^{\infty} a_{n+m} (z - z_0)^{n+m}$$

$$= (z - z_0)^m \sum_{n=m}^{\infty} a_{n+m} (z - z_0)^n$$

A zero of order one (m=1) is said to be a simple zero, the following theorem shows that the zeros.

Theorem1:- If f(z) is an analytic function in a domain D, then unless f(z) is identically zero. There exists a neighbourhood of each point in D throughout which the function has no zero except possibly at the point itself.

Or

The zeros of analytic function are isolated.

Proof:- Let z = zo be a zero of order m of the function f(z). then we can write

$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} a_{n+m} (z - z_0)^n$ ……………………… (1)

$= (z - z_0)^m \phi(z)$, say

Where $\phi(z) = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n$

Clearly $\phi(z) = a_m \neq 0$

Now the series (1) is uniformly convergent and its each term is continuous at z_0. Therefore, $\phi(z)$ is also continuous at z_0. Hence for $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\phi(z) - \phi(z_0)| < \varepsilon$$

Where $|z - z_0| < \delta$

Let us choose $\varepsilon = \frac{1}{2}|am|$ and let δ_1 be the corresponding values of δ.

Then from (2) and (3), we have

$$|\phi(z) - a_m| < \frac{1}{2}|a_m|$$

Where $|z - z_0| < \delta_1$. Now if we set $\phi(z) = 0$, (4) will not hold. Thus $\phi(z) = 0$ can not be zero at any neighbourhood of z_0. The argument also holds good when m = 0 in which case $\phi(z) = f(z)$ and $f(z_0) \neq 0$,

Hence the zeros of an analytic function are isolated.

4.10 Limit Points of Zeros and Poles:

Limit point of zeros: - we prove here the following result concerning limit point of the set of zeros of an analytic functions.

Suppose $z = a$ is a limit point of the sequence of poles of an analytic function f(z). then every neighbourhood of the point $z = a$ containing poles of the give function. Therefore the point $z = a$ is a singularity of $f(z)$. This singularity cannot be a pole, since it is not isolated. Such a singularity is called non-isolated essential singularity or essential singularity simply.

Theorem 1. If f(z) is an analytic function in a simply connected region D and a_1, a_2, a_3….. is a sequence of zeros of f(z), having a as its limit point, then eighter f(z) vanishes identically or else has an isolated essential singularity at z = a.

Proof. Let E be the set of sequence of zeros a_1, a_2, a_3, ….. if $a \in E$. Then we would have f(a) = 0. Since a is a limit point of the set E, every neighbourhood of a must contain infinitely many points of E. But this is contrary to the fact that zeros are isolated. Hence a can not be a zero of the function f(z) unless f(z) is identically zero in the region D.

On the other hand, if f(z) does not vanish identically in D, then a is not a zero of f(z) while being surrounded by many zeros. This shows that a is a singularity. This singularity can not be a pole since f(z) does not tend to infinity in the neighbourhood of a, the function f(z) is analytic (tending to zero everywhere in the neighbourhood). Hence a is an isolated essential singularity.

Working Rule:- If we are to show that a certain point a is an isolated essential singularity of f(z) it will do if we prove that a is the limit point of zero of f(z).

Limit point of Poles:- we prove below a useful result for limit point of poles.

Theorem 2. The limit point of a sequence of poles of a function f(z) is non-isolated essential singularity.

Proof- Let a be the limit point of a sequence of poles so that f(z) becomes unbounded there. Consequently, f(z) can not be analytic at a.

Thus a is a singularity of f(z) but is not isolated. Hence a must be a non-isolated essential singularity of f(z).

Working Rule: - If we are to show that a certain point a is a non-isolated essential singularity of f(z) it will do if we prove that a is the limit point of poles of f(z).

Example: - Show that the function e^{-1/z^2} has no singularities.

Solution. We have f(z) = e^{-1/z^2}. The zeros of f(z) are given by $e^{-1/z^2} = 0$, i.e. $z^2 = 0$

So z = 0 is a zero of order two, since these zeros have no limit point, there is no singularity of f(z).

Further, poles of f(z) are given $e^{-1/z^2} = 0$ which does not hold for any z. so there exist no poles.

Hence e^{-1/z^2} has no singularities.

4.11 SUMMARY

The results of this unit may be summarised as follows:

Maximum Modulus Theorem: The absolute value of a non-constant function f (z) cannot have a maximum at any point where the function is analytic. Further, if f (z) ≠ 0 inside C, then | f (z) | must assume its minimum value on C.

• Lioville’s Theorem: If f (z) is an integral function which satisfies inequality | f (z) | for all z, M being a constant, then f (z) is a constant.

• Taylor’s Theorem: If f (z) is an analytic function, regular in the neighbourhood | z – a | < R of the point z = a, it can be expressed in that neighbourhood as a convergent power series of the form $f(z) = f(a) + \sum_{n=1}^{\infty} f^n(a) \frac{(z-a)^n}{n!}$

The above expansion is uniformally convergent when | z – a | ≤ R_1, provided R_1 < R. When a = 0, in the above expansion, it becomes

$$f(z) = f(0) + \sum_{n=1}^{\infty} f^n(0) \frac{z^n}{n!}$$

which is called Maclaurin’s Series for f (z).

Laurent’s Theorem: If f (z) is analytic throughout the closed region bounded by two concentric circles, then at any point of the annulus region bounded by the circles, f (z) can be represented as $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$

where a is the centre of concentric circles and

$$a_n = \frac{1}{2\pi i} \int_c \frac{f(t)}{(t-a)^{n+1}} dt$$

each integral being taken in the counter clockwise direction around any curve C lying in the annulus and encircling the inner boundary. The Laurent’s expansion of a function over a given annulus, if it exists, is unique.

4.12 Terminal Questions

1. If f (z) is entire and satisfies an inequality $|f(z)| \leq \ |z|^n$ for some n and sufficiently large | z |, then prove that f (z) must be a polynomial.

2. Find Taylor Series of $f(z) = \frac{1}{z}$ about $z = -1, z = 1 \ and \ z = 2.$ Determine the circle of convergence in each case.

3. Develop the function $f(z) = \frac{1}{1-z-z^2}$ into Taylor series about 0.

4. Expand sin z in a Taylor series about the point $z = \frac{\pi}{2}$.

5. Find the Laurent's expansion of the function $f(z) = \frac{7z-2}{z(z+1)(z+2)}$ in the annulus
   1 < | z + 1 | < 3.

6. For the function $f(z) = \frac{2z^2+1}{z^2+z}$, find (a) Taylor's series expansion valid in the neighbourhood of the point z = i.
   (b) Laurent's series expansion within the annulus when centre is the origin.

7. A rational function has a no Singularities other than poles.

8. Find zeros and poles of $\left(\frac{z+1}{z^2+1}\right)^2$

9. What kind of Singularity has the function

   (a) $f(z) = \frac{1}{\cos\left(\frac{1}{z}\right)} \ at \ z = 0$

   (b) and cot z at $z = \infty$

10. Show that the function $e^{-\frac{1}{z^2}}$ has no Singularities.

Bachelor of Science
DCEMM -113

Function of Complex Variables

U. P. Rajarshi Tandon
Open University

Block

3 The Calculus of Residues (Integration) and Evaluation of real definite integrals by contour integration

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Unit 5

The Calculus OF Residues (Integration)

Unit- 6

Evaluation of Real Definite Integrals by Contour Integration

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Block – 3

The Calculus of Residues (Integration)and Evaluation of real definite integrals by contour integration

The calculus of residues (integration) is an important and useful tool in complex analysis. A zero of analytic function is the value of for which. A point at which a function is not analytic is known as a singular point or singularity of the function. Residue of at is defined as where the integration is taken round in anti-clockwise direction, where is a large circle containing all finite singularities of in the second unit we introduce Contour Integration of residue classes which is an important and useful tool in complex analysis. It gives many important techniques for finding the complex integration and it is useful to find many other important theories of complex analysis which are necessary for the development in mathematics. A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve, find the pole of the function and calculate residues at those poles only which lie within the curve. (Sum of the residues of at the pole within) We call the curve, a contour and the process of integration along a contour is called contour integration.

Unit-5

THE CALCULUS OF RESIDUES (INTEGRATION)

5.1. INTRODUCTION:

The calculus of residues (integration) is an important and useful tool in complex analysis. A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$. A point at which a function $f(z)$ is not analytic is known as a singular **point or singularity** of the function. For example, the function $\frac{1}{z-2}$ has a singular point at z – 2 = 0 or z = 2. If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding to the point $z = a$, then $z = a$ is said to be an **isolated singularity** of the function $f(z)$; otherwise it is called **non-isolated.** Let a function $f(z)$ have an isolated singular point $z = a, f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \frac{b_{m+2}}{(z-a)^{m+2}} + \cdots \ldots \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

In some cases it may happen that the coefficients $b_{m+1} = b_{m+2} = b_{m+3} = 0$, then (1) reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + b_3(z-a)^{m-2} + \cdots + b_m\}$$

When $z = a$ is said to be a pole of order m of the function $f(z)$, when $m = 1$, the pole is said to be **simple pole**. In this case $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \frac{b_1}{z-a}$

If the number of the terms of negative powers in expansion (1) is infinite, then $z = a$ is called to **essential singular point** of $f(z)$. Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z - a| < R$ and can be expanded within the annulus. The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. it is denoted by symbol, $Res.$ at $(z = a)$ b_1.Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_c f(z)\,dz$ where the integration is taken round C in anti-clockwise direction, where C is a large circle containing all finite singularities of $f(z)$.

5.2. Objectives

After studying this unit we should be able to:

- The definition of Zero and poles of Analytic function;
- Definition of Singular point ;
- Definition of the residue at a pole;
- The residue at infinity;
- Method of finding residues;
- Applications;

A point at which a function $f(z)$ is not analytic is known as a singular **point or singularity** of the function.

For example, the function $\frac{1}{z-2}$ has a singular point at z – 2 = 0 or z = 2.

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding to the point $z = a$, then $z = a$ is said to be an **isolated singularity** of the function $f(z)$; otherwise it is called **non-isolated.** Let a function $f(z)$ have an isolated singular point $z = a, f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \frac{b_{m+2}}{(z-a)^{m+2}} + \cdots \ldots \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

In some cases it may happen that the coefficients $b_{m+1} = b_{m+2} = b_{m+3} = 0$, then (1) reduces to $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + b_3(z-a)^{m-2} + \cdots + b_m\}$$

When $z = a$ is said to be a pole of order m of the function $f(z)$, when $m = 1$, the pole is said to be **simple pole**. In this case $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \frac{b_1}{z-a}$

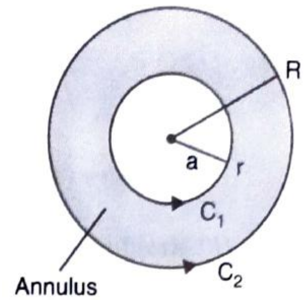
If the number of the terms of negative powers in expansion (1) is infinite, then $z = a$ is called to **essential singular point** of $f(z)$. Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 be circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z-a| < R$ and can be expanded within the annulus. Laurent's series:

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$ …………… (1)

Where $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^{n+1}}$ ……………..(2)

And $b_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)dz}{(z-a)^{-n+1}}$ ……………..(3)

$|z-a| = r$ being the circle C_1.



Particularly, $b_1 = \frac{1}{2\pi i}\int_{c_1} f(z)dz$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$.

it is denoted by symbol, $Res.$ at $(z = a)$ b_1.

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_c f(z)\,dz$ where the integration is taken round C in anti-clockwise direction, where C is a large circle containing all finite singularities of $f(z)$.

5.3. Zero of Analytic Function:

A zero of analytic function $f(z)$ is the value of z for which $f(z) = 0$,

Example 1: Find out the zeros and discuss the nature of the singularities of $f(z) = \frac{(z-2)}{z^2} sin\left(\frac{1}{z-1}\right)$

Solution: Pole of $f(z)$ are given by equating to zero the denominator of $f(z)$ $i.e. z = 0$ is a pole of order two.

Zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ $i.e.,(z - 2)\, sin\left(\frac{1}{z-1}\right) = 0$

$\Rightarrow$ Either $z - 2 = 0$ or $sin\left(\frac{1}{z-1}\right) = 0$

$\Rightarrow z = 2$ and $\frac{1}{z-1} = n\pi$

$\Rightarrow z = 2$ $z = \frac{1}{n\pi} + 1,\ n = \mp 1,\ \mp 2, \dots \dots$

Thus, $z = 2$ is a simple zero. The limit point of the zeros are given by

$z = \frac{1}{n\pi} + 1,\ (n = \mp 1,\ \mp 2, \dots \dots)$ is $z\ =\ 1$.

Hence $z\ =\ 1$ is an isolated essential singularity.

5.4 Singular Point:

A point at which a function $f(z)$ is not analytic is known as a singular **point or singularity** of the function.

For example, the function $\frac{1}{z-2}$ has a singular point at z – 2 = 0 or z = 2.

Isolated singular point: If $z\ =\ a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z\ =\ a$, then $z\ =\ a$ is said to be an **isolated singularity** of the function $f(z)$; otherwise it is called **non-isolated**.

Example 2: The function $\frac{1}{(z-1)(z-3)}$ has two isolated singular points, namely $z\ =\ 1$ and $z\ =\ 3$. $[(z - 1)(z - 3)\ =\ 0\ or\ z\ =\ 1, 3]$.

Example 3: Non-isolated singularity. Function $\frac{1}{sin\frac{\pi}{z}}$ is not analytic at the points where $sin\frac{\pi}{z} =$ 0i.e. at the point $\frac{\pi}{z} = n\pi$ i.e., the points $z = \frac{1}{n}(n\ =\ 1,2,3, \dots \dots)$. Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \dots \dots, z = 0$ are the points of singularity. $z\ =\ 0$ is the non-isolated singularity of the function $\frac{1}{sin\frac{\pi}{z}}$ because in the neighbourhood of $z\ =\ 0$, there are infinite number of other singularities $z = \frac{1}{n}$, where n is very large.

Pole of order m: Let a function $f(z)$ have an isolated singular point $z\ =\ a, f(z)$ can be expanded in a Laurent's series around $z\ =\ a$, giving

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m} + \frac{b_{m+1}}{(z-a)^{m+1}} + \frac{b_{m+2}}{(z-a)^{m+2}} + \cdots \dots \dots$ .... (1)

In some cases it may happen that the coefficients $b_{m+1} = b_{m+2} = b_{m+3} = 0$, then (1) reduces to

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{1}{(z-a)^m}\{b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + b_3(z-a)^{m-2} + \cdots + b_m\}$$

When $z\ =\ a$ is said to be a pole of order m of the function $f(z)$, when $m\ =\ 1$, the pole is said to be **simple pole**. In this case $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \frac{b_1}{z-a}$

If the number of the terms of negative powers in expansion (1) is infinite, then $z\ =\ a$ is called to **essential singular point** of $f(z)$.

Example 4: Define the singularity of a function. Find the singularity (ties) of the functions

(i) f(z) = $sin\frac{1}{z}$ (ii) $g(z) = e^z$

Solution: (i). We know that

$$sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} + \cdots \ldots. + (-1)^n \frac{1}{(2n+1)!\,z^{2n+1}}$$

Obviously, there is a number of singularity.

$sin\frac{1}{z}$ is not analytic at $z = 0$. $\left(\frac{1}{z} = \infty\ at\ z = 0\right)$

Hence $sin\frac{1}{z}$ has a singularity at $z = 0$.

(i) Here, we have $g(z) = \frac{\frac{1}{e^z}}{z^2}$

We know that, $\left(\frac{1}{z^2}\right)\left(\frac{1}{e^z}\right) = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots .. + \frac{1}{n!z^n} \ldots + \right)$

$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!z^4} + \frac{1}{3!z^5} + \cdots .. + \frac{1}{n!z^{n+2}} + \cdots +$

Here, $f(z)$ has infinite number of terms in negative powers of z.

Hence, $f(z)$ has essential singularity at $z = 0$.

Example 5: Find the pole of the function $\frac{e^{z-a}}{(z-a)^2}$

Solution: $\frac{e^{z-a}}{(z-a)^2} = \frac{1}{(z-a)^2}\left[1 + (z-a) + \frac{(z-a)^2}{2!} + \cdots\right]$

The given function has negative power 2 $of\ (z - a)$.

So, the given function has a pole at $z = a$ of order 2.

Example 6: Find the pole of $f(z) = sin\left(\frac{1}{z-a}\right)$

Solution: $sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} - \cdots.$

The given function $f(z)$ has infinite number of terms in the negative power of $z - a$. so $f(z)$ has essential singularity at $z = a$.

Example 7: Find the pole of $f(z) = \frac{\sin(z-a)}{(z-a)^4}$

Solution: $\frac{\sin(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \frac{(z-a)^7}{7!} + \cdots\right]$

$$= \frac{1}{(z-a)^3}\left[1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \frac{(z-a)^6}{7!} + \cdots\right]$$

The given function has a negative power 3 of $(z - a)$.

So, $f(z)$ has a pole at $z = a$ of order 3.

Example 8: Prove that $f(z) = \lim_{z\to a} e^{\frac{1}{z-a}}$ does not exist.

Solution: $\lim_{z\to a} e^{\frac{1}{z-a}}$

$$= \lim_{z\to a}\left(1 + \frac{1}{z-a} + \frac{1}{2!\,(z-a)^2} + \frac{1}{3!\,(z-a)^3} + \cdots + \frac{1}{n!\,(z-a)^n} + \cdots.\infty\right)$$

Here $z \to \infty$, $f(z)$ has infinite number of terms in negative power of $(z - a)$.

Thus, $f(z)$ has essential singularity at $z = a$.

Hence, f(z) = $\lim_{z\to a} e^{\frac{1}{z-a}}$ does not exist.

Example 9: Discuss singularity of $\frac{1}{1-e^z}$ $at\ z = 2\pi i$.

Solution: We have, $f(z) = \frac{1}{1-e^z}$

The poles are determined by putting the denominator equal to zero.

i.e., $1 - e^z = 0$

$\Rightarrow e^z = 1 = (\cos 2n\pi + i\sin 2n\pi) = e^{2n\pi i}$

$\Rightarrow z = 2n\pi i$ (n = 0, ±1, ±2, ……)

Clearly z = $2\pi i$ is a simple pole.

Example 10: Discuss singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and z = ∞

Solution: Let $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cot \pi z}{\sin \pi z (z-a)^2}$

The poles are given by putting the denominator equal to zero,

i.e., $\sin \pi z (z-a)^2 = 0 \Rightarrow (z-a)^2 = 0$ or $\sin \pi z = 0 = \sin n\pi$

$\Rightarrow \quad z = a,\ \pi z = n\pi,\ \ (n \in I)$

$\Rightarrow \quad z = a, n$

$f(z)$ has essential singularity at $z = \infty$.

Also, $z = a$ being repeated twice gives the double pole.

Example 11: Show that $e^{-\left(\frac{1}{z^2}\right)}$ has no singularities.

Solution: $f(z) = e^{-\left(\frac{1}{z^2}\right)} = \frac{1}{e^{\left(\frac{1}{z^2}\right)}}$

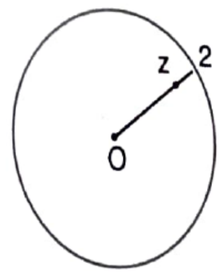
The poles are determined by putting the denominator $e^{\left(\frac{1}{z^2}\right)} = 0$

It is not possible to find the value of z which can satisfy equation (1).

Hence, there is no pole or singularity of the given function

Example 12: Define the Laurent series expansion of a function and expand $f(z) = e^{\left(\frac{z}{z-2}\right)}$ in a Laurent series about the point $z = 2$.

Solution: Here, we have



$$f(z) = e^{\left(\frac{z}{z-2}\right)} = e^{\left(\frac{z-2+2}{z-2}\right)}$$

$$= e.e^{\left(\frac{2}{z-2}\right)} = ee^{\frac{2}{-2\left(1-\frac{z}{2}\right)}}$$

$$= e.e^{-\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots.\right]}$$

$$= e^{1-1}.e^{-\left[\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots.\right]} = e^{-\left[\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots.\right]}$$

$$= \left[1 - \frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{8} + \frac{\left(\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\right)^2}{2!} - \frac{\left(\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}\right)^3}{3!} + \cdots.\right]$$

$$= \left[1 - \frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{8} + \frac{z^2}{8} + \frac{z^4}{32} + \frac{z^3}{8} + \frac{z^4}{16} - \frac{z^3}{48} \ldots..\right]$$

$$= 1 - \frac{z}{2} - \frac{z^2}{8} - \frac{z^3}{48} - \cdots \ldots..$$

Example 13: Find the nature of singularities of $f(z) = \frac{z-\sin z}{z^3}$ at $z = 0$.

Solution: $f(z) = \frac{z-\sin z}{z^3}$

$$= \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.\right)\right]$$

$$= \frac{1}{z^3}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right] = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \ldots$$

There is no negative power of z;

Hence, there is no pole.

Example**14**: Determine the pole of the function z, $f(z) = \frac{1}{z^4+1}$

Solution: $f(z) = \frac{1}{z^4+1}$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

i.e., $z^4 + 1 = 0 \ \Rightarrow \ z^4 = -1$

$z = (-1)^{\frac{1}{4}} = (\cos(2n+1)\pi + i\sin(2n+1)\pi)^{\frac{1}{4}}$

$= [\cos(2n+1)\pi + isin\,(2n+1)\pi]^{\frac{1}{4}}$

$= \left[\cos\frac{(2n+1)\pi}{4} + isin\,\frac{(2n+1)\pi}{4}\right]$[By De Moiver's theorem]

If n = 0, pole at $z = \left[\cos\frac{\pi}{4} + isin\,\frac{\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

If n = 1, pole at $z = \left[\cos\frac{3\pi}{4} + isin\,\frac{3\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

If n = 2, pole at $z = \left[\cos\frac{5\pi}{4} + isin\,\frac{5\pi}{4}\right] = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

If n = 3, pole at $z = \left[\cos\frac{7\pi}{4} + isin\,\frac{7\pi}{4}\right] = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

Example 15: Show that the function e^z has an isolated essential singularity at $z = \infty$.

Solution: Let $f(z) = e^z$. Putting $z = \frac{1}{t}$, we get

$$f\left(\frac{1}{t}\right) = e^{\frac{1}{t}} = 1 + \frac{1}{t} + \frac{1}{2!\,t^2} + \frac{1}{3!\,t^3} + \cdots$$

Here, the principal part of $f\left(\frac{1}{t}\right)$ is

$\frac{1}{t} + \frac{1}{2!t^2} + \frac{1}{3!t^3} + \cdots$

Contains infinite number of terms.

Hence $t = 0$ is an isolated essential singularity of $e^{\frac{1}{t}}$ and

$z = \infty$ is an isolated essential singularity of e^z.

Check your progress

Find the poles or singularity of the following functions:

1. $\frac{1}{(z-2)(z-3)}$ Ans: 2 simple poles at $z = 2$ and $z = 3$.
2. $\frac{e^z}{(z-2)^3}$ Ans: Pole at $z = 2$ of order 3,
3. $\frac{1}{\sin z - \cos z}$ Ans: Simple pole at $z = \frac{\pi}{4}$
4. $\cot\frac{1}{z}$ Ans: Essential singularity at $z = 0$
5. $z\ cosec\ z$ Ans: Non-isolated essential singularity
6. $\sin\frac{1}{z}$ Ans: Essential singularity

Theorem: If $f(z)$ has a pole at $z = a$, then $|f(z)| \to \infty \; as \; z = a$

Proof: Let $z = a$ be a pole of order m of $f(z)$. Then by Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{m=0}^{\infty} b_m (z-a)^{-m}$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{1}{(z-a)^m}[b_1 z - a)^{m-1} + b_2 z - a)^{m-2} + \cdots + b_{m-1}(z-a) + b_m]$$

$$= \sum_{0}^{\infty} a_n (z-a)^n + \frac{\varphi(z)}{(z-a)^m}$$

Now $\varphi(z) = b_m \; as \; z \to a$

Hence $|f(z)| \to \infty \; as \; z = a$

Example 16: If an analytic function f(z) has a pole of order m at $z = a$, then $\frac{1}{f(z)}$ has a zero of order m at z = a.

Solution: If f(z) has a pole of order m at $z = a$, then

$f(z) = \frac{\varphi(z)}{(z-a)^m}$ where $\varphi(z)$ is analytic and non-zero at $z = a$.

Hence, $\frac{1}{f(z)} = \frac{(z-a)^m}{\varphi(z)}$

Clearly, $\frac{1}{f(z)}$ has a zero of order m at $z = a$, since $\phi(a) \neq 0$.

5.5. The residue at a pole

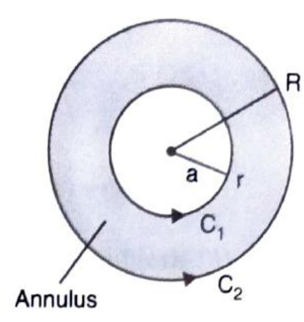
Let $z = a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$ then $f(z)$ is analytic within the annulus $r < |z-a| < R$ and can be expanded within the annulus. Laurent's series:

$f(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$ …………… (1)

Where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}$ ……………..(2)

And $b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)dz}{(z-a)^{-n+1}}$ ……………..(3)

$|z-a| = r$ being the circle C_1.



Particularly, $b_1 = \frac{1}{2\pi i}\int_{c_1} f(z)dz$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$.

it is denoted by symbol, $Res.$ at $(z = a)$ b_1.

5.6 Residue at infinity

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_c f(z)\,dz$ where the integration is taken round C in anti-clockwise direction, where C is a large circle containing all finite singularities of $f(z)$.

5.7. Method of Finding Residues

(a) Residue at simple pole

(i) if $f(z)$ has a simple pole at $z = a$, then

$$Res.\, f(a) = \lim_{z\to a}(z-a)f(z)$$

Proof: $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a}$

$\Rightarrow (z-a)f(z) = a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + \cdots + b_1$

$\Rightarrow b_1 = (z-a)f(z) - [a_0(z-a) + a_1(z-a)^2 + a_2(z-a)^3 + \cdots]$

Taking limit as z → a, we have $b_1 = \lim_{z\to a}(z-a)f(z)$

(ii) If $f(z)$ is of the form $f(z) = \frac{\phi(a)}{\psi(a)}$ where, $\psi(a) = 0$,

but $\phi(a) \neq 0$. $Res\ (at\ z = a) = \frac{\phi(a)}{\psi'(a)}$

Proof: $f(z) = \frac{\phi(a)}{\psi(a)}$, $Res\ (at\ z = a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)\frac{\phi(a)}{\psi(a)}$

$= \lim_{z\to a}\frac{(z-a)[\phi(a)+(z-a)\phi'(a)+\cdots]}{\psi(a)+(z-a)\psi'(a)+\frac{(z-a)^2}{2!}\psi''(a)+\cdots}$ (By Taylor's Theorem)

$= \lim_{z\to a}\frac{(z-a)[\phi(a)+(z-a)\phi'(a)+\cdots]}{(z-a)\psi'(a)+\frac{(z-a)^2}{2!}\psi''(a)+\cdots}$ [since $\psi(a) = 0$]

$= \lim_{z\to a}\frac{\phi(a)+(z-a)\phi'(a)+\cdots}{\psi'(a)+\frac{(z-a)}{2!}\psi''(a)+\cdots}$

$Res\ (at\ z = a) = \frac{\phi(a)}{\psi'(a)}$

(b) Residue at a pole of order n,

If $f(z)$ has a pole of order n at $z = a$, then

$$Res\ (at\ z = a) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right]_{z=a}$$

Proof: If $z = a$ is a pole of order n of function $f(z)$, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^n}$$

Multiplying by $(z-a)^n$, we get

$$(z-a)^n f(z) = a_0(z-a)^n + a_1(z-a)^{n+1} + a_2(z-a)^{n+2} + \cdots + b_1(z-a)^{n-1} + b_2(z-a)^{n-2} + b_3(z-a)^{n-3} + \cdots + b_n$$

Differentiating both sides w.r.t. 'z'$(n-1)$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\, b_1$$

$$\Rightarrow b_1 = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

$$Residue\ f(at\ z = a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

(c) **Residue at a pole z = a of any order (simple or of order m)**

$Res\ f(at\ z = a + t) =$ coefficient of $\frac{1}{t}$

Proof: If $f(z)$ has a pole of order m, then by Laurent's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

If we put $z - a = t$ $or\ z = a + t$, then

$$f(a+t) = a_0 + a_1 t + a_2(t)^2 + \cdots + \frac{b_1}{t} + \frac{b_2}{(t)^2} + \cdots + \frac{b_m}{(t)^m}$$

$Res\ f(a) = b_1$, $Res\ f(at\ z = a) =$ coefficient of $\frac{1}{t}$

Rule: Put $z = a + t$ in the function $f(z)$, expand it in powers of t. coefficient of $\frac{1}{t}$ is the residue of $f(z)$ $at\ z = a$.

(d) Residue of $f(z)$ $at\ z = \infty = \lim_{z\to\infty}\{-zf(z)\}$

Or, The residue of $f(z)$ at infinity $= -\frac{1}{2\pi i}\int_c f(z)\,dz$

Example 17: Find the residue at $z = 0$ of $z\cos\frac{1}{z}$.

Solution: Expanding the function in power of $\frac{1}{z}$ we have

$$zcos\frac{1}{z} = z\left[1 - \frac{1}{2z^2} + \frac{1}{4!\,z^4} - \cdots \ldots.\right]$$

$$= z - \frac{1}{2z} + \frac{1}{24z^3} - \cdots.$$

This is the Laurent's expansion about $z = 0$

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$. So the residue of $zcos\frac{1}{z}$ at $z = 0$ is $-\frac{1}{2}$

Example 18: Find the residue of $f(z) = \frac{z^3}{z^2-1}$ at $z = \infty$.

Solution: We have, $f(z) = \frac{z^3}{z^2-1}$

$f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1 - \frac{1}{z^2}\right)^{-1}$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \cdots$$

Residue at infinity = $-\left(coeff.of\,\frac{1}{z}\right) = -1.$

Example 19: Determine the pole and residue at the pole of the function $f(z) = \frac{z}{z-1}$

Solution: The pole of $f(z)$are given by putting the denominator equal to zero

$z - 1 = 0 \Rightarrow z = 1$

The function $f(z)$ has a simple pole at $z = 1$

Residue is calculated by the formula, $Residue$at $(z = a) = \lim_{z\to a}(z - a)f(z)$

$Residue\ of\ f(z)\ at\ (at\ z = 1) = \lim_{z\to 1}(z - 1)\left(\frac{z}{z-1}\right)$

$$= \lim_{z\to 1}(z) = 1$$

Hence $f(z)$ has a simple pole at $z = 1$ and residue at the pole is 1.

Example 20: Determine the pole and the residue at simple pole of the function $f(z) = \frac{z^2}{(z-1)^1(z+2)}$

Solution: The pole of $f(z)$ are given by putting the denominator equal to zero.

$(z - 1)^2(z + 2) = 0$ z = 1, 1, -2

The function $f(z)$ has simple pole at $z = -2\ and\ at\ z = 1$ pole of second order.

Residue of $f(z)\ at\ z = -2$ is $\lim_{z\to -2}(z + 2)f(z)$ [Residue = $\lim_{z\to a}(z - 2)f(z)$]

$$= \lim_{z\to -2}(z + 2)\frac{z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z \to -2} \frac{z^2}{(z-1)^2} = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

Hence, residue at simple pole is $\frac{4}{9}$

Example 21: Find the order of each pole and residue at it of $\frac{1-2z}{z(z-1)(z-2)}$.

Solution: Let $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

The poles of f(z) are given by $z(z-1)(z-2) = 0$

$Z = 0, 1, 2$ all are simple poles.

Residue of $f(z)\ at\ (z = 0) = \lim_{z \to 0}(z-0)\,f(z) = \lim_{z \to 0} \frac{z(1-2z)}{z(z-1)(z-2)}$

$$= \lim_{z \to 0} \frac{1-2z}{z(z-1)(z-2)} = \frac{1}{2}$$

$Residue\ of\ f(z) at\ (z = 1) = \lim_{z \to 1}(z-1)\,f(z)$

$$= \lim_{z \to 1} \frac{(z-1)(1-2z)}{z(z-1)(z-2)} = \lim_{z \to 1} \frac{1-2z}{z(z-2)} = 1$$

$Residue\ of\ f(z) at\ (z = 2) = \lim_{z \to 2}(z-2)\,f(z)$

$$= \lim_{z \to 2} \frac{(z-2)(1-2z)}{z(z-1)(z-2)} = \lim_{z \to 2} \frac{1-2z}{z(z-1)} = -\frac{3}{2}$$

Hence, the residue*s* of $f(z)$ at $z = 0,\ z = 1$

$and\ z = 2\ are\ \frac{1}{2}, 1\ and\ -\frac{3}{2}$ respectively.

Example 22: Determine the residue of $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at its simple poles.

Solution: The poles of $f(z)$ are determined by putting the denominator equal to zero.

i.e. $(z-1)^4(z-2)(z-3) = 0$

$z = 1, 1, 1, 1\ and\ z = 2\ and\ z = 3$

The simple poles of the function $f(z)\ are\ at\ z = 2\ and\ z = 3.$

Pole at $z = 2$

Residue $R(2) = \lim_{z \to 2}(z-2)\frac{z^3}{(z-1)^4(z-2)(z-3)}$ [Residue $(2) = \lim_{z \to 2}(z-2)\,f(z)$]

$$= \lim_{z \to 2}(z-2)\frac{z^3}{(z-1)^4(z-3)} = \frac{(2)^3}{(1)^4(-1)} = -8$$

Pole at $z = 3$

Residue $R(3) = \lim_{z \to 3}(z-3)\frac{z^3}{(z-1)^4(z-2)(z-3)}$

$= \lim_{z \to 2}(z-2)\frac{z^3}{(z-1)^4(z-2)}$

$= \frac{(2)^3}{(3-1)^4(3-2)} = \frac{27}{16}$

Hence, residue at $z = 2$ and $z = 3$ are -8 and $\frac{27}{16}$ respectively

Example 23: Evaluate the residues of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 1, 2, 3$ at infinity and show that their sum is zero.

Solution: Let $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

The poles of $f(z)$ are determined by putting the denominator equal to zero.

$(z-1)(z-2)(z-3) = 0$; $z = 1, 2, 3$

$Residue\ of\ f(z)\ at\ (z = 1) = \lim_{z \to 1}(z-1)f(z)$

$= \lim_{z \to 1}(z-1)\frac{z^2}{(z-1)(z-2)(z-3)}$

$= \lim_{z \to 1}\frac{z^2}{((z-2)(z-3)} = \frac{1}{2}$

Residue of f(z) at (z = 2) = $\lim_{z \to 2}(z-2)f(z)$

$= \lim_{z \to 2}(z-2)\frac{z^2}{(z-1)(z-2)(z-3)}$

$= \lim_{z \to 2}\frac{z^2}{((z-1)(z-3)} = -4$

Residue of $f(z)\ at\ (z = 3) = \lim_{z \to 3}(z-3)f(z)$

$= \lim_{z \to 3}(z-3)\frac{z^2}{(z-1)(z-2)(z-3)}$

$= \lim_{z \to 1}\frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$

Residue of f(z) at (z = ∞) = $\lim_{z \to \infty}(-z)f(z)$

$= \lim_{z \to \infty}\frac{-z\,z^2}{(z-1)(z-2)(z-3)}$

$= \lim_{z \to 1}\frac{-1}{\left(1-\frac{1}{z}\right)(1-\frac{2}{z})(1-\frac{3}{z})} = -1$

Sum of the residues at all the poles of f(z)

$= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$. Hence, the sum of the residues is zero.

(e). Residue of $\boldsymbol{f(at\ z = a)}$ $= \lim_{z\to a} \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$

Example 24: Find the residue of a function $f(z) = \frac{z^2}{(z+1)^2(z-2)}$ at its double pole.

Solution: We have, $f(z) = \frac{z^2}{(z+1)^2(z-2)}$

Poles are determined by putting denominator equal to zero.

i.e., $(z+1)^2(z-2) = 0$

$z = -1, -1\ and\ z = 2$

The function has a double pole at $(z = -1)$ then

Residue at $(z = -1) = \lim_{z\to -1} \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z^2}{(z+1)^2(z-2)}\right\}\right]$

$= \left[\frac{d}{dz}\left(\frac{z^2}{z-2}\right)\right]_{z=-1} = \left(\frac{(z-2)2z - z^2.1}{(z-2)^2}\right)_{z=-1}$

$= \left[\frac{z^2 - 4z}{(z-2)^2}\right]_{z=-1} = \frac{(-1)^2 - 4(-1)}{(-1-2)^2}$

$Residue\ at\ (z = -1) = \frac{1+4}{9} = \frac{5}{9}$

Example25: Find the residue of $\frac{1}{(z^2+1)^3}$ at $z = i$

Solution: Let $f(z) = \frac{1}{(z^2+1)^3}$

The pole of $f(z)$ are determined by putting denominator equal to zero.

i.e.; $(z^2+1)^3 = 0$

or, $(z^2+i)^3(z^2-i)^3 = 0$ or, $z = \mp i$

Here, $z = i$ is a pole of order 3 of $f(z)$

Residue at $z = i$: $\lim_{z\to i} \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3 \frac{1}{(z^2+1)^3}\right]\right\}$

$= \lim_{z\to i} \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]\right\} = \lim_{z\to i} \frac{1}{2}\left(\frac{3\times 4}{(z+i)^5}\right)$

$= \frac{1}{2} \times \frac{12}{(i+i)^5} = \frac{3}{32i} = \frac{3}{16i} = -\frac{3i}{16}$

Hence, the residue of the given function at z = i is $-\frac{3i}{16}$

(f). Residue (at $z = a$) $= \frac{\phi(a)}{\psi'(a)}$

Example 26: Determine the poles and residue at each pole of the function f(z) = cot z.

Solution: $f(z) = \cot z = \frac{\cos z}{\sin z}$

The poles of the function f(z) are given by $\sin z = 0,\; z = n\pi, where\; n = 0, \mp 1, \mp 2, \mp 3$ ...

Residue of $f(z)\; at\; z = n\pi$ is

$$= \frac{\cos z}{\frac{d}{dz}(\sin z)} = \frac{\cos z}{\cos z} = 1 \left[Res.at\;(z = a) = \frac{\phi(a)}{\psi'(a)}\right]$$

Example 27: Determine the poles of the function and residue at the poles $f(z) = \frac{z}{\sin z}$.

Solution: $f(z) = \frac{z}{\sin z}$

Poles are determined by putting $\sin z = 0 = \sin n\pi\; i.e.\;\; z = n\pi$

$$Residue = \left(\frac{z}{\cos z}\right)_{z=n\pi} = \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$$

Hence, the residue of the given function at pole $z = n\pi\; is = \frac{n\pi}{(-1)^n}$

(g). Residue with Coefficient of $\frac{1}{t}$, Where $z = \frac{1}{t}$

Example 28: Find the residue of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at a pole of order 4.

Solution: The poles of $f(z)$ are determined by $(z-1)^4(z-2)(z-3) = 0,\;\; z = 1, 2, 3$

Here $z = 1$ is a pole of order 4.

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} \;\ldots\ldots\ldots\ldots\ldots(1)$$

Putting $z - 1 = t\; or\; z = 1 + t$ in (1), we get

$$f(1+t) = \frac{(1+t)^3}{(t)^4(t-1)(t-2)}$$

$$= \frac{1}{t^4}(t^3 + 3t^2 + 3t + 1)(1-t)^{-1}\frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(\frac{1}{t} + \frac{3}{T^2} + \frac{3}{t^3} + \frac{1}{t^4}\right)(1 + t + t^2 + t^3 + ..) \times \left(1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} \ldots\right)$$

$$= \frac{1}{2}\left(\frac{1}{t} + \frac{3}{T^2} + \frac{3}{t^3} + \frac{1}{t^4}\right)\left(1 + \frac{3}{2}t^2 + \frac{7}{4}t^2 + \frac{15}{8}t^3 + ..\right)$$

$$= \frac{1}{2}\left(\frac{1}{t} + \frac{9}{2t} + \frac{21}{4}\frac{1}{t} + \frac{15}{8}\frac{1}{t}\right) + ..$$

$= \frac{1}{2}\left(1 + \frac{9}{2} + \frac{21}{4} + \frac{15}{8}\right)\frac{1}{t}$

Coefficient of $\frac{1}{t} = \frac{1}{2}\left(1 + \frac{9}{2} + \frac{21}{4} + \frac{15}{8}\right) = \frac{101}{16}$

Hence, the residue of the given function at a pole of order 4 is $\frac{101}{16}$.

Example 29: Find the residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at its pole.

Solution: The pole of $f(z)$ is given by $(z-a)^3 = 0$i.e. $z = a$

Here, $z = a$ is a pole of order 3.

Putting $z = a + t$ where t is small.

$$f(z) = \frac{ze^z}{(z-a)^3} = \frac{(a+t)e^{a+t}}{t^3}$$

$$= \left(\frac{a}{t^3} + \frac{1}{t^2}\right)e^{a+t} = e^a\left(\frac{a}{t^3} + \frac{1}{t^2}\right)e^t$$

$$= e^a\left(\frac{a}{t^3} + \frac{1}{t^2}\right)\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots\right)$$

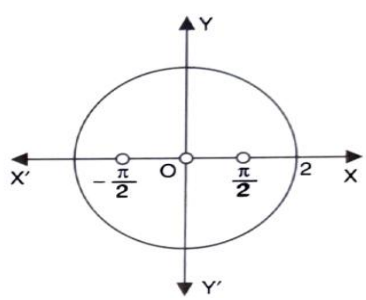
$$= e^a\left[\frac{a}{t^3} + \frac{1}{t^2} + \frac{a}{2t} + \frac{1}{t^2} + \frac{1}{t} + \frac{1}{2} + \cdots\right]$$

$$= e^a\left[\frac{1}{2} + \left(\frac{a}{2} + 1\right)\frac{1}{t} + (a+1)\frac{1}{t^2} + (a)\frac{1}{t^3} + \cdots\right]$$

Coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2} + 1\right)$

Hence the residue at $z = a$ is $e^a\left(\frac{a}{2} + 1\right)$

Example 30: Find the sum of the residues of the function $f(z) = \frac{sinz}{zcosz}$ at its poles inside the circle $|z| = 2$.



Solution: We have, $f(z) = \frac{sinz}{zcosz}$

The pole can be determined by putting denominator $zcosz = 0$

$\Rightarrow z = 0, \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots \ldots$

Of these poles only $z = 0,\ z = \mp\frac{\pi}{2}$ lie inside a circle $|z| = 2$

Residue of $f(z)\ at\ z = 0$ is $\lim_{z\to 0}|z.f(z)| = \lim_{z\to 0}\frac{sinz}{cosz} = 0$ …………………… (1)

Residue of $f(z)\ at\ z = \frac{\pi}{2}$ is

$\lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)f(z)=\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\sin z}{\cos z}$

$=\lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\cos z+\sin z}{\cos z-z\sin z}$[by L'Hopital's Rule]

$=\frac{1}{-\frac{\pi}{2}}=-\frac{2}{\pi}$ ………………. (2)

Similarly, residue of $f(z)$ at $z=-\frac{\pi}{2} is \frac{2}{\pi}$ …………….. (3)

∴ Sum of the residues $=0-\frac{2}{\pi}+\frac{2}{\pi}=0.$

Check your progress

1. Determine the poles of the following functions. Find the order of each pole.
   (i) $\frac{z^2}{(z-a)(z-b)(z-c)}$ Ans. Simple poles at $z = a, z = b, z = c$
   (ii) $\frac{(z-3)}{(z-2)^2(z+1)}$ Ans. Pole at $z = 2$ of second order and $z = 1$ of first order.
   (iii) $\frac{ze^{iz}}{z^2+a^2}$ Ans. Poles at $z = \pm ia$ order 1
   (iv) $\frac{1}{(z-1)(z-2)}$ Ans. $z = 2, z = 1$

Find the residue of the followings:

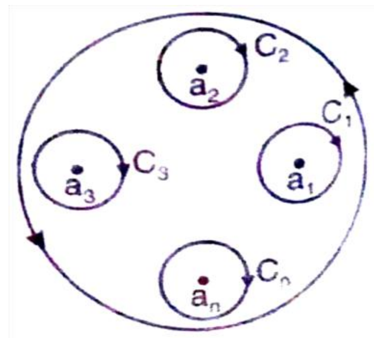
2. $\frac{z^3}{(z-2)(z-3)}$ at its poles. Ans. 27, -8
3. $\frac{z^2}{z^2+a^2}$ at z = ia Ans. $\frac{1}{2}ia$
4. $\frac{1}{(z^2+a^2)^2}$ at z = ia Ans. $-\frac{1}{4a^3}$
5. $\tan z$ at its pole Ans. -1 at its poles $f\left(n+\frac{\pi}{2}\right)$
6. $z^2e^{\frac{1}{z}}$ at the point $z = 0$ Ans. 1/6
7. $z^2 sin\left(\frac{1}{z}\right)$ at $z = 0$ Ans. 1/6
8. $\frac{1}{z^2(z-i)}$ at $z = i$ Ans. -1
9. $\frac{e^{2z}}{1-e^z}$ at its pole Ans. -1
10. $\frac{1+e^z}{\sin z+z cos\, z}$ at $z = 0$ Ans. 1
11. $\frac{1}{z(e^z-1)}$ at its poles Ans. ½

Choose the correct answers:

12. The function $f(z)=\left\{sin\left(\frac{1}{z}\right)\right\}^{-1}$ has multiple poles all of which are isolated singularity.
    (i) False (ii) True (iii) Partially true (iv) None of these

13. The residue of a function can be evaluated only if the pole is an isolated singularity.
    (i) False (ii) True (iii) Partially true (iv) None of these
    (ii)

Residue Theorem: If f(z) is analytic in a closed curve C, except at a finite number of poles within C, then $\int_c f(z)dz = 2\pi i$ (sum of residues at the poles within C).



Proof: Let $C_1, C_2, C_3, \ldots C_n$ be the non-intersecting circle with centres at $a_1, a_2, a_3, \ldots a_n$ respectively, and radii so small that they lie entirely within the closed curve C. Then f(z) is analytic in the multiple connected region lying between the curves C and $C_1, C_2, C_3, \ldots, C_n$.

Applying Cauchy's Theorem

$$\int_c f(z)dz = \int_{c_1} f(z)dz = \int_{c_2} f(z)dz + \int_{c_3} f(z)dz + \cdots .. + \int_{c_n} f(z)dz$$

$= 2\pi i[Res\, f(a_1) + Resf(a_2) + Resf(a_3) + \cdots . + Resf(a_n)]$.

Rouche's Theorem: Suppose that f and g are meromorphic in a neighborhood $B(a.R)$ with no zeroes or poles on the circle $\gamma = \{z: |z-a| = R\}$. If $Z_{f,}\ Z_g\ (P_f, P_g)$ are the number of zeroes(poles) of f *and* g inside γ counted according to their multiplicities and if $|f(z) +$ $gz<fz+|g(z)|$ on γ, then $Zf-Pf=Zg-Pg$

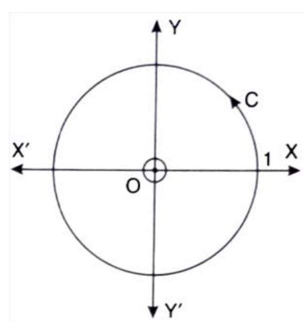
Proof: From the hypothesis $\left|\frac{f(z)}{g(z)} + 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$ on γ. If $\lambda = \frac{f(z)}{g(z)}$ and if λ is a positive real number then this inequality becomes $\lambda + 1 < \lambda + 1$ which is a contradiction. Hence, the meromorphic function $\frac{f(z)}{g(z)}$ maps γ on $\Lambda = \mathbb{C} - [0, \infty)$. If l is the branch of logarithm on Λ them $l\left(\frac{f(z)}{g(z)}\right)$ is a well defined primitive for $\left(\frac{f(z)}{g(z)}\right)' \left(\frac{f(z)}{g(z)}\right)^{-1}$ in a neighborhood of γ.

Thus $0 = \frac{1}{2\pi i}\int_\gamma \left(\frac{f(z)}{g(z)}\right)' \left(\frac{f(z)}{g(z)}\right)^{-1} = \frac{1}{2\pi i}\int_\gamma \left[\frac{f'(z)}{g(z)} - \frac{g'(z)}{g(z)}\right]$

$= Z_f - P_f = Z_g - P_g$.

5.8. Applications:

Example 31: Evaluate the following integral using residue theorem $\int_c \frac{1+z}{z(2-z)} dz$ Where C is the circle $|z| = 1$



Solution: The pole of the integrand are given by putting the denominator equal to zero.

$z(2-z) = 0\ or\ z = 0, 2$

The integrand is analytic on $|z| = 1$ and all points inside except $z = 0$, as a pole at $z = 0$ is inside the circle $|z| = 1$.

Hence by residue theorem

$\int_c \frac{1+z}{z(2-z)} dz = 2\pi i\ [Res\ f(0)]$ ……………. (1)

Residue $f(0) = \lim_{z\to 0} z \cdot \frac{1+z}{z(2-z)} = \lim_{z\to 0} \frac{1+z}{2-z} = \frac{1}{2}$

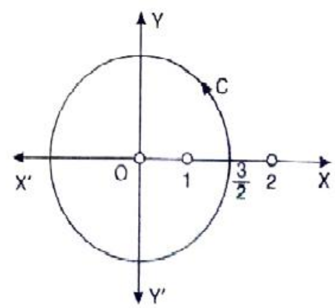
Putting the value of Residue $f(0)$ in (1), we get

$$\int_c \frac{1+z}{z(2-z)} dz = 2\pi i \left(\frac{1}{2}\right) = \pi i$$

Example 32: Evaluate the following integral using residue theorem

$\int_c \frac{4-3z}{z(z-1)(z-2)} dz$. Where c is the circle $|z| = \frac{3}{2}$

Solution: The pole of the function $f(z)$ are given by equating the denominator is zero.



$z(z-1)(z-2) = 0, z = 0, 1, 2$

The function has poles at $z = 0, z = 1\ and\ z = 2$ of which the given circle encloses the pole at $z = 0\ and\ z = 1$.

Residue of $f(z)$ at the simple pole $z = 0$ is

$$= \lim_{z\to 0} z \frac{4-3z}{z(z-1)(z-2)} = \lim_{z\to 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

Residue of $f(z)$ at the simple pole $z = 1$ is

$$= \lim_{z\to 0} z \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{4-3z}{z(z-2)} = \frac{4-3}{1(1-2)} = -1$$

By Cauchy's integral formula

$$\int_c f(z)\, dz = 2\pi i \times sum\ of\ the\ residue\ within\ C$$

$$= 2\pi i \times (2-1) = 2\pi i$$

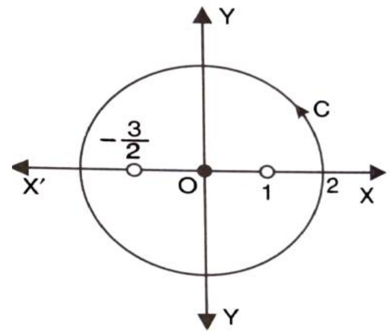
Example 33: Evaluate $\int_c \frac{12z-7}{(z-1)^2(2z+3)} dz$ where C is the circle:

(i) $|z| = 2$ (ii) $|z+i| = \sqrt{3}$

Solution: We have $f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$

Pole are given by

$z = 1$ (double pole) and $z = -\frac{3}{2}$ (simple pole)



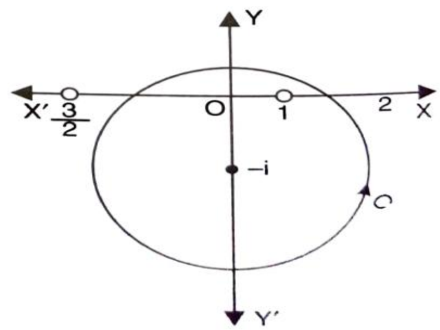
Residue at $(z = 1)$ is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-1)^2.\frac{12z-7}{(z-1)^2(2z+3)}\right\}\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)\right]_{z=1}$$

$$= \left[\frac{(2z+3).12-(12z-7).2}{(2z+3)^2}\right]_{z=1}$$

$= \frac{60-10}{25} = \frac{50}{25} = 2$



Residue at simple pole $\left(z = -\frac{3}{2}\right)$ is

$$R_2 = \lim_{z\to-3/2}\left(z+\frac{3}{2}\right).\frac{12z-7}{(z-1)^2(2z+3)}$$

$$= \lim_{z\to-3/2}\frac{1}{2}.\frac{12z-7}{(z-1)^2} = -2$$

(i) The contour $|z| = 2$ encloses both the poles 1 and $-\frac{3}{2}$.

$\therefore$ The given integral = $2\pi i(R_1+R_2) = 2\pi i(2-2) = 0$

(ii) The contour $|z+i| = \sqrt{3}$ is a circle of radius $\sqrt{3}$ and centre at $z = -i$. the distances of the centre from $z = 1$ and $-\frac{3}{2}$ are respectively $\sqrt{2}$ and $\sqrt{\frac{13}{4}}$. The first of these is $< \sqrt{3}$ and the second is $> \sqrt{3}$.

$\therefore$ The second contour includes only the first singularity $z = 1$.

Hence, the given integral $2\pi i(R_1) = 2\pi i(2) = 4\pi i$.

Example 34: Evaluate the complex integral $\int_c \frac{\coth z\, dz}{(z-i)}$, $c: |z| = 2$

Solution: $\int_c \frac{\coth z\, dz}{(z-i)} = \int_c \frac{e^z+e^{-z}}{(e^z-e^{-z})(z-i)}dz$

The pole of the integration are given by $(e^z - e^{-z})(z-i) = 0$

i.e. $e^z - e^{-z} = 0$ and $z - i = 0$

i.e. $e^{2z} = 1 \quad z = 0 \; and \; z = i$

Both the poles are inside $C : |z| = 2$.

Residue (at $z = i$) $= \lim_{z \to i} (z - i) \frac{e^z + e^{-z}}{(e^z - e^{-z})(z-i)}$

$$= \frac{e^i + e^{-1}}{e^i - e^{-1}} = \coth i$$

To find the residue at $z = 0$, we apply $\frac{\phi(z)}{\psi(z)}$ method

$$\frac{\phi(z)}{\psi(z)} = \frac{\frac{e^z + e^{-z}}{z-i}}{e^z - e^{-z}}, \frac{\phi(z)}{\psi'(z)} = \frac{\frac{e^z + e^{-z}}{z-i}}{e^z + e^{-z}}$$

Residue [at $(z = 0)$] $= \frac{\phi(z)}{\psi(z)} = \frac{\frac{1+1}{0-i}}{1+1} = -\frac{1}{i} = i$

$Sum \; of \; the \; residues \; = \; cothi \; + \; i$

By Cauchy's Residue Theorem $\int_c \; f(z)\, dz = (2\pi i \times Sum \; of \; the \; residues)$

$$\int_c \frac{\coth z}{z - i} dz = 2\pi i [cothi + i]$$

Example 35: Determine the poles of the following function and residue at each pole:

$f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and hence evaluate $\int_c \frac{z^2 dz}{(z-1)^2(z+2)}$ where $C : |z| = 3$.

Solution: $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

Pole of $f(z)$ are given by $(z-1)^2(z+2) = 0$ i.e. $z = 1, 1, -2$

The pole at $z = 1$ is of second order and the pole at $z = -2$ is simple.

Residue of $f(z)$ at $(z = 1)$

$= \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d}{dz} \frac{(z-1)^2 z^2}{(z-1)^2(z+2)}$

$$= \lim_{z \to 1} \frac{d}{dz} \frac{z^2}{(z+2)}$$

$$= \lim_{z \to 1} \frac{(z+2)2z - 1.z^2}{(z+2)^2}$$

$$\lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{1+4}{(1+2)^2} = \frac{5}{9}$$

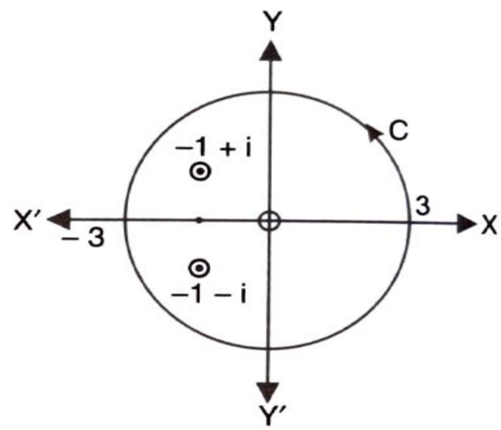
Residue of f(z) (at z = -2) $= \lim_{z \to 2} \frac{(z+2)z^2}{(z-1)^2(z+2)} = \lim_{z \to -2} \frac{z^2}{(z-1)^2}$

$$= \frac{4}{(-2-1)^2} = \frac{4}{9}$$

$$\int_c \frac{z^2}{(z-1)^2(z+2)} = 2\pi i\left(\frac{5}{9}+\frac{4}{9}\right) = 2\pi i$$

Example 36: Using residue theorem, evaluate $\frac{1}{2\pi i}\int_c \frac{e^{zt}dz}{z^2(z^2+2z+2)}$ where C is the circle $|z| = 3$.

Solution: Here, we have $\frac{1}{2\pi i}\int_c \frac{e^{zt}dz}{z^2(z^2+2z+2)}$



Pole are given by $z = 0$ (double pole)

And $z = -1 \mp i$ (simple poles)

all the four poles are inside the given circle $|z| = 3$

Residue (at $z = 0$) is $\lim_{z\to 0}\frac{d}{dz}(z)^2\frac{e^{zt}}{z^2(z^2+2z+2)}$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{e^{zt}}{z^2(z^2+2z+2)}\right)$$

$$= \lim_{z\to 0}\frac{(z^2+2z+2)te^{zt}-(2z+2)e^{zt}}{(z^2+2z+2)^2}$$

$$= \frac{2te^0-2e^0}{4} = \frac{(t-1)}{2}$$

Residue at $(z = -1 + i) = \lim_{z\to -1+i}\frac{(z+1-i)e^{zt}}{z^2(z+1-i)(z+1+i)}$

$$= \lim_{z\to -1+i}\frac{e^{zt}}{z^2(z+1+i)} = \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i-1)(2i)} = \frac{e^{(-1+i)t}}{4}$$

Similarly, Residue at $(z = -1 - i) = \frac{e^{(-1+i)t}}{4}$

$$\int \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i \text{ (sum of the residues)}$$

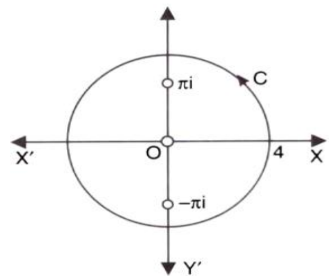
$$\frac{1}{2\pi i}\int \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2}+\frac{e^{(-1+i)t}}{4}+\frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2}+\frac{e^{-1}}{4}\left(e^{it}+e^{-it}\right) = \frac{t-1}{2}+\frac{e^{-1}}{4}(2\,cost)$$

$$= \frac{t-1}{2}+\frac{e^{-1}}{4}\cos t$$

Example 37: Evaluate $\oint_c \frac{1}{\sinh z} dz$, where C is the circle $|z| = 4$

Solution: Here $f(z) = \frac{1}{\sinh z}$



Poles are given by $\sinh z = 0$

$\sinh iz = 0$ So, $z = n\pi i$ where n is an integer.

Out of these, the poles $z = -\pi i$, 0 and πi lie inside the circle $|z| = 4$

The given function $\frac{1}{\sinh z}$ is of the form $\frac{\phi(z)}{\psi(z)}$

Its pole at $(z = a)$ is $\frac{\phi(a)}{\psi'(a)}$

Residue at $(z = -\pi i) = \frac{1}{\cosh(-\pi i)} = \frac{1}{\cos i(-\pi i)}$

$= \frac{1}{\cos \pi} = \frac{1}{-1} = -1$

Residue at $(z = 0) = \frac{1}{\cosh 0} = \frac{1}{1} = 1$

Residue at $(z = \pi i) = \frac{1}{\cosh(\pi i)} = \frac{1}{cosi\ (\pi i)}$

$= \frac{1}{\cos(-\pi)} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$

Residues $at\ -\pi i, 0, \pi i$ are respectively -1, 1 $and\ -1$.

Hence, the required integral $= 2\pi i(-1 + 1 - 1) = -2\pi i$

Example 38: Evaluate $\int_c \frac{dz}{z \sin z}$: C is the unit circle about origin.

Solution: $\frac{1}{zsin\ z} = \frac{1}{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]} = \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]}$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120} \dots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 + \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \left(\frac{z^2}{6} - \frac{z^4}{120}\right)^2 \dots\right]$$

$$= \frac{1}{z^2}\left[1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \cdots\right] = \frac{1}{z^2} + \frac{1}{6} - \frac{z^2}{120} + \frac{z^4}{36} - \cdots$$

$$= \frac{1}{z^2} + \frac{1}{6} - \frac{7z^2}{360} + \cdots$$

This shows that $z = 0$ is a pole of order 2 for the function $\frac{1}{z \sin z}$ and the residue at the pole is zero (coefficient of $\frac{1}{z}$). Pole at z = 0 lies within C.

$\int_c \frac{1}{z \sin z} dz = 2\pi i$ (sum of residues) = 0

Check your progress

1. Obtain Laurent's expansion for the junction $f(z) = \frac{1}{z^2 \sinh z}$ at the isolated singularity and hence evaluate $\oint_c \frac{1}{z^2 \sinh z} dz$, where C is the circle $|z - 1| = 2$.

2. Evaluate $\oint_c \frac{1}{z^2 \sin z} dz$ where C is triangle with vertices (0, 1), (2, -2), (7, 1).

Evaluate the following complex integrals:

3. $\oint_c \frac{1-2z}{z(z-1)(z-2)} dz$, where C is the circle $|z| = 1.5$

4. $\oint_c \frac{z^2 e^{2z}}{z^2+1} dz$, where C is the circle $|z| = 2$

5. $\oint_c \frac{z-1}{(z+1)^2(z-2)} dz$, where C is the circle $|z| = 2$

6. $\oint_c \frac{2z^2+z}{z^2-1} dz$, where C is the circle $|z - 1| = 1$

7. $\oint_c \frac{1}{(z^2-4)(z^2+1)} dz$, where C is the circle $|z| = 1.5$

8.The residue at the pole of the function f(z) = cot z, equals,

(i) 0 (ii) 1 (iii) -1 (iv) 2πi Ans (ii)

9.The function (z – 1)sin 1/z at z = 0 has

(i) A removable singularity (ii) a simple pole

(iii) an essential singularity (iv) a multiple pole Ans. (iii)

Conclusion: After studying this unit we should be able to know the definition of Zero and poles of Analytic function, definition of Singular point, definition of the residue at a pole, the residue at infinity, method of finding residues and its applications in brief.

Unit-6
Evaluation of Real Definite Integrals by Contour Integration

6.1. INTRODUCTION

Contour Integration of residue classes is an important and useful tool in complex analysis.It gives many important techniques for finding the complex integration and it is useful to find many other important theories of complex analysis which are necessary for the development in mathematics. A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the pole of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$\int_c f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the pole within C)

We call the curve, a contour and the process of integration along a contour is called contour integration.

6.2. Objectives

After studying this unit we should be able to:

- Evaluation of real definite integrals by contour integration;
- Integration round the unit circle of the type: $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$;
- Evaluate of $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x;
- Rectangular contour;

- Indented semi- circular contour;

Contour Integration of residue classes is an important and useful tool in complex analysis.It gives many important techniques for finding the complex integration and it is useful to find many other important theories of complex analysis which are necessary for the development in mathematics. A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the pole of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$\int_c f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the pole within C)

6.3. Evaluation of real definite integrals by contour integration

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using Cauchy's theorem of residues. For finding the integrals we take a closed curve C, find the pole of the function $f(z)$ and calculate residues at those poles only which lie within the curve C.

$\int_c f(z)dz = 2\pi i$ (sum of the residues of $f(z)$ at the pole within C)

We call the curve, a contour and the process of integration along a contour is called contour integration.

6.4. Integration round the unit circle of the type: $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

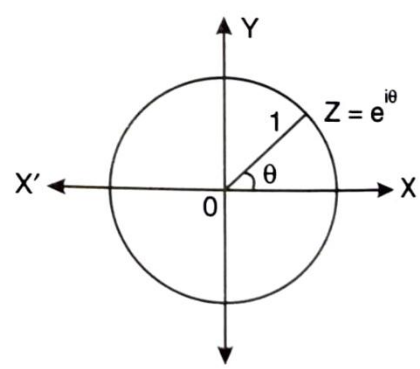
Where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Convert $\sin\theta, \cos\theta$ into z.

Consider a circle of unit radius with centre at origin, as contour.

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left[z-\frac{1}{z}\right]$, $\quad z = re^{i\theta} = 1.e^{i\theta} = e^{i\theta}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left[z+\frac{1}{z}\right]$$



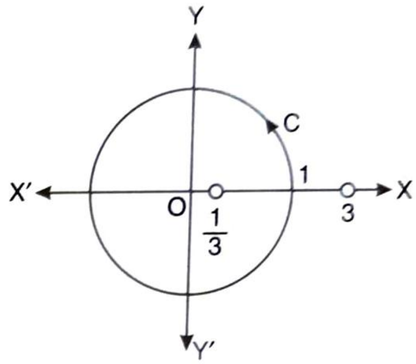
As we know $z = e^{i\theta}$, $dz = e^{i\theta} i d\theta$ or $d\theta = \frac{dz}{iz}$

The integrand is converted into a function of z.

Then apply Cauchy's residue theorem to evaluate the integral.

Some examples of these are illustrated below.

Example 1: Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$.



Solution: $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-3\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$

$$\int_0^{2\pi} \frac{2d\theta}{10-3e^{i\theta}-3e^{-i\theta}} \quad put\ e^{i\theta} = z,\ \ i.e^{i\theta}d\theta = dz$$

$= \int_c \frac{2}{10-3z-\frac{3}{z}} \frac{dz}{iz} =$, [C is the unit circle $|z| = 1$]

$$= \frac{2}{i}\int_c \frac{dz}{(3z-1)(z-3)}$$

$$= -\frac{2}{i}\int_c \frac{dz}{(3z-1)(z-3)}$$

$$= 2i\int_c \frac{dz}{(3z-1)(z-3)}$$

Let $\mathrm{I} = 2i\int_c \frac{dz}{(3z-1)(z-3)}$

Poles of the integrand are given by $(3z-1)(z-3) = 0 \ \Rightarrow z = \frac{1}{3},\ 3$

There is only one pole at $z = \frac{1}{3}$ inside the unit circle C.

Residue at $z = \frac{1}{3} = \lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)f(z) = \lim_{z\to\frac{1}{3}} \frac{2i\left(z-\frac{1}{3}\right)}{(3z-1)(z-3)} = \lim_{z\to\frac{1}{3}} \frac{2i}{3(z-3)}$

$$= \frac{2i}{3\left(\frac{1}{3}-3\right)} = -\frac{i}{4}$$

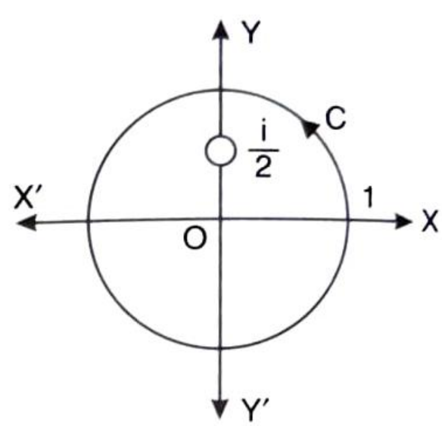
Hence, by Cauchy's Residue Theorem

I = 2πi (Sum of the residues within Contour) = $2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$

$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \frac{\pi}{2}$

Example 2: Use residue calculus to evaluate the following integral $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$

Solution: Let I = $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{1}{5-4\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)} d\theta$



$= \int_0^{2\pi} \frac{d\theta}{5+2ie^{i\theta}-2ie^{-i\theta}}$ [putting $e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$]

$\int_c \frac{1}{5+2iz=\frac{2i}{z}} \frac{dz}{iz}$ where c is the unit circle $|z| = 1$.

$$= \int_c \frac{dz}{5iz - 2z^2 + 2}$$

Pole of integrand are given by

$$-2z^2 + 5iz + 2 = 0 \ \ or \ z = \frac{-5i \mp \sqrt{-25+16}}{-4} = \frac{-5i \mp 3i}{-4} = 2i, \frac{i}{2}$$

Only $z = \frac{i}{2}$ lies inside C.

Residue at the simple pole at $z = \frac{i}{2}$ is

$$\lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \times \left[\frac{1}{(2z-i)(-z+2i)}\right] = \lim_{z \to \frac{i}{2}} \frac{1}{2(-z+2i)} = \frac{1}{2\left(-\frac{i}{2} + 2i\right)} = \frac{1}{3i}$$

Hence, by Cauchy's residue theorem

$I = 2\pi i \times Sum$ of residues within the contour $= 2\pi i \times \frac{1}{3i} = \frac{2\pi}{3}$

Hence, given integral = $\frac{2\pi}{3}$

Example 3: Evaluate $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} \ if \ a > |b|$

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

$$= \int_0^{2\pi} \frac{1}{a + b\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)} d\theta \left[e^{i\theta} = z, d\theta = \frac{dz}{iz}\right]$$

$= \int_c \frac{1}{a + \frac{b}{2i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$ (where C is the unit circle $|z| = 1$)

$$= \int_c \frac{2}{2iaz + bz^2 - b} dz$$

$$= \int_c \frac{2}{bz^2 + 2iaz - b} dz = \frac{1}{b} \int \frac{2dz}{z^2 + \frac{2aiz}{b} - 1}$$

$$= \frac{1}{b} \int_c \frac{2}{(z-\alpha)(z-\beta)} dz \ when \ bz^2 + 2aiz - b = b\left\{z^2 + \frac{2aiz}{b} - 1\right\}$$

and $\alpha + \beta = -\frac{2ai}{b}$

$$\alpha\beta = -1 \left[(\alpha - \beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = -\frac{4a^2}{b^2} + 4\right]$$

$|\alpha| < 1$ then $|\beta| > 1$
i.e., pole lies at z = α in the unit circle.

Residue at z = a = $\lim_{z\to\alpha}(z-\alpha)\frac{2}{(z-\alpha)(z-\beta)} = \frac{2}{\alpha-\beta} = \frac{b}{\sqrt{b^2-a^2}} = \frac{b}{i\sqrt{a^2-b^2}}$

$$\int_0^{2\pi}\frac{1}{a+bsin\,\theta} = \frac{1}{b}\int\frac{2dz}{z^2+\frac{2aiz}{b}-1} = 2\pi i\frac{b}{bi\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 4: Evaluate $\int_0^{\pi}\frac{d\theta}{3+2\cos\theta}$ by contour integration in the complex plane.

Solution: $\int_0^{\pi}\frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{\pi}\frac{d\theta}{3+2\cos\theta}$

$= \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{3+(e^{i\theta}+e^{-i\theta})}$ [putting $e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$]

$= \frac{1}{2}\int_c \frac{\frac{dz}{iz}}{3+z+\frac{1}{z}} = \frac{1}{2i}\int_c \frac{dz}{z^2+3z+1}$, where c is the unit circle $|z| = 1$.

Poles are given by $z^2+3z+1 = 0$ or $z = \frac{-3\mp\sqrt{9-4}}{2} = \frac{-3\mp\sqrt{5}}{2}$

There are two poles at $z = \frac{-3+\sqrt{5}}{2}$ and $z = \frac{-3-\sqrt{5}}{2}$

Only one of these poles at $z = \frac{-3+\sqrt{5}}{2}$ is inside the circle

Residue at $z = \frac{-3+\sqrt{5}}{2}$

$$\lim_{z\to\frac{-3+\sqrt{5}}{2}}\left(z-\frac{-3+\sqrt{5}}{2}\right)\frac{1}{\left(z-\frac{-3+\sqrt{5}}{2}\right)\left(z-\frac{-3-\sqrt{5}}{2}\right)} = \frac{1}{\frac{-3+\sqrt{5}}{2}-\frac{-3-\sqrt{5}}{2}} = \frac{1}{\sqrt{5}}$$

Hence by Cauchy Residue theorem

$$\frac{1}{2i}\int_c\frac{dz}{z^2+3z+1} = \frac{1}{2i}\left[2\pi i\times Residue\ at\ \left(z = \frac{-3+\sqrt{5}}{2}\right)\right] = \frac{1}{2i}\times 2\pi i\times\frac{1}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

$\int_0^{\pi}\frac{d\theta}{3+2\cos\theta} = \frac{\pi}{\sqrt{5}}$ Ans.

Example 5: Use the complex variable technique to find the value of the integral $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Solution: Let$I = \int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \int_0^{2\pi}\frac{d\theta}{2+\frac{e^{i\theta}+e^{-i\theta}}{2}} = \int_0^{2\pi}\frac{2d\theta}{4+e^{i\theta}+e^{-i\theta}}$

Put $e^{i\theta} = z$ so that $e^{i\theta}(id\theta) = dzizd\theta = dzd\theta = \frac{dz}{iz}$

$I = \int_c \frac{2\frac{dz}{iz}}{4+z+\frac{1}{z}}$ where c denotes the unit circle $|z| = 1$

$$= \frac{1}{i}\int_c \frac{2dz}{z^2+4z+1}$$

The poles are given by putting the denominator equal to zero.

$z^2+4z+1=0$ or $z=\frac{-4\mp\sqrt{16-4}}{2}=\frac{-4\mp 2\sqrt{3}}{2}=-2\mp\sqrt{3}$

The pole within the unit circle C is a simple pole at $z=-2+\sqrt{3}$.

Now we calculate the residue at this pole.

Residue at $z=-2+\sqrt{3}=\lim_{z\to(-2+\sqrt{3})}\frac{1}{i}\frac{(z+2-\sqrt{3})2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$$= \lim_{z\to(-2+\sqrt{3})}\frac{2}{i(z+2+\sqrt{3})}=\frac{2}{i(-2+\sqrt{3}+2+\sqrt{3})}=\frac{1}{\sqrt{3}i}$$

Hence by Cauchy's Residue theorem, we have

$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=2\pi i$ (sum of the residues within the contour)

$=2\pi i\frac{1}{i\sqrt{3}}=\frac{2\pi}{\sqrt{3}}$

Example 6: Using complex variable techniques evaluate the real integral $\int_0^{2\pi}\frac{sin^2\theta\, d\theta}{5-4\cos\theta}$

Solution: If we put $z=e^{i\theta}$

$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right),\quad d\theta=\frac{dz}{iz}$

And so $I=\int_0^{2\pi}\frac{sin^2\theta\, d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{1-cos\theta\, d\theta}{5-4\cos\theta}$ [where c is a circle of unit radius with centre z = 0]

$$I = Real\ part\ of\ \frac{1}{2}\int_0^{2\pi}\frac{(1-cos2\theta-isin2\theta)\,d\theta}{5-4\cos\theta}$$

$$= Real\ part\ of\ \frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}\,d\theta}{5-4\cos\theta}$$

$$= Real\ part\ of\ \frac{1}{2}\int_c \frac{1-z^2}{5-2\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= Real\ part\ of\ \frac{1}{2i}\int_c \frac{1-z^2}{5z-2z^2-2}dz$$

$$= Real\ part\ of\ \frac{1}{2i}\int_c \frac{z^2-1}{2z^2-5z+2}$$

Pole are determined by $2z^2 - 5z + 2 = 0$ or $(2z - 1)(z - 2) = 0 \ or \ z = ½, 2$
so inside the contour c there is a simple pole at $z = ½$

Residue at the simple pole $\left(z = \frac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^2-1}{(2z-1)(z-2)}$

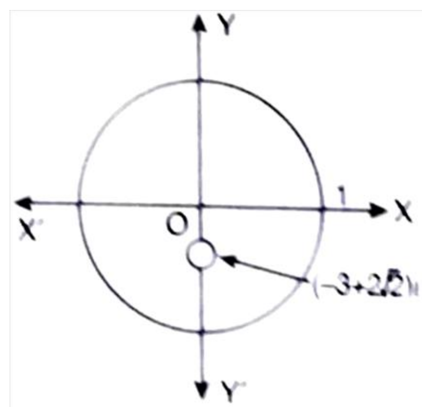
$$= \lim_{z \to \frac{1}{2}} \frac{z^2 - 1}{2(z-2)} = \frac{\frac{1}{4} - 1}{2\left(\frac{1}{2} - 2\right)} = \frac{1}{4}$$

Real part of $\frac{1}{2i} \int_c \frac{z^2-1}{2z^2-5z+2} dz = \frac{1}{2i} 2\pi i$ (sum of the residues)

$$Hence, \int_0^{2\pi} \frac{sin^2\theta \ d\theta}{5 - 4\cos\theta} d\theta = \pi \left(\frac{1}{4}\right) = \frac{\pi}{4}$$

Example 7: Using the complex variable techniques, evaluate the real integral $\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta} d\theta$

Solution: Let $I = \int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta} d\theta$



$$I = Real \ part \ of \int_0^{2\pi} \frac{e^{i\theta}}{3 + \sin\theta} d\theta$$

$$= Real \ part \ of \int_0^{2\pi} \frac{e^{i\theta}}{3 + \frac{e^{i\theta} - e^{-i\theta}}{2i}} d\theta$$

Putting $e^{i\theta} = z$ so that $e^{i\theta} i d\theta = dz iz d\theta = dz \ or \ d\theta = \frac{dz}{iz}$

$\therefore I = Real \ part \ of \int_c \frac{z}{3+\frac{1}{2i}\left(z - \frac{1}{z}\right) iz} \frac{dz}{iz}$

$$= Real \ part \ of \int_c \frac{z}{3iz + \frac{z^2}{2} - \frac{1}{2}} dz = Real \ part \ of \int_c \frac{2z}{z^2 + 6iz - 1} dz$$

The poles are given by putting the denominator equal to zero

$$z^2 + 6iz - 1 = 0 z = \frac{-6i \mp \sqrt{-36 + 4}}{2}$$

$z = -3i + 2\sqrt{2}i$ and $-3i - 2\sqrt{2}i$

The pole within the unit circle c is a simple pole at $z = \left(-3 + 2\sqrt{2}\right)i$

Now we calculate the residue at $z = \left(-3 + 2\sqrt{2}\right)i$

Residue = $\lim_{z \to (-3+2\sqrt{2})i} (z + 3i - 2\sqrt{2}i) \frac{2z}{z^2+6iz-1}$

$$= \lim_{z \to (-3+2\sqrt{2})i} \frac{\left(z + 3i - 2\sqrt{2}i\right).2z}{(z + 3i - 2\sqrt{2}i)(z + 3i + 2\sqrt{2}i)}$$

$$= \lim_{z \to (-3+2\sqrt{2})i} \frac{2z}{z + 3i + 2\sqrt{2}i}$$

$$= \frac{2(-3 + 2\sqrt{2})i}{-3i + 2\sqrt{2}i + 3i + 2\sqrt{2}i}$$

$$= \frac{-6i + 4\sqrt{2}i}{4\sqrt{2}i} = \frac{-3}{2\sqrt{2}} + 1$$

Hence by Residue theorem, we have

Real part of $\int_c \frac{2z}{z^2+6iz-1} dz$

$$= Real\ part\ of\ 2\pi i \left(\frac{-3}{2\sqrt{2}} + 1\right)$$

$$\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta} d\theta = 0$$

Example 8: Using contour integration, evaluate the real integral $\int_0^{\pi} \frac{1+2cos\theta}{5+4\cos\theta} d\theta$

Solution: Let $I = \int_0^{\pi} \frac{1+2cos\theta}{5+4\cos\theta} d\theta$

$$= \frac{1}{2}\int_0^{\pi} \frac{1 + 2cos\theta}{5 + 4\cos\theta} d\theta$$

$$= Real\ part\ of \frac{1}{2}\int_0^{\pi} \frac{1 + 2e^{i\theta}}{5 + 4\cos\theta} d\theta$$

$$= Real\ part\ of\ \frac{1}{2}\int_0^{\pi} \frac{1 + 2e^{i\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})} d\theta$$

Putting $e^{i\theta} = z,\ \ d\theta = \frac{dz}{iz}$ where C is the unit circle $|z| = 1$.

$$= Real\ part\ of \frac{1}{2}\int_c \frac{1 + 2z}{5 + 2\left(z + \frac{1}{z}\right)} \frac{dz}{iz},$$

$$= Real\ part\ of \frac{1}{2}\int_c \frac{-i(1 + 2z)}{2z^2 + 5z + 2} dz$$

$$= Real\ part\ of \frac{1}{2}\int_c \frac{-i(1 + 2z)}{(2z + 1)(z + 2)} dz$$

$= Real\ part\ of\ -\frac{i}{2}\int_c \frac{1}{z+2}dz$

Pole is given by $z + 2 = 0\ i.e., z = -2$

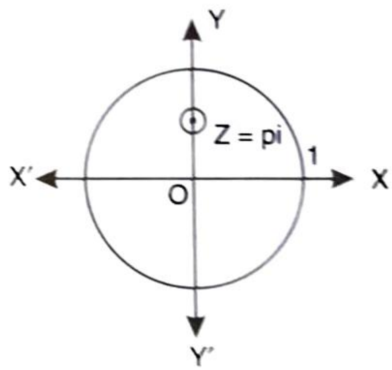
Thus there is no pole of f(z) inside the unit circle C. hence f(z) is analytic in C.

By Cauchy's Theorem $\int_c f(z)dz = 0$if f(z) is analytic in C.

∴ I = Real part of zero = 0. Hence the given integral = 0

Example 9: Using complex variable, evaluate the real integral $\int_0^{2\pi}\frac{d\theta}{1-2p\sin\theta+p^2}$, where $p^2 < 1$.

Solution: $\int_0^{2\pi}\frac{d\theta}{1-2p\sin\theta+p^2} = \int_0^{2\pi}\frac{d\theta}{1-2p\frac{\left(e^{i\theta}-e^{-i\theta}\right)}{2i}+p^2}$



Let $I = \int_0^{2\pi}\frac{d\theta}{1+ip\left(e^{i\theta}-e^{-i\theta}\right)+p^2}$

Writing $z = e^{i\theta},\ dz = ie^{i\theta}d\theta = izd\theta,\ d\theta = \frac{dz}{iz}$

$I = \int_C \frac{1}{1+ip\left(z-\frac{1}{z}\right)+p^2}\frac{dz}{zi}$ [where c is the unit circle $|z| = 1$]

$$= \int_c \frac{dz}{zi - pz^2 + p + p^2 zi}$$

$$\int_c \frac{dz}{-pz^2 + ip^2 + zi + p} = \int_c \frac{dz}{(iz+p)(izp+1)}$$

Poles are given by $(iz+p)(izp+1) = 0$

$$z = -\frac{p}{i} = ip\ and\ z = -\frac{1}{pi} = \frac{i}{p}|ip| < 1$$

$$and\ \left|\frac{1}{p}\right| > 1\ as\ p^2 < 1$$

pi is the only pole inside the unit circle.

Residue at $(z = pi) = \lim_{z\to pi}\frac{(z-pi)}{(iz+p)(izp+1)}$

$$= \lim_{z\to pi}\left[\frac{1}{i(izp+1)}\right] = \frac{1}{i}\frac{1}{i(-p^2+1)}$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi}\frac{d\theta}{1-2p\sin\theta+p^2} = 2\pi i\left(\frac{1}{i}\frac{1}{1-p^2}\right)$$

$$= \frac{2\pi}{1-p^2}$$

Example 10: Apply calculus of residue to prove that: $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2} \quad (a^2 < 1)$

Solution: Let $I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2}$

$$= \int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1 - a(e^{i\theta} + e^{-i\theta}) + a^2}$$

$$= Real\ part\ of\ \int_0^{2\pi} \frac{e^{2i\theta}}{(1-ae^{i\theta})(1-ae^{-i\theta})} d\theta$$

$$= Real\ part\ of\ \oint_c \frac{z^2}{(1-az)\left(1-\frac{a}{z}\right)} \frac{dz}{iz} \left[put\ e^{i\theta} = z\ so\ that\ d\theta = \frac{dz}{iz}\right]$$

$= Real\ part\ of\ \oint_c \frac{-iz^2}{(1-az)(z-a)} dz$ [C is the unit circle $|z| = 1$]

Poles of $\frac{-iz^2}{(1-az)(z-a)}$ are given by $(1-az)(z-a) = 0$

Thus, $z = \frac{1}{a}\ and\ z = a$ are the simple poles.

Only z = a lies within the unit circle C as a < 1.

The residue of f(z) at (z = a) = $\lim_{z\to a}(z-a)\frac{-iz^2}{(1-az)(z-a)}$

$$= \lim_{z\to a} \frac{-iz^2}{(1-az)} = -\frac{ia^2}{1-a^2}$$

Hence, by Cauchy's Residue Theorem, we have

$\oint_C f(z)dz = 2\pi i$ [Sum of residues within the contour]

$= 2\pi i\left(-\frac{ia^2}{1-a^2}\right) = \frac{2\pi a^2}{1-a^2}$, which is purely real.

Thus, I = Real part of $\oint_C f(z)dz = \frac{2\pi a^2}{1-a^2}$

Hence $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}$

Example 11: Using complex variable techniques, evaluate the integral $\int_0^{2\pi} \frac{sin^2\theta - 2\cos\theta}{2+\cos\theta} d\theta$

Solution: $\int_0^{2\pi} \frac{sin^2\theta-2\cos\theta}{2+\cos\theta} d\theta = \int_0^{2\pi} \frac{\frac{1}{2}-\frac{1}{2}\cos 2\theta - 2\cos\theta}{2+\cos\theta} d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos 2\theta - 2\cos\theta}{2 + \cos\theta}\, d\theta = Real\ part\ of\ \frac{1}{2}\int_0^{2\pi} \frac{1 - e^{2i\theta} - 4e^{i\theta}}{2 + \cos\theta}\, d\theta$$

Put $e^{i\theta} = z\ so\ that\ ie^{i\theta} d\theta = dz\ or\ izd\theta = dz\ or\ d\theta = \frac{dz}{iz}$

$$= Real\ part\ of\ \frac{1}{2}\int_c \frac{1 - z^2 - 4z}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= Real\ part\ of\ \frac{1}{i} \frac{(1 - z^2 - 4z)dz}{4z + z^2 + 1}$$

The poles are given by $4z + z^2 + 1 = 0$

$$z = \frac{-4 \mp \sqrt{16 - 4}}{2} = -2 \mp \sqrt{3}$$

The pole within the unit circle c is $= -2 \mp \sqrt{3}$

Residue at the simple pole $z = -2 + \sqrt{3}$

$$= \lim_{z \to 2 + \sqrt{3}} (z + 2 - \sqrt{3}) \frac{1 - z^2 - 4z}{(z + 2 - \sqrt{3})(z + 2 + \sqrt{3})}$$

$$= \lim_{z \to 2 + \sqrt{3}} \left[\frac{1 - z^2 - 4z}{(z + 2 + \sqrt{3})}\right]$$

$$= \frac{1 - \left(-2 + \sqrt{3}\right)^2 - 4(-2 + \sqrt{3})}{\left(-2 + \sqrt{3}\right) + 2 + \sqrt{3}} = \frac{1}{\sqrt{3}}$$

Real part of $\frac{1}{i}\int_c \frac{(1 - z^2 - 4z)dz}{4z + z^2 + 1} = Real\ part\ of\ \left(\frac{1}{i}\right) 2\pi\, i\ (Residue)$

$$= Real\ part\ of\ 2\pi\left(\frac{1}{\sqrt{3}}\right)\ or\ I = \frac{2\pi}{\sqrt{3}}$$

Hence, the given integral $= \frac{2\pi}{\sqrt{3}}$

Example 12: Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$ by using contour integration.

Solution: Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$

$$= Real\ part\ of\ \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5 + 4\cos\theta}\, d\theta$$

$$= Real\ part\ of\ \int_0^{2\pi} \frac{e^{2i\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})}\, d\theta \begin{bmatrix} e^{i\theta} = z\ i.e., \\ d\theta = dz, \\ d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} \end{bmatrix}$$

$= Real\ part\ of\ \oint_c \frac{z^2}{5+2\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$ [C is the unit circle $|z| = 1$]

$$= Real\ part\ of\ \oint_c \frac{z^2}{5z+2z^2+2}\frac{dz}{i}$$

$$= Real\ part\ of\ \oint_c \frac{-iz^2}{2z^2+5z+2}dz$$

$$= Real\ part\ of\ \oint_c \frac{-iz^2}{(2z+1)(z+2)}dz$$

Poles are determined by putting denominator equal to zero,

$$(2z+1)(z+2) = 0, \quad z = -\frac{1}{2}, -2$$

Only the simple pole at $z = -\frac{1}{2}$ is inside the contour.

Residue at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{-iz^2}{(2z+1)(z+2)}$$

$$\lim_{z\to -\frac{1}{2}}\frac{-iz^2}{2(z+2)} = \frac{-i\left(-\frac{1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)} = -\frac{i}{12}$$

By Cauchy's Integral Theorem

$\int_c f(z)dz = 2\pi i$ (Sum of the residues within C)

$= 2\pi i\left(-\frac{i}{12}\right) = \frac{\pi}{6}$, which is real. Hence, $\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{6}$

Example 13: Evaluate contour integration of the real integral $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta$

Solution: $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = Real\ part\ of\ \int_0^{2\pi}\frac{e^{3i\theta}}{5-4\cos\theta}d\theta$

$$= Real\ part\ of\ \int_0^{2\pi}\frac{e^{3i\theta}}{5-2(e^{i\theta}+e^{-i\theta})}d\theta$$

On putting z = $e^{i\theta}$ and $d\theta = \frac{dz}{iz}$

$= Real\ part\ of\ \int_c \frac{z^3}{5-2\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$ where c is the unit circle.

$$= Real\ part\ of\ \frac{1}{i}\int_c \frac{z^3}{5z-2z^2-2}dz$$

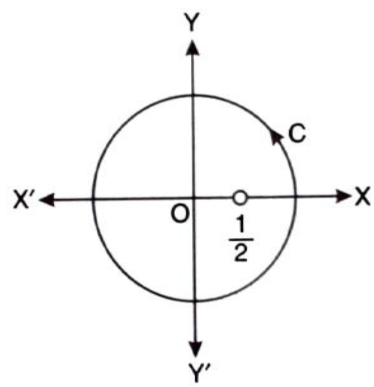
$$= Real\ part\ of\ -\frac{1}{i}\int_c \frac{z^3}{2z^2-5z+2}dz$$

$$= Real\ part\ of i\int_c \frac{z^3}{(2z-1)(z-2)}dz$$

Poles are given by $(2z-1)(z-2)=0$

$i.e., z=\frac{1}{2}, z=2$

$z=\frac{1}{2}$ is the only pole inside the unit circle.



Residue at $z=\frac{1}{2}=\lim_{z\to\frac{1}{2}}\frac{i\left(z-\frac{1}{2}\right)z^3}{(2z-1)(z-2)}$

$$=\lim_{z\to\frac{1}{2}}\frac{iz^3}{2(z-2)}=\frac{\frac{i}{8}}{2\left(\frac{1}{2}-2\right)}=-\frac{i}{24}$$

$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = Real\ part\ of\ 2\pi i\left(-\frac{i}{24}\right)=\frac{\pi}{4}$

Example 14: Use the residue theorem to show that $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}=\frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}$ $\quad where\ a>$ *0, b>0, a>b.*

Solution**:** $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}=\int_0^{2\pi}\frac{d\theta}{\left(a+b.\frac{e^{i\theta}+e^{i\theta}}{2}\right)^2}$

Put $e^{i\theta}=z\ so\ that\ e^{i\theta}(id\theta)=dz,\quad izd\theta=dz,\quad d\theta=\frac{dz}{iz}$

$=\int_c \frac{1}{\left\{a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right\}^2}\frac{dz}{iz}$ where c is the unit circle $|z|=1$

$$\int_c \frac{1}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{iz}=\int_c \frac{-4iz}{\left(a+\frac{bz}{2}+\frac{b}{2z}\right)^2}\frac{dz}{(2z)^2}$$

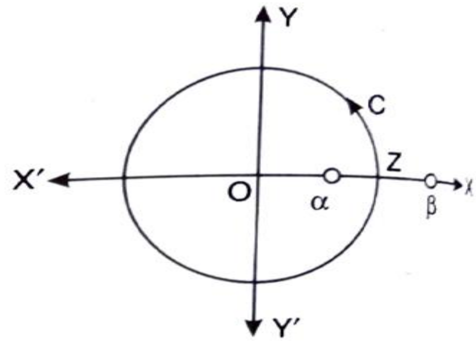
$$\int_c \frac{-4izdz}{(bz^2+2az+b^2)^2}$$

$$=\frac{-4i}{b^2}\int_c \frac{zdz}{\left(z^2+\frac{2az}{b}+1\right)^2}$$

The poles are given by putting the denominator equal to zero.

i.e., $\left(z^2+\frac{2az}{b}+1\right)^2=0$

$(z-\alpha)^2(z-\beta)^2 z=0$



Where, $\alpha=\frac{-\frac{2a}{b}+\sqrt{\frac{4a^2}{b^2}-4}}{2}=\frac{-a+\sqrt{a^2-b^2}}{b}$

$$\beta=\frac{-\frac{2a}{b}-\sqrt{\frac{4a^2}{b^2}-4}}{2}=\frac{-a-\sqrt{a^2-b^2}}{b}$$

There are two poles at z = α and z = β, each of order 2.

Since $|\alpha\beta=1|\ or\ |\alpha||\beta|=1\ if\ |\alpha|<1\ then\ |\beta|<1$

There is only one pole $|\alpha|<1$ of order 2 within the unit circle c.

Residue (at the double pole z = α) = $\lim_{z\to\alpha}\frac{d}{dz}(z-\alpha)^2\frac{-4iz}{b^2(z-\alpha)^2(z-\beta)^2}$

$$=\lim_{z\to\alpha}\frac{d}{dz}\frac{-4iz}{b^2(z-\beta)^2}$$

$$=\frac{-4i}{b^2}\lim_{z\to\alpha}\frac{(z-\beta)^2.1-2(z-\beta)z}{(z-\beta)^4}$$

$$=\frac{-4i}{b^2}\lim_{z\to\alpha}\frac{z-\beta-2z}{(z-\beta)^3}=\frac{-4i}{b^2}\lim_{z\to\alpha}\frac{-(z+\beta)}{(z-\beta)^3}$$

$$=\frac{4i}{b^2}\frac{(\alpha+\beta)}{(\alpha-\beta)^3}=\frac{4i}{b^2}\frac{\alpha+\beta}{[(\alpha+\beta)^2-4\alpha\beta]^{\frac{3}{2}}}$$

$$=\frac{4i}{b^2}\frac{-\frac{2a}{b}}{\left[(-\frac{2a}{b})^2-4(1)\right]^{\frac{3}{2}}}$$

$$=\frac{-8ai}{(4a^2-4b^2)^{\frac{3}{2}}}=-\frac{ai}{(a^2-b^2)^{\frac{3}{2}}}$$

Hence, $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}=2\pi i\times\frac{-ai}{(a^2-b^2)^{\frac{3}{2}}}=\frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}$

Example 15: Show by the method of residues, that $\int_0^{\pi}\frac{a d\theta}{a^2+sin^2\theta}=\frac{\pi}{\sqrt{1+a^2}}$

Solution: Let $I=\int_0^{\pi}\frac{a d\theta}{a^2+sin^2\theta}=\int_0^{\pi}\frac{2a d\theta}{2a^2+2sin^2\theta}\cos 2\theta=1-2sin^2\theta$

$=\int_0^{\pi}\frac{2a d\theta}{2a^2+1-\cos 2\theta}==\int_0^{\pi}\frac{a d\phi}{2a^2+1-cos\phi}$ [putting $2\theta=\phi,\ 2d\theta=d\phi$]

$$= \int_0^{\pi} \frac{a d\phi}{2a^2 + 1 - \frac{1}{2}(e^{i\phi} + e^{-i\phi})} = \int_0^{\pi} \frac{2a d\phi}{4a^2 + 2 - (e^{i\phi} + e^{-i\phi})}$$

Putting $e^{i\phi} = z, e^{i\phi}(id\phi) = dz \text{ or } z(id\phi) = dz, d\phi = \frac{dz}{iz}$

$= \int_c \frac{2a}{4a^2+2-\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$ [where, c is unit circle $|z| = 1$]

$$= \frac{2a}{i} \int_c \frac{dz}{(4a^2+2)z - z^2 - 1} = \frac{2a}{-i} \int_c \frac{dz}{z^2 - (4a^2+2)z + 1}$$

$$= 2ai \int_c \frac{dz}{z^2 - (4a^2+2)z + 1}$$

The poles are given by $z^2 - (4a^2+2)z + 1 = 0$

$$z = \frac{(4a^2+2) \mp \sqrt{(4a^2+2)^2 - 4}}{2}$$

$$= \frac{(4a^2+2) \mp \sqrt{16a^4 + 16a^2}}{2}$$

$$= 2a^2 + 1 \mp 2a\sqrt{a^2+1}$$

Let $\alpha = 2a^2 + 1 + 2a\sqrt{a^2+1}$

$\beta = 2a^2 + 1 - 2a\sqrt{a^2+1}$

$z^2 - (4a^2+2)z + 1 = (z-\alpha)(z-\beta)$

$$I = 2ai \int \frac{dz}{(z-\alpha)(z-\beta)}$$

Product of the roots = $\alpha\beta = 1 \text{ or } |\alpha\beta| = 1$

But $|\alpha| > 1$ $|\beta| = 1$

Only β lies inside the circle c.

Now we calculate the residue at z = β

Residue at (z = β) = $\lim_{z\to\beta}(z-\beta)\frac{2ai}{(z-\alpha)(z-\beta)} = \lim_{z\to\beta}\frac{2ai}{(z-\alpha)}$

$$= \frac{2ai}{(\beta-\alpha)} = \frac{2ai}{\left(2a^2+1-2a\sqrt{a^2+1}\right) - \left(2a^2+1+2a\sqrt{a^2+1}\right)}$$

$$= \frac{2ai}{-4a\sqrt{a^2+1}} = -\frac{i}{2\sqrt{a^2+1}}$$

Hence by Cauchy's residue theorem

I = 2πi (Sum of the residues within the contour c)

$$= 2\pi i \frac{-i}{2\sqrt{a^2+1}} = \frac{\pi}{\sqrt{a^2+1}}$$

Hence, $\int_0^{\pi} \frac{a d\theta}{a^2+sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$

Example 16: Evaluate by contour integration $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta) d\theta$

Solution: Let $I = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta) + isin(\sin\theta - n\theta) d\theta$

$$= \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)} d\theta = \int_0^{2\pi} e^{\cos\theta + isin\,\theta} . e^{-ni\theta} d\theta = \int_0^{2\pi} e^{e^{i\theta}} . e^{-in\theta} d\theta$$

Put $e^{i\theta} = z \; so\; that\; d\theta = \frac{dz}{iz}$

Then, $I = \int_c e^z . \frac{1}{z^n} . \frac{dz}{iz} = -\int_c \frac{e^z}{z^{n+1}} dz$

Pole is at $z = 0$ of order $(n + 1)$. It lies inside the unit circle

Residue of $f(z)\; at\; z = 0$ is $= \frac{1}{(n+1-1)!} \left[\frac{d^n}{dz^n} \left\{ z^{n+1} . \frac{-ie^z}{z^{n+1}} \right\} \right]$

$$= \frac{-i}{n!} \left[\frac{d^n}{dz^n} (e^z) \right]_{z=0} = \frac{-i}{n!} (e^z)_{z=0} = \frac{-i}{n!}$$

∴ By Cauchy's Residue theorem

$$I = 2\pi i \left(\frac{-i}{n!} \right) = \frac{2\pi}{n!}$$

Comparing real part of $\int_0^{2\pi} e^{\cos\theta} [\cos(\sin\theta - n\theta) + isin(\sin\theta - n\theta)] d\theta = \frac{2\pi}{n!}$

We have $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$

Check your progress

Evaluate the following integrals:

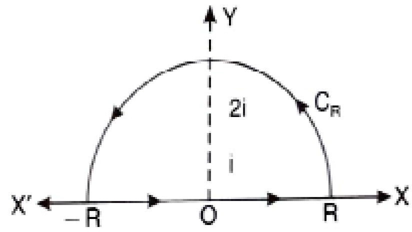
1. $\int_0^{2\pi} \frac{sin^2\theta}{a+bcos\,\theta} d\theta$ Ans. $\frac{2\pi}{b^2} \{ a - \sqrt{a^2 - b^2} \}, a > b > 0$
2. $\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta$ Ans. $\frac{2\pi}{\sqrt{5}} (3 - \sqrt{5})^n, n > 0$
3. $\int_0^{2\pi} \frac{4}{5+4\sin\theta} d\theta$ Ans. $\frac{8\pi}{3}$
4. $\int_0^{\pi} \frac{d\theta}{17-8\cos\theta}$ Ans. $\frac{\pi}{15}$
5. $\int_0^{\pi} \frac{d\theta}{a+bcos\,\theta}, where\; a > |b|$. Hence or otherwise evaluate $\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$. Ans. $\frac{\pi}{\sqrt{a^2-b^2}}$; π

6.5. Evaluate of $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x.

Such integrals can be reduced to contour integrals, if

(i) $f_2(x)$ has no real roots

(ii) The degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.



Procedure: Let $f(x) = \frac{f_1(x)}{f_2(x)}$

Consider $\int_C f(z)dz$, Where C is a curve, consisting of the upper half C_R of the circle $|z| = R$ and part of the real axis from $-R \; to \; R$.

If there are no poles of $f(z)$ on the real axis, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its circumference $C_R$in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's theorem of residues we have

$\int_C f(z)\,dz = 2\pi i \times$(sum of the residues of f(z) at the poles within C)

i.e. $\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i$ (sum of residues within C)

$\int_{-R}^{R} f(x)\,dx = -\int_{C_R} f(z)\,dz + 2\pi i$ (sum of residues within C)

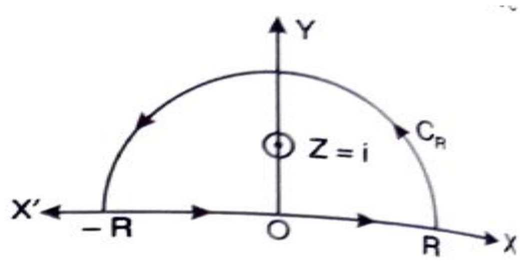
$\therefore \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = -\int_{C_R} f(z)\,dz + 2\pi i$ (sum of residues within C) …(1)

Now, $\lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz = \int_0^{\pi} f(Re^{i\theta})Rie^{i\theta}d\theta$

(1) reduces $\int_{-R}^{R} f(x)\,dx = 2\pi i$ (sum of residues within C)

Example 17: Evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+1}dx$

Solution: $\int_0^{\infty} \frac{\cos mx}{x^2+1}dx$



Consider the integral $\int_C f(z)\,dz$, where

$f(z) = \frac{e^{imz}}{z^2+1}$, taken round the closed contour C considting of the upper half of a large circle $|z| = R$ and the real axis from $-R \; to \; R$.

Poles of $f(z)$ are given by $z^2 + 1 = 0 \;\; i.e. \; z^2 = -1 \;\; i.e. \; z = \mp i$

The only pole which lies within the contour is at z = i .

The residue of $f(z)$ at $z = i = \lim_{z\to i} \frac{(z-i)e^{imz}}{z^2+1} = \lim_{z\to i} \frac{e^{imz}}{z+1} = \frac{e^{-m}}{2i}$

Hence by Cauchy's residue theorem, we have

$\int_{-R}^{R} f(z)\,dz = 2\pi i$ (sum of residues within C)

$$\int_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i \times \frac{e^{-m}}{2i} \int_{-R}^{R} \frac{e^{imz}}{x^2+1}\,dx = \pi e^{-m}$$

Equating real parts, we have

$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m} \int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx = \frac{\pi e^{-m}}{2}$

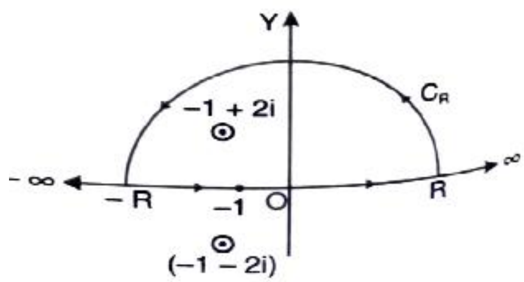
Example 18: Evaluate $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx$

Solution: Here, we have $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx$

Let us consider $\int_C \frac{z \sin \pi z}{z^2+2z+5}\,dz$

The pole can be determined by putting the denominator equal to zero.

$z^2 + 2z + 5 = 0 z = \frac{-2 \mp \sqrt{4-20}}{2} = -1 \mp 2i$



Out of two poles, only $z = -1 \mp 2i$ is inside the contour

Residue at $z = -1 \mp 2i$

$$\lim_{z \to -1 \mp 2i} (z+1-2i) \frac{z \sin \pi z}{z^2+2z+5}$$

$$= \lim_{z \to -1 \mp 2i} (z+1-2i) \frac{z \sin \pi z}{(z+1-2i)(z+1+2i)}$$

$$= \lim_{z \to -1 \mp 2i} \frac{z \sin \pi z}{(z+1+2i)} = \frac{(-1+2i) \sin \pi (-1+2i)}{(-1+2i+1+2i)}$$

$$= \frac{(-1+2i) \sin \pi (-1+2i)}{4i}$$

$\int_{-R}^{R} \frac{z \sin \pi z}{z^2+2z+5}\,dz = 2\pi i$ (Residue)

$$= 2\pi i \frac{(-1+2i) \sin \pi (-1+2i)}{4i} = \frac{\pi}{2}(2i-1)\sin(-\pi+2\pi i)$$

$$= \frac{\pi}{2}(2i-1)(-\sin 2\pi i) \left[\sin \begin{matrix} (-\pi+\theta) = -\sin(\pi-\theta) \\ = -sin\theta \end{matrix}\right]$$

$$= \frac{\pi}{2}(1-2i)\sin 2\pi i == \frac{\pi}{2}(1-2i) i \sinh 2\pi$$

$$= \frac{\pi}{2}(i+2)\sinh 2\pi$$

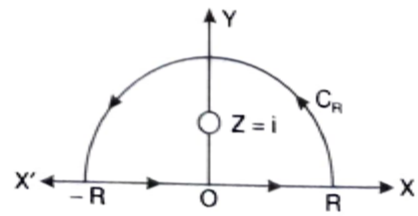
Hence, $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5} = \pi \sinh 2\pi$ (Taking real parts)

Example 19: Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx$

Solution: We consider $\int_C \frac{z^2-z+2}{(z^4+10z^2+9)} dz = \int_C \frac{z^2-z+2}{(z^2+1)(z^2+9)} dz = \int_C f(z)\, dz$

Poles at z = $\mp i$ and z = $\mp 3i$

Simple poles at z = i and z = 3i lie in the given ontour.



The residue at (z = i)

$$\lim_{z\to i}(z-i)\frac{z^2-z+2}{(z+i)(z-i)(z^2+9)}$$
$$= \lim_{z\to i}\frac{z^2-z+2}{(z+i)(z^2+9)} = \frac{i^2-i+2}{2i(-1+9)}$$
$$= \frac{-1-i+2}{2i(8)} = \frac{1-i}{16i} = \frac{1}{16i} - \frac{1}{16}$$

The residue at $(z = 3i)$

$$\lim_{z\to i}(z-3i)\frac{z^2-z+2}{(z+3i)(z-3i)(z^2+1)} = \lim_{z\to i}\frac{z^2-z+2}{(z+3i)(z^2+1)} = \frac{-9-3i+2}{(-9+1)(3i+3i)} = \frac{-7-3i}{48i}$$
$$= \frac{7}{48i} + \frac{1}{16}$$

By residue theorem,

$\int_C f(z)dz = 2\pi i$ [Res f(i) + Res f(3i)] = $2\pi i \left[\frac{1}{16i} - \frac{1}{16} + \frac{7}{48i} + \frac{1}{16}\right]$

$$= 2\pi i\left(\frac{1}{16i} + \frac{7}{48i}\right) = 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

i.e. $\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)\, dz = \frac{5\pi}{12}$

Now $R \to \infty, \int_{C_R} f(z)\, dz = 0$

$Hence, \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} = \frac{5\pi}{12}$

Example 20: Use contour integration to evaluate the real integral $\int_0^{\pi} \frac{dx}{(1+x^2)^3}$

Solution: Consider $\int_C f(z)dz$, where $f(z) = \frac{1}{(1+z^2)^3}$taken round the closed contour C consisting of real axis and upper half C_R of a large semi-circle $|z| = R$.

Poles of $f(z)$ are given by

$(1+z^2)^3 = 0 \; i.e. (z-i)^3(z+i)^3 = 0$

i.e. $z = \mp i$ are the poles each of order 3.

The only pole which lies within C is z = i of order 3.

$\therefore$ Residue of $\frac{1}{(z-i)^3} \cdot \frac{1}{(z+i)^3}$ $at \; (z = i)$

$$\frac{1}{2}\left[\frac{d^2}{dz^2}(z-i)^3 \cdot \frac{1}{(z-i)^3} \cdot \frac{1}{(z+i)^3}\right]_{z=i} = \frac{1}{2}\left[\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\right]_{z=i} = \frac{1}{2}\left[\frac{(-3)(-4)}{(z+1)^5}\right]_{z=i} = \frac{3}{16i}$$

Hence by Cauchy's residue theorem, we have

$$\int f(z)dz = 2\pi i \times sum \; of \; residues \; within \; C.$$

$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)\, dz = 2\pi i \times \frac{3}{16i} \int_{-R}^{R} \frac{1}{(x^2+1)^3} dx + \int_{C_R} \frac{1}{(z^2+1)^3} dz = \frac{3\pi}{8}$ ….. (1)

Now, $\left|\int_{C_R} \frac{1}{(z^2+1)^3} dz\right| \le \int_{C_R} \left|\frac{1}{(z^2+1)^3}\right| |dz| \le \int_{C_R} \frac{1}{(|z|^2+1)^3} |dz| = \int_0^{\pi} \frac{Rd\theta}{(R^2-1)^3}$

[since $z = Re^{i\theta}, |z| = Rd\theta$]

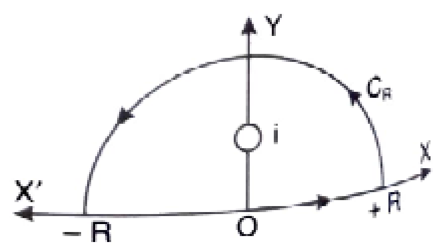
$= \frac{\pi R}{(R^2-1)^3}$, which $R \to 0 \; as \; R \to \infty$

Hence making $R \to \infty$, relation (1) becomes

$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx = \frac{3\pi}{8}$ or $\int_0^{\infty} \frac{1}{(x^2+1)^3} dx = \frac{3\pi}{16}$ Ans.

Example 21: Evaluate by the method of complex variables, the integral $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3} dx$

Solution: Consider $\int_C \frac{z^2}{(1+z^2)^3} dz$ where c is a closed contour consisting of the upper half C_R of a large circle $|z| = R$ and the real axis from -R to R.



Pole of $\frac{z^2}{(1+z^2)^3}$ are given by $(1+z^2)^3 = 0 \;\; or \; z^2 - 1$ $\quad z = \mp 1$

$\therefore \; z = i$ and $z = -i$ are the two poles each of order 3.But only $z = i$ lies within the contour.

To get residue at $z = i$, put $z = i + t$, then

$$\frac{z^2}{(1+z^2)^3} = \frac{(i+t)^2}{(1+(i+t)^2)^3} \frac{-1+2it+t^2}{[1-1+2it+t^2]^3}$$

$$= \frac{-1+2it+t^2}{(2it)^3\left(1+\frac{1}{2i}t\right)^3} = \frac{-1+2it+t^2}{(2it)^3}\left(1+\frac{1}{2i}t\right)^{-3}$$

$$= -\frac{1}{8i}\left(-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right)\left(1-\frac{3t}{2i}+\frac{(-3)(-4)}{2}\frac{t^2}{-4}+\cdots.\right)$$

$= -\frac{1}{8i}\left(-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right)\left(1-\frac{3t}{2i}-\frac{3t^2}{2}+\cdots.\right)$ Hence coefficient of

$\frac{1}{t}$ $is - \frac{1}{8i}\left(\frac{3}{2}-3+1\right)$ $or \frac{i}{8}\left(-\frac{1}{2}\right) or -\frac{i}{16}$ which is therefore the residue at $z = i$.

Hence by Cauchy's residue theorem we have

$\int f(z)\, dz = 2\pi i\, \times$(sum of the residues within c)

i.e. $\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)dz = 2\pi i\left(-\frac{1}{16}\right)$

$\int_{-R}^{R} \frac{x^2}{(1+x^2)^3}\, dx + \int_{C_R} \frac{z^2}{(1+z^2)^3}\, dz = \frac{\pi}{8}$ .... (1)

Now, $\left|\int_{C_R} \frac{z^2}{(1+z^2)^3}\, dz\right| \le \int_{C_R} \frac{|z^2|}{(|1+z^2|)^3}\, |dz| \le \frac{R^2}{(R^2-1)^3}\int_0^{\pi} R d\theta$

Since, $z = Re^{i\theta}, \ |dz| = Rd\theta$

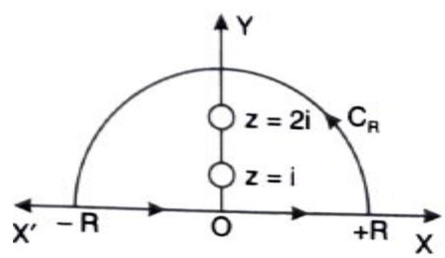
$= \frac{R^3\pi}{(R^2-1)^3}$, which $\rightarrow 0 \ as\ R \rightarrow \infty$

Hence, by making $R \rightarrow \infty$, equation (i) becomes $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\, dx = \frac{\pi}{8}$

Example 22: Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$.

Solution: We consider $\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C f(z)\, dz$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R\ to\ R$.



The integral has simple poles at $z = \mp i,\ z = \mp 2i$

Of which $z = i, 2i$ only lie inside C_R.

The residue at (z = i) = $\lim_{z\to i}(z-i)\frac{z^2}{(z+i)(z-i)(z^2+4)}$

$= \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{2i(-1+4)} = -\frac{1}{6i}$

The residue at (z = 2i) = $\lim_{z\to 2i}(z-2i)\frac{z^2}{(z+2i)(z-2i)(z^2+1)}$

$$= \lim_{z\to 2i}\frac{z^2}{(z+2i)(z^2+1)} = \frac{(2i)^2}{(2i+2i)(-4+1)} = \frac{1}{3i}$$

By theorem of residue;

$\int_c f(z)dz = 2\pi i\,[Res\, f(i) + Res\, f(2i)] = 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{3}$ ……….. (1)

i.e. $\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \frac{\pi}{3}$

Hence, by making $R \to \infty$ relation (1) becomes

$$\int_{-\infty}^{\infty} f(x)\,dx + \lim_{z\to\infty}\int_{C_R} f(z)\,dz = \frac{\pi}{3}$$

Now $R \to \infty, \int_{C_R} f(z)\,dz$ vanishes

For any point on CR as $|z| \to \infty, f(z) = 0$

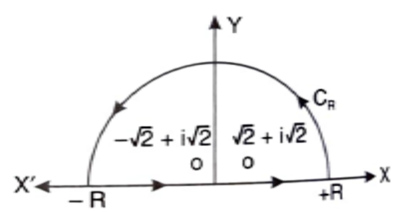
$$\lim_{|z|\to\infty}\int_{C_R} f(z)\,dz = 0, \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3}$$

Hence, $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$

Example 23: Using the complex variable techniques, evaluate the integral $\int_0^{\infty} \frac{1}{x^4+16}\,dx$

Solution: For $\int_0^{\infty} \frac{1}{x^4+16}\,dx$

Consider $\int_C f(z)\,dz\ where\ f(z) = \frac{1}{x^4+16}$



Taken around the closed contour consisting of real axis and upper half of C_R, i.e.$|z| = R$. Poles of f(z) are given by

$z^4 + 16 = 0\ \ i.e. z^4 = -16 = 16(\cos\pi + i\sin\pi)$

$z^4 = -16 = 16[\cos(2n+1)\pi + i\sin(2n+1)\pi]$

$$z = 2[\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}} = 2\left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right] = e^{i(2n+1)\frac{\pi}{4}}$$

If $n = 0$, $z_1 = 2e^{i\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}$

$$n = 1, \quad z_2 = 2e^{i\frac{3\pi}{4}} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} + i\sqrt{2}$$

$$n = 2, z_3 = 2e^{i\frac{5\pi}{4}} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - i\sqrt{2}$$

n = 3, $z_4 = 2e^{i\frac{7\pi}{4}} = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2} - i\sqrt{2}$

There are four poles, but only two poles at z_1 and z_2 lie within the contour.

Residue at $\left(z_1 = 2e^{i\frac{\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+16)}\right]_{z_1=2e^{i\frac{\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z_1=2e^{i\frac{\pi}{4}}} = \frac{1}{4\left(2e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{32e^{i\frac{\pi}{4}}}$

$$\frac{1}{32}e^{-i\frac{\pi}{4}} = \frac{1}{32}\left[\cos\frac{3\pi}{4} + isin\ \frac{3\pi}{4}\right] = \frac{1}{32}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

Residue at $\left(z_2 = 2e^{i\frac{3\pi}{4}}\right) = \frac{1}{4\left(2e^{i\frac{3\pi}{4}}\right)^3} = \frac{1}{32e^{i\frac{9\pi}{4}}}$

$$= \frac{1}{32}e^{-i\frac{9\pi}{4}} = \frac{1}{32}\left[\cos\frac{9\pi}{4} + isin\ \frac{9\pi}{4}\right] = \frac{1}{32}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

We know that $\int_C\ f(z)\,dz = 2\pi i$ (sum of residues at poles within C)

$\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i$ (sum of the residues)

$\int_{-R}^{R} \frac{1}{x^4+16}dz + \int_{C_R}\frac{1}{z^4+16}dz = 2\pi i$ (sum of the residues)

Now, $\left|\int_{C_R}\frac{1}{z^4+16}dz\right| \int_{C_R}\frac{1}{|z^4+16|}|dz| \leq \int_{C_R}\frac{1}{|z^4|+16}|dz|$

[since $z = Re^{i\theta}, |dz| = \left|Re^{i\theta}id\theta\right| = Rd\theta$]

$$\leq \int_0^{\pi}\frac{1}{R^4-16}Rd\theta \leq \frac{R}{R^4-16}\int_0^{\pi}d\theta$$

$\leq \frac{R\pi}{R^4-16} = \frac{\pi/R^3}{1-16/R^4}$ which $\rightarrow 0\ as\ R \rightarrow \infty$

Hence, $\int_{-R}^{R}\frac{1}{x^4+16}dz = 2\pi i$ (sum of the residues within contour) As $R \rightarrow \infty$

Hence $\int_{-R}^{R}\frac{1}{x^4+16}dz = 2\pi i$ (sum of the residues within contour)

$$\int_{-\infty}^{\infty}\frac{1}{x^4+16}dx = 2\pi i\left[\frac{1}{32}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{32}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{16}i\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{\pi}{16}i\left(-i\frac{2}{\sqrt{2}}\right) = \frac{\pi}{8\sqrt{2}} = \frac{\sqrt{2}\pi}{16}$$

$2\int_0^{\infty}\frac{1}{x^4+16}dx = \frac{\sqrt{2}\pi}{16}\int_0^{\infty}\frac{1}{x^4+16}dx = \frac{\sqrt{2}\pi}{32}$

Example 24: Using the complex variable techniques, evaluate the integral $\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$

Solution: $\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$

Consider $\int_C\ f(z)dz,\ where\ f(z) = \frac{1}{z^4+1}$

Taken around the closed contour consisting of real axis and upper half C_R, i.e. z = R.

Poles of f(z) are given by

$z^4 + 1 = 0 \; i.e. z^4 = -1 = (\cos\pi + i\sin\pi)$

$z^4 = (\cos(2n+1)\pi + i\sin(2n+1)\pi)$

$z = [\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{4}} = \left[\cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}\right]$

If $n = 0, \quad z_1 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{\pi}{4}}$

n = 1, $z_2 = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = e^{i\frac{3\pi}{4}}$

$n = 2, \qquad z_3 = \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$n = 3, \qquad z_4 = \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

There are four poles, but only two poles at z_1 and z_2 lie within the contour.

Residue at $\left(z = e^{i\frac{\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{i\frac{\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=e^{i\frac{\pi}{4}}} = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}}$

$= \frac{1}{4}e^{-i\frac{3\pi}{4}} = \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$

Residue at $\left(z = e^{i\frac{3\pi}{4}}\right) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{i\frac{3\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{9\pi}{4}}}$

$= \frac{1}{4}e^{-i\frac{9\pi}{4}} = \frac{1}{4}\left[\cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right] = \frac{1}{4}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$

$\int_C f(z)dz = 2\pi i$ (sum of residues at poles within c)

$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i$ (sum of the residues)

$\int_{-R}^{R} \frac{1}{x^4+1}dz + \int_{C_R} \frac{1}{z^4+1}dz = 2\pi i$ (sum of the residues)

Now, $\left|\int_{C_R} \frac{1}{z^4+1}dz\right| \le \int_{C_R} \frac{1}{|z^4+1|}|dz| \le \int_{C_R} \frac{1}{|z^4|+1}|dz|$

[since $z = Re^{i\theta}, |dz| = \left|Re^{i\theta}id\theta\right| = Rd\theta$]

$\le \int_0^{\pi} \frac{1}{R^4-1}Rd\theta \le \frac{R}{R^4-1}\int_0^{\pi} d\theta$

$\leq \frac{R\pi}{R^4-1} = \frac{\pi/R^3}{1-1/R^4}$ which $\rightarrow 0$ $as\ R \rightarrow \infty$

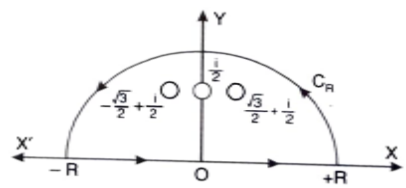
Hence $\int_{-R}^{R} \frac{1}{x^4+1} dz = 2\pi i$ (sum of the residues within contour)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$= \frac{\pi}{2} i \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2} i \left(-i\frac{2}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$. Hence the given integral = $\frac{\pi}{\sqrt{2}}$

Example 25: Using complex variable techniques, evaluate the real integral $\int_0^{\infty} \frac{dx}{1+x^6}$

Solution: Let $f(z) = \int_0^{\infty} \frac{dz}{1+z^6}$



We consider $\int_C \frac{dz}{1+z^6}$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of real axis from $-R\ to\ R$.

Poles of $f(z)$ are given by

$z^6 + 1 = 0\ i.e. z^6 = -1 = (\cos\pi + i\sin\pi)$

$z^6 = (\cos(2n+1)\pi + i\sin(2n+1)\pi) = e^{(2n+1)\pi i}$

$z = e^{\frac{2n+1}{6}\pi i} = \left[\cos(2n+1)\frac{\pi}{6} + i\sin(2n+1)\frac{\pi}{6}\right]$ where, $n = 0, 1,2,3, 4, 5$

If $n = 0$, $z = e^{\frac{\pi i}{6}} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$n = 1$, $z = e^{\frac{\pi i}{2}} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i$

n = 2, $z = e^{\frac{5\pi i}{6}} = \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{i}{2}$

$n = 3$, $z = e^{\frac{7\pi i}{6}} = \left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \frac{-\sqrt{3}}{2} - \frac{i}{2}$

$n = 4$, $z = e^{\frac{3\pi i}{2}} = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -i$

$n = 5$, $z = e^{\frac{11\pi i}{6}} = \left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$

Only, first three poles i.e.$e^{\frac{\pi i}{6}}, e^{\frac{\pi i}{2}}, e^{\frac{5\pi i}{6}}$ are inside the contour

Residue at $\left(z = e^{i\frac{\pi}{6}}\right) = \lim_{z \to e^{i\frac{\pi}{6}}} \frac{1}{\frac{d}{dz}(z^6+1)} = \lim_{z \to e^{i\frac{\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{-i\frac{5\pi}{6}}$

Residue at $\left(z = e^{i\frac{\pi}{2}}\right) = \lim_{z \to e^{i\frac{\pi}{2}}} \frac{1}{\frac{d}{dz}(z^6+1)} = \lim_{z \to e^{i\frac{\pi}{2}}} \frac{1}{6z^5} = \frac{1}{6} e^{-i\frac{5\pi}{2}}$

Residue at $\left(z = e^{i\frac{5\pi}{6}}\right) = \lim_{z\to e^{i\frac{5\pi}{6}}} \frac{1}{\frac{d}{dz}(z^6+1)} = \lim_{z\to e^{i\frac{5\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6}e^{-i\frac{25\pi}{6}}$

Sum of residues $= \frac{1}{6}\left[e^{-i\frac{5\pi}{6}} + e^{-i\frac{\pi}{2}} + e^{-i\frac{25\pi}{6}}\right] = \frac{1}{6}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} + 0 - i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{1}{6}(-2i) = -\frac{i}{3}$

$$\int_C \frac{dz}{1+z^6} = 2\pi i\,(residue) = 2\pi i\left(-\frac{i}{3}\right) = \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{dz}{1+x^6} = \frac{2\pi}{3}$$

$Hence, \int_0^{\infty} \frac{dz}{1+x^6} = \frac{\pi}{3}$

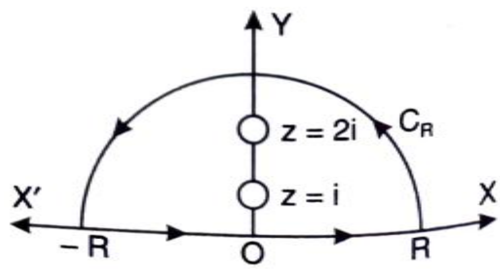
Example 26: Using complex variable, evaluate the real integral $\int_0^{\infty} \frac{\cos 3x dx}{(x^2+1)(x^2+4)}$

Solution: Let $f(z) = \frac{e^{3iz}}{(z^2+1)(z^2+4)}$

Poles are given by $(z^2+1)(z^2+4) = 0$

i.e. $z^2 + 1 = 0$ or $z = \mp i$

$z^2 + 4 = 0$ or $z = \mp 2i$



Let C be a closed contour consisting of the upper half C_R of a large circle $|z| = R$ and the real axis from $-R$ to R. the poles at $z = i$ and $z = 2i$ lie within the contour.

Residue at $(z = i) = \lim_{z\to i} \frac{(z-i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to i} \frac{e^{3iz}}{(z+i)(z^2+4)} = \frac{e^{-3}}{6i}$

Residue at $(z = 2i) = \lim_{z\to 2i} \frac{(z-2i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to 2i} \frac{e^{3iz}}{(z+2i)(z^2+1)} = \frac{e^{-6}}{12i}$

By theorem of Residue $\int_C f(z)dz = 2\pi i$ [sum of residues]

$$\int_{-R}^{R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} + \int_{C_R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} = 2\pi i\left[\frac{e^{-3}}{6i} + \frac{e^{-6}}{12i}\right]$$

$$\left[\int_{C_R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} = 0 \text{ } as \text{ } z = Re^{i\theta} \text{ } and \text{ } R \to \infty\right]$$

$$\int_{-R}^{R} \frac{e^{3ix}dx}{(x^2+1)(x^2+4)} = \pi\left[\frac{e^{-3}}{3} + \frac{e^{-6}}{6}\right]$$

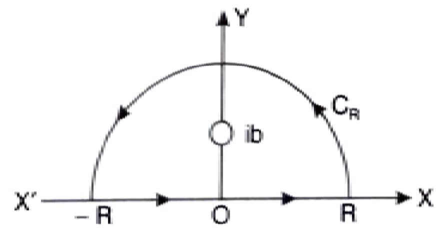
$$\int_0^{\infty} \frac{\cos 3x\, dx}{(x^2+1)(x^2+4)} = Real\ part\ of\ \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{3ix}dx}{(x^2+1)(x^2+4)}$$

$= Real\ part\ of\ \frac{\pi}{2}\left(\frac{e^{-3}}{3}+\frac{e^{-6}}{6}\right)$

Hence, given integral $= \frac{\pi}{2}\left(\frac{e^{-3}}{3}+\frac{e^{-6}}{6}\right)$

Example 27: Using the calculus of residues, evaluate the integral given by the following:

$\int_0^{\infty} \frac{\cos ax}{(z^2+b^2)^2} dx,\ a>0, b>0$



Solution: Consider the integral $\int_C\ f(z)\, dz$

Where $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$

Taken around the closed contour C consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R\ to\ R$.

Poles of $f(z)$ are given by $(z^2+b^2) = 0$

i.e., $z\ =\ ib\ and\ z\ =\ -ib$ are two poles of order two. The only pole which lies within the contour is $z\ =\ ib$ of order two.

Residue at $(z\ =\ ib)\ = \lim_{z\to ib} \frac{d}{dz}(z-ib)^2 \frac{e^{iaz}}{(z^2+b^2)^2} = \lim_{z\to ib} \frac{d}{dz} \frac{e^{iaz}}{(z+ib)^2}$

$$= \lim_{z\to ib} \frac{(z+ib)^2 iae^{iaz} - e^{iaz}2(z+ib)}{(z+ib)^4} = \lim_{z\to ib} \frac{[(z+ib)ia-2]e^{iaz}}{(z+ib)^3}$$

$$= \frac{[(2ib)ia-2]e^{iaz}}{(2ib)^3} = \frac{(-2ab-2)e^{-ab}}{-8ib^3} = \frac{(ab+1)e^{-ab}}{4ib^3}$$

Hence, by Cauchy's residue theorem, we have

$\int_c\ f(z)\, dz = 2\pi i\ \times$sum of the residues within C

$$\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \frac{(ab+1)e^{-ab}}{4ib^3}$$

$\int_{-R}^{R} \frac{e^{iaz}}{(x^2+b^2)^2} dx + \int_{C_R} \frac{e^{iaz}}{(z^2+b^2)^2} dz = \pi \frac{(ab+1)e^{-ab}}{2b^3}$ …….(1)

Now, $\left|\int_{C_R} \frac{e^{iaz}dz}{(z^2+b^2)^2}\right| \le \int_{C_R} \frac{|e^{iaz}||dz|}{(z^2+b^2)^2} \le \int_{C_R} \frac{|e^{iaz}||dz|}{(|z|^2-b^2)^2}$

$$\le \int_0^{\pi} \frac{e^{-aR\sin\theta} R d\theta}{(R^2+b^2)^2} \le \frac{R}{(R^2+b^2)^2} \int_0^{\pi} e^{-aR\sin\theta} d\theta \le \frac{R}{(R^2+b^2)^2} \int_0^{\frac{\pi}{2}} e^{-aR\frac{2\theta}{\pi}} d\theta$$

$\le \frac{R}{a(R^2+b^2)^2}(1-e^{-aR})$ which $\to 0, as\ R \to \infty$

Hence, by making $R \to \infty$, (1) becomes.

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{(x^2+b^2)^2}\,dx = \pi\frac{(ab+1)e^{-ab}}{2b^3}$$

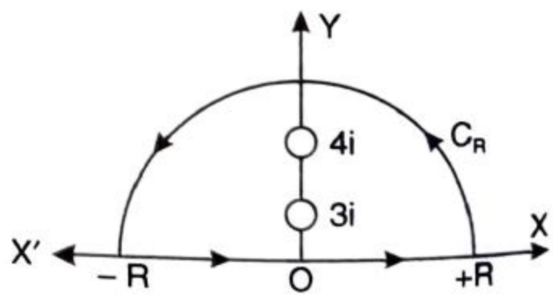
Equating real parts we have

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \pi\frac{(ab+1)e^{-ab}}{2b^3}$$

$$\int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \pi\frac{(ab+1)e^{-ab}}{2b^3}$$

Example 28: Using complex variable techniques, evaluate the real integral $\int_0^\infty \frac{\cos 2x}{(x^2+9)^2(x^2+16)}\,dx$

Solution: Consider the integral $\int_C f(z)\,dz$



Where, $f(z) = \frac{e^{2iz}}{(z^2+9)^2(z^2+16)}$

Taken around the closed contour C consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R\ to\ R$.

Poles of $f(z)$ are given by

$(z^2+9)^2(z^2+16) = 0$

$= (z+3i)^2(z-3i)^2(z+4i)(z-4i) = 0$

i.e. $z = 3i, -3i, 4i, -4i$

The poles which lie within the contour are $z = 3i$ of the second order and $z = 4i$ simple pole. Residue of $f(z)\ at\ z = 3i$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left\{(z-3i)^2\frac{e^{2iz}}{(z-3i)^2(z+3i)^2(z^2+16)}\right\}\right]_{z=3i} = \left[\frac{d}{dz}\left\{\frac{e^{2iz}}{(z+3i)^2(z^2+16)}\right\}\right]_{z=3i}$$

$$= \left[\frac{(z+3i)^2(z^2+16)2ie^{2iz} - e^{2iz}[2(z+3i)(z^2+16) + 2z(z+3i)^2]}{(z+3i)^4(z^2+16)^2}\right]_{z=3i}$$

$$= \left[\frac{(z+3i)(z^2+16)2ie^{2iz} - e^{2iz}[2(z^2+16) + 2z(z+3i)]}{(z+3i)^3(z^2+16)^2}\right]_{z=3i}$$

$$= \frac{6i \times 7 \times 2ie^{-6} - e^{-6}(2 \times 7 + 6i \times 6i)}{(6i)^3(7)^2} = \frac{e^{-6}[-84+22]i}{216 \times 47} = \frac{e^{-6}(-62)i}{216 \times 49} = \frac{i31e^{-6}}{108 \times 49}$$

Residue of f(z) at (z = 4i) = $\lim_{z\to 4i}(z-4i)\,\frac{e^{2iz}}{(z^2+9)^2(z-4i)(z+4i)}$

$$= \frac{e^{-8}}{(-16+9)^2(4i+4i)} = \frac{e^{-8}}{49 \times 8i} = \frac{-ie^{-8}}{392}$$

Sum of the residues = $\frac{i31e^{-6}}{108 \times 49} - \frac{ie^{-8}}{392}$

Hence by Cauchy's residue theorem, we have

$\int_C f(z)\, dz = 2\pi i \times$(sum of the residues within C)

i.e. $\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)\, dz = 2\pi i \times$(sum of residues)

or, $\int_{-R}^{R} \frac{e^{2ix}}{(x^2+9)^2(x^2+16)} dx + \int_{C_R} \frac{e^{2iz}}{(z^2+9)^2(z^2+16)} dz = 2\pi i \times$(sum of residues) ……(1)

Now let $R \to \infty$, so as to show that the second integral in above relation vanishes.

For any point on C_R. As $|z| \to \infty$

Let, $f(z) = \frac{1}{z^6} \frac{e^{2iz}}{\left(1+\frac{9}{z^2}\right)^2\left(1+\frac{16}{z^2}\right)}$

$$\lim_{|z|\to\infty} f(z) = 0 \int_{C_R} \frac{e^{2iz}}{(z^2+9)^2(z^2+16)} dz = 0$$

Hence, by making $R \to \infty$ relation (1) becomes

$\therefore \int_{-\infty}^{\infty} \frac{e^{2iz}}{(z^2+9)^2(z^2+16)} dx = 2\pi i \left[\frac{-i31e^{-6}}{108 \times 49} - i\frac{e^{-8}}{392}\right] = \frac{2\pi}{196}\left[\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right]$

Equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+9)^2(x^2+16)} = \frac{\pi}{98}\left[\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right]$$

$$\int_{0}^{\infty} \frac{\cos 2x}{(x^2+9)^2(x^2+16)} = \frac{\pi}{196}\left[\frac{31e^{-6}}{27} + \frac{e^{-8}}{2}\right]$$

6.6. Rectangular contour

Example 29: Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1} dx$

Solution: We consider $\int_C \frac{e^{az}}{e^z+1} dz = \int_C f(z)\, dz$

Where C is the rectangle $ABCD$ with vertices at $(R,0), (R,2\pi), (-R,2\pi)$ and $(-R,0)$

$f(z)$ has simple poles $e^z = -1$

$= cos\,(2n+1)\pi + isin\,(2n+1)\pi = e^{i(2n+1)\pi}$

$\Rightarrow\ z = (2n+1)\pi i,$ where n = 0, ±1, ±2, …

The only pole inside the rectangle is z = πi. Therefore, By Residue theorem

$$\int_C f(z)dz = 2\pi i \ \ Residue\, f(\pi i) = 2\pi i \left[\frac{e^{az}}{\frac{d}{dz}(e^z+1)}\right]_{z=\pi i} \quad \left[R(a) = \frac{\phi(a)}{\psi'(a)}\right]$$

$$= 2\pi i \ \left[\frac{e^{az}}{e^z}\right]_{z=\pi i} = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i} \begin{bmatrix} e^{\pi i} = \cos\pi + i\sin\pi \\ = -1+0 \\ = -1 \end{bmatrix}$$

Also $\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$ ………. (1)

$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^{0}(-R+iy)idy + \int_{-R}^{R} f(x)dx$…(2)

$[z = R+iy \ along\ AB, z = x+2\pi i\ along\ BC, z = -R+iy\ along\ CD\ and\ z = x\ along\ DA]$.

$$\int_C f(z)\,dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}}dy + \int_R^{-R} \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}}dx + i\int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^{R}\frac{e^{ax}}{e^x+1}dx$$

Now for any two complex number z_1, z_2, $|z_1| \geq |z_2|$

We have, $|z_1+z_2| \geq |z_1| - |z_2|$

So that, $\left|e^{R+iy}+1\right| \geq e^R - 1$. Also $\left|e^{a(R+iy)}\right| = e^{aR}$

∴ For the integrand of first integral in (2), we have

$$\left|\frac{e^{a(R+iy)}}{e^{R+iy}+1}\right| \leq \frac{e^{aR}}{e^R-1} \quad which \to 0\ as\ R \to \infty\ [\because a > 1]$$

Similarly, for the integrand of the third integral in (2), we get

$$\left|\frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\right| \leq \frac{e^{-aR}}{1-e^{-R}} \quad which\ also \to 0\ as\ R \to \infty\ [\because a < 0]$$

Hence as R → since the first and third integrals in (2) approach zero, we get

$$\int_C f(z)\,dz = -e^{2a\pi i}\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx$$

$= (1-e^{2a\pi i})\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx$ …………… (3)

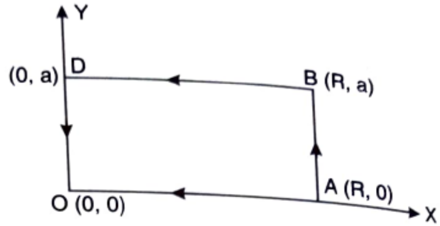
Thus, from (1) and (3) we obtain

$$\left(1-e^{2a\pi i}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i} \ or \ \int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx = \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}}$$

$\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$

Example 30: By integrating e^{-z^2} round the rectangle whose vertices are $0, R, R+ia, ia$. Show that (i). $\int_0^{\pi} e^{-x^2}\cos 2ax dx = \frac{e^{-a^2}}{2}\sqrt{\pi}$ and (ii). $\int_0^{\pi} e^{-x^2}\sin 2ax dx = e^{-a^2}\int_0^a e^{y^2}dy$

Solution: (i). Let $f(z) = e^{-z^2}$



$$\int_C f(z)dz = \int_C e^{-z^2}\, dz$$

Here C is the closed contour, a rectangle $OABD$.

Since $f(z)$ is analytic within and on the contour. There is no pole within rectangle $OABD$. Hence by Cauchy's residue theorem we have

$\int_{OABD} e^{-z^2}dz = 0$ or, $\int_{OA} e^{-z^2}dz + \int_{AB} e^{-z^2}dz + \int_{BD} e^{-z^2}dz + \int_{DO} e^{-z^2}\, dz = 0$.....(1)

Since, on $OA, z = x,\ dz = dx$. And On $AB, z = R + iy, dz = idy$

Also, On $BD, z = x + ia,\ dz = dx$ and On $DO, z = iy, dz = idy$

Hence (1) becomes

$\int_0^R e^{-x^2}dx + \int_0^a e^{-(R+iy)^2} idy + \int_R^0 e^{-(x+iy)^2}dx + \int_a^0 e^{-(iy)^2}\, idy = 0$.....(2)

Now, $\left|\int_0^a e^{-(R+iy)^2} idy\right| \le \left|\int_0^a e^{-(R+iy)^2}\right| |idy| \le \int_0^a e^{-R^2+y^2}\, dy \le \int_0^a e^{-R^2+a^2}dy \le e^{-R^2+a^2}.a = 0$

As $R \to \infty$

Hence by making $R \to \infty$ equation (2) becomes

$$\int_0^{\infty} e^{-(x+iy)^2}dx = \int_0^{\infty} e^{-x^2}\, dx - i\int_0^a e^{y^2}dy$$

$\Rightarrow \int_0^{\infty} e^{(-x^2+a^2-2aix)}dx = \frac{\sqrt{\pi}}{2} - i\int_0^a e^{y^2}dy$

$$\int_0^{\infty} e^{(-x^2+a^2)}.e^{-2aix}dx = \frac{\sqrt{\pi}}{2} - i\int_0^a e^{y^2}dy$$

$$\int_0^{\infty} e^{(-x^2+a^2)}.(\cos 2ax - i\sin 2ax)dx = \frac{\sqrt{\pi}}{2} - i\int_0^a e^{y^2}dy$$

$$\int_0^{\infty} e^{(-x^2)}.(\cos 2ax - i\sin 2ax)dx = \frac{\sqrt{\pi}}{2}e^{-a^2} - ie^{-a^2}\int_0^a e^{y^2}dy$$

Now we equating real and imaginary parts we have

(i) $\int_0^{\infty} e^{-x^2}\cos 2ax\, dx = \frac{\sqrt{\pi}}{2}e^{-a^2}$

(ii) $\int_0^{\infty} e^{-x^2}\sin 2ax\, dx = e^{-a^2}\int_0^a e^{y^2}dy$

Check your progress

Evaluate the following:

1. $\int_0^\infty \frac{1}{1+x^2}\,dx$ Ans. $\frac{\pi}{2}$
2. $\int_{-\infty}^\infty \frac{1}{(x^2+1)^2}\,dx$ Ans. $\frac{\pi}{2}$
3. $\int_0^\infty \frac{x^3\sin x}{(x^2+a^2)(x^2+b^2)}\,dx$ Ans. $\frac{\pi}{2(a^2+b^2)}[a^2e^{-a}-b^2e^{-b}]$
4. $\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx\,, a>b>0$ Ans. $\frac{\pi}{(a^2+b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$
5. Show that $\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a}$
6. Evaluate $\int_{-\infty}^\infty \frac{\cos 3x}{(x^2+1)(x^2+4)}\,dx$ Ans. $\frac{\pi}{3}\left(\frac{e^{-3}}{1}-\frac{e^{-6}}{2}\right)$
7. Show that $\int_{-\infty}^\infty \frac{x^3\sin x}{(x^2+a^2)}\,dx = -\frac{\pi}{4}(a-2)a^{-a}, a>0$

Evaluate the following

8. $\int_{-\infty}^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx\,, m>0, a>0$ Ans. $\frac{\pi}{a^2}(2-e^{-ma})$
9. $\int_0^\infty \frac{x^2}{x^6+1}\,dx$ Ans. $\frac{\pi}{6}$
10. $\int_0^\infty \frac{x\sin ax}{x^4+a^4}\,dx$ Ans. $\frac{\pi}{2a^2}e^{-\frac{a^2}{\sqrt{2}}}\sin\frac{a^2}{\sqrt{2}}$
11. $\int_0^\infty \frac{x^6}{(x^4+a^4)^2}\,dx$ Ans. $\frac{3\pi\sqrt{2}}{16a}, a>0$
12. $\int_0^\infty \frac{\cos x^2+\sin x^2-1}{x^2}\,dx$ Ans. 0
13. $\int_0^\infty \frac{\cos mx}{x^4+x^2+1}\,dx$ Ans. $\frac{\pi}{\sqrt{3}}\,sin\,\frac{1}{2}\left(m+\frac{\pi}{3}\right)e^{-\frac{1}{2}m\sqrt{3}}$
14. $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx$ Ans. $\pi\log 2$
15. Using contour integration, show that $\int_0^\infty \frac{x^6\,dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}\pi}{16a}$, (a > 0).
16. Using method of contour integration, evaluate $\int_0^\infty \frac{x\sin ax}{x^4+4}\,dx$ Ans. $\frac{\pi}{8}e^{-a}\sin a$
17. Integration $\frac{e^{iz}}{z+a}$ along the boundary of the square defined by $x\ 0, x\ R, y=\ 0, y=\ R$.
    Prove that (i) $\int_0^\infty \frac{\cos x}{x+a}\,dx = \int_0^\infty \frac{xe^{-ax}}{1+x^2}\,dx$
    (ii) $\int^\infty \frac{\sin x}{x+a} = \int_0^\infty \frac{e^{-ax}}{1+x^2}\,dx$
18. Evaluate using Cauchy's integral formula $\oint_c \frac{\cos\pi z}{z^2-1}\,dz$ around a rectangle
    (i) 2 ± i, -2 ± i Ans. 0 (ii) –i, 2 – i , 2 ± i and i
19. By integrating $\frac{e^{iaz^2}}{\sinh\pi z}$ round the rectangle with vertices $\mp R\mp\frac{i}{2}$, show that

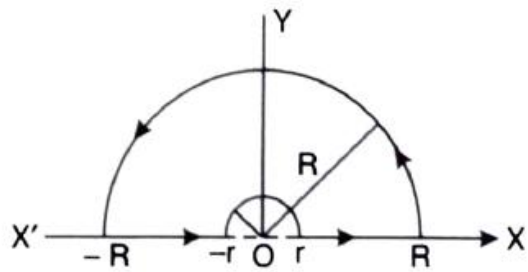
$$\int_0^\infty \frac{\cos ax^2\cosh ax}{\cosh\pi x}\,dx = \frac{1}{2}\cos\left(\frac{a}{4}\right)\ and\ \int_0^\infty \frac{\sin ax^2\cosh ax}{\cosh\pi x}\,dx = \frac{1}{2}\sin\left(\frac{a}{4}\right)\ (0<a\le\pi)$$

6.7. Indented semi-circular Contour

When the integrand has a simple pole on real axis, it is deleted from the region by indenting the contour (a small semi-circle having pole is drawn)

Example 31: By contour integration, prove that $\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}$

Solution: Consider the integral $\int_C \frac{e^{miz}}{z} dz$



When C is a large semi circle $|z| = R$ indented at $z = 0$ (pole), let r be the radius of indentation.

There is no singularity within the given contour.

Hence by Cauchy Theorem.

$$\int_C \frac{e^{miz}}{z} dz = 0$$

i.e., $\int_{-R}^{-r} \frac{e^{mix}}{x} dx + \int_{C_r} \frac{e^{miz}}{z} dz + \int_r^R \frac{e^{mix}}{x} dx + \int_{C_R} \frac{e^{miz}}{z} dz = 0$ …………(1)

Substituting z for x in the first integral and combining it with the third integral, we get

$$\int_r^R \frac{e^{mix} - e^{-mix}}{x} dx + \int_{C_2} \frac{e^{miz}}{z} dz + \int_{C_1} \frac{e^{miz}}{z} dz = 0 \left[z = Re^{i\theta}, \quad dz = Rie^{i\theta} d\theta\right]$$

$2i \int_r^R \frac{\sin mx}{x} dx + \int_{C_2} \frac{e^{miz}}{z} dz + \int_{C_1} \frac{e^{miz}}{z} dz = 0$ ……..(2)

Now, $\int_{C_2} \frac{e^{miz}}{z} dz = \int_{C_2} \frac{1}{Z} dz + \int_{C_2} \frac{e^{imx}-1}{z} dz$ ……………….(3)

On C_2: $\quad z = re^{i\theta}$

Therefore, $\int_c \frac{1}{z} dz = \int_\pi^0 \frac{re^{i\theta} i d\theta}{re^{i\theta}} = \int_0^\pi i d\theta = -i\pi$

Also, $\left|\int_{C_2} \frac{e^{imx}-1}{z} dz\right| \leq M \int_{C_2} \frac{|dz|}{|z|} = \pi M$

When M is the maximum value on C_2 of $\left|e^{imx} - 1\right| = \left|e^{imr(\cos\theta + i\sin\theta)} - 1\right|$

Clearly, $M = 0 \; as \; r = 0$

From (3), $\int_{C_2} \frac{e^{imz}}{z} dz = -i\pi$ …………… (4)

Putting $z = Re^{i\theta}$ in the integral over C_1, we get

$$\int_{c_1} \frac{e^{imz}}{z} dz = \int_0^\pi \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} R\, e^{i\theta} i d\theta = i \int_0^\pi e^{imR\cos\theta} . e^{-mR\sin\theta} d\theta$$

Since $\left|e^{imR\cos\theta}\right| \le 1$

$$\left|\int_{C_1} \frac{e^{imz}}{z} dz\right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} e^{-mR\sin\theta} d\theta$$

Also, $\frac{\sin\theta}{\theta}$ continuously decreases from 1 to $\frac{2}{\pi}$ $as\ \theta$ increases from 0 to $\frac{\pi}{2}$.

for $0 \le \theta \le \frac{\pi}{2}, \frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ $or\ \sin\theta \ge \frac{2\theta}{\pi}$

$$\left|\int \frac{e^{imz}}{z} dz\right| \le 2\int_0^{\frac{\pi}{2}} e^{-2mR\,\theta/\pi} d\theta = \left[-\frac{\pi}{mR} e^{-2mR\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{mR}(1 - e^{-mR})$$

As $R \to \infty, \frac{\pi}{mR}(1 - e^{-mR}) \to 0$

$$\int_{C_1} \frac{e^{imz}}{z} dz = 0$$

Hence from (2), on taking the limit as $r = 0\ and\ R = 0$, we get

$$2i\int_0^{\infty} \frac{\sin mx}{x} dx - i\pi = 0 \quad or \quad \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

Example 32: Evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$.

Solution: Consider the integral $\int_C \frac{e^{iz}}{z} dz$

Where C is a large semi circle $|z| = R$ indented at $z = 0$ (pole), let r be the radius of indentation. There is no singularity within the given contour.

Hence, by Cauchy theorem, $\int_C \frac{e^{iz}}{z} dz = 0$

i.e. $\int_{-R}^{-r} \frac{e^{iz}}{z} dz + \int_{C_r} \frac{e^{iz}}{z} dz + \int_r^R \frac{e^{iz}}{z} dz + \int_{C_R} \frac{e^{iz}}{z} dz = 0$….. (1)

Now, $\int_{C_R} \frac{e^{iz}}{z} dz = \int_0^{\pi} \frac{e^{iR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} . Re^{i\theta} i d\theta \quad [z = Re^{i\theta}]$

$$= i\int_0^{\pi} e^{iR(\cos\theta + i\sin\theta)} d\theta$$

$$\left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{-R\cos\theta + iR\sin\theta}\right| = e^{-R\sin\theta}$$

$$\left|\int_{C_R} \frac{e^{iz}}{z} dz\right| \le \int_0^{\pi} e^{-R\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} e^{\frac{-2R\theta}{\pi}} d\theta \begin{pmatrix} 0 < \theta \le \frac{\pi}{2} \\ \frac{\sin\theta}{\theta} \ge \frac{2}{\pi} \end{pmatrix}$$

$$= \frac{-2\pi}{2R}\left[e^{\frac{-2R\theta}{\pi}}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{R}(1 - e^{-R}) = 0 \qquad \text{as } R \to \infty$$

$\int_{C_r} \frac{e^{iz}}{z} dz = i\int_{\pi}^{0} e^{ir(\cos\theta + i\sin\theta)} d\theta = i\int_{\pi}^{0} d\theta \qquad$ as $r \to \infty$

$= -i\pi$

Equation (1) is reduced to

$\int_0^{\infty} \frac{e^{ix}}{x} dx + 0 + \int_{-\infty}^{0} \frac{e^{ix}}{x} dx - i\pi = 0 \quad [As\ r \to \infty, R \to \infty]$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\cos x + isinx}{x} dx = i\pi$$

Equating imaginary parts, we get

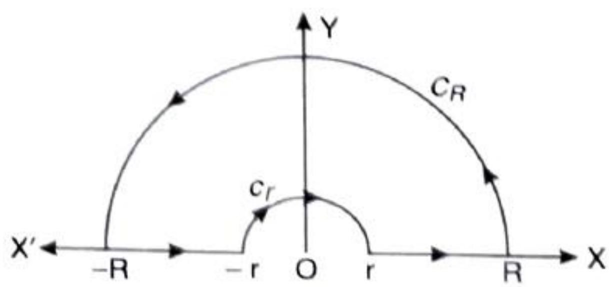
$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$ or $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Example 33: Show that, if $a \geq b \geq 0$ then $\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b-a)$

Solution: Consider the integral $\int_c f(z)\, dz$

Where, $f(z) = \frac{e^{i2az} - e^{i2bz}}{z^2}$

And C is a large semi-circle $|z| = R$ indented at $z = 0$ (pole), let r be the radius of indentation. Now there is no singularity within the given contour.



$\int_c f(z)\, dz = 0$ (By Cauchy Integral Theorem)

$\int_{-R}^{-r} f(x)\, dx + \int_{C_R} f(z)\, dz + \int_r^R f(x)\, dx + \int_{C_R} f(z)\, dz = 0 \qquad$ ………… (1)

Now, $\left|\int_{C_R} f(z)\, dz\right|$

$$\leq \int_{C_R} \frac{|e^{2iaz} - e^{2ibz}|}{|z|^2} |dz|$$

$$\leq \int_{C_R} \frac{|e^{2iaz}| + |e^{2ibz}|}{|Z|^2} |dz|$$

$$= \int_0^{\pi} \frac{e^{-2aR\sin\theta} + e^{-2bR\sin\theta}}{R^2} R d\theta$$

$\leq \frac{2}{R}\int_0^{\frac{\pi}{2}} \left[e^{\frac{-4aR\theta}{\pi}} + e^{\frac{-4bR\theta}{\pi}}\right] d\theta \qquad$ [By Jordan's inequality]

$$= \frac{2}{R}\left[\frac{\pi}{4aR}(1 - e^{-2aR}) + \frac{\pi}{4aR}(1 - e^{-2bR})\right]$$

$= 0$ or $R \to \infty$

We have, $\lim_{z\to 0}\{z(f(z))\} = \lim_{z\to 0}\left\{z\frac{e^{2iaz}-e^{2ibz}}{z^2}\right\}$

$$= \lim_{z\to 0}\{2i(a-b) - 2(a^2-b^2)z^2 \ldots.\} = 2i(a-b)$$

$$\lim_{r\to 0}\int_c f(z)dz = -i(\pi - 0) \times 2i(a-b) = -2\pi(b-a)$$

Hence, by making $R \to \infty$ *and* $r \to \infty$, equation (1) reduces to

$$\int_{-\infty}^{0} f(x)dx - 2\pi(b-a) + \int_{0}^{\infty} f(x)\,dx + 0$$

$$= 0 \;+\; \int_{-\infty}^{\infty} f(x)\,dx = 2\pi(b-a)$$

$$\int_{-\infty}^{\infty} \frac{e^{2iax} - e^{i2bx}}{x^2}dx = 2\pi(b-a)$$

$$\int_{-\infty}^{\infty} \frac{(\cos 2ax + i\sin 2ax) - (\cos 2bx + i\sin 2bx)}{x^2}dx = 2\pi(b-a)$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}dx = 2\pi(b-a)$$

$$\left[\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx\,, if\ f(x) is\ even\ function\right]$$

Hence, $\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2}dx = \pi(b-a)$

Example 34: Using contour integration method, prove the integral

(i) $\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a}$. (0 < a < 1)

(ii) $\int_0^\infty \frac{x^{a-1}}{1-x}dx = \pi\cot\pi a.$

Solution: Let the integral be $\int_c f(z)\,dz$, where $f(z) = \frac{z^{a-1}}{1-z}$

Taken around the closed contour C consisting of real axis from $-R$ *to* R, and upper half of a circle $|z| = R$ indented at $z = 0, z = 1$, the radii of indentations being r *and* r' respectively.

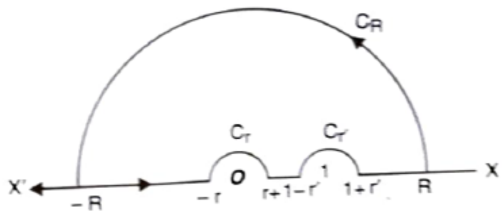
The singularities of $f(z)$ are $z = 0, z = 1$ which have been avoided by the indentation, so there no singularities within the contour.

Hence, by Cauchy's residue theorem, we have

$$\int_{-R}^{-r} f(x)\,dx + \int_{C_r} f(z)\,dz + \int_{r}^{1-r} f(x)\,dx + \int_{C_r} f(z)\,dz + \int_{1+r'}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 0 \quad (1)$$

Since $\lim_{z\to\propto} zf(z) = \lim_{z\to\propto} z\frac{z^{a-1}}{1-z} = \lim_{z\to\propto} \frac{z^a}{1-z} = 0, 0 < a < 1.$

$\lim_{R\to\propto} \int_{C_R} f(z)\,dz = i(\pi - 0)0 = 0$



Again $\lim_{z\to 0}\{zf(z)\} = \lim_{z\to 0}\left\{\frac{z.z^{a-1}}{1-z}\right\} =$
$\lim_{z\to 0}\left(\frac{z^a}{1-z}\right) = 0, a > 0\,.$

$$\lim_{Z\to 0}\int_{C_R} f(z)\,dz = -i(\pi - 0).0 = 0$$

Also, $\lim_{r\to 1}\{(z-1)f(z)\} = \lim_{z\to 1}\left\{(z-1)\frac{z^{a-1}}{1-z}\right\} = -1$

$$\lim_{r\to 0}\int_{C_r} f(z)\,dz = -(\pi - 0)(-1) = i\pi$$

Hence making $R \to \infty, r \to 0, r' \to 0$, we have from (1)

$$\int_{-\propto}^{\propto} f(x)\,dx + \int_0^1 f(x)dz + \pi i + \int_1^{\propto} f(x)\,dx = 0$$

$$or, \int_{-\propto}^{\propto} f(x)dx + \pi i = 0$$

$\int_{-\propto}^{0} \frac{x^{a-1}}{1-x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$ or, $-\int_0^{-\propto} \frac{x^{a-1}}{1-x}dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}dx = -\pi i$

Putting –x for x in the first integral, we have

$$\int_0^{\propto} \frac{(-1)^{a-1}x^{a-1}}{1+x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$$

$$\int_0^{\propto} \frac{(e^{ix})^{a-1}x^{a-1}}{1+x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$$

$$\int_0^{\propto} \frac{e^{-ix}e^{iax}x^{a-1}}{1+x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$$

$-e^{iax}\int_0^{\propto} \frac{x^{a-1}}{1+x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$ [since, $e^{-ix} = -1$]

$$-(\cos a\pi + i\sin a\pi)\int_0^{\propto} \frac{x^{a-1}}{1+x}dx + \int_0^{\propto} \frac{x^{a-1}}{1-x}dx = -\pi i$$

Equating imaginary and real parts, we have

$-sina\pi \int_0^{\propto} \frac{x^{a-1}}{1+x} dx = -\pi \quad \Rightarrow \quad \int_0^{\propto} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin a\pi}$…..(1)

And $-\cos a\pi \int_0^{\infty} \frac{x^{a-1}}{1+x} dx + \int_0^{\infty} \frac{x^{a-1}}{1-x} dx$

$-\cos a\pi \int_0^{\infty} \frac{x^{a-1}}{1+x} dx = -\int_0^{\infty} \frac{x^{a-1}}{1-x} dx \Rightarrow \cos a\pi \times \frac{\pi}{\sin a\pi} = \int_0^{\infty} \frac{x^{a-1}}{1-x} dx$ [from (1)]

Thus, $\int_0^{\infty} \frac{x^{a-1}}{1-x} dx = \pi \cot a\pi$

Check your progress

Using the method of contour integration, evaluate the following:

1. $\int_0^{\infty} \frac{\cos x}{x} dx$ Ans. 0
2. $\int_0^{\infty} \frac{\log(1+x^2)}{x^{1+\alpha}} dx,\ 0 < \alpha < 1$ Ans. 0
3. $\int_0^{\infty} \frac{\log x}{(a+x)^3} dx$ Ans. ½
4. $\int_{-\infty}^{\infty} \frac{1}{x^3+1}$ Ans. $\frac{\pi}{\sqrt{3}}$
5. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$ Ans. $\frac{\pi}{2}$
6. $\int_0^{\infty} \frac{1}{x+1} dx$ Ans. $\frac{\pi}{2}$
7. $\int_0^{\infty} \frac{\log x}{1+x^2} dx$ Ans. $\frac{\pi^3}{8}$

Conclusion: After the study of this chapter we are able to get evaluation of real definite integrals by contour integration, Integration round the unit circle of the type: $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$, evaluate of $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x, Rectangular contour and Indented semi- circular contour.

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Open University

Bachelor of Science
DCEMM -113
Function of Complex Variables

Block

4 Conformal Representation

Unit 7

Conformal Representation

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Block -4

Conformal Representation

Conformal mapping is an important and useful tool in complex analysis. To draw a curve of complex variable on plane we take two axes i.e., one real axis and the other imaginary axis. A number of point are plotted on plane, by taking different value of (different values). The curve is drawn by joining the plotted points. The diagram obtained is called Argand diagram in -plane. But a complex function involves four variables A figure of only three dimensions is possible in a plane. A figure of four dimensional regions for is not possible. So, we choose two complex planes plane and plane. In the plane we plot the corresponding point By joining these points we have a corresponding curve in plane.

Unit -7

Conformal Representation

7.1. INTRODUCTION

Conformal mapping is an important and useful tool in complex analysis. To draw a curve of complex variable (x, y) on z −plane we take two axes i.e., one real axis and the other imaginary axis. A number of point (x, y) are plotted on z −plane, by taking different value of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called Argand diagram in z-plane.

But a complex function $w = f(z)$ $i.e., (u + iv) = f(x + iy)$ involves four variables x, y and u, v.

A figure of only three dimensions (x, y, z) is possible in a plane. A figure of four dimensional region for x, y, u, v is not possible.

So, we choose two complex planes z −plane and w −plane. In the w −plane we plot the corresponding point $w = u + iv$. By joining these points we have a corresponding curve C' in w −plane.

7.2. Objectives

After studying this unit we should be able to:

- Define Mapping and Conformal Mapping ;
- Translation $w = z + c$ and Rotation $w = ze^{i\theta}$;
- Magnification and Rotation $w = cz$;
- Inversion and Reflection;
- Mobius Transformation;
- Invariant points of Bilinear Transformation;
- Properties of Bilinear Transformation;
- Methods to find Bilinear Transformation;
- Inverse point with respect to a circle;
- Transformation $w = z^2$;

Conformal mapping is an important and useful tool in complex analysis.To draw a curve of complex variable (x, y) on z −plane we take two axes i.e., one real axis and the other imaginary

axis. A number of point (x, y) are plotted on z −plane, by taking different value of z (different values of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called Argand diagram in z-plane.

But a complex function $w = f(z)$ $i.e.$, $(u + iv) = f(x + iy)$ involves four variables x, y and u, v. We know that a figure of only three dimensions (x, y, z) is possible in a plane. A figure of four dimensional region for x, y, u, v is not possible.

So, we choose two complex planes z −plane and w −plane. In the w −plane we plot the corresponding point $w = u + iv$. By joining these points we have a corresponding curve C' in w −plane. For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this "transformation or mapping of z −plane into w −plane". If a pointz_0 maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along a curve C in z −plane, the point $P'(u, v)$ will move along a corresponding curve C' in w −plane, then we say that a curve C in the z −plane is mapped into the corresponding curve C' in the w −plane by the relation $w = f(z)$.

7.3. Mapping

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this "transformation or mapping of z −plane into w −plane". If a point maps into the point w_0, w_0 is also known as the image of z_0.

If the point $P(x, y)$ moves along a curve C in z −plane, the point $P'(u, v)$ will move along a corresponding curve C' in w −plane, then we say that a curve C in the z −plane is mapped into the corresponding curve C' in the w −plane by the relation $w = f(z)$.

Example 1: Transform the rectangular region $ABCD$ in z-plane bounded by x = 1, x = 3; y = 0 $and \quad y = 3$. Under the transformation $w = z + (2 + i)$.

Solution: Here, $\quad w = z + (2 + i)$

$u + iv = x + iy + (2 + i) = (x + 2) + i(y + 1)$

By equating real and imaginary quantities, we have $u = x + 2$ and $v = y + 1$

$z-plane$	$w-plane$	$z-plane$	$w-plane$
X	$u = x + 2$	y	$v = y + 1$
1	$= 1 + 2 = 3$	0	$= 0 + 1 = 1$
3	$= 3 + 2 = 5$	3	$= 3 + 1 = 4$

Here the lines $x = 1, x = 3$; $y = 0$ and $y = 1$ in the z −plane are transformed onto the line $u = 3, u = 5$; $v = 1$ and $v = 4$ in the w −plane. The region $ABCD$ in z −plane is transformed into the region $EFGH$ in w −plane.

Example 2: Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$.

Solution: $w = z^2 = (x + iy)^2, u + iv = x^2 - y^2 + 2ixy$

This gives $u = x^2 - y^2 \ and\ v = 2xy$

Table of (x, y) and (u, v)

x	2	2.5	3	3.5	4	4.5	5
$y = \mp\sqrt{x^2 - 4}$	0	±1.5	±2.2	±2.9	±3.5	±4.1	±4.6
$u = x^2 - y^2$	4	4	4	4	4	4	4
$v = 2xy$	0	±7.5	±13.2	±20.3	±28	±36.0	±46

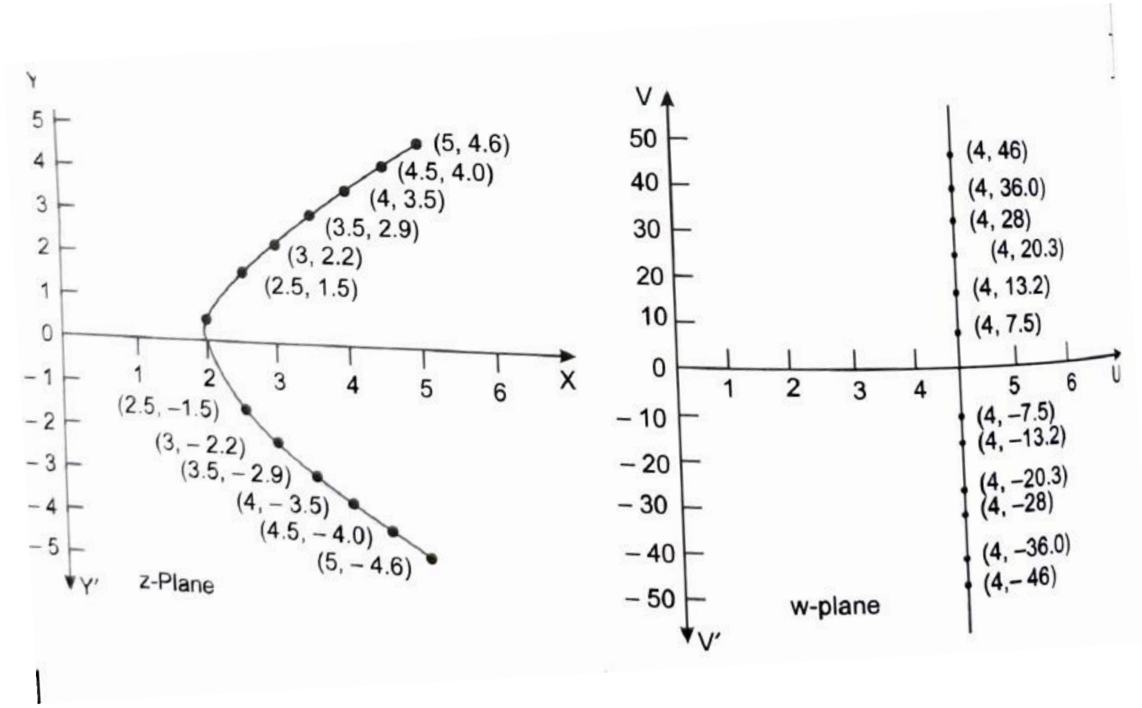


Image of the curve $x^2 - y^2 = 4$ is a straight line, $u = 4$ parallel to the v-axis in w −plane axis.

7.4. Conformal Mapping

Let two curves C, C_1 in the z −plane intersect at the point P and the corresponding plane $C', C_1{}'$ in the w −plane intersect at P'. If the angle of intersection of the curves at P in z −plane is the same as the angle of intersection of the curves of w −plane at P' in magnitude and same then the transformation is called conformal.

Conditions: (i) $f(z)$ is analytic (ii) $f'(z) \neq 0$ or

If the sense of the rotation as well as the magnitude of the angle is preserved, the transformation is said to be **conformal.**

If only the magnitude of the angle is preserved, transformation is **Isogonal**.

Theorem: If $f(z)$ is analytic, mapping is conformal

Proof: Let C_1 and C_2 be the two curves in the z −plane intersecting at the point z_0 and let the tangents at this point make angles α_1 and α_2 with the real axis, Let z_1 and z_2 be the points on the curves C_1 and C_2 near to z_0 at the same distance r fromz_0, so that we have

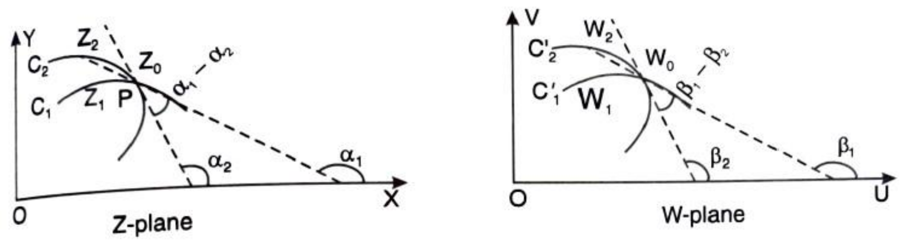
$z_1 - z_0 = re^{i\theta_1}, \quad z_2 - z_0 = re^{i\theta_2}$

As $\qquad r \to 0, \ \theta_1 \to \alpha_1 \ and \ \theta_2 \to \alpha_2$

Let the image of the curves C_1 and C_2 be C'_1 and C'_2 in w −plane and image of z_0, z_1 and z_2 be w_0, w_1 and w_2. Let

$w_1 - w_0 = re^{i\phi_1}, \quad w_2 - w_0 = re^{i\phi_2}$

$$f'(z_0) = \lim_{z_1 - z_0} \frac{w_1 - w_0}{w_2 - w_0}$$



$Re^{i\lambda} = \lim_{r\to 0} \frac{r_1 e^{i\phi_1}}{re^{i\theta_1}}$ $\qquad$ (since, $f'(z_0) = Re^{i\lambda}$)

$$Re^{i\lambda} = \frac{r_1}{r} e^{i\phi_1 - i\theta_1} = \frac{r_1}{r} e^{i(\phi_1 - \theta_1)}$$

Hence $\lim_{r\to 0} \left[\frac{r_1}{r}\right] = R = |f'(z_0)| \ and$

$\lim\,(\phi_1 - \theta_1) = \lambda$

$\Rightarrow \lim\phi_1 - \lim\theta_1 = \lambda$ or $\beta_1 - \alpha_1 = \lambda \ that \ is \ \beta_1 = \alpha_1 + \lambda$

Similarly, it can be proved $\beta_1 = \alpha_1 + \lambda$ curve C_1 has a definite tangent atw_0 making angles $\alpha_1 + \lambda$ and $\alpha_2 + \lambda$ so curve C_2.

Angle between two curves C'_1 and C'_2

$$= \beta_1 - \beta_2 = (\alpha_1 + \lambda) - (\alpha_2 + \lambda) = (\alpha_1 - \alpha_2)$$

So the transformation is conformal at each point where $f'(z) = 0$.

Note 1: The point at which $f'(z) = 0$ is called a **critical point** of the transformation. Also the points where $\frac{dw}{dz} \neq 0$ are called **ordinary points**.

2. Let $\phi = \alpha_1 - \alpha_2$ or $\alpha_1 = \alpha_2 + \phi$ shows that the tangent at P to the curve is rotated through an $\angle\phi = amp\{f'(z)\}$ under the given transformation.

Angle of rotation = $tan^{-1}\frac{v}{u}$

3. In formal transformation, element of are passing through P is magnified by the factor $|f'(z)|$. The area element is also magnified by the factor $|f'(z)|$ or $J = \frac{\partial(u,v)}{\partial(x,y)}$ in a conformal transformation.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2 = |f'(x+iy)|^2$$

$|f'(z)|$ is called the coefficient of magnification.

4. Conjugate functions remain conjugate functions after conformal transformation. A function which is the solution of Laplace's equation, its transformed function again remains the solution of Laplace's equation after conformal transformation.

Theorem: An analytic function $f(z)$ ceases to be conformal at the points where $f'(z) = 0$

Proof: Let $f'(z) = 0$ and $f'(z_0) = 0$ at $z = z_0$

Suppose that $f'(z_0)$ has a zero of order $(n-1)$ at the point z_0, then first $(n - 1)$ derivative of $f(z)$ vanish at z_0 but f"'(z_0) ≠0, hence at any point z in the neighbourhood of z_0, we have by Taylor's Theorem.

$$f(z) = f(z_0) + a_n(z-z_0)^n + \cdots ..$$

Where, $a_n = \frac{f^n(z_0)}{n!}$. So, that $a_n \neq 0$

Hence, $\quad f(z_1) - f(z_0) = a_n(z_1 - z_0)^n + \cdots$

i.e. $\quad w_1 - w_0 = a_n(z_1 - z_0)^n + \cdots$

or $\quad \rho_1 e^{1\phi_1} = |a_n|.r^n e^{i(n\theta_1+\lambda)} + \cdots \quad$ where $\lambda = amp\ a_n$

hence $\quad lim\phi_1 = lim(n\theta_1 + \lambda) = n\alpha_1 + \lambda$

Similarly, $\quad lim\phi_2 = n\alpha_2 + \lambda$

Thus the curves γ_1 and γ_2 still have definite tangents at w_0

But the angle between the tangents = $lim\phi_1 - lim\phi_2 = n(\alpha_2 - \alpha_1)$.

So magnitude of the angle is not preserved.

Also the linear magnification R = $Lim(\rho_1/r) = 0$

Hence, the conformal property does not hold good at a point where $f'(z) = 0$

Example 3: If $u = 2x^2 + y^2$ and $v = \frac{y^2}{x}$, show that the curves $u = constant$ and $v = constant$ cut orthogonally at all intersections but that the transformation $w = u + iv$ is not conformal.

Solution: For the curve $u = 2x^2 + y^2$

$$2x^2 + y^2 = \text{constant} = k_1 \text{ (say)} \ldots\ldots\ldots\ldots(1)$$

Differentiating (1), we get

$$4x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}\ldots..(2)$$

$$\frac{y^2}{x} = v$$

For the curve, $\frac{y^2}{x} = constant = k_2$(say)

$$\Rightarrow \qquad y^2 = k_2 x \qquad \ldots\ldots\ldots(3)$$

Differentiating (3), we get

$$2y\frac{dy}{dx} = k_2$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{k_2}{2y} = \frac{y^2}{x} \times \frac{1}{2y} = \frac{y}{2x}\ldots.(4)$$

From (2) and (4) we see that

$$m_1 m_2 = \left(\frac{-2x}{y}\right)\left(\frac{y}{2x}\right) = -1$$

Hence, two curves cut orthogonally

However, since $\qquad \frac{\partial u}{\partial x} = 4x \qquad \frac{\partial u}{\partial y} = 2y$

and $\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$ $\frac{\partial v}{\partial y} = \frac{2y}{x}$

The Cauchy-Reimann equations are not satisfied by $u \, and \, v$.

Hence, the function $u + iv$ is not analytic. So the transformation is not conformal.

Example 4: (i) for the conformal transformation $w = z^2$, show that

(a) The coefficient of magnification at $z = 2 + i \; is \; 2\sqrt{5}$

(b) The angle of rotation at $z = 2 + i \; is \; tan^{-1} 0.5$

(ii) For the conformal transformation $w = z^2$, show that

(a) The co-efficient of magnification at $z = 1 + i \; is \; 2\sqrt{2}$

(b) The angle of rotation at $z = 1 + i \; is \; \frac{\pi}{4}$.

Solution: (i) Let $w = f(z) = z^2$, $f'(z) = 2z$

$$f'(2+i) = 2(2+i) = 4 + 2i$$

(a) Coefficient of magnification at $z = 2 + i$ is $|f'(2+i)| = |4 + 2i| = \sqrt{16+4} = 2\sqrt{5}$

(b) Angle of rotation at $z = 2 + i$ is $amp\, f'(2+i) = amp.(4 + 2i)$

$$= tan^{-1}\left(\frac{2}{4}\right) = tan^{-1}(0.5)$$

(ii) Here $f(z) = w = z^2$, $f'(z) = 2z$

And $f'(1+i) = 2(1+i) = 2 + 2i$

(a) The co-efficient of magnification at $z = 1 + i$ is $|f'(1+i)| = |2 + 2i| = \sqrt{4+4} = 2\sqrt{2}$

(b) The angle of rotation at $z = 1 + i$ is $amp\, f'(1+i)$

$$= amp.\, 2(1+i) = 2 + 2i = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

Some Standard Transformations:

7.5. Translation $w = z + C$,

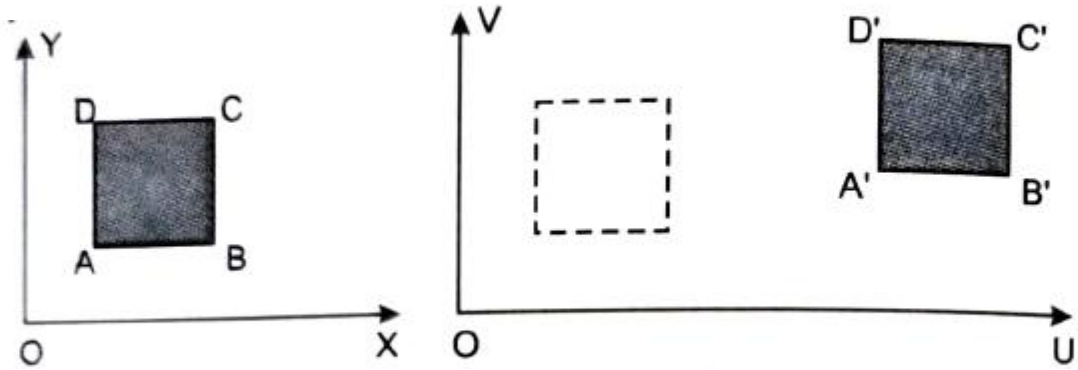
Where $C = a + ib$

$u + iv = x + iy + a + ib$

$u = x + a$ and $v = y + b$ or, $x = u - a$ and $y = v - b$

On substituting the values of x and y in the equation of the curve to be transformed, we get the equation of the image in the w −plane.

The point $P(x, y)$ in the z −plane is mapped onto the point $P' = (x + a, y + b)$ in the w −plane. Similarly other points of z −plane are mapped onto w −plane. Thus if w −plane is superimposed on the z −plane, the figure of w −plane is shifted through a vector C.



In other words the transformation is mere translation of the axes.

7.6. Rotation $w = ze^{i\theta}$

The figure in z −plane rotates through an angle θ in anticlockwise in w −plane.

Example 5: Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w −plane corresponding to the triangular region R bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in z −plane.

Solution: $w = ze^{i\pi/4}$

$$w = (x + iy)\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$$

$$\Rightarrow u + iv = (x + iy)\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}[x - y + i(x + y)]$$

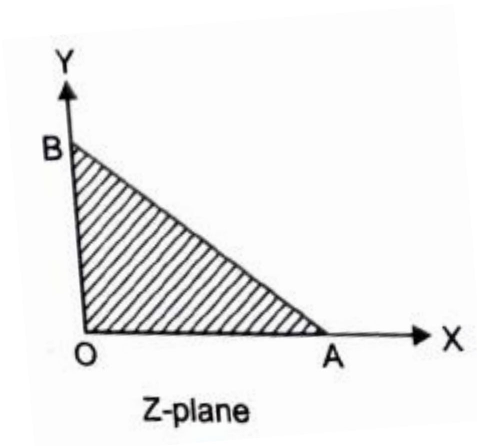
Equating real and imaginary parts, we get

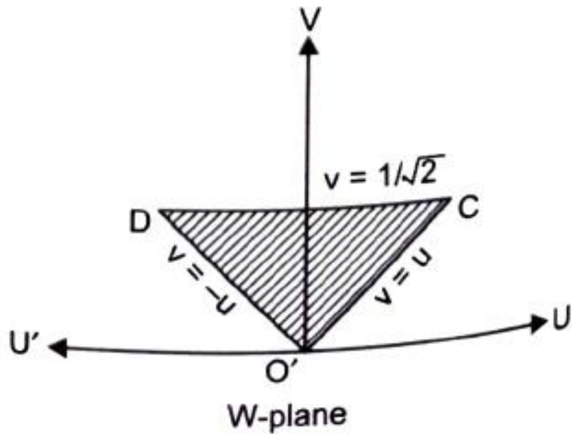
$\Rightarrow \quad u = \frac{1}{\sqrt{2}}(x - y),$ and v$= \frac{1}{\sqrt{2}}(x+y)$

(i) Put $x = 0$ $u = \frac{1}{\sqrt{2}}y,$ v$= \frac{1}{\sqrt{2}}y$ or $v = -u$

(ii) Put $y = 0$ $u = \frac{1}{\sqrt{2}}x,$ v$= \frac{1}{\sqrt{2}}x$ or $v = u$

(iii) Putting $x + y = 1$ in (1), we get v$= \frac{1}{\sqrt{2}}$





Hence the triangular region OAB in z −plane is mapped on a triangular region $O'CD$ of w −plane bounded by the lines $v = u,\ v = -u,\ v = \frac{1}{\sqrt{2}}$

$$f'(z) = \frac{1}{\sqrt{2}}(1+i)$$

$$f(z) = \frac{1}{\sqrt{2}}[(x-y) + i(x+y)$$

Amp. $f'(z) = tan^{-1}(1) = \frac{\pi}{4}$

The mapping $w = ze^{i\pi/4}$ performs a rotation of R through an angle $\frac{\pi}{4}$

7.7. Magnification: $w = cz$, Where c is a real quantity

(1) The figure in $w - p$lane is magnified c-times the size of the figure in z −plane.

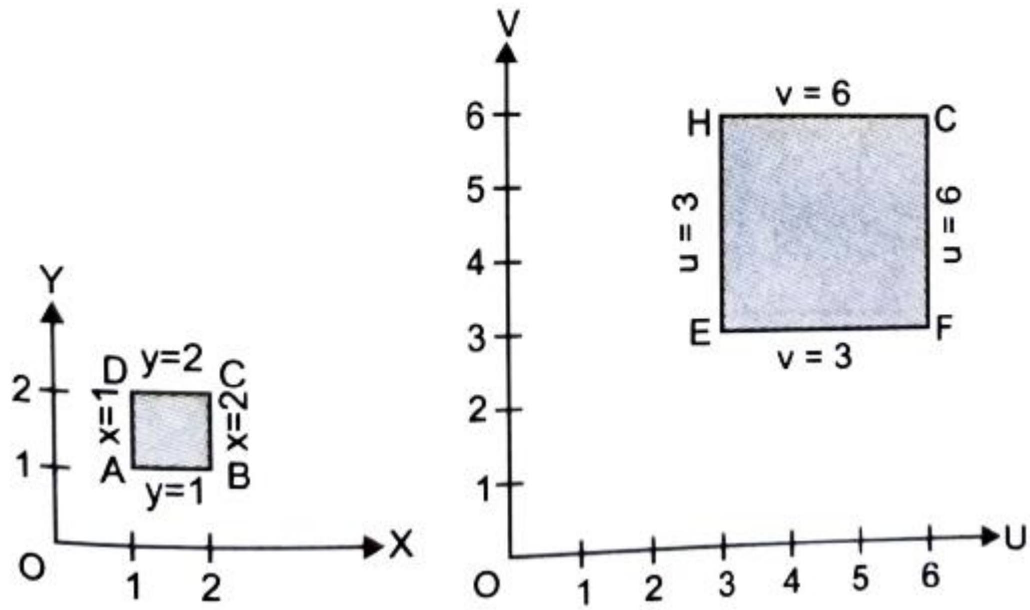
(2) Both figures in z −plane and w −plane are singular.

Example 6: Determine the region in w −plane on the transformation of rectangular region enclosed by $x = 1, y = 1, x = 2$ and $y = 2$ in the z-plane. The transformation is $w = 3z$.

Solution: We have, $w = 3z$ that is $u + iv = 3(x + iy)$

Equating the real and imaginary parts, we get, $u = 3x \quad and\ v = 3y$

$z - plane$		$w - plane$	
x	y	$u = 3x$	$v = 3y$
1	1	3	3
2	2	6	6



7.8. Magnification and Rotation $w = cz$ ....(1)

Where, c, z, w all are complex numbers.

$$c = ae^{i\alpha}, \qquad z = re^{i\theta}, \qquad w = Re^{i\phi}$$

Putting these values in (1), we have

$$Re^{i\phi} = \left(ae^{i\alpha}\right)\left(re^{i\theta}\right) = are^{i(\theta+\alpha)}. \ R = ar \ \text{ and } \ \phi = \theta + \alpha$$

Thus we see that the transform $w = cz$ corresponding to a rotation, together with magnification.

Algebraically $\quad w = cz$

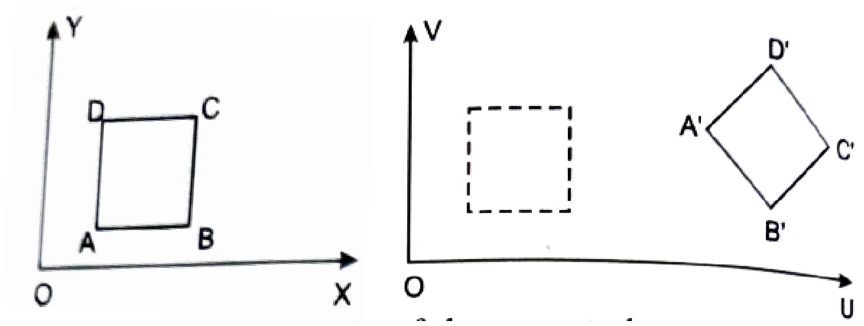
or, $u + iv = (a + ib)(x + iy)$

$\Rightarrow u + iv = ax - by + i(ay + bx)$

or, $u = ax - by \;\; and \;\; v = ay + bx$

On solving these equations, we can get the values of $x \; and \; y$.

$$x = \frac{au+bv}{a^2+b^2}, \quad y = \frac{-bu+bv}{a^2+b^2}$$



On putting values of $x \; and \; y$ in the equation of the curve to be transformed we get the equation of the image.

Example 7: Find the image of the triangle with vertices $at \; i, 1 \, + \, i, \;\; 1 - i$ in the z −plane under the transformation

(i) $w = 3z + 4 - 2i,$

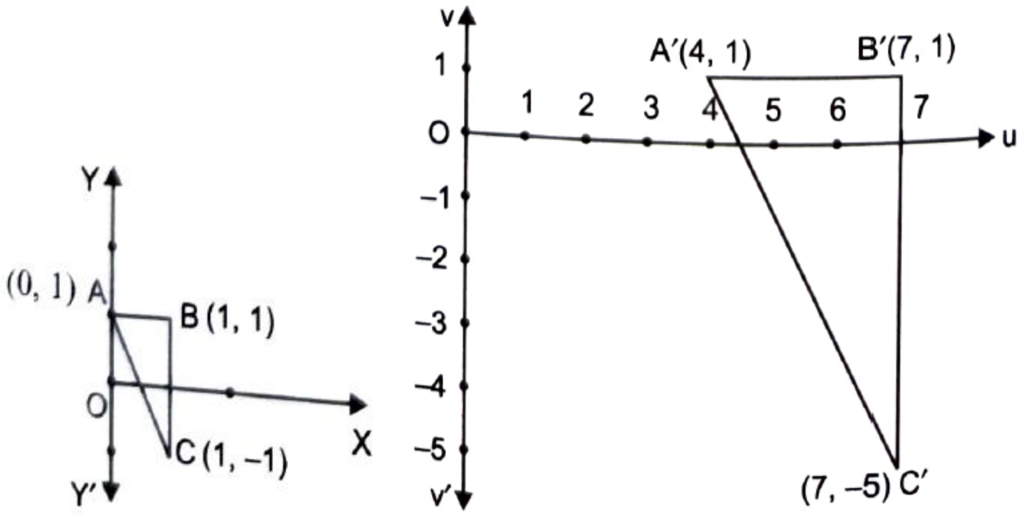
(ii) $w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$

Solution. (i) $w = 3z + 4 - 2i,$

$\Rightarrow u + iv = 3(x + iy) + 4 - 2i$

$\Rightarrow u = 3x + 4, \;\; v = 3y - 2$

	$z - plane$		$w - plane$	
$S.No.$	x	y	$u = 3x + 4$	$v = 3y - 2$
1	0	1	4	1
2.	1	1	7	1
3.	1	-1	7	-5



(ii) $\quad w = e^{\frac{5\pi i}{3}}.z - 2 + 4i$

$$u + iv = \left(cos\frac{5\pi}{3} + isin\frac{5\pi}{3}\right)(x + iy) - 2 + 4i$$

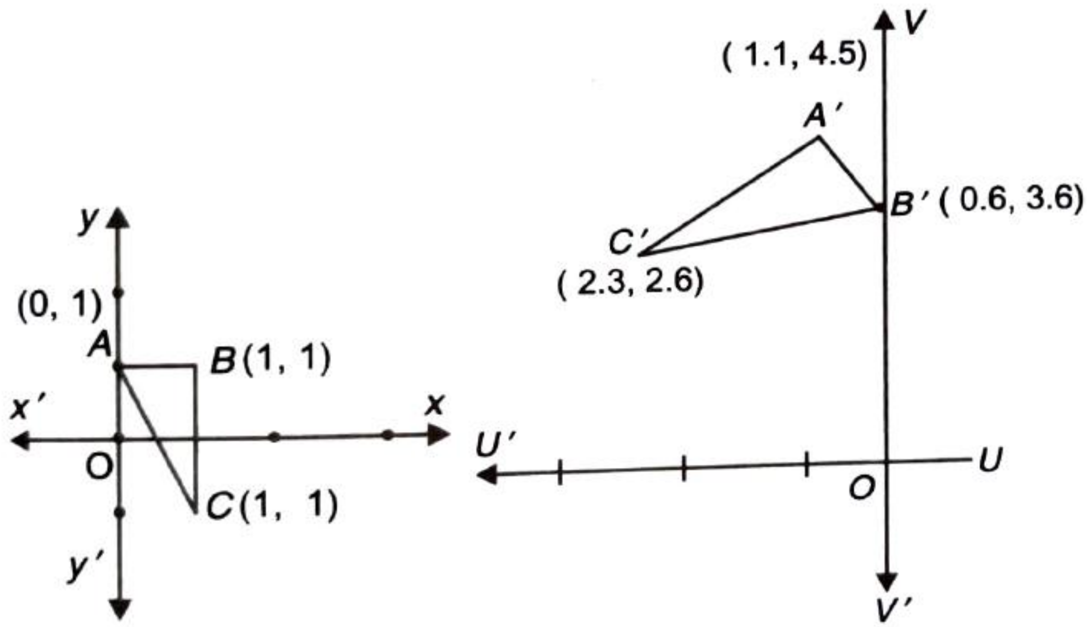
$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(x + iy) - 2 + 4i$$

$$u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y + i\left(-\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4\right)$$

$\Rightarrow u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y$ and $v = -\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4$

$S.$ $No.$	$z - plane$			$w - plane$		
	x	y	$Point$	u	$Points$	v
				$u = \frac{x}{2} - 2 + \frac{\sqrt{3}}{2}y$		$v = -\frac{\sqrt{3}}{2}x + \frac{y}{2} + 4$
1.	0	1	A	$-2 + \frac{\sqrt{3}}{2} = -1.1$	A'	$\frac{9}{2} = 4.5$
2	1	1	B	$-\frac{3}{2} + \frac{\sqrt{3}}{2} = 0.6$	B'	$-\frac{\sqrt{3}}{2} + \frac{9}{2} = 3.6$

3.	1	-1	C	$-\frac{3}{2}-\frac{\sqrt{3}}{2}=-2.3$	C’	$-\frac{\sqrt{3}}{2}+\frac{7}{2}=2.6$



7.9. **Inversion and reflection**: $w = \frac{1}{z}$ .....(1)

If $z = re^{i\theta}$ and $w = Re^{i\phi}$

Putting these values in (1), we get

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

On equating, we get $R = \frac{1}{r}$ and $\phi = -\theta$

Thus the point $P(r,\theta)$ in the z −plane is mapped into the point $P'(1/r, -\theta)$ in the w −plane. Hence the transformation is an inversion of z and followed by reflection into the real axis. The points inside the unit circle ($|z| = 1$) map into points outside it, and points outside the unit circle into points inside it.

Algebraically $w = \frac{1}{z}$ or $z = \frac{1}{w}$

$x + iy = \frac{1}{u+iv} \Rightarrow x + iy = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$

$$x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$$

Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0 \ldots.(1)$ be in z −plane.

On substituting the values of x and y in (1), we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\frac{u}{u^2+v^2} + 2f\frac{(-v)}{u^2+v^2} + c = 0$$

This is the equation of circle in w −plane. This shows that a circle in z −plane transforms another circle in w −plane. But a circle through origin transforms into a straight line.

Example 8: Find the image of $|z - 3i| = 3$ under the mapping $w = \frac{1}{z}$.

Solution: $w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$

$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$ ……….. (1)

The given curve is $|z - 3i| = 3$

$\Rightarrow |x + iy - 3i| = 3 \Rightarrow x^2 + (y-3)^2 = 9$….(2)

On substituting the values of x and y from (1) into (2), we get

$$\frac{u^2}{(u^2+v^2)^2} + \left(-\frac{v}{u^2+v^2} - 3\right)^2 = 9$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{(-v - 3u^2 - 3v^2)^2}{(u^2+v^2)^2} = 9$$

$$\Rightarrow u^2 + (-v - 3u^2 - 3v^2)^2 = 9(u^2+v^2)^2$$

$\Rightarrow u^2 + v^2 + 9u^4 + 9v^4 + 6u^2v + 6v^3 + 18u^2v^2$

$= 9u^4 + 18u^2v^2 + 9v^4$

$\Rightarrow \quad u^2 + v^2 + 6u^2v + 6v^3 = 0$

$\Rightarrow \quad u^2 + v^2 + 6v(u^2 + v^2) = 0$

$(u^2 + v^2)(6v + 1) = 0$

$6v + 1 = 0$ is the equation of the image

Also $|z - 3i| = 3, \quad z = \frac{1}{w}$

Or $\left|\frac{1}{w} - 3i\right| = 3 \Rightarrow |1 - 3iw| = 3|w|$

$|1 - 3i(u + iv)| = 3|u + iv|$

$\Rightarrow |1 - 3iu + 3v)| = 3|u + iv|$

$(1 + 3v)^2 + 9u^2 = 9(u^2 + v^2)$

$\Rightarrow 1 + 6v + 9v^2 + 9u^2 = 9(u^2 + v^2)$

Or, $1 + 6v = 0$

Another method: $|z - 3i| = 3$

$\Rightarrow z - 3i = 3e^{i\theta}$

$\Rightarrow z = 3i + 3e^{i\theta}$

$w = \frac{1}{z} = \frac{1}{3i + 3e^{i\theta}} \Rightarrow 3w = \frac{1}{i + e^{i\theta}}$

$3(u + iv) = \frac{cos\theta - i(1 + sin\theta)}{cos^2\theta + (1 + sin\theta)^2} \Rightarrow 3v = -\frac{1 + sin\theta}{2 + 2sin\theta} = -\frac{1}{2}$ Ans.

Example 9: Image of $|z + 1| = 1$ under the mapping $w = \frac{1}{z}$ is

(a) $2v + 1 = 1$ (b) $2v - 1 = 0$ (c) $2u + 1 = 0$ (d) $2u - 1 = 0$

Solution: $w = \frac{1}{z}$

$\Rightarrow u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$\Rightarrow u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$

Given $|z+1| = 1 \Rightarrow |x+iy+1| = 1$

$\Rightarrow (x+1)^2 + y^2 = 1$

$\Rightarrow x^2 + y^2 + 2x = 0$

$\Rightarrow x^2 + y^2 = -2x$

$\Rightarrow \frac{1}{2} = \frac{-x}{x^2+y^2} = -u$

$\Rightarrow \frac{1}{2} = -u \Rightarrow 2u + 1 = 0.$

Hence c is correct answer.

Example 10: Show that under the transformation $w = \frac{1}{z}$ the image of the hyperbola $x^2 + y^2 = 1$ is the lemniscate $R = cos2\phi$.

Solution: $x^2 + y^2 = 1$

Putting $x = rcos\theta$ $\quad and \quad$ $y = rsin\theta$

$\Rightarrow r^2cos^2\theta - r^2sin^2\theta = 1$

$\Rightarrow \quad r^2(cos^2\theta - sin^2\theta) = 1$

$$r^2cos2\theta = 1 \qquad \ldots\ldots\ldots\ldots (1)$$

And $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\Rightarrow re^{i\theta} = \frac{1}{Re^{i\phi}} \Rightarrow re^{i\theta} = \frac{1}{R}e^{-i\phi}$

$\Rightarrow\ r = \frac{1}{R}$ and $\theta = -\phi$

Putting the values of r and θ in (1), we get

$$\frac{1}{R^2}cos2(-\phi) = 1 \Rightarrow R^2 = cos2\phi$$

Check your progress

1. Find the image of the semi infinite, strip $x > 0, 0 < y < 2$ under the transformation $w = 1 + iz$
2. Determine the region in the w-plane in which the rectangle bounded by the lines $x = 0, y = 0, x = 2$ and $y = 1$ is mapped under the transformation $w = \sqrt{2}e^{i\pi/4}z$.
3. Show that the condition for transformation $w = a^2 + blcz + d$ to make the circle $|w| = |c|$ respond to a straight line in the z −plane is (a) = (c).
4. What is the region of the w-plane in two ways the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - c)$.
5. For the mapping $w(z) = \frac{1}{z}$, find the image of the family of circle $x^2 + y^2 = ax$, where a is real.
6. Show that the function $w = \frac{4}{z}$ transforms the straight line $x = c$ in the z −plane into a circle in the w −plane.
7. If $(w + 1)^2 = \frac{4}{z}$, then prove that the unit circle in the w-plane corresponds to a parabola in the z-plane, and the inside of the circle to the outside of the parabola.
8. Find the image of $|z - 2i| = 2$ under the mapping $w = \frac{1}{z}$.
9. Determine the region in the z-plane by $4 < |z + i| < 8$.

7.10. Bilinear transformation (Mobius Transformation): $w = \frac{az+b}{cz+d}$ ……(1)

is known as **bilinear transformation i**f then $\frac{dw}{dz} \neq 0$ i.e. transformation is conformal.

From (1), $z = \frac{-dw+b}{cw-a}$. This is also bilinear except $w = \frac{a}{c}$

Note: From (1) every point of z −plane is mapped into unique point in w −plane except $z = \frac{b}{c}$

From (2) every point of w −plane is mapped into unique point in z-plane except $w = \frac{a}{c}$

7.11. Invariant points of Bilinear Transformation:

We know that $w = \frac{az+b}{cz+d}$ ………(1)

If z maps into itself, then $w = z$

(1) becomes $z = \frac{az+b}{cz+d}$ ……….(2)

Roots of (2) are the invariants of fixed points of the bilinear transformation.

If the roots are equal, the bilinear transformation is said to be parabolic.

Cross Ratio: If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called cross ratio of z_1, z_2, z_3, z_4.

Theorem: A bilinear transformation preserves cross ratio of four points.

Proof: We know that $w = \frac{az+b}{cz+d}$.

As w_1, w_2, w_3, w_4 are image of z_1, z_2, z_3, z_4

$w_1 = \frac{az_1+b}{cz_2+d}$, $w_2 = \frac{az_2+b}{cz_2+d}$, $w_3 = \frac{az_3+b}{cz_3+d}$ and $w_4 = \frac{az_4+b}{cz_4+d}$

$w_1 - w_2 = \frac{(ad-bc)}{(cz_1+d)(cz_2+d)}(z_1 - z_2)$…………………(1)

Similarly $w_2 - w_3 = \frac{(ad-bc)}{(cz_2+d)(cz_3+d)}(z_2 - z_3)$…………………(2)

$w_3 - w_4 = \frac{(ad-bc)}{(cz_3+d)(cz_4+d)}(z_3 - z_4)$…………………(3)

$w_4 - w_1 = \frac{(ad-bc)}{(cz_4+d)(cz_1+d)}(z_4 - z_1)$…………………(4)

From (1), (2), (3) and (4), we have

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$\Rightarrow$ $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

7.12. Properties of bilinear transformation:

1. A bilinear transformation maps circles into circles.

2. A bilinear transformation preserves cross ratio of four points.

If four points $z_1,\ z_2,\ z_3,\ z_4$ of the z −plane map onto the points w_1, w_2, w_3, w_4 of the w −plane respectively.

$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Hence under the bilinear transform of four points cross ratio is preserved.

7.13. Methods to find bilinear transformation:

1. by finding a, b, c, d for $w = \frac{az+b}{cz+d}$ with the given conditions.

2. with the help of cross-ratio

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Example 11: Find the bilinear transformation which maps the points $z\ =\ 1$, i, -1 into the points $w\ =\ i,, 0, -i$. Hence find the image of $|z| < 1$.

Solution: Let the required transformation be $w = \frac{az+b}{cz+d}$

Or, $w = \frac{\frac{a}{d}z+\frac{b}{d}}{\frac{c}{d}z+1}\ =\ \frac{pz+q}{rz+1}$ …………….. (1) $\left[p = \frac{a}{d}, q = \frac{b}{d},\ r = \frac{c}{d}\right]$

On substituting the values of z and corresponding values of w in (1), we get

$i = \frac{p+q}{r+1} \Rightarrow p + q = ir + i$ …….. (2)

$0 = \frac{pi+q}{ri+1} \quad \Rightarrow pi + q = 0$ ……. (3)

$-i = \frac{-p+q}{-r+1} \Rightarrow -p + q = ir - i$ (4)

z	w
1	i
i	0
-1	i

On subtracting (4) from (2), we get $2p = 2i$ or $p = i$

On putting the value of p in (3), we have $i(i) + q = 0$ or $q = 1$

On substituting the values of p and q in (2), we obtain

$$i + 1 = ir + i \quad or\ 1 = ir \quad or\ r = -i$$

by using the values of p, q, r and (1), we have $w = \frac{iz+1}{-iz+1}$

$$u + iv = \frac{i(x+iy)+1}{-i(x+iy)+1} = \frac{(ix-y+1)(ix+y+1)}{(-ix+y+1)(ix+y+1)} = \frac{-x^2-y^2+1+2ix}{x^2+(y+1)^2}$$

Equating real and imaginary parts, we get $u = \frac{-x^2-y^2+1}{x^2+(y+1)^2}$ and $v = \frac{2x}{x^2+(y+1)^2}$

But $|z| < 1 \Rightarrow x^2 + y^2 < 1 \Rightarrow 1 - x^2 - y^2 > 0$

From (5) $u > 0$ as denominator is positive

Example 12: Find a bilinear transformation which maps the points $i, -i, 1$ of the z −plane into $0, 1, \infty$ of the w −plane respectively.

Example 13: Find the bilinear transformation which maps the points $z = 0, -1, i$ into

$w = i, 0, \infty$. Also find the image of the unit circle $|z| = 1$.

Solution: On putting $z = 0, -1, i$ into $w = i, 0, \infty$ respectively in

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-w_1)(\frac{w_2}{w_3}-1)}{(\frac{w}{w_3}-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-i)(-1)}{(-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)} \Rightarrow \left(\frac{w-i}{-i}\right) = \frac{z(1+i)}{z-i}$$

$$\Rightarrow w - i = \frac{(-i+1)z}{z-i} \Rightarrow w = \frac{(1+i)z}{z-i} + i = \frac{(1-i)z+iz+1}{z-i} \Rightarrow \quad \mathrm{w} = \frac{z+1}{z-i} \quad \ldots\ldots.(2)$$

From (2), $z = \frac{iw+1}{w-1}$….. (3) [Inverse transformations is $z = \frac{-dw+b}{cw-a}$]

And $|z| = 1 \Rightarrow \left|\frac{iw+1}{w-1}\right| = 1 \Rightarrow |1+iw| = |w-1|$

$$\Rightarrow |1 + i(u+iv)| = |u + iv - 1|$$

$$\Rightarrow |1 - v + iu| = |u - 1 + iv| \Rightarrow (1+v)^2 + u^2 = (u-1)^2 + v^2$$

$$\Rightarrow 1 + v^2 - 2v + u^2 = u^2 + 1 - 2u + v^2 \Rightarrow u - v = 0 \Rightarrow v = u$$

Example 14: Find the fixed points and the normal form of the following bilinear transformations (a). $w = \frac{3z-4}{z-1}$ and (b) $w = \frac{z-1}{z+1}$. Discuss the nature of these transformations.

Solution: (a) The fixed points are obtained by

$z = \frac{3z-4}{z-1}$ or, $z^2 - 4z + 4 = 0$ or, $(z-2)^2 = 0 \Rightarrow$ z = 2

$Z = 2$ is the only fixed point. This transformation is parabolic.

Normal form:

$$w = \frac{3z-4}{z-1} \Rightarrow \frac{1}{w-2} = \frac{1}{\frac{3z-4}{(z-1)} - 2} = \frac{z-1}{3z-4-2z+2} = \frac{z-1}{z-2}$$

$\Rightarrow \frac{1}{w-2} = \frac{1}{z-2}+1$

Example 15: Show that $w = \frac{i-z}{i+z}$ maps the real axis of the z −plane into (i). The circle $|w| = 1$ and (ii) the half plane $y > 0$ into the interior of the unit circle $|w| < 1$ in the w −plane.

Solution: We have $w = \frac{i-z}{i+z}$

$$|\omega| = \left|\frac{i-z}{i+z}\right| = \frac{|i-z|}{|i+z|} = \frac{|i-x-iy|}{|i+x+iy|}$$

$$\omega = \left|\frac{-x+i(1-y)}{x+i(1+y)}\right|, \quad \omega = \frac{\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+(1+y)^2}}$$

Now the real axis in z −plane i.e., $y = 0$ transform into

$$|\omega| = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1, \ \ |\omega| = 1$$

Hence the real axis in the z −plane is mapped into the circle, $|\omega| = 1$.

(ii) The interior of the circle i.e., $|\omega| < 1$ gives.

$$\frac{\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+(1+y)^2}} < 1$$

$$\Rightarrow \frac{x^2+(1-y)^2}{x^2+(1+y)^2} < 1 \Rightarrow x^2 + (1-y)^2 < x^2 + (1+y)^2$$

$$1 + y^2 - 2y < 1 + y^2 + 2y \Rightarrow -4y < 0, \qquad \Rightarrow y > 0$$

Thus the upper half of the z −plane corresponds to the interior of the circle $|w| = 1$.

Example 16: Show that the transformation $w = \frac{3-z}{z-2}$ transforms the circle with centre$\left(\frac{5}{2}, \ 0\right)$ and radius ½ in the z −plane into the imaginary axis in the w-plane and the interior of the circle into the right half of the plane.

Solution: $w = \frac{3-z}{z-2} \Rightarrow u + iv = \frac{3-x-iy}{x+iy-2} \Rightarrow (u+iv)(x+iy-2) = 3 - x - iy$

$\Rightarrow ux + iuy - 2u + ivx - vy - 2iv = 3 - x - iy$

$\Rightarrow ux - 2u - vy + i(uy + vx - 2v) = 3 - x - iy$

Equating real and imaginary quantities, we have

$ux - vy - 2u = 3 - x$ and $vx + (u+1)y = 2v$

$\Rightarrow (u+1)x - vy = 2u + 3$ and $vx + (u+1)y = 2v$

On solving the equation for x and y, we have

$x = \frac{2u^2+2v^2+5u+3}{u^2+v^2+2u+1}$, $\quad y = \frac{-v}{u^2+v^2+2u+1}$

Here, the equation of the given circle is $\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{1}{4}$ ………………(1)

Putting the values of x and y in (1), we have

$$\left(\frac{2u^2 + 2v^2 + 5u + 3}{u^2 + v^2 + 2u + 1} - \frac{5}{2}\right)^2 + \left(\frac{-v}{u^2 + v^2 + 2u + 1}\right)^2 = \frac{1}{4}$$

$\Rightarrow \left(\frac{-u^2-v^2+1}{2(u^2+v^2+2u+1)}\right)^2 + \left(\frac{-v}{u^2+v^2+2u+1}\right)^2 = \frac{1}{4}$

$\Rightarrow (-u^2 - v^2 + 1)^2 + 4v^2 = (u^2 + v^2 + 2u + 1)^2$

$\Rightarrow (u^2 + v^2 - 1)^2 + 4v^2 = [(u^2 + v^2 - 1) + (2u + 2)]^2$

$\Rightarrow (u^2 + v^2 - 1)^2 + 4v^2 = (u^2 + v^2 - 1) + (2u + 2)^2 + 2(u^2 + v^2 - 1)(2u + 2)$

$\Rightarrow \qquad v^2 = (u+1)^2 + (u^2 + v^2 - 1)(u + 1)$

$\Rightarrow \qquad v^2 = u^2 + 2u + 1 + u^3 + uv^2 - u + u^2 + v^2 - 1$

$\Rightarrow \qquad 0 = u^3 + 2u^2 + u + uv^2 \Rightarrow u(u^2 + 2u + 1 + v^2) = 0$

$\Rightarrow u = 0$ i.e. equation of imaginary axis.

Equation of the interior of the circle is $\left(x - \frac{5}{2}\right)^2 + y^2 < \frac{1}{4}$.

Then corresponding equation in u, v, is

$u(u^2 + 2u + 1 + v^2) > 0$ Or, $u[(u + 1)^2 + v^2] > 0$

As $(u + 1)^2 + v^2 > 0$, so, u > 0 i.e. equation of the right half plane.

7.14. Inverse point with respect to a circle:

Two points P and Q are said to be the **inverse points** with respect a circle S if they are collinear with the centre C on the same side of it, and if the product of their distances from the centre is equal to $r^2$where r is the radius of the circle.

Thus when P and Q are the inverse points of the circle, then the three points C, P, Q are collinear, and also $CP.CQ = r^2$

Example 17: Show that the inverse of a point a, with respect to the circle $|z - c| = R$ is the point $(c + \frac{R^2}{\bar{a}-\bar{c}})$

Solution: Let b be the inverse point of the point a' with respect to the circle $|z - c| = R$.

Condition I: The points a, b, c are collinear. Hence $\arg(\bar{b} - \bar{c}) = \arg(\bar{a} - \bar{c}) = -\arg(\bar{a} - \bar{c})$

$\Rightarrow \arg(\bar{b} - \bar{c}) = \arg(\bar{a} - \bar{c}) = 0$ or, $\arg(\bar{b} - \bar{c})(\bar{a} - \bar{c}) = 0$

$\therefore (\bar{b} - \bar{c})(\bar{a} - \bar{c})$ is real, so that, $(\bar{b} - \bar{c})(\bar{a} - \bar{c}) = |(\bar{b} - \bar{c})(\bar{a} - \bar{c})|$

Condition II: $|\bar{b} - \bar{c}||\bar{a} - \bar{c}| = R^2 \Rightarrow |\bar{b} - \bar{c}||\bar{a} - \bar{c}| = R^2$

$\Rightarrow |(\bar{b} - \bar{c})(\bar{a} - \bar{c})| = R^2 \Rightarrow (\bar{b} - \bar{c})(\bar{a} - \bar{c}) = R^2 \Rightarrow \bar{b} - \bar{c} = \frac{R^2}{\bar{a}-\bar{c}}$

$\Rightarrow b = c + \frac{R^2}{\bar{a}-\bar{c}}$

Example 18: Find a Mobius transformation which maps the circle $|w| \leq 1$ into the circle $|z - 1| < 1$ and maps $w = 0, w = 1$ respectively into $z = \frac{1}{2}, z = 0$.

Solution: Let the transformation be, $w = \frac{az+b}{cz+d}$ ……..(1)

Since, $w = 0$ maps into $z = \frac{1}{2}$

From (1) we get

z	w
½	0
0	1

$$0 = \frac{\frac{a}{2}+b}{\frac{c}{2}+d} \quad \Rightarrow \frac{a}{2} + b = 0 \text{ ....(2)}$$

Since $w = 1$ maps into $z = 0$, from (1), we get

$$1 = \frac{0+b}{0+d} \Rightarrow \quad b = d\text{...........(3)}$$

Here $|w| = 1$ corresponding to $|z - 1| = 1$

Therefore points w, $\frac{1}{w}$ inverse with respect to the circle $|w| = 1$ correspond to the points z, $1 + \frac{1}{z-1}$ inverse with respect to the circle $|z - 1| = 1$.

[z and $a + \frac{R^2}{z-a}$ are the inverse points on the circle$|z - 1| = R$]

Particular $w = 0$ and α correspond to $z = \frac{1}{2}, 1 + \frac{1}{\frac{1}{2}-1} \Rightarrow z = \frac{1}{2}, -1$

Since, $w = 0$maps into $z = -1$ from (1), we get $w = \frac{-a+b}{-c+d} \Rightarrow -\mathrm{c} + \mathrm{d} = 0 \Rightarrow \mathrm{c} = \mathrm{d}$......(4)

From (2), (3), and (4) we get $b = -\frac{a}{2}, \; b = c = d$

From (1), $w = \frac{az+b}{cz+d} = \frac{-2bz+b}{bz+b} = \frac{-2z+1}{z+1}$.

Example 19: Show that the transformation $w = \frac{5-4z}{4z-2}$ transform the circle $|z| = 1$ into a circle of radius unity in $w -$ plane and find the centre of the circle.

Solution: Here. $w = \frac{5-4z}{4z-2}$

$\Rightarrow z = \frac{2w+5}{4w+4} \Rightarrow |z| = \left|\frac{2w+5}{4w+4}\right| = 1$

$\Rightarrow |2w + 5| = |4w + 4|$

$\Rightarrow |2u + 5 + 2iv| = |4u + 4 + 4iv| \quad [\because w = u + iv]$

$\Rightarrow (2u+5)^2 + 4v^2 = (4u+4)^2 + (4v)^2$ ....(1)

$\Rightarrow 4u^2 + 25 + 20u + 4v^2 = 16u^2 + 16 + 32u + 16v^2$

$\Rightarrow 12u^2 + 12v^2 + 12u - 9 = 0 \Rightarrow u^2 + v^2 + u - \frac{3}{4} = 0$ ....(2)

This is the equation of circle in $w -$ plane.

Now we have to find its centre

$u^2 + v^2 + 2gu + 2fv + c = 0$........(3)

From (2) and (3) $\quad g = \frac{1}{2}, \quad f = 0, \quad c = \frac{3}{4}$

Centre is $(-g, -f)$ i.e. $\left(-\frac{1}{2},\ 0\right)$ and Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{\frac{1}{4} + 0 + \frac{3}{4}} = 1$

Thus (2) is circle with its centre at $\left(-\frac{1}{2},\ 0\right)$ and of radius unity in w −plane.

Example 20: Find the image of $x^2 + y^2 - 4y + 2 = 0$ under the mapping $w = \frac{z-i}{iz-1}$

Solution: $\quad w = \frac{z-i}{iz-1} \Rightarrow w(iz-1) = z - i$ that is $x^2 + y^2 - 4y + 2 = 0$ ....(1)

$\Rightarrow z = \frac{w-i}{iw-1} \Rightarrow x + iy = \frac{u+i(v-1)}{iu-(v+1)}$ ....(2)

$\Rightarrow x - iy = \frac{u-i(v-1)}{-iu-(v+1)}$ ....(3)

Multiplying (2) and (3) we get $\Rightarrow x^2 + y^2 = \frac{u^2+(v-1)^2}{u^2+(v+1)^2}$ .... (4)

Subtracting (3) from (2), we get $2iy = \frac{-2iu^2-2i(v^2-1)}{u^2+(v+1)^2}$ ....(5)

Putting the values of $x^2 + y^2$ and y in (1), we get $\frac{u^2+(v-1)^2}{u^2+(v+1)^2} + 4\frac{u^2+(v^2-1)}{u^2+(v+1)^2} + 2 = 0$

$\Rightarrow u^2 + (v-1)^2 + 4[u^2 + (v^2 - 1)] + 2[u^2 + (v+1)^2] = 0$

$\Rightarrow 7(u^2+v^2)+2v-1=0$. This is the image of $x^2+y^2-4y+2=0$ under the mapping $= \frac{z-i}{iz-1}$.

Check your progress

1. Find the bilinear transformation that maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the points $w_1 = 1, w_2 = i, and\ w_3 = -1$ respectively. Ans. $w = \frac{3z+2i}{iz+6}$
2. Determine the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = i, w_2 = -1, w_3 = -i$ respectively. Ans. $w = \frac{z-i}{iz-1}$
3. Verify that the equation $w = \frac{1+iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half plane $v > 0$.
4. Find the bilinear transformation which maps $1, i, -1\ to\ 2, i, -2$ respectively. Find the fixed and critical points of the transformation. Ans. i, 2i
5. Show that the transformation $w = \frac{i(1-z)}{1+z}$ maps the circle $|z| = 1$ into the real axis of the w −plane and the interior of the circle $|z| < 1$ into the upper half of the w −plane.
6. Show that the transformation $w = \frac{iz+2}{4z+i}$ transforms the real axis in the z −plane into circle in the w −plane. Find the centre and the radius of this circle. Ans. $\left(0, \frac{7}{8}\right), \frac{9}{8}$
7. If z_0 is the upper half of the z-plane show that the bilinear transformation $w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right)$ maps the upper half of the z −plane into the interior of the unit circle at the origin in the w −plane.
8. Find the condition that the transformation $w = \frac{az+b}{cz+d}$ transforms the unit circle in the w −plane into straight lines the z −plane.
9. Prove that $w = \frac{z}{z+i}$ maps the upper half of the z −plane into the upper half of the w −plane. What is the image of the circle $|z| = 1$ under this transformation?
10. Show that the map of the real axis of the z −plane on the w −plane by the transformation $w = \frac{1}{z-i}$ is a circle and find its centre and radius.

11. Find the invariant points of the transformation $w = -\left(\frac{2z+4i}{iz+1}\right)$. Prove also that these two points together with any point z and its image w, form a set of four points having a contant cross ratio.

12. Show that under the transformation $w = \frac{z-i}{z+i}$ the real axis in z −plane is mapped into the circle $|w| = 1$. What portion of the z −plane corresponds to the interior of the circle? (Ans. the half z −plane above the real axis corresponds to the interior of the circle $|w| = 1$.)

13. Discuss the application of the transformation $w = \frac{iz+1}{z+i}$ to the areas in the z −plane which are respectively inside and outside the unit circle with its centre at the origin.

14. What is the form of a bilinear transformation which has one fixed point α and the other fixed point ∞?

Choose the correct alternative:

The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are

(a) $-\frac{4}{5},\ -5$ (b) 2, 2 (c) -2, -2 (d) 2, -2

The invariant points of the bilinear transformation are

(a) $1 \mp 2i$ (b) $-1 \mp 2i$ (c) $\mp 2i$ (d) invariant point does not exist

7.15. Transformation $w = z^2$

Solution: $w = z^2$ that is $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

Equating real and imaginary parts, we get $u = x^2 - y^2, \quad v = 2xy$

(i)(a). Any line parallel to x −axis, i.e., $y = c$, maps into

$u = x^2 - c^2, \quad v = 2cx$

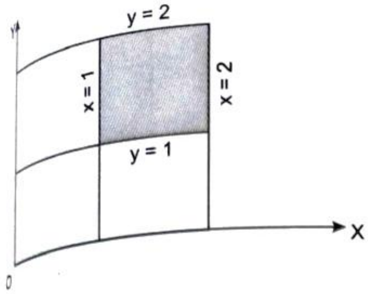
Eliminating, x, we get $v^2 = 4c^2(u + c^2)$……. (1) which is a parabola.

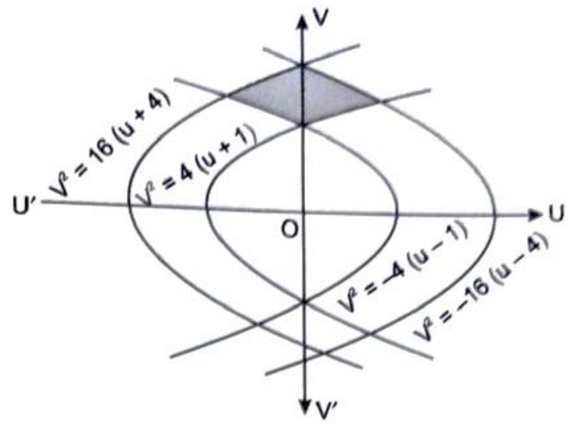
(b). Any line parallel to y −axis, i.e., $x = b$, maps into a curve

$u = b^2 - y^2, \quad v = 2by$

Eliminating, y, we get $v^2 = -4b^2(u - b^2)$, …….. (2) which is a parabola

(c). The rectangular region bounded by the lines $x = 1, x = 2$ and $y = 1, y = 2$ maps into the region bounded by the parabolas.





By Putting $x = 1 = b$ in (2) we get $v^2 = -4(u - 1)$

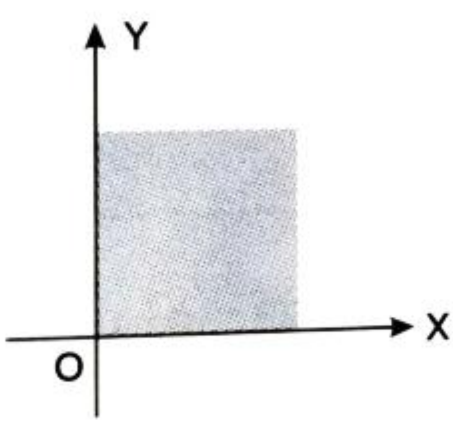
By putting $x = 2 = b$ in (2) we get $v^2 = -16(u - 4)$

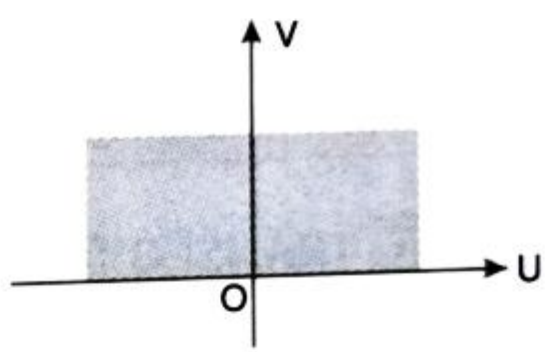
By putting $y = 1 = c$ in (1) we get $v^2 = 4(u + 1)$

By putting $y = 2 = c$ in (1) we get $v^2 = 16(u + 4)$

(ii) (a). In polar co-ordinates: $z = re^{i\theta}$,, $w = Re^{i\theta}$

$w = z^2$, $Re^{i\theta} = r^2e^{2i\theta}$





Then $R = r^2$, $\emptyset = 2\theta$

In z −plane a circle $r = a$ maps $R = a^2$ in w −plane.

Thus, circles with centre at origin map into circles with at the origin.

(b). if $\theta = 0, \emptyset = 0$, i.e. real axis in z −plane maps into real axis in w −plane.

If $\theta = \frac{\pi}{2}$, $\emptyset = \pi$, i.e. the positive imaginary axis in z −plane maps into negative real axis in same. Thus, the first quadrant in z −plane $0 \leq \theta \leq \frac{\pi}{2}$, maps into upper half of w − plane $0 \leq \emptyset \leq \pi$.

The angle in z − plane at origin maps into double angle in w −plane at origin.

Hence the mapping $w = z^2$ is not conformal at the origin.

It is conformal in the entire z −plane except origin. Since, $\frac{dw}{dz} = 2z = 0$ for $z = 0$, therefore critical point of mapping.

Example 21: For the conformal transformation $w = z^2$. Show that the circle $|z - 1| = 1$ transforms into the cardioid $R = 2(1 + cos\emptyset)$ where $w = Re^{i\theta}$ in the w −plane.

Solution: $|z - 1| = 1$………………… (1)

Equation (1) represents a circle with centre at (1, 0) and radius 1.

Shifting the pole to the point (1, 0), any point on (1) is $1 + e^{i\theta}$

Transformation is under $w = z^2$.

$Re^{i\emptyset} = (1 + e^{i\theta})^2 = e^{i\theta}(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}})^2 = e^{i\theta}(2\cos(\frac{\theta}{2}))^2 = 4e^{i\theta}cos^2\left(\frac{\theta}{2}\right)$

This gives that $R = 4cos^2\left(\frac{\theta}{2}\right) \Rightarrow \mathrm{R} = 2(2cos^2\left(\frac{\emptyset}{2}\right)) \Rightarrow \mathrm{R} = 2(1 + \cos\emptyset)$

Summary

After studying this unit we will be able to define Mapping and Conformal Mapping ,translation $w = z + c$ and Rotation $w = ze^{i\theta}$, magnification and Rotation $w = cz$, inversion and reflection, Mobius transformation, invariant points of Bilinear transformation, properties of Bilinear transformation, methods to find Bilinear transformation , inverse point with respect to a circle and transformation $w = z^2$.